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Abstract

We analyse the most general bosonic supersymmetric solutions of type IIB supergravity whose metrics are warped products of five-dimensional anti-de Sitter space (AdS$_5$) with a five-dimensional Riemannian manifold M_5. All fluxes are allowed consistent with $SO(4, 2)$ symmetry. We show that the necessary and sufficient conditions can be phrased in terms of a local identity structure on M_5. For a special class, with constant dilaton and vanishing axion, we reduce the problem to solving a second-order nonlinear ODE. We find an exact solution of the ODE which reproduces a solution first found by Pilch and Warner. A numerical analysis of the ODE reveals an additional class of local solutions giving complete metrics on S^5 but with three-form flux and Killing spinors that are not globally defined.

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1. Introduction

The AdS/CFT correspondence [1] is one of the most important developments in string theory. It is therefore an important issue to understand the geometric structures underpinning the correspondence. On the one hand such an understanding can lead to new explicit examples where one can make detailed comparisons with the dual field theory and which can also suggest further generalizations. On the other hand, and more generally, a precise statement of the underlying geometry is the foundation for progress without recourse to explicit examples. By analogy, recall that our understanding of Calabi–Yau manifolds has been developed without a single non-trivial explicit compact Calabi–Yau 3-fold metric having been constructed.

In [2] we analysed the most general kind of solutions of $D = 11$ supergravity that can be dual to a four-dimensional superconformal field theory. These bosonic supersymmetric solutions have a metric that is a warped product of AdS$_5$ with a six-dimensional Riemannian manifold M_6. In order that the $SO(4, 2)$ isometry group of AdS$_5$ is a symmetry group of the full solution, the four-form field strength has non-vanishing components only on M_6. We used the, by now, standard technique of analysing the canonical G-structure dictated by supersymmetry [3–5] in order to obtain necessary and sufficient conditions for supersymmetry. We showed that the geometry on M_6 admits a local $SU(2)$-structure and that this implies that M_6 is determined, in part, by a one-parameter family of four-dimensional Kähler metrics.

We further analysed a special sub-class of solutions by imposing the condition that M_6 is complex and we used the results to construct several new classes of compact examples of M_6 in explicit form. We showed that one sub-class of solutions leads to new type IIA and type IIB solutions with AdS$_5$ factors, via dimensional reduction and T-duality, respectively. In particular, the type IIB solutions turn out to be direct product backgrounds AdS$_5 \times X_5$ with X_5 a Sasaki–Einstein manifold and only the self-dual five-form non-vanishing and proportional to the sum of the volume forms on AdS$_5$ and X_5—see [6–9] for a general discussion of such backgrounds. This is an interesting class of solutions since the dual SCFT can be identified as that arising on a stack of D3-branes transverse to the Calabi–Yau three-fold cone based on X_5. The most well-known examples of five-dimensional Sasaki–Einstein manifolds, and until recently the only explicit examples, are S^5 and $T^{1,1}$; the corresponding IIB solutions are dual to $N = 4$ super Yang–Mills theory and an $N = 1$ superconformal field theory discussed in [6, 9], respectively. The solutions found in [2] led to an infinite number of new explicit Sasaki–Einstein metrics on $S^2 \times S^3$ called $Y^{p,q}$ [10]. The dual conformal field theories have now been identified [11–13] and there have been many further checks and developments. The $Y^{p,q}$ metrics were generalized to all dimensions in [14–16] and were recently further generalized to the $L^{a,b,c}$ metrics in [17, 18] (see also [19]).

The analysis of [2] covered AdS$_5$ geometries in $D = 11$ supergravity preserving $N = 1$ supersymmetry. A refinement of this analysis, considering the additional conditions imposed by $N = 2$ supersymmetry, recently appeared in [20] as a corollary to the description of 'bubbling' Ads spacetimes.

In this paper we will generalize the M-theory analysis of [2] to type IIB string theory. In particular, we go beyond the Sasaki–Einstein class and analyse the most general bosonic supersymmetric solutions of type IIB supergravity with a metric that is a warped product of AdS$_5$ with M_5. In addition, we allow all of the NS–NS and R–R bosonic fields consistent with $SO(4, 2)$ symmetry. Once again, following [3–5], we analyse the G-structure defined by the Killing spinors. We find that the most general geometries have a local identity structure, or equivalently a canonically defined frame, and we use this to determine the necessary and sufficient conditions for supersymmetry. The geometries have a canonically defined Killing vector, which corresponds to the $U(1)$ R-symmetry of the dual SCFT. We also show that for these solutions supersymmetry implies the equations of motion, just as we saw in [2].

To construct explicit solutions we further restrict our considerations to the special case of constant dilaton and vanishing axion with some additional restrictions imposed on the geometry. We can then reduce the entire problem to solving a second-order nonlinear ODE. We find one solution in closed form, which turns out to be a solution first obtained by Pilch and Warner [21] (constructed by uplifting a solution first found in five-dimensional gauged supergravity [22]). This solution has constant dilaton and vanishing axion, but non-vanishing three-forms and self-dual five-form; it has been identified [23, 24] as being dual to an $N = 1$ supersymmetric fixed point discovered by Leigh and Strassler [25]. A numerical analysis of our ODE leads to a continuous family of local solutions. We show that they lead to complete

metrics on S^5, but a detailed analysis indicates that neither the three-form fluxes nor the spinors are globally defined. It is not clear to us whether or not these solutions can be given a physical interpretation. It is also possible that other solutions of the ODE lead to interesting solutions, but we leave this for future work.

The plan of the rest of paper is as follows. In section 2 we outline our conventions for type IIB supergravity. Section 3 derives the necessary and sufficient conditions for the most general supersymmetric solutions with AdS$_5$ factors. For the convenience of the reader, we have summarized the main results, in a somewhat self-contained way, in section 3.6. Section 4 continues the analysis by introducing local coordinates. The discussion of the special class of solutions, including vanishing axion and constant dilaton, and the recovery of the Pilch–Warner solution, is presented in section 5. Section 6 briefly concludes. We have relegated some technical material to several appendices.

2. Type IIB equations and conventions

We begin by presenting the equations of motion and supersymmetry transformations for bosonic configurations of type IIB supergravity [26–28] in the conventions given in appendix A. Essentially we are following [27], with some minor changes, including the signature of the metric.

The conditions for a bosonic geometry to preserve some supersymmetry are

$$\delta\psi_M \approx D_M\epsilon - \frac{1}{96}\big(\Gamma_M{}^{P_1P_2P_3}G_{P_1P_2P_3} - 9\Gamma^{P_1P_2}G_{MP_1P_2}\big)\epsilon^c + \frac{i}{192}\Gamma^{P_1P_2P_3P_4}F_{MP_1P_2P_3P_4}\epsilon = 0$$

$$\delta\lambda \approx i\Gamma^M P_M\epsilon^c + \frac{i}{24}\Gamma^{P_1P_2P_3}G_{P_1P_2P_3}\epsilon = 0.$$

$$(2.1)$$

We are working in the formalism where $SU(1,1)$ is realized linearly. In particular, there is a local $U(1)$ invariance and Q_M acts as the corresponding gauge field. Note that Q_M is a composite gauge field with field strength given by

$$dQ = -iP \wedge P^*. \tag{2.2}$$

Also note that D is the covariant derivative with respect to local Lorentz transformations and local $U(1)$ transformations. The spinor ϵ has $U(1)$ charge $1/2$ so that

$$D_M\epsilon = \left(\nabla_M - \frac{i}{2}Q_M\right)\epsilon. \tag{2.3}$$

The field P has charge 2, while G has charge 1. We also have the chirality conditions $\Gamma_{11}\psi = -\psi$, $\Gamma_{11}\lambda = \lambda$ and $\Gamma_{11}\epsilon = -\epsilon$.

The equations of motion are[6]

$$R_{MN} = P_M P_N^* + P_N P_M^* + \frac{1}{96}F_{MP_1P_2P_3P_4}F_N{}^{P_1P_2P_3P_4}$$

$$+ \frac{1}{8}\left(G_M{}^{P_1P_2}G_{NP_1P_2}^* + G_N{}^{P_1P_2}G_{MP_1P_2}^* - \frac{1}{6}g_{MN}G^{P_1P_2P_3}G_{P_1P_2P_3}^*\right)$$

$$D^P G_{MNP} = P^P G_{MNP}^* - \frac{i}{6}F_{MNP_1P_2P_3}G^{P_1P_2P_3} \tag{2.4}$$

$$D^M P_M = -\frac{1}{24}G_{P_1P_2P_3}G^{P_1P_2P_3}$$

$$F = *_{10}F.$$

[6] The sign in the third equation differs from that of [27]: we fixed it here by studying the integrability conditions for supersymmetry, as discussed in appendix D.

We also need to impose the Bianchi identities

$$DP = 0 \qquad DG = -P \wedge G^* \qquad \mathrm{d}F = \frac{\mathrm{i}}{2} G \wedge G^*. \tag{2.5}$$

Note that in the usual string theory variables we have, following [29],

$$P = \frac{\mathrm{i}}{2} \, \mathrm{e}^\phi \, \mathrm{d}C^{(0)} + \frac{1}{2} \, \mathrm{d}\phi \qquad Q = -\frac{1}{2} \, \mathrm{e}^\phi \, \mathrm{d}C^{(0)}, \tag{2.6}$$

and we observe that the Binachi identity $DP = 0$ is identically satisfied. In addition,

$$G = \mathrm{i}\mathrm{e}^{\phi/2}(\tau \, \mathrm{d}B - \mathrm{d}C^{(2)}) \tag{2.7}$$

(taking into account a sign difference between our G and that in [29]). In these conventions, according to [29], the $SL(2, \mathbb{R})$ action is

$$\tau \to \frac{p\tau + q}{r\tau + s}, \qquad \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \tag{2.8}$$

where $\tau \equiv C^{(0)} + \mathrm{i}\mathrm{e}^{-\phi}$, with the Einstein metric and the five-form left unchanged.

3. The conditions for supersymmetry in $d = 5$

We consider the most general class of bosonic supersymmetric solutions of type IIB supergravity with $SO(4, 2)$ symmetry. The $d = 10$ metric in Einstein frame is taken to be a warped product

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{2\Delta}\big[\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s_5^2\big] \tag{3.1}$$

where $\mathrm{d}s^2(\mathrm{AdS}_5)$ denotes the metric on AdS_5, normalized so that its Ricci tensor is $-4m^2$ times the metric, and $\mathrm{d}s_5^2$ denotes an arbitrary five-dimensional metric on the internal space M_5. Δ is a real function on this space, $\Delta \in \Omega^0(M_5, \mathbb{R})$. We also take $P \in \Omega^1(M_5, \mathbb{C})$, $Q \in \Omega^1(M_5, \mathbb{R})$, $G \in \Omega^3(M_5, \mathbb{C})$ and

$$F = (\mathrm{vol}_{AdS_5} + \mathrm{vol}_5)f \tag{3.2}$$

where vol_5 denotes the volume form on M_5 and f is a real constant to ensure that the five-form Bianchi identity (or equation of motion), $\mathrm{d}F = 0$, is satisfied.

For the geometry to preserve supersymmetry it must admit solutions to the Killing spinor equations (2.1). To proceed we construct the most general ansatz for the spinor ϵ consistent with minimal supersymmetry in AdS_5. As explained in detail in appendix A, ϵ is constructed from two spinors, ξ_i, of $\mathrm{Spin}(5)$ combined with a $\mathrm{Spin}(4, 1)$ spinor ψ satisfying $\nabla_\mu \psi = \frac{1}{2} m \rho_\mu \psi$ on AdS_5, where ρ_μ generate $\mathrm{Cliff}(4, 1)$. After substituting this spinor ansatz into (2.1), one eventually obtains two differential conditions

$$D_m \xi_1 + \frac{\mathrm{i}}{4}(\mathrm{e}^{-4\Delta} f - 2m)\gamma_m \xi_1 + \frac{1}{8} \, \mathrm{e}^{-2\Delta} G_{mnp}\gamma^{np}\xi_2 = 0 \tag{3.3}$$

$$\bar{D}_m \xi_2 - \frac{\mathrm{i}}{4}(\mathrm{e}^{-4\Delta} f + 2m)\gamma_m \xi_2 + \frac{1}{8} \, \mathrm{e}^{-2\Delta} G^*_{mnp}\gamma^{np}\xi_1 = 0 \tag{3.4}$$

and four algebraic conditions

$$\gamma^m \partial_m \Delta \xi_1 - \frac{1}{48} \, \mathrm{e}^{-2\Delta} \gamma^{mnp} G_{mnp}\xi_2 - \frac{\mathrm{i}}{4}(\mathrm{e}^{-4\Delta} f - 4m)\xi_1 = 0 \tag{3.5}$$

$$\gamma^m \partial_m \Delta \xi_2 - \frac{1}{48} \, \mathrm{e}^{-2\Delta} \gamma^{mnp} G^*_{mnp}\xi_1 + \frac{\mathrm{i}}{4}(\mathrm{e}^{-4\Delta} f + 4m)\xi_2 = 0 \tag{3.6}$$

$$\gamma^m P_m \xi_2 + \frac{1}{24} e^{-2\Delta} \gamma^{mnp} G_{mnp} \xi_1 = 0 \tag{3.7}$$

$$\gamma^m P_m^* \xi_1 + \frac{1}{24} e^{-2\Delta} \gamma^{mnp} G_{mnp}^* \xi_2 = 0 \tag{3.8}$$

where γ^m generate Cliff(5) with $\gamma_{12345} = +1$.

It is interesting to consider first the special case where one of the two spinors, ξ_2 say, is identically zero. It is then easy to see that the warp factor must be constant and related to f by $f = 4m\, e^{4\Delta}$. Hence the metrics are direct products of AdS₅ with a five-manifold. In addition, we deduce that

$$G_{mnp}^* \gamma^{np} \xi_1 = G_{mnp} \gamma^{mnp} \xi_1 = 0 \qquad \gamma^m P_m^* \xi_1 = 0 \qquad D_m \xi_1 + i \frac{m}{2} \gamma_m \xi_1 = 0. \tag{3.9}$$

The first two conditions imply[7] that $G = 0$. Next, the third condition implies $P^2 = 0$. Writing this out in terms of the axion and dilaton, using (2.6), the imaginary part says that $\partial C^{(0)} \cdot \partial \phi = 0$. The equation of motion for $C^{(0)}$ then says that it is harmonic. Now on a compact manifold, which is the case of most interest for AdS/CFT applications, we deduce that C^0 is constant. The equation of motion for the dilaton then implies that the dilaton is also constant, for the same reason. The last condition in (3.9) then leads us back to the well-known AdS₅ × X_5 solutions, where X_5 is Sasaki–Einstein.

More generally, we can enquire whether it is possible to have solutions preserving supersymmetry with both ξ_i, non-vanishing but linearly dependent. In fact this is not possible, as we show in appendix C. Note that this implies that the only solutions with compact M_5 having a local $SU(2)$ structure (determined by supersymmetry), rather than an identity structure to be considered next, are Sasaki–Einstein.

3.1. The identity structure

We now turn to the main focus of the paper: supersymmetric solutions with ξ_i generically linearly independent. We first note that in neighbourhoods where ξ_i are generic they define, locally, an identity structure[8], or equivalently, a canonical orthonormal frame e^a. One way to see this is that the set of spinors $\{\xi_1, \xi_2, \xi_1^c, \xi_2^c\}$ generically form a complete basis for the spinor representation of Spin(5).

Equivalently, this structure can easily be seen by noting that there are six real vectors that can be constructed from two non-vanishing spinors. These can be written as

$$K^m \equiv \bar{\xi}_1^c \gamma^m \xi_2 \qquad\qquad K_3^m \equiv \bar{\xi}_2 \gamma^m \xi_1$$
$$K_4^m \equiv \tfrac{1}{2}(\bar{\xi}_1 \gamma^m \xi_1 - \bar{\xi}_2 \gamma^m \xi_2) \qquad K_5^m \equiv \tfrac{1}{2}(\bar{\xi}_1 \gamma^m \xi_1 + \bar{\xi}_2 \gamma^m \xi_2), \tag{3.10}$$

where the first two are complex and the last two are real. Since we are in a five-dimensional space they cannot be linearly independent. Using Fierz identities one finds that there is a single linear relation,

$$\epsilon^{ik} \epsilon^{jl} (\bar{\xi}_i \xi_j)(\bar{\xi}_k \gamma^m \xi_l) - 2 \operatorname{Re}(\bar{\xi}_2 \xi_1^c)(\bar{\xi}_1^c \gamma^m \xi_2) = 0 \tag{3.11}$$

leaving five independent vectors. From these, given the norms of the vectors, one can build an orthonormal basis e^a defining the identity structure. The relation between the vectors and a particular useful basis e^a is given in appendix B. Again by Fierz identities, one can

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[7] It is the same calculation that is used to derive (3.18) below.
[8] An alternative, but equivalent, point of view is that the spinors $\xi_1 \otimes \xi_2$ define an $SU(2) \times SU(2)$ structure on $TM_5 \oplus T^*M_5$, in the sense of Hitchin [30] (see also [31–33]). However, we will not adopt this language in the present paper.

write the norms of the vectors in terms of the six independent scalar bilinears. These can be parametrized as

$$A \equiv \tfrac{1}{2}(\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2) \qquad A \sin \zeta \equiv \tfrac{1}{2}(\bar{\xi}_1 \xi_1 - \bar{\xi}_2 \xi_2)$$
$$S \equiv \bar{\xi}_2^c \xi_1 \qquad\qquad Z \equiv \bar{\xi}_2 \xi_1, \tag{3.12}$$

where the first two are real and the second two are complex. In summary, these vector and scalar bilinears define the identity structure.

We now aim to find the conditions on the identity structure and on the fluxes that are equivalent to supersymmetry. This calculation falls into two parts. First, one considers the differential conditions (3.3) and (3.4). This is equivalent to giving the intrinsic torsion, or here since we have an identity structure, the torsion itself, in terms of the flux, f, m and the warp factor Δ. The same information is contained in the exterior derivatives of the canonical orthonormal frame e^m, which is in turn encoded in the exterior derivatives of the vector and the scalar bilinears. The second step is then to find necessary and sufficient constraints on the structure due to the algebraic conditions (3.5)–(3.8).

3.2. Torsion conditions

In calculating the torsion conditions it is convenient to work not with the exterior derivatives of a particular orthonormal basis e^m, but rather the exterior derivatives of the vector and scalar bilinears defined above, which are completely equivalent. The results of appendix B then provide a translation to e^m if required.

We start by calculating the derivatives of the scalar bilinears. Making use of the algebraic conditions (3.5)–(3.8), one finds first that A is constant, $dA = 0$. Thus we can consistently set

$$A = 1. \tag{3.13}$$

The remaining scalars then satisfy

$$d(e^{4\Delta} \sin \zeta) = 0 \tag{3.14}$$

$$e^{-4\Delta} d(e^{4\Delta} S) = 3imK \tag{3.15}$$

$$e^{-2\Delta} D(e^{2\Delta} Z) = -PZ^*. \tag{3.16}$$

Next we turn to the vectors. Again with some judicious use of conditions (3.5)–(3.8), after some work we find that the differential constraints (3.3) and (3.4) imply

$$d(e^{4\Delta} K) = 0 \tag{3.17}$$

$$e^{-6\Delta} D(e^{6\Delta} K_3) = P \wedge K_3^* - 4imW - e^{-2\Delta} * G \tag{3.18}$$

$$e^{-4\Delta} d(e^{4\Delta} K_4) = -2mV \tag{3.19}$$

$$e^{-8\Delta} d(e^{8\Delta} K_5) = e^{-4\Delta} fV - 6mU - \mathrm{Re}(e^{-2\Delta} Z^* * G), \tag{3.20}$$

where we have introduced the two-forms

$$iU_{mn} \equiv \tfrac{1}{2}(\bar{\xi}_1 \gamma_{mn} \xi_1 + \bar{\xi}_2 \gamma_{mn} \xi_2)$$
$$iV_{mn} \equiv \tfrac{1}{2}(\bar{\xi}_1 \gamma_{mn} \xi_1 - \bar{\xi}_2 \gamma_{mn} \xi_2) \tag{3.21}$$
$$W_{mn} \equiv -\bar{\xi}_2 \gamma_{mn} \xi_1$$

which can, of course, be rewritten in terms of the basis e^m (see appendix B) and hence the scalar and vector bilinears. Doing so, or using Fierz identities, and given that $A = 1$, one finds the identity

$$\sin \zeta \, V - U - \frac{i}{2} K^* \wedge K + \text{Re}[iZ^*W] = 0. \tag{3.22}$$

We note first that the first differential condition (3.17) is in fact implied by the scalar condition (3.15). Next we recall that the six vector bilinears are not independent. The linear relation (3.11) implies that

$$K_5 = \sin \zeta \, K_4 + \text{Re}[Z^* K_3] - \text{Re}[S^* K] \tag{3.23}$$

where we have again used the fact that $A = 1$. Taking the exterior derivative of (3.23) and comparing with dK_5 in (3.20) gives the consistency condition

$$(e^{-4\Delta} f + 2m \sin \zeta) V = 6mU - 4m \, \text{Re}[iZ^*W] + 3im K^* \wedge K. \tag{3.24}$$

However, the two-forms above are linearly related; in particular, they obey the identity (3.22). To be consistent with this identity we require, first, that

$$e^{-4\Delta} f = 4m \sin \zeta, \tag{3.25}$$

fixing the integration constant in the differential condition (3.14). (In fact, it is straightforward to show this relation holds, directly from the algebraic constraints (3.5) and (3.6).) Secondly, we also require $\text{Re}[iZ^*W] = 0$. Using the explicit expression for W (see appendix B), it is easy to show that this implies the important condition

$$Z = 0. \tag{3.26}$$

This condition simplifies considerably the algebraic and differential conditions obeyed by the bilinears.

In summary, the torsion conditions (3.3) and (3.4) are equivalent[9] to

$$e^{-4\Delta} f = 4m \sin \zeta, \qquad A = 1, \qquad Z = 0 \tag{3.27}$$

together with the differential conditions

$$e^{-4\Delta} \, d(e^{4\Delta} S) = 3im K \tag{3.28}$$

$$e^{-6\Delta} \, D(e^{6\Delta} K_3) = P \wedge K_3^* - 4im W - e^{-2\Delta} * G \tag{3.29}$$

$$e^{-8\Delta} \, d(e^{8\Delta} K_5) = 4m \sin \zeta \, V - 6mU. \tag{3.30}$$

(We drop the dK_4 condition since the linear dependence means it is implied by the other vector bilinear conditions.) As expected, starting with the work [34] and others following this (in particular, see [5, 35–38]), we note that these differential conditions are written in a form reminiscent of 'generalized calibrations' [39, 40].

3.3. Algebraic conditions

Next we turn to the algebraic conditions (3.5)–(3.8). We would like to find the equivalent algebraic conditions relating the identity structure, P, G, f, m and Δ. The simplest way to do this is to note that, as mentioned above, generically the set $\eta_\alpha \in \{\xi_1, \xi_2, \xi_1^c, \xi_2^c\}$ forms a complete basis in the Spin(5) spinor representation space. Thus we can construct the identity operator

$$\mathbf{1} = \eta_\alpha (m^{-1})^{\alpha\beta} \bar{\eta}_\beta, \tag{3.31}$$

where $m_{\alpha\beta} = \bar{\eta}_\alpha \eta_\beta$.

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[9] Note we have also used some of the algebraic conditions (3.5)–(3.8).

Next, using the fact that $\gamma_{mnp}G^{mnp} = -3*G_{mn}\gamma^{mn}$ one rewrites the algebraic conditions in the form

$$\mathrm{e}^{-2\Delta}*G_{mn}\gamma^{mn}\eta_\alpha = \Sigma_\alpha{}^\beta\eta_\beta. \tag{3.32}$$

Using the completeness relation (3.31), we see that the algebraic conditions are equivalent to an operator equation

$$\mathrm{e}^{-2\Delta}*G_{mn}\gamma^{mn} = \Sigma_\alpha{}^\gamma(m^{-1})^{\alpha\beta}\eta_\gamma\bar{\eta}_\beta. \tag{3.33}$$

Performing Fierz identities on $\eta_\gamma\bar{\eta}_\beta$ one gets three types of relations. From the **1** coefficient one finds

$$i_{K_3^*}P = 2i_{K_3}\,\mathrm{d}\Delta. \tag{3.34}$$

The γ^m coefficient gives three additional conditions,

$$i_{K_5}\,\mathrm{d}\Delta = 0 \tag{3.35}$$

$$i_{K_5}P = 0 \tag{3.36}$$

$$\mathrm{e}^{-4\Delta}\,f = 4m\sin\zeta, \tag{3.37}$$

where the final expression has already appeared as a consistency condition (3.25). The γ^{mn} coefficient meanwhile gives an expression for the flux $*G$

$$\begin{aligned}
(\cos^2\zeta - |S|^2)\,\mathrm{e}^{-2\Delta}*G = {}&2P\wedge K_3^* - (4\,\mathrm{d}\Delta + 4\mathrm{i}m\,K_4 - 4\mathrm{i}m\sin\zeta\,K_5)\wedge K_3 \\
&+ 2*(P\wedge K_3^*\wedge K_5 - 2\,\mathrm{d}\Delta\wedge K_3\wedge K_5).
\end{aligned} \tag{3.38}$$

In deriving this last expression one uses the identities

$$\begin{aligned}
S^*\xi_1^c\gamma_{(2)}\xi_1 &= (1 + \sin\zeta)W - (K_4 + K_5)\wedge K_3 \\
S^*\xi_2^c\gamma_{(2)}\xi_2 &= (1 - \sin\zeta)W^* - (K_4 - K_5)\wedge K_3^*.
\end{aligned} \tag{3.39}$$

For completeness, we also note that, with $Z = 0$, the two-forms U, V and W are given by

$$\begin{aligned}
U &= \frac{1}{2(\cos^2\zeta - |S|^2)}(\mathrm{i}\sin\zeta\,K_3\wedge K_3^* + \mathrm{i}K\wedge K^* - 2\,\mathrm{Im}\,S^*K\wedge K_5), \\
V &= \frac{1}{2\sin\zeta(\cos^2\zeta - |S|^2)}(\mathrm{i}\sin\zeta\,K_3\wedge K_3^* + \mathrm{i}[\sin^2\zeta + |S|^2]K\wedge K^* - 2\,\mathrm{Im}\,S^*K\wedge K_5), \\
W &= \frac{1}{\sin\zeta(\cos^2\zeta - |S|^2)}(\cos^2\zeta\,K_5 + \mathrm{Re}\,S^*K + \mathrm{i}\sin\zeta\,\mathrm{Im}\,S^*K)\wedge K_3.
\end{aligned} \tag{3.40}$$

In summary, the algebraic conditions (3.34)–(3.38), together with (3.27) and (3.28)–(3.30), are equivalent to the Killing spinor equations (3.3)–(3.8).

3.4. The Killing vector K_5

We now show that K_5 is a Killing vector and, moreover, it generates a symmetry of the full solution. This corresponds to the fact that the dual $D = 4$ superconformal field theories have a global $U(1)_R$ symmetry. While the Killing condition is implied by the necessary and sufficient conditions (3.28)–(3.30) and (3.34)–(3.38), the simplest derivation is directly from the spinor conditions (3.3)–(3.8). Calculating ∇K_5, one can easily show that the symmetric part vanishes and hence K_5 is Killing.

We next compute its action on the remaining bosonic fields. From (3.35) we immediately see that

$$\mathcal{L}_{K_5} \Delta = 0 \tag{3.41}$$

and hence, by (3.37), $\mathcal{L}_{K_5} \zeta = 0$. From (3.36), given the expression (2.6) for P, one also immediately has

$$\mathcal{L}_{K_5} \phi = \mathcal{L}_{K_5} C^{(0)} = 0 \quad \Leftrightarrow \quad \mathcal{L}_{K_5} P = 0. \tag{3.42}$$

Finally, we need to consider $\mathcal{L}_{K_5} G$. This can be calculated directly from the expression (3.38) for $*G$. To do so we need to know the action of the Lie derivative $\mathcal{L}_{K_5}$ on the scalar and vector bilinears. One finds that all the bilinears are invariant except for

$$\mathcal{L}_{K_5} S = -3\mathrm{i}mS, \tag{3.43}$$

(and hence also $\mathcal{L}_{K_5} K = -3\mathrm{i}mK$). This implies $\mathcal{L}_{K_5}(SS^*) = 0$ and thus, from (3.38),

$$\mathcal{L}_{K_5} G = 0. \tag{3.44}$$

We thus see that the Killing vector K_5 does indeed generate a symmetry of the full solution.

3.5. Equations of motion

We now show that the conditions we have derived for supersymmetry automatically imply the equations of motion and the Bianchi identities.

We first recall that $DP = 0$ follows automatically from the expression for P in terms of the variables (2.6). Also, our ansatz has $\mathrm{d}F = 0$ by construction. Next, from (3.29) (and using (3.37)) we find

$$D(\mathrm{e}^{4\Delta} * G) = \mathrm{e}^{4\Delta} P \wedge *G^* - \mathrm{i}fG \tag{3.45}$$

which is just the G equation of motion. The easiest way to show that the G Bianchi identity is also satisfied is to derive a differential condition for W directly from spinor conditions (3.3)–(3.8). One finds

$$D(\mathrm{e}^{6\Delta} W) = -\mathrm{e}^{6\Delta} P \wedge W^* + (f/4m)G. \tag{3.46}$$

Taking a derivative with D then reproduces the Bianchi identity for G.

In appendix D we consider the integrability conditions for the Killing spinor equations. Assuming that the P, G and F Bianchi identities together with the G equation of motion are satisfied, one finds that a supersymmetric background necessarily satisfies the P equations of motion. Moreover, all but one component of Einstein's equations is automatically satisfied. In appendix D we also show that this component is satisfied in the present case, so we can conclude the following:

> For the class of solutions with metric of the form (3.1) and fluxes respecting $SO(4, 2)$ symmetry, all the equations of motion and Bianchi identities are implied by supersymmetry.

A similar situation was found to hold for the supersymmetric AdS$_5$ M-theory solutions of [2]. Clearly, this is very useful for constructing solutions. In fact, it is often the Bianchi identities that are the difficult equations to satisfy.

3.6. Summary

Let us end by summarizing the necessary and sufficient conditions for the generic supersymmetric solution with metric of the form (3.1) and fluxes respecting $SO(4, 2)$ symmetry. M_5 must admit an identity structure defined by two spinors ξ_1, ξ_2 which determine the preserved supersymmetry. The scalars A and Z defined in (3.12) are given by $A = 1$ and $Z = 0$. The identity structure can then be specified by a real vector K_5, and two complex vectors K, K_3 defined in (3.10), along with a real scalar ζ and a complex scalar S defined in (3.12). These satisfy the following conditions,

$$e^{-4\Delta} \, d(e^{4\Delta} S) = 3im K \tag{3.47}$$

$$e^{-6\Delta} \, D(e^{6\Delta} K_3) = P \wedge K_3^* - 4im W - e^{-2\Delta} * G \tag{3.48}$$

$$e^{-8\Delta} \, d(e^{8\Delta} K_5) = 4m \sin \zeta \, V - 6m U, \tag{3.49}$$

and the additional algebraic constraint

$$i_{K_3^*} P = 2i_{K_3} \, d\Delta. \tag{3.50}$$

The five-form flux is given by (3.2) with

$$f = 4m \, e^{4\Delta} \, \sin \zeta \tag{3.51}$$

while the three-form flux is given by

$$(\cos^2 \zeta - |S|^2) \, e^{-2\Delta} * G = 2P \wedge K_3^* - (4 \, d\Delta + 4im K_4 - 4im \sin \zeta \, K_5) \wedge K_3$$
$$+ 2 * (P \wedge K_3^* \wedge K_5 - 2 \, d\Delta \wedge K_3 \wedge K_5). \tag{3.52}$$

The metric can be written (using results in appendix B)

$$ds^2 = \frac{(K_5)^2}{\sin^2 \zeta + |S|^2} + \frac{K_3 \otimes K_3^*}{\cos^2 \zeta - |S|^2} + \frac{|S|^2}{\cos^2 \zeta - |S|^2} (\mathrm{Im} \, S^{-1} K)^2$$
$$+ \frac{|S|^2}{\sin^2 \zeta} \frac{\sin^2 \zeta + |S|^2}{\cos^2 \zeta - |S|^2} \left(\mathrm{Re} \, S^{-1} K + \frac{1}{\sin^2 \zeta + |S|^2} K_5 \right)^2. \tag{3.53}$$

The conditions imply that K_5 is a Killing vector field that generates a symmetry of the full solution: $\mathcal{L}_{K_5} \Delta = i_{K_5} P = \mathcal{L}_{K_5} G = 0$. Furthermore, all equations of motion and the Bianchi identities are satisfied.

4. Reducing the conditions

It is now useful to introduce some convenient local coordinates and hence reduce the conditions to a simpler set. We will first reduce on the Killing direction K_5 and then use condition (3.47) to write the resulting four-dimensional metric as a product of a one-dimensional metric and a three-dimensional metric $\tilde{g}$. The problem then reduces to a set of conditions on the local identity structure on $\tilde{g}$.

We begin by choosing a coordinate ψ that is adapted to the Killing direction K_5. As a vector, we write

$$K_5^\# = 3m \frac{\partial}{\partial \psi} \tag{4.1}$$

and therefore as a one-form

$$K_5 = \frac{1}{3m} \cos^2 \eta (d\psi + \rho), \tag{4.2}$$

where $\cos \eta$ is the norm of K_5, given by $\cos^2 \eta = \sin^2 \zeta + |S|^2$. (Note that in the conventions of appendix B $\eta = 2\phi$.)

Let us now turn to (3.47). The dependence of S on ψ is given by the Lie derivative (3.43) which is solved by $S = e^{-i\psi} \hat{S}$, where the complex scalar $\hat{S}$ is independent of ψ. Noting that there is a gauge freedom in shifting the coordinate ψ by a function of the remaining coordinates, we take $\hat{S}$ to be real, without loss of generality. It is then natural to introduce a coordinate λ such that

$$S = \sin \zeta \, \lambda \, e^{-i\psi}, \tag{4.3}$$

where the factor of $\sin \zeta$ is added for convenience so that (3.47) now reads, given (3.51),

$$S^{-1} K = -\frac{1}{3m} \left(d\psi + i \, d \ln \lambda \right). \tag{4.4}$$

Note that in these coordinates we have

$$\sin \zeta = \frac{\cos \eta}{(1 + \lambda^2)^{1/2}}, \tag{4.5}$$

and it is convenient to switch to η, ψ and λ instead of the scalars ζ and S.

For convenience let us also define a new complex one-form σ by

$$K_3 = \sigma / 3m. \tag{4.6}$$

Using the results contained in appendix B, one can write the underlying orthonormal frame as

$$3me^1 = \cos \eta (d\psi + \rho), \qquad 3me^2 = \lambda \cot \eta \rho,$$
$$3m(-ie^3 + e^4) = \frac{1}{\sin \eta} \sigma, \qquad 3me^5 = \frac{\cot \eta}{(1 + \lambda^2)^{1/2}} \, d\lambda. \tag{4.7}$$

Now since $\mathcal{L}_{K_5} \rho = \mathcal{L}_{K_5} \sigma$ we can always choose coordinates (ψ, λ, x^i) such that ρ and σ are independent of $d\lambda$. (One first reduces on the Killing direction to a four-dimensional metric, independent of ψ, spanned by e^2, e^3, e^4 and e^5. Then, given $e^5 \sim d\lambda$, one can always make a four-dimensional coordinate transformation such that there are no cross-terms $d\lambda \, dx^i$ in the metric.) Thus the five-dimensional metric has the form

$$9m^2 \, ds_5^2 = \cos^2 \eta (d\psi + \rho)^2 + \frac{\cot^2 \eta}{1 + \lambda^2} \, d\lambda^2 + \tilde{g}_{ij}(\lambda, x^i) \, dx^i \, dx^j, \tag{4.8}$$

where the three-dimensional metric $\tilde{g}$ is given in terms of σ and ρ

$$\tilde{g}_{ij}(\lambda, x^i) \, dx^i \, dx^j = \lambda^2 \cot^2 \eta \rho^2 + \frac{\sigma \otimes \sigma^*}{\sin^2 \eta}. \tag{4.9}$$

In summary, we have reduced the problem to a three-dimensional metric $\tilde{g}$ with a local identity structure given by (ρ, σ, σ^*), which also depends on the coordinate λ. In addition, there is one remaining scalar η. (Note that Δ is given in terms of η and λ using (3.51) and (4.5).) In making this reduction we have used (3.47) and the fact that K_5 is Killing. It remains to translate the remaining conditions (3.48) and (3.49) into conditions on ρ and σ.

Let us first split

$$d = \tilde{d} + d\lambda \, \frac{\partial}{\partial \lambda} + d\psi \, \frac{\partial}{\partial \psi}, \tag{4.10}$$

where $\tilde{d} = dx^i \, \partial / \partial x^i$. Similarly we write

$$P = \tilde{P} + P_\lambda \, d\lambda, \tag{4.11}$$

recalling that $\mathrm{i}_{K_5} P = 0$. Writing $\partial_\lambda = \partial/\partial\lambda$, condition (3.49) is equivalent to

$$
\begin{aligned}
\partial_\lambda \rho &= -\frac{2(1 + 2\sin^2 \eta)\lambda}{3\sin^2 \eta (1 + \lambda^2)} \rho, \\
\tilde{\mathrm{d}}\rho &= -\frac{\mathrm{i}}{3\sin^2 \eta \cos \eta (1 + \lambda^2)^{1/2}} \sigma \wedge \sigma^*.
\end{aligned}
\tag{4.12}
$$

Similarly, condition (3.48) reduces to

$$
\begin{aligned}
\sin^2 \eta\, \mathrm{e}^{-6\Delta}\, D_\lambda(\mathrm{e}^{6\Delta}\, \sigma) &= \left(4\partial_\lambda \Delta - \frac{4\lambda \cos^2 \eta}{3(1 + \lambda^2)} \right) \sigma - (1 + \cos^2 \eta) P_\lambda \sigma^* \\
&\quad - \frac{2\cos^2 \eta}{\sin \eta (1 + \lambda^2)^{1/2}} \tilde{*}\, (2\,\tilde{\mathrm{d}}\Delta \wedge \sigma - \tilde{P} \wedge \sigma^*),
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
\sin^2 \eta\, \mathrm{e}^{-6\Delta}\, \tilde{D}(\mathrm{e}^{6\Delta}\, \sigma) &= 4\,\tilde{\mathrm{d}}\Delta \wedge \sigma - (1 + \cos^2 \eta) \tilde{P} \wedge \sigma^* \\
&\quad - 2\mathrm{i}\lambda(1 + \lambda^2)^{1/2} \cos \eta \rho \wedge (2\partial_\lambda \Delta \sigma + P_\lambda \sigma^*).
\end{aligned}
$$

The only remaining condition is the algebraic relation (3.50) which reads

$$
\mathrm{i}_{\sigma^*} P = 2\mathrm{i}_\sigma\, \mathrm{d}\Delta.
\tag{4.14}
$$

In summary, one needs to solve (4.12) and (4.13) subject to (4.14). This concludes our analysis of the most general AdS$_5$ geometries arising in type IIB supergravity.

5. A simplifying ansatz

In order to find explicit solutions to these equations we will now make a particular, very natural, ansatz. First, we assume that the dilaton is constant and the axion zero, $P = 0$. Then we assume that the one-forms (ρ, σ) are (locally) proportional to the left-invariant one-forms on S^3, that is

$$
\rho = A\sigma_3, \qquad \sigma = B(\sigma_2 - \mathrm{i}\sigma_1).
\tag{5.1}
$$

(Note that with this choice $(\sigma_1, \sigma_2, \sigma_3)$ define the same orientation as (e^2, e^3, e^4).) Explicitly we can introduce coordinates $\sigma_3 = \mathrm{d}y - \cos\alpha\, \mathrm{d}\beta$ and $\sigma_1 = -\sin y\, \mathrm{d}\alpha - \cos y \sin\alpha\, \mathrm{d}\beta$, $\sigma_2 = \cos y\, \mathrm{d}\alpha - \sin y \sin\alpha\, \mathrm{d}\beta$. In addition, we assume that the functions A, B and η all depend only on λ. As we will see, this ansatz means the metric has a local $SU(2) \times U(1) \times U(1)$ isometry group.

We find that the entire analysis then boils down to solving a second-order nonlinear ordinary differential equation. Furthermore, we find one exact solution to this ODE which after a change of coordinates turns out to be precisely a solution first found by Pilch and Warner [21]. Our numerical investigations of the ODE lead to a one-parameter family of local solutions, which do not extend to globally defined solutions, as we will discuss.

We start by introducing two functions

$$
h = -A(1 + \lambda^2), \qquad g = \frac{1}{\sin \zeta} = \frac{(1 + \lambda^2)^{1/2}}{\cos \eta}.
\tag{5.2}
$$

To satisfy the $\tilde{\mathrm{d}}$-equation in (4.12) one requires

$$
B = \left[\frac{3h(g^2 - 1 - \lambda^2)}{2g^3} \right]^{1/2}.
\tag{5.3}
$$

This implies that the metric takes the form

$$9m^2 \, \mathrm{d}s_5^2 = \frac{1 + \lambda^2}{g^2} \left(\mathrm{d}\psi - \frac{h}{1 + \lambda^2} \sigma_3 \right)^2$$

$$+ \frac{1}{g^2 - 1 - \lambda^2} \left(\mathrm{d}\lambda^2 + \frac{\lambda^2}{1 + \lambda^2} h^2 \sigma_3^2 \right) + \frac{3h}{2g} (\sigma_1^2 + \sigma_2^2), \tag{5.4}$$

and it is clear that the metric has a local $SU(2) \times U(1) \times U(1)$ isometry group. (Note that $\sigma_1^2 + \sigma_2^2$ is just the round metric on S^2.) The ∂_λ-equations in (4.12) and conditions (4.13) all reduce to a pair of coupled first-order differential equations for g and h, namely

$$\dot{h} = -\frac{2\lambda h}{3} \frac{1}{g^2 - 1 - \lambda^2}, \qquad \dot{g} = \frac{1}{\lambda h} (g^2 - 1 - \lambda^2), \tag{5.5}$$

where the dot denotes ∂_λ. These are equivalent to a second-order ODE for g, which reads

$$\ddot{g}\lambda(g^2 - 1 - \lambda^2) + \dot{g}\left(g^2 - 1 + \tfrac{1}{3}\lambda^2 - 2\lambda g \dot{g}\right) = 0. \tag{5.6}$$

Any solution to these equations gives rise to a (local) supersymmetric solution with an AdS$_5$ factor and non-trivial three-form flux.

For completeness, we note that the flux is given by

$$G = \left(\frac{f}{4m} \right)^{1/2} mg^{1/2} \left[\frac{4\lambda^2 gh - 3(1 + \lambda^2)(g^2 - 1 - \lambda^2)}{\lambda(1 + \lambda^2)^{1/2} gh} e^{15} \right.$$

$$\left. - 4 \frac{(g^2 - 1 - \lambda^2)^{1/2}}{g(1 + \lambda^2)^{1/2}} e^{25} + 3\mathrm{i} \frac{g^2 - 1 - \lambda^2}{\lambda h} e^{12} \right] \wedge (e^4 - \mathrm{i}e^3). \tag{5.7}$$

One can also integrate this expression to give the complex potential A in a relatively simple form (E.7). Note also that equations (5.5) are symmetric under $\lambda \to -\lambda$, $g \to -g$ and $h \to -h$.

5.1. Pilch–Warner solution

We managed to find a single analytic solution to (5.5) given by

$$g = 1 + \tfrac{1}{\sqrt{3}}\lambda \qquad h = 2\left(1 - \tfrac{1}{\sqrt{3}}\lambda\right). \tag{5.8}$$

We now show that this is locally the same solution first found by Pilch and Warner [21]. To see this we take the range of λ to be $0 \leqslant \lambda \leqslant \sqrt{3}$ and change coordinates via

$$\lambda = \sqrt{3}\sin^2\theta \qquad \psi = 2\phi \qquad y = \gamma + 2\phi. \tag{5.9}$$

We also define the corresponding set of left-invariant forms $\hat{\sigma}_3 = \mathrm{d}\gamma - \cos\alpha \, \mathrm{d}\beta$ and $\hat{\sigma}_1 = -\sin\gamma \, \mathrm{d}\alpha - \cos\gamma \sin\alpha \, \mathrm{d}\beta$, $\hat{\sigma}_2 = \cos\gamma \, \mathrm{d}\alpha - \sin\gamma \sin\alpha \, \mathrm{d}\beta$.

Then the metric can be written as

$$(9m^2) \, \mathrm{d}s_5^2 = 6 \, \mathrm{d}\theta^2 + \frac{6\sin^2(2\theta)}{(3 - \cos(2\theta))^2} \hat{\sigma}_3^2 + \frac{6\cos^2\theta}{3 - \cos(2\theta)} (\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + 4\left[\mathrm{d}\phi + \frac{2\cos^2\theta}{3 - \cos(2\theta)} \hat{\sigma}_3 \right]^2 \tag{5.10}$$

which is the form of the metric as written by Pilch and Warner. Note that in the new coordinates the canonical Killing vector takes the form $\partial_\psi = (1/2)\partial_\phi - \partial_\gamma$. The warp factor is given by

$$e^{2\Delta} = \left(\frac{f}{4m} \right)^{1/2} \frac{(3 - \cos 2\theta)^{1/2}}{\sqrt{2}}. \tag{5.11}$$

In the new orthonormal frame with $\hat{e}^1 \propto [d\phi + \cdots]$, $\hat{e}^2 \propto \hat{\sigma}_3$, $\hat{e}^3 \propto \hat{\sigma}_1$, $\hat{e}^4 \propto \hat{\sigma}_2$, the flux is given by

$$G = m\sqrt{3}\,e^{2\Delta}\,e^{2i\phi}\left(\hat{e}^1 + i\frac{\sqrt{2}\sin 2\theta}{\sqrt{3}(3 - \cos 2\theta)}\hat{e}^5\right)(\hat{e}^5 - i\hat{e}^2)(\hat{e}^4 - i\hat{e}^3) \qquad (5.12)$$

which coincides with [21] (up to possible factors).

The fibration defined by ∂_ϕ defines, locally, a four-dimensional base space. However, it is easy to see that there is no choice of the range of coordinates ϕ and $\hat{\psi}$ that makes it a regular four-dimensional manifold. The same is true of the base space defined by the foliation using ∂_y. We therefore introduce a new set of coordinates defined by

$$\phi = \delta \qquad \gamma = \gamma' + \delta. \qquad (5.13)$$

Then the metric takes the form

$$9m^2\,ds_5^2 = 6\,d\theta^2 + \frac{12\sin^2 2\theta}{35 - 3\cos^2 2\theta}(\sigma_3')^2 + \frac{3(1 + \cos 2\theta)}{3 - \cos 2\theta}[(\sigma_1')^2 + (\sigma_2')^2]$$
$$+ \frac{2(35 - 3\cos^2 2\theta)}{(3 - \cos 2\theta)^2}(d\delta + A)^2, \qquad (5.14)$$

where σ_i' are the left invariant one-forms $\hat{\sigma}_i$ above with γ replaced with γ' and the one-form A is given by

$$A = \frac{(1 + \cos 2\theta)(11 - 3\cos(2\theta))}{(35 - 3\cos^2(2\theta))}\sigma_3'. \qquad (5.15)$$

If we choose the period of γ' to be 4π, then it is not difficult to see that the four-dimensional base orthogonal to ∂_δ is diffeomorphic to $\mathbb{C}P^2$. In particular, at $\theta = 0$ the metric has a two-sphere bolt, with normal neighbourhood being that of the chiral spin bundle of S^2, while at $\theta = \pi/2$ the metric has a NUT, i.e. it smoothly approaches $\mathbb{R}^4$. The full space is obtained by gluing these together which gives $\mathbb{C}P^2$. Furthermore, we note that the single non-trivial two-cycle of the base space is represented by the two-sphere bolt at $\theta = 0$. We next analyse the fibre direction ∂_δ. First note that the norm of this Killing vector field is nowhere vanishing. The one-form A is a bona-fide connection one-form; its first Chern class, defined by the integral of $dA/(2\pi)$ over the two-sphere bolt, is one. After recalling the Hopf fibration of S^5 over $\mathbb{C}P^2$, we conclude that if we choose the period[10] of δ to be 2π the topology of the five-dimensional space is in fact S^5.

5.2. Numerical analysis

A numerical investigation of the ODE seems to reveal a continuous family of solutions containing the PW solution and all with topology S^5. We summarize the main points first and then discuss how the three-form flux and the spinors are not globally defined.

Following on from our discussion of the PW solution, we first consider the general coordinate transformation

$$\psi = 2\delta \qquad y = \gamma' + c\delta. \qquad (5.16)$$

The metric then takes the form

$$(9m^2)\,ds^2 = A[d\delta + D\sigma_3']^2 + \frac{d\lambda^2}{g^2 - 1 - \lambda^2} + Q(\sigma_3')^2 + \frac{3h}{2g}((\sigma_1')^2 + (\sigma_2')^2), \qquad (5.17)$$

---

[10] For completeness, we note that the periodicities of δ and γ' imply that y is a periodic coordinate with period 4π while the range of ψ is 4π.

where

$$A = \frac{-4\lambda^4 - 8\lambda^2 + 4\lambda^2 g^2 + 4\lambda^2 hc + 4g^2 - 4 + 4hc + h^2c^2g^2 - 4hcg^2 - h^2c^2}{(g^2 - 1 - \lambda^2)g^2}$$

$$Q = \frac{4\lambda^2 h^2}{-4\lambda^4 - 8\lambda^2 + 4\lambda^2 g^2 + 4\lambda^2 hc + 4g^2 - 4 + 4hc + h^2c^2g^2 - 4hcg^2 - h^2c^2} \quad (5.18)$$

$$D = \frac{(2\lambda^2 - 2g^2 + 2 - hc + hcg^2)h}{-4\lambda^4 - 8\lambda^2 + 4\lambda^2 g^2 + 4\lambda^2 hc + 4g^2 - 4 + 4hc + h^2c^2g^2 - 4hcg^2 - h^2c^2},$$

where c is an arbitrary constant.

Now at $\lambda = 0$ we have the two-parameter family of approximate solutions

$$g = 1 + \beta\lambda^p + \cdots$$

$$h = \frac{2}{p} - \frac{2}{3p(2 - p)\beta}\lambda^{2-p} + \cdots \quad (5.19)$$

for $0 < p < 2$. This includes the exact solution when $p = 1$ and $\beta = 1/\sqrt{3}$.

For these solutions, near $\lambda = 0$ we get

$$9m^2 \, \mathrm{d}s^2 \approx 4\frac{(p - c)^2}{p^2}\left[\mathrm{d}\delta + \frac{1}{c - p}\sigma_3'\right]^2 + \frac{1}{2\beta\lambda^p}\,\mathrm{d}\lambda^2 + \frac{3}{p}[(\sigma_1')^2 + (\sigma_2')^2] + \frac{2\lambda^{2-p}}{(p - c)^2\beta}(\sigma_3')^2. \quad (5.20)$$

We see that, for a given solution specified by p, β, this is regular, with a two-sphere bolt, provided that the period of γ' is correlated with c. For example, it will be useful to observe shortly that the period of γ' can be taken to be 4π provided that $c = 3p - 4$ or $4 - p$.

In order to mimic the PW solution, we would like to match these solutions onto solutions with $h(\lambda_c) = 0$ for some λ_c. Consider then the one-parameter family of solutions[11]

$$g = \left(1 + \lambda_c^2\right)^{1/2} - \frac{2\lambda_c}{3\left(1 + \lambda_c^2\right)^{1/2}}\epsilon + \frac{3 - \lambda_c^2}{27\left(1 + \lambda_c^2\right)^{3/2}}\epsilon^2 + \cdots$$

$$h = \frac{\left(1 + \lambda_c^2\right)^{1/2}}{\lambda_c}\epsilon + \frac{3 - \lambda_c^2}{18\lambda_c^2\left(1 + \lambda_c^2\right)^{1/2}}\epsilon^2 + \cdots \quad (5.21)$$

with $\epsilon = \lambda_c - \lambda$, which also includes the exact solution when $\lambda_c = \sqrt{3}$.

Now consider the behaviour of the metric for these solutions (5.21) near $\lambda = \lambda_c$. We get, for all α,

$$(9m^2) \, \mathrm{d}s^2 = 4[\mathrm{d}\delta + D\sigma_3']^2 + \frac{3}{2\lambda_c\epsilon}\,\mathrm{d}\epsilon^2 + \frac{3\epsilon}{2\lambda_c}[(\sigma_1')^2 + (\sigma_2')^2 + (\sigma_3')^2] \quad (5.22)$$

with $D \propto \epsilon$. This is regular provided that we take the period of γ' to be 4π. We can now numerically integrate these back to $\lambda = 0$. The numerical analysis indicates that they map onto a one-parameter subset of the solutions (5.19), with $3/2 < h(0)$, i.e. $0 < p < 4/3$. Thus we see that if we choose $c = 3p - 4$ or $4 - p$, then the base of the fibration defined by ∂_δ is regular and has the same topology as $\mathbb{C}P^2$ (see the discussion above for the Pilch–Warner

---

[11] Note that there is also a two-parameter family,

$$g = \left(1 + \lambda_c^2\right)^{1/2} - \frac{6}{5A}\epsilon^{5/3} + \cdots \qquad h = A\epsilon^{1/3} + \cdots,$$

but h/g, and hence the size of the two-sphere, diverges at $\epsilon = 0$.

solution). Furthermore, the numerical analysis reveals that A never vanishes and that D only vanishes at λ_c in the way described above. We next note that integrating $(1/2\pi)\,\mathrm{d}(D\sigma_3')$ over the two-sphere bolt at $\lambda = 0$ gives $2D(0) = \pm 1/(p-2)$. Thus by choosing the period of δ to be $2\pi/(2-p)$ we deduce that the topologies of this one-parameter family of solutions, generalizing the Pilch–Warner solution, are all S^5.

There are some problems with this family of solutions, however. Firstly, from (4.3), the spinor bilinear S satisfies $S = (\lambda/g)\,\mathrm{e}^{-2\mathrm{i}\delta}$. For this to be well defined we need the period of δ to be an integer times π, i.e., $2/(2-p)$ is an integer. However, for $0 < p < 4/3$, this is only possible for the PW solution with $p = 1$. The second problem concerns the expression for the flux (5.7). For this to be globally well defined we should be able to write it in terms of globally defined one-forms $\mathrm{d}\delta$ and σ_i'. Note, however, $e^4 - \mathrm{i}e^3 \sim \sigma_2 - \mathrm{i}\sigma_1 = \mathrm{e}^{\mathrm{i}c\delta}(\sigma_2' - \mathrm{i}\sigma_1')$, which requires that the period of $c\delta$ be an integer times 2π to be globally defined. For $c = 3p - 4$ or $4 - p$, this is again only possible for the PW solution. In fact the condition that the flux be well defined is equivalent to the condition that the bilinear S is globally defined. It is also worth noting that this phase in the expression for the complex three-form flux cancels out in the energy momentum tensor and this is consistent with the fact that the metric is globally defined on S^5. We note that in appendix E, blindly ignoring these problems, we have calculated the central charge of the putative dual conformal field theories by determining the effective five-dimensional Newton's constant.

Thus, to summarize, we conclude that these numerical solutions for the ODE give rise to a regular metric on S^5 but they do not give rise to a globally defined solution since the three-form flux is not globally defined. Moreover, the Killing spinors are also not globally defined. As discussed in appendix E, the five-dimensional Newton's constant is, remarkably, analytic for this family. In particular, Newton's constant is a monotonic decreasing function of p for $0 < p < 4/3$. However, the results of a-maximization [47] in four-dimensional superconformal field theories imply that the central charges are always algebraic numbers. Indeed, in the current setting it is natural to expect a quantization condition on p to come from imposing well-definedness of the spinors and flux. As we have shown, there are in fact no solutions to these conditions. It seems possible that, nevertheless, there is some physical interpretation of these solutions. Alternatively, we hope that by slightly relaxing our assumptions new globally defined solutions can be found.

6. Conclusions

The main result of this paper is a determination of the necessary and sufficient conditions on supersymmetric solutions of type IIB supergravity that can be dual to four-dimensional superconformal field theories. The ten-dimensional metric is taken to be a warped product of AdS_5 with a five-dimensional Riemannian metric, and we allowed for the most general fluxes consistent with $SO(4, 2)$ symmetry. Excluding the well-known $\mathrm{AdS}_5 \times X_5$ solutions where X_5 is Sasaki–Einstein and only the self-dual five-form is non-vanishing, we showed that the generic compact M_5 admits a canonical local identity structure. We showed how supersymmetry restricts the torsion of this structure and how it determines the fluxes.

By imposing some additional restrictions, including that the dilaton is constant and the axion vanishes, we reduced the conditions to solving a second-order nonlinear ODE. We managed to find an analytic solution of this ODE and showed that it reproduces a solution found previously in [21]. A numerical analysis revealed a class of solutions that are defined on S^5 but with the flux and the Killing spinors not being globally defined. It would be interesting to know if these local solutions have a physical interpretation. More generally, it may well be possible to find new exact solutions by slightly relaxing some of our assumptions.

While this work was being completed, two papers appeared where new classes of AdS$_5$ solutions of type IIB were discovered. In fact the construction of these solutions was one of the original motivations of this work. In [41] numerical evidence for a family of solutions interpolating between the PW solution and the AdS$_5 \times T^{1,1}$ solution were found. In [42] a powerful technique to generate new AdS$_5$ solutions from old ones, which describe the so-called β-deformations of the original conformal field theory, was presented. It would be interesting to see how these solutions fit into the formalism presented here. It would be particularly interesting if the results of this paper could be used to find AdS$_5$ solutions corresponding to exactly marginal deformations more general than the β-deformations.

We only considered solutions preserving minimal $N = 1$ supersymmetry. This includes geometries preserving $N = 2$ supersymmetry as a special case, but it would be interesting to determine the additional restrictions on the identity structure that are imposed by $N = 2$ supersymmetry. Hopefully, these will be strong enough that further exact solutions can be found. Recall that in the context of $D = 11$ supergravity, the analysis of [2] covered AdS$_5$ geometries preserving $N = 1$ supersymmetry. A refinement of this analysis was carried out in [20], where the additional conditions imposed by $N = 2$ supersymmetry were studied. It is interesting to note that a double wick rotation of these geometries in [20] was shown to be related to quite different physical phenomena, and this may also be the case for the analogous type IIB supergravity solutions.

Our analysis has focused on the local identity G-structure on M_5, as this is most useful for obtaining explicit solutions. Of course, the category of families of solutions that can ultimately be found in explicit form is presumably quite small. We also view our work as providing the foundation for studying more general aspects of conformal field theories with type IIB duals. For example, it would be interesting to know what topological restrictions supersymmetry imposes on M_5. To tackle this, one could try to determine the global G-structure that M_5 admits. A converse result of the form that M_5 satisfying certain topological restrictions always admits a solution would be most desirable. It would be also very interesting to see if there is a generalization of Z-minimization [43] (see also [44–46]), a geometrical version of a-maximization [47] in the toric Sasaki–Einstein setting, to the more general class of geometries analysed here.

Acknowledgments

Appendix A. Conventions and useful formulae for reduction

The ten-dimensional metric has the signature $(-, +, \ldots, +)$. The ten-dimensional gamma matrices Γ^A satisfy

$$[\Gamma^A, \Gamma^B]_+ = 2\eta^{AB} \tag{A.1}$$

and generate the $D = 10$ Clifford algebra Cliff(9, 1), where A, $B = 0, 1, \ldots, 10$ are frame indices. We define $\Gamma_{11} \equiv \Gamma_0 \Gamma_1 \ldots \Gamma_9$.

For the configurations that are a warped product of AdS$_5$ with M_5, it is useful to decompose Cliff(9, 1) by writing

$$\Gamma^a = \rho^a \otimes 1 \otimes \sigma^3 \qquad \Gamma^i = 1 \otimes \gamma^i \otimes \sigma^1 \tag{A.2}$$

where a, $b = 0, 1, \ldots, 4$ and i, $j = 1, 2, \ldots, 5$ are frame indices on AdS$_5$ and M_5 respectively, and we have

$$[\rho^a, \rho^b]_+ = 2\eta^{ab}, \qquad [\gamma^i, \gamma^j]_+ = 2\delta^{ij} \tag{A.3}$$

with $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1, 1)$. The ρ^a satisfy $\rho_{01234} = \mathrm{i}$ and generate Cliff(4, 1), while the γ^m satisfy $\gamma_{12345} = 1$ and generate Cliff(5). In addition, σ^i, $i = 1, 2, 3$, are the Pauli matrices. We then have

$$\Gamma_{11} = 1 \otimes 1 \otimes \sigma^2. \tag{A.4}$$

Let us work out a consistent set of conventions for the various intertwiner operators in the relevant dimensions (see, e.g. [48]). The A-intertwiners operate as follows,

$$A_{10} \Gamma^M A_{10}^{-1} = \Gamma^{M\dagger} \qquad A_{1,4} \rho^\mu A_{1,4}^{-1} = -\rho^{\mu\dagger} \qquad A_5 \gamma^i A_5^{-1} = \gamma^{i\dagger}, \tag{A.5}$$

and can be chosen to be Hermitian:

$$A_\bullet^\dagger = A_\bullet. \tag{A.6}$$

The charge conjugation matrices, or C-intertwiners, operate as follows,

$$C_{10}^{-1} \Gamma^M C_{10} = -\Gamma^{MT} \qquad C_{1,4}^{-1} \rho^\mu C_{1,4} = \rho^{\mu T} \qquad C_5^{-1} \gamma^i C_5 = \gamma^{iT}, \tag{A.7}$$

and in the given dimensions are all antisymmetric:

$$C_\bullet = -C_\bullet^T. \tag{A.8}$$

Finally, we have the following D-intertwiners:

$$\begin{aligned}
\Gamma^{*A} &= \tilde{D}_{10}^{-1} \Gamma^A \tilde{D}_{10} & \tilde{D}_{10} \tilde{D}_{10}^* &= 1 \\
\gamma^{*m} &= \tilde{D}_5^{-1} \gamma^m \tilde{D}_5 & \tilde{D}_5 \tilde{D}_5^* &= -1 \\
\rho^{*a} &= -D_{1,4}^{-1} \rho^a D_{1,4} & D_{1,4} D_{1,4}^* &= -1.
\end{aligned} \tag{A.9}$$

Also recall that, by definition, $D_{10} = C_{10} A_{10}^T$ and that $\tilde{D}_{10} = \Gamma_{11} D_{10}$. It turns out that one can take the following decompositions:

$$A_{10} = 1 \otimes 1 \otimes \sigma^1 \qquad C_{10} = C_{1,4} \otimes C_5 \otimes \sigma^2 \qquad \tilde{D}_{10} = D_{1,4} \otimes \tilde{D}_5 \otimes \sigma^1 \tag{A.10}$$

with

$$\begin{aligned}
A_{1,4} &= 1, & C_{1,4} &= D_{1,4} \\
A_5 &= 1, & C_5 &= \tilde{D}_5.
\end{aligned} \tag{A.11}$$

We now consider decomposing a $D = 10$ Majorana–Weyl spinor ϵ' as $\epsilon' = \psi \otimes \chi \otimes \theta$. The chirality condition in $D = 10$ is

$$\Gamma_{11} \epsilon' = -\epsilon' \tag{A.12}$$

which implies

$$\sigma^2 \theta = -\theta. \tag{A.13}$$

Moreover, $\epsilon'^c = \tilde{D}_{10} \epsilon'^*$, which now reads

$$\epsilon'^c = \psi^c \otimes \chi^c \otimes \sigma^1 \theta^* \tag{A.14}$$

where

$$\psi^c = C_{1,4}\psi^* \qquad \chi^c = C_5\chi^* \tag{A.15}$$

and we note that $\psi^{cc} = -\psi$ and $\chi^{cc} = -\chi$. To impose the Majorana condition in $D = 10$, $\epsilon'^c = \epsilon'$, we take

$$\theta = \sigma^1\theta^* \tag{A.16}$$

which we note is consistent with the chirality condition already imposed on θ.

We now want to construct the most general spinor ansatz that is consistent with minimal supersymmetry in AdS$_5$. Since type IIB supersymmetry is parametrized by two $D = 10$ Majorana–Weyl spinors, ϵ_i, we take

$$\epsilon_i = \psi \otimes \chi_i \otimes \theta + \psi^c \otimes \chi_i^c \otimes \theta \tag{A.17}$$

where we assume that the spinor ψ satisfies

$$\nabla_\mu\psi = \tfrac{1}{2}m\rho_\mu\psi \tag{A.18}$$

to ensure that supersymmetry is preserved on AdS$_5$. Note that ψ^c then satisfies this equation with $m \mapsto -m$. Note also that each spinor has 16 real components, realized as the real part of 4 complex $\times$ 4 complex components. We may then complexify

$$\epsilon \equiv \epsilon_1 + \mathrm{i}\epsilon_2 \equiv \psi \otimes \xi_1 \otimes \theta + \psi^c \otimes \xi_2^c \otimes \theta \tag{A.19}$$

where $\xi_1 = \chi_1 + \mathrm{i}\chi_2, \xi_2^c = \chi_1^c + \mathrm{i}\chi_2^c$. Then

$$\epsilon^c = \psi \otimes \xi_2 \otimes \theta + \psi^c \otimes \xi_1^c \otimes \theta. \tag{A.20}$$

In fact, to derive (3.3)–(3.8), we rescaled by a convenient power of the warp factor. Indeed, the ansatz we used is

$$\epsilon \equiv \psi \otimes \mathrm{e}^{\Delta/2}\xi_1 \otimes \theta + \psi^c \otimes \mathrm{e}^{\Delta/2}\xi_2^c \otimes \theta. \tag{A.21}$$

In order to analyse the Killing spinor equations, we shall also need the following result. Suppose we have two complex vector spaces V and W, such that V comes equipped with an anti-unitary operation c, mapping $v \in V$ to $v^c \in V$, with $(av)^c = a^*v^c \, \forall a \in \mathbb{C}$, which also squares to -1: $v^{cc} = -v$. Then for non-zero $v \in V$, we have $v \otimes w_1 + v^c \otimes w_2 = 0$ implies that $w_1 = w_2 = 0$. To see this, first note that $v \otimes w_1 + v^c \otimes w_2 = 0$ implies that either $w_1 = w_2 = 0$, or else $v = av^c$, $w_1 = w_2/a$ for some $a \in \mathbb{C}^*$. Suppose the latter case holds. Taking the conjugate of $v - av^c = 0$ gives $v^c + a^*v = 0$. These two equations imply $1 + |a|^2 = 0$, which is impossible. Hence the result. Using this algebraic lemma one can show that the equations are then a sum of two terms, each of which is separately zero.

Much of our analysis of the supersymmetry conditions (3.3)–(3.8) comes from analysing bilinears that can be constructed from ξ_i. Note that there are two kinds of bilinears that can be constructed,

$$\bar{\chi}\gamma_{(n)}\xi = \chi^\dagger\gamma_{(n)}\xi \qquad \bar{\chi}^c\gamma_{(n)}\xi = \chi^\mathrm{T}C^{-1}\gamma_{(n)}\xi, \tag{A.22}$$

where we have used $A_5 = 1$, defined $C \equiv C_5 = \tilde{D}_5$ and $\gamma_{(n)}$ is the antisymmetrized product of n gamma matrices. For convenience, we record once again, for reference,

$$C^* = -C^{-1} \qquad C^\mathrm{T} = -C \tag{A.23}$$

and

$$\gamma^i = \gamma^{i\dagger} \qquad C^{-1}\gamma^iC = \gamma^{i\mathrm{T}}. \tag{A.24}$$

Finally, we note that the Fierz identity for Cliff(5) reads

$$\bar{\xi}_1\xi_2\bar{\xi}_3\xi_4 = \tfrac{1}{4}\bar{\xi}_1\xi_4\bar{\xi}_3\xi_2 + \tfrac{1}{4}\bar{\xi}_1\gamma_m\xi_4\bar{\xi}_3\gamma^m\xi_2 - \tfrac{1}{8}\bar{\xi}_1\gamma_{mn}\xi_4\bar{\xi}_3\gamma^{mn}\xi_2. \tag{A.25}$$

Appendix B. Algebraic conditions satisfied by the bilinears

There are a number of algebraic conditions satisfied by the various bilinears that we use in the main text that can be derived using Fierz identities. However, we find it most useful to construct them using a convenient basis of γ-matrices of Cliff(5). Specifically, we start by taking

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbf{1} \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1} \qquad \gamma^a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \tau^a \tag{B.1}$$

where $\tau^a = -\mathrm{i}\sigma^a$ and σ^a are the Pauli matrices. The intertwiner $\gamma^{*m} = \tilde{D}^{-1}\gamma^m\tilde{D}$, used in the definition $\xi^c = \tilde{D}\xi^*$, is given by $D = \mathbf{1} \otimes -\mathrm{i}\sigma^2$. The corresponding bases of one-forms are labelled e^i.

We write the two spinors as $\xi_i = s_i \otimes \theta_i$. We demand that the two vectors K_4 and K_5 defined in (3.10) are chosen to lie in the $(e^1$–$e^2)$-plane. This requires $\bar{\xi}_i \gamma^a \xi_i = 0$ and constrains the s_i. If, in addition, we require K_5 to be parallel to e^1 we find

$$s_1 = \sqrt{2} \begin{pmatrix} \cos\theta\cos\phi \\ -\sin\theta\sin\phi \end{pmatrix} \qquad s_2 = \sqrt{2} \begin{pmatrix} \sin\theta\cos\phi \\ \cos\theta\sin\phi \end{pmatrix} \tag{B.2}$$

where, without loss of generality, we have $\bar{\theta}_i\theta_i = 1$. Note that we have imposed (3.13):

$$\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2 = 2. \tag{B.3}$$

One then finds that the θ and ϕ functions are related to the scalar bilinears $\sin\zeta$, Z and S defined in (3.12) by

$$\sin\zeta = \cos 2\theta\cos 2\phi \qquad Z = \sin 2\theta\cos 2\phi\,\bar{\theta}_2\theta_1 \qquad S = \sin 2\theta\cos 2\phi\,\bar{\theta}_2^c\theta_1. \tag{B.4}$$

The vectors defined in (3.10) are given by

$$\begin{aligned}
K_5 &= (\cos 2\phi)e^1 \\
K_4 &= (\cos 2\theta)e^1 - (\sin 2\theta\sin 2\phi)e^2 \\
K_3 &= (\sin 2\theta\bar{\theta}_2\theta_1)e^1 + (\cos 2\theta\sin 2\phi\,\bar{\theta}_2\theta_1)e^2 + (\sin 2\phi)\bar{\theta}_2\tau_a\theta_1 e^a \\
K &= (\sin 2\theta\bar{\theta}_1^c\theta_2)e^1 + (\cos 2\theta\sin 2\phi\,\bar{\theta}_1^c\theta_2)e^2 - (\sin 2\phi)\bar{\theta}_1^c\tau_a\theta_2 e^a.
\end{aligned} \tag{B.5}$$

It is similarly straightforward to write out the two-forms. In particular, we find

$$\begin{aligned}
W &= Z(\csc 2\theta\tan 2\phi)e^{12} + (\cos 2\theta\sin 2\phi e^1 - \sin 2\theta e^2) \wedge \bar{\theta}_2\tau_a\theta_1 e^a \\
&\quad + \tfrac{1}{2}(\sin 2\theta\cos 2\phi)\epsilon_{abc}\bar{\theta}_2\tau^c\theta_1 e^{ab}.
\end{aligned} \tag{B.6}$$

We then find that $\mathrm{Re}[\mathrm{i}Z^*W] = 0$ implies $Z = 0$, as claimed in the text.

We now put $Z = 0$ by setting $\bar{\theta}_1\theta_2 = 0$. Choosing K_3 to suitably lie just within the $(e^3$–$e^4)$-plane, we can choose

$$\theta_1 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} \\ 0 \end{pmatrix} \qquad \theta_2 = \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\alpha} \end{pmatrix} \tag{B.7}$$

and hence

$$\begin{aligned}
K_5 &= (\cos 2\phi)e^1 \\
K_4 &= (\cos 2\theta)e^1 - (\sin 2\theta\sin 2\phi)e^2 \\
K_3 &= (\sin 2\phi)(e^4 - \mathrm{i}e^3) \\
\mathrm{e}^{-2\mathrm{i}\alpha}K &= (\sin 2\theta)e^1 + (\cos 2\theta\sin 2\phi)e^2 - \mathrm{i}(\sin 2\phi)e^5,
\end{aligned} \tag{B.8}$$

where $\bar{\theta}_1^c\theta_2 = \mathrm{e}^{2\mathrm{i}\alpha}$, which is the phase of S, so that the scalars are now given by

$$\sin\zeta = \cos 2\theta\cos 2\phi \qquad S = -\sin 2\theta\cos 2\phi\,\mathrm{e}^{2\mathrm{i}\alpha}. \tag{B.9}$$

Similarly, the two-forms are

$$
\begin{aligned}
U &= -\sin 2\theta \sin 2\phi e^{15} - \cos 2\theta e^{25} + \cos 2\theta \cos 2\phi e^{34} \\
V &= -\cos 2\phi e^{25} + e^{34} \\
W &= (\cos 2\theta \sin 2\phi e^1 - \sin 2\theta e^2 + \mathrm{i} \sin 2\theta \cos 2\phi e^5) \wedge (e^4 - \mathrm{i} e^3)
\end{aligned}
\tag{B.10}
$$

and this leads to a quick derivation of (3.40).

Appendix C. Absence of solutions with ξ_i linearly dependent and $\xi_i \neq 0$

Let us consider the possibility

$$
\xi_2 = u\xi_1 + v\xi_1^c
\tag{C.1}
$$

for some functions u, v, which defines a local $SU(2)$ structure in five dimensions. We will use conditions (3.22) and (3.23) that can be derived directly from Fierz relations and in particular do not rely on any aspects of the identity structure that we considered in the text. We also use the differential conditions (3.17)–(3.20). Recall from section 3.2 that we can then deduce (3.25) and $\mathrm{Re}[\mathrm{i}Z^*W] = 0$.

A calculation shows that $\mathrm{Re}[\mathrm{i}Z^*W] = 0$ implies that

$$
2|u|^2 \bar\xi_1 \gamma_{(2)} \xi_1 + uv^* \bar\xi_1^c \gamma_{(2)} \xi_1 + u^* v \bar\xi_1 \gamma_{(2)} \xi_1^c = 0.
\tag{C.2}
$$

In the special case that $v = 0$ we deduce, for non-trivial ξ_1, that $u = 0$ and we return to the Sasaki–Einstein case analysed just before the start of section 3.1. We continue, therefore, with $v \neq 0$. To proceed we derive the following expression for V:

$$
V = -\frac{1 + |u|^2 + |v|^2}{2} \mathrm{i} \bar\xi_1 \gamma_{(2)} \xi_1.
\tag{C.3}
$$

Next observe that $K = -SK_5$ and using (3.15) we get

$$
-3\mathrm{i}mK_5 = \mathrm{d}[\ln(\mathrm{e}^{4\Delta} S)]
\tag{C.4}
$$

and hence $\mathrm{d}K_5 = 0$. We also have $K_4 = \sin \zeta K_5$ and then using (3.37) we get

$$
\mathrm{e}^{4\Delta} K_4 = \frac{f}{4m} K_5.
\tag{C.5}
$$

From (3.19) we then conclude that $V = 0$. However, from (C.3) we see that this is not possible unless $\xi_1 = 0$.

Appendix D. Integrability of IIB supersymmetry conditions

Writing the variation of the gravitino appearing in (2.1) as $\delta\psi_M = \mathcal{D}_M \epsilon$, we calculate[12] that

$$
\mathcal{D}_{[M}\mathcal{D}_{N]}\epsilon = I_{MN}^{(1)}\epsilon + I_{MN}^{(2)}\epsilon^c
\tag{D.1}
$$

where

$$
\begin{aligned}
I_{MN}^{(1)} = {}& \frac{1}{8} R_{MNS_1S_2} \Gamma^{S_1S_2} - \frac{1}{2} P_{[M} P_{N]}^* + \frac{\mathrm{i}}{192} D_{[M} F_{N]S_1S_2S_3S_4} \Gamma^{S_1S_2S_3S_4} \\
& + \frac{1}{768} \left(F_{MNS_1}{}^{R_1R_2} F_{S_2S_3S_4R_1R_2} \Gamma^{S_1S_2S_3S_4} - 2F_{[M|S_1|}{}^{R_1R_2R_3} F_{N]S_2R_1R_2R_3} \Gamma^{S_1S_2} \right) \\
& + \frac{1}{9216} \left(-\Gamma_{MN}{}^{S_1S_2S_3S_4S_5S_6} G_{S_1S_2S_3} G_{S_4S_5S_6}^* - 9\Gamma_{MN}{}^{S_1S_2S_3S_4} G_{S_1S_2}{}^{R} G_{S_3S_4R}^* \right. \\
& \left. + 12\Gamma_{[M}{}^{S_1S_2S_3S_4S_5} G_{N]S_1S_2} G_{S_3S_4S_5}^* - 6\Gamma_{[M}{}^{S_1S_2S_3S_4S_5} G_{|S_1S_2S_3|} G_{N]S_4S_5}^* \right.
\end{aligned}
$$

[12] The IIB integrability conditions were also discussed recently in [50].

$$+18\Gamma_{MN}{}^{S_1S_2}G_{S_1}{}^{R_1R_2}G^*_{S_2R_1R_2}+36\Gamma_{[M}{}^{S_1S_2S_3}G_{|S_1S_2|}{}^RG^*_{N]S_3R}$$

$$+72\Gamma_{[M}{}^{S_1S_2S_3}G_{N]S_1}{}^RG^*_{S_2S_3R}+72\Gamma^{S_1S_2S_3S_4}G_{[M|S_1S_2|}G^*_{N]S_3S_4}$$

$$-18\Gamma^{S_1S_2S_3S_4}G_{S_1S_2S_3}G^*_{MNS_4}+18\Gamma^{S_1S_2S_3S_4}G_{MNS_1}G^*_{S_2S_3S_4}$$

$$+6\Gamma_{MN}G^{R_1R_2R_3}G^*_{R_1R_2R_3}-72\Gamma_{[M}{}^SG_{N]}{}^{R_1R_2}G^*_{SR_1R_2}$$

$$+36\Gamma_{[M}{}^SG_{|S|}{}^{R_1R_2}G^*_{N]R_1R_2}-288\Gamma^{S_1S_2}G_{[M|S_1|}{}^RG^*_{N]S_2R}-54\Gamma^{S_1S_2}G_{S_1S_2}{}^RG^*_{MNR}$$

$$+54\Gamma^{S_1S_2}G_{MN}{}^RG^*_{S_1S_2R}-144G_{[M}{}^{R_1R_2}G^*_{N]R_1R_2}\Big) \tag{D.2}$$

and

$$I^{(2)}_{MN}=\frac{1}{96}\Big(\Gamma_{[M}{}^{S_1S_2S_3}D_{N]}G_{S_1S_2S_3}+9\Gamma^{S_1S_2}D_{[M}G_{N]S_1S_2}\Big)$$

$$-\frac{i}{1536}\Big(3G_{MN}{}^RF_{RS_1S_2S_3S_4}\Gamma^{S_1S_2S_3S_4}+6G_{[M}{}^{R_1R_2}F_{R_1R_2S_1S_2S_3}\Gamma_{N]}{}^{S_1S_2S_3}$$

$$-6G_{S_1}{}^{R_1R_2}F_{[M|R_1R_2S_2S_3|}\Gamma_{N]}{}^{S_1S_2S_3}-6G_{S_1S_2}{}^RF_{MNRS_3S_4}\Gamma^{S_1S_2S_3S_4}$$

$$-12G_{[M}{}^{R_1R_2}F_{N]R_1R_2S_1S_2}\Gamma^{S_1S_2}+4G^{R_1R_2R_3}F_{[M|R_1R_2R_3S|}\Gamma_{N]}{}^S$$

$$+4G^{R_1R_2R_3}F_{MNR_1R_2R_3}\Big) \tag{D.3}$$

and we have used the self-duality of F. Note that this result is consistent with that in [49]. Setting this to zero, contracting with $8\Gamma^N$, we deduce that

$$2\Bigg[-R_{MS}+P_MP^*_S+P_SP^*_M+\frac{1}{96}F_M{}^{R_1R_2R_3R_4}F_{SR_1R_2R_3R_4}$$

$$+\frac{1}{8}\left(G_M{}^{R_1R_2}G^*_{SR_1R_2}+G_S{}^{R_1R_2}G^*_{MR_1R_2}-\frac{1}{6}g_{MS}G^{R_1R_2R_3}G^*_{R_1R_2R_3}\right)\Bigg]\Gamma^S\epsilon$$

$$-\frac{i}{48}*_{10}\left[dF-\frac{i}{2}G\wedge G^*\right]_{S_1S_2S_3S_4}\Gamma_M{}^{S_1S_2S_3S_4}\epsilon$$

$$+\frac{i}{12}*_{10}\left[dF-\frac{i}{2}G\wedge G^*\right]_{MS_1S_2S_3}\Gamma^{S_1S_2S_3}\epsilon$$

$$-\frac{1}{96}(DG+P\wedge G^*)_{S_1S_2S_3S_4}\Gamma_M{}^{S_1S_2S_3S_4}\epsilon^c+\frac{1}{8}(DG+P\wedge G^*)_{MS_1S_2S_3}\Gamma^{S_1S_2S_3}\epsilon^c$$

$$-\frac{1}{8}\left(D_RG^R{}_{S_1S_2}+\frac{i}{6}G_{R_1R_2R_3}F^{R_1R_2R_3}{}_{S_1S_2}-P^RG^*_{S_1S_2R}\right)\Gamma_M{}^{S_1S_2}\epsilon^c$$

$$+\frac{3}{4}\left(D_RG^R{}_{MS}+\frac{i}{6}G_{R_1R_2R_3}F^{R_1R_2R_3}{}_{MS}-P^RG^*_{MSR}\right)\Gamma^S\epsilon^c$$

$$=-\frac{i}{24}G^*_{S_1S_2S_3}\Gamma_M{}^{S_1S_2S_3}\delta\lambda+\frac{3i}{8}G^*_{MS_1S_2}\Gamma^{S_1S_2}\delta\lambda+4iP_M\delta\lambda^*. \tag{D.4}$$

Similarly, again using the self-duality of F, a calculation reveals that

$$i\left(D_MP^M+\frac{1}{24}G^{M_1M_2M_3}G_{M_1M_2M_3}\right)\epsilon^c+iD_{[S_1}P_{S_2]}\Gamma^{S_1S_2}\epsilon^c$$

$$+\frac{i}{24}\left(D_{[S_1}G_{S_2S_3S_4]}+P_{[S_1}G^*_{S_2S_3S_4]}\right)\Gamma^{S_1S_2S_3S_4}\epsilon$$

$$+\frac{i}{8}\left(D_MG^M{}_{S_1S_2}-P_MG^{*M}{}_{S_1S_2}+\frac{i}{6}F_{S_1S_2M_1M_2M_3}G^{M_1M_2M_3}\right)\Gamma^{S_1S_2}\epsilon$$

$$=\Gamma^SD_S\delta\lambda-\frac{i}{960}\Gamma^{S_1S_2S_3S_4S_5}F_{S_1S_2S_3S_4S_5}\delta\lambda-i\Gamma^M\Gamma^SP_S\delta\psi^*_M$$

$$-\frac{i}{24}\Gamma^M\Gamma^{S_1S_2S_3}G_{S_1S_2S_3}\delta\psi_M. \tag{D.5}$$

Suppose we have a supersymmetric configuration satisfying $\delta\psi_M = \delta\lambda = 0$. If we demand that it also satisfies the equation of motion and the Bianchi identity for G, the Bianchi identity for P and the Bianchi identity for the self-dual five-form F, then we conclude from (D.5) that the equation of motion for P is automatically satisfied and from (D.4) that

$$E_{MS}\Gamma^S\epsilon = 0 \tag{D.6}$$

where $E_{MS} = 0$ is equivalent to Einstein's equations. Now the vector bilinear, $K^M \equiv \bar{\epsilon}\Gamma^M\epsilon$, that can be constructed from a spinor of Spin(9, 1) is null. We can use it to set up a frame

$$\mathrm{d}s^2 = 2e^+e^- + e^ae^a \tag{D.7}$$

for $a = 1, \ldots, 9$ with K, as a one-form, equal to e^+. Following the argument of section 2.3 of [5], we conclude that (D.6) implies that the only component of E_{MS} that is not automatically zero is E_{++}, which thus is the only extra condition that needs to be imposed in order to get a full supersymmetric solution to the equations of motion.

For the class of solutions considered in the text, we have $E_{++} = 0$. To see this, we first note that the spinor ansatz (A.21) implies that the vector K^M only has non-vanishing components in the AdS$_5$ directions. Next we observe that the Ricci tensor of the ten-dimensional metric has components in the AdS$_5$ directions given by

$$R_{\mu\nu} = -\bar{g}_{\mu\nu}(4m^2 + 8(\bar{\nabla}\Delta)^2 + \bar{\nabla}^2\Delta) \tag{D.8}$$

where $\bar{g}$ is the metric on AdS$_5$. In addition, the right-hand side of the Einstein equation in (2.1) is also proportional to $\bar{g}$. Since $\bar{g}_{++} = 0$, in the frame (D.7), we conclude that $E_{++} = 0$ is trivially satisfied.

Appendix E. Central charges

Recall that the central charge of the conformal field theory dual is determined by the five-dimensional Newton's constant [51]. The type IIB action takes the form

$$S = \frac{1}{2\kappa_{10}^2}\int_M \sqrt{G}[R(G) + \cdots], \tag{E.1}$$

where G is the ten-dimensional metric. Assuming $M = M_{4,1} \times M_5$ and using our warped-product ansatz (3.1), we get

$$S = \frac{1}{2\kappa_{10}^2}\int_{M_5} \sqrt{g_5}\,\mathrm{e}^{8\Delta}\int_{M_{4,1}} \sqrt{g_E}[R(g_E) + 12m^2 + \cdots], \tag{E.2}$$

where g_E and g_5 are the metrics on $M_{4,1}$ and M_5, respectively. The m^2 term in the integrand appears since our AdS$_5$ metric on $M_{4,1}$ is normalized so that $\mathrm{Ric}(g_E) = -4m^2 g_E$. We will also need the quantization condition on the five-form which reads

$$\int_{M_5} F - \frac{\mathrm{i}}{2}A \wedge G^* = c_0 N_{\mathrm{D}3} \tag{E.3}$$

where $N_{\mathrm{D}3}$ is the number of $D3$-branes and c_0 is a constant the precise form of which we will not need.

Let us first consider the simplest case of where M_5 is Sasaki–Einstein. We then have $\mathrm{e}^{8\Delta} = (f/4m)^2$, and the quantization condition gives $f = c_0 N_{\mathrm{D}3}m^5/\mathrm{vol}'(M_5)$ where $\mathrm{vol}'(M_5)$ is the volume of the Sasaki–Einstein metric normalized so that $\mathrm{Ric}(g_5) = 4g_5$. In particular, $\mathrm{vol}'(S^5) = \pi^3$. Thus the type IIB dimensionally reduced action reads

$$S = \frac{1}{16\pi G_5^{\mathrm{SE}}}\int_{M_{4,1}} \sqrt{g_E}[R(g_E) + 12m^2 + \cdots], \tag{E.4}$$

where

$$G_5^{\text{SE}} = \frac{2\kappa_{10}^2 \, \text{vol}'(M_5)}{\pi m^3 c_0^2 N_{\text{D3}}^2}. \tag{E.5}$$

Next we consider the special class of solutions that we discussed in section 5. We first observe

$$\int_{M_5} \sqrt{g_5} e^{8\Delta} = -\frac{2f^2}{16^2 \times 27 m^7} \int_{M_5} \partial_\lambda (h^2) \, d\lambda \, d\psi \, dy \, \sigma_1 \wedge \sigma_2 \tag{E.6}$$

noting the remarkable fact that the integrand can be trivially integrated. Equally remarkable is the fact that the quantization condition on the flux also takes a simple form. To carry out the integral, we first observe that a two-form potential for the three-form flux G is given by

$$A = \frac{1}{3m^2} \left(\frac{f}{4m}\right)^{1/2} \frac{h^{1/2}(g^2 - 1 - \lambda^2)^{1/2}}{\sqrt{6}\lambda g^2} (-i \, d\lambda + g\lambda \, d\psi) \wedge (\sigma_2 - i\sigma_1). \tag{E.7}$$

It is then straightforward to deduce that

$$f^{-1} = \frac{1}{27 \times 8 m^5 c_0 N_{\text{D3}}} \int_{M_5} \partial_\lambda (h^2/g^2) \, d\lambda \, d\psi \, dy \, \sigma_1 \wedge \sigma_2. \tag{E.8}$$

Substituting this into (E.6) gives an analytic expression for Newton's constant for this class of solutions.

We finally focus on the local solutions that we found numerically that include the PW solution as a special case. In particular, $0 \leqslant \lambda \leqslant \lambda_c$ with $h(\lambda_c) = 0$ and $g(0) = 1$. In this case we get

$$G_5^{(\text{sec } 5)} = -\frac{\kappa_{10}^2 h^2(0)}{4 \times 27\pi m^3 c_0^2 N_{\text{D3}}^2} \int d\psi \, dy \, \sigma_1 \wedge \sigma_2. \tag{E.9}$$

For the numerical solutions, $\sigma_1^2 + \sigma_2^2$ gives the round metric on S^2 so $\int \sigma_1 \wedge \sigma_2 = 4\pi$, while $d\psi \wedge dy = 2 d\delta \wedge d\gamma'$ and so when integrated contributes a factor of $16\pi^2/(2 - p)$. Recall that $h(0) = 2/p$ and that the value of p is determined by λ_c which specifies the numerical solution and that for the PW solution we have $p = 1$.

As a check on these formulae we can calculate the ratio of the central charges of the theories dual to $\text{AdS}_5 \times S^5$ and the PW solution. We get

$$\frac{G_5^{\text{PW}}}{G_5^{S^5}} = \frac{32}{27} \tag{E.10}$$

in agreement with [22].

More generally, the expression (E.9) shows that the ratio of central charges of two solutions in our new family of local solutions depends on the ratio of their values of $1/p^2(2 - p)$. As this is not constant it indicates that the local solutions could not possibly represent exactly marginal deformations of the PW solution. Furthermore, if the local solutions were to somehow make physical sense, some restrictions on p would have to be imposed to ensure an algebraic central charge as implied by the general results on a-maximization [47].

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