Soliton Solutions of Noncommutative Anti-Self-Dual Yang-Mills Equations

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Abstract

We present exact soliton solutions of anti-self-dual Yang-Mills equations for $G = GL(N)$ on noncommutative Euclidean spaces in four-dimension by using the Darboux transformations. Generated solutions are represented by quasideterminants of Wronski matrices in compact forms. We give special one-soliton solutions for $G = GL(2)$ whose energy density can be real-valued. We find that the soliton solutions are the same as the commutative ones and can be interpreted as one-domain walls in four-dimension. Scattering processes of the multi-soliton solutions are also discussed.

[1] Jonathan Nimmo sadly passed away on 20 June 2017. We owe section 4 to his unpublished note.

1 Introduction

Noncommutative extension of integrable systems has been studied for many years and various integrability aspects have been revealed. (See, e.g. [1, 5, 6, 7, 10, 11, 25, 26, 36, 38] and references therein.) In particular, exact soliton solutions have been found to be constructed in terms of quasideterminants in compact forms. The quasideterminants are introduced first by Gelfand and Retakh [9] toward a unification of theory of noncommutative determinants. They actually play crucial roles in the theory of noncommutative solitons and simplify proofs in commutative theories. This suggests that quasideterminants might be fundamental objects which lead to a new essential formulation of soliton theory.

Darboux transformations are one of powerful methods to construct exact Wronskian-type solutions of integrable systems [33]. This can be applied to wide class of noncommutative integrable equations in three-dimension of space-time or less. (See, e.g. [12, 13, 14, 15, 16, 19, 29, 42] and references therein.) For example, the noncommutative potential KP equation is derived in the framework of the noncommutative integrable hierarchy as follows:

$$(v_t + v_{xxx} + 3v_x v_x)_x - 3v_{yy} - 3[v_x, v_y] = 0, \tag{1.1}$$

where the subscripts denote partial derivatives. Every variable belongs to a noncommutative ring (e.g. quarternion). Differentiation and intergration are assumed to be as in commutative case. The Wronskian solutions are first given [7] and derived by using the Darboux transformation in a natural way [13].

Extension of integrable equations to noncommutative (NC) spaces is also a hot topic. (See, e.g. [4, 17, 20, 22, 27, 28, 30, 31, 37, 44, 45] and references therein.) In gauge theories, noncommutative extensions are equivalent to the presence of background $U(1)$ flux. In four dimension, anti-self-dual Yang-Mills (ASDYM) equations are important in both mathematics and physics, including integrable systems as well [2, 32, 40, 43, 46]. Noncommutative anti-self-dual Yang-Mills equations have attracted a lot of interest in recent development of gauge theories and string theories. By using noncommutative Penrose-Ward transformations, exact solutions are constructed in terms of quasideterminants which are not Wronskian-type [10, 11]. The solutions include not only instantons but soliton-type solutions. Two soliton scatterings are discussed for $G = GL(2)$ [21], where energy densities (the second Chern classes) are in general complex valued. For n-soliton scatterings, Wronskian-type solutions are suitable. In order to interpret physical meaning of the solutions, energy densities should be real-valued.

In this paper, we construct exact multi soliton solutions of anti-self-dual Yang-Mills equations for $G = GL(N)$ in four-dimensional noncommutative Euclidean spaces by generalizing the Darboux transformations [35] to noncommutative settings. The generated solutions are described in terms of quasideterminants in compact forms. The proofs need only a few formula of quasideterminants. We focus on soliton solutions for $G = GL(2)$ where energy densities of them can be real. We find that the one-soliton solutions have the same configurations as commutative ones which have localized energy densities on three-dimensional hyperplanes. These are domain-walls in $\mathbb{R}^4$. We show that asymptotic behaviors of multi-soliton solutions are the same as commutative ones. Therefore the n soliton solutions would be n intersecting domain-walls with phase shifts. In discussion of the soliton scatterings, the properties of quasideterminants play a key role.

This paper is organized as follows. In section 2, we introduce noncommutative anti-self-dual Yang-Mills equations in the star-product formalism. In section 3, we review definitions of quasi-determinants and summarize some properties of them. In section 4, we present a noncommutative version of the Darboux transformation for $G = GL(N)$ noncommutative anti-self-dual Yang-Mills equations to generate exact solutions of them. We obtain the solutions of Yang's J and K matrices in terms of quasideterminants. The results are all new and reduce to the commutative one [35]. In section 5, we give exact soliton solutions of the noncommutative anti-self-dual Yang-Mills equation for $G = GL(N)$ which are Wronskian-type. We focus on $G = GL(2)$ case and present one soliton solutions with real-valued energy densities. We also give multi soliton solutions of the same type and discuss the asymptotic behavior. The results are also brand new. Section 6 is devoted conclusion and discussion.

2 ASDYM Equations in Noncommutative Spaces

Noncommutative spaces are defined by the noncommutativity of the coordinates:

$$[x^\mu, x^\nu] = i\vartheta^{\mu\nu}, \tag{2.1}$$

where the constant $\vartheta^{\mu\nu}$ is called the noncommutative parameter. If the coordinates are real, the noncommutative parameters should be real because of hermicity of the coordinates. We note that the noncommutative parameter $\vartheta^{\mu\nu}$ is anti-symmetric with respect to μ and ν which implies that the rank of it is even.

Noncommutative field theories are given by the replacement of ordinary products in the commutative field theories with the star-products. The star-product is defined for ordinary c-number functions. On flat spaces, is it represented explicitly by

$$\begin{aligned} f\star g(x) &:= \exp\left(\frac{i}{2}\vartheta^{\mu\nu}\partial_\mu^{(x_1)}\partial_\nu^{(x_2)}\right) f(x_1)g(x_2)\Big|_{x_1=x_2=x} \\ &= f(x)g(x)+\frac{i}{2}\vartheta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x)+\mathcal{O}(\vartheta^2), \end{aligned} \tag{2.2}$$

where $\partial_\mu^{(x)} := \partial/\partial x^\mu$. This is known as the Moyal product [34]. The ordering of fields in nonlinear terms are determined so that some structures such as gauge symmetries should be preserved.

The star-product has associativity: $f\star(g\star h)=(f\star g)\star h$. It reduces to the ordinary product in the commutative limit: $\vartheta^{\mu\nu}\to 0$. In this sense, the noncommutative field theories are deformed theories from the commutative ones. The replacement of the product makes the ordinary spatial coordinates "noncommutative," in the sense that $[x^\mu, x^\nu]_\star := x^\mu\star x^\nu - x^\nu\star x^\mu = i\vartheta^{\mu\nu}$.

We note that the fields themselves take c-number values. The differentiation and the integration for them are the same as commutative ones. Field equations are deformed and hence solutions of them are also deformed from commutative ones.

Now let us introduce the noncommutative anti-self-dual Yang-Mills equations in 4-dimensional noncommutative Euclidean spaces whose real coordinates are denoted by x^μ $(\mu=1,2,3,4)$. The gauge group is denoted by G. In this paper we consider $G = GL(N,\mathbb{C})$ or subgroups of it.

Complex coordinates are introduced as follows:

$$y := \frac{1}{\sqrt{2}}\left(x^1+ix^2\right),\ z := \frac{1}{\sqrt{2}}\left(x^3+ix^4\right). \tag{2.3}$$

The noncommutative anti-self-dual Yang-Mills equation is derived from the compatibility condition of the following linear system:

$$\begin{aligned} L\star\phi &:= (D_y-\zeta D_{\overline{z}})\star\phi = (\partial_y + A_y - \zeta(\partial_{\overline{z}}+A_{\overline{z}}))\star\phi(x;\zeta)=0, \\ M\star\phi &:= (D_z+\zeta D_{\overline{y}})\star\phi = (\partial_z + A_z + \zeta(\partial_{\overline{y}}+A_{\overline{y}}))\star\phi(x;\zeta)=0, \end{aligned} \tag{2.4}$$

where $A_z, A_y, A_{\overline{z}}, A_{\overline{y}}$ and $D_z, D_y, D_{\overline{z}}, D_{\overline{y}}$ denote gauge fields and covariant derivatives in the Yang-Mills theory, respectively. The constant $\zeta\in\mathbb{C}P_1$ is called the *spectral parameter*.

The compatibility condition $[L,M]_\star=0$, gives rise to a quadratic polynomial of ζ and each coefficient yields the anti-self-dual Yang-Mills equation:

$$F_{yz}^\star=0,\quad F_{\overline{y}\,\overline{z}}^\star=0,\quad F_{z\overline{z}}^\star+F_{y\overline{y}}^\star=0, \tag{2.5}$$

where $F_{\mu\nu}^\star := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star$ denotes the field strength. This is equivalent to self-duality in the sense of the Hodge dual operator $*$: $F_{\mu\nu}^\star = -*F_{\mu\nu}^\star$.

Gauge transformations act on the linear system (2.4) as

$$L \mapsto g^{-1} \star L \star g, \;\; M \mapsto g^{-1} \star M \star g, \;\; \phi \mapsto g^{-1} \star \phi, \quad g(x) \in G. \tag{2.6}$$

Here we discuss the potential forms of the noncommutative anti-self-dual Yang-Mills equations such as noncommutative J, K-matrix formalism and the noncommutative Yang's equation [44].

The *J-matrix formalism* is given as follows. The first equation of noncommutative anti-self-dual Yang-Mills equation (2.5) is the compatible condition of the linear system $D_z \star h = 0, D_y \star h = 0$, where h is a $N \times N$ matrix. Hence we get $A_z = -(\partial_z h) \star h^{-1}, \; A_y = -(\partial_y h) \star h^{-1}$. Similarly, the second equation of noncommutative anti-self-dual Yang-Mills equation (2.5) leads to $A_{\overline{z}} = -(\partial_{\overline{z}} \widetilde{h}) \star \widetilde{h}^{-1}, \; A_{\overline{y}} = (\partial_{\overline{y}} \widetilde{h}) \star \widetilde{h}^{-1}$, where $\widetilde{h}$ is also a $N \times N$ matrix. We note that $h(x) = \phi(x, \zeta = 0), \widetilde{h}(x) = \widetilde{\phi}(x, \zeta = \infty)$.

By defining a new matrix $J = \widetilde{h} \star h^{-1}$, the third equation of the noncommutative anti-self-dual Yang-Mills equation (2.5) becomes the noncommutative Yang's equation

$$\partial_{\overline{z}}(\partial_z J \star J^{-1}) + \partial_{\overline{y}}(\partial_y J \star J^{-1}) = 0. \tag{2.7}$$

Gauge transformations act on h and $\widetilde{h}$ as $h \mapsto g^{-1} \star h, \; \widetilde{h} \mapsto g^{-1} \star \widetilde{h}, \; g(x) \in G$. Hence the Yang's J-matrix is gauge invariant. We note that all $h, \widetilde{h}$ and J take values in G.

There is another potential form of the noncommutative anti-self-dual Yang-Mills equation, known as the *K-matrix formalism*. In the gauge $A_{\overline{y}} = A_{\overline{z}} = 0$, the third equation of (2.5) becomes $\partial_{\overline{z}} A_z + \partial_{\overline{y}} A_y = 0$. This implies the existence of a potential K such that $A_z = -\partial_{\overline{y}} K, A_y = \partial_{\overline{z}} K$. Then the second equation of (2.5) becomes

$$\partial_y \partial_{\overline{y}} K + \partial_z \partial_{\overline{z}} K + [\partial_{\overline{z}} K, \partial_{\overline{y}} K]_\star = 0. \tag{2.8}$$

Then, we get

$$A_z = -\partial_z J \star J^{-1} = -\partial_{\overline{y}} K, \quad A_y = -\partial_y J \star J^{-1} = \partial_{\overline{z}} K. \tag{2.9}$$

In gauge theory, gauge invariant quantities are important. In this paper, we will discuss the following quantity, which is called the energy density,

$$\mathrm{Tr} F_\star^2 \;\; := \;\; \mathrm{Tr} F_{\mu\nu}^\star \star F_\star^{\mu\nu}. \tag{2.10}$$

We follow the Einstein convention. In the right hand side, the summation over $\mu, \nu = 1, 2, 3, 4$ is taken.

3 Review of Quasi-determinants

In this section, we give a quick review of quasi-determinants introduced by Gelfand and Retakh [9] and present a few properties of them which play key roles in the following sections. Detailed discussion is seen in e.g. [8].

Quasi-determinants are not just a generalization of usual commutative determinants but rather related to inverse matrices. From now on, we assume existence of the inverses in any case.

Let $A = (a_{ij})$ be a $N \times N$ matrix and $B = (b_{ij})$ be the inverse matrix of A: $A \star B = B \star A = 1$. In this paper, all products of matrix elements are assumed to be star-products.

Quasi-determinants of A are defined formally as the inverse of the elements of $B = A^{-1}$:

$$|A|^{\star}_{ij} := b_{ji}^{-1}. \tag{3.1}$$

In the commutative limit, this is reduced to

$$|A|^{\star}_{ij} \stackrel{\theta\to 0}{\longrightarrow} (-1)^{i+j} \frac{\det A}{\det A^{ij}}, \tag{3.2}$$

where A^{ij} is the matrix obtained from A by deleting the i-th row and the j-th column.

We can write down a more explicit form of quasi-determinants. In order to see it, let us recall the following formula for the inverse 2×2 block matrix:

$$\begin{bmatrix} A & B \\ C & d \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1} \star B \star s^{-1} \star C \star A^{-1} & -A^{-1} \star B \star s^{-1} \\ -s^{-1} \star C \star A^{-1} & s^{-1} \end{bmatrix},$$

where A is a square matrix and d is a single element and $s := d - C \star A^{-1} \star B$ is called the Schur complement. We note that any matrix can be decomposed as a 2×2 matrix by block decomposition where one of the diagonal parts is 1×1. By choosing an appropriate partitioning, any element in the inverse of a square matrix can be expressed as the inverse of the Schur complement. Hence quasi-determinants can be defined iteratively by:

$$\begin{aligned} |A|^{\star}_{ij} &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star ((A^{ij})^{-1})_{i'j'} \star a_{j'j} \\ &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star (|A^{ij}|_{j'i'})^{-1} \star a_{j'j}. \end{aligned} \tag{3.3}$$

It is convenient to represent the quasi-determinant as follows:

$$|A|^{\star}_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & \boxed{a_{ij}} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}_{\star}. \tag{3.4}$$

Examples of quasi-determinants are, for a 1×1 matrix $A = a$

$$|A|^\star = a,$$

and for a 2×2 matrix $A = (a_{ij})$

$$|A|^\star_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_\star = a_{11} - a_{12} \star a_{22}^{-1} \star a_{21}, \quad |A|^\star_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix}_\star = a_{12} - a_{11} \star a_{21}^{-1} \star a_{22},$$
$$|A|^\star_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix}_\star = a_{21} - a_{22} \star a_{12}^{-1} \star a_{11}, \quad |A|^\star_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}_\star = a_{22} - a_{21} \star a_{11}^{-1} \star a_{12},$$

and for a 3×3 matrix $A = (a_{ij})$

$$\begin{aligned} |A|^\star_{11} &= \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_\star = a_{11} - (a_{12}, a_{13}) \star \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \star \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} \\ &= a_{11} - a_{12} \star \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_\star^{-1} \star a_{21} - a_{12} \star \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}_\star^{-1} \star a_{31} \\ &\qquad - a_{13} \star \begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}_\star^{-1} \star a_{21} - a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}_\star^{-1} \star a_{31}, \end{aligned}$$

and so on.

Quasideterminants have various interesting properties similar to those of determinants. Among them, the following ones are relevant to the discussion on soliton scattering.

Proposition 3.1 [9] Let $A = (a_{ij})$ be a square matrix of order n.
(i) Permutation of Rows and Columns.

The quasi-determinant $|A|^\star_{ij}$ does not depend on permutations of rows and columns in the matrix A that do not involve the i-th row and j-th column.
(ii) The multiplication of rows and columns.

Let the matrix $M = (m_{ij})$ be obtained from the matrix A by multiplying the i-th row by $f(x)$ from the left, that is, $m_{ij} = f \star a_{ij}$ and $m_{kj} = a_{kj}$ for $k \neq i$. Then

$$|M|^\star_{kj} = \begin{cases} f \star |A|^\star_{ij} & \text{for } k = i \\ |A|^\star_{kj} & \text{for } k \neq i \ \text{ and } \ f \text{ is invertible} \end{cases} \tag{3.5}$$

Let the matrix $N = (n_{ij})$ be obtained from the matrix A by multiplying the j-th column by $f(x)$ from the right, that is, $n_{ij} = a_{ij} \star f$ and $n_{il} = a_{il}$ for $l \neq j$. Then

$$|N|^\star{}_{il} = \begin{cases} |A|^\star_{ij} \star f & \text{for } l = j \\ |A|^\star_{il} & \text{for } l \neq j \ \text{ and } \ f \text{ is invertible} \end{cases} \tag{3.6}$$

(iii) The addition of rows and columns.

Let the matrix $M = (m_{ij})$ be obtained from the matrix A by replacing the k-th row of A with the sum of the k-th row and l-th row, that is, $m_{kj} = a_{kj} + a_{lj}$ and $m_{ij} = a_{ij}$ for $k \neq i$. Then

$$|A|^{\star}_{ij} = |M|^{\star}_{ij}, \quad \text{for } i \neq k. \tag{3.7}$$

Let the matrix $N = (n_{ij})$ be obtained from the matrix A by replacing the k-th column of A with the sum of the k-th column and l-th column, that is, $n_{ik} = a_{ik} + a_{il}$ and $n_{ij} = a_{ij}$ for $k \neq j$. Then

$$|A|^{\star}_{ij} = |N|^{\star}_{ij}, \quad \text{for } j \neq k. \tag{3.8}$$

Proposition 3.2 [9] If the quasi-determinant $|A|_{ij}$ is defined, then the following statements are equivalent.
(i) $|A|^{\star}_{ij} = 0$.
(ii) the i-th row of the matrix A is a left linear combination of the other rows of A.
(iii) the j-th column of the matrix A is a right linear combination of the other columns of A.

Proposition 3.3 In the block matrices given in the following results, lower case letters denote single entries and upper case letters denote matrices of compatible dimensions so that the overall matrix is square.
(i) NC Jacobi identity [13] (A useful special case of the NC Sylvester's Theorem[9]):

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}_{\star} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix}_{\star} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix}_{\star} \star \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}_{\star}^{-1} \star \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}_{\star}. \tag{3.9}$$

(ii) Homological relations [9]

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix}_{\star} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}_{\star} \star \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}, \quad \begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix}_{\star} = \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix}_{\star} \star \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}_{\star} \tag{3.10}$$

(iii) A derivative formula for quasideterminants [13]

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}_{\star}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix}_{\star} + \sum_{k=1}^{n} \begin{vmatrix} A & e_k \\ C & \boxed{0} \end{vmatrix}_{\star} \star \begin{vmatrix} A & B \\ (A_k)' & \boxed{(B_k)'} \end{vmatrix}_{\star}, \tag{3.11}$$

$$= \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix}_{\star} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix}_{\star} \star \begin{vmatrix} A & B \\ e_k^t & \boxed{0} \end{vmatrix}_{\star}, \tag{3.12}$$

where A_k is the kth column and A^k is the kth row of a matrix A and e_k is the column n-vector (δ_{ik}) (i.e. 1 in the kth row and 0 elsewhere).

We note that the definition of the quasideterminants are valid in the case that the elements of matrices belong to noncommutative associative algebras, for example, the case that all elements a_{ij} in (3.4) are $N \times N$ matrices. Propositions 3.1-3.3 still hold. In the next section, we will consider such situations.

4 Darboux Transformation for NC ASDYM equation

In this section, we give the Darboux transformation for the $G = GL(N)$ noncommutative anti-self-dual Yang-Mills equation.

Let us start with the following linear systems:

$$
\begin{aligned}
L \star \phi &:= J \star \partial_y (J^{-1} \star \phi) - (\partial_{\bar{z}} \phi)\zeta = 0, \\
M \star \phi &:= J \star \partial_z (J^{-1} \star \phi) + (\partial_{\bar{y}} \phi)\zeta = 0,
\end{aligned}
\tag{4.1}
$$

where ζ is a matrix generalization of the spectral parameter and $N \times N$ constant matrix. We note that this matrix acts on ϕ from right side where ϕ is an $N \times N$ matrix whose columns consist of N-independent solutions of the linear systems. We can find that the compatibility condition $L \star M \star \phi - M \star L \star \phi = 0$ gives the noncommutative anti-self-dual Yang-Mills equation.

Here the existence of N-independent solutions of the linear systems is assumed because of lack of Frobenius theorem. We will see in Theorem 5.1 that there exists $N \times N$ matrix ϕ which leads to soliton-type solutions when the spectral parameter matrix ζ is diagonal.

Theorem 4.1 The linear system (4.1) is covariant under the following Darboux transformation:

$$
\widetilde{\phi} = \phi\zeta - \theta\Lambda \star \theta^{-1} \star \phi, \tag{4.2}
$$

$$
\widetilde{J} = -\theta\Lambda \star \theta^{-1} \star J, \tag{4.3}
$$

when θ is an eigenfunction of the linear system (4.1) for the choice of eigenvalue $\zeta = \Lambda$. We use the notation $\phi^{(k)} := \phi\zeta^k$, where ϕ and ζ are any eigenfunction-eigenvalue pair. (Hence $\theta^{(k)} := \theta\Lambda^k$.)
(Proof) It is proved by straightforward calculation by using original conditions $J \star \partial_y (J^{-1} \star$

$\phi) - (\partial_{\bar{z}}\phi)\zeta = 0$ and $J \star \partial_y(J^{-1} \star \theta) - (\partial_{\bar{z}}\theta)\Lambda = 0$ and Eqs.(4.2), (4.3):

$$\begin{aligned}
\widetilde{L}\widetilde{\phi} &= \widetilde{J} \star \partial_y(\widetilde{J}^{-1} \star \widetilde{\phi}) - (\partial_{\bar{z}}\widetilde{\phi})\zeta \\
&= -\theta\Lambda \star \theta^{-1} \star J \star \partial_y(-J^{-1} \star \theta\Lambda^{-1} \star \theta^{-1} \star \phi\zeta + J^{-1} \star \phi) - \partial_{\bar{z}}(\phi\zeta - \theta\Lambda \star \theta^{-1} \star \phi)\zeta \\
&= \theta\Lambda \star \theta^{-1} \star [J \star (\partial_y J^{-1}) \star \theta + \partial_y\theta]\Lambda^{-1} \star \theta^{-1} \star \phi\zeta + \theta \star (\partial_y\theta^{-1}) \star \phi\zeta + \partial_y(\phi\zeta) \\
&\quad -\theta\Lambda \star \theta^{-1} \star J \star \partial_y(J^{-1} \star \phi) \\
&\quad -(\partial_{\bar{z}}\phi)\zeta^2 + (\partial_{\bar{z}}\theta)\Lambda \star \theta^{-1} \star \phi\zeta + \theta\Lambda \star (\partial_{\bar{z}}\theta^{-1}) \star \phi\zeta + \theta\Lambda \star \theta^{-1} \star (\partial_{\bar{z}}\phi)\zeta \\
&= \theta\Lambda \star \theta^{-1} \star \left\{-J \star \partial_y(J^{-1} \star \phi) + (\partial_{\bar{z}}\phi)\zeta\right\} \\
&\quad +\theta\Lambda \star \theta^{-1} \star \left\{\partial_y\theta - (\partial_y J) \star J^{-1} \star \theta - (\partial_{\bar{z}}\theta)\Lambda\right\} \Lambda^{-1} \star \theta^{-1} \star \phi\zeta \\
&\quad + \left\{-\partial_y\theta - J \star (\partial_y J^{-1}) \star \theta + (\partial_{\bar{z}}\theta)\Lambda\right\} \star \theta^{-1} \star \phi\zeta = 0
\end{aligned}$$

In the last line, we substitute $\partial_y\phi - \partial_{\bar{z}}\phi\zeta$ with $-J \star (\partial_y J^{-1}) \star \phi$ by using the original condition. It is similar to show $\widetilde{M}\widetilde{\phi} = 0$. $\Box$

Iterating the Darboux transformation (4.2) and (4.3), we get solutions of the linear systems.

Theorem 4.2 Let us prepare eigenfunction-eigenvalue pairs (θ_i, Λ_i) $i = 1, 2, \cdots, n$, and consider n iteration of the Darboux transformation (4.2):

$$\phi_{[k+1]} = \phi_{[k]}^{(1)} - \theta_{[k]}^{(1)} \star \theta_{[k]}^{-1} \star \phi_{[k]} = \begin{vmatrix} \theta_{[k]} & \phi_{[k]} \\ \theta_{[k]}^{(1)} & \boxed{\phi_{[k]}^{(1)}} \end{vmatrix}_\star, \quad \phi_{[1]} = \phi, \tag{4.4}$$

$$J_{[k+1]} = -\theta_{[k]}^{(1)} \star \theta_{[k]}^{-1} \star J_{[k]} = \begin{vmatrix} \theta_{[k]} & 1 \\ \theta_{[k]}^{(1)} & \boxed{0} \end{vmatrix}_\star \star J_{[k]}, \quad J_{[1]} = J, \tag{4.5}$$

where $\theta_{[k]} = \phi_{[k]}|_{\phi \to \theta_k}$. As is commented at the end of section 3, the quasideterminants are well-defined in this situation with all elements $\theta_{[k]}, \phi_{[k]}, \theta_{[k]}^{(1)}, \phi_{[k]}^{(1)}$ being $N \times N$ matrices. Then the solution to the linear system (4.1) is

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}_\star, \tag{4.6}$$

where Θ depends on how many times the Darboux transformation is iterated. It is defined by the $N \times iN$ matrix $\Theta := (\theta_1, \cdots, \theta_i)$ for i iterations. The simbol $\Theta^{(k)}$ is defined by $\Theta^{(k)} := \Theta\Lambda^k$ and $\Lambda := \text{diag}(\Lambda_1, \cdots, \Lambda_i)$. Note that in (4.6), $\Theta^{(k)}$ is an $N \times nN$ matrix with $\Theta = (\theta_1, \cdots, \theta_n)$ and $\Lambda = \text{diag}(\Lambda_1, \cdots, \Lambda_n)$.

(Proof) Proof is made by induction. It holds for $n=1$ by definition. Assuming that it holds for some k $(1\le k\le n)$, we consider

$$Q := \phi_{[k+2]} = \phi^{(1)}_{[k+1]} - \theta^{(1)}_{[k+1]}\star\theta^{-1}_{[k+1]}\star\phi_{[k+1]} = \begin{vmatrix} \theta_{[k+1]} & \phi_{[k+1]} \\ \theta^{(1)}_{[k+1]} & \boxed{\phi^{(1)}_{[k+1]}} \end{vmatrix}_\star .$$

(Here we change the lower indices $k \longrightarrow k+1$ and $\widetilde{\phi}_{[k+1]} \longrightarrow \phi_{[k+1]}$)

The first term in Q is

$$\phi^{(1)}_{[k+1]} = \phi_{[k+1]}\zeta = \begin{vmatrix} \Theta & \phi\zeta \\ \vdots & \vdots \\ \Theta^{(k-1)} & \phi^{(k-1)}\zeta \\ \Theta^{(k)} & \boxed{\phi^{(k)}\zeta} \end{vmatrix}_\star = \begin{vmatrix} \Theta\Lambda & \phi\zeta \\ \vdots & \vdots \\ \Theta^{(k-1)}\Lambda & \phi^{(k-1)}\zeta \\ \Theta^{(k)}\Lambda & \boxed{\phi^{(k)}\zeta} \end{vmatrix}_\star = \begin{vmatrix} \Theta^{(1)} & \phi^{(1)} \\ \vdots & \vdots \\ \Theta^{(k)} & \phi^{(k)} \\ \Theta^{(k+1)} & \boxed{\phi^{(k+1)}} \end{vmatrix}_\star ,$$

where we have applied proposition 3.1. Note that $\Theta = (\theta_1,\cdots,\theta_k)$ here. Next by using homological relation,

$$\begin{aligned}
\theta^{-1}_{[k+1]}\star\phi_{[k+1]} &= \begin{vmatrix} \Theta & \theta_{k+1} \\ \vdots & \vdots \\ \Theta^{(k-1)} & \theta^{(k-1)}_{k+1} \\ \Theta^{(k)} & \boxed{\theta^{(k)}_{k+1}} \end{vmatrix}_\star^{-1} \star \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(k-1)} & \phi^{(k-1)} \\ \Theta^{(k)} & \boxed{\phi^{(k)}} \end{vmatrix}_\star \\
&= \left(\begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \star \begin{vmatrix} \Theta & \boxed{\theta_{k+1}} \\ \vdots & \vdots \\ \Theta^{(k-1)} & \theta^{(k-1)}_{k+1} \\ \Theta^{(k)} & \theta^{(k))}_{k+1} \end{vmatrix}_\star \right)^{-1} \star \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(k-1)} & \phi^{(k-1)} \\ \Theta^{(k)} & \boxed{\phi^{(k)}} \end{vmatrix}_\star \\
&= \begin{vmatrix} \Theta & \boxed{\theta_{k+1}} \\ \vdots & \vdots \\ \Theta^{(k-1)} & \theta^{(k-1)}_{k+1} \\ \Theta^{(k)} & \theta^{(k)}_{k+1} \end{vmatrix}_\star^{-1} \star \begin{vmatrix} \Theta & \boxed{\phi} \\ \vdots & \vdots \\ \Theta^{(k-1)} & \phi^{(k-1)} \\ \Theta^{(k)} & \phi^{(k)} \end{vmatrix}_\star .
\end{aligned}$$

It then follows immediately by using NC Jacobi identity that

$$Q = \begin{vmatrix} \Theta & \theta_{k+1} & \phi \\ \Theta^{(1)} & \theta^{(1)}_{k+1} & \phi^{(1)} \\ \vdots & \vdots & \vdots \\ \Theta^{(k)} & \theta^{(k)}_{k+1} & \phi^{(k)} \\ \Theta^{(k+1)} & \theta^{(k+1)}_{k+1} & \boxed{\phi^{(k+1)}} \end{vmatrix}_\star . \qquad \square$$

Theorem 4.3 Yang's J matrix generated from a trivial seed solution $J = 1$ is represented by a single quasideterminant as follows:

$$J_{[n+1]} = \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}_{\star}, \tag{4.7}$$

(Proof) This follows immediately by induction and the definition (4.3) and the formula

$$-\theta^{(1)}_{[k+1]}\theta^{-1}_{[k+1]} = \begin{vmatrix} \Theta & \theta_{k+1} & 1 \\ \Theta^{(1)} & \theta^{(1)}_{k+1} & 0 \\ \vdots & & \vdots \\ \Theta^{(k)} & \theta^{(k)}_{k+1} & 0 \\ \Theta^{(k+1)} & \theta^{(k+1)}_{k+1} & \boxed{0} \end{vmatrix}_{\star} \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_{\star}^{-1}. \qquad \square$$

Theorem 4.4 Yang's K matrix generated from a trivial seed solution $K = 0$ is represented by a single quasideterminant as follows:

$$K_{[n+1]} = -\begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}_{\star}. \tag{4.8}$$

(Proof) First we note that the recursion relation on Θ holds by definition:

$$\partial_y\Theta - \partial_{\tilde{z}}\Theta \cdot \Lambda = 0, \quad \partial_z\Theta + \partial_{\tilde{y}}\Theta \cdot \Lambda = 0. \tag{4.9}$$

By using the derivative formula (3.11) and the recursion relation (4.9), we have

$$
\begin{aligned}
\partial_y J_{[k+1]} \star J_{[k+1]}^{-1} &= \left(\begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \partial_y \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star + \sum_{p=0}^{k-1} \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(p)} & 1 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \partial_y \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \right) \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star^{-1} \\
&= \left(\begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \partial_{\bar{z}} \Theta^{(k+1)} & \boxed{0} \end{vmatrix}_\star + \sum_{p=0}^{k-1} \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(p)} & 1 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \partial_{\bar{z}} \Theta^{(k+1)} & \boxed{0} \end{vmatrix}_\star \right) \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star^{-1} \\
&= - \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 1 \\ \partial_{\bar{z}} \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star - \sum_{p=0}^{k-1} \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(p)} & 1 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \star \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 1 \\ \partial_{\bar{z}} \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star = -\partial_{\bar{z}} \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 1 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star
\end{aligned}
$$

where the final line uses the following formula, obtained by mimicking part of the proof of the derivative formula (3.11)

$$
\begin{aligned}
\begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \partial_{\bar{z}} \Theta^{(k+1)} & \boxed{0} \end{vmatrix}_\star &= -\partial_{\bar{z}} \Theta \Lambda^{k+1} \star \begin{pmatrix} \Theta \\ \Theta^{(1)} \\ \vdots \\ \Theta^{(k-1)} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= -\partial_{\bar{z}} \Theta \Lambda^{k} \star \begin{pmatrix} \Theta \\ \Theta^{(1)} \\ \vdots \\ \Theta^{(k-1)} \end{pmatrix}^{-1} \star \sum_{p=1}^{k} e_p e_p^t \begin{pmatrix} \Theta \\ \Theta^{(1)} \\ \vdots \\ \Theta^{(k-1)} \end{pmatrix} \Lambda \star \begin{pmatrix} \Theta \\ \Theta^{(1)} \\ \vdots \\ \Theta^{(k-1)} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= -\sum_{p=0}^{k-1} \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(p)} & 1 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \partial_{\bar{z}} \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(p+1)} & \boxed{0} \end{vmatrix}_\star = - \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 1 \\ \partial_{\bar{z}} \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star \star \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(k-1)} & 0 \\ \Theta^{(k)} & \boxed{0} \end{vmatrix}_\star .
\end{aligned}
$$

Gauge fields are obtained from these potentials:

$$A_z^{[n+1]} = -\partial_z J_{[n+1]} \star J_{[n+1]}^{-1} = -\partial_{\overline{y}} K_{[n+1]}, \ A_y^{[n+1]} = -\partial_y J_{[n+1]} \star J_{[n+1]}^{-1} = \partial_{\overline{z}} K_{[n+1]}. \quad (4.10)$$

5 Soliton Solutions of NC ASDYM equation

In this section, we discuss exact soliton solutions for $G = GL(N)$ focusing on $N = 2$ solutions. In order to discuss energy densities of the solutions, we put a suitable reduction condition so that the energy densities can be real-valued. In subsection 5.1, we discuss one soliton solutions and calculate the energy density explicitly. We prove that the energy density is real-valued and localized on a three-dimensional hyperplane in $\mathbb{R}^4$. Hence this soliton is a domain-wall in the 4-dimensional Euclidean space. In subsection 5.2, we study asymptotic behavior of multi soliton solutions of the same type where the configurations take real values in the asymptotic region.

Let us first introduce star-exponential functions by

$$e_\star^{f(x)} := \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f(x) \star \cdots \star f(x)}_{n\text{times}}.$$

When $f(x)$ is a linear function of x^μ, the star exponential function $e_\star^{f(x)}$ is tractable. For example,

$$(e_\star^{k_\mu x^\mu})^{-1} = e_\star^{-k_\mu x^\mu}, \quad (5.1)$$

$$\partial_\nu e_\star^{k_\mu x^\mu} = k_\nu e_\star^{k_\mu x^\mu}. \quad (5.2)$$

These formula play crucial roles in discussion on behavior of noncommutative soliton solutions. Soliton solutions for $G = GL(N)$ are given as follows.

Theorem 5.1 Soliton-type solutions for $G = GL(N)$ can be constructed when the $N \times N$ matrices $\Lambda_s = \mathrm{diag}(\lambda_1^{(s)}, \cdots, \lambda_N^{(s)})$ $(s = 1, 2, \cdots, n)$ and the associated $N \times N$ eigenfunctions are

$$\theta_s = \begin{pmatrix} a_{11}^{(s)} e_\star^{K_1^{(s)}} + a_{11}'^{(s)} e_\star^{-K_1^{(s)}} & \cdots & a_{1N}^{(s)} e_\star^{K_N^{(s)}} + a_{1N}'^{(s)} e_\star^{-K_N^{(s)}} \\ \vdots & & \vdots \\ a_{N1}^{(s)} e_\star^{K_1^{(s)}} + a_{N1}'^{(s)} e_\star^{-K_1^{(s)}} & \cdots & a_{NN}^{(s)} e_\star^{K_N^{(s)}} + a_{NN}'^{(s)} e_\star^{-K_N^{(s)}} \end{pmatrix}, \quad (5.3)$$

$$K_p^{(s)} := \lambda_p^{(s)} \beta_p^{(s)} z + \alpha_p^{(s)} \overline{z} + \lambda_p^{(s)} \alpha_p^{(s)} y - \beta_p^{(s)} \overline{y}. \quad (5.4)$$

where $a_{pq}^{(s)}, a_{pq}'^{(s)}, \alpha_p^{(s)}, \beta_p^{(s)}$ $(p, q = 1, 2, \cdots, N)$ are complex constants. We can easily check that this solves (4.9).

Corollary 5.2 There is no non-trivial soliton solution (5.3) for $G = GL(1)$ as in theorem 5.1.

(Proof) For $G = GL(1)$, the eigenvalue matrix Λ_s is a 1×1 scalar function which is commutative. Then the solution of the noncommutative Yang's equation becomes constant:

$$J_{[n+1]} = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1\Lambda_1 & \cdots & \theta_n\Lambda_n & 0 \\ \vdots & & & \vdots \\ \theta_1\Lambda_1^{n-1} & \cdots & \theta_n\Lambda_n^{n-1} & 0 \\ \theta_1\Lambda_1^n & \cdots & \theta_n\Lambda_n^n & \boxed{0} \end{vmatrix}_\star = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \Lambda_1\theta_1 & \cdots & \Lambda_n\theta_n & 0 \\ \vdots & & & \vdots \\ \Lambda_1^{n-1}\theta_1 & \cdots & \Lambda_n^{n-1}\theta_n & 0 \\ \Lambda_1^n\theta_1 & \cdots & \Lambda_n^n\theta_n & \boxed{0} \end{vmatrix}_\star = \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Lambda_1 & \cdots & \Lambda_n & 0 \\ \vdots & & & \vdots \\ \Lambda_1^{n-1} & \cdots & \Lambda_n^{n-1} & 0 \\ \Lambda_1^n & \cdots & \Lambda_n^n & \boxed{0} \end{vmatrix}_\star .$$

In the final step, we eliminate a common factor θ_s on the right side of the s-th column of the quasideterminant. □

Finally we make a comment on an important formula which play crucial roles in the following discussion. Let x^ρ, x^σ be noncommutative space-time coordinates $[x^\rho, x^\sigma]_\star = i\vartheta$. Introducing new noncommutative coordinates as $w := x^\rho + kx^\sigma, \overline{w} := x^\rho - kx^\sigma$ $(k \in \mathbb{C})$, we can easily find

$$f(w) \star g(w) = f(w)g(w) \tag{5.5}$$

because the star-product (2.2) is rewritten in terms of $(w, \overline{w})$ as

$$f(w,\overline{w}) \star g(w,\overline{w}) = e^{ik\vartheta\left(\partial_{\overline{w}_1}\partial_{w_2} - \partial_{w_1}\partial_{\overline{w}_2}\right)} f(w_1,\overline{w}_1)g(w_2,\overline{w}_2)\Big|_{\substack{w_1 = w_2 = w \\ \overline{w}_1 = \overline{w}_2 = \overline{w}.}} \tag{5.6}$$

In four-dimensional noncommutative spaces, for any functions f which have the forms of $f(k_\mu x^\mu)$, the property (5.6) holds for any pair of noncommutative coordinates x^ρ and x^σ among the four real coordinates x^μ $(\mu = 1,2,3,4)$ and therefore the star-products is reduced to ordinary commutative products of functions.

5.1 One-Soliton Solutions of NC ASDYM equation

In this subsection, we focus on one-soliton solutions for $J \in U(2)$. In commutative spaces, they are obtained and energy densities of them are real-valued [23]. A noncommutative candidate is given by:

$$\theta = \begin{pmatrix} ae_\star^K & be_\star^{-\overline{K}} \\ -be_\star^{-K} & ae_\star^{\overline{K}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \lambda = e^{i\varphi} \tag{5.7}$$

where $K := \lambda\beta z + \alpha\overline{z} + \lambda\alpha y - \beta\overline{y}$, $a, b, \varphi \in \mathbb{R}$, and $\alpha, \beta \in \mathbb{C}$. In this subsection we prove that this solution leads to real-valued one-soliton solution in every region of $\mathbb{R}^4$.

Yang's J-matrix becomes

$$
\begin{aligned}
J &= -\theta\Lambda\star\theta^{-1} = \begin{vmatrix} \theta & 1 \\ \theta\Lambda & \boxed{0} \end{vmatrix}_\star \\
&= \begin{vmatrix} ae_\star^{K} & be_\star^{-\overline{K}} & \\ -be_\star^{-K} & ae_\star^{\overline{K}} & 1_{2\times 2} \\ \lambda ae_\star^{K} & -\lambda be_\star^{-\overline{K}} & \\ -\lambda be_\star^{-K} & -\lambda ae_\star^{\overline{K}} & \boxed{O_{2\times 2}} \end{vmatrix}_\star = \begin{vmatrix} ae_\star^{K+\overline{K}+i\Delta} & be_\star^{-(K+\overline{K})-i\Delta} & \\ -be_\star^{\overline{K}-K-i\Delta} & ae_\star^{\overline{K}-K+i\Delta} & 1_{2\times 2} \\ \lambda ae_\star^{K+\overline{K}+i\Delta} & -\lambda be_\star^{-(K+\overline{K})-i\Delta} & \\ -\lambda be_\star^{\overline{K}-K-i\Delta} & -\lambda ae_\star^{\overline{K}-K+i\Delta} & \boxed{O_{2\times 2}} \end{vmatrix}_\star, \\
&\text{where } \Delta := (1/2)k_\mu \overline{k}_\nu \vartheta^{\mu\nu}, \quad K = k_\mu x^\mu. \qquad (5.8)
\end{aligned}
$$

In the final step, we use invariance of quasideterminant by right-multiplication of $e_\star^{\overline{K}}$ on the first column and $e_\star^{-K}$ on the second column, and use the formula $e_\star^{K}\star e_\star^{\overline{K}} = e_\star^{K+\overline{K}+i\Delta}$.

Here let us take the following gauge transformation on θ:

$$
\theta \mapsto g^{-1}\star\theta, \quad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & e_\star^{K-\overline{K}}. \end{pmatrix}, \quad g\star g^\dagger = g^\dagger \star g = 1
$$

Then we get

$$
J = \begin{vmatrix} ae_\star^{K+\overline{K}+i\Delta} & be_\star^{-(K+\overline{K})-i\Delta} & \\ -be^{-i\Delta} & ae^{i\Delta} & 1_{2\times 2} \\ \lambda ae_\star^{K+\overline{K}+i\Delta} & -\lambda be_\star^{-(K+\overline{K})-i\Delta} & \\ -\lambda be^{-i\Delta} & -\lambda ae^{i\Delta} & \boxed{O_{2\times 2}} \end{vmatrix}_\star. \qquad (5.9)
$$

Now we can see that the coordinate dependence has the form of $K+\overline{K} = (k_\mu + \overline{k}_\mu)x^\mu$ and the star-products disappear after this step due to (5.6). In order to get real-valued energy density, J should be unitary, which is equivalent that in the final form (5.9) of J, $ae^{i\Delta}$ and $be^{-i\Delta}$ should be real. This is realized by redefinition of $a \mapsto ae^{-i\Delta}$ and $b \mapsto be^{i\Delta}$.

Results are summarized in the following:

Theorem 5.3 One soliton solution for $J \in U(2)$ is given by

$$
\theta = \begin{pmatrix} ae_\star^{K-i\Delta} & be_\star^{-(\overline{K}-i\Delta)} \\ -be_\star^{-(K-i\Delta)} & ae_\star^{\overline{K}-i\Delta} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & -e^{i\varphi} \end{pmatrix}, \quad a, b, \varphi \in \mathbb{R} \qquad (5.10)
$$

where $K = e^{i\varphi}\beta z + \alpha\overline{z} + e^{i\varphi}\alpha y - \beta\overline{y}$, $\alpha, \beta \in \mathbb{C}$.

This solution θ leads to the final form (5.9) of J without the constant Δ. Gauge fields are calculated as in (4.10) from this J and field strengths are obrtained. In the calculations, the star-products always disappear due to the property (5.6) and the energy density is reduced to the commutative one [23]:

$$
\mathrm{Tr}F_\star^2 = \mathrm{Tr}F^2 = 32(|\alpha|^2 + |\beta|^2)\left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right), \qquad (5.11)
$$

where $X := K + \overline{K} + \log|a/b|$. This is actually real-valued at every point in $\mathbb{R}^4$. We can find that the energy density has its peak on a three-dimensional hyperplane defined by $X = K + \overline{K} + \log|a/b| = 0$ whose normal vector is $k_\mu + \overline{k}_\mu$. (k_μ is defined in (5.8).) This is a domain-wall in $\mathbb{R}^4$.

5.2 Multi-Soliton Solutions of NC ASDYM equation

In this subsection, we present multi-soliton solutions of the same type as the previous subsection and discuss the asymptotic behavior of them.

The n-soliton solution is given by ($s = 1, 2, \cdots, n$):

$$\theta_s = \begin{pmatrix} a_s e_\star^{K_s - i\Delta_s} & b_s e_\star^{-(\overline{K_s} - i\Delta_s)} \\ -b_s e^{-(K_s - i\Delta_s)} & a_s e^{\overline{K_s} - i\Delta_s} \end{pmatrix}, \quad \Lambda_s = \begin{pmatrix} e^{i\varphi_s} & 0 \\ 0 & -e^{i\varphi_s} \end{pmatrix}, \tag{5.12}$$

where $K_s = e^{i\varphi_s}\beta_s z + \alpha_s \overline{z} + e^{i\varphi_s}\alpha_s y - \beta_s \overline{y} =: k_\mu^{(s)} x^\mu$, $\Delta_s := (1/2) k_\mu^{(s)} \overline{k}_\nu^{(s)} \vartheta^{\mu\nu}$, and $\alpha_s, \beta_s \in \mathbb{C}$, $a_s, b_s, \varphi_s \in \mathbb{R}$. We assume that K_s ($s = 1, 2, \cdots, n$) are independent and there is no special relation among them, which corresponds to pure-soliton scattering, that is, no resonance.

As in the standard discussion, let us consider the asymptotic limit $r := ((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2)^{1/2} \to \infty$ keeping both K_I and $\overline{K_I}$ finite. (Later this condition will be loosed.) Then for $s \neq I$, $|e_\star^{K_s}|$ and $|e_\star^{\overline{K_s}}|$ go to positive infinity or zero. Then we can get the asymptotic form of J by eliminating common factors which go to infinite in a column of the quasideterminant J:

$$J \to \begin{vmatrix} C_1 & \cdots & \theta_I & \cdots & C_n & 1 \\ C_1\Lambda_1 & \cdots & \theta_I\Lambda_I & \cdots & C_n\Lambda_n & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ C_1\Lambda_1^n & \cdots & \theta_I\Lambda_1^n & \cdots & C_n\Lambda_n^n & \boxed{0} \end{vmatrix}_\star \quad \text{where } C_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Focusing on the θ_I-column in the above representation of J, let us multiply $e_\star^{\overline{K}_I}$ on the first column of θ_I and $e_\star^{-K_I}$ on the second column of θ_I, and further take a gauge transformation $\Theta \mapsto g_I^{-1} \star \Theta$, $g_I^{-1} = \mathrm{diag}(1, e_\star^{K_I - \overline{K}_I})$, $\Theta = (\theta_1, \cdots, \theta_n)$ as in subsection 5.1. Then we get

$$J \to \begin{vmatrix} C_1 & \cdots & \widetilde{\theta}_I & \cdots & C_n & 1 \\ C_1\Lambda_1 & \cdots & \widetilde{\theta}_I\Lambda_I & \cdots & C_n\Lambda_n & 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ C_1\Lambda_1^n & \cdots & \widetilde{\theta}_I\Lambda_1^n & \cdots & C_n\Lambda_n^n & \boxed{0} \end{vmatrix}_\star \quad \widetilde{\theta}_I = \begin{pmatrix} a_I e_\star^{K_I + \overline{K_I}} & b_I e_\star^{-(K_I + \overline{K_I})} \\ -b_I & a_I \end{pmatrix}.$$

We note that the factors $e_\star^{K_I - \overline{K}_I}$ in $C_1, \cdots, C_n$ disappear because it is a common factor in a column of the quasideterminant. Now we can see that the coordinate dependence has the form of $K_I + \overline{K}_I \in \mathbb{R}$ and the star-products disappear after this step. All elements are real now and the energy density should be real.

Hence we can conclude that in the asymptotic behavior is all the same as commutative case in the pure soliton process. As a result, the n soliton solutions would have n isolated localized lumps of energy. They preserve their shapes and velocities of the localized solitary wave lumps. It is worthwhile to compare commutative limits of our solutions with commutative results [3]. Details will be reported elsewhere.

6 Conclusion and Discussion

In this paper, we gave multi soliton solutions of noncommutative anti-self-dual Yang-Mills equations for $G = GL(N)$ by the Darboux transformations and discussed soliton solutions for $G = GL(2)$ in detail. The generated solutions were represented in terms of quasideterminants in compact forms. The Yang J-matrices obtained here are different from those obtained by the Penrose-Ward transform [10, 11]. K matrices are new. We found that special one-soliton solutions for $G = GL(2)$ have the same configuration as commutative one which have localized energy density on a three-dimensional hyperplane. This is a domain-wall in $\mathbb{R}^4$. We also presented multi-soliton solutions of the same type and discussed asymptotic behaviors of them.

The present discussion is straightforwardly generalized to other signature such as ultra-hyperbolic signature $(++--)$. The anti-self-dual Yang-Mills equations with the ultra-hyperbolic signature can be reduced to various lower-dimensional integrable systems such as KdV, NLS equations. This is first conjectured in [46] (known as Ward's conjecture) and summarized in [32]. Various examples of the noncommutative Ward conjecture are seen in [18]. This Darboux technique could be applied to wide class of noncommutative integrable systems.

In commutative spaces, a binary Darboux transformation is developed in [35]. This gives Grammian-type solutions. Noncommutative extension of it would be possible and interesting (cf. [16]). Noncommutative binary Darboux transformations are successful for KP and KdV equations [13] and modified KP equations [15]. Noncommutative extension of the bilinear approach to anti-self-dual Yang-Mills equations [40] is also worthwhile.

In this paper, we discussed pure soliton scatterings. It is an interesting question whether there are resonance processes for (noncommutative) anti-self-dual Yang-Mills equation with respect to the energy density. It might lead to higher-dimensional extension of Kodama's Grassmannian approach to KP soliton scatterings including all possible resonance processes [24].

It would be also worth clarifying the relation between the Wronskian-type solutions in the present paper and the non-Wronskian-type solutions in [10, 11]. Wronskian-type solutions play crucial roles in the Sato theory of the KP equations which reveals hidden infinite symmetry on infinite-dimensional Grassmannians from the viewpoint of Plücker relations of the Wronskians [41]. Hence the present formulation might be a hint for a higher-dimensional extension of Sato theory. On the other hand, the non-Wronskian solutions relate to the twistor theory [32, 47]. The geometrical viewpoints give explanation of the origin of integrability and can be applied to construction of soliton solutions as well. The present Wronskian solutions could be represented by the non-Wronskian-type solutions with a suitable parametrization.[2] This might shed light on a profound relationship between Sato theory and twistor theory.

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