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Definable Strategies and Equilibria for Games

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Abstract

We consider logical languages which express strategies when games are given in Normal form, Extensive form and with a Symbolic representation. Many practical applications in Electronic Commerce can be formulated as games and the relationship between complexity and strategies is an important issue. Descriptive complexity allows to directly link the Logic used to define a strategy with its computational complexity. We give examples based on the the Prisoner's dilemma in extensive form and the North-East game in a symbolic representation. We then study the computational complexity for computing the Nash equilibria under various restrictions and we prove that for games in symbolic form, it can reach arbitrary levels of the polynomial hierarchy.

1 Introduction

Many natural problems in decision theory can be formulated as games and the study of effective solutions of these games has led Game theory and Computer Science to converge on similar problems [2]. In Electronic Commerce, agents follow various strategies to achieve a global goal and the study of effective strategies becomes important. In mechanism design such as combinatorial auctions, there is direct connection between the game representation, the complexity of the strategies and equilibria and the various approximations. We concentrate on two specific aspects of games in extensive and symbolic forms. What happens to the Nash equilibria when we restrict ourselves to strategies computable in polynomial time or within other complexity bounds? Are the equilibria easier to compute?

One of the fundamental observation of [8] is that natural restrictions on strategies based on the size of automata change the equilibria of the game. We wish to study how restrictions on definable strategies change the equilibria and the difficulty to compute them.

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Our basic tool is descriptive complexity [5] which allows to directly link the logical language used to define a strategy with its computational complexity. For games in extensive forms, we need to define a class of finite structures for which a strategy appears as a global function or a query. We take the classical repeated Prisoner's dilemma as an example and show how uniform recursive definition characterize strategies computable in polynomial time. We then extend the notion to games in symbolic form, such as the game North-East.

We then recall the basic techniques for computing equilibria and prove a standard result : for games in symbolic form, the complexity of the equilibria can be as high as any level of the polynomial hierarchy. In section 2, we describe the definable strategies for games in Normal, Extensive and Symbolic form. In section 3, we study the complexity of the Equilibria.

2 Definable strategies

There are two classical representations [7] of two-person games which also generalize to N-person. The *Normal form* where matrices A, B with values in $\mathbb{Z}$ are given such that $a(i, j)$ (resp. $b(i, j)$) is the gain obtained by Player I (resp. Player II) when I takes the decision i and II the decision j. The *Extensive form* where a tree whose nodes are colored (the Information sets of the players) and whose edges are labeled by the player's decisions.

Yet in many games (checkers, chess or North-East used in this paper), the extensive form is too large and the game is only known with rules, expressed in some formal language. Such rules define a class of models but also underline the notion of definable strategies, which we will study in detail.

2.1 Normal and Extensive forms

For simplicity, suppose A and B are square (n, n) matrices and let us call E the vector $(1, 1, ...1)$ of size n. In the case of normal forms, a (mixed) strategy for I (resp. II) is a vector x (resp. y) of size n.

The strategy we may want to consider for this Normal form is the one such that :

$$Max \;\; x^{\top}(Ay) \text{ subject to } Ex = 1 \text{ and } x \geq 0.$$

It is definable by a linear program and the language considered is linear algebra. In the case of a game in Extensive form, there are several possible definitions of strategies. Let $C_{i,j}$ be the node color j (or information set j) of player i and let $C = \bigcup_{i,j} C_{i,j}$. Let $L_{i,j}$ be the set of decisions of player i on a node colored[1] j.

- A deterministic strategy [1] μ for player i is a function $C^* \rightarrow L_{i,j}$ which determines given a history of node colors $C^1, ...C^k$ where $C^k = C_{i,j}$ a decision $l \in L_{i,j}$.
- A behavior strategy [4] π for player i is a function $C_{i,j} \rightarrow L_{i,j}$. It describes the decision the player i takes on each node colored j.
- A mixed strategy in the normal form of the game associated with the extensive form.

In the first two cases, randomized strategies assign a probability distribution on $L_{i,j}$. Under the hypothesis of *perfect recall* and *perfect history*, the first two definitions coincide. The normal form can be exponential in the size of the extensive form, so that mixed strategies are less interesting. It is proved in [4], that any mixed strategy is equivalent from the game's point of view to a behavioral strategy.

Example 1: Prisoner's dilemma in Normal form and the two-round repeated version in extensive form.

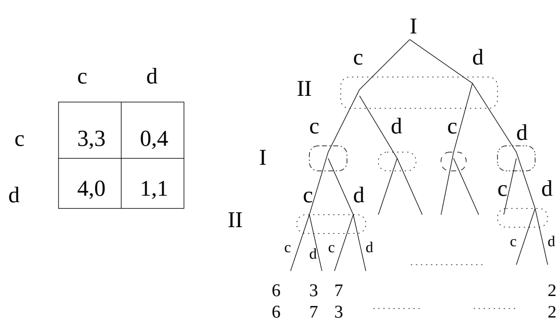


Figure 1 : in the first round of the extensive form, I plays first then II plays but does not see his opponent's move. In the second round they both see the moves of the first round. If the players play on the left branch, they both collaborate and their gain is 6.

[1] All nodes colored by the same color have the same decision set.

In this example, $L_{1,j} = L_{2,j} = \{c, d\}$ for collaborate or defect. A classical strategy for player I is: *collaborate unless player II played d in the last move.* Let us set the basic notations for definability and show how to effectively define this strategy.

Let $\mathcal{U}_k = (\{1, .., k\}, \{c, d\}, <, h_1, h_2, k, c, d)$ be a structure with two domains ($\{1, .., k\}$ and $\{c, d\}$), distinguished elements k, c, d, the order relation $<$ on $\{1, .., k\}$ and two functions $h_1, h_2 : \{1, .., k\} \rightarrow \{c, d\}$ representing the history of players I and II, i.e. $h_1(i) = c$ if the decision of I at round i is c. Let K be the class of structures $\mathcal{U}_k$ and let us define a global function[2] $\mu()$ with values in $\{c, d\}$ representing the strategy of I at round $k + 1$.

A **definable strategy** on K is a fixed term or expression which defines the global function, i.e. the strategy on each structure $\mathcal{U}_k \in K$. For example, the expression:

$$\mu_1() \Leftarrow if \;\; h_2(k) = d \; then \; d \; else \; c$$

defines the previous strategy with a term t in the language[3] of K. It is a classical observation [3, 9, 5] of descriptive complexity that a global function is P-computable, i.e. computable in polynomial time iff it is defined by a recursive term. It allows to restrict strategies based on computational bounds and not only on the number of states of automata [8].

Complexity classes such as NC, L, NL, P, NP, PH have similar logical characterizations so that for each of these classes, we can find a logic $\mathcal{L}$ such that μ is definable in $\mathcal{L}$ iff it is computable in the class. In practical E-commerce situations, it allows to restrict agents to feasible strategies by only considering these logical languages.

2.2 Symbolic forms

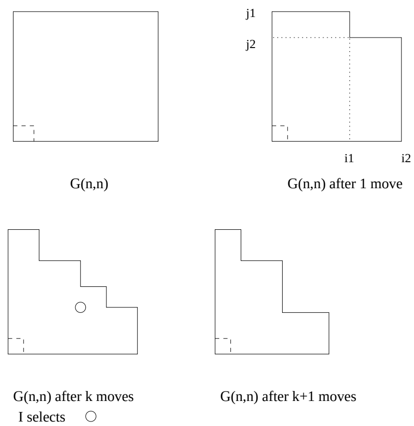
For many games, the extensive form is too large and we can only have a symbolic representation of the state of the game. A state gives a complete representation of the game at a given time and a transition is a binary relation between states.

Example 2: North-East. Start with a (n, n) grid $G(n, n)$ viewed as a square and I and II select alternatively a valid point (i, j) where $1 \leq i, j \leq n$. The North-East corner of (i, j) is removed after each play and the new valid region shrinks. The player left with the move $(1, 1)$ loses.

[2] A global function on a class K is a function wich associates with any structure $\mathcal{U}_k \in K$ a function on that structure.

[3] The language of a class K of structures of the same signature has constant symbols for the distinguished elements is closed by function composition, equality and definition by cases (if then else)

Figure 2 : The initial square of North-East, a state after the first move of I, a state after k and $k+1$ moves.



The state of the game is a sequence $i_1, j_1 ..., i_k, j_k$ such that: $i_l < i_{l+1}$ and $j_l > j_{l+1}$ for $l = 1, ...k$ and let S be the set of states. It determines the North-East border of the partial grid, i.e. the line $(0, j_1), (i_1, j_1), (i_1, j_2), (i_2, j_2) .... (i_k, j_k), (i_k, 0)$. The extensive form is a tree of degree n^2 whose size is n^{2k} after k moves.

In the case of perfect information, each state is a color (Information set) for each player. A strategy μ is a function $S^* \rightarrow \{1, ..., n\} \times \{1, ..., n\}$ such that $\mu(S^1, ..., S^k) = (i, j)$ such that the point (i, j) is inside the North-East border. This condition can be written as the disjunction of two linear constraints. A Markovian strategy is a function π: $S \rightarrow \{1, ..., n\} \times \{1, ..., n\}$ which does not take the history into account.

Let $\mathcal{V}_k = (\{1, .., k\}, \{1, .., n\}, <, h_1, h_2)$ be a structure with two domains ($\{1, .., k\}$ and $\{1, ..., n\}$), the order relation $<$ on $\{1, .., k\}$ and $\{1, .., n\}$ and two functions $h_1, h_2 : \{1, .., k\} \rightarrow \{1, .., n\} \times \{1, .., n\}$ representing the history of players I and II, i.e. $h_1(l) = (i, j)$ if the decision of I at round l is (i, j). The state of the game is captured by the two functions h_1, h_2. Let K' be the class of structures $\mathcal{V}_k$. As before, we can capture P computable strategies by restricting the language of definition of a strategy μ.

If we consider global primitive recursive definitions on the class K' as a logic for defining global functions, we will obtain L (or $LOGSPACE$) computable functions. We can also consider global relations, in particular the graph of the strategies. We obtain the classical first-order hierarchies based on Σ_k and Π_k formulas, all inside the class L.

3 Definable Equilibria

If strategies are restricted, the equilibria change. What can be said about the computation of the equilibria? In the case of games in Normal form, the equilibria are found by the Lemke-Howson algorithm [6], which solves a Complementary Linear program. The exact complexity of the problem is not known.

In the case of games in Extensive normal form, the equilibria can be found [10, 4] efficiently with Lemke's algorithm (a version of Lemke-Howson) by considering only behavioral strategies and the so called sequence forms. The dimension of the LCP program is linear in the size of the tree but the general complexity is not known either. For symbolic forms, the complexity of finding an equilibrium can be more precisely measured and we can prove:

Theorem 1. *There is a game in symbolic form for which the complexity of computing an equilibrium is at any level of the polynomial hierarchy.*

The argument reduces any problem of the form $SAT^{SAT^{SAT}}$ to an equilibrium in a game.

Note: The full paper is available from the CD of conference proceedings.

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