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Toric Sasaki–Einstein metrics on $S^2 \times S^3$

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Abstract

We show that by taking a certain scaling limit of a Euclideanised form of the Plebanski–Demianski metrics one obtains a family of local toric Kähler–Einstein metrics. These can be used to construct local Sasaki–Einstein metrics in five dimensions which are generalisations of the $Y^{p,q}$ manifolds. In fact, we find that these metrics are diffeomorphic to those recently found by Cvetic, Lu, Page and Pope. We argue that the corresponding family of smooth Sasaki–Einstein manifolds all have topology $S^2 \times S^3$. We conclude by setting up the equations describing the warped version of the Calabi–Yau cones, supporting $(2,1)$ three-form flux.

Recently Sasaki–Einstein geometry has been the focus of much attention. The interest in this subject has arisen due to the discovery in [1,2] of an infinite family $Y^{p,q}$ of explicit Sasaki–Einstein metrics on $S^2 \times S^3$, and the subsequent identification of the corresponding family of AdS/CFT dual quiver gauge theories in [3,4].

The construction of [2] was immediately generalised to higher dimension in Ref. [5] and a further generalisation subsequently appeared in [6,7]. However, dimension five is the most interesting dimension physically and the purpose of this work was to investigate if there exist other local Kähler–Einstein metrics in dimension four from which one can construct complete Sasaki–Einstein manifolds in one dimension higher.

As we show, one can obtain a family of local toric Kähler–Einstein metrics by taking a certain scaling limit of a Euclideanised form of the Plebanski–Demianski metrics [8]. Here toric refers to the fact that the metric has two commuting holomorphic Killing vector fields. In fact the resulting metrics were found independently by Apostolov and collaborators in [9] using rather different methods. In the latter reference it is shown that this family of metrics constitute the most general local Kähler–Einstein metric which is *orthotoric*, a term that we define later. We also show that

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these Kähler–Einstein metrics are precisely those used in the recent construction of Sasaki–Einstein manifolds generalising $Y^{p,q}$ [10]. Higher-dimensional orthotoric Kähler–Einstein metrics are given in explicit form in Ref. [11].

Our starting point will be the following family of local Einstein metrics in dimension four

$$\begin{aligned}\mathrm{d}s_4^2 &= \left(p^2-q^2\right)\left[\frac{\mathrm{d}p^2}{P}+\frac{\mathrm{d}q^2}{Q}\right] \\ &\quad+\frac{1}{p^2-q^2} \\ &\quad\times\left[P\left(\mathrm{d}\tau+q^2\,\mathrm{d}\sigma\right)^2+Q\left(\mathrm{d}\tau+p^2\,\mathrm{d}\sigma\right)^2\right], \qquad (1)\end{aligned}$$

where P and Q are the fourth order polynomials

$$\begin{aligned}P(p) &= -\kappa(p-r_1)(p-r_2)(p-r_3)(p-r_4)\\ Q(q) &= \kappa(q-r_1)(q-r_2)(q-r_3)(q-r_4)+cq, \qquad (2)\\ 0 &= r_1+r_2+r_3+r_4. \qquad (3)\end{aligned}$$

As shown in [12], these metrics arise by taking a scaling limit of the well-known Plebanski–Demianski metrics [8]. The Weyl tensor is anti-self-dual if and only if $c=0$.

The natural almost Kähler two-form associated to the metric (1) is

$$J=\mathrm{d}p\wedge\left(\mathrm{d}\tau+q^2\,\mathrm{d}\sigma\right)+\mathrm{d}q\wedge\left(\mathrm{d}\tau+p^2\,\mathrm{d}\sigma\right). \qquad (4)$$

Our strategy will be to obtain a scaling limit for which (4) becomes closed. Thus, consider the following change of coordinates

$$\begin{aligned}&p=1-\epsilon\xi, \qquad q=1-\epsilon\eta,\\ &\Phi=\epsilon(\tau+\sigma), \qquad \Psi=-2\epsilon^2\sigma \qquad (5)\end{aligned}$$

and redefinition of the metric constants

$$\begin{aligned}&r_i=1-\epsilon\alpha_i, \quad i=1,2,3,\\ &r_4=-3+(\alpha_1+\alpha_2+\alpha_3)\epsilon,\\ &c=4\epsilon^3\gamma. \qquad (6)\end{aligned}$$

The latter ensures that the constraint (3) is satisfied. Defining

$$\begin{aligned}F(\xi) &= -\kappa(\alpha_1-\xi)(\alpha_2-\xi)(\alpha_3-\xi),\\ G(\eta) &= \kappa(\alpha_1-\eta)(\alpha_2-\eta)(\alpha_3-\eta)+\gamma \qquad (7)\end{aligned}$$

it is straightforward to see that, upon sending $\epsilon\to 0$, the metric (1) becomes

$$\begin{aligned}\mathrm{d}s_4^2 &= \frac{(\eta-\xi)}{2F(\xi)}\,\mathrm{d}\xi^2+\frac{2F(\xi)}{(\eta-\xi)}(\mathrm{d}\Phi+\eta\,\mathrm{d}\Psi)^2\\ &\quad+\frac{(\eta-\xi)}{2G(\eta)}\,\mathrm{d}\eta^2+\frac{2G(\eta)}{(\eta-\xi)}(\mathrm{d}\Phi+\xi\,\mathrm{d}\Psi)^2. \qquad (8)\end{aligned}$$

In fact it is also immediate to see that $J=2\,\mathrm{d}A$ where

$$-A=\frac{1}{2}(\xi+\eta)\,\mathrm{d}\Phi+\frac{1}{2}\xi\eta\,\mathrm{d}\Psi. \qquad (9)$$

One can verify that this metric is Kähler–Einstein with curvature

$$\mathrm{Ric}=3\kappa g. \qquad (10)$$

In particular, setting $\kappa=2$ the metric

$$\mathrm{d}s_5^2=\mathrm{d}s_4^2+(\mathrm{d}\psi'+A)^2 \qquad (11)$$

is then locally Sasaki–Einstein with curvature 4. (See, e.g., [5] for curvature conventions.)

Having found these metrics we subsequently discovered that the same solutions had been obtained independently, and in a completely different manner, in reference [9]. In fact the metric, as presented, is essentially already in the form given in [9] and moreover is the most general orthotoric Kähler–Einstein metric. Here orthotoric means that the Hamiltonian functions $\xi+\eta$, $\xi\eta$ for the Killing vector fields $\partial/\partial\Phi$, $\partial/\partial\Psi$, respectively, have the property that the one-forms $\mathrm{d}\xi$, $\mathrm{d}\eta$ are orthogonal. The metric is self-dual if and only if $\gamma=0$. Moreover these metrics have been generalised to arbitrary dimension in [11] which thus gives a more general construction of local Sasaki–Einstein metrics.

One should now proceed to analyse when the local Sasaki–Einstein metrics extend to complete metrics on a smooth manifold. As the metrics generically possess three commuting Killing vectors $\partial/\partial\Phi$, $\partial/\partial\Psi$, $\partial/\partial\psi'$, the resulting five-dimensional manifolds should be toric, as was the case in [2]. In particular, real codimension two fixed point sets correspond to toric divisors [3] in the Calabi–Yau cone, and it is a very simple matter to find such vector fields for the metric (11).

A generic Killing vector can be written as

$$V=S\frac{\partial}{\partial\Phi}+T\frac{\partial}{\partial\Psi}+U\frac{\partial}{\partial\psi'}, \qquad (12)$$

where S,T and U are constants. A short calculation then shows that its norm is given by

$$\|V\|^2 = \frac{2}{(\eta-\xi)}\big[F(\xi)(S+T\eta)^2 + G(\eta)(S+T\xi)^2\big] + \frac{1}{4}\big[S(\xi+\eta)+T\xi\eta-2U\big]^2. \tag{13}$$

Now, crucially, since $F(\xi)/(\eta-\xi)>0$, $G(\eta)/(\eta-\xi)>0$ for a positive definite metric, this is a sum of positive functions. Therefore it can vanish in codimension two if and only if $\xi=\xi_i$ (or $\eta=\eta_i$) are at the roots of F (or G) and at the same time the remaining terms manage to vanish for generic values of η (or ξ). In fact, it is easy to see that this is true. We therefore see that there are *four* codimension two fixed point sets, so that if the metrics extend onto complete smooth toric manifolds, the Calabi–Yau cones must be a $\mathbb{T}^3$ fibration over a four faceted polyhedral cone in $\mathbb{R}^3$.

However, we will not complete the details of this argument because it turns out that these metrics are diffeomorphic to those found by Cvetic, Lu, Page, and Pope in [10]. These authors have performed the global analysis in detail. Therefore, we instead exhibit an explicit change of coordinates, demonstrating the equivalence of the two metrics.

The Kähler–Einstein metrics in Ref. [10] were given in the form

$$\begin{aligned}\mathrm{d}s_4^2 &= \frac{\rho^2\,\mathrm{d}x^2}{4\Delta_x} + \frac{\rho^2\,\mathrm{d}\theta^2}{\Delta_\theta} \\ &\quad + \frac{\Delta_x}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}\,\mathrm{d}\phi + \frac{\cos^2\theta}{\beta}\,\mathrm{d}\psi\right)^2 \\ &\quad + \frac{\Delta_\theta\sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha-x}{\alpha}\,\mathrm{d}\phi - \frac{\beta-x}{\beta}\,\mathrm{d}\psi\right)^2,\end{aligned} \tag{14}$$

where

$$\Delta_x = x(\alpha-x)(\beta-x)-\mu, \qquad \rho^2=\Delta_\theta - x,$$
$$\Delta_\theta = \alpha\cos^2\theta+\beta\sin^2\theta. \tag{15}$$

Consider the coordinate transformation[1]

$$\eta=\alpha-x, \qquad \xi=(\alpha-\beta)\sin^2\theta,$$
$$\Phi=\frac{1}{2\beta}\psi, \qquad \Psi=\frac{\phi}{2(\alpha-\beta)\alpha}-\frac{\psi}{2(\alpha-\beta)\beta}. \tag{16}$$

It is a simple exercise to show that, in these coordinates, the metric (14) takes the form (8) where α,β and μ parametrise the cubic function. Explicitly, we have

$$F(\xi)=2\xi(\alpha-\xi)(\alpha-\beta-\xi),$$
$$G(\eta)=-2\eta(\alpha-\eta)(\alpha-\beta-\eta)-2\mu. \tag{17}$$

As shown in [10], the complete metrics $L^{a,b,c}$ with local form (11) are specified by three integers a,b,c. One recovers the $Y^{p,q}$ metrics in the limit $a=p-q$, $b=p+q$, $c=p$. Moreover, as explained, there are precisely four Killing vector fields V_i, $i=1,2,3,4$, that vanish on codimension 2 submanifolds. This means that the image of the Calabi–Yau cone under the moment map for the $\mathbb{T}^3$ action is a four faceted polyhedral cone in $\mathbb{R}^3$—see [3] for a review. Indeed, using the linear relation among the vectors in [10] one can show that the normal vectors to this polyhedral cone satisfy the relation

$$av_1+bv_2-cv_3-(a+b-c)v_4=0, \tag{18}$$

where v_i, $i=1,2,3,4$, are the primitive vectors in $\mathbb{R}^3$ that define the cone. From the Delzant theorem in [13] it follows that, for a,b,c relatively prime, the Sasaki–Einstein metrics $L^{a,b,c}$ are equivariantly contactomorphic to the link of the symplectic quotient

$$\mathbb{C}^4//(a,b,-c,-a-b+c). \tag{19}$$

Note that indeed in the $Y^{p,q}$ limit we obtain charges $(p-q,p+q,-p,-p)$ which is the result of [3]. The base Y of the cone is non-singular if a and b are pairwise prime to each of c and $a+b-c$, and these integers are strictly positive. By the results of [14] we then have $\pi_2(Y)\cong\mathbb{Z}$. Since Y is simply-connected, spin and has no torsion in $H_2(Y)$ it follows from Smale's theorem that Y is diffeomorphic to $S^2\times S^3$.

We conclude with some technical computations that may be of use in future developments. In particular, following [3], we first introduce complex coordinates on the Calabi–Yau cone. We then write down the equations for a warped version of the Calabi–Yau cone, thus generalising the solution of [15]. Note that the form of the metric that we have presented here is more symmetric than in the coordinate system of [10].

The holomorphic $(3,0)$ form on the cone can be written in the standard fashion

$$\Omega = e^{i\psi'}r^2\Omega_4\wedge\big[\mathrm{d}r+ir(\mathrm{d}\psi'+A)\big] \tag{20}$$

[1] Note that this is degenerate when $\alpha=\beta$, which corresponds to the $Y^{p,q}$ limit [10].

with appropriate Ω_4 [3]. Introducing the one-forms

$$\hat{\eta}_1 = \frac{1}{2F}\,\mathrm{d}\xi + \frac{i}{\eta-\xi}(\mathrm{d}\Phi + \eta\,\mathrm{d}\Psi),$$
$$\hat{\eta}_2 = \frac{1}{2G}\,\mathrm{d}\eta + \frac{i}{\eta-\xi}(\mathrm{d}\Phi + \xi\,\mathrm{d}\Psi),$$
$$\hat{\eta}_3 = \frac{\mathrm{d}r}{r} + i(\mathrm{d}\psi' + A) \tag{21}$$

this may be written as

$$\Omega = 2(\eta-\xi)\sqrt{FG}r^3 e^{i\psi'}\,\hat{\eta}_1\wedge\hat{\eta}_2\wedge\hat{\eta}_3. \tag{22}$$

By construction the $\hat{\eta}_i$ are such that $\hat{\eta}_i \wedge \Omega = 0$, i.e., they are $(1,0)$ forms. However, one must take combinations of these to obtain integrable (closed) forms. These are given by

$$\eta_1 = \hat{\eta}_1 - \hat{\eta}_2 = \frac{1}{2F}\,\mathrm{d}\xi - \frac{1}{2G}\,\mathrm{d}\eta + i\,\mathrm{d}\Psi,$$
$$\eta_2 = \xi\hat{\eta}_1 - \eta\hat{\eta}_2 = \frac{\xi}{2F}\,\mathrm{d}\xi - \frac{\eta}{2G}\,\mathrm{d}\eta - i\,\mathrm{d}\Phi,$$
$$\begin{aligned}\eta_3 &= \xi^2\hat{\eta}_1 - \eta^2\hat{\eta}_2 - 2\hat{\eta}_3\\ &= \frac{\xi^2}{2F}\,\mathrm{d}\xi - \frac{\eta^2}{2G}\,\mathrm{d}\eta - 2\frac{\mathrm{d}r}{r} - 2i\,\mathrm{d}\psi'.\end{aligned} \tag{23}$$

The effect of this change of basis is to simplify (22) slightly

$$\Omega = \sqrt{FG}r^3 e^{i\psi'}\,\eta_1\wedge\eta_2\wedge\eta_3. \tag{24}$$

Integrating these one-forms, thus introducing $\eta_i = \mathrm{d}z_i/z_i$, we obtain the following set of complex coordinates:

$$z_1 = \prod_{i=1}^{3}(\xi-\xi_i)^{\frac{1}{4\prod_{j\neq i}(\xi_i-\xi_j)}}(\eta-\eta_i)^{\frac{1}{4\prod_{j\neq i}(\eta_i-\eta_j)}}e^{i\Psi},$$
$$z_2 = \prod_{i=1}^{3}(\xi-\xi_i)^{\frac{\xi_i}{4\prod_{j\neq i}(\xi_i-\xi_j)}}(\eta-\eta_i)^{\frac{\eta_i}{4\prod_{j\neq i}(\eta_i-\eta_j)}}e^{-i\Phi},$$
$$\begin{aligned}z_2 &= \prod_{i=1}^{3}(\xi-\xi_i)^{\frac{\xi_i^2}{4\prod_{j\neq i}(\xi_i-\xi_j)}}\\ &\quad\times(\eta-\eta_i)^{\frac{\eta_i^2}{4\prod_{j\neq i}(\eta_i-\eta_j)}}r^{-2}e^{-2i\psi'}.\end{aligned} \tag{25}$$

For convenience we have written the cubic polynomials as

$$F(\xi) = 2(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3),$$
$$G(\eta) = -2(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3). \tag{26}$$

These coordinates generalise those introduced in [3] for the metric cone over the $Y^{p,q}$ manifolds.

Next we turn to the problem of finding warped solutions which arise after placing fractional branes at the apex of the cone. These solutions are expected to be relevant for the study of cascades in the dual gauge theories. In the following we follow the logic of reference [15]. Recall that in type IIB supergravity it is possible to turn on a complex three-form flux preserving supersymmetry, provided this is of Hodge type $(2,1)$ with respect to the complex structure of the Calabi–Yau cone. Such a three-form is easily constructed in terms of a local closed primitive $(1,1)$ form on the Kähler–Einstein space. It is straightforward to see that such a two-form is

$$\omega = \frac{1}{(\xi-\eta)^2}\big[\mathrm{d}(\xi-\eta)\wedge\mathrm{d}\Phi + (\eta\,\mathrm{d}\xi - \xi\,\mathrm{d}\eta)\wedge\mathrm{d}\Psi\big]. \tag{27}$$

Then, by construction

$$\Omega_{2,1} = \left[\frac{\mathrm{d}r}{r} + i(\mathrm{d}\psi' + A)\right]\wedge\omega \tag{28}$$

is $(2,1)$ and closed. Next, let us give the scalar Laplacian operator on the Calabi–Yau cone, acting on $\mathbb{T}^3$-invariant functions. Again, this is highly symmetric in ξ,η due to the particularly simple form of the metric:

$$\begin{aligned}\Delta_{\mathrm{CY}} = \frac{1}{r^5(\eta-\xi)}\big[&(\eta-\xi)\partial_r\big(r^5\partial_r\big)\\ &+ 2r^3\big(\partial_\eta(G\partial_\eta) + \partial_\xi(F\partial_\xi)\big)\big].\end{aligned} \tag{29}$$

One is then interested in finding solutions of the type

$$\mathrm{d}s^2 = h^{-1/2}\,\mathrm{d}s^2\big(\mathbb{R}^4\big) + h^{1/2}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}s_5^2\big),$$
$$F_3 + iH_3 \propto \Omega_{2,1}, \tag{30}$$

where F_3 and H_3 are the RR and NS three-forms, respectively, [15], for which the only non-trivial equation reduces to

$$\Delta_{\mathrm{CY}}h = -\frac{1}{6}|H_3|^2. \tag{31}$$

After the substitutions [15]

$$h(r,\xi,\eta) = r^{-4}\left[\frac{A}{2}t + s(\xi,\eta)\right],\qquad t = \log r \tag{32}$$

we are left with the following PDE

$$\frac{\partial}{\partial\eta}\left(G(\eta)\frac{\partial}{\partial\eta}s(\xi,\eta)\right)+\frac{\partial}{\partial\xi}\left(F(\xi)\frac{\partial}{\partial\xi}s(\xi,\eta)\right)$$
$$=-\frac{C^2}{(\xi-\eta)^3}+A(\xi-\eta), \tag{33}$$

where C is a proportionality constant. Of course this is still dependent on two variables, but it seems that solutions generalising those of [15] should exist.

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References

[1] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Class. Quantum Grav. 21 (2004) 4335, hep-th/0402153.
[2] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Adv. Theor. Math. Phys. 8 (2004) 711, hep-th/0403002.
[3] D. Martelli, J. Sparks, hep-th/0411238, Commun. Math. Phys., in press.
[4] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, J. Sparks, JHEP 0506 (2005) 064, hep-th/0411264.
[5] J.P. Gauntlett, D. Martelli, J.F. Sparks, D. Waldram, hep-th/0403038, Adv. Theor. Math. Phys., in press.
[6] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Supersymmetric AdS backgrounds in string and M-theory, hep-th/0411194, in: Proceedings of the 73rd Meeting between Physicists and Mathematicians (A)dS/CFT Correspondence, Strasbourg, 11–13 September 2003, in press.
[7] W. Chen, H. Lu, C.N. Pope, J.F. Vazquez-Poritz, hep-th/0411218.
[8] J.F. Plebanski, M. Demianski, Ann. Phys. (N.Y.) 98 (1976) 98.
[9] V. Apostolov, D.M. Calderbank, P. Gauduchon, math.DG/0104233.
[10] M. Cvetic, H. Lu, D.N. Page, C.N. Pope, hep-th/0504225.
[11] V. Apostolov, D.M.J. Calderbank, P. Gauduchon, math.DG/0202280.
[12] K. Behrndt, G. Dall'Agata, D. Lust, S. Mahapatra, JHEP 0208 (2002) 027, hep-th/0207117.
[13] E. Lerman, J. Symplectic Geom. 1 (4) (2003) 785, math.SG/0107201.
[14] E. Lerman, Trans. Amer. Math. Soc. 356 (10) (2004) 4075, math.SG/0204064.
[15] C.P. Herzog, Q.J. Ejaz, I.R. Klebanov, JHEP 0502 (2005) 009, hep-th/0412193.