The Fractionary Schrödinger Equation, Green Functions and Ultradistributions *

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Abstract

In this work, we generalize previous results about the Fractionary Schrödinger Equation within the formalism of the theory of Tempered

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Ultradistributions. Several examples of the use of this theory are given. In particular we evaluate the Green's function for a free particle in the general case, for an arbitrary order of the derivative index.

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1 Introduction

Fractional calculus has found motivations in a growing area concerning general stochastic phenomena. These include the appearance of alternative diffusion mechanisms other than Brownian, as well as classical and quantum mechanics formalisms including dissipative forces, and therefore allowing an extension of the quantization schemes for non-conservative systems [1]-[4]. In particular it is interesting to study the time fractional Schrödinger equation, which describes the time irreversibility present in all processes in nature. This inhomogeneity in time is particularly important at the quantum level, where damping is inherent to all non equilibrium evolution. Our aim here is to extend a previous study about this equation, introduced in [3]. With this purpose we use an analytical definition of fractional derivative [5], where we show that it is possible to obtain a general solution for the time fractional equation, for any complex value of the derivative index. Furthermore the associated Green functions can be evaluated in a straightforward way. This analytical generalization is possible within the framework of Ultradistributions.

Ultradistribution's theory (ref.[11, 12]) proofs to be a mathematical adequate theory to approach several branches of theoretical physics, helping to overcome certain common subtleties arising from the formal handling of analytical formulas. For instance, using this tool in ref.[7] we have realized the mathematical treatment of higher order field theories, and in ref.[8] we have solved the problem of normalization of Gamow states or resonances in quantum mechanics.

Ultradistributions have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with them.

They have interesting properties. One of those properties is that Schwartz tempered distributions are canonical and continuously injected into tempered ultradistributions and as a consequence the Rigged Hilbert Space with tempered distributions is canonical and continuously included in the Rigged Hilbert Space with tempered ultradistributions.

Therefore the properties of ultradistributions are also well adapted for their use in fractional calculus. In this respect we have shown that it is possible (ref.[5]) to define a general fractional calculus with the use of them.

This paper is organized as follow:

In section 2 we define the fractional Schrödinger equation for all ν complex with the use of the fractional derivative defined via the theory of tempered ultradistributions. In section 3 we solve this equation for the free particle and give three examples:$\nu = 1/2, \nu = 1$ and $\nu = 2$. In section 4 we realize the treatment of the potential well and we analyze the cases $\nu = 1/2, \nu = 1$ and $\nu = 2$. In section 5 we study the Green fractional functions for the free particle in three cases: the retarded Green function, the advanced Green function and the Wheeler-Green function. As an example we prove that for $\nu = 1$ these functions coincide with the Green functions of usual Quantum Mechanics. In section 6 we discuss the results obtained in the previous sections. Finally we have included three appendixes: a first appendix on distributions of exponential type, a second appendix on tempered ultradistributions and a third appendix on fractional calculus using ultradistributions.

2 The Fractional Schrödinger Equation

Our starting point in the study of the fractional Schrödinger equation is the current known Schrödinger equation:

$$i\hbar\partial_t\psi(t,x) = -\frac{\hbar^2}{2m}\partial_x^2\psi(t,x) + V(x)\psi(t,x) \tag{2.1}$$

According to ref.[3], (2.1) can be written as:

$$iT_p\partial_t\psi(t,x) = -\frac{L_p^2 M_p}{2m}\partial_x^2\psi(t,x) + \frac{V(x)}{E_p}\psi(t,x) \tag{2.2}$$

where $L_P = \sqrt{G\hbar/c^3}$, $T_p = \sqrt{G\hbar/c^5}$, $M_p = \sqrt{\hbar c/G}$ and $E_p = M_p c^2$, and G is the gravitational constant.

If we define $N_m = m/M_p$ and $N_\nu = V/E_p$ we obtain for (2.2)

$$iT_p \partial_t \psi(t,x) = -\frac{L_p^2}{2N_m}\partial_x^2 \psi(t,x) + N_\nu \psi(t,x) \tag{2.3}$$

By analogy with ref.[3] we define the fractional Schrödinger equation for all ν complex as:

$$(iT_p)^\nu \partial_t^\nu \psi(t,x) = -\frac{L_p^2}{2N_m}\partial_x^2 \psi(t,x) + N_\nu \psi(t,x) \tag{2.4}$$

where the temporal fractionary derivative is defined following ref.[5] (see Appendix III)

3 The Free Particle

From (2.4) for the free particle the fractionary equation is:

$$(i\partial_t)^\nu \psi(t,x) + \frac{L_p^2}{2T_p^\nu N_m}\partial_x^2 \psi(t,x) = 0 \tag{3.1}$$

By the use of the Fourier transform (complex in the temporal variable and real as usual in the spatial variable) the corresponding equation is (see Appendix II and ref.[5])

$$\left(k_0^\nu - \frac{L_p^2}{2T_p^\nu N_m}k^2\right)\hat{\psi}(k_0,k) = b(k_0,k) \tag{3.2}$$

whose solution is:

$$\hat{\psi}(k_0, k) = \frac{b(k_0, k)}{k_0^\nu - \frac{L_p^2}{2T_p^\nu N_m} k^2} \tag{3.3}$$

and in the configuration space (anti-transforming)

$$\psi(t, x) = \oint_\Gamma \int_{-\infty}^{\infty} \frac{a(k_0, k)}{k_0^\nu - \frac{L_p^2}{2T_p^\nu N_m} k^2} e^{-i(k_0 t + kx)} dk_0 \ dk \tag{3.4}$$

where:

$$a(k_0, k) = \frac{b(k_0, k)}{4\pi^2}$$

As $a(k_0, k)$ is analytic entire in the variable k_0 we have for it the development:

$$a(k_0, k) = \sum_{m=0}^{\infty} a_m(k) k_0^m$$

Inserting this development in (3.4) we obtain

$$\psi(t, x) = \sum_{m=0}^{\infty} \oint_\Gamma \int_{-\infty}^{\infty} \frac{a_m(k) k_0^m}{k_0^\nu - \frac{L_p^2}{2T_p^\nu N_m} k^2} e^{-i(k_0 t + kx)} dk_0 \ dk \tag{3.5}$$

In ref.[6] is given the Laplace transform of the generalized Mittag-Leffler's function

$$\int_0^{\infty} e^{-\mu x} x^{\beta - 1} E_{\alpha,\beta}(x^\alpha z) \ dx = \frac{\mu^{\alpha - \beta}}{\mu^\alpha - z} \tag{3.6}$$

and as a consequence

$$\int_0^{\infty} e^{-i\frac{\pi}{2}(\nu - m)} (x + i0)^{\nu - m - 1} E_{\nu, \nu - m}[(x + i0)^\nu e^{-i\frac{\pi\nu}{2}} z] e^{ikx} \ dx = \frac{k^m}{k^\nu - z} \tag{3.7}$$

with $\Im(k) > 0$, and

$$\int_{-\infty}^{0} e^{-i\frac{\pi}{2}(\nu-m)}(x+i0)^{\nu-m-1}E_{\nu,\nu-m}[(x+i0)^{\nu}e^{-i\frac{\pi\nu}{2}}z]e^{ikx}\,dx = -\frac{k^m}{k^\nu - z} \quad (3.8)$$

with $\Im(k) < 0$ From which we obtain for the complex Fourier transform

$$H[\Im(k)]\int_{0}^{\infty} e^{-i\frac{\pi}{2}(\nu-m)}(x+i0)^{\nu-m-1}E_{\nu,\nu-m}[(x+i0)^{\nu}e^{-i\frac{\pi\nu}{2}}z]e^{ikx}\,dx -$$

$$H[-\Im(k)]\int_{-\infty}^{0} e^{-i\frac{\pi}{2}(\nu-m)}(x+i0)^{\nu-m-1}E_{\nu,\nu-m}[(x+i0)^{\nu}e^{-i\frac{\pi\nu}{2}}z]e^{ikx}\,dx$$

$$= \frac{k^m}{k^\nu - z} \quad (3.9)$$

and then we have for the inverse complex Fourier transform:

$$\frac{1}{2\pi}\oint_{\Gamma}\frac{k^m}{k^\nu - z}e^{-ikx}\,dk =$$

$$e^{-i\frac{\pi}{2}(\nu-m)}(x+i0)^{\nu-m-1}E_{\nu,\nu-m}[(x+i0)^{\nu}e^{-i\frac{\pi\nu}{2}}z] \quad (3.10)$$

Thus we have from (3.5)

$$\psi(t,x) = \sum_{m=0}^{\infty} e^{-i\frac{\pi}{2}(\nu-m)}(t+i0)^{\nu-m-1} \times$$

$$\int_{-\infty}^{\infty} a_m(k)E_{\nu,\nu-m}[(t+i0)^{\nu}e^{-i\frac{\pi\nu}{2}}w]e^{-ikx}\,dk \quad (3.11)$$

where we have made the rescaling $\frac{1}{2\pi}a_m(k) \to a_m(k)$ and

$$w = \frac{L_p^2 k^2}{2T_p^\nu N_m}$$

When $a(k_0, k) = a(k)$ formula (3.11) simplifies:

$$\psi(t,x) = e^{-i\frac{\pi}{2}(\nu)}(t+i0)^{\nu--1}\int_{-\infty}^{\infty} a(k)E_{\nu,\nu}[(t+i0)^{\nu}e^{-i\frac{\pi\nu}{2}}w]e^{-ikx}\, dk \quad (3.12)$$

When we consider the solution only for $t \geq 0$, i.e. writing $\psi(t,x)H(t)$ in (3.1), we obtain the result of ref.[3]. In fact, the equation for this case is:

$$(i\partial_t)^{\nu-1}\{[(i\partial_t)\psi(t,x)]H(t)\} + \frac{L_p^2}{2N_m T_p^{\nu}}\partial_x^2[\psi(t,x)H(t)] = 0 \quad (3.13)$$

With the initial condition $\psi(0,x) = \psi(x)$ we have for (3.12)

$$(i\partial_t)^{\nu-1}\{(i\partial_t)[\psi(t,x)H(t)] - i\psi(x)\delta(t)\} + \frac{L_p^2}{2N_m T_p^{\nu}}\partial_x^2[\psi(t,x)H(t)] = 0 \quad (3.14)$$

Let $\hat{\psi}(k_0, k)$ and $\hat{\psi}(k)$ the Fourier transforms of $\psi(t,x)H(t)$ and $\psi(x)$ respectively. Then (3.14) transforms into:

$$k_0^{\nu}\hat{\psi}(k_0,k) - ik_0^{\nu-1}\hat{\psi}(k)H[\Im(k_0)] - w\hat{\psi}(k_0,k) = 0 \quad (3.15)$$

whose solution is

$$\hat{\psi}(k_0,k) = i\frac{k_0^{\nu-1}}{k_0^{\nu} - w}\hat{\psi}(k)H[\Im(k_0)] \quad (3.16)$$

Using again (3.6) we have:

$$H[\Im(k)]\int_0^{\infty} E_{\nu}[(-ix)^{\nu}z]e^{ikx}\, dx = H[\Im(k)]i\frac{k^{\nu-1}}{k^{\nu} - z} \quad (3.17)$$

and anti-transforming

$$\frac{1}{2\pi}\oint_{\Gamma} H[\Im(k)]i\frac{k^{\nu-1}}{k^{\nu} - z}e^{-ikx}\, dk = H(x)E_{\nu}[(-ix)^{\nu}z] \quad (3.18)$$

Then taking the inverse Fourier transform of (3.16) we obtain

$$H(t)\psi(t,x) = \frac{H(t)}{2\pi}\int_{-\infty}^{\infty} E_\nu[(-it)^\nu w]\psi(k)\, dk \tag{3.19}$$

Thus, for $t \geq 0$ we have:

$$\psi(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_\nu[(-it)^\nu w]\psi(k)\, dk \tag{3.20}$$

which is the result obtained in ref.[3].

We proceed to analyze the solutions of (3.4) for some typical cases of the derivative index ν in the following section.

Examples for free particle fractionary time evolution

As a first example we consider the case $\nu = 1/2$

Let α be given by:

$$\alpha = \frac{L_p^2}{2T_p^{\frac{1}{2}} N_m} \tag{3.21}$$

From (3.4) we obtain

$$\psi(t,x) = \oint_\Gamma \int_{-\infty}^{\infty} \frac{a(k_0,k)}{k_0^{\frac{1}{2}} - \alpha k^2} e^{-i(k_0 t + kx)} dk_0\, dk \tag{3.22}$$

or equivalently:

$$\psi(t,x) = \int_{-\infty}^{\infty} a(k) e^{-i(\alpha^2 k^4 t + kx)} dk + \int_{-\infty}^{0}\int_{-\infty}^{\infty} a(k_0,k)$$

$$\left[\frac{1}{(k_0+i0)^{\frac{1}{2}}-\alpha k^2}-\frac{1}{(k_0-i0)^{\frac{1}{2}}-\alpha k^2}\right]e^{-i(k_0t+kx)}dk_0\,dk \tag{3.23}$$

where:

$$a(k)=-4\pi i\alpha k^2 a(\alpha^2k^4,k)$$

With some of algebraic calculus we obtain for (3.7):

$$\psi(t,x)=\int_{-\infty}^{\infty}a(k)e^{-i(\omega^2t+kx)}dk+$$

$$\int_0^{\infty}\int_{-\infty}^{\infty}\frac{a(k_0,k)}{k_0+\omega^2}e^{i(k_0t-kx)}dk_0\,dk \tag{3.24}$$

with:

$$\omega=\alpha k^2$$

and where we have made the re-scaling:

$$-2ik_0^{\frac{1}{2}}a(-k_0,k)\rightarrow a(k_0,k)$$

The first term in (3.8) represent free particle on-shell propagation and the second term describes the contribution of off-shell modes.

As a second example we consider the case $\nu=1$.

In this case (3.4) takes the form:

$$\psi(t,x)=\oint_{\Gamma}\int_{-\infty}^{\infty}\frac{a(k_0,k)}{k_0-\omega}e^{-i(k_0t+kx)}dk_0\,dk \tag{3.25}$$

Evaluating the integral in the variable k_0 we have:

$$\psi(t,x) = \int_{-\infty}^{\infty} a(k) e^{-i(\omega t + kx)} dk \tag{3.26}$$

where $a(k) = -2\pi i a(\omega, k)$. Thus we recover the usual expression for the free-particle wave function.

Finally we consider the case $\nu = 2$. For it we have

$$\psi(t,x) = \oint_{\Gamma} \int_{-\infty}^{\infty} \frac{a(k_0,k)}{k_0^2 - \omega^2} e^{-i(k_0 t + kx)} dk_0 \, dk \tag{3.27}$$

After to perform the integral in the variable k_0 we obtain from (3.27):

$$\psi(t,x) = \int_{-\infty}^{\infty} a(k) e^{-i(\omega t + kx)} + b^{+}(k) e^{i(\omega t + kx)} dk \tag{3.28}$$

with $a(k) = -2\pi i a(\omega, k)$ and $b^{+}(k) = -2\pi i a(-\omega, -k)$

4 The Potential Well

We consider in this section the potential well. The fractionary equation for a particle confined to move within interval $0 \le x \le a$ is:

$$(iT_p)^{\nu} \partial_t^{\nu} \psi(t,x) = -\frac{L_p^2}{2N_m} \partial_x^2 \psi(t,x) \tag{4.1}$$

To solve this equation we use the method of separation of variables. Thus if we write:

$$\psi(t,x) = \psi_1(t)\psi_2(x) \tag{4.2}$$

As is usual we obtain:

$$\frac{(iT_p)^\nu \partial_t^\nu \psi_1(t)}{\psi_1(t)} = -\frac{\frac{L_p^2}{2N_m}\partial_x^2 \psi_2(x)}{\psi_2(x)} = \lambda \tag{4.3}$$

Then we conclude that $\psi_2(x)$ satisfies:

$$\partial_x^2 \psi_2(x) + \frac{2\lambda N_m}{L_p^2}\psi_2(x) = 0 \tag{4.4}$$

The solution of (4.4) is the habitual one:

$$\psi_{2n}(x) = b_n \sin\left(\frac{n\pi}{a}x\right) \tag{4.5}$$

with:

$$\lambda_n = \frac{1}{2N_m}\left(\frac{n\pi L_p}{a}\right)^2 \tag{4.6}$$

and the boundary conditions satisfied by $\psi_{2n}(x)$ are:

$$\psi_{2n}(0) = \psi_{2n}(a) = 0$$

As a consequence of (4.3),(4.5) and (4.6) the Fourier transform $\hat{\psi}_1(k_0)$ of $\psi_1(t)$ should be satisfy:

$$(k_0^\nu - w_n)\hat{\psi}_{1n}(k_0) = 0 \tag{4.7}$$

($w_n = \lambda_n / T_p^\nu$) whose solution is:

$$\hat{\psi}_{1n}(k_0) = \frac{c_n(k_0)}{k_0^\nu - w_n} \tag{4.8}$$

Therefore the final general solution for $\psi(t,x)$ is:

$$\psi(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \oint_{\Gamma} \frac{a_n(k_0)e^{-ik_0 t}}{k_0^{\nu} - w_n} dk_0 \tag{4.9}$$

where we have defined:

$$a_n(k_0) = \frac{b_n c_n(k_0)}{2\pi}$$

which is an entire analytic function of k_0. As a consequence we can develop $a_n(k_0)$ in a power series already of $k_0 = 0$

$$a_n(k_0) = \sum_{0}^{\infty} a_{nm} k_0^m$$

and obtain for $\psi(t,x)$ the expression:

$$\psi(t,x) = \sum_{n=1,m=0}^{\infty} a_{nm} \sin\left(\frac{n\pi}{a}x\right) \oint_{\Gamma} \frac{k_0^m}{k_0^{\nu} - w_n} e^{-ik_0 t}\, dk_0 \tag{4.10}$$

According to (3.10) we have then:

$$\psi(t,x) = \sum_{n=1,m=0}^{\infty} a_{nm} \sin\left(\frac{n\pi}{a}x\right) e^{-i\frac{\pi}{2}(\nu-m)}(t+i0)^{\nu-m-1} \times$$
$$E_{\nu,\nu-m}[(t+i0)^{\nu} e^{-i\frac{\pi}{2}\nu} w_n] \tag{4.11}$$

If we select $a_n(k_0) = a_n$ independent of the variable k_0, we obtain:

$$\psi(t,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{a}x\right) e^{-i\frac{\pi}{2}(\nu)}(t+i0)^{\nu--1} \times$$
$$E_{\nu,\nu}[(t+i0)^{\nu} e^{-i\frac{\pi}{2}\nu} w_n] \tag{4.12}$$

When we consider the solution only for $t \geq 0$, i. e. $\psi(t,x)H(t)$, we recover the result given in ref.[3], since the equation for this case is:

$$(i\partial_t)^{\nu-1}\{[(i\partial_t)\psi(t,x)]H(t)\} + \frac{L_p^2}{2N_m T_p^\nu}\partial_x^2[\psi(t,x)H(t)] = 0 \tag{4.13}$$

With the initial condition $\psi(0,x) = \psi(x)$ we have for (4.13)

$$(i\partial_t)^{\nu-1}\{(i\partial_t)[\psi(t,x)H(t)] - i\psi(x)\delta(t)\} + \frac{L_p^2}{2N_m T_p^\nu}\partial_x^2[\psi(t,x)H(t)] = 0 \tag{4.14}$$

Purposing again

$$\psi(t,x) = \psi_1(t)\psi_2(x)$$

we obtain two equations:

$$T_p^\nu(i\partial_t)^{\nu-1}\{(i\partial_t)[\psi_1(t)H(t)] - i\psi_1(0)\delta(t)\} - \lambda\psi_1(t)H(t) = 0 \tag{4.15}$$

$$\partial_x^2\psi_2(x) + \frac{2N_m\lambda}{L_p^2}\psi_2(x) = 0 \tag{4.16}$$

Solution of (4.16) is again:

$$\psi_{2n}(x) = b_n \sin\left(\frac{n\pi}{a}x\right) \tag{4.17}$$

with:

$$\lambda_n = \frac{1}{2N_m}\left(\frac{n\pi L_p}{a}\right)^2 \tag{4.18}$$

Using the Fourier transform for (4.15) we obtain:

$$T_p^\nu k_0^\nu \hat{\psi}_1(k_0) - ik_0^{\nu-1}\psi_1(0)H[\Im(k_0)] - \lambda_n\hat{\psi}_1(k_0) \tag{4.19}$$

whose solution is:

$$\hat{\psi}_1(k_0) = \frac{i\psi_1(0)}{T_p^\nu} H[\Im(k_0)] \frac{k_0^{\nu-1}}{k_0^\nu - w_n} \tag{4.20}$$

and then

$$\psi_1(t)H(t) = \frac{i\psi_1(0)}{2\pi T_p^\nu} \oint_\Gamma H[\Im(k_0)] \frac{k_0^{\nu-1}}{k_0^\nu - w_n} e^{-ik_0 t}\, dk_0 \tag{4.21}$$

In the configuration space (4.21) reads:

$$\psi_1(t)H(t) = \frac{i\psi_1(0)}{2\pi T_p^\nu} H(t) E_\nu[(-it)^\nu w_n] \tag{4.22}$$

Then we have for $\psi(t,x)$:

$$\psi(t,x) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi}{a}x\right) E_\nu[(-it)^\nu w_n] \tag{4.23}$$

Examples for Potential Well fractionary time evolution

As a first example we consider the case $\nu = 1/2$. For it the solution (4.9) takes the form:

$$\psi(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \oint_\Gamma \frac{a_n(k_0)e^{-ik_0 t}}{k_0^{\frac{1}{2}} - w_n} dk_0 \tag{4.24}$$

or equivalently:

$$\psi(t,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{a}x\right) e^{-iw_n^2 t} +$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \int_{-\infty}^{0} \left[\frac{1}{(k_0 + i0)^{\frac{1}{2}} - w_n} - \frac{1}{(k_0 - i0)^{\frac{1}{2}} - w_n}\right] \times$$

$$a_n(k_0)e^{-ik_0t}\, dk_0 \tag{4.25}$$

After performing some algebra we have for (4.11) the expression:

$$\psi(t,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{a}x\right) e^{-iw_n^2 t} +$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \int_0^{\infty} \frac{a_n(k_0)}{k_0 + w_n^2} e^{-ik_0t}\, dk_0 \tag{4.26}$$

Analogously as before, the second term in (4.12) represents of-shell stationary modes.

As a second example we consider $\nu = 1$. In this case:

$$\psi(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \oint_{\Gamma} \frac{a_n(k_0)e^{-ik_0t}}{k_0 - w_n} dk_0 \tag{4.27}$$

Performing the integral in the variable k_0 we have:

$$\psi(t,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{a}x\right) e^{-iw_n t} \tag{4.28}$$

Which is the familiar general solution for the infinite well.

Finally for $\nu = 2$:

$$\psi(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \oint_{\Gamma} \frac{a_n(k_0)e^{-ik_0t}}{k_0^2 - w_n} dk_0 \tag{4.29}$$

and after to compute the integral:

$$\psi(t,x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right) \left(a_n e^{-i\sqrt{w_n}t} + b_n e^{+i\sqrt{w_n}t}\right) \tag{4.30}$$

with $a_n = a_n(\sqrt{w_n})$ and $b_n^+ = a_n(-\sqrt{w_n})$

5 The Green Function for The Free Particle

As other application that shows the generality of the fractional calculus defined with the use of ultradistributions, we give the evaluation of the Green function corresponding to the free particle. Let β be defined as:

$$\beta^2 = \frac{L_p^2}{2T_p^\nu N_m} \tag{5.1}$$

Then $G(t-t', x-x')$ should be satisfy the equation:

$$(i\partial_t)^\nu G(t-t', x-x') + \beta^2 \partial_x^2 G(t-t', x-x') = \delta(t-t')\delta(x-x') \tag{5.2}$$

As G is function of $(t-t', x-x')$ it is sufficient to consider G as function of (t, x):

$$(i\partial_t)^\nu G(t, x) + \beta^2 \partial_x^2 G(t, x) = \delta(t)\delta(x) \tag{5.3}$$

For the Fourier transform $\hat{G}$ of G we have:

$$(k_0^\nu - \beta^2 k^2)\hat{G}(k_0, k) = \frac{Sgn[\Im(k_0)]}{2} + a(k_0, k) \tag{5.4}$$

where $a(k_0, k)$ is as usual a rapidly decreasing analytic entire function of the variable k_0 . Selecting:

$$a(k_0, k) = \frac{1}{2}$$

we obtain the equation for the retarded Green function:

$$(k_0^\nu - \beta^2 k^2)\hat{G}_{ret}(k_0, k) = H[\Im(k_0)] \tag{5.5}$$

and then:

$$G_{ret}(t,x) = \frac{1}{4\pi^2} \oint_{\Gamma} \int_{-\infty}^{\infty} \frac{H[\Im(k_0)]}{k_0^{\nu} - \beta^2 k^2} e^{-i(k_0 t + kx)}\, dk_0\, dk \tag{5.6}$$

If we take:

$$a(k_0, k) = -\frac{1}{2}$$

we obtain the advanced Green function:

$$G_{adv}(t,x) = -\frac{1}{4\pi^2} \oint_{\Gamma} \int_{-\infty}^{\infty} \frac{H[-\Im(k_0)]}{k_0^{\nu} - \beta^2 k^2} e^{-i(k_0 t + kx)}\, dk_0\, dk \tag{5.7}$$

For the Wheeler Green function (half advanced plus half retarded):

$$G_W(t,x) = \frac{1}{2}[G_{adv}(t,x) + G_{ret}(t,x)] \tag{5.8}$$

we have:

$$G_W(t,x) = \frac{1}{8\pi^2} \oint_{\Gamma} \int_{-\infty}^{\infty} \frac{Sgn[\Im(k_0)]}{k_0^{\nu} - \beta^2 k^2} e^{-i(k_0 t + kx)}\, dk_0\, dk \tag{5.9}$$

Using (3.7), (3.8) with $m = 0$ we obtain:

$$G_{ret}(t,x) = \frac{1}{2\pi} e^{-i\frac{\pi\nu}{2}} t_+^{\nu-1} \int_{-\infty}^{\infty} E_{\nu,\nu}(e^{-i\frac{\pi\nu}{2}} t_+^{\nu} \beta^2 k^2) e^{-ikx}\, dk \tag{5.10}$$

$$G_{adv}(t,x) = \frac{1}{2\pi} e^{-i\pi(\frac{\nu}{2}-1)} t_-^{\nu-1} \int_{-\infty}^{\infty} E_{\nu,\nu}(e^{i\frac{\pi\nu}{2}} t_-^{\nu} \beta^2 k^2) e^{-ikx}\, dk \tag{5.11}$$

$$G_w(t,x) = \frac{1}{4\pi} e^{-i\frac{\pi\nu}{2}} (t+i0)^{\nu-1} \int_{-\infty}^{\infty} E_{\nu,\nu}[e^{-i\frac{\pi\nu}{2}} (t+i0)^{\nu} \beta^2 k^2] e^{-ikx}\, dk \tag{5.12}$$

Example for the free particle Green function

When we select $\nu = 1$ we obtain the usual Green functions of Quantum Mechanics. For example for G_{ret} we have:

$$G_{ret}(t,x) = \frac{1}{4\pi^2} \oint_{\Gamma} \int_{-\infty}^{\infty} \frac{H[\Im(k_0)]}{k_0 - \beta^2 k^2} e^{-i(k_0 t + kx)} \, dk_0 \, dk \tag{5.13}$$

or equivalently:

$$G_{ret}(t,x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(k_0 + i0) - \beta^2 k^2} e^{-i(k_0 t + kx)} \, dk_0 \, dk \tag{5.14}$$

After the evaluation of the integral in the variable k_0, G_{ret} takes the form:

$$G_{ret}(t,x) = -\frac{i}{2\pi} H(t) \int_{-\infty}^{\infty} e^{-i(\beta^2 k^2 t + kx)} \, dk \tag{5.15}$$

With a square's completion (5.12) transforms into:

$$G_{ret}(t,x) = -\frac{iH(t)}{2\pi\beta\sqrt{t}} e^{\frac{ix^2}{4\beta^2 t}} \int_{-\infty}^{\infty} e^{is^2} ds \tag{5.16}$$

From the result of ref.[9]

$$\int_{-\infty}^{\infty} e^{is^2} ds = \sqrt{\pi} e^{-i\frac{\pi}{4}} \tag{5.17}$$

we have

$$G_{ret}(t,x) = -iH(t) \left(\frac{m}{2\pi i \hbar t} \right)^{\frac{1}{2}} e^{\frac{imx^2}{2\hbar t}} \tag{5.18}$$

Taking into account that for $\nu = 1$:

$$\beta^2 = \frac{\hbar}{2m}$$

we obtain the usual form of G_{ret} (see ref.[10])

$$G_{ret}(t-t^{'},x-x^{'}) = -iH(t-t^{'})\left(\frac{m}{2\pi i\hbar(t-t^{'})}\right)^{\frac{1}{2}} e^{\frac{im(x-x^{'})^2}{2\hbar(t-t^{'})}} \tag{5.19}$$

With a similar calculus we have for G_{adv}:

$$G_{adv}(t-t^{'},x-x^{'}) = iH(t^{'}-t)\left(\frac{m}{2\pi i\hbar(t^{'}-t)}\right)^{\frac{1}{2}} e^{\frac{im(x-x^{'})^2}{2\hbar(t-t^{'})}} \tag{5.20}$$

and for G_W:

$$G_W(t-t^{'},x-x^{'}) = -\frac{i}{2}Sgn(t-t^{'})\left(\frac{m}{2\pi i\hbar|t-t^{'}|}\right)^{\frac{1}{2}} e^{\frac{im(x-x^{'})^2}{2\hbar(t-t^{'})}} \tag{5.21}$$

6 Discussion

We have defined the fractionary Schrödinger equation for all values of the complex variable ν and treated the cases of the free particle and the potential well. We have performed this task based in a earlier paper (ref.[5] where we have shown the existence of a general fractional calculus defined via tempered ultradistributions. In fact, all ultradistributions provide integrands that are analytic functions along the integration path. Therefore these properties show that tempered ultradistributions provide an appropriate framework for applications to fractional calculus. With the use of this calculus we have generalized in the present work the results obtained by

Naber (ref.[3]). As special examples, for $\nu = 1$ the results obtained coincide with the usual Quantum Mechanics, and the cases $\nu = 1/2$ and $\nu = 2$ have shown the appearance of extra terms, besides to those with the usual ($\nu = 1$) framework. In addition we have obtained a general expression for the Green function of the free particle, and shown that for $\nu = 1$ this Green function coincide with the obtained in ref.[10]. For the benefit of the reader we give in this paper two Appendixes with the main characteristics of n-dimensional tempered ultradistributions and their Fourier anti-transformed distributions of the exponential type, and a third Appendix about the general fractional calculus defined via the use of tempered ultradistributions.

7 Appendix I: Distributions of Exponential Type

For the sake of the reader we shall present a brief description of the principal properties of Tempered Ultradistributions.

Notations. The notations are almost textually taken from ref[12]. Let $\mathbb{R}^n$ (res. $\mathbb{C}^n$) be the real (resp. complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp $z = (z_1, z_2, ..., z_n)$). We shall use the notations:

(i) $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$

(ii)$x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, ..., x_n \geqq 0$

(iii)$x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$

(iV)$| x |= \sum\limits_{j=1}^{n} | x_j |$

Let $\mathbb{N}^n$ be the set of n-tuples of natural numbers. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, ..., p_n)$, and p_j is a natural number, $1 \leqq j \leqq n$. $p + q$ denote $(p_1 + q_1, p_2 + q_2, ..., p_n + q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, ..., p_n \geqq q_n$. x^p means $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We shall denote by $| p |= \sum\limits_{j=1}^{n} p_j$ and by D^p we denote the differential operator $\partial^{p_1+p_2+...+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} ... \partial {x_n}^{p_n}$

For any natural k we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k ... \partial x_n^k$

The space $\mathcal{H}$ of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any p and q is defined (ref.[12]) by means of the countably set of norms:

$$\|\hat{\phi}\|_p = \sup_{0\leq q\leq p,\, x} e^{p|x|} \left|D^q\hat{\phi}(x)\right| \quad , \quad p = 0, 1, 2, ... \tag{7.1}$$

According to reference[14] $\mathcal{H}$ is a $\mathcal{K}\{\boldsymbol{M_p}\}$ space with:

$$M_p(x) = e^{(p-1)|x|} \quad , \quad p = 1, 2, ... \tag{7.2}$$

$\mathcal{K}\{\boldsymbol{e}^{(p-1)|x|}\}$ satisfies condition $(\mathcal{N})$ of Guelfand (ref.[13]). It is a countable Hilbert and nuclear space:

$$\mathcal{K}\{\boldsymbol{e}^{(p-1)|x|}\} = \mathcal{H} = \bigcap_{p=1}^{\infty} \mathcal{H}_p \tag{7.3}$$

where $\mathcal{H}_p$ is obtained by completing $\mathcal{H}$ with the norm induced by the scalar product:

$$< \hat{\phi}, \hat{\psi} >_p = \int\limits_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q\overline{\hat{\phi}}(x) D^q\hat{\psi}(x)\; dx \quad ; \quad p = 1, 2, ... \tag{7.4}$$

where $dx = dx_1\; dx_2 ... dx_n$

If we take the usual scalar product:

$$< \hat{\phi}, \hat{\psi} >= \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\; dx \tag{7.5}$$

then $\mathcal{H}$, completed with (7.5), is the Hilbert space $\mathbf{H}$ of square integrable functions.

The space of continuous linear functionals defined on $\boldsymbol{\mathcal{H}}$ is the space $\boldsymbol{\Lambda}_\infty$ of the distributions of the exponential type (ref.[12]).

The "nested space"

$$\mathsf{H} = (\boldsymbol{\mathcal{H}}, \mathsf{H}, \boldsymbol{\Lambda}_\infty) \tag{7.6}$$

is a Guelfand's triplet (or a Rigged Hilbert space [13]).

In addition we have: $\boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{S}} \subset \mathsf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\Lambda}_\infty$, where $\boldsymbol{\mathcal{S}}$ is the Schwartz space of rapidly decreasing test functions (ref[15]).

Any Guelfand's triplet $\mathsf{G} = (\boldsymbol{\Phi}, \mathsf{H}, \boldsymbol{\Phi}')$ has the fundamental property that a linear and symmetric operator on $\boldsymbol{\Phi}$, admitting an extension to a self-adjoint operator in H, has a complete set of generalized eigen-functions in $\boldsymbol{\Phi}'$ with real eigenvalues.

8 Appendix II: Tempered Ultradistributions

The Fourier transform of a function $\hat{\phi} \in \boldsymbol{\mathcal{H}}$ is

$$\phi(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\ e^{iz\cdot x}\ dx \tag{8.1}$$

$\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $\mathfrak{H}$ the set of all such functions.

$$\mathfrak{H} = \mathcal{F}\{\boldsymbol{\mathcal{H}}\} \tag{8.2}$$

It is a $\mathcal{Z}\{\mathbf{M_p}\}$ space (ref.[14]), countably normed and complete, with:

$$M_p(z) = (1 + |z|)^p \tag{8.3}$$

$\mathfrak{H}$ is also a nuclear space with norms:

$$\|\phi\|_{pn} = \sup_{z \in V_n} (1 + |z|)^p |\phi(z)| \tag{8.4}$$

where $V_k = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : | \operatorname{Im} z_j | \leqq k, 1 \leqq j \leqq n\}$

We can define the usual scalar product:

$$< \phi(z), \psi(z) >= \int\limits_{-\infty}^{\infty} \phi(z)\psi_1(z)\ dz = \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\ dx \tag{8.5}$$

where:

$$\psi_1(z) = \int\limits_{-\infty}^{\infty} \hat{\psi}(x)\ e^{-iz\cdot x}\ dx$$

and $dz = dz_1\ dz_2 ... dz_n$

By completing $\mathfrak{H}$ with the norm induced by (8.5) we get the Hilbert space of square integrable functions.

The dual of $\mathfrak{H}$ is the space $\boldsymbol{\mathcal{U}}$ of tempered ultradistributions (ref.[12]). In other words, a tempered ultradistribution is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis.

The set $U = (\mathfrak{H}, H, \boldsymbol{\mathcal{U}})$ is also a Guelfand's triplet.

Moreover, we have: $\mathfrak{H} \subset \boldsymbol{\mathcal{S}} \subset \mathsf{H} \subset \boldsymbol{\mathcal{S}}' \subset \boldsymbol{\mathcal{U}}$.

$\boldsymbol{\mathcal{U}}$ can also be characterized in the following way (ref.[12]): let $\boldsymbol{\mathcal{A}}_\omega$ be the space of all functions $F(z)$ such that:

I- $F(z)$ is analytic for $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

II- $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let $\boldsymbol{\Pi}$ be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\boldsymbol{\mathcal{U}}$ is the quotient space:

III- $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}}_\omega / \boldsymbol{\Pi}$

By a pseudo-polynomial we understand a function of z of the form $\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$ with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \boldsymbol{\mathcal{A}}_\omega$

Due to these properties it is possible to represent any ultradistribution as (ref.[12]):

$$F(\phi) = < F(z), \phi(z) > = \oint_\Gamma F(z)\phi(z) \, dz \tag{8.6}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup ...\Gamma_n$ where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).

Formula (8.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac formula" for ultradistributions (ref.[11]):

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \, dt \tag{8.7}$$

where the "density" $f(t)$ is such that

$$\oint_{\Gamma} F(z)\phi(z) \, dz = \int_{-\infty}^{\infty} f(t)\phi(t) \, dt \tag{8.8}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (8.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z (ref.[12]):

$$|F(z)| \leq C|z|^p \tag{8.9}$$

where C and p depend on F.

The representation (8.6) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ do not alter the ultradistribution:

$$\oint_{\Gamma} \{F(z) + P(z)\}\phi(z) \, dz = \oint_{\Gamma} F(z)\phi(z) \, dz + \oint_{\Gamma} P(z)\phi(z) \, dz$$

But:

$$\oint_{\Gamma} P(z)\phi(z) \, dz = 0$$

as $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing),

$$\therefore \quad \oint\limits_{\Gamma}\{F(z)+P(z)\}\phi(z)\, dz = \oint\limits_{\Gamma} F(z)\phi(z)\, dz \tag{8.10}$$

9 Appendix III: Fractional Calculus

The purpose of this sections is to introduce definition of fractional derivation and integration given in ref. [11]. This definition unifies the notion of integral and derivative in one only operation. Let $\hat{f}(x)$ a distribution of exponential type and $F(\Omega)$ the complex Fourier transformed Tempered Ultradistribution. Then:

$$F(k) = H[\Im(k)]\int\limits_{0}^{\infty}\hat{f}(x)e^{ikx}\, dx - H[-\Im(k)]\int\limits_{-\infty}^{0}\hat{f}(x)e^{ikx}\, dx \tag{9.1}$$

($H(x)$ is the Heaviside step function) and

$$\hat{f}(x) = \frac{1}{2\pi}\oint\limits_{\Gamma} F(k)e^{-ikx}\, dk \tag{9.2}$$

where the contour Γ surround all singularities of $F(k)$ and runs parallel to real axis from $-\infty$ to ∞ above the real axis and from ∞ to $-\infty$ below the real axis. According to [11] the fractional derivative of $\hat{f}(x)$ is given by

$$\frac{d^{\lambda}\hat{f}(x)}{dx^{\lambda}} = \frac{1}{2\pi}\oint\limits_{\Gamma}(-ik)^{\lambda}F(k)e^{-ikx}\, dk + \oint\limits_{\Gamma}(-ik)^{\lambda}a(k)e^{-ikx}\, dk \tag{9.3}$$

Where $a(k)$ is entire analytic and rapidly decreasing. If $\lambda = -1$, d^λ/dx^λ is the inverse of the derivative (an integration). In this case the second term of the right side of (9.3) gives a primitive of $\hat{f}(x)$. Using Cauchy's theorem the additional term is

$$\oint \frac{a(k)}{k} e^{-ikx} dk = 2\pi a(0) \tag{9.4}$$

Of course, an integration should give a primitive plus an arbitrary constant. Analogously when $\lambda = -2$ (a double iterated integration) we have

$$\oint \frac{a(k)}{k^2} e^{-ikx} dk = \gamma + \delta x \tag{9.5}$$

where γ and δ are arbitrary constants.

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