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A Time Integration Method of Approximate Particular Solutions for Nonlinear Ordinary Differential Equations

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A TIME INTEGRATION METHOD OF APPROXIMATE PARTICULAR SOLUTIONS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

by

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ABSTRACT

We consider a time-dependent method which is coupled with the method of approximate particular solutions (MAPS) of Delta-shaped basis functions and the method of fundamental solutions (MFS) to solve nonlinear ordinary differential equations. Firstly, we convert a nonlinear problem into a sequence of time-dependent non-homogeneous boundary value problems through a fictitious time integration method. The superposition principle is applied to split the numerical solution at each time step into an approximate particular solution and a homogeneous solution. Delta-shaped basis functions are used to provide an approximation of the source function at each time step. The purpose of this is to allow a convenient derivation of an approximate particular solution. The corresponding homogeneous boundary value problem is solved using the method of fundamental solutions. Numerical results support the accuracy and validity of this computational method.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure

LIST OF TABLES

Table

LIST OF ABBREVIATIONS

MAPS	-	Method of Approximate Particular Solutions
MFS	-	Method of Fundamental Solutions
FTIM	-	Fictitious Time Integration Method
FEM	-	Finite Element Method
FVM	-	Finite Volume Method
FDM	-	Finite Difference Method

NOTATION AND GLOSSARY

General Usage and Terminology

The notation used in this text represents fairly standard mathematical and computational usage. In many cases these fields tend to use different preferred notation to indicate the same concept, and these have been reconciled to the extent possible, given the interdisciplinary nature of the material. In particular, the notation for partial derivatives varies extensively, and the notation used is chosen for stylistic convenience based on the application. While it would be convenient to utilize a standard nomenclature for this important symbol, the many alternatives currently in the published literature will continue to be utilized.

The blackboard fonts are used to denote standard sets of numbers: $\mathbb{R}$ for the field of real numbers, $\mathbb{C}$ for the complex field, $\mathbb{Z}$ for the integers, and $\mathbb{Q}$ for the rationals. The capital letters, $A, B, \cdots$ are used to denote matrices, including capital greek letters, e.g., Λ for a diagnonal matrix. Functions which are denoted in boldface type typically represent vector valued functions, and real valued functions usually are set in lower case roman or greek letters. Caligraphic letters, e.g., Ω, denote a bounded domain, $\partial\Omega$ denoting the boundaries of the Ω, or $\mathcal{F}$ denoting a general function space. Lower case letters such as i, j, k, l, m, n and sometimes p and d are used to denote indices.

Vectors are typset in square brackets, e.g., $[\cdot]$, and matrices are typeset in parenthesese, e.g., $(\cdot)$. In general the norms are typeset using double pairs of lines, e.g., $||\cdot||$, and the abolute value of numbers is denoted using a single pairs of lines, e.g., $|\cdot|$. Single pairs of lines around matrices indicates the determinant of the matrix.

Chapter 1

INTRODUCTION

Nonlinear differential equations have been extensively used to mathematically model many of the interesting and important phenomena that are observed in areas of science and technology. They are inspired by problems which arise in diverse fields such as economics, biology, fluid dynamics, physics, engineering and materials science. Nonlinear differential equations are challenging problems since the general classes have no known forms of exact solutions. Numerical approximation is an alternative approach for solving these problems.

Over several decades, there has been increased research in developing efficient computational methods for finding numerical solutions of differential equations. These computational methods are either mesh-free or mesh-based methods.

Mesh-free methods use a set of nodes scattered in the domain or on the boundary of the domain of a problem. They include methods such as the radial basis functions (RBFs) method, the moving least squares (MLS) method, the method of fundamental solutions (MFS), the method of approximate fundamental solutions (MAFS), the element free Galerkin method (EFG), and the finite point (FP) method. [5, 6, 7, 24, 27]. Alternatively, the mesh-based methods, also known as the traditional methods, include finite-element methods (FEM), finite-difference methods (FDM) and finite-volume methods (FVM) [11, 21]. These methods require the interaction and connection of each node with its neighbors inside the computational domain or on the boundary.

In recent decades, meshless methods have been developed for solving nonlinear partial differential equations [1, 8, 26]. These techniques aim to produce accurate and efficient algorithms with least amount of time spent on its domain or boundary discretization.

In this paper, we study a general class of nonlinear differential equations subjected to boundary conditions as follows:

$$-u''(x) = H(x,u,u'), \qquad x \in \Omega, \tag{1.1}$$

$$u(x) = f(x), \qquad x \in \partial\Omega, \tag{1.2}$$

where H is a known function of x,u or u', f is a known function of x , Ω is a bounded domain and $\partial\Omega$ is the boundary of Ω.

Here, we implement a numerical method to approximate the solution to the nonlinear problem iteratively. Firstly, by incorporating a fictitious time function and numerical integration [9, 26], we transform a given nonlinear differential equation into a sequence of time-dependent non-homogeneous equations. We then apply the principle of superposition at each time step to split the numerical solution into an approximate particular solution and a homogeneous solution. Delta-shaped basis functions are then used to provide an accurate approximation of the source function at each time step. This leads to also a convenient derivation of the approximate particular solution of the non-homogeneous equation. The method of fundamental solutions [10] is used to find an approximate solution to the corresponding homogeneous problem.

This paper is organized as follows. In section 2, we apply the fictitious time integration method to convert the nonlinear ordinary differential equation into a sequence of time dependent non-homogeneous differential equations [8, 9, 17, 26]. In Section 3, an approximate particular solution at each time step is derived by using Delta-shaped basis. The method of fundamental solutions is provided in Section 4 to approximate the homogeneous problems and its effectiveness is shown coupled with MAPS for non-homogeneous problems. In Section 5, we provide numerical results for various nonlinearities to illustrate the effectiveness of the method. A conclusion is given in Section 6.

Chapter 2

CONVERSION USING FICTITIOUS TIME INTEGRATION METHOD

A fictitious time integration method (FTIM) was originally developed by Liu and Alturi in 2008 to solve large systems of nonlinear algebraic equations [17] where a fictitious time is introduced to convert the original nonlinear system of algebraic equations into evolutionary ordinary differential equations. Since then, the FTIM has been applied to solve mixed complementarity problems with applications to nonlinear optimization [14], nonlinear obstacle problems [18], two-dimensional quasilinear elliptic boundary value problems [12], discretized inverse sturm-louiville problems, for specified eigenvalues [15], m-point boundary value problems [13], the Fredholm Integral equation [16], nonlinear algebraic equations with multiple solutions [19], and non-linear Poisson-type boundary value problems [8, 26].

We apply this method in our solution process. Firstly, a fictitious time is introduced to convert the nonlinear differential equation problem (1.1) and (1.2) into a time-dependent differential equation. Secondly, a numerical integration is performed on the time-dependent problem to obtain a sequence of linear non-homogeneous boundary value problems. A time-dependent function $w(x,t)$ is defined as:

$$w(x,t) = u(x,t)q(t), \tag{2.1}$$

or equivalently written as

$$u(x,t) = \frac{w(x,t)}{q(t)}, \tag{2.2}$$

where t is a fictitious time and $q(t)$ is the time function

$$q(t) = (1+t)^{\beta}, \tag{2.3}$$

with $0 < \beta \leq 1$, a parameter for the convergence of the time integration method.

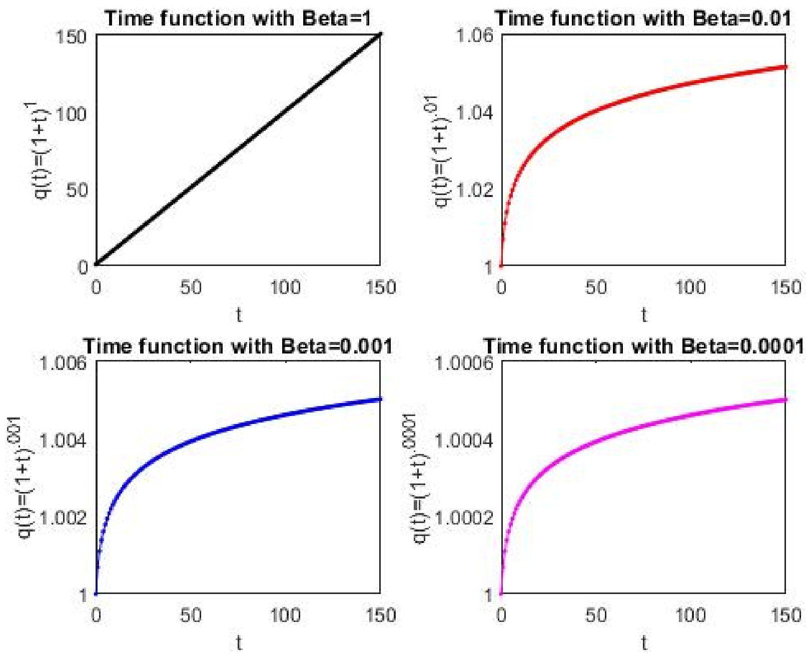

Figure 2.1: *Plot of the time function $q(t) = (1+t)^{\beta}$ with various β.*

Substituting Eq. (2.2) into Eq. (1.1), we obtain

$$-\frac{d^2}{dx^2}\left(\frac{w(x,t)}{q(t)}\right) = H\left(x, \frac{w(x,t)}{q(t)}, \frac{w_x(x,t)}{q(t)}\right). \tag{2.4}$$

Differentiating Eq. (2.1) with respect to t and utilizing Eqs. (2.2) and (2.4), we have

$$\frac{dw}{dt} = u\frac{dq}{dt},$$

$$\frac{dw}{dt} - \frac{d^2}{dx^2}(\frac{w}{q}) = \frac{w}{q}\frac{dq}{dt} + H(x, \frac{w}{q}, \frac{w_x}{q}),$$

$$\frac{1}{q}\frac{dw}{dt} - \frac{w}{q^2}\frac{dq}{dt} - \frac{1}{q}\frac{d^2}{dx^2}\left(\frac{w}{q}\right) = \frac{1}{q}H\left(x, \frac{w}{q}, \frac{w_x}{q}\right). \tag{2.5}$$

Since

$$\frac{d}{dt}\left(\frac{w}{q}\right) = \frac{1}{q}\frac{dw}{dt} - \frac{w}{q^2}\frac{dq}{dt},$$

the Eq. (2.5) is reduced to

$$\frac{du}{dt} - \frac{1}{q}\frac{d^2u}{dx^2} = \frac{1}{q}H(x, u, u_x). \tag{2.6}$$

We then use forward Euler integration technique to discretize the time component of the ordinary differential equation defined in Eq. (2.6) with du/dt being approximated as follows:

$$\frac{du}{dt} \approx \frac{u^{I+1} - u^{I}}{\triangle t} = \frac{u(x,t^{I+1}) - u(x,t^{I})}{\triangle t}$$

.

Therefore, the Eq. (2.6) becomes

$$\frac{u^{I+1} - u^{I}}{\triangle t} - \frac{1}{q}\frac{d^2}{dx^2}u^{I+1} = \frac{1}{q}H(x,u^{I},u_x^{I}),$$

which can be written as

$$\frac{q^{I+1}u^{I+1}}{\triangle t} - \frac{q^{I}u^{I}}{\triangle t} - \frac{d^2}{dx^2}u^{I+1} = H(x,u^{I},u_x^{I})$$

.

Now, we consider solving the following problems

$$\frac{d^2}{dx^2}u^{I+1}(x,t) - \frac{q^{I+1}(t)}{\triangle t}u^{I+1}(x,t) = -\frac{q^{I}(t)}{\triangle t}u^{I}(x,t) - H(x,u^{I},u_x^{I}), \quad x \in \Omega, \tag{2.7}$$

$$u^{I+1}(x,t) = f(x), \qquad x \in \partial\Omega. \tag{2.8}$$

for $I = 0,1,2,...$, where $u^{I}(x,t) = u(x,t^{I})$, $q^{I}(t) = q(I\triangle t)$, $t^{I} = I\triangle t$ and $\triangle t$ is the step size for each time step.

The numerical procedure for solving Eqs. (2.7) and (2.8) begins with a chosen initial value of $u^{0}(x,t)$. During the solution process, Eq. (2.7) is integrated from $t = 0$ to some final time $t^{n} = n\triangle t$. The inequality

$$\| u^{I+1}(x,t) - u^{I}(x,t) \|_{\infty} < \varepsilon$$

is used as the criterion to stop the iteration where $\| \bullet \|_{\infty}$ is the maximum norm and ε is a small positive number.

Chapter 3

APPROXIMATING PARTICULAR SOLUTIONS

3.1 Superposition Principle

In chapter 2, we show that at each time step our original problem takes the form of Eqs. (2.7) - (2.8). In order to solve the problem, we apply the superposition principle to split the solution $u^{I+1}(x,t)$ as

$$u^{I+1}(x,t) = u_p^{I+1}(x,t) + u_h^{I+1}(x,t),$$

where $u_p^{I+1}(x,t)$ is the approximate particular solution satisfying

$$\frac{d^2}{dx^2}u_p^{I+1}(x,t) - \frac{q^{I+1}(t)}{\triangle t}u_p^{I+1}(x,t) = -\frac{q^I(t)}{\triangle t}u^I(x,t) - H(x,u^I,u_x^I), \tag{3.1}$$

and $u_h^{I+1}(x,t)$ is the corresponding homogeneous solution satisfying

$$\frac{d^2}{dx^2}u_h^{I+1}(x,t) - \frac{q^{I+1}(t)}{\triangle t}u_h^{I+1}(x,t) = 0, \quad x \in \Omega, \tag{3.2}$$

$$u_h^{I+1}(x,t) = f(x,t) - u_p^{I+1}(x,t), \qquad x \in \partial\Omega, \tag{3.3}$$

for $I = 0,1,2,3,....$

3.2 Delta-shaped Basis and Source Function Approximation

Now it is necessary for us to find an accurate approximation of the source function in Eq. (3.1) in order to derive an accurate approximation to the particular solution $u_p^{I+1}(x,t)$ for each time step I.

Let's consider the right hand side of Eq. (3.1),

$$h^I(x,t) = -\frac{q^I(t)}{\triangle t}u^I(x,t) - H(x,u^I,u_x^I). \tag{3.4}$$

Our first objective here is to obtain a good approximation of the source function. This is to achieve an easy derivation of a closed form approximate particular solution $u_p^{I+1}(x,t)$. One-dimensional Delta-shaped basis function $I_{M,\chi}(x;\xi)$ are used for this purpose because of their specific characteristics [22, 23, 25]. A one-dimensional Delta-shaped basis function is given as

$$I_{M,\chi}(x;\xi) = \sum_{n=1}^{M} c_n(\xi)\varphi_n(x),$$

where

$$c_n(\xi) = r_n(M,\chi)\varphi_n(\xi), \tag{3.5}$$

and ξ is the center of the basis function, and the φ_n are the eigenfunctions of the following Sturm- Liouville problem on the interval [-1,1]:

$$-\varphi_n'' = \lambda_n \varphi_n, \tag{3.6}$$

$$\varphi_n(-1) = \varphi_n(1) = 0. \tag{3.7}$$

More specifically, the eigenfunctions and eigenvalues of the problem (3.6)-(3.7) are

$$\varphi_n(x) = sin\left(\frac{n\pi(x+1)}{2}\right),$$

and

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2,$$

for $n = 1,2,3,....$

The regularizing coefficients $r_n(M,\chi)$ are determined by the Riesz regularization technique [23] with

$$\begin{aligned} r_n(M,\chi) &= \left[1 - \left(\frac{\lambda_n}{\lambda_{M+1}}\right)^2\right]^{\chi} \\ &= \left[1 - \left(\frac{n}{M+1}\right)^2\right]^{\chi}, \end{aligned}$$

where the parameters M and χ are positive integers with M playing the role of scaling and χ playing the role of regularization. The coupled parameters are taken as $\chi = 4,6,9,14,22$ for $M = 10,20,30,50,100$ respectively.

Now we use a linear combination of the one-dimensional Delta-shaped basis functions to approximate Eq. (3.4) for each time step sampled at the scattered data points $\{x_i\}_{i=1}^{N}$ in Ω. To obtain accurate approximation, we use Delta-shaped basis functions of two different shapes.

Now, let $\tilde{h}^I(x_i,t)$ denote the approximation of h^I in Eq. (3.4) at each data point x_i which takes the form

$$\tilde{h}^I(x_i,t) = \sum_{j=1}^{R_1} p_j I_{M_1,\chi_1}(x_i;\xi_j) + \sum_{j=R_1+1}^{R_1+R_2} p_j I_{M_2,\chi_2}(x_i;\xi_j), \tag{3.8}$$

for $i = 1,...,N$, where

$$\gamma_1 = \sum_{j=1}^{R_1} p_j I_{M_1,\chi_1}(x;\xi_j) \tag{3.9}$$

is the contribution of R_1 type-one basis functions and

$$\gamma_2 = \sum_{j=R_1+1}^{R_1+R_2} p_j I_{M_2,\chi_2}(x;\xi_j) \tag{3.10}$$

is a contribution of R_2 type-two basis functions.
Let $R = R_1 + R_2$.The above system of equations contains as many equations as scattered data points $\{(x_i),h_i^I\}_{i=1}^{N}$ and as many unknown as center points $\{(\xi_j)\}_{j=1}^{R}$, so the system contains N equations and R variables. This implies that, if $R = N$, we obtain a square matrix but if $R \leq N$, an over-determined system is obtained of which least squares method can be used to solve.
Therefore, the system of equations in Eq. (3.8) is set up for each time step I to determine the coefficients $\{p_j\}_{j=1}^{R}$.
Let

$$A = [a_{i,j}]_{N\times R}$$

whose entry at the i-th row and j-th column is as follows

$$a_{ij} = \begin{cases} I_{M_1,\chi_1}(x_i;\xi_j),\ for\ 1 \leq j \leq R_1, \\ \\ I_{M_2,\chi_2}(x_i;\xi_j),\ for\ R_1+1 \leq j \leq R_1+R_2, \end{cases}$$

and

$$b = [h_i^I]_{N\times 1},$$

where h_i^I is $h^I(x,t)$ in Eq. (3.4) evaluated at each point x_i when $t = I \triangle t$.
The column vector $\mathbf{p}$ is defined as

$$\mathbf{p} = [p_j]_{R\times 1}$$

with $j = 1, ..., R$. The system in matrix form is

$$A_{N\times R} \cdot \mathbf{p}_{R\times 1} = b_{N\times 1}.$$

3.3 The Method of Approximate Particular Solutions

After approximating the source function, we now derive an approximate particular solution $u_p^{I+1}(x,t)$ for each I satisfying Eq. (3.1). Under the framework of the dual reciprocity method (DRM)[20], an approximate particular solution associated with the Delta-shaped basis function $I_{M,\chi}(x;\xi)$ is required. That is, at each time step I, a function $\phi^{I+1}(x;\xi)$ is desired satisfying,

$$\frac{d^2}{dx^2}\phi^{I+1}(x;\xi) - \frac{q^{I+1}}{\triangle t}\phi^{I+1}(x;\xi) = I_{M\chi}(x;\xi) = \sum_{n=1}^{M} c_n(\xi)\varphi_n(x). \tag{3.11}$$

Let an approximate particular solution $\phi^{I+1}(x,\xi)$ takes the form

$$\phi^{I+1}(x;\xi) = \sum_{n=1}^{M} d_n^{I+1}(\xi)\varphi_n(x).$$

Utilizing Eq. (3.11), we obtain

$$\phi''(x;\xi) - \frac{q^{I+1}(t)}{\triangle t}\phi(x;\xi) = \sum_{n=1}^{M} c_n(\xi)\varphi_n(x),$$

which is

$$\sum_{n=1}^{M} d_n^{I+1}(\xi)\Big[\varphi_n''(x) - \frac{q^{I+1}(t)}{\triangle t}\varphi_n(x)\Big] = \sum_{n=1}^{M} c_n(\xi)\varphi_n(x),$$

Now, solving for the coefficients $d_n^{I+1}(\xi)$, we obtain

$$d_n^{I+1}(\xi)\Big[-\Big(\frac{n\pi}{2}\Big)^2 - \frac{q^{I+1}(t)}{\triangle t}\Big]\varphi_n(x) = c_n(\xi)\varphi_n(x),$$

thus

$$d_n^{I+1}(\xi) = -\frac{c_n(\xi)}{\left(\frac{n\pi}{2}\right)^2 + \frac{q^{I+1}(t)}{\triangle t}}$$

$$= -\frac{\left[1-\left(\frac{n}{M+1}\right)^2\right]^{\chi} sin\left(\frac{n\pi(\xi+1)}{2}\right)}{\left(\frac{n\pi}{2}\right)^2 + \frac{q^{I+1}(t)}{\triangle t}}$$

$$= -\frac{r_n(M,\chi)\varphi_n(\xi)}{\left(\frac{n\pi}{2}\right)^2 + \frac{q^{I+1}(t)}{\triangle t}}.$$

Since the source function Eqs. (3.4) is approximated by Eq. (3.8) as a linear combination of two types of Delta- shaped basis functions, the approximate particular solution $u_p^{I+1}(x,t)$ can be represented as

$$\Phi^{I+1}(x,t) = -[\Phi_1^{I+1}(x,t) + \Phi_2^{I+1}(x,t)],$$

where

$$\Phi_1^{I+1} = \sum_{j=1}^{R_1}\sum_{n=1}^{M_1} \frac{p_j r_n(M_1,\chi_1)\varphi_n(\xi_j)\varphi_n(x)}{\left(\frac{n\pi}{2}\right)^2 + \frac{q^{I+1}(t)}{\triangle t}},$$

and

$$\Phi_2^{I+1} = \sum_{j=R_1+1}^{R_1+R_2}\sum_{n=1}^{M_2} \frac{p_j r_n(M_2,\chi_2)\varphi_n(\xi_j)\varphi_n(x)}{\left(\frac{n\pi}{2}\right)^2 + \frac{q^{I+1}(t)}{\triangle t}},$$

are the approximate particular solutions corresponding to Eqs. (3.9) and (3.10) respectively.

Chapter 4

METHOD OF FUNDAMENTAL SOLUTIONS

4.1 The Method of Fundamental Solutions

Once we have found our approximation of u_p, we are prepared to solve the corresponding homogeneous problem u_h at each time step using the method of fundamental solutions. The MFS method was first proposed by Kupradze and Aleksidze [10]. It is a flexible and efficient mesh-free boundary method provided the fundamental solution for the differential operator is known [2, 3, 4, 10]. The method of fundamental solutions is used for solving the homogeneous boundary valued problem (3.2) - (3.3). Our homogeneous problem takes the form,

$$\frac{d^2u_h}{dx^2} - \lambda^2 u_h = 0, \qquad x \in \Omega, \tag{4.1}$$

$$\mathbf{B}u_h(x) = f(x) - u_p(x), \qquad x \in \partial\Omega. \tag{4.2}$$

The solution to the homogeneous equation at each time step can be approximated as a linear combination of fundamental solutions

$$u_h^{I+1}(x,t) = \sum_{j=1}^{K} c_j G^{I+1}(x,t,\xi_j), \qquad x \in \partial\Omega, \tag{4.3}$$

where $\{c_j\}_{j=1}^{K}$ are coefficients to be determined and $G(x,t,\xi)$ is the fundamental solution to Eq. (3.2) which is represented as

$$G^{I+1}(x,t,\xi) = sinh(\lambda(x-\xi)).$$

Now, the homogeneous solution u_h^{I+1} satisfies the boundary conditions which results in the following system of linear equations

$$\sum_{j=1}^{K} c_j G(x_i,t,\xi_j) = f(x_i) - u_p(x_i,t),$$

for $K = 2$ and $i = 1,2$.

$$c_1 sinh(\lambda(x_1-\xi_1)) + c_2 sinh(\lambda(x_1-\xi_2)) = f(x_1) - u_p(x_1)$$
$$c_1 sinh(\lambda(x_2-\xi_1)) + c_2 sinh(\lambda(x_2-\xi_2)) = f(x_2) - u_p(x_2)$$

where ξ_1, ξ_2 are points outside the bounded domain and x_1, x_2 are points on the boundary of the domain.

4.2 A Homogeneous Case using MFS

Consider the homogeneous problem,

$$y'' - \lambda^2 y = 0, \quad x \in (-0.5, 0.5),$$
$$y(0.5) \ \ and \ \ y(-0.5) \ \ given,$$

such that the exact solution is $y = e^x + 3e^{-x}$. The homogeneous solution u_h is expressed as a linear combination of fundamental solutions

$$u_h(x) = \sum_{j=1}^{2} c_j G(x, \xi_j),$$
$$u_h = c_1 G(x, \xi_1) + c_2 G(x, \xi_2),$$

where $G(x, \xi)$ is the fundamental solution given as

$$G(x, \xi) = sinh\lambda(x - \xi).$$

Now, we solve the system of linear equations to obtain the coefficients c_1 and c_2 for the homogeneous equation:

$$\sum_{j=1}^{2} c_j G(x_i, \xi_j) = \mathbf{B}y(x_i).$$

Choosing ξ_1 and ξ_2 outside the domain $(-0.5, 0.5)$ as -1 and 1 respectively, and x_1 and x_2 as boundary points -0.5 and 0.5 respectively, the matrix

$$\begin{pmatrix} sinh((x_1-\xi_1)) & sinh((x_1-\xi_2)) \\ sinh((x_2-\xi_1)) & sinh((x_2-\xi_2)) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y(x_1) \\ y(x_2) \end{pmatrix},$$

becomes

$$\begin{pmatrix} sinh((-0.5+1)) & sinh((-0.5-1)) \\ sinh((0.5+1)) & sinh((0.5-1)) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y(-0.5) \\ y(0.5) \end{pmatrix},$$

which is

$$\begin{pmatrix} 0.52109530549 & \text{-}2.12927945509 \\ 2.12927945509 & \text{-}0.52109530549 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5.552694472 \\ 3.46831325 \end{pmatrix}.$$

Solving this linear system, we obtain $c_1 = 1.0538$and $c_2 = -2.3499$.
Therefore the homogeneous solution is

$$u_h = 1.0538 sinh((x+1)) - 2.3499 sinh((x-1)).$$

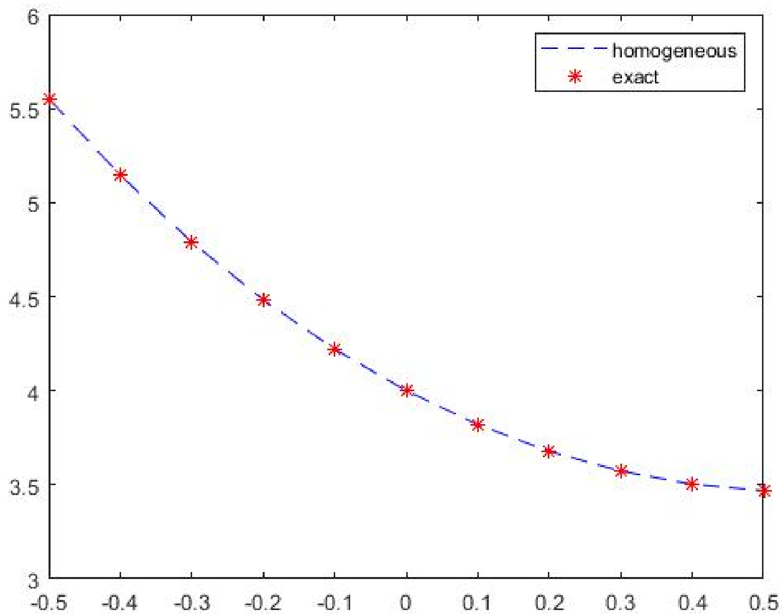

Figure 4.1: *Plot of the exact solution and homogeneous solution* .

Figure (4.1) shows that the exact solution (star line) and the homogeneous solution (double dash line) lay closely to each other making the error between the two solutions to be really small, which shows the accuracy of the MFS.

4.3 A Non-homogeneous Case Coupled with MAPS

To demonstrate the use of the MFS for solving a general ODE boundary value problem, we consider the non-homogeneous problem

$$y'' - \lambda^2 y = 6 - 3x^2 - x, \quad x \in (-0.5, 0.5),$$

$$y(0.5) \;\; and \;\; y(-0.5) \;\; given,$$

such that the exact solution is $y = 3x^2 + x$.

N_1	N_2	*MSE*	*RMSE*
50	100	$1.7668 \cdot 10^{-11}$	$3.9667 \cdot 10^{-11}$
100	150	$4.6451 \cdot 10^{-11}$	$1.0563 \cdot 10^{-11}$
150	200	$7.3097 \cdot 10^{-12}$	$1.6205 \cdot 10^{-11}$
201	200	$3.3114 \cdot 10^{-11}$	$7.6328 \cdot 10^{-11}$

Table 4.1: *MFS results for the non-homogeneous problem.*

The boundary of this problem is $\partial\Omega = \{-0.5, 0.5\}$. The number of collocation points $N = N_1 + N_2$ is selected to be twice the number of source points. To determine the accuracy of the approximate solution, the collocation points are also used as test points within the domain. The approximate solution $\tilde{u}$ is compared to the exact solution u at the collocation points using the mean square root error (MSE) and relative mean square root error (RMSE) which is respectively represented as follow:

$$MSE = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}[\tilde{u}(x_i) - u(x_i)]^2},$$

and

$$RMSE = \frac{\sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}[\tilde{u}(x_i) - u(x_i)]^2}}{\sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}[u(x_i)]^2}},$$

where N_t is the number of test points. The results displayed in Table (4.1) demonstrate the accuracy of MFS coupled MAFS for solving the non-homogeneous boundary value problem.

Chapter 5

NUMERICAL EXAMPLES

We provide numerical examples in this section to measure and determine the accuracy and validity of the proposed method for solving nonlinear ordinary differential equations. To measure the accuracy of an approximate solution at each time step, we use the mean square root error defined as follows,

$$MSE^I = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[u^I(x_i) - u^{exact}(x_i)]^2}$$

where u^I is the numerical solution at each time step which is compared with the exact solution u^{exact}. The test points $\{(x_i)_{k=1}^N\}$ are the points used for collocation inside the domain. To see how the error is relative to the magnitude of the solution, we also provide the relative mean square root error,

$$RMSE^I = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}[u^I(x_i) - u^{exact}(x_i)]^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}[u^{exact}(x_i)]^2}}.$$

Let Ω be the interval $-0.5 < x < 0.5$ and $\partial\Omega$ be the boundary of Ω given as $|x| = 0.5$. If a problem is not originally defined on the interval $[-0.5, 0.5]$ we can always use proper scaling or translation to make it a problem over this interval. Hence in this chapter, we assume all problems are defined on this interval.

Example 1. We consider the nonlinear differential equation,

$$\begin{aligned} -u'' &= -\frac{3}{2}u^2, && x \in \Omega, \\ \mathbf{B}u &= u(x) \;\; given, && x \in \partial\Omega, \end{aligned}$$

such that its exact solution is

$$u = \frac{4}{(3+x)^2}.$$

For this example, we choose 50 of the type I basis and 150 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 400, twice the total number of basis centers. The regularization parameters are chosen as $m_1 = 10$, $\chi_1 = 4$, $m_2 = 20$ and $\chi_2 = 6$. The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. We recorded 50 time steps. Figure (5.1) displays the error after various numbers of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. It can be observed from Table (5.1) that the iteration corresponding to a larger Δt tends to converge rapidly, thus has smaller error. The numerical results tends to be more accurate for smaller β.

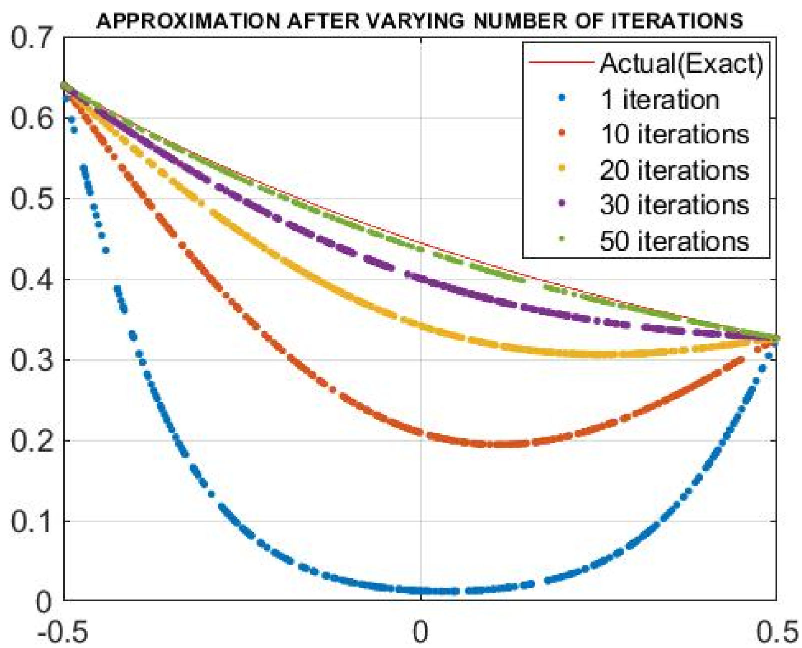

Figure 5.1: *(Example 1) Plot of exact solutions and approximate solutions at different number of iterations.*

β	ΔT	*MSE*	*RMSE*
10^{-02}	2^{-05}	$1.4728 \cdot 10^{-04}$	$3.1650 \cdot 10^{-04}$
10^{-02}	2^{-06}	$3.1963 \cdot 10^{-04}$	$6.8572 \cdot 10^{-04}$
10^{-02}	2^{-07}	$6.1164 \cdot 10^{-03}$	$1.3296 \cdot 10^{-02}$
10^{-02}	2^{-08}	$4.9734 \cdot 10^{-02}$	$1.0508 \cdot 10^{-01}$
10^{-04}	2^{-05}	$1.5134 \cdot 10^{-06}$	$3.3023 \cdot 10^{-06}$
10^{-04}	2^{-06}	$1.0157 \cdot 10^{-04}$	$2.1657 \cdot 10^{-04}$
10^{-04}	2^{-07}	$5.7693 \cdot 10^{-03}$	$1.2516 \cdot 10^{-02}$
10^{-04}	2^{-08}	$4.8963 \cdot 10^{-02}$	$1.0722 \cdot 10^{-01}$

Table 5.1: *(Example 1) MSE & RMSE for time function $\beta = 10^{-2}$ and $\beta = 10^{-4}$; Basis 1: $m_1 = 10$ and $\chi_1 = 4$, Basis 2: $m_2 = 20$ and $\chi_2 = 6$.*

Example 2. For our next example, we take a look at the nonlinear differential equation

$$-u'' = -2u^3, \qquad x \in \Omega,$$

$$\mathbf{B}u = u(x) \;\; given, \qquad x \in \partial\Omega,$$

such that its exact solution is

$$u = \frac{1}{(4+x)}.$$

For this example, we choose 75 of the type I basis and 125 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 400, twice the total number of basis centers. We choose the regularization parameters $m_1 = 20$, $\chi_1 = 6$, $m_2 = 30$ and $\chi_2 = 9$. The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. Figure (5.2) displays the error after various numbers of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. It can be observed from Table (5.2) that the iteration corresponding to a larger Δt tends to converge rapidly, thus after 50 steps the MSE with $\beta = 10^{-02}$ and $\Delta t = 2^{-08}$ reaches 3.1509×10^{-02} while the MSE with $\Delta t = 2^{-05}$ reaches 9.1636×10^{-05}. Also, smaller β coupled with a large Δt tends to produce a more accurate numerical results.

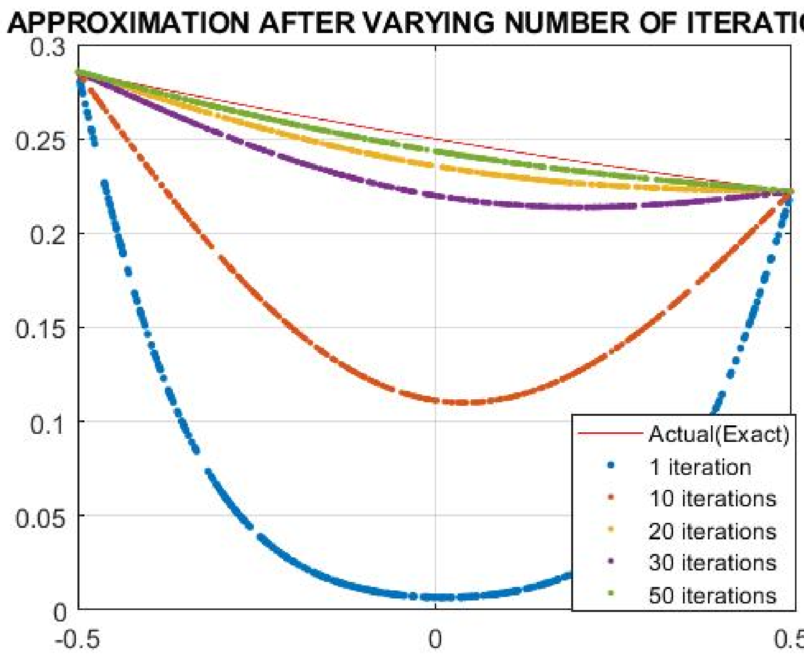

Figure 5.2: (Example 2) Plot of exact solutions and approximate solutions at different number of iterations.

β	ΔT	*MSE*	*RMSE*
10^{-02}	2^{-05}	$9.1636 \cdot 10^{-05}$	$3.6466 \cdot 10^{-04}$
10^{-02}	2^{-06}	$2.4905 \cdot 10^{-04}$	$9.8796 \cdot 10^{-04}$
10^{-02}	2^{-07}	$4.6832 \cdot 10^{-03}$	$1.8670 \cdot 10^{-02}$
10^{-02}	2^{-08}	$3.1509 \cdot 10^{-02}$	$1.2542 \cdot 10^{-01}$
10^{-04}	2^{-05}	$1.0346 \cdot 10^{-06}$	$4.0797 \cdot 10^{-06}$
10^{-04}	2^{-06}	$1.1705 \cdot 10^{-04}$	$4.6416 \cdot 10^{-04}$
10^{-04}	2^{-07}	$4.6558 \cdot 10^{-03}$	$1.8496 \cdot 10^{-02}$
10^{-04}	2^{-08}	$3.1913 \cdot 10^{-02}$	$1.2622 \cdot 10^{-01}$

Table 5.2: (Example 2) MSE & RMSE for time function $\beta = 10^{-2}$ and $\beta = 10^{-4}$; Basis 1: $m_1 = 20$ and $\chi_1 = 6$, Basis 2: $m_2 = 30$ and $\chi_2 = 9$.

Example 3. To demonstrate flexibility of the methods, consider the nonlinear problem that also includes spatial variables in the source function.

$$
\begin{aligned}
-u'' &= u^2 - e^x - e^{2x}, && x \in \Omega,\\
\mathbf{B}u &= u(x) \ \ given, && x \in \partial\Omega,
\end{aligned}
$$

such that its exact solution is $u = e^x$.

We choose 50 of the type I basis and 150 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 500, twice the total number of basis centers. The regularization parameters are chosen as $m_1 = 10$, $\chi_1 = 4$, $m_2 = 20$ and $\chi_2 = 6$.The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. Figure (5.3) displays the error after various number of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. Again, it can be observed from Table (5.3) that the iteration corresponding to a larger Δt tends to converge rapidly.

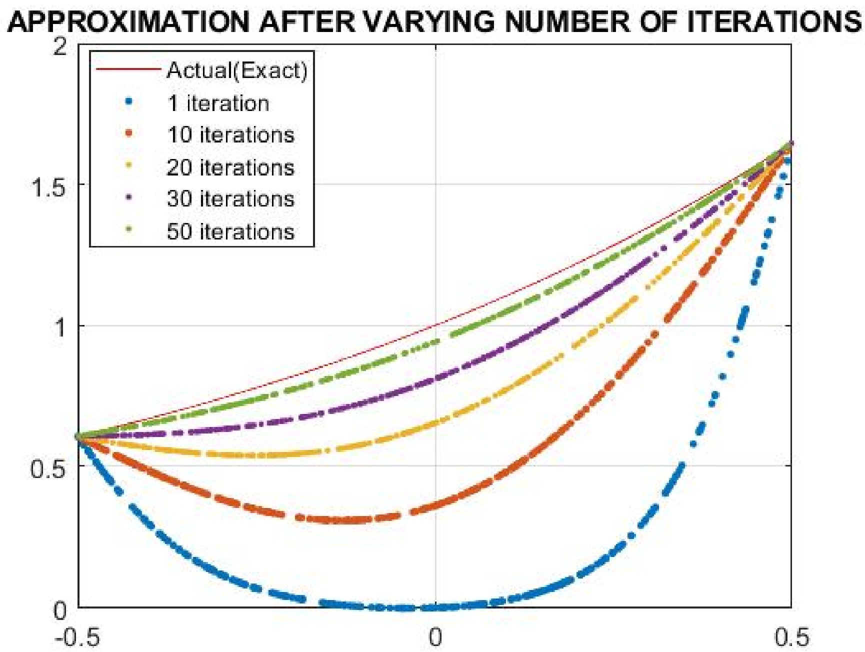

Figure 5.3: *(Example 3) Plot of exact solutions and approximate solutions at different number of iterations.*

β	ΔT	MSE	$RMSE$
10^{-02}	2^{-05}	$5.2178 \cdot 10^{-04}$	$4.8614 \cdot 10^{-04}$
10^{-02}	2^{-06}	$3.3885 \cdot 10^{-03}$	$3.1413 \cdot 10^{-03}$
10^{-02}	2^{-07}	$4.1809 \cdot 10^{-02}$	$3.9181 \cdot 10^{-02}$
10^{-02}	2^{-08}	$1.7904 \cdot 10^{-01}$	$1.6165 \cdot 10^{-01}$
10^{-04}	2^{-05}	$2.5818 \cdot 10^{-05}$	$2.3808 \cdot 10^{-05}$
10^{-04}	2^{-06}	$2.6338 \cdot 10^{-03}$	$2.4183 \cdot 10^{-03}$
10^{-04}	2^{-07}	$4.0345 \cdot 10^{-02}$	$3.6956 \cdot 10^{-02}$
10^{-04}	2^{-08}	$1.8309 \cdot 10^{-01}$	$1.7098 \cdot 10^{-01}$

Table 5.3: *(Example 3) MSE & RMSE for time function $\beta = 10^{-2}$ and $\beta = 10^{-4}$; Basis 1: $m_1 = 10$ and $\chi_1 = 4$, Basis 2: $m_2 = 20$ and $\chi_2 = 6$.*

Example 4. To further demonstrate the applicability of the methods, we consider a nonlinear problem with a source function that depends on u, $u\prime$ and spatial variable x.

$$
\begin{aligned}
-u'' &= -2u^3 + (u')^2 - \frac{1}{(4+x)^4}, && x \in \Omega, \\
\mathbf{B}u &= u(x) \;\; given, && x \in \partial\Omega,
\end{aligned}
$$

such that its exact solution is

$$
u = \frac{1}{(4+x)}.
$$

For this example, we choose 100 of the type I basis and 150 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 500, twice the total number of basis centers. The regularization parameters are chosen as $m_1 = 10$, $\chi_1 = 4$, $m_2 = 30$ and $\chi_2 = 9$. The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. Figure (5.4) displays the error after various numbers of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. It can be observed from Table (5.4) that the iteration corresponding to a larger Δt tends to converge rapidly.

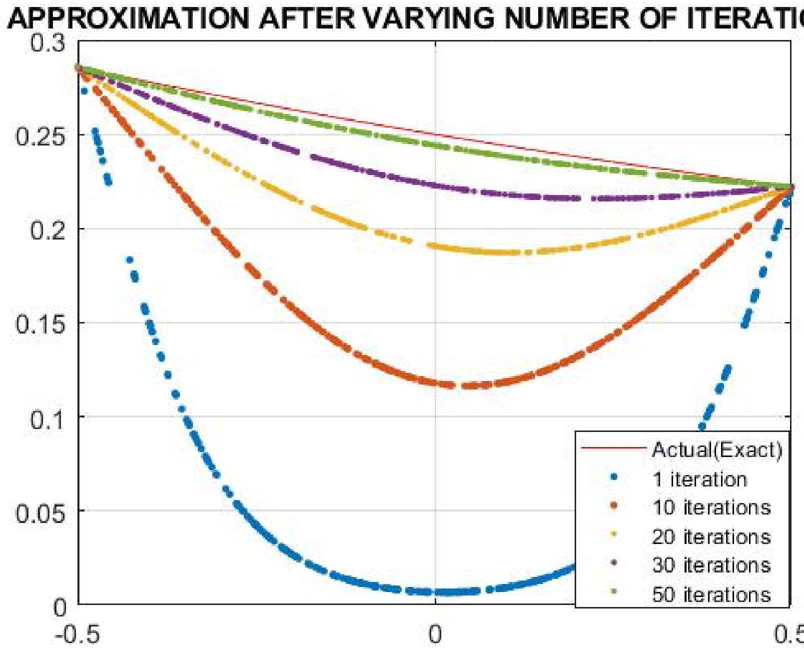

Figure 5.4: *(Example 4) Plot of exact solutions and approximate solutions at different number of iterations.*

β	ΔT	MSE	$RMSE$
10^{-02}	2^{-05}	$9.0013 \cdot 10^{-05}$	$3.5579 \cdot 10^{-04}$
10^{-02}	2^{-06}	$2.2890 \cdot 10^{-04}$	$9.1579 \cdot 10^{-04}$
10^{-02}	2^{-07}	$4.3144 \cdot 10^{-03}$	$1.7059 \cdot 10^{-02}$
10^{-02}	2^{-08}	$2.8341 \cdot 10^{-02}$	$1.1275 \cdot 10^{-01}$
10^{-04}	2^{-05}	$1.0017 \cdot 10^{-06}$	$3.9329 \cdot 10^{-06}$
10^{-04}	2^{-06}	$1.0249 \cdot 10^{-04}$	$4.0596 \cdot 10^{-04}$
10^{-04}	2^{-07}	$3.9891 \cdot 10^{-03}$	$1.5846 \cdot 10^{-02}$
10^{-04}	2^{-08}	$2.8985 \cdot 10^{-02}$	$1.1471 \cdot 10^{-01}$

Table 5.4: *(Example 4) MSE & RMSE for time function* $\beta = 10^{-2}$ *and* $\beta = 10^{-4}$*; Basis 1:* $m_1 = 10$ *and* $\chi_1 = 4$ *, Basis 2:* $m_2 = 30$ *and* $\chi_2 = 9$.

Example 5. For our next example, we choose a more difficult nonlinear problem which incorporates trigonometry functions.

$$
\begin{aligned}
-u'' &= 4u' + 2u - \frac{2u}{sin^2(u)+1} - cos(x) + 4sin(x) + \frac{2cos(x)}{sin^2(cos(x))+1}, \quad && x \in \Omega, \\
\mathbf{B}u &= u(x) \ \ given, && x \in \partial\Omega,
\end{aligned}
$$

such that its exact solution is $u = cos(x)$.
For this example, we choose 100 of the type I basis and 150 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 500, twice the total number of basis centers. The regularization parameters are chosen as $m_1 = 10$, $\chi_1 = 4$, $m_2 = 20$ and $\chi_2 = 6$. The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. Figure (5.5) displays the error after various number of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. It can be observed from Table (5.5) that the iteration corresponding to a larger Δt tends to converge rapidly. Table (5.5) shows the accurate results with smaller β coupled with larger Δt.

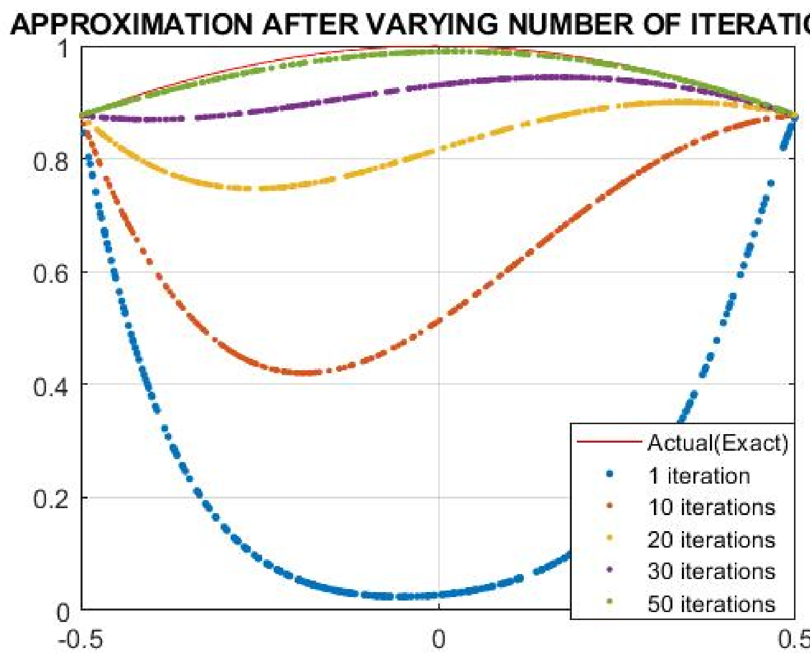

Figure 5.5: *(Example 5) Plot of the exact solution and approximate solutions at different number of iterations.*

β	ΔT	MSE	$RMSE$
10^{-02}	2^{-05}	$3.2402 \cdot 10^{-04}$	$3.3818 \cdot 10^{-04}$
10^{-02}	2^{-06}	$5.2581 \cdot 10^{-04}$	$5.4834 \cdot 10^{-04}$
10^{-02}	2^{-07}	$9.1234 \cdot 10^{-03}$	$9.4851 \cdot 10^{-03}$
10^{-02}	2^{-08}	$9.1766 \cdot 10^{-02}$	$9.5725 \cdot 10^{-02}$
10^{-04}	2^{-05}	$3.2267 \cdot 10^{-06}$	$3.3681 \cdot 10^{-06}$
10^{-04}	2^{-06}	$6.0586 \cdot 10^{-05}$	$6.3211 \cdot 10^{-05}$
10^{-04}	2^{-07}	$7.9037 \cdot 10^{-03}$	$8.2584 \cdot 10^{-03}$
10^{-04}	2^{-08}	$9.2965 \cdot 10^{-02}$	$9.6628 \cdot 10^{-02}$

Table 5.5: *(Example 5) MSE & RMSE for time function* $\beta = 10^{-2}$ *and* $\beta = 10^{-4}$*; Basis 1: m1=10 and* $\chi_1 = 4$ *, Basis 2:* $m_2 = 20$ *and* $\chi_2 = 6$.

Example 6. We consider a nonlinear problem with both derivative and logarithm incorporated in it.

$$
\begin{aligned}
-u'' &= x^2u' - u - x \cdot u \cdot log(u), && x \in \Omega, \\
\mathbf{B}u &= u(x) \;\; given, && x \in \partial\Omega,
\end{aligned}
$$

such that its exact solution is $u = e^x$.

For this example, we choose 75 of the type I basis and 125 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 400, twice the total number of basis centers. The regularization parameters are chosen as $m_1 = 30$, $\chi_1 = 9$, $m_2 = 50$ and $\chi_2 = 14$. The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. Figure (5.6) displays the error after various number of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. It can also be observed from Table (5.6) that the iteration corresponding to a larger Δt tends to converge rapidly.

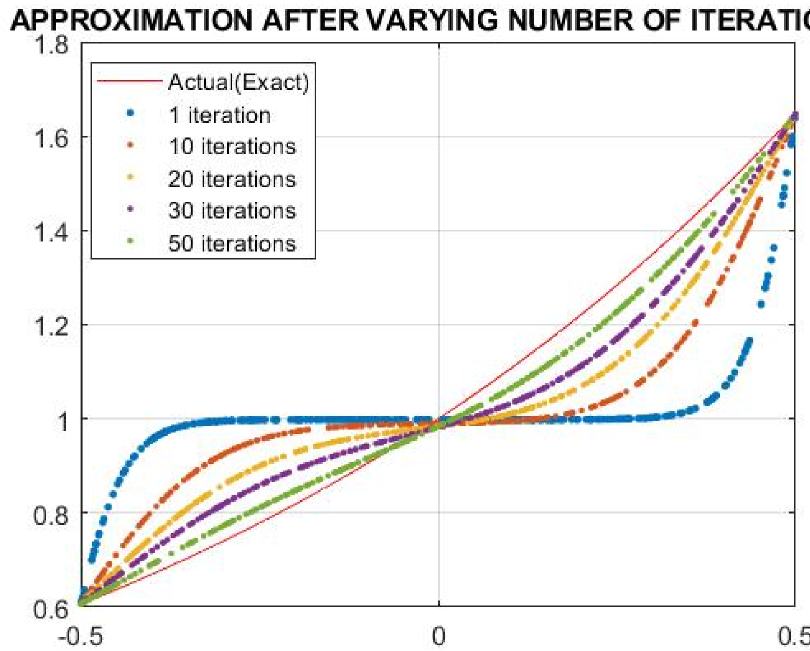

Figure 5.6: *(Example 6) Plot of exact solutions and approximate solutions at different number of iterations.*

β	ΔT	*MSE*	*RMSE*
10^{-02}	2^{-05}	$3.4285 \cdot 10^{-04}$	$3.1641 \cdot 10^{-04}$
10^{-02}	2^{-06}	$5.1540 \cdot 10^{-04}$	$4.7900 \cdot 10^{-04}$
10^{-02}	2^{-07}	$9.3589 \cdot 10^{-04}$	$8.5241 \cdot 10^{-04}$
10^{-02}	2^{-08}	$3.0150 \cdot 10^{-03}$	$2.8734 \cdot 10^{-03}$
10^{-04}	2^{-05}	$3.4130 \cdot 10^{-06}$	$3.0813 \cdot 10^{-06}$
10^{-04}	2^{-06}	$1.0425 \cdot 10^{-05}$	$9.7449 \cdot 10^{-06}$
10^{-04}	2^{-07}	$2.9076 \cdot 10^{-04}$	$2.6422 \cdot 10^{-04}$
10^{-04}	2^{-08}	$2.3294 \cdot 10^{-03}$	$2.1113 \cdot 10^{-03}$

Table 5.6: *(Example 6) MSE & RMSE for time function $\beta = 10^{-2}$ and $\beta = 10^{-4}$; Basis 1: $m_1 = 30$ and $\chi_1 = 9$, Basis 2: $m_2 = 50$ and $\chi_2 = 14$.*

Example 7. Finally, we choose a nonlinear problem which incorporates trigonometry functions.

$$-u'' = 4u' + 2u - \frac{2u}{sin^2(u)+1} - 98cos(10x) + 40sin(10x) + \frac{2cos(10x)}{sin^2(cos(10x))+1}, \quad x \in \Omega,$$

$$\mathbf{B}u = u(x) \;\; given, \qquad x \in \partial\Omega,$$

such that its exact solution is $u = cos(10x)$.
For this example, we choose 50 of the type I basis and 150 of the type II basis centers randomly inside the bounded domain. The number of collocation point is 500, twice the total number of basis centers. The regularization parameters are chosen as $m_1 = 20$, $\chi_1 = 6$, $m_2 = 30$ and $\chi_2 = 9$. The arbitrarily chosen initial solution for $I = 0$ is the constant function $u^0 = 0$. Figure (5.7) displays the error after various number of iteration. We choose different Δt with $\beta = 10^{-2}$ and $\beta = 10^{-4}$. It can be observed from Table (5.7) that the iteration corresponding to a larger Δt tends to converge rapidly. Table (5.7) shows the accurate results with smaller β coupled with larger Δt. In general , the numerical results tends to be more accurate for smaller β with larger Δt.

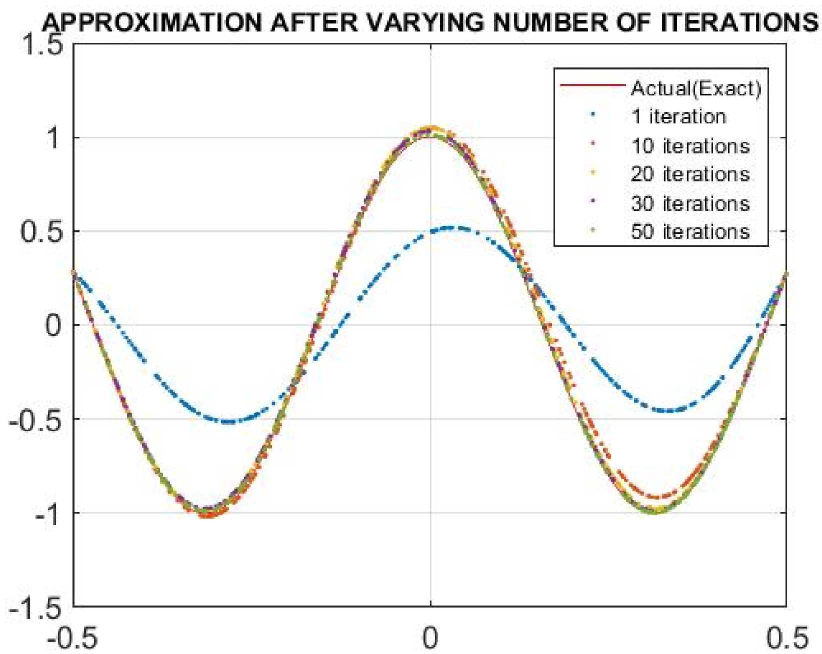

Figure 5.7: *(Example 7) Plot of the exact solution and approximate solutions at different number of iterations.*

β	ΔT	*MSE*	*RMSE*
10^{-02}	2^{-05}	$3.2636 \cdot 10^{-05}$	$4.6867 \cdot 10^{-05}$
10^{-02}	2^{-06}	$5.3332 \cdot 10^{-05}$	$7.6249 \cdot 10^{-05}$
10^{-02}	2^{-07}	$9.4165 \cdot 10^{-04}$	$1.4100 \cdot 10^{-04}$
10^{-02}	2^{-08}	$8.7518 \cdot 10^{-03}$	$1.3300 \cdot 10^{-02}$
10^{-04}	2^{-05}	$3.3549 \cdot 10^{-07}$	$4.8031 \cdot 10^{-07}$
10^{-04}	2^{-06}	$7.0426 \cdot 10^{-06}$	$1.0569 \cdot 10^{-05}$
10^{-04}	2^{-07}	$8.7337 \cdot 10^{-04}$	$1.2829 \cdot 10^{-03}$
10^{-04}	2^{-08}	$9.2667 \cdot 10^{-03}$	$1.3907 \cdot 10^{-02}$

Table 5.7: *(Example 7) MSE & RMSE for time function* $\beta = 10^{-2}$ *and* $\beta = 10^{-4}$*; Basis 1: m1=20 and* $\chi_1 = 6$ *, Basis 2:* $m_2 = 30$ *and* $\chi_2 = 9$.

Chapter 6

CONCLUSION

A general class of nonlinear differential equations is solved by incorporating a fictitious time integration method to transform the nonlinear ODE into a sequence of time-dependent linear non-homogeneous differential equations. Delta-shaped basis functions are used to approximate the source function at each time step. An approximate particular solution is obtained at each time step by using the Delta-shaped basis functions. The corresponding homogeneous problem at each time step is solved using the MFS. The proposed method is applicable to a general class of nonlinear ODEs provided the fundamental solution for the differential operator is known. The validity and accuracy of this computational method is supported with numerical examples with different kinds of nonlinear source functions. The numerical results demonstrate that with proper parameter values approximate solutions converge rapidly to exact solution.

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