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Abstract

In this paper we will examine some implications of online data for a classical operations management model, vis. the Economic Order Quantity model. Customer waiting behavior on individual orders (which occur during stockouts) forms the basis for evaluating the potential backorders. The potential attraction of reducing inventory holding costs must be balanced with the loss due to lost sales. We clearly delineate the conditions under which it is profitable to stock out every ordering cycle, and the conditions under which the traditional economic order quantity model still holds. In order to allow practical application of the model, we develop a number of different approaches to the problem of estimating the backorder function from available on-line transaction data.

1. Introduction

The emergence of web based online retailing has changed the manner in which customer behavior can be tracked and profiled. A recent article profiling Yahoo presents the potential and pitfalls in utilizing this data for new business models [10]. New tools are available that allow one to target promotions and product offerings to customers predisposed to purchasing them as indicated by customers' observed past behavior.

The availability of extensive customer purchase behavior data is changing the way in which different functional areas can formulate optimal tactics and strategies. This has particular relevance for marketing. For instance; Active Buyers Guide helps visitors select models of various consumer electronic items (see http://www.activebuyersguide.com). They evaluate the customer's preferences for attributes by conducting a factor analysis using dynamically generated flash card comparisons. Similarly, in retailing, some sites track the online browsing behavior and have a history of items that a particular customer finds interesting. This data is used to suggest similar items for purchase. Amazon (http://www.amazon.com) is a popular retailer with extremely sophisticated customer management. "What Amazon.com has done is invent and implement a model for interacting with millions of customers, one at a time" (see [1]). "Customers love Amazon not because it offers the lowest prices--it doesn't--but because the experience has been crafted so carefully that most of us actually enjoy it" (see [6]).

There are implications for inventory management as well. As an example, this data is also being used to suggest alternate items in case a particular item is out of stock, or to provide an updated estimate for the waiting time till new stock arrives. This allows the customer to make informed choices about substitution and/or backordering. This behavior is now visible to the retailer, and can form the basis for better business decisions. The challenge is not merely to suggest replacement items or provide waiting times, but to use operational policies and management strategies to better support this interaction between the firm and the customer.

In this paper we will examine some implications of online data for a classical operations management model, vis. the Economic Order Quantity model. Customer waiting behavior on individual orders (which occur during stockouts) forms the basis for evaluating the potential backorders. The potential attraction of reducing inventory holding costs must be balanced with the loss due to lost sales. We clearly delineate the conditions under which it is profitable to stock out every ordering cycle, and the conditions under which the traditional economic order quantity model still holds. In order to allow practical application of the model, we develop a number of different approaches to the problem of estimating the backorder function from available on-line transaction data.

2. Literature Survey

There is a significant body of recent literature on online retailing. The increasing popularity of the web has led to an explosion of research related to retail substitution behavior (see [2] [3] [8] [14] [21]). These papers illustrate the effect of detailed customer behavior data on the Operations Management literature.

The EOQ formula was initially derived in [11], and the related literature is voluminous, see e.g. [9] [23]. The EOQ model with backorders has been discussed by [9] as well. They consider a linear backorder function, and derive a policy based on a backorder cost per unit backordered per time unit.

A number of authors have extended the EOQ model by considering backorders. Backorder models where only a fraction of the demand is backordered when a stockout occurs are examined in [15] [16] [20] [22]. Our model easily handles these kinds of mixtures between backorders (pent-up demand) and lost sales. Our model does not have explicit costs of not satisfying demand from on-hand stock. Instead, some of the demand during the stockout period turns into lost sales, which reduces revenue for the retailer.

Backorders have been the subject of extensive research in the context of stochastic models. An order point order quantity model with a mixture of backorders and lost sales is investigated in [18]. Lost sales will occur when backorders exceed a certain threshold level. General backorder costs in stochastic inventory models are discussed in [5].

The cost structure of backorders has been extended by some authors. Inventory policies when the backorder costs have fixed and proportional components are examined in [4]. Another approach to modeling the effect of stock-outs can be found in [17], where it is assumed that the demand rate is influenced by backorders. However, to the best of our knowledge, the approach to modeling the effect of stockouts presented in our model is new, as is the characterization of conditions under which periodic availability is an optimal policy.

There is also extensive research on perishable inventories, when the stock at hand may decay, or become obsolete. One representative paper [13] integrates the stocking decision with backordering. While we do not have perishable inventories, we have "perishable" backorders, in the sense that backorders grow less than proportionately with the elapsed stockout time.

Finally, in [12] a recent paper uses an exponential pent-up demand function in the context of stochastic lead times. Lee focuses on algorithms to obtain the optimal stockout period without deriving any structural results. This is the only paper we have found so far which examines a non-linear pent-up demand function. In contrast, our paper presents the structural results for any pent-up demand function, as well as the parametric conditions under which three different policies are optimal: the EOQ stocking policy, the periodic stocking policy and the no-inventory policy.

Many EOQ inventory models focus on inventory as a cost, as the original work [11] did. The decision maker controls a cost center, and attempts to minimize costs while delivering a certain service level. Often, this leads to situations where the optimal decision is unprofitable, as already observed in [22]. In contrast, our objective is to maximize profits, which allows us to choose not to operate if it is not profitable.

A different way to think of the periodic availability policies in this paper is as a bridge between classical "supply from inventory" policies (where a retailer attempts to fill demand from on-hand stock) and stockless retailing policies (where the retailer acts purely as an order taker who passes the demand on to his supplier). Indeed, both extremes are special cases of our general model.

We use fractional programming to solve the discounted cash flow optimization. A good introduction to non-linear fractional programming can be found in [7]. Our algorithm is similar, but specifically crafted for our needs.

3. The Inventory Model

We make all the standard assumptions of the Economic Order Quantity (EOQ) model, but with some additional assumptions regarding what happens when the retailer runs out of stock (which is not allowed in the EOQ model). The retailer experiences a constant deterministic demand of D units of product per unit time for a given product. The retailer can place an order at any time for any quantity of product desired, and the delivery will be made after a constant lead time. There is a fixed order cost of F per order placed, a purchasing cost of c per unit purchased, non-financial inventory holding cost of h per unit kept in inventory per unit time, and since we will use continuous time discounting, a discount rate of r per time unit. The retailer sells items at a price of p per unit sold. The retailer seeks to maximize total discounted profit.

Contrary to the standard EOQ assumptions, we assume that when the retailer runs out of stock, some of the demand is backordered and some of the demand is lost. In particular, if the retailer has been out of stock for t time units, the total "pent-up" or backordered demand equals $\bar{D}(t)$. The function $\bar{D}(t)$ is assumed to satisfy the following properties: $\bar{D}(0)=0$, $\bar{D}(t)$ is concave and non-decreasing, and $\bar{D}(t)-\bar{D}(t)\leq(t-s)D$ for all $t>s\geq 0$. Of course if $\bar{D}(t)$ is differentiable, these conditions can be stated as $0\leq\bar{D}'(t)\leq D$, and $\bar{D}'(t)$ is non-increasing. It will be convenient to use a scaled version of the pent-up demand function, so we define $k(t) \; =\bar{D}(t)/D$. This function will therefore satisfy the following assumptions:

$k(0)=0$,

$k(t)$ is non-decreasing and concave.

Define the subdifferential $\partial k(t)$ of k at t as the set of all values x such that $k(s)\leq k(t)+x(s-t)$ for all $s\geq 0$. In particular the assumptions imply that k is almost everywhere differentiable, and that the subdifferential ∂k always exists and is decreasing in the obvious sense, i.e., if $t<s$, then $k'(t)\geq k'(s)$ for every $k'(t)\in\partial k(t)$ and $k'(s)\in\partial k(s)$.

Using standard dynamic programming arguments, it is easy to see that a cyclic policy is optimal. For convenience, we will assume that the retailer starts with no stock, and with no backorders. The retailer must then decide, first, how long to allow backorders to build up before taking delivery (t), and second, how long (x) to keep the item in stock after that. Either of these time periods can be zero: if t = 0, the classical EOQ situation results, if x = 0, the retailer follows a "stockless" policy, i.e., the retailer is basically just an order taker.

Hence during a single cycle the following events occur. At time 0 (the start of the cycle), the retailer runs out of stock. Pent-up demand accumulates during the next t time units. At that moment, an order of Q units of product is received. The pent-up demand $\bar{D}(t)$ is satisfied as soon as the delivery of product is made at time t, so the resulting inventory is $Q-\bar{D}(t)$. Note that it doesn't make sense to not satisfy all the pent-up demand at time *t*: the retailer could just order $\bar{D}(t)-Q$ more units and sell them immediately at time *t*. This would lead to a positive cash flow of $(p-c)(\bar{D}(t)-Q)$ at time *t*. If this pent-up demand is not satisfied at time *t*, some of it may turn into lost sales, and the net present value of the remainder is reduced because of discounting.

During the interval $(t, t+x)$, the inventory of $Q-\bar{D}(t)$ is drawn down at a rate of D by the regular demand, and at time $t+x$ the retailer runs out of stock again. Hence

$$x = \left(Q-\bar{D}(t)\right)/D = Q/D - k(t) \quad (1)$$

or

$$Q = D\left(x+k(t)\right). \quad (2)$$

Finally, to avoid pathological cases, we assume $p > c \geq 0$, $F > 0$, $h > 0$.

4. The Optimal Inventory Policy

We first derive an expression for the net present value associated with a policy (x,t). We assume that at time 0 the retailer runs out of stock. Then during the interval $(0,t)$ there are no cash flows. At time t, the retailer purchases $D\left(x+k(t)\right)$ units of product at discounted cost $e^{-rt}\left(cD\left(x+k(t)\right)+F\right)$. At the same time, the retailer sells $Dk(t)$ units with a discounted revenue of $e^{-rt}pDk(t)$. During the interval $(t,t+x)$ the retailer sells D units of product per time period, which leads to total discounted revenue of

$$pD\int_t^{t+x} e^{-ry}dy = p\tfrac{D}{r}e^{-rt}(1-e^{-rx}).$$

During this time interval inventory decreases from Dx to 0, which means that the discounted holding cost over this period equals

$$hD\int_t^{t+x}(t+x-y)e^{-ry}dy = h\tfrac{D}{r}e^{-rt}\left(x+\tfrac{e^{-rx}-1}{r}\right).$$

Define $\alpha = D(p-c)$, $\beta = (D/r)(p+h/r)$, and $\gamma = (D/r)(c+h/r)$. Then the discounted value of all cash flows during the first cycle can be expressed as

$$e^{-rt}\left(\alpha k(t) - F - \gamma rx + \beta(1-e^{-rx})\right).$$

Finally the total discounted value of all cash flows for an infinite horizon is the single cycle value multiplied by $1/(1-e^{-r(t+x)})$, which equals

$$\pi(x,t) = \frac{\alpha k(t)+\beta(1-e^{-rx})-\gamma rx - F}{e^{rt}-e^{-rx}}. \quad (3)$$

It is not hard to show that in maximizing (3) we can confine ourselves to the region

$$\Psi = \left\{(x,t): x\geq 0, t\geq 0, (x,t)\neq(0,0)\right\}$$

Hence we want to find

$$G^* = \sup\left\{\pi(x,t):(x,t)\in\Psi\right\} \quad (4)$$

Define the function

$$f(x,t,G) = \alpha k(t)+\beta(1-e^{-rx})-\gamma rx - F - G(e^{rt}-e^{-rx}),$$

then we can write

$$G^* = \sup\left\{G: f(x,t,G)=0, (x,t)\in\Psi\right\}. \quad (5)$$

We will exploit the fact that *f* is (jointly) concave in *x* and *t* to develop an efficient algorithm for solving problem (4).

Before we can characterize the optimal policy, we need some lemmas.

Lemma 1 $G^* < \beta-\gamma$.

Proof: First, note that for all $t\geq 0$ we have

$$\alpha k(t) \leq \alpha t = (\beta-\gamma)rt \leq (\beta-\gamma)(e^{rt}-1),$$

and for all $x \geq 0$ we have

$$\beta(1-e^{-rx})-\gamma rx \leq (\beta-\gamma)(1-e^{-rx}),$$

hence for every $(x,t)\in\Psi$ we have

$$\pi(x,t) \leq \frac{(\beta-\gamma)(e^{rt}-e^{-rx})-F}{e^{rt}-e^{-rx}} < \beta-\gamma. \qquad \square$$

Lemma 2 Let $0 < G < \beta-\gamma$. *The problem*

$$z_G = \max\left\{f(x,t,G): x\geq 0, t\geq 0\right\} \quad (6)$$

has a unique solution (x_G, t_G) *given by*

$$rx_G = \ln(\beta-G) - \ln\gamma, \quad (7)$$

$$t_G = \begin{cases} 0 \text{ if } \alpha k'(0) \leq rG, \\ \text{the unique value } t \text{ for which} \\ \quad \alpha k'(t) = rGe^{rt} \text{ otherwise.}\end{cases} \quad (8)$$

(If k is not continuously differentiable, then $k'(t)$ *should be interpreted as a suitably chosen subdifferential of k at t).*

Proof: Note that *f* is separable in *x* and *t*. It is thus straightforward to show that *f* is jointly concave in *x* and *t*, and that (7) and (8) are the first order conditions for a global maximum. $\square$

Lemma 3 Let $0 < G < \beta-\gamma$. *Then*

$$sign(z_G) = sign(G^*-G).$$

Proof: We consider the obvious three cases separately.

Case 1: $z_G > 0$. Define $\varepsilon = z_G/(e^{rt_G}-e^{-rx_G}) > 0$, then $f(x_G,t_G,G+\varepsilon) = f(x_G,t_G,G)-\varepsilon(e^{rt}-e^{-rx}) = 0$, and hence $G^* \geq \pi(x_G,t_G) = G+\varepsilon > G$.

Case 2: $z_G = 0$. Then $f(x_G,t_G,G)=0$, and since $f(0,0,G) = -F < 0$, this implies $(x_G,t_G)\neq 0$, so $(x_G,t_G)\in\Psi$ and $G^*\geq\pi(x_G,t_G)=G$. Furthermore, for arbitrary $(x',t')\in\Psi$ we know $f(x',t',G)\leq 0$, and since *f* is strictly decreasing in G on Ψ, this implies by (5) that $G^*\leq G$. We conclude that in this case $G^*=G$.

Case 3: $z_G < 0$. We need to show $G^* < G$. If $G^*\leq 0$, we are done, since the lemma assumes $G>0$. So assume $\varepsilon = \pi(x',t') > 0$ for some $(x',t')\in\Psi$. We will next show that $\Phi = \{(x,t)\in\Psi:\pi(x,t)\geq\varepsilon\}$ is compact. Note $\Phi = \{(x,t)\in\Psi: f(x,t,\varepsilon)\geq 0\}$, and since $f(0,0,\varepsilon) = -F < 0$, $\Phi = \{(x,t): f(x,t,\varepsilon)\geq 0, x\geq 0, t\geq 0\}$. Since *f* is continuous in *x* and *t*, it follows that Φ is closed. Using that *f* is separable and concave and that *f* tends to $-\infty$ when either *x* or *t* tends to ∞, it is not hard to show that Φ is bounded. Hence

$$\begin{aligned} G^* &= \sup\{\pi(x,t) : (x,t) \in \Psi\} \\ &= \sup\{\pi(x,t) : (x,t) \in \Phi\} \\ &= \pi(x^*, t^*) \end{aligned}$$

for some $(x^*, t^*) \in \Psi$. But since $f(x^*, t^*, G) \le z_G < 0$, this implies $G^* < G$.
The three cases together imply the lemma. □

Lemma 4 *Define* $k(\infty) = \lim_{t\to\infty} k(t)$, *then*

$$\alpha k(\infty) + \beta - \gamma(1 + \ln(\beta/\gamma)) > F \Leftrightarrow G^* > 0. \quad (9)$$

Proof: Note $f(x,t,0) \le \alpha k(\infty) + \beta - \gamma(1+\ln(\beta/\gamma)) - F$ for every $(x,t) \in \Psi$, and that the bound is tight (this is easily shown using the same approach as the proof of lemma 2). Hence if the condition on the left in (9) is true, there exists a $(x',t') \in \Psi$ such that $f(x',t',0) > 0$. But this implies $G^* \ge \pi(x',t') > 0$. If on the other hand the condition on the left in (9) is false, then $f(x,t,0) \le 0$ for every $(x,t) \in \Psi$, and this implies $G^* \le 0$. □

Theorem 1 *If* $\alpha k(\infty) + \beta - \gamma(1+\ln(\beta/\gamma)) > F$, *then the optimal policy* (x^*, t^*) *and* G^* *satisfy*

$$x^* = \tfrac{1}{r} \ln\left(\tfrac{\beta - G^*}{\gamma}\right), \quad (10)$$

$$\begin{cases} (\beta-\gamma)e^{-rt^*} k'(t^*) = G^* & \text{if } (\beta-\gamma)k'(0) > G^*, \\ t^* = 0 & \text{otherwise}, \end{cases} \quad (11)$$

$$\alpha k(t^*) - G^* e^{rt^*} + \beta - \gamma - F - \gamma \ln\left(\tfrac{\beta-G^*}{\gamma}\right) = 0. \quad (12)$$

If $\alpha k(\infty) + \beta - \gamma(1+\ln(\beta/\gamma)) \le F$, *then* $G^* = 0$ *(i.e., it is optimal never to purchase the item).*

Proof: Lemmas 2 and 3 imply that the values x^*, t^* and G^* satisfy equations (10), (11) and

$$\alpha k(t^*) - G^* e^{rt^*} + \beta(1 - e^{-rx^*}) + Ge^{-rx^*} - \gamma r x^* - F = 0.$$

Substituting (10) into this last equation gives (12). The last statement follows from lemma 4 and the fact that $\pi(x,t) \to -\infty$ as $t \to \infty$ for any fixed x. □

Theorem 1 gives exact optimality conditions. An immediate consequence is

Corollary 1 *Assume* $\alpha k(\infty) + \beta - \gamma(1+\ln(\beta/\gamma)) > F$, *and let* $\tilde{G}$ *satisfy the equation*

$$\tilde{G} + \gamma \ln\left(\tfrac{\beta-\tilde{G}}{\gamma}\right) = \beta - \gamma - F, \quad (13)$$

then $t^* = 0$ *if and only if* $k'(0) \le \tfrac{r}{\alpha}\tilde{G}$.

Corollary 1 gives the precise conditions under which the classical EOQ policy (of not planning to run out of stock) is optimal. Theorem 1 inspires the following very efficient algorithm to identify the optimal policy and its cost. To simplify notation, define

$$\begin{aligned} \hat{f}(t,G) &= f(\tfrac{1}{r}\ln\tfrac{\beta-G}{\gamma}, t, G) \\ &= \alpha k(t) + \gamma(\tfrac{\beta - Ge^{rt}}{\gamma} - \ln\tfrac{\beta-G}{\gamma}) - \gamma - F, \end{aligned}$$

Algorithm A:

IF $\lim_{t\to\infty} \hat{f}(t,0) \le 0$ THEN
  $t^* := \infty; G^* := 0$
ELSEIF $\hat{f}(0, (\beta-\gamma)k'(0)) < 0$ THEN
  let t^* be the unique solution to the equation

$$\hat{f}(t, (\beta-\gamma)e^{-rt}k'(t)) = 0 \quad (14)$$

  (if k is not differentiable, then $k'(t)$ is a suitably chosen subgradient of k at t);
  $G^* := (\beta-\gamma)e^{-rt^*}k'(t^*)$
ELSE
  $t^* := 0$;
  let G^* be the solution to the equation (in G)

$$\hat{f}(0,G) = 0 \quad (15)$$

END IF
$x^* := \tfrac{1}{r}\ln\tfrac{\beta-G^*}{\gamma}$

To prove the correctness of the algorithm, we need the following technical lemma.

Lemma 5 *The function* $h(t) = \hat{f}(t, (\beta-\gamma)e^{-rt}k'(t))$ *is increasing in t, and by appropriate choices of* $k'(t)$ *assumes all values between* $\hat{f}(0,(\beta-\gamma)k'(0))$ *and* $\lim_{t\to\infty}\hat{f}(t,0) \le 0$ *as t varies from 0 to* ∞.

Proof: Note that we can write

$$h(t) = h_1(t, k'(t)) + h_3(h_2(t, k'(t))), \quad (16)$$

where

$$h_1(t) = \alpha\left(k(t) - \tfrac{1-e^{-rt}}{r}k'(t)\right) - \gamma - F,$$

$$h_2(t,k'(t)) = \tfrac{1}{\gamma}\left(\beta - (\beta-\gamma)e^{-rt}k'(t)\right),$$

$$h_3(x) = \gamma(x - \ln x).$$

Note that $h_2(t,k'(t))$ is increasing in *t*, and it can assume all values between $(\beta - (\beta-\gamma)k'(0))/\gamma \ge 1$ and β/γ by choosing appropriate values for the subgradient $k'(t)$ of *k* at *t*. Note also that $h_3(x)$ is increasing in *x* for $x \ge 1$. We conclude that the second term of (16) is increasing in *t* and can assume all values between $h_3(h_2(0,k'(0))$ and $h_3(\beta/\gamma)$.
Next, we will show that $h_1(t,k'(t))$ is increasing in *t* as well. Choose $s > t \ge 0$. Note that the concavity of *k* implies that $k(s) - k(t) \ge (s-t)k'(s)$ and $k'(s) \le k'(t)$ for every choice of subgradients of *k* at *s* and *t*. Hence

$$\begin{aligned} &\tfrac{1}{\alpha}\left[h_1(s,k'(s)) - h_1(t,k'(t))\right] \\ &= k(s) + \tfrac{e^{-rs}-1}{r}k'(s) - k(t) - \tfrac{e^{-rt}-1}{r}k'(t) \\ &\ge (s-t)k'(s) + \left(\tfrac{e^{-rs}-1}{r} - \tfrac{e^{-rt}-1}{r}\right)k'(s) \\ &= \left(1 - e^{-rt}\tfrac{1-e^{-r(s-t)}}{r(s-t)}\right)(s-t)k'(s) \\ &\ge (s-t)(1-e^{-rt})k'(s) \ge 0, \end{aligned}$$

where the next to last inequality follows since $1 - e^{-x} \le x$ for all *x*. Hence $h_1(t,k'(t))$ is increasing in *t* and assumes all values between $-\gamma - F$ (inclusive) and

$\alpha k(\infty) - \gamma - F$ (the latter value possibly excluded). ▯

<u>Theorem 2</u> *Algorithm A correctly solves problem (4).*

Proof: The IF condition of the algorithm is simply the condition under which the retailer can't make a positive NPV on the item (see the last part of theorem 1), so $t^* = \infty; G^* = 0$ if it is satisfied.

Define $\tilde{G} = (\beta - \gamma)k'(0)$ then by lemma 2, $t_{\tilde{G}} = 0$ and $x_{\tilde{G}} = (\ln(\beta - \tilde{G}) - \ln\gamma)/r$, and the ELSEIF condition of the algorithm is equivalent to $z_{\tilde{G}} < 0$ which in turn is equivalent to $G^* < \tilde{G}$ by lemma 3. Hence if the ELSEIF condition is satisfied, $G^* = (\beta - \gamma)e^{-rt^*}k'(t^*)$ by (11) and substituting this into (12) gives (14). If it is not satisfied, $t^* = 0$ by (11) and substituting this into (12) gives (15). Finally, lemma 5 guarantees that (14) has a unique solution $(t^*, k'(t^*))$ whenever it needs to be solved, and it is easy to show that the same holds for (15). ▯

A few comments are in order. First, note that the optimality of planned stockouts (the ELSEIF test in the algorithm) depends only on $k'(0)$, the fraction of demand that is not immediately lost at the moment the retailer first runs out of stock. The condition is always satisfied when $k'(0) = 1$, so some stockouts will always be optimal in this case. On the other hand, when $k'(0) < 1$, the optimality of having some stockouts depends on the value of the fixed order costs *F*, and for small enough values of *F* it will be optimal to avoid stockouts altogether. When $k'(0) = 0$, the IF and ELSEIF tests in the algorithm actually coincide (note that $k'(0) = 0$ implies $k(t) \equiv 0$ since k is normalized, non-decreasing and concave), and hence either the retailer doesn't sell the item at all, or he wants to avoid stockouts altogether.

The decision on whether to sell the item at all (the IF test in the algorithm) depends only on $k(\infty)$, the maximum (normalized) fraction of demand that is not lost when the retailer has no stock for a very long time. Hence the specific form of the function *k* plays a very limited role in making these basic decisions. Of course in order to derive an optimal stocking policy in the case that planned stockouts are optimal, more information about the pent-up demand function is necessary.

Note that equation (15) can be solved efficiently by using Newton's method to find the unique root y^* of the equation $y - \ln y = 1 + F/\gamma$ (a convenient starting point is $y = 1 + (2F/\gamma)^{1/2}$, the solution to the approximate equation obtained from $\ln y \approx y - 1 + \frac{1}{2}(y-1)^2$), and then calculating $G^* = \beta - \gamma y^*$.

Equation (14) can of course easily be solved using bisection. When *k* is differentiable, the conditions under which the LHS of (14) is concave involve the third derivative of *k*, so a simple implementation of Newton's method is not recommended.

When *k* is piece-wise linear, the LHS of (14) is convex in *t* between breakpoints of $k(t)$, while it is concave in $k'(t)$ at the breakpoints. Hence once the breakpoint or segment which contains t^* has been identified, a couple of Newton-Raphson steps will quickly yield the solution with great accuracy.

5. Estimating the backorder function

In this section we turn to the issue of implementing the model in practice. In particular, one needs to estimate the backorder function. We will sketch a number of different approaches to this estimation problem that could be useful in various circumstances.

Assume that a web retailer uses a web site that provides potential customers with information on how long it will be until the retailer can ship the product. Customers then either place an order for one unit of the product (which will be shipped as soon as it becomes available for shipping), or they exit without ordering. The retailer collects the following information related to a particular product of interest. Over a period of *M* days, the retailer responded to n_i inquiries that the shipping delay would be *i* days, and these inquires resulted in s_i sales, $i = 0,1,...,N$. Note that even when the product is available for shipment, some potential customers still don't place an order. From this data, we wish to estimate the underlying demand rate *D* and the (scaled) pent-up demand function $k(t)$.

We assume that inquiries are generated according to some stationary process with a rate of μ per day. Let the random variable S_{ij} denote the number of units sold resulting from the *j*-th inquiry that occurs when the shipping delay is *i* days. We assume that $\{S_{ij} : j = 1,2,...\}$ are iid fandom variables with mean τ_i. If estimates $\hat{\tau}_i$ for τ_i $(i = 0,...,N)$ are available, one can estimate the scaled pen-up demand function by linear interpolation on these estimates:

$$\widehat{k(\ell)} = \frac{1}{\hat{\tau}_0}\sum_{i=1}^{\ell}\hat{\tau}_i,\ \ell = 0,1,2,...,N.$$

The straightforward approach to estimating the average number of units sold per inquiry as a function of the number of days until shipment is of course

$$\hat{\tau}_i^{naive} = s_i / n_i.$$

The problem with this approach is that the resulting scaled pent-up demand function need not be concave, since it is quite possible that in the available data $s_i / n_i < s_j / n_j$ for some $j > i$. Of course it is counter-intuitive (to say the least) that a longer shipping delay would lead to a higher expected number of units sold per inquiry, so we will require that our estimates satisfy

$$0 \le \hat{\tau}_N \le \hat{\tau}_{N-1} \le ... \le \hat{\tau}_0 \le 1.$$

In the remainder of this section we discuss several ways to handle this estimation problem.

5.1 Parameter Estimation from a Specific Functional Form

The exponential pent-up demand function is given by

$$\bar{D}(t) = \tfrac{\theta D}{\kappa}(1 - e^{-\kappa t}).$$

This form arises if we assume that pent-up demand satisfies the differential equation $\bar{D}'(t) = \theta D - \kappa \bar{D}(t)$ with starting condition $\bar{D}(0) = 0$. Here κ is the rate at which pent-up demand dissipates (e.g. because it is satisfied by a competitor or because substitutes are acquired by consumers), and θ is the fraction of demand that is not immediately lost when the retailer runs out of stock. So there is exponential decay of pent-up demand, comparable to radioactive decay.

It is not hard to show that the exponential pent-up demand function satisfies our assumptions if $\kappa > 0$ and $0 \le \theta \le 1$. To be complete, we have $k(t) = \theta(1 - e^{-\kappa t}) / \kappa$, $k(\infty) = \theta / \kappa$, $k'(t) = \theta e^{-\kappa t}$ and $k'(0) = \theta$.

A simple approach to estimating values for κ and θ is to minimize

$$\sum_{\ell=1}^{N} \left(\frac{n_0}{s_0} \sum_{i=1}^{\ell} \frac{s_i}{n_i} - \frac{\theta}{\kappa}(1 - e^{-k\ell}) \right)^2,$$

the squared sum of errors between the values of k calculated using the naïve estimates $\hat{\tau}_i^{naive}$ and the values of k calculated with the specified function.

An alternative specific functional form is the logarithmic function given by

$$\bar{D}(t) = \tfrac{\theta D}{\kappa} \ln(1 + \kappa t).$$

It is easy to verify that this implies $k(t) = (\theta / \kappa) \ln(1 + \kappa t)$, $k(\infty) = \infty$, $k'(t) = \theta / (1 + \kappa t)$ and $k'(0) = \theta$.

5.2 The Customer Utility Approach

Define the average residual utility $\bar{R} = \bar{U} - p$, where $\bar{U}$ is the utility that the average customer derives from the product if it is immediately available, and p is the purchase price. In addition, a customer has a disutility for waiting for the item, so let $b(x)$ be the backorder disutility, if the customer has to wait x time units. Then a customer decides to purchase the item if and only if $\bar{R} - b(x) + \varepsilon > 0$, where ε is a customer specific random variable with mean 0, and x is the time the customer has to wait for the item. For specific functional forms of $b(x)$ one can use probit or logit regression analysis to estimate $\Pr\{\bar{R} - b(x) + \varepsilon > 0\}$, and the normalized pent-up demand function is calculated as

$$k(t) = \int_0^t \Pr\left(\varepsilon > b(x) - \bar{R}\right) dx$$

Clearly, since $\Pr(\varepsilon > b(x) - \bar{R})$ is non-increasing in x and cannot exceed the value of 1, the resulting function $k(t)$ satisfies the standard assumptions of section 2.

If the function $b(x)$ is assumed to be linear, then probit estimation leads to

$$k(t) = \tfrac{1}{\kappa}[\Phi^1(-\nu) - \Phi^1(\kappa t - \nu)],$$
$$k'(t) = \Phi(\kappa t - \nu),\ k'(0) = \Phi(-\nu),$$
$$k(\infty) = \tfrac{1}{\kappa}\Phi^1(-\nu),$$

where Φ denotes the standard normal cumulative distribution function, and $\Phi^1(y) = \Phi'(y) - y(1 - \Phi(y))$ is the familiar standard normal linear loss function (see e.g. [23, page 458]), and ν and κ are parameters obtained from the probit regression. Clearly, the estimated parameter κ needs to be positive for this approach to make sense.

If the function $b(x)$ is assumed to be linear, then logit estimation leads to

$$k(t) = \tfrac{1}{\kappa}[\ln(1 + e^{\nu}) - \ln(1 + e^{\nu - \kappa t}]$$
$$k'(t) = 1/(1 + e^{\kappa t - \nu})$$
$$k(\infty) = \tfrac{1}{\kappa}\ln(1 + e^{\nu})$$

where ν and κ are parameters obtained from the logit regression and again we need κ to be positive.

The utility approach has as additional advantages that on can correct for price changes (and possibly additional environmental factors) in the estimation, as well as say something about the impact of price changes (and possibly additional environmental factors) on optimal policy, profit, etc.

5.3 Piecewise Linear Approximation Using Isotonic Regression

We can use a constrained maximum-likelihood approach to obtain estimates $\hat{\tau}_i$ for the values τ_i that satisfy the constraints $0 \le \hat{\tau}_N \le \hat{\tau}_{N-1} \le ... \le \hat{\tau}_0 \le 1$. This amounts to finding the isotonic regression of the points $(i, s_i / n_i)$ with weights n_i (see [19, p. 32]). The result is a piece-wise linear approximation with breakpoints $t_0 = 0 < t_1 < \cdots < t_n$, and slopes $1 \ge \theta_0 > \theta_1 > \cdots > \theta_n \ge 0$. If we define $j(t) = \max\{i : t \ge t_i\}$ for any $t \ge 0$, then

$$k(t) = \sum_{i=0}^{j(t)-1} (t_{i+1} - t_i)\theta_{j(t)},$$
$$k'(t) = \theta_{j(t)} \text{ if } t \ne t_{j(t)},$$
$$\partial k(t) = [\theta_{j(t)}, \theta_{j(t)-1}] \text{ if } t = t_{j(t)}.$$

6. Implications

For on-line retailers, this is an ideal opportunity to examine their inventory management policies in the context of customer behavior. Most sites already have the technology to collect extremely detailed data about the real-time shopping behavior of visitors. The policy algorithm and estimation techniques presented here lend themselves to automation quite easily, in the same manner that simple EOQ and stocking level calculations are standard routines in many existing enterprise software solutions. They can be used directly to change the stocking policy as well as parameters on an automatic basis, or more realistically, be used to suggest changes to a decision-maker.

In case of stockouts, many savvy retailers list similar items to promote substitution, and also provide the restocking date to allow the customer to wait. Thus, substitution and waiting by customers allows them to attract a larger market (or at a lower cost) than if they had a fixed assortment of stocked items. Hence there is a growing trend to allow customers to make their own substitution/wait/leave decisions.

Whatever be the (arguable) benefits of this policy in the “bricks” world of retailing, there are many more benefits for the “clicks” world of on-line retailing. In many cases customers are conditioned to a shipping delay, and may be willing to wait for a short time in order to get their first choice. The on-line world provides a wealth of tracking information about customer preferences, stockout and substitution behavior. In addition, it is possible to ensure that when stocks arrive, all backlogs are instantaneously cleared. To these organizations an understanding of customer waiting behavior is critical in order to implement profitable intermittent stocking policies - clearly a strong argument for systematic analysis of stockout behavior (as well as substitution behavior) in the on-line retailing industry.

6.1 Enterprise Resource Planning (ERP) Software

Existing ERP software today has pre-programmed routines that allow calculation of inventory parameters, e.g. safety stock (using previous order data), reorder points, and order quantities. We have included the algorithms that are implementable as additional routines to find out the optimal stocking policy for items of interest, as well as an algorithm to calculate maximum likelihood estimators (in linear running time). This allows interested firms to directly implement the policies that we describe here.

Just as most stocking level calculation procedures are used not to directly change values but to suggest changes to a decision maker, we envisage that the additional algorithms presented here can be used to suggest changes in policy to the decision maker. This allows for a degree of flexibility that is currently not available in stocking policy decisions.

6.2 Other Considerations

The periodic availability policies that we have described have some other implications as well. Since we decrease the average inventory holding time in such policies (as compared to a pure EOQ policy), this has an impact on accounting measures, like Return on Equity. For online firms this may be an important consideration if their performance, incentives and even existence depends upon common accounting measures of profitability.

Since each order is larger then the pure EOQ policy order, there may be quantity discounts or shipping economies that kick in if a periodic availability policy is adopted. While this is not explicitly modeled in this paper, it is plausible that quantity discounts or shipping economies makes such a policy even more attractive.

While we have looked at the inventory policy that can capitalize on the increased visibility of customer behavior, it is equally important to have business processes that can support it. For instance, it is necessary to have logistics functions that can integrate well with a regular activity of filling backorders. Online business models have to be supported by all functional areas in tandem. It is certainly not enough to merely apply a periodic availability inventory policy at the warehouse.

7. Conclusion

Our model is just a start however. In the future, sophisticated on-line retailers will use their customer data to model customer behavior in the face of stockouts, price changes, available substitutes in the retailer's collection, and possibly competitive actions. Hence there will be an increasing need for models that integrate the traditional emphasis on replenishment decisions with other aspects of logistics systems.

In this paper we start from a different perspective on stockouts and the cost of backorders. We have extended the EOQ model in an intuitive manner which shows the specific conditions under which the EOQ model is optimal, and conditions under which EOQ is not optimal (or even profitable) but periodic availability is optimal and profitable. This represents a simple but powerful extension of this classic inventory model.

Potential extensions to this research include considering many items to be stocked at one location, the issue of substitution between multiple items, and the case of competition between retailers using different stocking policies. Each of these occurs in practice, and is relevant to both research and industry.

There are important applications to the field of on-line retail inventory, and there are implications for all inventory systems, including supply chains. In any situation where a stockout causes a partial non-linear loss of sales we assert that it is important to asses the profitability of periodic availability.

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