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Generalised Schwarzschild metric from double copy of point-like charge solution in Born-Infeld theory

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ABSTRACT

We discuss possible application of the classical double copy procedure to construction of a generalisation of the Schwarzschild metric starting from an α'-corrected open string analogue of the Coulomb solution. The latter is approximated by a point-like charge solution of the Born-Infeld action, which represents the open string effective action for an abelian vector field in the limit when derivatives of the field strength are small. The Born-Infeld solution has a regular electric field which is constant near the origin suggesting that corrections from the derivative terms in the open string effective action may be small there. The generalization of the Schwarschild metric obtained by the double copy construction from the Born-Infeld solution looks non-singular but the corresponding curvature invariants still blow up at $r=0$. We discuss the origin of this singularity and comment on possible generalizations.

Contents

1. Introduction

The (classical) double copy is a procedure to construct gravity solutions from gauge theory ones. It originated from the KLT relations in string theory and BCJ duality associated to scattering amplitudes in field theory (for a review see [1]).

A simple example of the double copy is a relation between the Schwarzschild metric and the Coulomb potential $A_\mu = (\phi(r), 0, 0, 0)$, $\phi = Q/r$, created by a point charge [2]. After a gauge transformation we get $A_\mu = \phi(r) k_\mu$ where $k_\mu = (1,\ x_i/r)$ is null. Then the metric in the Kerr-Schild form $g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$ becomes the Schwarzschild metric with mass $M = 2Q$.

So far almost all examples of the double copy started with linear Maxwell fields. The validity and physical origins of the classical double copy construction at the full non-linear, quantum and string theory levels are not clearly understood at present but one might

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speculate that it may extend beyond the leading order in α' and relate exact open-string and closed-string backgrounds. Here we will make a first naive attempt to study such an extension.

Gauge theory equations of motion appear as the leading order approximation to the effective field equations for the massless vector field in the open string theory [3]. The tree-level open string effective action is given, in the abelian case, by the Born-Infeld [4] term $\sqrt{\det(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}$ [5] plus terms depending on derivatives of the field strength $F_{\mu\nu}$ (for a review see [6]). We may attempt to first ignore all derivative corrections and generalize the Maxwell's theory Coulomb solution to its Born-Infeld counterpart [4]. The corresponding electric field is $E = Q/\sqrt{r_0^4 + r^4}$, where $r_0^2 = T^{-1}Q$ and $T = \frac{1}{2\pi\alpha'}$ is string tension. In contrast to the Coulomb case here the field is non-singular at the origin. This may be interpreted as a consequence of the inclusion of the α' corrections that are expected to "regularize" point-like singularities in string theory [7]. Since the field of the Born-Infeld solution is approximately constant near the origin, this suggests that it may be possible to consider it as an approximation to a solution of the full (tree-level) open string effective field equations in the region close to $r = 0$.

One may wonder whether this regular Born-Infeld solution may double-copy to a generalization of the Schwarzschild metric that will also be non-singular at the origin.[2] Making the simplest assumption that the form $g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$ of the standard "leading-order" double copy ansatz is not modified by the α'-corrections, the resulting metric with the potential ϕ corresponding to the regular Born-Infeld solution will look formally non-singular at $r = 0$. However, as we will find below, the corresponding curvature invariants happen to diverge at the origin. This has to do with too slow $\phi \sim r$ decay of the scalar potential at $r \to 0$.

We do not expect this singular α'-dependent double-copy metric to solve a closed-string generalization of the Einstein equations. First, the string-theory generalization of the double copy ansatz may require its non-trivial α'-modification. One may also need to generalize the double copy ansatz to allow for a non-zero dilaton field [8,9] which is expected to be non-trivial for the closed-string generalization of the Schwarzschild solution beyond the leading order in α'. Finally, our use of the Born-Infeld solution as an approximation to the exact open-string solution may be too naive: it is possible that (a resummation of) the derivative corrections in the open-string equations may lead to a subtle modification of the Born-Infeld solution resulting in a non-singular double-copy metric.

This paper is organised as follows. In Section 2 we will discuss the structure of the open string effective action and the Born-Infeld solution that we will use. In Section 3 we will recall how the classical double copy procedure may be applied to get the Schwarzschild metric from the Coulomb potential. In Section 4 we will present the double copy metric corresponding to the Born-Infeld solution and discuss the singularity of the corresponding curvature invariants. Section 5 will contain some concluding remarks. There are also three technical appendices.

2. Open string effective action and the Born-Infeld solution

The effective action for the abelian gauge field in the bosonic open string theory has the following structure [5,10,11] (we consider reduction to 4 dimensions; $T^{-1} \equiv 2\pi\alpha'$)[3]

$$S = c\int d^4x\,\sqrt{-\det(\eta_{\mu\nu}+T^{-1}F_{\mu\nu})}\Big[1 + T^{-3}\,B^{\mu\nu\rho\sigma\lambda\gamma}(T^{-1}F)\,\partial_\mu F_{\nu\rho}\partial_\sigma F_{\lambda\gamma} + \mathcal{O}(\partial^4 F)\Big], \tag{2.1}$$

where the ∂F-independent part is the Born-Infeld action and B is a particular function of the field-strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Explicitly, the leading order α'^5 derivative terms are [11]

$$\begin{aligned} S =& c\int d^4x\Big(\sqrt{-\det(\eta+T^{-1}F)} - \frac{1}{48\pi}\,T^{-5}\Big[(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu})F_{\rho\sigma}F^{\rho\sigma} \\ &+ 8(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\nu\lambda})F_{\lambda\rho}F^{\rho\mu} + 4(\partial_\alpha F_{\mu\nu})(\partial^\beta F^{\mu\nu})F_{\beta\lambda}F^{\alpha\lambda}\Big] + \mathcal{O}(T^{-7})\Big)\,. \end{aligned} \tag{2.2}$$

The resulting equation for $F_{\mu\nu}$ may be written as:

$$\begin{aligned} &2\partial_\mu\Big[\frac{\partial\sqrt{-\det(\eta+T^{-1}F)}}{\partial F_{\mu\nu}}\Big] - \frac{1}{12\pi}T^{-5}\Big[(\partial_\alpha F_{\lambda\gamma})(\partial^\alpha F^{\lambda\gamma})(\partial_\mu F^{\mu\nu}) \\ &\quad + 2(\partial_\mu\partial_\alpha F_{\lambda\gamma})(\partial^\alpha F^{\lambda\gamma})F^{\mu\nu} + 4\partial_\mu\big[(\partial_\alpha F_{\sigma\gamma})(\partial^\mu F^{\sigma\gamma})F^{\alpha\nu}\big] \\ &\quad + 4\partial_\mu\big[(\partial_\alpha F_{\beta\gamma})(\partial^\alpha F^{\gamma\mu})F^{\nu\beta} + (\partial_\alpha F_{\gamma\lambda})(\partial^\alpha F^{\nu\gamma})F^{\lambda\mu}\big]\Big] + \mathcal{O}(T^{-7}) = 0\,. \end{aligned} \tag{2.3}$$

The Born-Infeld equation corresponding to the vanishing of the first term here is equivalent to $(\eta - T^{-2}F^2)^{-1}_{\lambda\mu}\,\partial^\lambda F^{\mu\nu} = 0$.

Ignoring the contributions of the derivative correction terms in (2.2) let us look for a point-like charge solution of the Born-Infeld term in (2.3). In the purely electric case the Born-Infeld part of (2.3) reduces to $\partial_i\Big(E_i/\sqrt{1-T^{-2}E^2}\Big) = 0$. If the electric field is spherically symmetric (corresponding to a point-like charge), i.e. has only the radial component depending on r one finds [4][4]

$$E_r = F_{0r} = -\partial_r A_0(r) = \frac{Q}{\sqrt{r_0^4 + r^4}}\,, \qquad r_0^2 \equiv T^{-1}Q\,. \tag{2.4}$$

[2] One may argue that to discuss a possibility of a double copy for Born-Infeld fields one should be assuming that there exists its non-abelian version that satisfies some form of color/kinematics duality.

[3] In superstring case derivative corrections start with 4-derivative terms.

[4] For some applications of this solution see [12].

In contrast to the standard Coulomb solution the Born-Infeld solution is regular at $r = 0$. Since the electric field (2.4) is approximately constant near $r = 0$, one may hope that at least near the origin this background may be trusted as a solution to the full open string effective action, including the derivative corrections. A further discussion of this point is presented in Appendix A.

Our aim below will be to construct the double copy metric corresponding to the scalar potential in (2.4) that generalizes the Schwarzschild metric which is the double copy of the Coulomb potential.

3. Schwarzschild metric as the double copy of the Coulomb solution

Let us first briefly review the application of the classical double copy procedure to the Schwarzschild solution [2]. The Schwarzschild metric is a particular case of

$$ds^2 = -\big[1-\phi(r)\big]dt^2 + \frac{dr^2}{1-\phi(r)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ , \tag{3.1}$$

with $\phi = 2M/r$. Changing coordinates to $(\bar t, x_i)$, $\bar t \equiv t + 2M\ln(r-2M)$, the Schwarzschild metric can be written in the Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu\ , \qquad k_\mu \equiv \big(1, \frac{x^i}{r}\big)\ , \qquad k_\mu k^\mu = 0\ . \tag{3.2}$$

This may be interpreted as a double copy corresponding to an abelian gauge potential

$$A_\mu = \phi(r)\, k_\mu\ , \qquad \phi = \frac{Q}{r}\ , \tag{3.3}$$

assuming that $Q \equiv 2M$.[5] The potential (3.3) is gauge-equivalent to the Coulomb potential $A_\mu = \phi(1,0,0,0)$

For general $\phi(r)$, the change of coordinates bringing the metric (3.1) to the Kerr-Schild form (3.2) can be found by looking for radial null geodesics of (3.1). Setting $-(1-\phi)dt^2 + \frac{dr^2}{1-\phi} = 0$ gives the following integral representation for t (denoted by $t^*(r)$):

$$t^*(r) = \pm \int \frac{dr}{1-\phi(r)}\ . \tag{3.4}$$

In the Schwarzschild case of $\phi = 2M/r$ this gave $t^*(r) = r + 2M\ln(r-2M)$. The Kerr-Schild form of (3.1) is then obtained by changing from (t, r, θ, ϕ) to $(\bar t, x_i)$ coordinates where x_i are the standard cartesian ones and $\bar t \equiv t - r + t^*(r)$. To perform the change of coordinates it is sufficient to use the differential of $\bar t = t - r + t^*$, i.e.

$$d\bar t = dt + \frac{\phi(r)}{1-\phi(r)}\, dr. \tag{3.5}$$

4. Double copy of the Born-Infeld solution

To construct the classical double copy metric for the Born-Infeld solution in (2.4) we need the corresponding gauge potential A_μ. Integrating (2.4) over r with the boundary condition $A_0|_{r\to\infty} \to 0$ gives

$$A_0(r) \equiv \phi(r) = \int_r^\infty dr'\, E_r(r') = \frac{Q}{r}\, {}_2F_1\Big(\frac14, \frac12, \frac54, -\frac{r_0^4}{r^4}\Big) = \frac{Q}{r}\Big[1 - \frac{r_0^4}{10 r^4} + \mathcal{O}(\frac{r_0^8}{r^8})\Big]\ , \tag{4.1}$$

where ${}_2F_1$ is the standard hypergeometric function. By a gauge transformation $A_\mu = (\phi, 0, 0, 0)$ can be transformed into (cf. (3.3))

$$A_\mu = \phi(r)\, k_\mu = \phi(r)\Big(1, \frac{x_i}{r}\Big)\ . \tag{4.2}$$

The corresponding double copy metric is then (3.2) with $\phi(r)$ given by (4.1). Here $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ with $x^\mu = (\bar t, x_i)$ and $\bar t$ related to t as in (3.5). Using this relation and the transformation between the cartesian and the spherical coordinates we find that the metric takes the same "Schwarschild" form (3.1) now with Coulomb $\phi = \frac{Q}{r}$ replaced by $\phi(r)$ in (4.1). It thus generalizes the Schwarschild metric to the case when $r_0^2 = 2\pi\alpha' Q$ is non-zero.

In contrast to the Schwarschild metric the components of the resulting metric (3.1) look non-singular since $\phi(r)$ in (4.1) has a regular expansion for small r:

$$\phi(r) = c_0 + c_1 r + c_5 r^5 + \mathcal{O}(r^9) = \frac{Q}{r_0}\Gamma(\tfrac54)\Big[\Gamma(\tfrac14)\sqrt{\pi} - \frac{r}{\Gamma(\frac14)\, r_0} + \frac{r^5}{5 r_0^5}\Big] + \mathcal{O}(r^9)\ . \tag{4.3}$$

Somewhat surprisingly, the corresponding curvature invariants still turn out to be singular at $r = 0$. For example, the scalar curvature is given by

$$R = \frac{2\phi(r)}{r^2} - \frac{2Q\,(r^4 + 2r_0^4)}{r(r^4 + r_0^4)^{3/2}} = \frac{2Q\ \Gamma(\frac14)\Gamma(\frac54)}{r_0\sqrt{\pi}}\ \frac{1}{r^2} - \frac{4Q\,[\Gamma(\frac14) + 2\Gamma(\frac54)]}{r_0^2\ \Gamma(\frac14)}\ \frac{1}{r} + \mathcal{O}(r^3)\ . \tag{4.4}$$

[5] We shall ignore normalization constants in the definition of mass and charge.

This singularity is due to the presence of the first two (c_0 and $c_1 r$) terms in the $r \to 0$ expansion of ϕ in (4.3).

If $\phi\big|_{r\to 0} = c_0 \neq 0$ then the metric (3.1) has a conical singularity at $r = 0$. This is not, however, a serious issue as we can set $c_0 = 0$ by changing the integration constant in (4.1) (or by a gauge transformation of the potential (4.2)) and then define the double copy metric (3.2) using ϕ in this gauge.[6] The real problem is that $\phi\big|_{r\to 0} = c_1 r = c_1\sqrt{x_i^2}$ is non-analytic in cartesian coordinates and this effectively produces singularity in the curvature invariants. This $c_1 r$ term can not be eliminated by a gauge transformation as it is responsible for the non-zero constant value of the Born-Infeld electric field $E_r = -\partial_r \phi$ in (2.4) at $r = 0$ (see also Appendix C).

In general, if we start with the metric (3.1) with ϕ having a regular $r \to 0$ expansion

$$\phi(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4 + c_5 r^5 + \mathcal{O}(r^6), \tag{4.5}$$

then the $r \to 0$ expansion of the curvature squared invariant is found to be

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big|_{r\to 0} = \frac{4c_0^2}{r^4} + \frac{8c_0c_1}{r^3} + \frac{4c_1^2 + 8c_0c_2}{r^2} + \frac{8(c_1c_2 + c_0c_3)}{r} \tag{4.6}$$

$$+ 4(c_2^2 + 2c_1c_3 + 2c_0c_4) + 8(c_2c_4 + c_1c_4 + c_0c_5)r + \mathcal{O}(r^2)\,. \tag{4.7}$$

Thus it is non-vanishing c_0 and c_1 that are, indeed, responsible for the singularity. Explicitly, in the case of $\phi(r)$ in (4.3) we find (see also (B.5))

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4[\phi(r)]^2}{r^4} + \frac{8Q^2(r^8 + r_0^4 r^4 + \frac{1}{2}r_0^8)}{r^2(r^4 + r_0^4)^3} = \frac{4Q^2\,\Gamma(\frac{1}{4})^2\Gamma(\frac{5}{4})^2}{\pi r_0^2}\,\frac{1}{r^4}$$

$$- \frac{32Q^2\Gamma(\frac{5}{4})^2}{\sqrt{\pi}\,r_0^3}\,\frac{1}{r^3} + \frac{4Q^2}{r_0^4}\,\frac{\Gamma(\frac{1}{4})^2 + 16\Gamma(\frac{5}{4})^2}{\Gamma(\frac{1}{4})^2}\,\frac{1}{r^2} + \frac{16Q^2\,\Gamma(\frac{5}{4})^2}{5r_0^7\sqrt{\pi}}\,r + \mathcal{O}(r^3)\,. \tag{4.8}$$

Note that the expressions in (4.4) and (4.8) (before expanding near $r \to 0$) reduce to the standard Schwarzschild values ($R = 0$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12Q^2}{r^6}$) once we set $r_0 = 2\pi\alpha' Q = 0$ for fixed r. For non-zero r_0 the corresponding metric (3.1) has a non-trivial Ricci tensor (see Appendix B). It is not clear if there is some generalization of the Einstein equations for which the metric (3.1) with ϕ in (4.1) is a solution.

5. Concluding remarks

Our aim in this note was to explore if the simplest classical double copy ansatz may produce a non-singular generalization of the Schwarzschild metric if applied to the exact open-string analog of the Coulomb solution. The latter was assumed to be approximated by the Born-Infeld solution. We suggested that since the Born-Infeld action is the leading term in the open string effective action expansion in powers of field strength derivatives and since the electric field of the Born-Infeld analog of the Coulomb solution is approximately constant near $r = 0$ this solution may be trusted near the origin.

The resulting double copy metric reduces to the Schwarzschild one in the $\alpha' \to 0$ limit and at first sight seems regular near $r = 0$. However, the decay of the Born-Infeld scalar potential ϕ for $r \to 0$ happens to be too slow (reflecting the non-vanishing value of the Born-Infeld field at the origin) for the corresponding curvature invariants to be regular. This does not of course imply the singularity of a closed string generalization of the Schwarzschild solution since there is no a priori reason to expect this double copy construction to produce a solution of the closed string effective equations and also given that the Born-Infeld field (2.4) is not an exact solution of the open-string theory.

One direction to investigate further is the influence of derivative corrections in the open string effective action on the behaviour of the corresponding solution near $r = 0$, going beyond a simplified analysis in Appendix A. In particular, one may wonder if a resummation of derivative corrections may alter the $\phi\big|_{r\to 0} \to r$ behaviour of the scalar potential that may resolve the singularity of the double copy metric. It is also interesting to study a possible generalization of the double copy ansatz [9] that allows for a non-trivial dilaton. More generally, the status of the double copy construction beyond the leading order in α' expansion and whether it may provide a map between exact open-string and closed-string solutions remains to be explored.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[6] In this case, however, instead of $\phi\big|_{r\to\infty} = 0$ we will have $\phi\big|_{r\to\infty} = -c_0$ so that will change the standard Minkowski asymptotic form of the metric (3.1).

Appendix A. Born-Infeld solution as an approximation to open-string solution

Assuming the same ansatz for $F_{\mu\nu}$ (no magnetic field, time-independent electric field) that led to the Born-Infeld solution in (2.4), only the $\nu = 0$ component of the equations (2.3) is non-trivial and may be written as (ignoring higher order terms in (2.3))

$$\partial_i\Big(\frac{E_i}{\sqrt{1-T^{-2}E^2}}\Big) + \frac{1}{6\pi T^3}\Big[\frac{1}{2}(\partial_k E_i)(\partial_k E_i)\partial_j E_j + (\partial_i\partial_j E_k)(\partial_j E_k)E_i$$
$$+ 2\partial_j[(\partial_i E_k)(\partial_j E_k)E_i] + \partial_i[(\partial_j E_k)(\partial_j E_i)E_k] + \partial_i[(\partial_j E_k)(\partial_j E_k)E_i]\Big] = 0. \tag{A.1}$$

Assuming further that E_i is spherically-symmetric we get ($E \equiv E_r(r)$)

$$\partial_r\Big[\frac{r^2 E}{\sqrt{1-T^{-2}E^2}}\Big] + \frac{3}{4\pi T^3}\partial_r E\Big[2rE^2 + r^3(\partial_r E)^2 + 2r^2 E(\partial_r E + r\partial_r^2 E)\Big] = 0\,. \tag{A.2}$$

From here we may find the leading correction to the Born-Infeld solution coming from the presence of the field strength derivative terms in the open string effective action. Setting $E(r) = E^{(0)}(r) + E^{(1)}(r)$, where $E^{(0)}(r)$ is the Born-Infeld solution (2.4) we obtain from (A.2) the following first-order differential equation for $E^{(1)}$ ($r_0^2 = T^{-1}Q$):

$$\frac{dE^{(1)}}{dr} + \frac{2(r^4 - 2r_0^4)}{r(r^4 + r_0^4)}E^{(1)} = \frac{3r_0^6 r^7(7r^8 - 6r_0^4 r^4 + r_0^8)}{\pi(r^4 + r_0^4)^6}\,. \tag{A.3}$$

Its solution may be written as:

$$E^{(1)} = -\frac{r_0^6 r^4(7r^8 + 2r_0^4 r^4 + r_0^8)}{2\pi(r^4 + r_0^4)^5} = -\frac{r^4}{2\pi r_0^6} + \frac{3r^8}{2\pi r_0^{10}} + \mathcal{O}(r^{12})\,. \tag{A.4}$$

Its expansion for $r \to 0$ starts at order r^4 so it does not change the $E\big|_{r\to 0} = \frac{Q}{r_0^2} =$ const behaviour of the Born-Infeld field (2.4) near the origin, suggesting it can be trusted near $r = 0$. Equivalently, the derivative terms do not alter the leading $c_1 r$ term in the scalar potential (4.3) that was found to be responsible for the singularity of the double-copy metric.

Appendix B. Curvature tensor for the double copy metric

The curvature tensor for the metric of the form (3.1) can be computed for general function $\phi(r)$ with the non-trivial components being

$$R^t{}_{rtr} = \frac{\phi''}{2(1-\phi)}, \qquad R^t{}_{\theta\theta t} = R^r{}_{\theta\theta r} = -\frac{r}{2}\phi', \qquad R^t{}_{\varphi\varphi t} = R^r{}_{\varphi\varphi r} = R^t{}_{\theta\theta t}\sin^2\theta,$$
$$R^r{}_{ttr} = \frac{1}{2}(1-\phi)\phi'', \qquad R^\theta{}_{tt\theta} = R^\varphi{}_{tt\varphi} = \frac{1-\phi}{2r}\phi', \qquad R^\theta{}_{r\theta r} = R^\varphi{}_{r\varphi r} = \frac{\phi'}{2r(1-\phi)},$$
$$R^\theta{}_{\varphi\theta\varphi} = R^\varphi{}_{\theta\varphi\theta}\sin^2\theta = \phi(r)\sin^2\theta. \tag{B.1}$$

For the Ricci tensor and scalar we get:

$$R_{tt} = -\frac{1-\phi}{2r}(2\phi' + r\phi''), \qquad R_{rr} = \frac{2\phi' + r\phi''}{2r - 2r\phi}, \qquad R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2\theta} = \phi + r\phi'\,, \tag{B.2}$$
$$R = \frac{2\phi}{r^2} + \frac{4\phi'}{r} + \phi''\,. \tag{B.3}$$

The explicit form of the Ricci tensor corresponding to the metric (3.1) with ϕ in (4.1) is

$$R_{tt} = \frac{Q\,r_0^4\,(1-\phi)}{r^7(1+\frac{r_0^4}{r^4})^{3/2}} = \frac{Q\,r_0\sqrt{\pi} - Q^2\,\Gamma(\frac14)\Gamma(\frac54)}{r_0^3\sqrt{\pi}}\,\frac{1}{r} + \frac{4Q^2\,\Gamma(\frac54)}{r_0^4\Gamma(\frac14)} + \mathcal{O}(r^3),$$
$$R_{rr} = \frac{Q\,r_0^4}{(1-\phi)(1+\frac{r_0^4}{r^4})^{3/2}} = -\frac{Q\sqrt{\pi}}{r_0[r_0\sqrt{\pi} - Q\,\Gamma(\frac14)\Gamma(\frac54)]}\,\frac{1}{r} + \frac{4Q^2\pi\,\Gamma(\frac54)\Gamma(\frac14)}{r_0^2[r_0\sqrt{\pi} - Q\,\Gamma(\frac14)\Gamma(\frac54)]^2} + \mathcal{O}(r),$$
$$R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2\theta} = \phi(r) - \frac{Q}{r}\Big(1 + \frac{r_0^4}{r^4}\Big)^{-1/2} = \frac{Q\,\Gamma(\frac14)\Gamma(\frac54)}{r_0\sqrt{\pi}} - \frac{r_0^2\,\Gamma(\frac14) + 8Q\,\Gamma(\frac54)}{2r_0^2\,\Gamma(\frac14)}\,r + \mathcal{O}(r^3). \tag{B.4}$$

The curvature squared invariant is

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4\phi^2}{r^4} + \frac{4\phi'^2}{r^2} + \phi''^2 = \frac{4\phi^2}{r^4} + \frac{8Q^2(r^8 + r_0^4 r^4 + \frac12 r_0^8)}{r^2(r^4 + r_0^4)^3}\,, \tag{B.5}$$

with its expansion at $r \to 0$ given in (4.8). The Weyl tensor squared is also singular at $r \to 0$

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = \frac{(2\phi - 2r\phi' + r^2\phi'')^2}{3r^4} = \frac{4Q^2\Gamma(\frac{1}{4})^2\Gamma(\frac{5}{4})^2}{3\pi r_0^2}\,\frac{1}{r^4} + \mathcal{O}(r^{-3}). \tag{B.6}$$

Appendix C. Gauge transformation of the vector potential near $r = 0$

Given the vector potential $A_\mu = \phi(r)(1,\ x_i/r)$ with $\phi\big|_{r\to 0} = c_0 + c_1 r + c_5 r^5 + \ldots$ as in (4.2), (4.3), let us see if there is a gauge transformation that eliminates c_0 and c_1 terms, i.e. if A_μ can be transformed into

$$\tilde{A}_\mu = \tilde{\phi}(r)\big(1, \frac{x_i}{r}\big), \qquad \tilde{\phi}(r)\big|_{r\to 0} = \tilde{c}_5 r^5 + \ldots . \tag{C.1}$$

The relation $\tilde{A}_\mu = A_\mu - \partial_\mu \chi$ implies

$$\partial_0\chi = c_0 + c_1 r + (c_5 - \tilde{c}_5)r^5 + \ldots, \qquad \partial_i\chi = \frac{c_0 x_i}{r} + c_1 x_i + (c_5 - \tilde{c}_5)x_i r^4 + \ldots . \tag{C.2}$$

These equations lead to

$$\chi(t, x) = \big[c_0 + c_1 r + (c_5 - \tilde{c}_5)r^5\big]\,t + f(x)\,, \tag{C.3}$$

$$\partial_i f(x) = \frac{c_0 x_i}{r} + c_1 x_i + (c_5 - \tilde{c}_5)x_i r^4 - \Big[\frac{c_1 x_i}{r} + 5(c_5 - \tilde{c}_5)x_i r^3\Big]\,t. \tag{C.4}$$

The left-hand side of (C.4) is time-independent, so it is consistent only if $c_5 = \tilde{c}_5$ and $c_1 = 0$. Thus c_1 cannot be eliminated by a gauge transformation.

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