Is estimation error overhyped? Simulations with different optimization procedures

A Work Project based on the CEMS MIM Business Project *Strategic Asset Allocation*, developed with *BPI Gestão de Activos*

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I – The Business Project

The current Work Project builds upon a CEMS MIM Business Project developed on the topic of strategic asset allocation with BPI Gestão de Activos, the asset management division of Grupo BPI, one of the major Portuguese banking groups. The nature of the underlying Business Project was essentially methodological, and it was not specific to a particular company or point in time. Although it focused on BPI's main methodological concerns, eligible asset classes and investment restrictions, the study was general enough to apply to any other asset manager and in any other market conditions.

Asset allocation is a crucial issue for BPI GA, which manages multi-asset portfolios worth € 2 billion, out of the total € 8 billion in assets under management. Its multi-asset portfolios are composed of highly liquid asset classes: equities (from Europe, US and Emerging Markets), bonds (corporate investment grade, corporate high yield, Eurozone governments, and non-Euro governments), commodities (gold and futures on other commodities) and cash. Investment restrictions include the prohibition on the use of leverage or short positions, as well as a preference for a low weight on cash in the portfolio.

Asset allocation is a key determinant of portfolio returns, and the Business Project focused on strategic asset allocation, which concerns the optimal balance between risk and return in the long run[1]. The strategic asset allocation process is closely related to modern portfolio theory: it involves the estimation of the expected returns, volatilities and correlations of the various asset classes and the use of these inputs to build a portfolio with the highest expected return for the desired level of risk. The Business Project challenge was precisely to review the methodology for the estimation of each of the inputs and the optimization process. It included a critical assessment of BPI's current practices and the recommendation of alternative models to improve the strategic asset allocation process in an intuitive and sensible way. The structure and methodology of the project are summarized in exhibit 1.

In the first stage of the project, the estimation of expected returns, we started by backtesting two methodologies that constituted an improvement over traditional predictive regressions: the Sum of the Parts (Ferreira, Santa-Clara, 2011) and Carry (Koijen, Moskowitz, Pedersen, Vrugt, 2013). We concluded that these methodologies

[1] This is of extreme importance to ensure that a fund does not stray from its long run goals and sustainability, especially as the sophistication of market participants increases market efficiency, and it is increasingly difficult to make short-term gains due to the difficulty in anticipating events amidst the complex economic environment.

have value as they contain fundamental drivers of returns, although the use of regressions and backtests for returns can be misleading due to the predominance of volatility around the slowly time-varying conditional mean. By reducing estimation error, these models could be combined with structural models to explain expected returns. Ultimately, for equities we modified BPI's implied internal rate of return approach into an extension of the Sum of the Parts method. Returns would be driven by the dividend yield (for which we proposed a more reliable estimate based on the carry method), the net buyback yield (newly introduced) and analyst forecasts of nominal economic growth (also polished from BPI's approach). For bonds, besides the yield to maturity, we introduced rolldown as an additional component of expected returns, and made the previously static estimate of expected credit losses a dynamic variable, ensuring consistency with business cycles and changes in yields. For commodities, we built expected returns from scratch, as BPI's approach consisted of a mix of several models lacking robustness. We ultimately used inflation as the driver of prices in the long-term, as well as the return obtained when rolling futures contracts, corresponding to the convenience yield. We also proposed a new estimate for exchange rate fluctuations, consistent with the empirical finding of no predictability. On top of this, we suggested the introduction of other important variables with impact in the medium term, according to the asset manager's market views.

The estimation of volatilities and correlations was separated in two stages, as opposed to the previously used exponentially weighted moving average for the covariance matrix. We estimated volatilities with the asymmetric MIDAS model (Ghysels, Santa-Clara, Valkanov, 2005), featuring higher frequency data, flexible weights and different treatment of asymmetric shocks. For correlations, we used the parsimonious DCC model (Engle, 2002) that allows to model time-varying correlations separately from volatilities while ensuring that the covariance matrix is positive definite.

In the optimization stage, we started with mean-variance optimization and suggested two methods to deal with the problem of estimation error: constraints in the partial contribution to risk of each asset class and a transparent form of shrinkage drawing information from market weights, as opposed to the opaque shrinkage method BPI was previously using. The choice between constraints and shrinkage should be based on the trade-off between increased diversification and decreased efficiency.

We concluded by stressing the importance of frequently rebalancing portfolios, to exploit new information on time-varying expected returns, volatilities and correlations.

II – Is estimation error overhyped? Simulations with different optimization procedures

- **Introduction**

The landmark mean-variance approach to optimization proposed by Markowitz (1959) features a vital limitation in the way it deals with error in the estimation of expected returns, volatilities and correlations. The optimizer is highly sensitive to changes in inputs and, by assuming that they are known with certainty, overfits the estimates and magnifies estimation error, returning portfolios with extreme weights. This issue is well documented by Jobson and Korkie (1980) and Michaud (1989). In the Business Project, two solutions were proposed to mitigate the impact of estimation error, at the cost of introducing a bias in the weight estimates: limiting the concentration of portfolio risk in individual asset classes and shrinking the estimated weights towards a fixed portfolio that contained no estimation error.

In the Business Project, the choice of one method over the other depended on the trade-off between the increase in diversification and the perceived loss in efficiency, as illustrated in exhibit 2. This is a problematic decision rule, as it does not really identify the ideal method to curb noise. It is instead a qualitative criterion that is not necessarily linked to the goal of optimizing the portfolio. Concentration can be optimal from a mean-variance standpoint and thus diversification is not a goal by itself, it is rather a way to be protected from excessive exposure to risk factors that do not yield the highest premia. Moreover, the estimate of the loss in efficiency illustrated by the downward shift of the allocation frontier contains estimation error from the inputs (especially expected returns), thus it does not represent the true loss in efficiency. In fact, there may be no loss in efficiency, as truly optimal portfolios may not be ruled out by constraints.

- **The experiment**

This Work Project is motivated by the pursuit of the approach that best reduces noise without losing valuable information, resulting in the closest portfolios to the optimal. Backtesting the out-of-sample performance of different optimizers is not the ideal method to identify the best optimizer, as available data series are not long enough to conduct proper backtests. Realized returns can be very different from the expected returns and as it is possible that optimal portfolios from a mean-variance standpoint are not the best performers best over the testing horizon, and the identification of the ideal optimization procedure would be incorrect. The solution to this problem is to conduct this study with simulations, in a controlled environment where the true inputs are known and there is no need to test out of sample.

Returns on 8 asset classes are simulated for a period of 30 years[2] comprising 250 working days each. Asset prices are assumed to follow a geometric Brownian motion:

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i z_{i,t}\sqrt{dt} \tag{1}$$

μ_i is the true expected return and σ_i is the true standard deviation of asset class i, in annual terms. dt is the daily time step (equal to 0,004 years), and $z_{i,t}$ is taken from a standard multivariate normal distribution. Because the time step is very short, it is not necessary to assume that asset prices follow a lognormal distribution. As such, daily simple returns correspond to normally distributed variables. The risk-free rate is assumed to be zero, such that the risk premium corresponds to the expected return[3]. Moreover, true expected returns, standard deviations and correlations are constant throughout time (returns are i.i.d. across time but correlated across assets on each day), meaning that the best estimators of these parameters are their sample counterparts.

Each optimizer uses the estimated inputs to build the tangency portfolio: the portfolio that maximizes the Sharpe ratio. It is assumed that leverage is available at the risk-free rate, hence the tangency portfolio is the only relevant portfolio of risky assets. The Sharpe ratio of a portfolio is defined as usual, with a risk-free rate equal to zero:

$$SR = \frac{\mu_p}{\sigma_p} = \frac{\sum_i w_i \mu_i}{\sqrt{\sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}}} = \frac{W^T \mu}{\sqrt{W^T \Omega W}} \tag{2}$$

ρ_{ij} represents the correlation between the returns of i and j. In matrix notation, W is the vector of weights, μ is the vector of expected returns and Ω is the covariance matrix.

Defining the 'true' expected returns, standard deviations and correlations is potentially problematic. Indeed, different sets of inputs could result in very different 'true' optimal portfolios. Three different scenarios will be considered here: (1) a 'balanced' scenario, where inputs are such that the tangency portfolio features weights between 5% and 25% for all asset classes; (2) a 'tilted' scenario, where it is optimal to be concentrated in 3 asset classes with weights up to 40% of the portfolio while other classes have negligible (but positive) weights; (3) an 'extreme' scenario, where the optimal portfolio includes long and short positions and weights range from -30% to over 50%. To ensure that

[2] The choice of this sample size could be seen as the average time span for which there is reliable data on realized returns for the various asset classes, for inference purposes

[3] This simplification does not affect the results of this experiment

inputs are realistic, each scenario has been obtained by adjusting the estimates from the Business Project (expected returns, which are the most uncertain inputs, were the most modified). Expected returns have also been scaled such that the Sharpe ratio of the optimal portfolio is 1 in all scenarios[4]. Furthermore, inputs have been set such that the global minimum variance portfolio features only positive weights, as would be expected from a non-aggressive portfolio. Exhibit 3 depicts the inputs of each scenario and the composition of the optimal portfolios in each case. Depending on the scenario, different optimizers may have different performances[5].

The list of optimizers to be tested in this experiment is the following: (1) unconstrained mean-variance optimization; (2) non-negative weights only; (3) individual weights between 0% and 20%; (4) maximum individual contribution to risk of 30%; (5) maximum individual contribution to risk of 40%; (6) maximum individual contribution to risk of 50%; (7) shrinkage towards equal-weighted portfolio; (8) minimum variance portfolio[6]; (9) shrinkage towards minimum variance portfolio. Shrinkage uses weights of 2/3 to the unconstrained portfolio and 1/3 to the target portfolio. The partial contribution of an asset class i to the total risk of the portfolio is defined as:

$$PCR_i = \frac{w_i Cov(r_i, r_p)}{\sigma_p^2} = \frac{w_i \sum_j w_j \sigma_{i,j}}{\sigma_p^2} \tag{3}$$

As the optimizers are targeting the maximum Sharpe ratio, they should be assessed based on the true Sharpe ratio of resulting portfolios. In each scenario, the simulation is run 1000 times and the distributions of true and perceived (according to true and estimated inputs, respectively) Sharpe ratios for each optimizer are taken. Exhibit 4 presents histograms of true Sharpe ratios and statistics for each optimizer and scenario.

In all three scenarios, the minimum variance optimizer stands out as very different from all other optimizers. It features a considerably lower average Sharpe ratio, and this ratio is very stable, since risk is measured accurately and all simulations result in similar portfolio weights. This similarity in the resulting portfolios also eliminates the negative skewness (the presence of outliers on the downside) that exists for all other optimizers.

[4] This proportional adjustment of expected returns does not have an impact on the weights of the optimal portfolio, but it makes the interpretation of results of different scenarios more intuitive – although results should not be compared across scenarios

[5] For instance, constraining maximum investment weights or setting equal weights is expected to work better in the 'balanced' scenario, while in other scenarios these constraints do not allow optimal weights to be reached

[6] This optimizer is the only one that does not attempt to maximize the Sharpe ratio. Nonetheless, the minimum variance portfolio is typically a stable portfolio with an acceptable performance and is included here for comparison purposes

The true Sharpe ratio, however, is never accurately known, since there is a high level of uncertainty over expected returns. For all optimizers except the minimum variance portfolio and corresponding shrinkage, the true Sharpe ratio is lower than the perceived ratio[7]. Although these optimizers result in a portfolio with a Sharpe ratio below the investors' perception, it is still higher than that of the minimum variance portfolio in most cases. As such, even though the minimum variance portfolio has performed well in the past, it should not be seen as a good portfolio optimizer.

In the 'balanced' scenario, the best average Sharpe ratios are obtained with weight constraints and shrinkage to equal weights. This is not surprising, since the optimal portfolio features moderate weights for all asset classes. The constrained optimizer can be deemed the best, as it also features lower standard deviation of the Sharpe ratios and lower (less negative) skewness. The long-only optimizer is not far behind.

In the 'tilted' scenario, risk-constrained optimizers produce Sharpe ratios below those of the unconstrained optimizer. Weight constraints and shrinkage to equal weights repeat the good performances (although the former is now highly skewed). This time, the long-only optimizer takes the pole position, with the highest average Sharpe ratio, lower skewness compared to constrained portfolios and an acceptable standard deviation. Again, this is not surprising, as the optimal weights are now positive but not moderate.

In the 'extreme' scenario, the weight-constrained optimizer is not a top performer. Instead, good choices are the long-only optimizer and shrinkage, either to equal weights or to the minimum variance portfolio.

In practice, one does not know which scenario truly holds. Nevertheless, long-only weights or shrinkage to equal weights are good options to curb estimation error in any scenario, even though optimal portfolios may become out of reach (e.g. long-only in the 'extreme' scenario). It is also worthwhile to note that constraining the exposure to individual sources of risk, despite its appealing rationale, is not particularly useful.

Perhaps the biggest surprise in these results is the fact that, in all scenarios, average Sharpe ratios of all optimizers are so close to the optimal ratio of 1 and to each other. Indeed, considering how volatile returns are, Sharpe ratios of 0,90 and 0,92 are not substantially different in practice. This suggests that estimation error is not a major problem in optimization. Although optimizers are likely to return extreme weights due

[7] This is another illustration of the impact of estimation error– the optimizer overfits estimated inputs (e.g. shifting most of the weight to the asset classes with higher in-sample return) and the resulting portfolio is not truly optimal. This does not hold for the minimum variance portfolio, which does not constitute an attempt to maximize the Sharpe ratio

to estimation error, in most cases the resulting portfolios will still have a Sharpe ratio close to the optimal, even with a very different composition. Constraining the optimizer properly is helpful but does not add as much value as could be believed.

- **The case of high correlations**

The aforementioned findings contrast with the experiment of Jobson and Korkie (1980), who highlighted a substantial benefit of moving from unconstrained to long-only optimization. There is an important difference between the two experiments: while that study dealt with equity portfolios segmented by industries, the present study deals with asset classes, which have lower correlations. Higher correlations pose a higher risk of overfitting, as the gains from diversification are lower and the optimizer is more likely to attribute all the weight to the assets with more appealing estimates of risk and return.
A new scenario is introduced to attempt to reconcile this paradox. In the 'high correlations' scenario (exhibit 5), correlations are shrunk from the previously used correlations to a target of 1, with a weight of 0,5 to each. Expected returns and standard deviations are adjusted to avoid extreme weights, and for this it is necessary that they become more similar across asset classes, compared to the previously used scenarios. Results are the same as for the case of lower correlations: the optimizers that restrict weights to long-only or up to 20%, in addition to shrinkage to equal weights, perform better than all others, but not by a large margin. Although extreme positions do exist, on average the impact on the Sharpe ratio is not substantial.

- **The impact of daily *versus* monthly observations**

In both the Business Project and these simulations, high frequency (daily) data has been used to forecast inputs. The estimates of expected returns do not change with the periodicity of the observations, but those of risk can become more accurate. This section will aim to measure the benefit from using high frequency data to measure risk. For the same simulations, the optimization inputs will be re-estimated using lower frequency data – each year is divided into 10 'months' of 25 days each. Monthly returns are computed from the returns of all days d in each month, as follows:

$$r_{monthly} = \prod_{d=1}^{25}(1 + r_d) - 1 \tag{4}$$

Exhibit 6 presents the statistics for each optimizer in each scenario. It can be seen that there is a benefit from estimating volatility with high frequency data: the resulting Sharpe ratio is on average higher and has lower variability if inputs are measured with daily observations. The minimum variance portfolio displays the most noticeable difference: in every scenario, the standard deviation of the Sharpe ratio using monthly observations is several times higher than the standard deviation with daily observations. This optimizer uses only risk estimates and, with daily data, risk estimates are extremely accurate and all simulations yield almost the same weights. This is not the case for monthly observations, where weights vary more as risk measurement is less precise. Still, risk is already estimated with reasonable accuracy on a monthly basis and the use of daily data does not substantially increase average obtained Sharpe ratios.

- **Conclusions**

This study, while aiming to find the best optimization procedure, produced other interesting findings as well. Constraints such as non-negative weights or shrinkage to equal weights, in addition to the use of high frequency data to forecast low frequency volatility, address the issue of estimation error, but do not represent a substantial improvement. Very different portfolio compositions, resulting from different optimizers or just from different input estimates, can have Sharpe ratios that are similar to each other and not far from the true optimal, even with high concentration of weights, suggesting that the impact of estimation error is not overly harmful.

It is undisputable that asset allocation is a crucial driver of returns, as low correlations determine that different asset classes have different returns at each point in time. However, from a mean-variance standpoint, it appears not to be very difficult to construct an adequate portfolio with a Sharpe ratio close to the optimal, although it may happen (with low probability) that optimized portfolios are outliers with lower Sharpe ratios. The key to exploiting asset allocation may therefore lie in tactical asset allocation: trying to anticipate at each moment which asset classes will perform best and setting portfolio weights accordingly, instead of relying solely on numerical methods.

The validity of these conclusions is conditional on the assumptions of this study, which may not be broad enough. For a more complete analysis, the experiment ought to be extended to cases with a different sampling horizon (equivalent to a different magnitude of estimation error) and different inputs. Realistic time-varying inputs should be introduced, and a different (lower) optimal Sharpe ratio should be tested. This flexible simulation framework can also incorporate further interesting extensions to deepen the analysis, such as the study of the entire efficient frontier instead of just the tangency portfolio, the use of different optimizers (possibly combining several procedures at once) and the use of other distributions for returns.

III – Reflection on learning

This technically demanding project constituted an opportunity to apply an extensive array of concepts and tools that I learned throughout my academic studies. In this context, I drew mostly from the courses of Investments (risk/return concepts, portfolio theory and return predictability) and Financial Econometrics (linear regressions, time series, maximum likelihood estimation, forecasting, volatility models), while also using tools from Risk Management, Hedge Funds and my BSc in Economics (calculus, statistics, matrix algebra and economics concepts). This was complemented with a selection of academic papers that I read, adapted, implemented and used to reinforce assumptions and conclusions throughout the project. At the same time, I became a more proficient user of Bloomberg and developed my programming skills with Matlab.

I was given autonomy to manage the project, and proposed to work on a daily basis at BPI's office, taking advantage of a continuous contact with the client to easily obtain information, discuss opinions, adapt to their goals and concerns and meet their expectations. We held every week one formal meeting with BPI to discuss the progress more thoroughly and one meeting with the academic advisor to ask questions arising during the week and plan the next stages of the project. The main difference between this project and my previous consulting experience was my *de facto* leadership of the team. This entailed difficulties, as I am not a renowned academic or practitioner in this field, and at times it was difficult to make the client believe me. People at BPI were reluctant to accept new ideas that would improve upon theirs, especially in front of their new boss, insisting that we should hold on to 'standard industry practices' that were wrong according to the academic advisor, raising an issue of who we should be loyal to.

Throughout the project, I displayed and further developed some of my strengths at different stages, including planning next steps, asking the client the right questions, building financial models, building, formatting and delivering the presentation and answering questions. I also exhibited initiative, attention to detail, rigor, reliability, time management, ability to work under pressure and to deal with shifts in the project goals and missing information. The academic advisor's distant approach to the project echoed his trust in my autonomy and ability to properly steer the project. At the same time, the difficulty of the working conditions was beyond anything I had ever experienced: I had to focus on developing the project alone while dealing with a team that was not prepared for it and kept raising conflicts, showing no willingness to learn and unprofessional attitude. I had to cut my way through a dense atmosphere, devoting time to help my colleagues and giving them tasks that they could (but would not) fulfill.

Amidst the multitude of situations out of my control that materialized even though I did not believe that they could happen at this level, I can also identify areas of improvement for myself: I should firstly enhance my ability to give simple explanations about complex concepts to people who are not familiar with them. I could have also possibly done a better job in empowering (although it is difficult when people are not committed), saying ‘no’ more often to my colleagues (my primary concern was to avoid further conflicts, prompting them to exploit this situation) and scheduling. I am now more aware of these issues, and I believe my new job at a top consulting firm will allow me to develop these skills with the help of more experienced and professional people.

Looking back, I strongly believe that developing the project with (and not just to) the client was a major value-enhancing factor. Even though it was a methodological project, the constant exchange of information, knowledge and concerns between the two parties contributed to a tailor-made project. BPI’s main concerns were highlighted, and the client was particularly satisfied with the treatment of the dividend yields, net buybacks, rolldown, credit losses, measurement of contribution to risk and the needs to introduce *ad-hoc* variables and to frequently rebalance optimal portfolios. I also appropriated value, as I learned new hard skills, developed soft skills and took advantage of the opportunity to contact closely with the corporate world and network with senior people, as well as to become more acquainted with the day-to-day issues of an asset manager.

Not everything was perfect, and some things could have been done differently. A more detailed schedule of the project should have been done with the academic advisor, as the first stage (which was the most important) could have been done faster if we had known beforehand the methodology and time required for later stages, avoiding the delay in the project delivery. The delay could also have been avoided if we had sought help earlier regarding the contribution imbalance, possibly changing the group composition. Stricter internal deadlines could have possibly compelled my colleagues to work more, although I could never enforce them as I lacked the power to punish in case of breach. Other setbacks could have been avoided if the client team was more open to alternative views. Finally, an additional section with backtests would have added value, but it was not done due to lack of time and infeasibility of the analysis in certain stages.

Overall, it is undeniable that this project provided me with a unique experience that will be an asset for my future professional and personal life.

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Appendix

Exhibit 1 – Business Project structure

Stage 1: Expected returns

Equities:
$E(r) = \frac{D}{P} + NBY + g_{GDP(real)} + \pi$

Bonds:
$E(r) = Yield + Rolldown - E(CL)$

Commodities:
$E(r) = \pi + \delta$

Exchange rate adjustment
- Hedged: use forwards
- Non-hedged: expected appreciation is zero

How to obtain each input

Additional variables with impact on returns over the medium term: introduce according to the investor's market views

Stage 2: Volatilities and correlations

Asymmetric MIDAS → Dynamic Conditional Correlation

Stage 3: Optimization

Investment restrictions

Mean-variance optimization

Constraints on individual partial contribution to risk

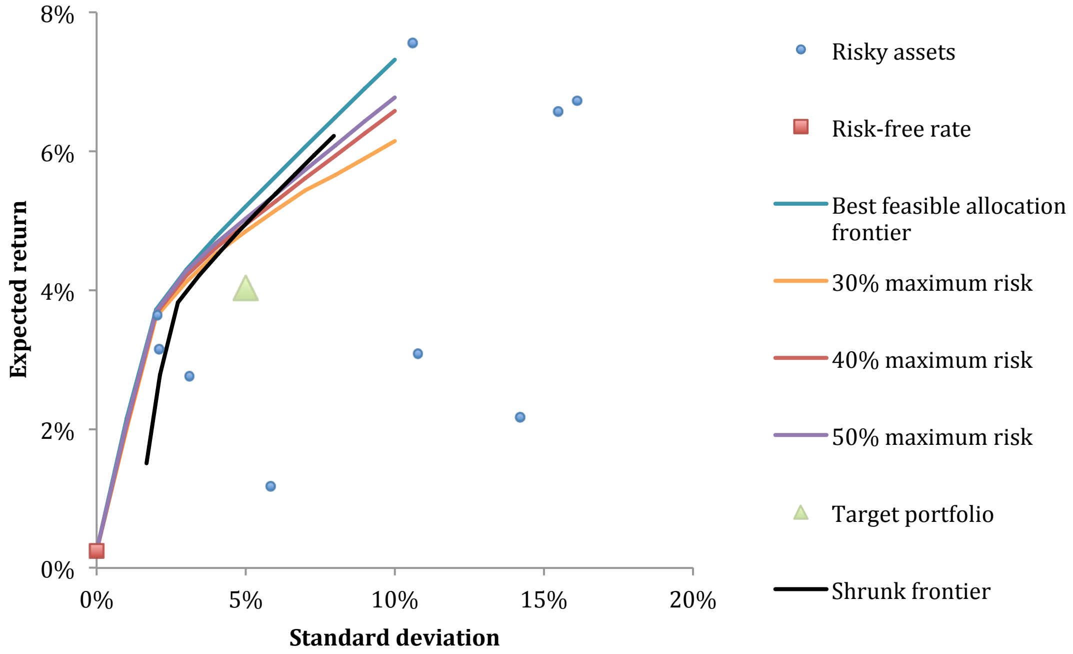
Shrinkage towards market portfolio

Increased diversification vs Decreased efficiency

Desired level of risk → **Optimal portfolio**

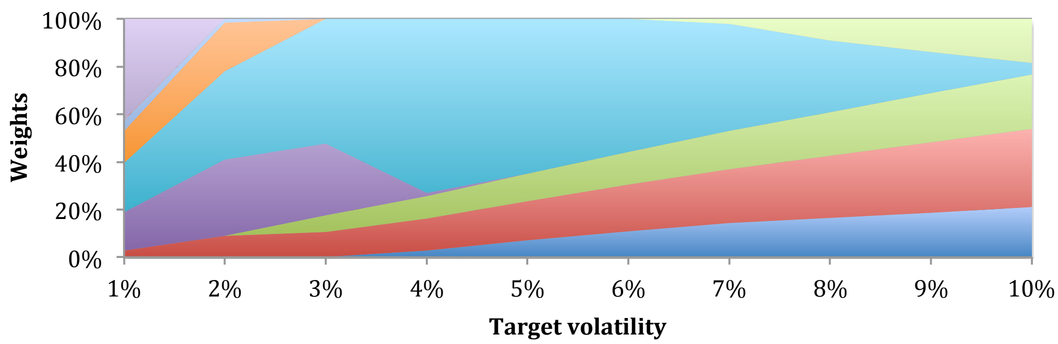
<u>Exhibit 2</u> – Choice between risk constraints or shrinkage in the Business Project

2.a: Perceived impact of each alternative in terms of risk and return

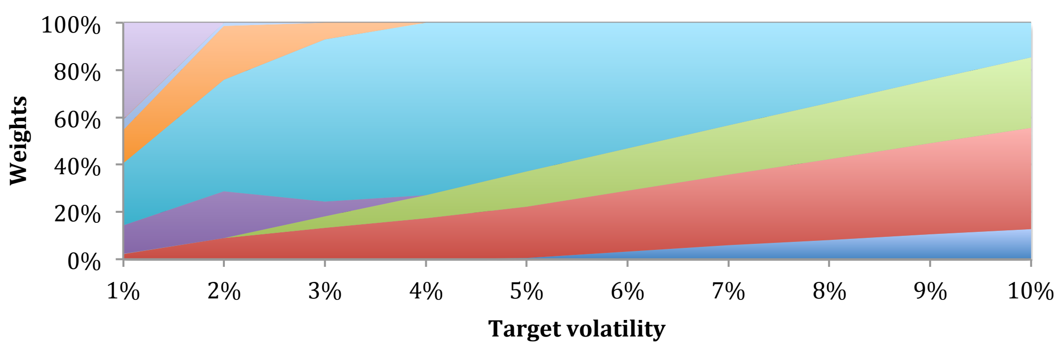


2.b: Portfolio composition under each alternative

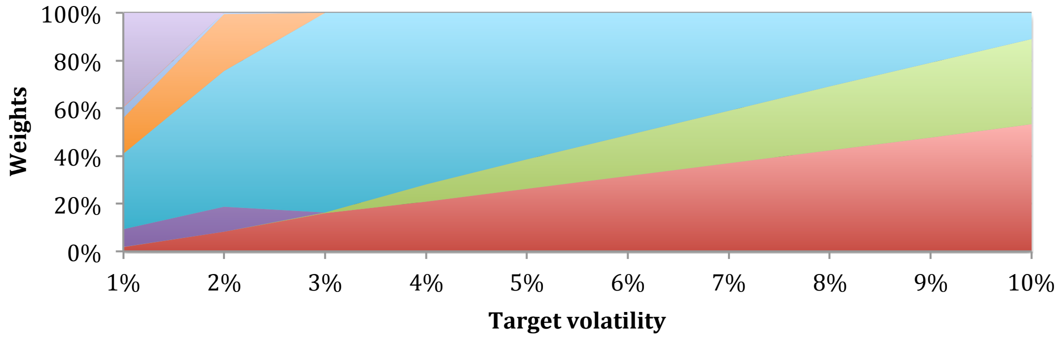
- Maximum 30% individual contribution to risk



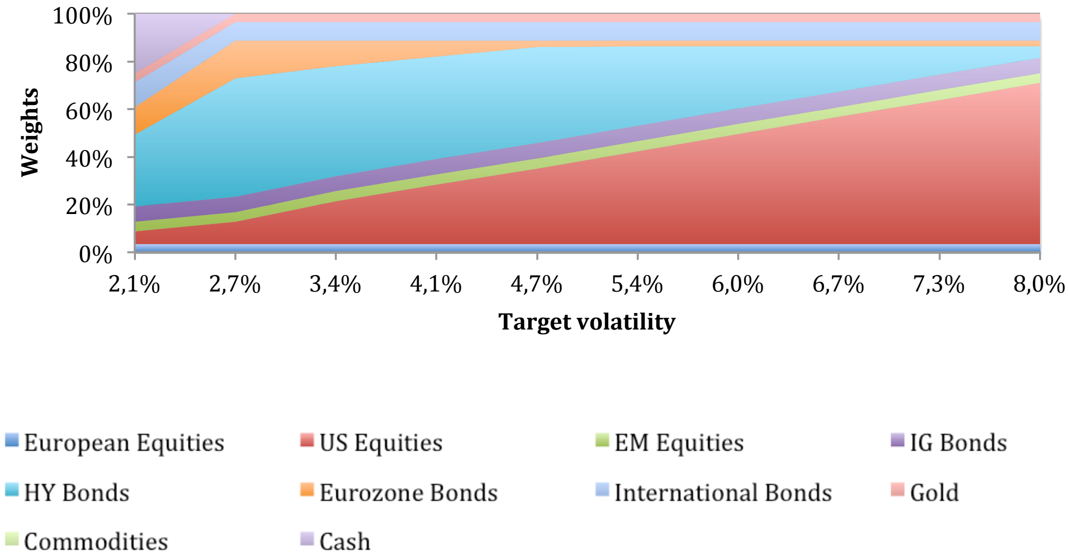
- Maximum 40% individual contribution to risk



- Maximum 50% individual contribution to risk



- Shrinkage of the best feasible allocation frontier (no leverage and no short positions allowed) towards the world portfolio



The choice:

- For low target volatilities, shrinkage seems to underperform substantially. The risk/return difference between the different intensities of the risk constraint is negligible at low volatilities, therefore the tightest constraint (30%) is chosen, as it ensures more diversification
- For higher volatilities, shrinkage is chosen, as it appears to be more efficient and includes more asset classes (even though at lower weights)

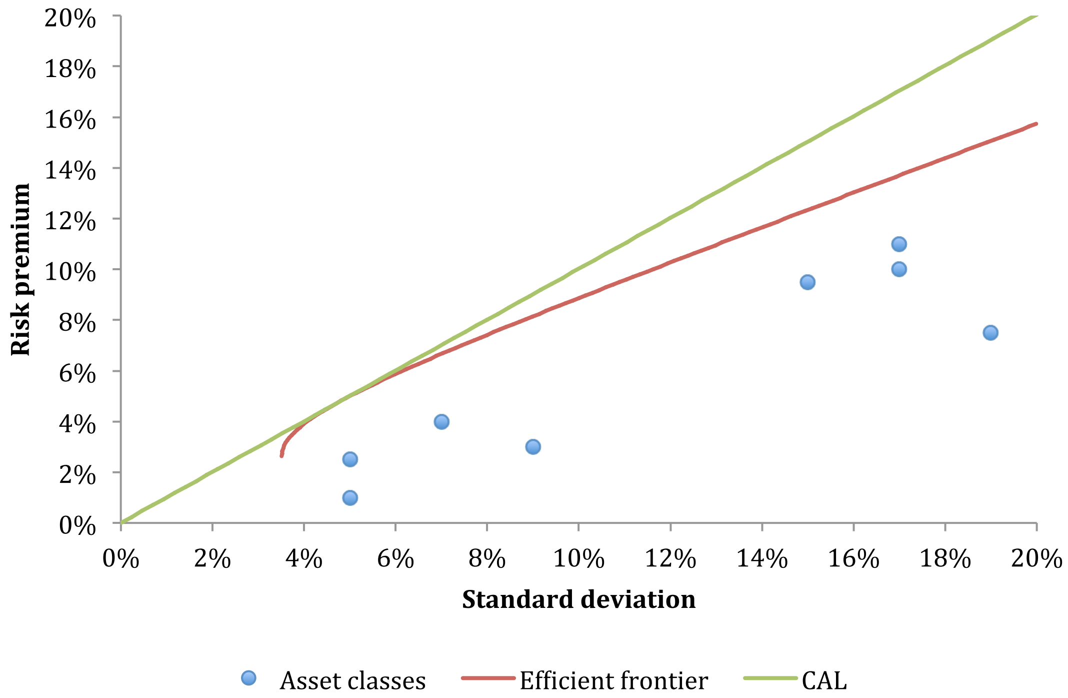
Exhibit 3 – Expected returns, volatilities, correlations and optimal portfolios in each scenario

3.a: Balanced scenario

Correlations	A	B	C	D	E	F	G	H
A	1	0,7	0,4	0,12	0,5	-0,1	-0,14	-0,07
B	0,7	1	0,5	0,11	0,5	-0,1	-0,1	0,04
C	0,4	0,5	1	0,22	0,2	-0,17	0,17	0,15
D	0,12	0,11	0,22	1	0,4	0,5	0,07	0,11
E	0,5	0,5	0,2	0,4	1	-0,03	-0,17	0,03
F	-0,1	-0,1	-0,17	0,5	-0,03	1	0,08	0,05
G	-0,14	-0,1	0,17	0,07	-0,17	0,08	1	-0,08
H	-0,07	0,04	0,15	0,11	0,03	0,05	-0,08	1

Standard deviation	17%	15%	17%	5%	7%	5%	9%	19%
Expected return	10%	9,5%	11%	2,5%	4%	1%	3%	7,5%

Optimal weight	8,2%	5,3%	7,5%	8,3%	20,9%	19,5%	21,2%	9,2%

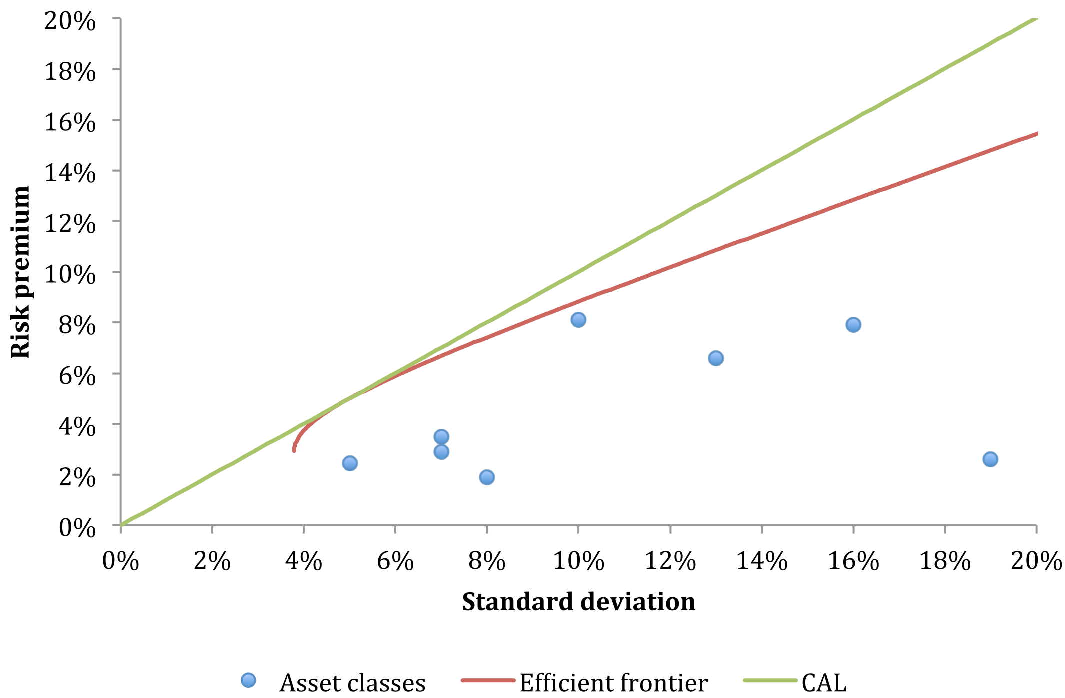


3.b: Tilted scenario

Correlations	A	B	C	D	E	F	G	H
A	1	0,78	0,5	0,12	0,6	-0,2	-0,14	-0,07
B	0,78	1	0,5	0,11	0,57	-0,2	-0,1	0,04
C	0,5	0,5	1	0,22	0,49	-0,3	0,17	0,15
D	0,12	0,11	0,22	1	0,51	0,5	0,07	0,11
E	0,6	0,57	0,49	0,51	1	-0,03	-0,17	0,03
F	-0,2	-0,2	-0,3	0,5	-0,03	1	0,08	0,05
G	-0,14	-0,1	0,17	0,07	-0,17	0,08	1	-0,08
H	-0,07	0,04	0,15	0,11	0,03	0,05	-0,08	1

Standard deviation	16%	13%	10%	5%	7%	8%	7%	19%
Expected return	7,9%	6,6%	8,1%	2,45%	3,5%	1,9%	2,9%	2,6%

Optimal weight	4,0%	5,1%	37,0%	1,0%	2,3%	30,0%	20,0%	0,6%



3.c: Extreme scenario

Correlations	A	B	C	D	E	F	G	H
A	1	0,78	0,5	0,12	0,6	-0,2	-0,14	-0,07
B	0,78	1	0,5	0,11	0,57	-0,2	-0,1	0,04
C	0,5	0,5	1	0,22	0,49	-0,3	0,17	0,15
D	0,12	0,11	0,22	1	0,51	0,5	0,07	0,11
E	0,6	0,57	0,49	0,51	1	-0,03	-0,17	0,03
F	-0,2	-0,2	-0,3	0,5	-0,03	1	0,08	0,05
G	-0,14	-0,1	0,17	0,07	-0,17	0,08	1	-0,08
H	-0,07	0,04	0,15	0,11	0,03	0,05	-0,08	1

Standard deviation	16%	13%	10%	5%	7%	8%	7%	19%
Expected return	8,3%	5,5%	8,3%	2,75%	2,8%	0,7%	2,8%	2,1%

Optimal weight	15,0%	-3,2%	40,0%	50,4%	-28,7%	8,2%	18,0%	0,2%

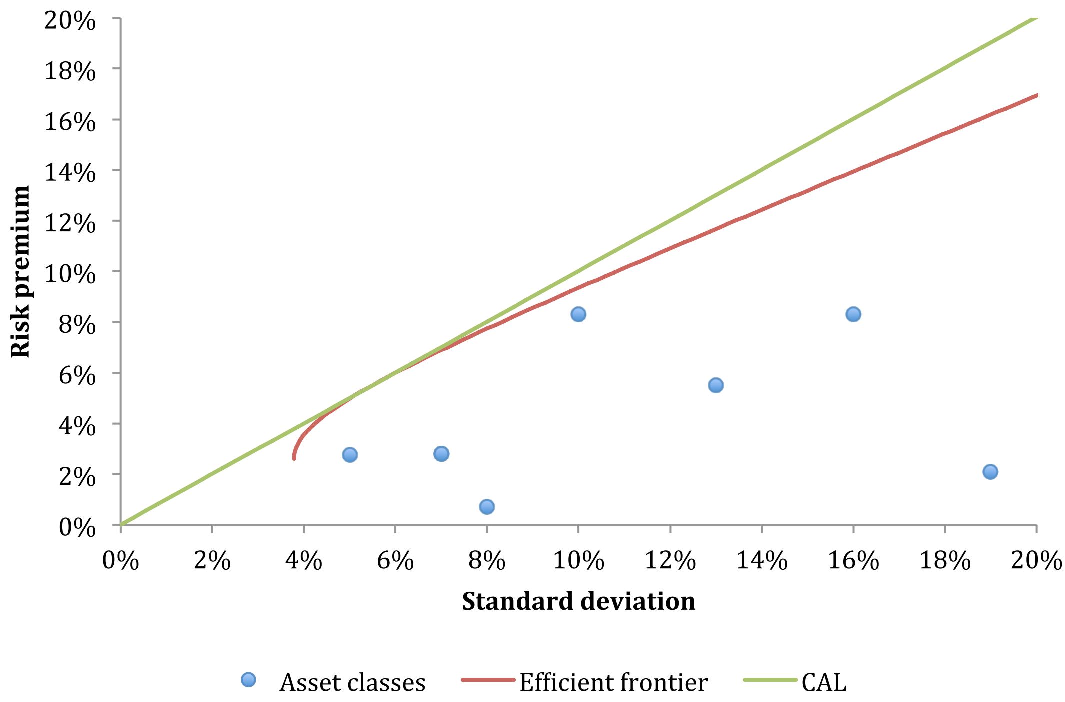
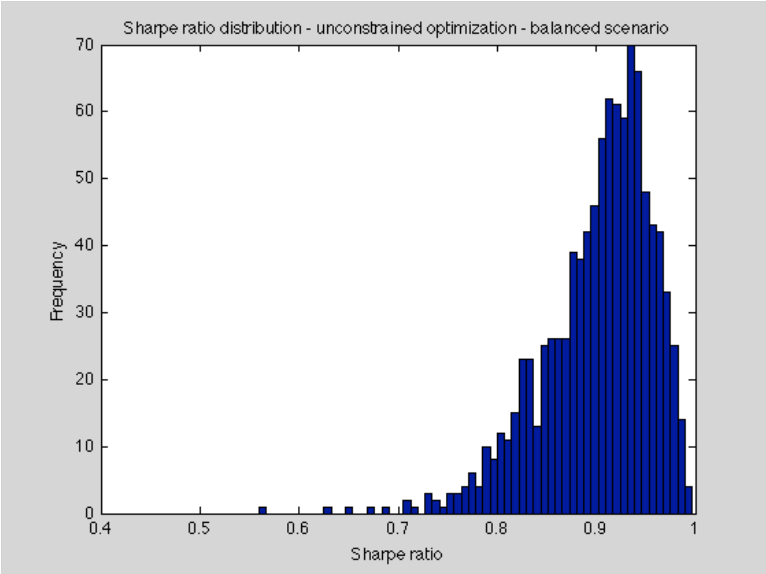
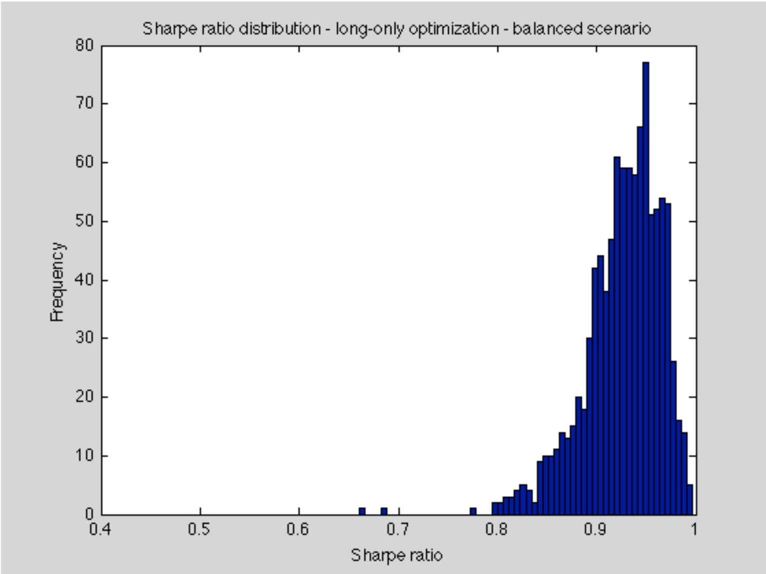
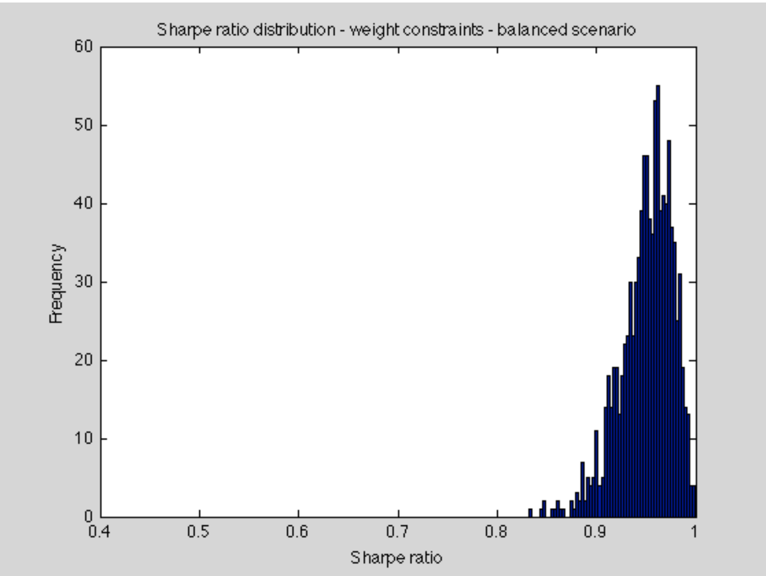


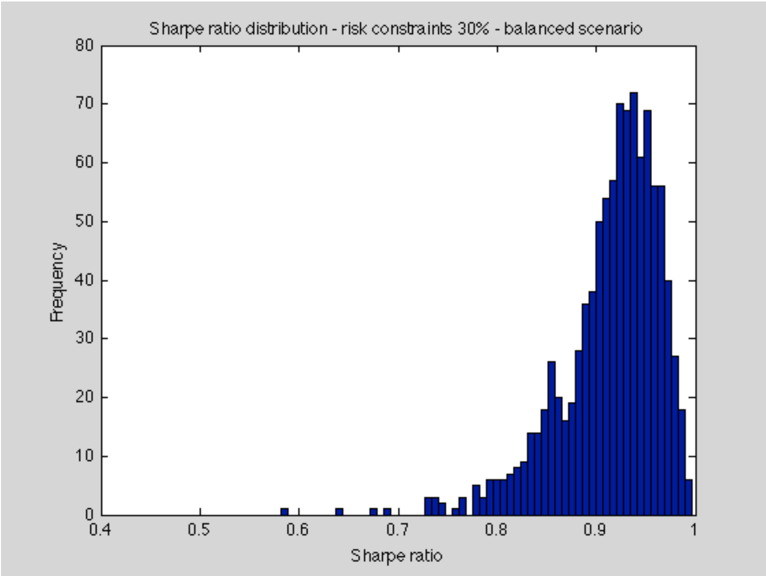
Exhibit 4 – Results of the simulations

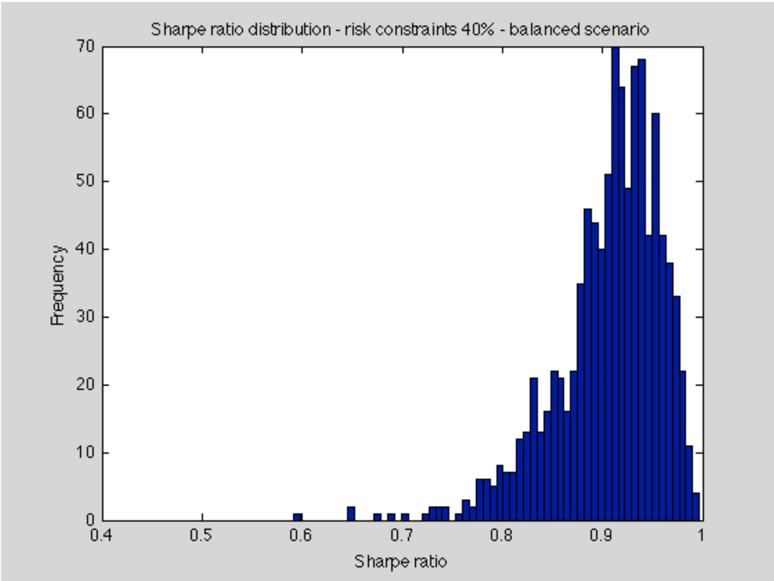
4.a: True Sharpe ratios for each optimizer and scenario

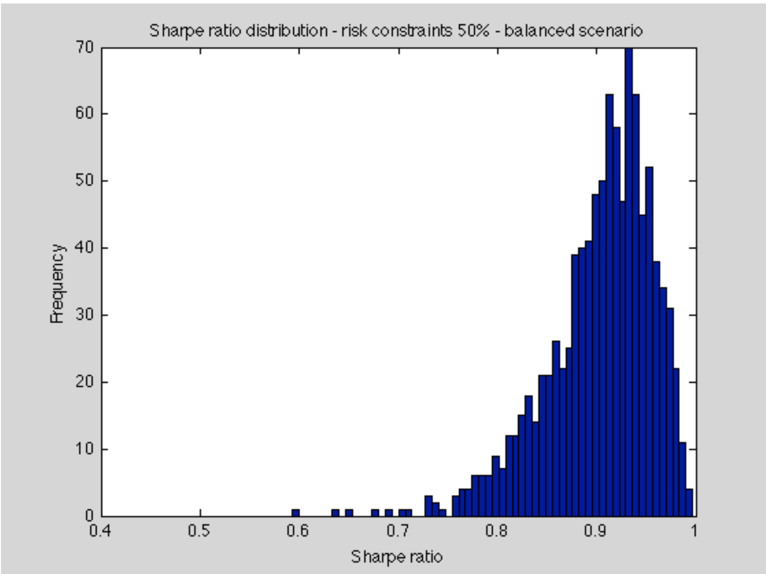


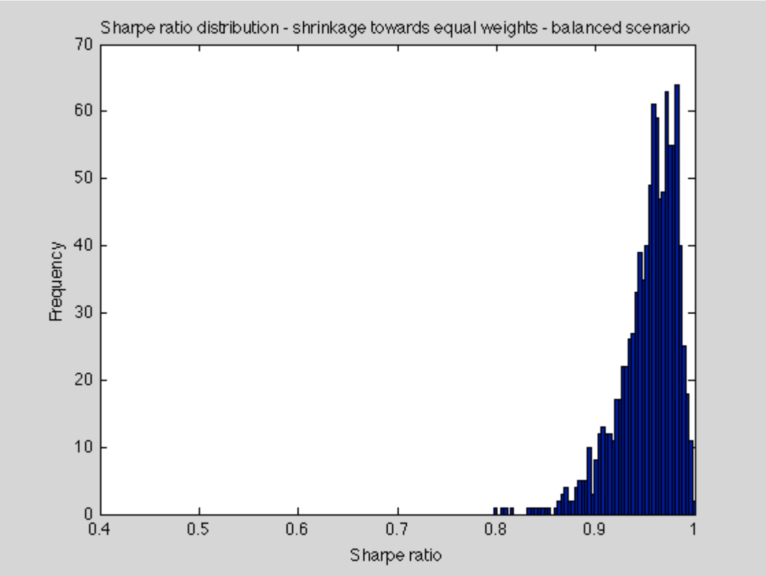


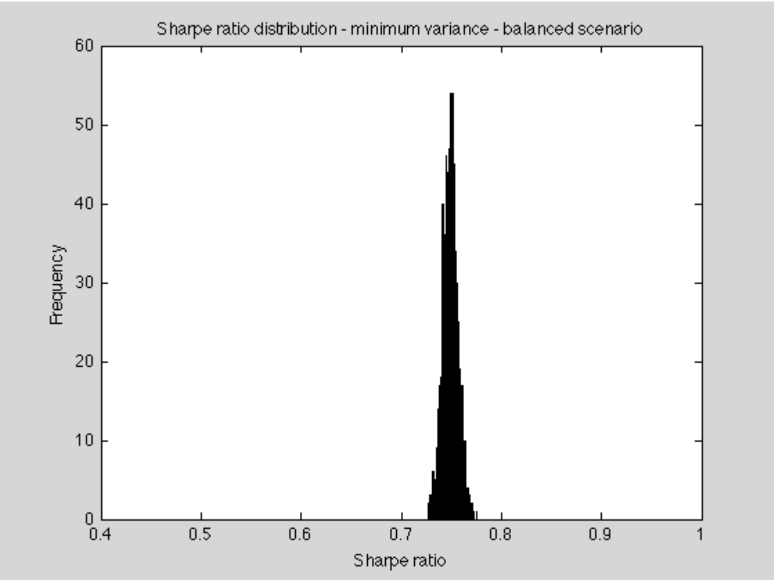


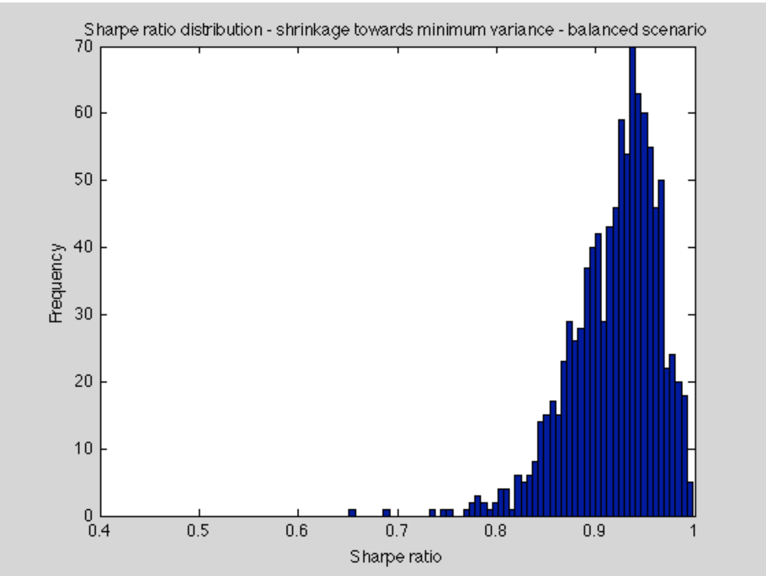

Sharpe ratio distribution - risk constraints 30% - balanced scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - risk constraints 40% - balanced scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - risk constraints 50% - balanced scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - shrinkage towards equal weights - balanced scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - minimum variance - balanced scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - shrinkage towards minimum variance - balanced scenario
Frequency
Sharpe ratio
0
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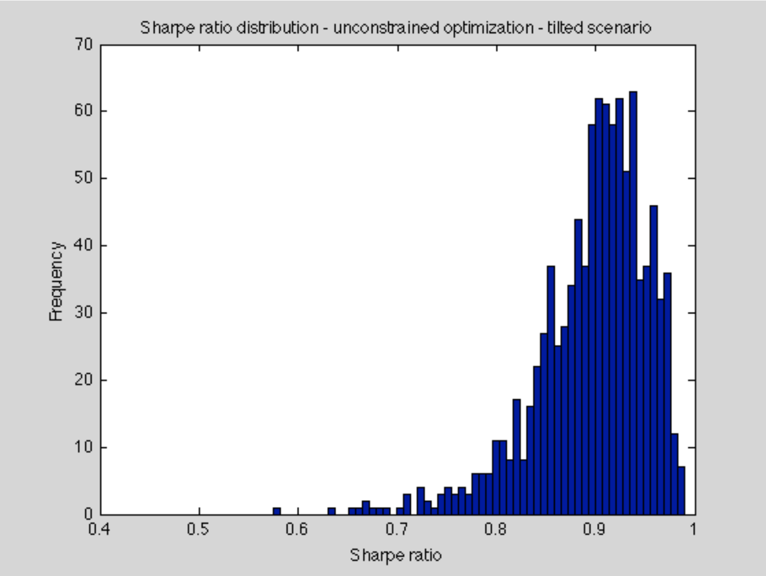
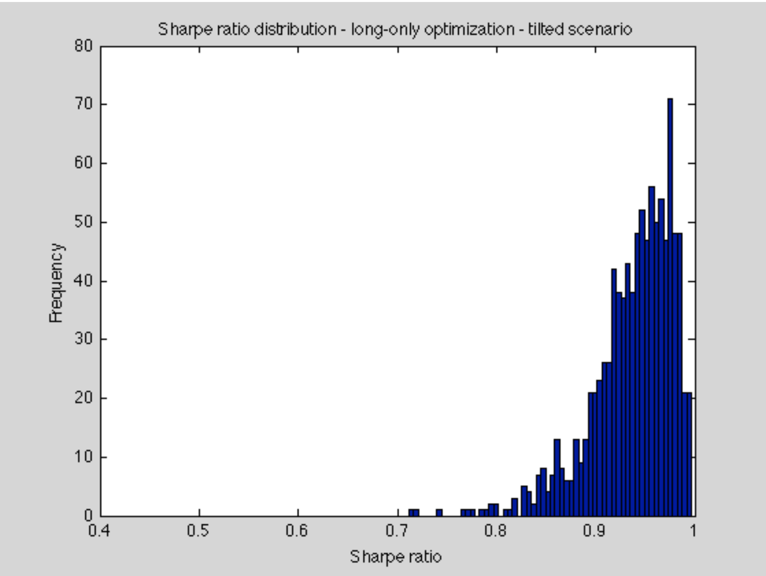
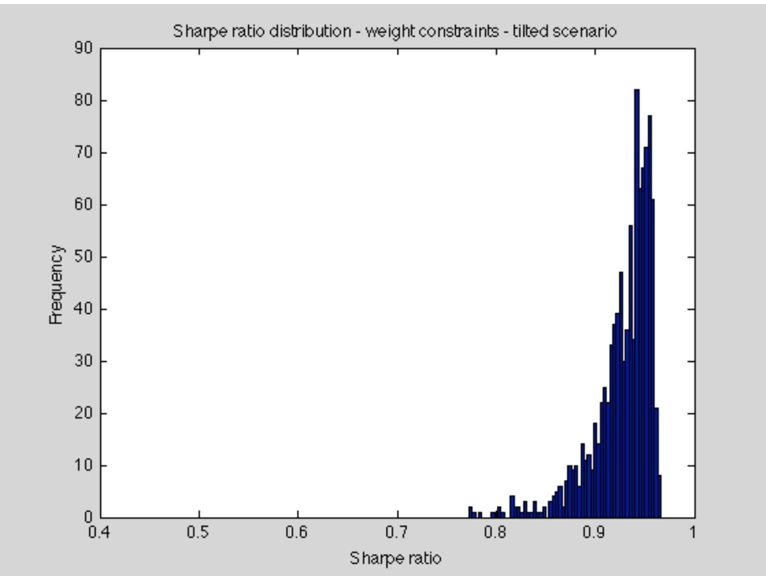
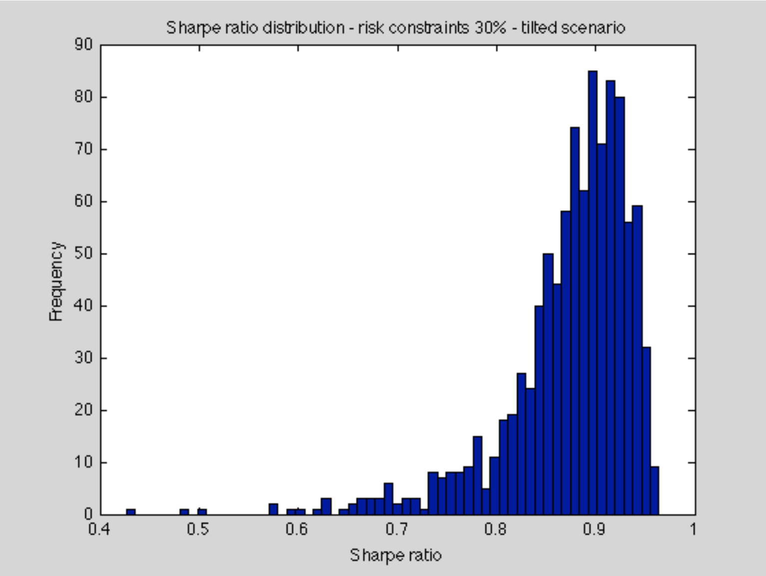
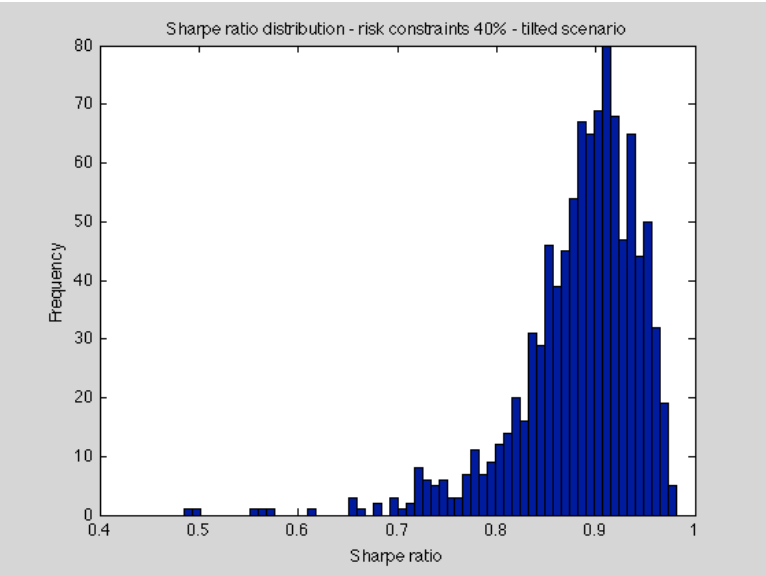
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1

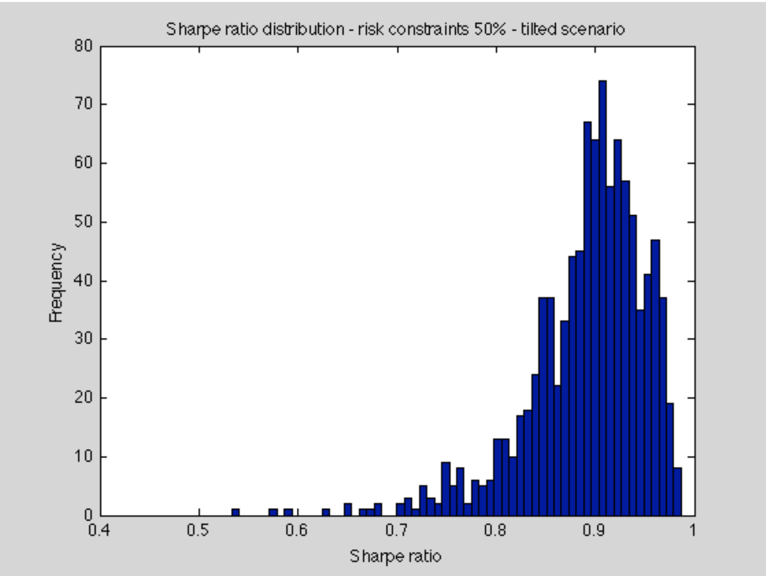

Sharpe ratio distribution - unconstrained optimization - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1

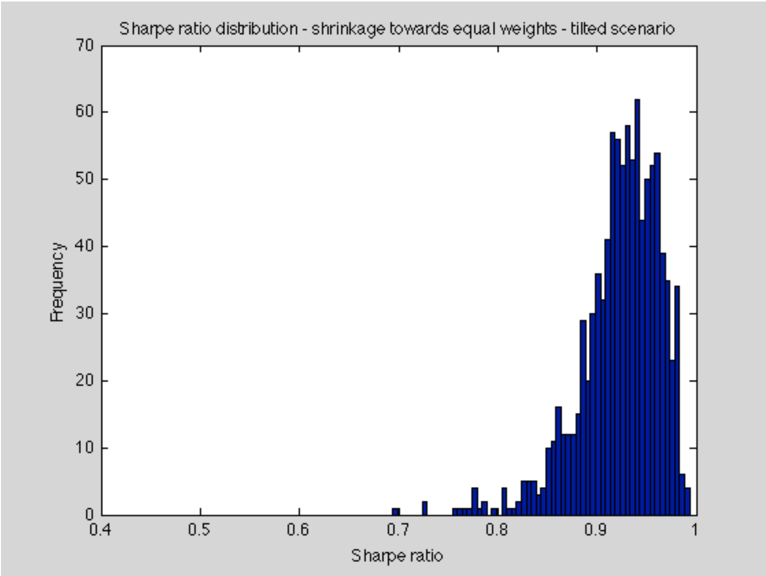

Sharpe ratio distribution - long-only optimization - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
0.4
0.5
0.6
0.7
0.8
0.9
1

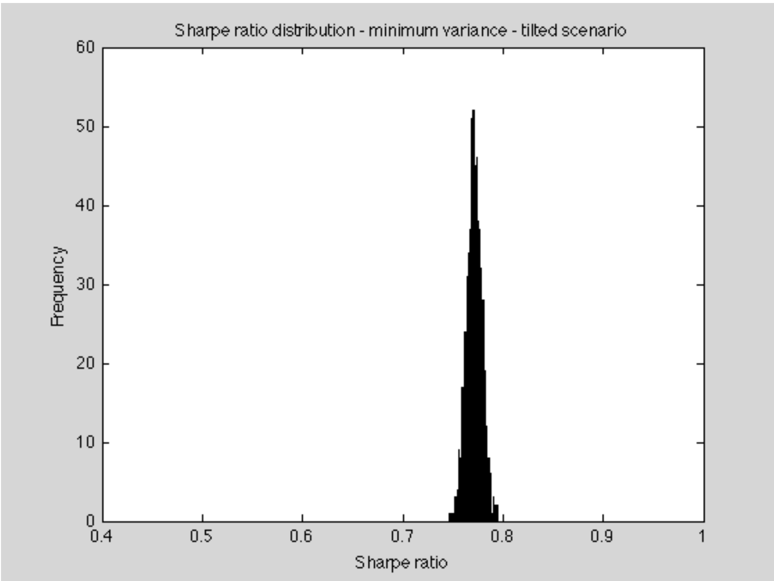

Sharpe ratio distribution - weight constraints - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
90
0.4
0.5
0.6
0.7
0.8
0.9
1

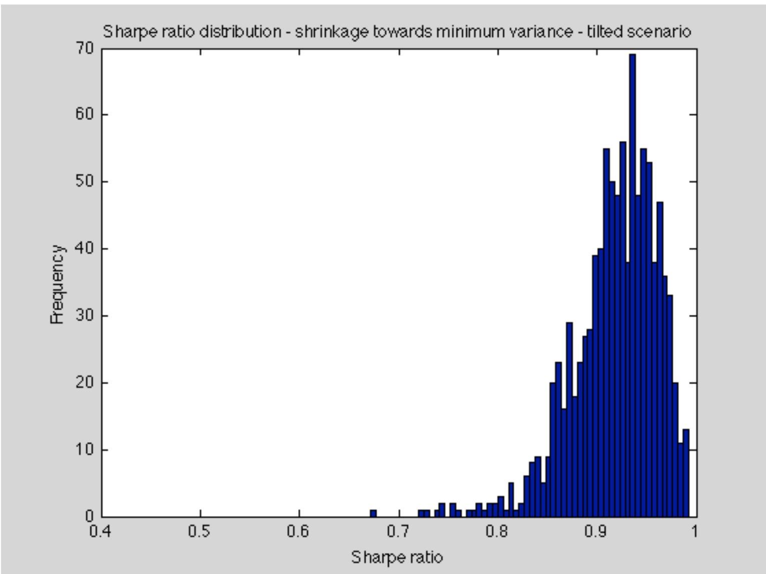

Sharpe ratio distribution - risk constraints 30% - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
90
0.4
0.5
0.6
0.7
0.8
0.9
1

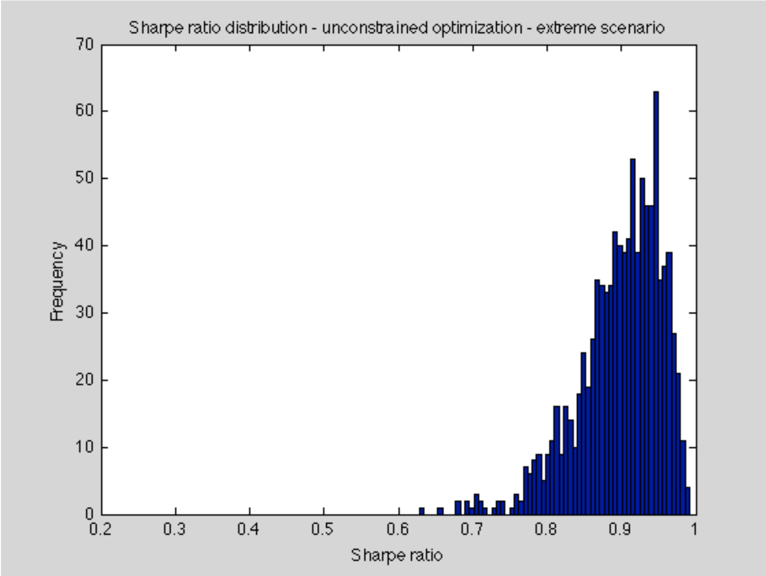

Sharpe ratio distribution - risk constraints 40% - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
0.4
0.5
0.6
0.7
0.8
0.9
1

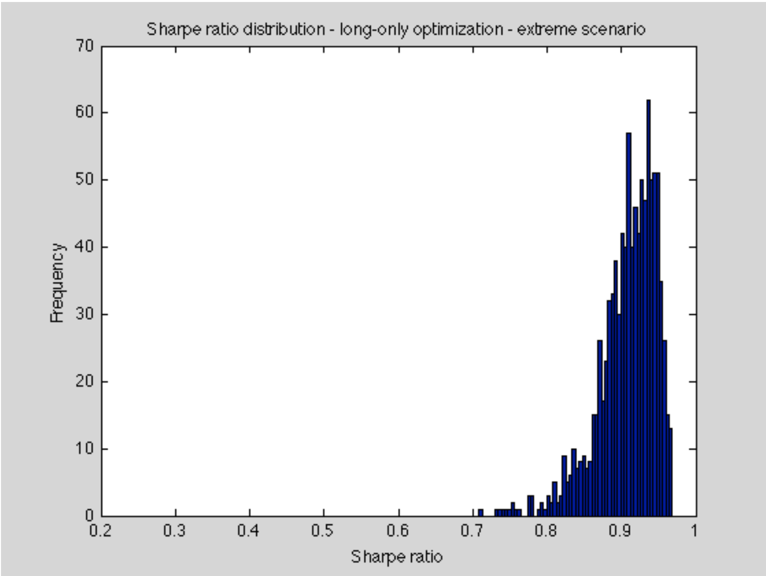

Sharpe ratio distribution - risk constraints 50% - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
0.4
0.5
0.6
0.7
0.8
0.9
1

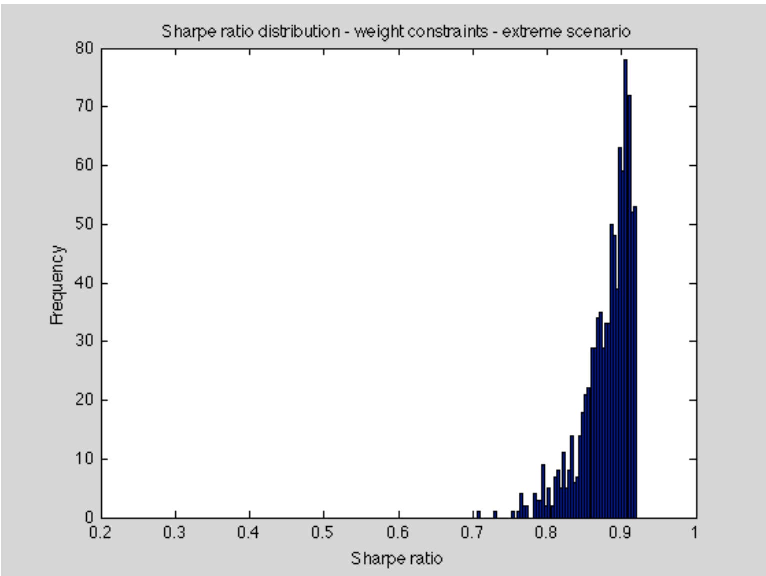

Sharpe ratio distribution - shrinkage towards equal weights - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1

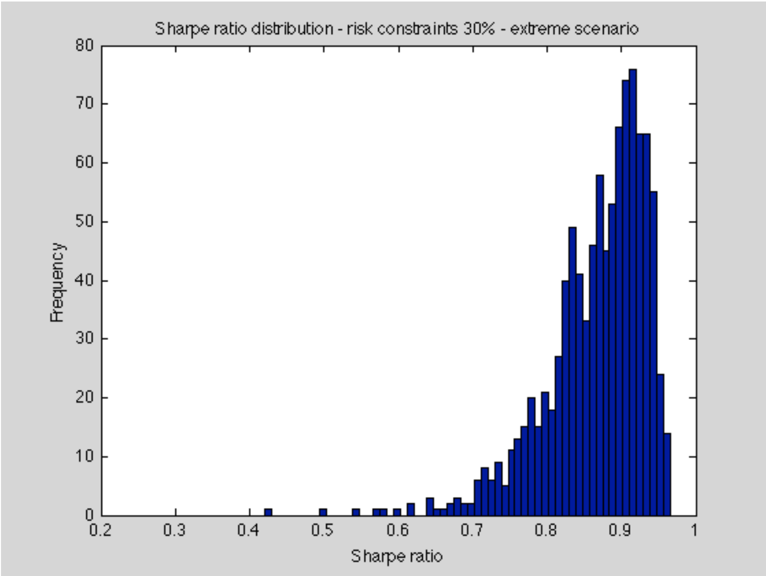

Sharpe ratio distribution - minimum variance - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
0.4
0.5
0.6
0.7
0.8
0.9
1

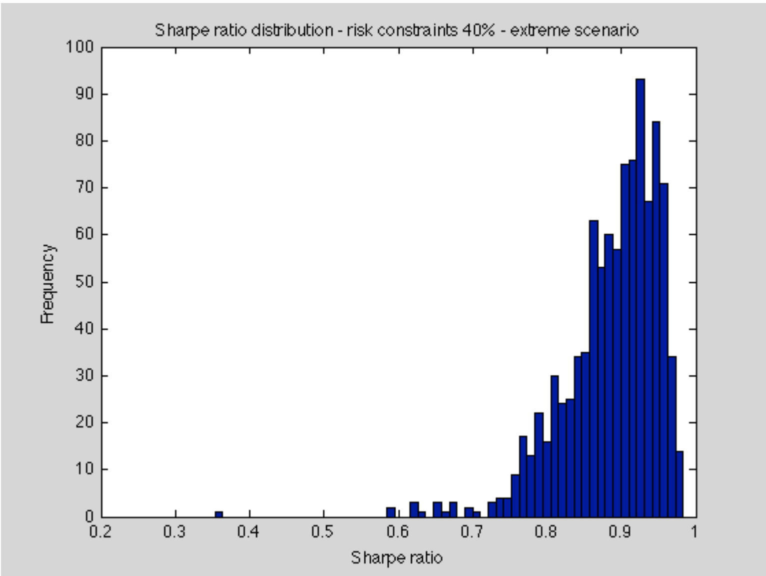

Sharpe ratio distribution - shrinkage towards minimum variance - tilted scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.4
0.5
0.6
0.7
0.8
0.9
1

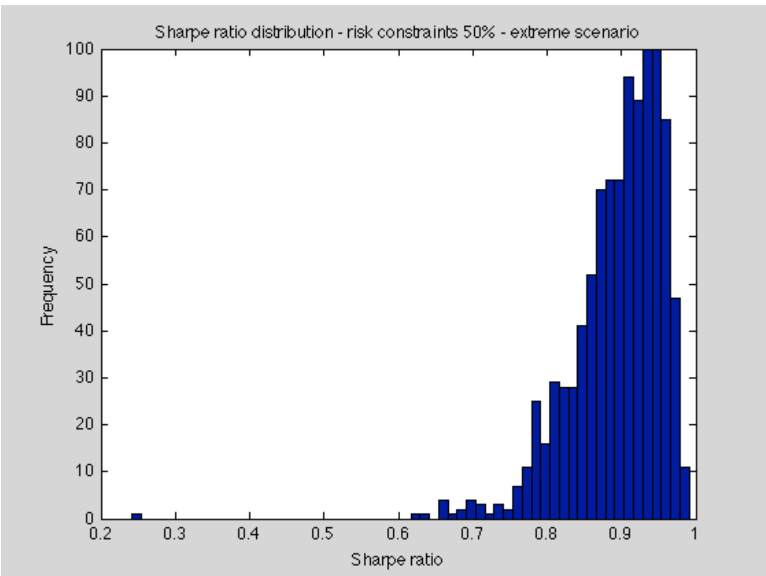

Sharpe ratio distribution - unconstrained optimization - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

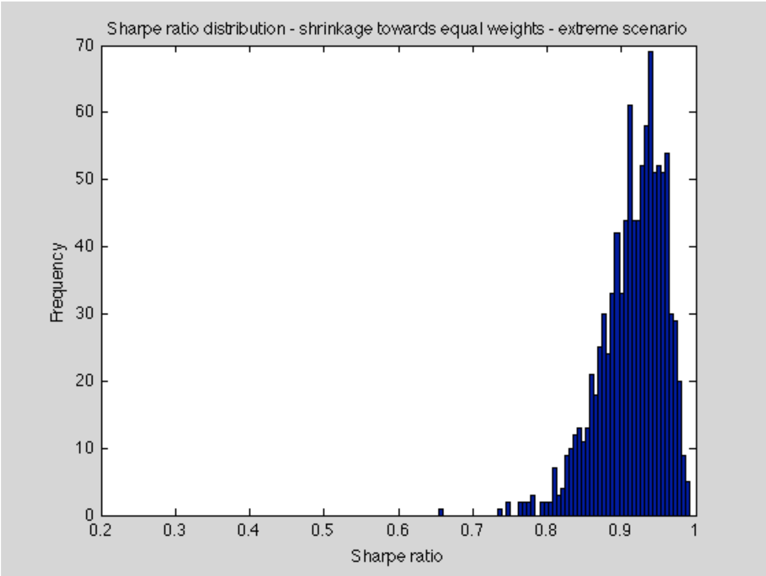

Sharpe ratio distribution - long-only optimization - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

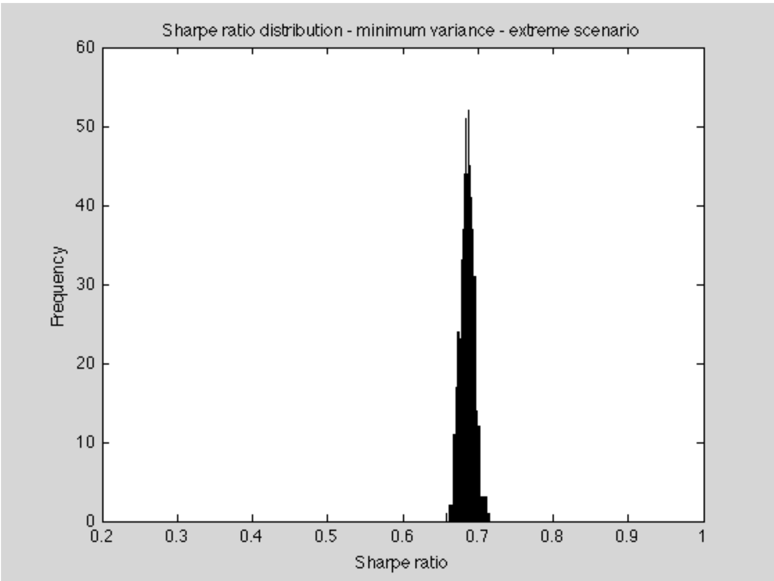

Sharpe ratio distribution - weight constraints - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

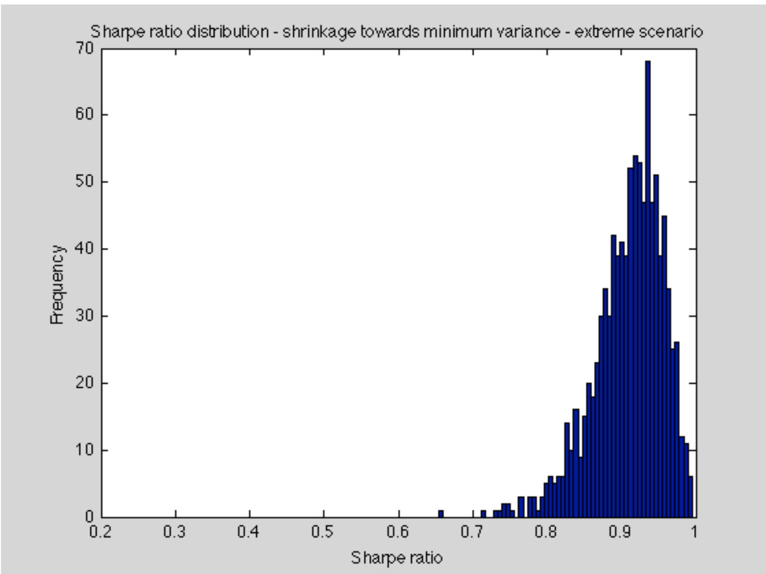

Sharpe ratio distribution - risk constraints 30% - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - risk constraints 40% - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
90
100
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - risk constraints 50% - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
80
90
100
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - shrinkage towards equal weights - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - minimum variance - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Sharpe ratio distribution - shrinkage towards minimum variance - extreme scenario
Frequency
Sharpe ratio
0
10
20
30
40
50
60
70
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

4.b: Statistics of the distributions of Sharpe ratios

Legend:

- Optimizers:

(1) Unconstrained optimization
(2) Non-negative weights
(3) Weight constraints: between 0% and 20%
(4) Partial contribution to risk limited to 30%
(5) Partial contribution to risk limited to 40%
(6) Partial contribution to risk limited to 50%
(7) Shrinkage towards equal-weighted portfolio
(8) Minimum variance portfolio
(9) Shrinkage towards minimum variance portfolio

- Statistics:

(a) Average Sharpe ratio
(b) Standard deviation of the Sharpe ratio
(c) Skewness of the Sharpe ratio
(d) Mean squared deviation of the Sharpe ratio from the optimal ratio of 1
(e) Average difference between the true Sharpe ratio and the perceived Sharpe ratio (in percentage of the perceived ratio)
(f) Standard deviation of the difference described above

- Balanced scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)
(1)	0,902	0,056	-1,197	0,115	-17,2%	13,2%
(2)	0,927	0,039	-1,265	0,085	-13,9%	13,9%
(3)	0,951	0,026	-0,890	0,058	-9,2%	15,8%
(4)	0,915	0,051	-1,474	0,101	-15,5%	13,6%
(5)	0,907	0,053	-1,299	0,109	-16,6%	13,3%
(6)	0,903	0,054	-1,206	0,113	-17,0%	13,2%
(7)	0,953	0,030	-1,397	0,058	-10,4%	14,6%
(8)	0,749	0,008	0,079	0,253	6,0%	28,7%
(9)	0,919	0,044	-1,097	0,094	29,9%	34,2%

- Tilted scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)
(1)	0,897	0,057	-1,315	0,118	-18,5%	12,5%
(2)	0,937	0,042	-1,368	0,076	-11,9%	14,8%
(3)	0,928	0,030	-1,770	0,080	-6,9%	17,5%
(4)	0,874	0,066	-2,016	0,143	-18,1%	12,9%
(5)	0,884	0,062	-1,806	0,132	-18,5%	12,6%
(6)	0,892	0,060	-1,466	0,125	-18,4%	12,6%
(7)	0,924	0,041	-1,319	0,088	-13,2%	13,6%
(8)	0,771	0,007	-0,004	0,230	6,3%	32,4%
(9)	0,918	0,044	-1,163	0,094	26,1%	35,6%

- Extreme scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)
(1)	0,898	0,057	-1,080	0,118	-17,8%	13,0%
(2)	0,908	0,040	-1,329	0,101	-11,1%	15,6%
(3)	0,881	0,033	-1,406	0,124	-6,2%	18,9%
(4)	0,867	0,068	-1,506	0,151	-17,6%	13,7%
(5)	0,887	0,065	-1,666	0,131	-17,5%	13,4%
(6)	0,895	0,063	-2,071	0,124	-17,6%	13,3%
(7)	0,915	0,044	-1,056	0,096	-13,4%	14,0%
(8)	0,686	0,009	-0,010	0,316	9,4%	53,0%
(9)	0,910	0,047	-0,999	0,103	44,2%	58,5%

4.c: Comparison of the average Sharpe ratios

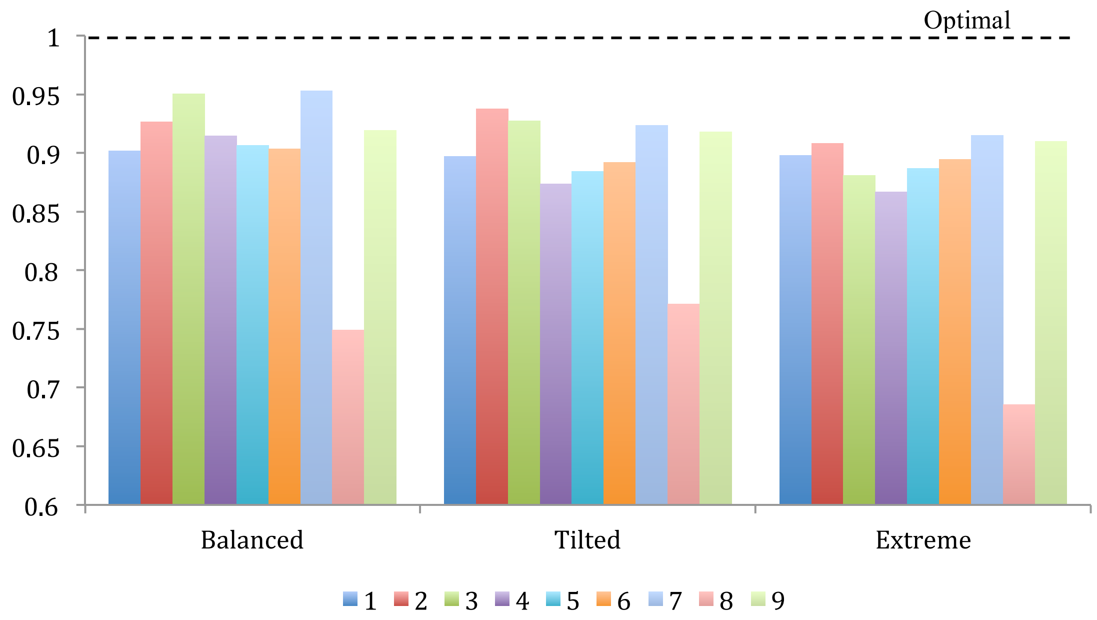


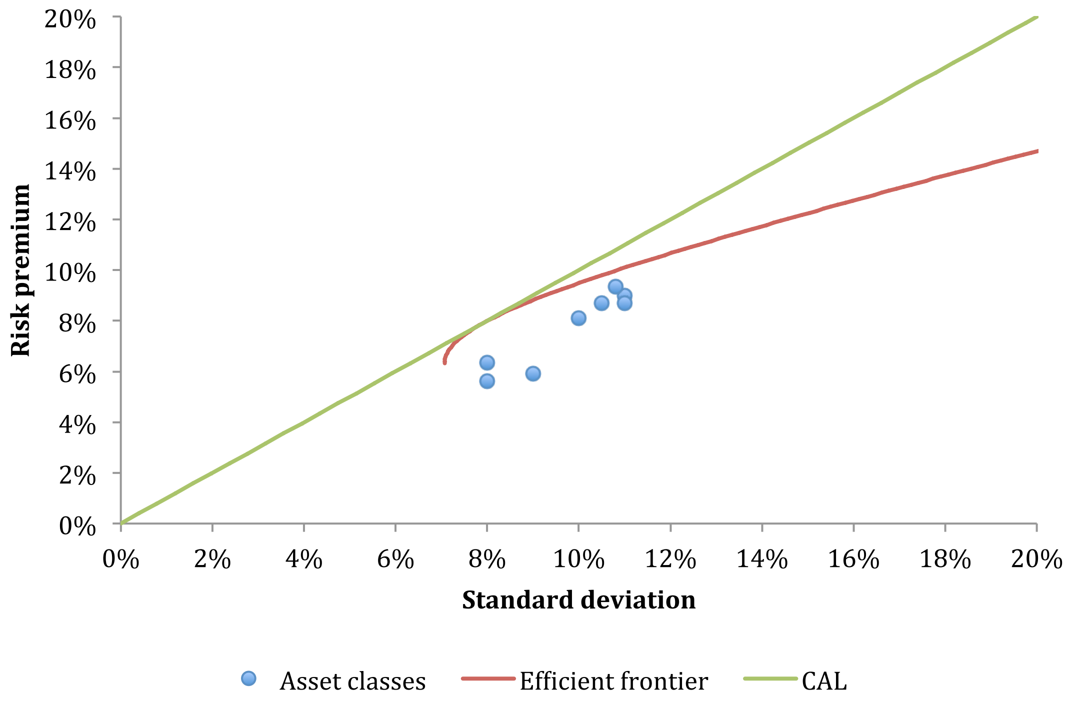
Exhibit 5 – High correlations scenario

5.a: Scenario overview

Correlations	A	B	C	D	E	F	G	H
A	1	0,85	0,7	0,56	0,75	0,45	0,43	0,465
B	0,85	1	0,75	0,555	0,75	0,45	0,45	0,52
C	0,7	0,75	1	0,61	0,6	0,415	0,585	0,575
D	0,56	0,555	0,61	1	0,7	0,75	0,535	0,555
E	0,75	0,75	0,6	0,7	1	0,485	0,415	0,515
F	0,45	0,45	0,415	0,75	0,485	1	0,54	0,525
G	0,43	0,45	0,585	0,535	0,415	0,54	1	0,46
H	0,465	0,52	0,575	0,555	0,515	0,525	0,46	1

Standard deviation	11%	10,8%	10,5%	8%	10%	8%	9%	11%
Expected return	9,0%	9,35%	8,7%	6,35%	8,1%	5,6%	5,9%	8,7%

Optimal weight	8,1%	19,8%	10,2%	10,6%	8,9%	13,0%	8,6%	20,7%



5.b: Statistics (same legend as in 4.b)

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)
(1)	0,895	0,058	-1,131	0,120	-17,0%	13,8%
(2)	0,954	0,026	-1,313	0,053	-8,3%	16,8%
(3)	0,980	0,010	-0,821	0,022	-2,9%	19,0%
(4)	0,926	0,049	-2,231	0,088	-12,4%	15,4%
(5)	0,912	0,051	-1,643	0,102	-14,7%	14,5%
(6)	0,903	0,053	-1,396	0,110	-15,9%	14,0%
(7)	0,946	0,033	-1,387	0,063	-11,3%	14,8%
(8)	0,894	0,005	0,115	0,106	5,6%	24,8%
(9)	0,931	0,038	-1,335	0,078	9,6%	24,2%

<u>Exhibit 6</u> – Statistics for monthly observation of returns

Legend: same as in 4.b plus:

(g) Difference between the average Sharpe ratio with daily and monthly observations

(h) Difference between the standard deviation of the Sharpe ratio with daily and monthly observations

- Balanced scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)
(1)	0,892	0,062	-1,352	0,127	-18,8%	13,7%
(2)	0,921	0,042	-1,305	0,091	-15,0%	14,3%
(3)	0,949	0,028	-1,025	0,060	-9,6%	16,1%
(4)	0,905	0,060	-2,588	0,114	-17,0%	14,2%
(5)	0,896	0,060	-1,739	0,122	-18,2%	13,9%
(6)	0,893	0,061	-1,458	0,125	-18,6%	13,8%
(7)	0,949	0,034	-1,824	0,063	-11,4%	15,0%
(8)	0,738	0,038	-0,284	0,267	4,2%	29,8%
(9)	0,908	0,051	-1,269	0,107	28,3%	36,7%

(g)	(h)
0,010	-0,006
0,006	-0,003
0,002	-0,001
0,010	-0,009
0,010	-0,007
0,010	-0,007
0,004	-0,004
0,011	-0,031
0,011	-0,007

- Tilted scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
(1)	0,887	0,062	-1,153	0,129	-19,9%	12,9%	0,010	-0,005
(2)	0,932	0,045	-1,441	0,082	-12,6%	15,1%	0,005	-0,003
(3)	0,926	0,032	-1,887	0,082	-7,2%	17,6%	0,002	-0,002
(4)	0,865	0,071	-1,899	0,154	-19,4%	13,4%	0,009	-0,005
(5)	0,875	0,066	-1,536	0,143	-19,9%	13,1%	0,010	-0,004
(6)	0,881	0,069	-2,297	0,138	-19,9%	13,1%	0,011	-0,009
(7)	0,919	0,044	-1,295	0,093	-14,1%	13,8%	0,005	-0,003
(8)	0,760	0,037	-0,274	0,244	4,8%	33,6%	0,011	-0,029
(9)	0,907	0,049	-0,996	0,105	24,8%	37,7%	0,011	-0,005

- Extreme scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
(1)	0,888	0,061	-1,041	0,129	-19,2%	13,6%	0,010	-0,004
(2)	0,903	0,043	-1,403	0,107	-11,9%	16,2%	0,005	-0,003
(3)	0,879	0,034	-1,509	0,126	-6,3%	19,6%	0,002	-0,001
(4)	0,856	0,073	-1,333	0,163	-19,0%	14,4%	0,011	-0,005
(5)	0,876	0,070	-1,552	0,143	-18,9%	14,1%	0,011	-0,005
(6)	0,884	0,067	-2,005	0,135	-19,0%	13,9%	0,011	-0,004
(7)	0,909	0,047	-1,226	0,103	-14,4%	14,5%	0,006	-0,003
(8)	0,674	0,042	-0,219	0,330	10,9%	125,6%	0,011	-0,033
(9)	0,899	0,053	-1,078	0,115	46,3%	123,0%	0,011	-0,006

- High correlations scenario

Optimizer	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
(1)	0,887	0,062	-1,071	0,129	-18,2%	14,1%	0,008	-0,004
(2)	0,953	0,026	-1,194	0,054	-8,5%	17,0%	0,001	0,000
(3)	0,980	0,010	-0,867	0,022	-2,8%	19,1%	0,000	0,000
(4)	0,921	0,052	-2,052	0,094	-13,1%	15,5%	0,005	-0,003
(5)	0,905	0,055	-1,542	0,110	-15,7%	14,7%	0,006	-0,004
(6)	0,896	0,057	-1,303	0,119	-17,0%	14,3%	0,007	-0,005
(7)	0,941	0,035	-1,360	0,068	-12,1%	14,9%	0,005	-0,003
(8)	0,884	0,027	-0,467	0,119	4,0%	24,7%	0,010	-0,022
(9)	0,923	0,042	-1,143	0,088	8,3%	24,6%	0,009	-0,004