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ADVANCES IN COMBINATORICS, 2019:4, 16 pp.
www.advancesincombinatorics.com

arXiv:1808.03006v3 [math.CO] 23 Oct 2019

Upper Density of Monochromatic Infinite Paths

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Received 10 August 2018; *Revised* 24 December 2018; *Published* 30 October 2019

Abstract: We prove that in every 2-colouring of the edges of $K_{\mathbb{N}}$ there exists a monochromatic infinite path P such that $V(P)$ has upper density at least $(12+\sqrt{8})/17 \approx 0.87226$ and further show that this is best possible. This settles a problem of Erdős and Galvin.

Key words and phrases: infinite graph, Ramsey, upper density, regularity lemma.

1 Introduction

Given a complete graph K_n, whose edges are coloured in red and blue, what is the longest monochromatic path one can find? Gerencsér and Gyárfás [4] proved that there is always a monochromatic path on $\lceil(2n+1)/3\rceil$ vertices, which is best possible. It is natural to consider a density analogue of this result for 2-colourings of $K_{\mathbb{N}}$. The *upper density* of a graph G with $V(G) \subseteq \mathbb{N}$ is defined as

$$\bar{d}(G) = \limsup_{t\to\infty} \frac{|V(G)\cap\{1,2,\dots,t\}|}{t}.$$

The *lower density* is defined similarly in terms of the infimum and we speak of the *density*, whenever lower and upper density coincide.

Erdős and Galvin [3] described a 2-colouring of $K_{\mathbb{N}}$ in which every monochromatic infinite path has lower density 0 and thus we restrict our attention to upper densities. Rado [9] proved that every

*Research supported by an LSE PhD scholarship.

†Research supported in part by Simons Foundation Collaboration Grant # 283194.

‡Research supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

§Research supported by EPSRC, grant no. EP/P002420/1.

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DOI: 10.19086/aic.10810

r-edge-coloured $K_{\mathbb{N}}$ contains r vertex-disjoint monochromatic paths which together cover all vertices. In particular, one of them must have upper density at least $1/r$. Erdős and Galvin [3] proved that in every 2-colouring of $K_{\mathbb{N}}$ there is a monochromatic path P with $\bar{d}(P) \geq 2/3$. Moreover, they constructed a 2-colouring of $K_{\mathbb{N}}$ in which every monochromatic path P has upper density at most $8/9$. DeBiasio and McKenney [2] recently improved the lower bound to $3/4$ and conjectured the correct value to be $8/9$. Progress towards this conjecture was made by Lo, Sanhueza-Matamala and Wang [8], who raised the lower bound to $(9+\sqrt{17})/16 \approx 0.82019$.

We prove that the correct value is in fact $(12+\sqrt{8})/17 \approx 0.87226$.

Theorem 1.1. *There exists a 2-colouring of the edges of $K_{\mathbb{N}}$ such that every monochromatic path has upper density at most $(12+\sqrt{8})/17$.*

Theorem 1.2. *In every 2-colouring of the edges of $K_{\mathbb{N}}$, there exists a monochromatic path of upper density at least $(12+\sqrt{8})/17$.*

Now that we have solved the problem for two colours, it would be very interesting to make any improvement on Rado's lower bound of $1/r$ for $r \geq 3$ colours (see [2, Corollary 3.5] for the best known upper bound). In particular for three colours, the correct value is between $1/3$ and $1/2$.

2 Notation

We write $\mathbb{N}$ to be the positive integers with the standard ordering. Throughout the paper when referring to a finite graph on n vertices, it is always assumed that the vertex set is $[n] = \{1, 2, \ldots, n\}$ and that it is ordered in the natural way. An *infinite path* P is a graph with vertex set $V(P) = \{v_i : i \in \mathbb{N}\}$ and edge set $E(P) = \{v_i v_{i+1} : i \in \mathbb{N}\}$. While paths are defined to be one-way infinite, all of the results mentioned above on upper density of monochromatic infinite paths apply equally well to two-way infinite paths. For a graph G with $V(G) \subseteq \mathbb{N}$ and $t \in \mathbb{N}$, we define

$$d(G,t) = \frac{|V(G) \cap [t]|}{t}.$$

Thus we can express the upper density of G as $\bar{d}(G) = \limsup_{t \to \infty} d(G,t)$.

3 Upper bound

In this section, we will prove Theorem 1.1. Let $q > 1$ be a real number, whose exact value will be chosen later on. We start by defining a colouring of the edges of the infinite complete graph. Let $A_0, A_1, \ldots$ be a partition of $\mathbb{N}$, such that every element of A_i precedes every element of A_{i+1} and $|A_i| = \lfloor q^i \rfloor$. We colour the edges of $G = K_{\mathbb{N}}$ such that every edge uv with $u \in A_i$ and $v \in A_j$ is red if $\min\{i, j\}$ is odd, and blue if it is even. A straightforward calculation shows that for $q = 2$, every monochromatic path P in G satisfies $\bar{d}(P) \leq 8/9$ (see Theorem 1.5 in [3]). We will improve this bound by reordering the vertices of G and then optimizing the value of q.

For convenience, we will say that the vertex $v \in A_i$ is red if i is odd and blue if i is even. We also denote by B the set of blue vertices and by R be the set of red vertices. Let b_i and r_i denote the i-th blue vertex and the i-th red vertex, respectively. We define a monochromatic red matching M_r by forming a matching between A_{2i-1} and the first $|A_{2i-1}|$ vertices of A_{2i} for each $i \geq 1$. Similarly, we define a monochromatic blue matching M_b by forming a matching between A_{2i} and the first $|A_{2i}|$ vertices of A_{2i+1} for each $i \geq 0$.

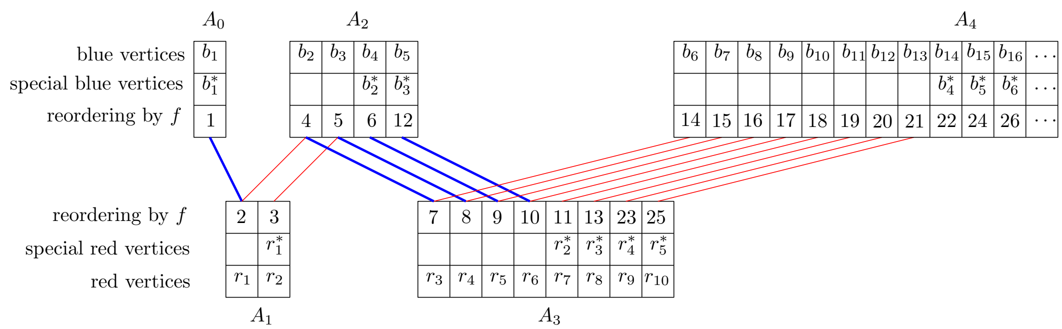

Figure 1: The colouring for $q = 2$ and the reordering by f.

Next, let us define a bijection $f\colon \mathbb{N} \to V(G)$, which will serve as a reordering of G. Let r_t^* denote the t-th red vertex not in M_b, and b_t^* denote the t-th blue vertex not in M_r. The function f is defined as follows. We start enumerating blue vertices, in their order, until we reach b_1^*. Then we enumerate red vertices, in their order, until we reach r_1^*. Then we enumerate blue vertices again until we reach b_2^*. We continue enumerating vertices in this way, changing colours whenever we find an r_t^* or a b_t^*. (See Figure 1.) Finally, for every $H \subseteq G$, we define

$$\bar{d}(H; f) = \limsup_{t\to\infty} \frac{|V(H) \cap f([t])|}{t}.$$

Note that $\bar{d}(H; f)$ is the upper density of H in the reordered graph $f^{-1}(G)$.

Claim 3.1. *Let P_r and P_b be infinite monochromatic red and blue paths in G, respectively. Then $\bar{d}(P_r; f) \leq \bar{d}(M_r; f)$ and $\bar{d}(P_b; f) \leq \bar{d}(M_b; f)$.*

Claim 3.2. *We have*

$$\bar{d}(M_r; f),\ \bar{d}(M_b; f) \leq \frac{q^2 + 2q - 1}{q^2 + 3q - 2}.$$

We can easily derive Theorem 1.1 from these two claims. Note that the rational function in Claim 3.2 evaluates to $(12 + \sqrt{8})/17$ at $q := \sqrt{2} + 1$. It then follows from Claim 3.1 and 3.2, that every monochromatic path P in G satisfies $\bar{d}(P; f) \leq (12 + \sqrt{8})/17$. Thus we can define the desired colouring of $K_{\mathbb{N}}$, by colouring each edge ij with the colour of the edge $f(i)f(j)$ in G.

It remains to prove Claim 3.1 and 3.2. The intuition behind Claim 3.1 is that in every monochromatic red path P_r there is a red matching with the same vertex set, and that M_r has the largest upper density among all red matchings, as it contains every red vertex and has the largest possible upper density of blue vertices. Note that the proof of Claim 3.1 only uses the property that f preserves the order of the vertices inside R and inside B.

Proof of Claim 3.1. We will show $\bar{d}(P_r;f) \leq \bar{d}(M_r;f)$. (The other case is analogous.) We prove that, for every positive integer k, we have $|V(P_r)\cap f([k])| \leq |V(M_r)\cap f([k])|$. Assume, for contradiction, that this is not the case and let k be the minimum positive integer for which the inequality does not hold. Every red vertex is saturated by M_r, so $|V(P_r)\cap f([k])\cap B| > |V(M_r)\cap f([k])\cap B|$. By the minimality of k, $f(k)$ must be in P_r but not in M_r, and in particular it must be blue.

Let $f(k) \in A_{2i}$. Since $f(k) \notin M_r$, we know that $f(k)$ is not among the first $|A_{2i-1}|$ vertices of A_{2i}. Therefore, since f preserves the order of the vertices inside B, $f([k])$ contains the first $|A_{2i-1}|$ blue vertices in A_{2i}, and hence

$$|V(P_r)\cap f([k])\cap B| > |V(M_r)\cap f([k])\cap B| = \sum_{j=1}^{i} |A_{2j-1}|. \tag{3.1}$$

On the other hand, every edge between two blue vertices is blue, so the successor of every blue vertex in P_r is red, and in particular there is a red matching between $V(P_r)\cap B$ and R saturating $V(P_r)\cap B$. So by (3.1), the number of red neighbours of $V(P_r)\cap f([k])\cap B$ is at least $|V(P_r)\cap f([k])\cap B| > \sum_{j=1}^{i}|A_{2j-1}|$. Observe that by the definition of f, we have $V(P_r)\cap f([k])\cap B \subseteq \bigcup_{j=0}^{i} A_{2j}$. Hence the red neighbourhood of $V(P_r)\cap f([k])\cap B$ is contained in $\bigcup_{j=1}^{i} A_{2j-1}$, a contradiction. □

Proof of Claim 3.2. Let $\ell_r(t)$ and $\ell_b(t)$ denote the position of r_t^* among the red vertices and of b_t^* among the blue vertices, respectively. In other words, let $\ell_r(t) = i$ where $r_t^* = r_i$ and $\ell_b(t) = j$ where $b_t^* = b_j$ (so for example in Figure 1, $\ell_r(4) = 9$ and $\ell_b(4) = 14$). Note that $f(\ell_b(t)+\ell_r(t)) = r_t^*$, so for $\ell_b(t-1)+\ell_r(t-1) \leq k \leq \ell_b(t)+\ell_r(t)-1$, $f([k])$ has exactly $t-1$ vertices outside of M_b and at least $t-1$ vertices outside of M_r. As a consequence, we obtain

$$\bar{d}(M_r;f),\ \bar{d}(M_b;f) \leq \limsup_{k\to\infty}(1-h(k)) = \limsup_{t\to\infty}\left(1-\frac{t-1}{\ell_r(t)+\ell_b(t)-1}\right), \tag{3.2}$$

where $h(k) = (t-1)/k$ if $\ell_b(t-1)+\ell_r(t-1) \leq k \leq \ell_b(t)+\ell_r(t)-1$. It is easy to see that

$$\ell_r(t) = t + \sum_{j=0}^{i} |A_{2j}| \quad \text{for} \quad \sum_{j=0}^{i-1}(|A_{2j+1}|-|A_{2j}|) < t \leq \sum_{j=0}^{i}(|A_{2j+1}|-|A_{2j}|), \text{ and}$$

$$\ell_b(t) = t + \sum_{j=1}^{i} |A_{2j-1}| \quad \text{for} \quad \sum_{j=1}^{i-1}(|A_{2j}|-|A_{2j-1}|) < t-|A_0| \leq \sum_{j=1}^{i}(|A_{2j}|-|A_{2j-1}|).$$

Note that $\ell_r(t)-t$ and $\ell_b(t)-t$ are piecewise constant and non-decreasing. We claim that, in order to compute the right hand side of (3.2), it suffices to consider values of t for which $\ell_r(t)-t > \ell_r(t-1)-(t-1)$ or $\ell_b(t)-t > \ell_b(t-1)-(t-1)$. This is because we can write

$$1-\frac{t-1}{\ell_r(t)+\ell_b(t)-1} = \frac{1}{2} + \frac{(\ell_r(t)-t)+(\ell_b(t)-t)+1}{2(\ell_r(t)+\ell_b(t)-1)}.$$

In this expression, the second fraction has a positive, piecewise constant numerator and a positive increasing denominator. Therefore, the local maxima are attained precisely at the values for which the numerator increases. We will do the calculations for the case when $\ell_r(t)-t > \ell_r(t-1)-(t-1)$ (the other case is similar), in which we have

$$t = 1+\sum_{j=0}^{i-1}(|A_{2j+1}|-|A_{2j}|) = 1+\sum_{j=0}^{i-1}(1+o(1))q^{2j}(q-1) = (1+o(1))\frac{q^{2i}}{q+1},$$

$$\ell_r(t) = t+\sum_{j=0}^{i}|A_{2j}| = (1+o(1))\left(\frac{q^{2i}}{q+1}+\sum_{j=0}^{i}q^{2j}\right) = (1+o(1))\frac{(q^2+q-1)q^{2i}}{q^2-1}, \text{ and}$$

$$\ell_b(t) = t+\sum_{j=1}^{i}|A_{2j-1}| = (1+o(1))\left(\frac{q^{2i}}{q+1}+\sum_{j=1}^{i}q^{2j-1}\right) = (1+o(1))\frac{(2q-1)q^{2i}}{q^2-1}.$$

Plugging this into (3.2) gives the desired result. □

4 Lower bound

This section is dedicated to the proof of Theorem 1.2. A *total colouring* of a graph G is a colouring of the vertices and edges of G. Due to an argument of Erdős and Galvin, the problem of bounding the upper density of monochromatic paths in edge coloured graphs can be reduced to the problem of bounding the upper density of monochromatic path forests in totally coloured graphs.

Definition 4.1 (Monochromatic path forest)**.** Given a totally coloured graph G, a forest $F \subseteq G$ is said to be a *monochromatic path forest* if $\Delta(F) \leq 2$ and there is a colour c such that all leaves, isolated vertices, and edges of F receive colour c.

Lemma 4.2. *For every $\gamma > 0$ and $k \in \mathbb{N}$, there is some $n_0 = n_0(k,\gamma)$ so that the following is true for every $n \geq n_0$. For every total 2-colouring of K_n, there is an integer $t \in [k,n]$ and a monochromatic path forest F with $d(F,t) \geq (12+\sqrt{8})/17-\gamma$.*

Some standard machinery related to Szemerédi's regularity lemma, adapted to the ordered setting, will allow us to reduce the problem of bounding the upper density of monochromatic path forests to the problem of bounding the upper density of monochromatic simple forests.

Definition 4.3 (Monochromatic simple forest)**.** Given a totally coloured graph G, a forest $F \subseteq G$ is said to be a *monochromatic simple forest* if $\Delta(F) \leq 1$ and there is a colour c such that all edges and isolated vertices of F receive colour c and at least one endpoint of each edge of F receives colour c.

Lemma 4.4. *For every $\gamma > 0$, there exists $k_0, N \in \mathbb{N}$ and $\alpha > 0$ such that the following holds for every integer $k \geq k_0$. Let G be a totally 2-coloured graph on kN vertices with minimum degree at least $(1-\alpha)kN$. Then there exists an integer $t \in [k/8, kN]$ and a monochromatic simple forest F such that $d(F,t) \geq (12+\sqrt{8})/17-\gamma$.*

The heart of the proof is Lemma 4.4, which we shall prove in Section 4.3. But first, in the next two sections, we show how to deduce Theorem 1.2 from Lemmas 4.2 and 4.4.

4.1 From path forests to paths

In this section we use Lemma 4.2 to prove Theorem 1.2. Our exposition follows that of Theorem 1.6 in [2].

Proof of Theorem 1.2. Fix a 2-colouring of the edges of $K_{\mathbb{N}}$ in red and blue. We define a 2-colouring of the vertices by colouring $n \in \mathbb{N}$ red if there are infinitely many $m \in \mathbb{N}$ such that the edge nm is red and blue otherwise.

Case 1. Suppose there are vertices x and y of the same colour, say red, and a finite set $S \subseteq \mathbb{N}$ such that there is no red path disjoint from S which connects x to y.

We partition $\mathbb{N} \setminus S$ into sets X, Y, Z, where $x' \in X$ if and only if there is a red path, disjoint from S, which connects x' to x and $y' \in Y$ if and only if there is a red path disjoint from S which connects y to y'. Note that every edge from $X \cup Y$ to Z is blue. Since x and y are coloured red, both X and Y are infinite, and by choice of x and y all edges in the bipartite graph between X and $Y \cup Z$ are blue. Hence there is a blue path with vertex set $X \cup Y \cup Z = \mathbb{N} \setminus S$.

Case 2. Suppose that for every pair of vertices x and y of the same colour c, and every finite set $S \subseteq \mathbb{N}$, there is a path from x to y of colour c which is disjoint from S.

Let γ_n be a sequence of positive reals tending to zero, and let a_n and k_n be increasing sequences of integers such that

$$a_n \geq n_0(k_n, \gamma_n) \text{ and } k_n/(a_1 + \cdots + a_{n-1} + k_n) \to 1,$$

where $n_0(k, \gamma)$ is as in Lemma 4.2. Let $\mathbb{N} = (A_i)$ be a partition of $\mathbb{N}$ into consecutive intervals with $|A_n| = a_n$. By Lemma 4.2 there are monochromatic path forests F_n with $V(F_n) \subseteq A_n$ and initial segments $I_n \subseteq A_n$ of length at least k_n such that

$$|V(F_n) \cap I_n| \geq \left(\frac{12 + \sqrt{8}}{17} - \gamma_n \right) |I_n|.$$

It follows that for any $G \subseteq K_{\mathbb{N}}$ containing infinitely many F_n's we have

$$\bar{d}(G) \geq \limsup_{n \to \infty} \frac{|V(F_n) \cap I_n|}{a_1 + \cdots + a_{n-1} + |I_n|} \geq \limsup_{n \to \infty} \frac{12 + \sqrt{8}}{17} - \gamma_n = \frac{12 + \sqrt{8}}{17}.$$

By the pigeonhole principle, there are infinitely many F_n's of the same colour, say blue. We will recursively construct a blue path P which contains infinitely many of these F_n's. To see how this is done, suppose we have constructed a finite initial segment p of P. We will assume as an inductive hypothesis that p ends at a blue vertex v. Let n be large enough that $\min(A_n)$ is greater than every vertex in p, and F_n is blue. Let $F_n = \{P_1, \ldots, P_s\}$ for some $s \in \mathbb{N}$ and let w_i, w_i' be the endpoints of the path P_i (note that w_i and w_i' could be equal) for every $i \in [s]$. By the case assumption, there is a blue path q_1 connecting v to w_1, such that q_1 is disjoint from $A_1 \cup \cdots \cup A_n$. Similarly, there is a blue path q_2 connecting w_1' to w_2, such that q_2 is disjoint from $A_1 \cup \cdots \cup A_n \cup \{q_1\}$. Continuing in this fashion, we find disjoint blue paths $q_3, \ldots, q_s$ such that q_i connects w_{i-1}' to w_i. Hence, we can extend p to a path p' which contains all of the vertices of F_n and ends at a blue vertex. □

4.2 From simple forests to path forests

In this section we use Lemma 4.4 to prove Lemma 4.2. The proof is based on Szemerédi's Regularity Lemma, which we introduce below. The main difference to standard applications of the Regularity Lemma is, that we have to define an ordering of the reduced graph, which approximately preserves densities. This is done by choosing a suitable initial partition.

Let $G=(V,E)$ be a graph and A and B be non-empty, disjoint subsets of V. We write $e_G(A,B)$ for the number of edges in G with one vertex in A and one in B and define the *density* of the pair (A,B) to be $d_G(A,B)=e_G(A,B)/(|A||B|)$. The pair (A,B) is ε*-regular* (in G) if we have $|d_G(A',B')-d_G(A,B)|\leq\varepsilon$ for all $A'\subseteq A$ with $|A'|\geq\varepsilon|A|$ and $B'\subseteq B$ with $|B'|\geq\varepsilon|B|$. It is well-known (see for instance [5]) that dense regular pairs contain almost spanning paths. We include a proof of this fact for completeness.

Lemma 4.5. *For $0<\varepsilon<1/4$ and $d\geq 2\sqrt{\varepsilon}+\varepsilon$, every ε-regular pair (A,B) with density at least d contains a path with both endpoints in A and covering all but at most $2\sqrt{\varepsilon}|A|$ vertices of $A\cup B$.*

Proof. We will construct a path $P_k=(a_1b_1\dots a_k)$ for every $k=1,\dots,\lceil(1-\sqrt{\varepsilon})|A|\rceil$ such that $B_k:=N(a_k)\setminus V(P_k)$ has size at least $\varepsilon|B|$. As $d\geq\varepsilon$, this is easy for $k=1$. Assume now that we have constructed P_k for some $1\leq k<(1-\sqrt{\varepsilon})|A|$. We will show how to extend P_k to P_{k+1}. By ε-regularity of (A,B), the set $\bigcup_{b\in B_k}N(b)$ has size at least $(1-\varepsilon)|A|$. So $A':=\bigcup_{b\in B_k}N(b)\setminus V(P_k)$ has size at least $(\sqrt{\varepsilon}-\varepsilon)|A|\geq\varepsilon|A|$. Let $B'=B\setminus V(P_k)$ and note that $|B'|\geq\sqrt{\varepsilon}|B|$ as $k<(1-\sqrt{\varepsilon})|A|$ and $|A|=|B|$. By ε-regularity of (A,B), there exists $a_{k+1}\in A'$ with at least $(d-\varepsilon)|B'|\geq 2\varepsilon|B|$ neighbours in B'. Thus we can define $P_{k+1}=(a_1b_1\dots a_kb_ka_{k+1})$, where $b_k\in B_k\cap N(a_{k+1})$. □

A family of disjoint subsets $\{V_i\}_{i\in[m]}$ of a set V is said to *refine* a partition $\{W_j\}_{j\in[\ell]}$ of V if, for all $i\in[m]$, there is some $j\in[\ell]$ with $V_i\subseteq W_j$.

Lemma 4.6 (Regularity Lemma [6, 10])**.** *For every $\varepsilon>0$ and $m_0,\ell\geq 1$ there exists $M=M(\varepsilon,m_0,\ell)$ such that the following holds. Let G be a graph on $n\geq M$ vertices whose edges are coloured in red and blue and let $d>0$. Let $\{W_i\}_{i\in[\ell]}$ be a partition of $V(G)$. Then there exists a partition $\{V_0,\dots,V_m\}$ of $V(G)$ and a subgraph H of G with vertex set $V(G)\setminus V_0$ such that the following holds:*

(i) *$m_0\leq m\leq M$;*

(ii) *$\{V_i\}_{i\in[m]}$ refines $\{W_i\cap V(H)\}_{i\in[\ell]}$;*

(iii) *$|V_0|\leq\varepsilon n$ and $|V_1|=\dots=|V_m|\leq\lceil\varepsilon n\rceil$;*

(iv) *$\deg_H(v)\geq\deg_G(v)-(d+\varepsilon)n$ for each $v\in V(G)\setminus V_0$;*

(v) *$H[V_i]$ has no edges for $i\in[m]$;*

(vi) *all pairs (V_i,V_j) are ε-regular and with density either 0 or at least d in each colour in H.*

Before we start with the proof, we will briefly describe the setup and proof strategy of Lemma 4.2. Consider a totally 2-coloured complete graph $G=K_n$. Denote the sets of red and blue vertices by R and B, respectively. For $\ell\geq 4$, let $\{W_j\}_{j\in[\ell]}$ be a partition of $[n]$ such that each W_j consists of at most $\lceil n/\ell\rceil$ subsequent vertices. The partition $\{W'_j\}_{j\in[2\ell]}$, with parts of the form $W_i\cap R$ and $W_i\cap B$, refines both $\{W_j\}_{j\in[\ell]}$ and $\{R,B\}$. Suppose that $V_0\cup\cdots\cup V_m$ is a partition obtained from Lemma 4.6 applied to G and $\{W'_j\}_{j\in[2\ell]}$ with parameters ε, m_0, 2ℓ and d. We define the (ε,d)-*reduced graph* G' to be the graph with vertex set $V(G')=[m]$ where ij is an edge of G' if and only if if (V_i,V_j) is an ε-regular pair of density at least d in the red subgraph of H or in the blue subgraph of H. Furthermore, we colour ij red if (V_i,V_j) is an ε-regular pair of density at least d in the red subgraph of H, otherwise we colour ij blue. As $\{V_i\}_{i\in[m]}$ refines $\{R,B\}$, we can extend this to a total 2-colouring of G' by colouring each vertex i red, if $V_i\subseteq R$, and blue otherwise. By relabelling the clusters, we can furthermore assume that $i<j$ if and only if $\max\{V_i\}<\max\{V_j\}$. Note that, by choice of $\{W_j\}_{j\in[\ell]}$, any two vertices in V_i differ by at most n/ℓ. Moreover, a simple calculation (see [7, Proposition 42]) shows that G' has minimum degree at least $(1-d-3\varepsilon)m$.

Given this setup, our strategy to prove Lemma 4.2 goes as follows. First, we apply Lemma 4.4 to obtain $t'\in[m]$ and a, red say, simple forest $F'\subseteq G'$ with $d(F',t')\approx(12+\sqrt{8})/17$. Next, we turn F' into a red path forest $F\subseteq G$. For every isolated vertex $i\in V(F')$, this is straightforward as $V_i\subseteq R$ by the refinement property. For every edge $ij\in E(F')$ with $i\in R$, we apply Lemma 4.5 to obtain a red path that almost spans (V_i,V_j) and has both ends in V_i. So the union F' of these paths and vertices is indeed a red path forest. Since the vertices in each V_i do not differ too much, it will follow that $d(F,t)\approx(12+\sqrt{8})/17$ for $t=\max\{V_{t'}\}$.

Proof of Lemma 4.2. Suppose we are given $\gamma>0$ and $k\in\mathbb{N}$ as input. Let $k_0,N\in\mathbb{N}$ and $\alpha>0$ be as in Lemma 4.4 with input $\gamma/4$. We choose constants $d,\varepsilon>0$ and $\ell,m_0\in\mathbb{N}$ satisfying

$$2\sqrt{\varepsilon}+\varepsilon\le 1/\ell, d\le\alpha/8 \text{ and } m_0\ge 4N/d, 2k_0N.$$

We obtain M from Lemma 4.6 with input ε,m_0 and 2ℓ. Finally, set $n_0=16k\ell MN$.

Now let $n\ge n_0$ and suppose that K_n is an ordered complete graph on vertex set $[n]$ and with a total 2-colouring in red and blue. We have to show that there is an integer $t\in[k,n]$ and a monochromatic path forest F such that $|V(F)\cap[t]|\ge((12+\sqrt{8})/17-\gamma)t$.

Denote the red and blue vertices by R and B, respectively. Let $\{W'_j\}_{j\in[\ell]}$ refine $\{R,B\}$ as explained in the above setting. Let $\{V_0,\dots,V_m\}$ be a partition of $[n]$ with respect to $G=K_n$ and $\{W'_j\}_{j\in[\ell]}$ as detailed in Lemma 4.6 with totally 2-coloured (ε,d)-reduced graph G'' of minimum degree $\delta(G'')\ge(1-4d)m$. Set $k'=\lfloor m/N\rfloor\ge k_0$ and observe that the subgraph G' induced by G'' in $[k'N]$ satisfies $\delta(G')\ge(1-8d)m\ge(1-\alpha)m$ as $m\ge 4N/d$. Thus we can apply Lemma 4.4 with input G', k', $\gamma/4$ to obtain an integer $t'\in[k'/8,k'N]$ and a monochromatic (say red) simple forest $F'\subseteq G'$ such that $d(F',t')\ge(12+\sqrt{8})/17-\gamma/4$.

Set $t=\max V_{t'}$. We have that $V_{t'}\subseteq W_j$ for some $j\in[\ell]$. Recall that $i<j$ if and only if $\max\{V_i\}<\max\{V_j\}$ for any $i,j\in[m]$. It follows that $V_i\subseteq[t]$ for all $i\le t'$. Hence

$$t\ge t'|V_1|\ge\frac{k'}{8}|V_1|\ge\left\lfloor\frac{m}{N}\right\rfloor\frac{(1-\varepsilon)n}{8m}\ge\frac{n}{16N}. \tag{4.1}$$

This implies $t \geq k$ by choice of n_0. Since $[t]$ is covered by $V_0 \cup W_j \cup \bigcup_{i\in[t']} V_i$, it follows that

$$\begin{aligned} t'|V_1| &\geq t - |V_0| - |W_j| \\ &\geq \left(1 - \varepsilon\frac{n}{t} - \frac{4}{\ell}\frac{n}{t}\right)t \\ &\geq \left(1 - 16\varepsilon N - \frac{64N}{\ell}\right)t \qquad \text{(by (4.1))} \\ &\geq \left(1 - \frac{\gamma}{2}\right)t. \end{aligned} \tag{4.2}$$

For every edge $ij \in E(F')$ with $V_i \subseteq R$, we apply Lemma 4.5 to choose a path P_{ij} which starts and ends in V_i and covers all but at most $2\sqrt{\varepsilon}|V_1|$ vertices of each V_i and V_j. We denote the isolated vertices of F' by I'. For each $i \in I'$ we have $V_i \subseteq R$. Hence the red path forest $F := \bigcup_{i\in I'} V_i \cup \bigcup_{ij\in E(F')} P_{ij} \subseteq K_n$ satisfies

$$\begin{aligned} |V(F)\cap[t]| &= \sum_{i\in I'} |V_i \cap [t]| + \sum_{ij\in E(F')} |V(P_{ij}) \cap [t]| \\ &\geq \sum_{i\in I'\cap[t']} |V_i| + \sum_{i\in V(F'-I')\cap[t'],} (|V_i| - 2\sqrt{\varepsilon}|V_1|) \\ &\geq (1-2\sqrt{\varepsilon})|V_1||V(F')\cap[t']| \\ &\geq (1-2\sqrt{\varepsilon})\left(\frac{12+\sqrt{8}}{17} - \frac{\gamma}{4}\right)t'|V_1| \\ &\overset{(4.2)}{\geq} \left(\frac{12+\sqrt{8}}{17} - \gamma\right)t \end{aligned}$$

as desired. □

4.3 Upper density of simple forests

In this section we prove Lemma 4.4. For a better overview, we shall define all necessary constants here. Suppose we are given $\gamma' > 0$ as input and set $\gamma = \gamma'/4$. Fix a positive integer $N = N(\gamma)$ and let $0 < \alpha \leq \gamma/(8N)$. The exact value of N will be determined later on. Let $k_0 = \lceil 8/\gamma \rceil$ and fix a positive integer $k \geq k_0$. Consider a totally 2-coloured graph G' on $n = kN$ vertices with minimum degree at least $(1-\alpha)n$.

Denote the sets of red and blue vertices by R and B, respectively. As it turns out, we will not need the edges inside R and B. So let G be the spanning bipartite subgraph, obtained from G' by deleting all edges within R and B. For each red vertex v, let $d_b(v)$ be the number of blue edges incident to v in G. Let $a_1 \leq \cdots \leq a_{|R|}$ denote the degree sequence taken by $d_b(v)$. The whole proof of Lemma 4.4 revolves around analysing this sequence.

Fix an integer $t = t(\gamma, N, k)$ and subset $R' \subseteq R$, $B' \subseteq B$. The value of t and nature of R', B' will be determined later. The following two observations explain our interest in the sequence $a_1 \leq \cdots \leq a_{|R|}$.

Claim 4.7. *If $a_j > j-t$ for all $1 \leq j \leq |R'|-1$, then there is a blue simple forest covering all but at most t vertices of $R' \cup B$.*

Proof. We write $R' = \{v_1, \ldots, v_{|R'|}\}$ such that $d_b(v_i) \leq d_b(v_j)$ for every $1 \leq i \leq j \leq |R'|$. By assumption, we have $d_b(v_j) \geq a_j > j-t$ for all $1 \leq j \leq |R'|-1$. Thus we can greedily select a blue matching containing $\{v_t, v_{t+1}, \ldots, v_{|R'|-1}\}$, which covers all but t vertices of R'. Together with the rest of B, this forms the desired blue simple forest. □

Claim 4.8. *If $a_i < i+t$ for all $1 \leq i \leq |B'|-t$, then there is a red simple forest covering all but at most $t+\alpha n$ vertices of $R \cup B'$.*

Proof. Let X' be a minimum vertex cover of the red edges in the subgraph of G induced by $R \cup B'$. If $|X'| \geq |B'|-t-\alpha n$, then by König's theorem there exists a red matching covering at least $|B'|-t-\alpha n$ vertices of B'. This together with the vertices in R yields the desired red simple forest.

Suppose now that $|X'| < |B'|-t-\alpha n$. Since every edge between $R \setminus (X' \cap R)$ and $B' \setminus (X' \cap B')$ is blue, we have for every vertex v in $R \setminus (X' \cap R)$,

$$d_b(v) \geq |B'| - |X' \cap B'| - \alpha n = |X' \cap R| + |B'| - |X'| - \alpha n > |X' \cap R| + t.$$

In particular, this implies $a_i \geq i+t$ for $i = |X' \cap R|+1$. So $|B'|-t+1 \leq |X' \cap R|+1$ by the assumption in the statement. Together with

$$|X' \cap R| + 1 \leq |X'| + 1 < |B'| - t - \alpha n + 1 < |B'| - t + 1,$$

we reach a contradiction. □

Motivated by this, we introduce the following definitions.

Definition 4.9 (Oscillation, $\ell^+(t)$, $\ell^-(t)$)**.** Let $a_1, \ldots, a_n$ be a non-decreasing sequence of non-negative real numbers. We define its *oscillation* as the maximum value T, for which there exist indices $i, j \in [n]$ with $a_i - i \geq T$ and $j - a_j \geq T$. For all $0 < t \leq T$, set

$$\ell^+(t) = \min\{i \in [n] \colon \ a_i \geq i+t\},$$
$$\ell^-(t) = \min\{j \in [n] \colon \ a_j \leq j-t\}.$$

Suppose that the degree sequence $a_1, \ldots, a_{|R|}$ has oscillation T and fix some positive integer $t \leq T$. We define ℓ and λ by

$$\ell = \ell^+(t) + \ell^-(t) = \lambda t. \tag{4.3}$$

The next claim combines Claims 4.7 and 4.8 into a density bound of a monochromatic simple forest in terms of the ratio $\ell/t = \lambda$. (Note that, in practice, the term αn will be of negligible size.)

Claim 4.10. *There is a monochromatic simple forest $F \subseteq G$ with*

$$d(F, \ell+t) \geq \frac{\ell - \alpha n}{\ell + t} = \frac{\lambda t - \alpha n}{(1+\lambda)t}.$$

Proof. Let $R' = R \cap [\ell + t]$ and $B' = B \cap [\ell + t]$ so that $\ell^+(t) + \ell^-(t) = \ell = |R'| + |B'| - t$. Thus we have either $\ell^-(t) \geq |R'|$ or $\ell^+(t) > |B'| - t$. If $\ell^-(t) \geq |R'|$, then $a_j > j - t$ for every $1 \leq j \leq |R'| - 1$. Thus Claim 4.7 provides a blue simple forest F covering all but at most t vertices of $[\ell + t]$. On the other hand, if $\ell^+(t) > |B'| - t$, then $a_i < i + t$ for every $1 \leq i \leq |B'| - t$. In this case Claim 4.8 yields a red simple forest F covering all but at most $t + \alpha n$ vertices of $[\ell + t]$. □

Claim 4.10 essentially reduces the problem of finding a dense linear forest to a problem about bounding the ratio ℓ/t in integer sequences. It is, for instance, not hard to see that we always have $\ell \geq 2t$ (which, together with the methods of the previous two subsections, would imply the bound $\bar{d}(P) \geq 2/3$ of Erdős and Galvin). The following lemma provides an essentially optimal lower bound on $\ell/t = \lambda$. Note that for $\lambda = 4 + \sqrt{8}$, we have $\frac{\lambda}{\lambda+1} = (12 + \sqrt{8})/17$.

Lemma 4.11. *For all $\gamma \in \mathbb{R}^+$, there exists $N \in \mathbb{N}$ such that, for all $k \in \mathbb{R}^+$ and all sequences with oscillation at least kN, there exists a real number $t \in [k, kN]$ with*

$$\ell := \ell^+(t) + \ell^-(t) \geq \left(4 + \sqrt{8} - \gamma\right) t.$$

The proof of Lemma 4.11 is deferred to the last section. We now finish the proof of Lemma 4.4. Set $N = N(\gamma)$ to be the integer returned by Lemma 4.11 with input $\gamma = \gamma'/4$. In order to use Lemma 4.11, we have to bound the oscillation of $a_1, \ldots, a_{|R|}$:

Claim 4.12. *The degree sequence $a_1, \ldots, a_{|R|}$ has oscillation $T \geq kN/8$ or there is a monochromatic simple forest $F \subseteq G$ with $d(F, n) \geq (12 + \sqrt{8})/17 - \gamma$.*

Before we prove Claim 4.12, let us see how this implies Lemma 4.4.

Proof of Lemma 4.4. By Claim 4.12, we may assume that the sequence $a_1, \ldots, a_{|R|}$ has oscillation at least $kN/8$. By Lemma 4.11, there is a real number $t' \in [k/8, kN/8]$ with

$$\ell = \ell^+(t') + \ell^-(t') \geq (4 + \sqrt{8} - \gamma)t'.$$

Let $t = t(\gamma, N, k) = \lceil t' \rceil$. Since the a_i's are all integers, we have $\ell^+(t) = \ell^+(t')$ and $\ell^-(t) = \ell^-(t')$. Let $F \subseteq G$ be the monochromatic simple forest obtained from Claim 4.10. As $n = kN$, $\ell \geq t' \geq k/8 \geq 1/\gamma$, $\alpha \leq \gamma/(8N)$, and by (4.3), it follows that

$$\begin{aligned}
d(F, \ell + t) &\geq \frac{\ell - \alpha n}{\ell + t} = \frac{1 - \alpha n/\ell}{1 + \frac{t}{\ell}} \geq \frac{1 - 8\alpha N}{1 + \frac{t'}{\ell} + \frac{1}{\ell}} \geq \frac{1}{1 + \frac{t'}{\ell}} - 2\gamma \\
&\geq \frac{1}{1 + \frac{1}{4 + \sqrt{8} - \gamma}} - 2\gamma = \frac{4 + \sqrt{8} - \gamma}{5 + \sqrt{8} - \gamma} - 2\gamma \\
&\geq \frac{4 + \sqrt{8}}{5 + \sqrt{8}} - 4\gamma = \frac{12 + \sqrt{8}}{17} - \gamma',
\end{aligned}$$

as desired. □

To finish, it remains to show Claim 4.12. The proof uses König's theorem and is similar to the proof of Claim 4.8.

Proof of Claim 4.12. Let X be a minimum vertex cover of the red edges. If $|X| \geq |B| - (1/8+\alpha)n$, then König's theorem implies that there is a red matching covering all but at most $(1/8+\alpha)n$ blue vertices. Thus adding the red vertices, we obtain a red simple forest F with $d(F,kN) \geq 7/8-\alpha \geq (12+\sqrt{8})/17-\gamma$. Therefore, we may assume that $|X| < |B| - (1/8+\alpha)n$. Every edge between $R\setminus(X\cap R)$ and $B\setminus(X\cap B)$ is blue. So there are at least $|R|-|X\cap R|$ red vertices v with

$$d_b(v) \geq |B| - |X\cap B| - \alpha n = |X\cap R| + |B| - |X| - \alpha n > |X\cap R| + n/8.$$

This implies that $a_i \geq i + n/8$ for $i = |X\cap R|+1$. (See Figure 2.)

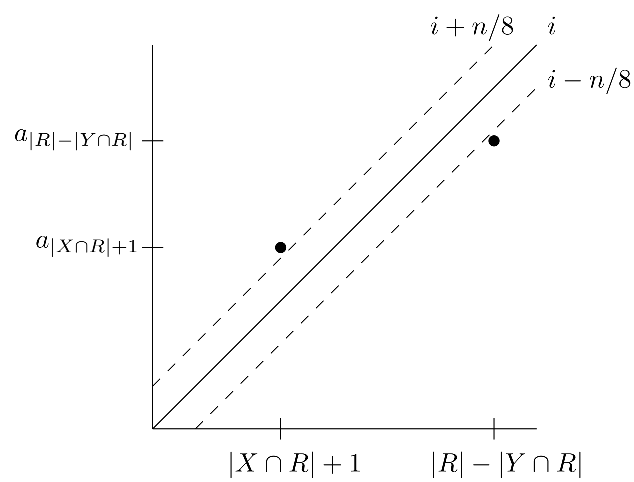

Figure 2: The sequence $a_1,\ldots,a_{|R|}$ has oscillation at least $kN/8$.

Let Y be a minimum vertex cover of the blue edges. Using König's theorem as above, we can assume that $|Y| \leq |R| - n/8$. Every edge between $R\setminus(Y\cap R)$ and $B\setminus(Y\cap B)$ is red. It follows that there are at least $|R|-|Y\cap R|$ red vertices v with

$$d_b(v) \leq |Y\cap B| = |Y| - |Y\cap R| \leq |R| - |Y\cap R| - \frac{n}{8}.$$

This implies that $a_j \leq j - n/8$ for $j = |R|-|Y\cap R|$. Thus $a_1,\ldots,a_{|R|}$ has oscillation at least $n/8 = kN/8$. $\square$

4.4 Sequences and oscillation

We now present the quite technical proof of Lemma 4.11. We will use the following definition and related lemma in order to describe the oscillation from the diagonal.

Definition 4.13 (k-good, $u_\mathrm{o}(k)$, $u_\mathrm{e}(k)$)**.** Let $a_1,\dots,a_n$ be a sequence of non-negative real numbers and let k be a positive real number. We say that the sequence is *k-good* if there exists an odd i and an even j such that $a_i \ge k$ and $a_j \ge k$. If the sequence is k-good, we define for all $0 < t \le k$

$$\begin{aligned} u_\mathrm{o}(t) &= a_1 + \cdots + a_{i_o-1} \quad \text{where } i_o = \min\{i\colon\ a_i \ge t, i \text{ odd}\}, \\ u_\mathrm{e}(t) &= a_1 + \cdots + a_{i_e-1} \quad \text{where } i_e = \min\{i\colon\ a_i \ge t, i \text{ even}\}. \end{aligned}$$

Lemma 4.14. *For all $\gamma \in \mathbb{R}^+$ there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{R}^+$ and all (kN)-good sequences, there exists a real number $t \in [k, kN]$ with*

$$u_o(t) + u_e(t) \ge \left(3 + \sqrt{8} - \gamma\right) t.$$

First we use Lemma 4.14 to prove Lemma 4.11.

Proof of Lemma 4.11. Given $\gamma > 0$, let N be obtained from Lemma 4.14. Let $k \in \mathbb{R}^+$ and $a_1,\dots,a_n$ be a sequence with oscillation at least kN. Suppose first that $a_1 \ge 1$. Partition $[n]$ into a family of non-empty intervals $I_1,\dots,I_r$ with the following properties:

- For every odd i and every $j \in I_i$, we have $a_j \ge j$.
- For every even i and every $j \in I_i$, we have $a_j < j$.

Define $s_i = \max\left\{|a_j - j|\colon\ j \in I_i\right\}$. Intuitively, this is saying that the values in the odd indexed intervals are "above the diagonal" and the values in the even indexed intervals are "below the diagonal" and s_i is the largest gap between sequence values and the "diagonal" in each interval.

Since $a_1,\dots,a_n$ has oscillation at least kN, the sequence $s_1,\dots,s_r$ is (kN)-good and thus by Lemma 4.14, there exists $t \in [k, kN]$ such that

$$u_\mathrm{o}(t) + u_\mathrm{e}(t) \ge \left(3 + \sqrt{8} - \gamma\right) t. \tag{4.4}$$

Since the sequence $a_1, a_2, \dots, a_n$ is non-decreasing, $a_j - j$ can decrease by at most one in each step and thus we have $|I_i| \ge s_i$ for every $i \in [r-1]$. Moreover, we can find bounds on $\ell^+(t)$ and $\ell^-(t)$ in terms of the s_i:

- $\ell^+(t)$ must lie in the interval I_i with the smallest odd index i_o such that $s_{i_o} \ge t$, therefore $\ell^+(t) \ge s_1 + \cdots + s_{i_o-1} = u_\mathrm{o}(t)$.
- $\ell^-(t)$ must lie in the interval I_j with the smallest even index i_e such that $s_{i_e} \ge t$. Moreover, it must be at least the t-th element in this interval, therefore $\ell^-(t) \ge s_1 + \cdots + s_{i_e-1} + t = u_\mathrm{e}(t) + t$.

Combining the previous two observations with (4.4) gives

$$\ell^+(t) + \ell^-(t) \ge u_\mathrm{o}(t) + u_\mathrm{e}(t) + t \ge \left(4 + \sqrt{8} - \gamma\right) t,$$

as desired.

If $0 \le a_1 < 1$, we start by partitioning $[n]$ into a family of non-empty intervals $I_1,\dots,I_r$ with the following properties:

- For every even i and every $j \in I_i$, we have $a_j \geq j$.
- For every odd i and every $j \in I_i$, we have $a_j < j$.

From this point, the proof is analogous. □

Finally, it remains to prove Lemma 4.14. The proof is by contradiction and the main strategy is to find a subsequence with certain properties which force the sequence to become negative eventually.

Proof of Lemma 4.14. Let $\rho = 3+\sqrt{8}-\gamma$ and let $m := m(\rho)$ be a positive integer which will be specified later. Suppose that the statement of the lemma is false for $N = 6\cdot 4^m$ and let $a_1,\ldots,a_n$ be an (Nk)-good sequence without t as in the statement. We first show that a_i has a long strictly increasing subsequence. Set

$$I = \{i\colon\ a_i \geq k, a_i > a_j \text{ for all } j < i\},$$

denote the elements of I by $i_1 \leq i_2 \leq \cdots \leq i_r$ and let $a'_j = a_{i_j}$. Consider any $j \in [r-1]$ and suppose without loss of generality that i_{j+1} is odd. For δ small enough, this implies $u_{\mathrm{o}}(a'_j+\delta) = a_1+\cdots+a_{i_{j+1}-1} \geq a'_1+\cdots+a'_j$, and $u_{\mathrm{e}}(a'_j+\delta) \geq a_1+\cdots+a_{i_{j+1}} \geq a'_1+\cdots+a'_{j+1}$. By assumption we have $u_{\mathrm{o}}(a'_j+\delta)+u_{\mathrm{e}}(a'_j+\delta) < \rho(a'_j+\delta)$. Hence, letting $\delta \to 0$ we obtain $2\left(a'_1+\cdots+a'_j\right)+a'_{j+1} \leq \rho a'_j$, which rearranges to

$$a'_{j+1} \leq (\rho-2)a'_j - 2\left(a'_1+\cdots+a'_{j-1}\right). \tag{4.5}$$

In particular, this implies $a'_{j+1} \leq (\rho-2)a'_j < 4a'_j$. Moreover, we have $a'_1 \leq u_{\mathrm{o}}(k)$ if i_1 is even and $a'_1 \leq u_{\mathrm{e}}(k)$ if i_1 is odd. Therefore,

$$6k\cdot 4^m = kN \leq a'_r < 4^r\cdot a'_1 \leq 4^r\max\{u_{\mathrm{o}}(k),u_{\mathrm{e}}(k)\} \leq 4^r(u_{\mathrm{o}}(k)+u_{\mathrm{e}}(k)) < 4^r\cdot\rho k < 6k\cdot 4^r$$

and thus $r \geq m$.

Finally, we show that any sequence of reals satisfying (4.5), will eventually become negative, but since a'_i is non-negative this will be a contradiction.

We start by defining the sequence $b_1,b_2,\ldots$ recursively by $b_1 = 1$ and $b_{i+1} = (\rho-2)b_i - 2(b_1+\cdots+b_{i-1})$. Note that

$$\begin{aligned}
b_{i+1} &= (\rho-2)b_i - 2(b_1+\cdots+b_{i-1}) \\
&= (\rho-1)b_i - b_i - 2(b_1+\cdots+b_{i-1}) \\
&= (\rho-1)b_i - ((\rho-2)b_{i-1} - 2(b_1+\cdots+b_{i-2})) - 2(b_1+\cdots+b_{i-1}) \\
&= (\rho-1)b_i - \rho b_{i-1}
\end{aligned}$$

So equivalently the sequence is defined by,

$$b_1 = 1,\ b_2 = \rho-2, \text{ and } b_{i+1} = (\rho-1)b_i - \rho b_{i-1} \text{ for } i \geq 2.$$

It is known that a second order linear recurrence relation whose characteristic polynomial has non-real roots will eventually become negative (see [1]). Indeed, the characteristic polynomial $x^2-(\rho-1)x+\rho$ has discriminant $\rho^2-6\rho+1 < 0$ and so its roots $\alpha,\bar{\alpha}$ are non-real. Hence the above recursively defined sequence has the closed form of $b_i = z\alpha^i + \bar{z}\bar{\alpha}^i = 2\mathrm{Re}\left(z\alpha^i\right)$ for some complex number z. By expressing $z\alpha^i$ in polar form we can see that $b_m < 0$ for some positive integer m. Note that the calculation of m only depends on ρ.

Now let $a'_1, \dots, a'_m$ be a sequence of non-negative reals satisfying (4.5). We will be done if we can show that $a'_j \le a'_1 b_j$ for all $1 \le j \le m$; so suppose $a'_s > a'_1 b_s$ for some s, and such that $\{a'_j\}_{j=1}^m$ and $\{a'_1 b_j\}_{j=1}^m$ coincide on the longest initial subsequence. Let p be the minimum value such that $a'_p \neq a'_1 b_p$. Clearly $p > 1$. Applying (4.5) to $j = p-1$ we see that

$$\begin{aligned} a'_p \le (\rho-2)a'_{p-1} - 2(a'_1 + \cdots + a'_{p-2}) &= (\rho-2)a'_1 b_{p-1} - 2(a'_1 b_1 + \cdots + a'_1 b_{p-2}) \\ &= a'_1((\rho-2)b_{p-1} - 2(b_1 + \cdots + b_{p-2})) = a'_1 b_p \end{aligned}$$

and thus $a'_p < a'_1 b_p$.

Let $\beta = (a'_1 b_p - a'_p)/a'_1 > 0$. Now consider the sequence a''_j where $a''_j = a'_j$ for $j < p$ and $a''_j = a'_j + \beta a'_{j-p+1}$ for $j \ge p$. Then $a''_p = a'_1 b_p = a''_1 b_p$. Clearly, this new sequence satisfies (4.5) for every $j < p$. Furthermore, we have

$$\begin{aligned} a''_{p+j} &= a'_{p+j} + \beta a'_{j+1} \\ &\le (\rho-2)a'_{p+j-1} - 2\left(a'_1 + \cdots + a'_{p+j-2}\right) + \beta(\rho-2)a'_j - 2\beta\left(a'_1 + \cdots + a'_{j-1}\right) \\ &= (\rho-2)a''_{p+j-1} - 2\left(a''_1 + \cdots + a''_{p+j-2}\right) \end{aligned}$$

for every $j \ge 0$. Hence, the whole sequence satisfies (4.5). We also have $a''_s \ge a'_s > a'_1 b_s = a''_1 b_s$. This contradicts the fact that a'_j was such a sequence which coincided with $a'_1 b_j$ on the longest initial subsequence. □

Acknowledgments

This project began as part of the problem session of the “Extremal Graph Theory and Ramsey Theory” focus week of the “Rio Workshop on Extremal and Structural Combinatorics” held at IMPA, Rio de Janeiro, Brazil in January 2018. We thank the organisers of the workshop and IMPA for the stimulating working environment.

We also thank the referees for their careful reading of the paper and their helpful suggestions.

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