Quantile Analysis of "Hazard-Rate" Game Models *

Andreea ENACHE† and Jean-Pierre FLORENS‡

June 2020

Abstract

This paper consists of an econometric analysis of a broad class of games of incomplete information. In these games, a player's action depends both on her unobservable characteristic (the private information), as well as on the ratio of the distribution of the unobservable characteristic and its density function (which we call the "hazard-rate"). The goal is to use data on players' actions to recover the distribution of private information. We show that the structural parameter (the distribution of the unobservable characteristic) can be related to the reduced form parameter (the distribution of the data) through a quantile relation that avoids the inversion of the players' strategy function. We estimate non-parametrically the density of the unobserved variables and we show that this is the solution of a well-posed inverse problem. Moreover, we prove that the density of the private information is estimated at a $\sqrt{n}$ speed of convergence. Our results have several policy applications, including better design of auctions and public good contracts.

Keywords: quantile estimation, well-posed inverse problems, auctions, regulation models, monotone hazard-rate

JEL Classification: $C7, C57, C14$

*We are indebted to Stéphane Bonhomme, Christian Bontemps, Xavier D'Haultfoeuille, Lars Peter Hansen, Christophe Hurlin, David Martimort, Isabelle Perrigne, Peter C.B. Phillips, Anna Simoni and Quang Vuong for many helpful discussions and constructive comments. Andreea Enache acknowledges financial support from the Carlo Giannini Research Fellowship in Econometrics (2016-2018). Jean-Pierre Florens acknowledges funding from the French National Research Agency (ANR) under the Investments for the Future program (Investissements díAvenir, grant ANR-17-EURE-0010).

†Stockholm School of Economics. Email: andreea.enache@hhs.se *(corresponding author).*

‡Toulouse School of Economics. Email: jean-pierre.florens@tse-fr.eu.

1 Introduction

This paper contributes to the econometric literature devoted to the study of information economics and empirical game theory models. Games of incomplete information are a special class of games in which the payoffs of other players, their possible actions, information, risk attitudes, identity etc. are not common knowledge.[1] The actions of the players are the result of a transformation of their unobserved characteristic[2] and of the distribution of this characteristic. Being able to estimate the distribution of hidden information of the players based on their actions and on a notion of equilibrium is essential in the policy-making process. For example, knowing the distribution of private values can help the auctioneer choose a better format of auction, set the reserve price, assess the effect of increasing or decreasing the number of participants in an auction or change the rules of the game. Once the distribution of latent variables is known, one can conduct different counterfactuals to evaluate which configuration of auction yields a higher revenue for the auctioneer. Another example is the principal-agent model where the agent could be a firm providing a public service while being regulated by a local municipality. In this particular case, the asymmetric information comes from the fact that the firm has more information about its productivity and technological process than the principal. Recovering these unknowns would help the principal in designing the contracts of public good provision and setting the fares.

We perform a quantile-based analysis of a general class of games (auctions, delegation models etc.) where the strategy function does not have an explicit functional form, but nevertheless is a function of the latent variable and of the "hazard-rate" of this unobservable.[3] The main characteristics of the previously mentioned problems can be summarized as follows. The econometrician observes the realization of a random element x (the action or the bids), which is generated by a transformation of a latent variable ξ ($x = \sigma(\xi)$), where ξ denotes the private signal or type of the player and σ is the strategy function. The strategic component of the model is formalized by the dependence of σ on the distribution of hidden types,ξ, denoted by F. The interpretation is the following: a player knows his information ξ, the distribution from which this information is drawn, and chooses x in function of the

[1] By contrast with games of imperfect information where the information about the game being played is actually complete. In games of imperfect informations, the uncertainty surrounds only the actions chosen by the other players.

[2] We are going to use interchangeably the terms of unobserved characteristic, latent variable and private information.

[3] Actually what we name in this paper "hazard-rate" is in fact the inverse of the reversed hazard-rate, i.e. the ratio between the distribution of unobservables and their density function. Therefore this concept is slightly different from the classical hazard rate defined as the ratio between the density function and the survivor function.

relative position of ξ to the distribution F. The model is then written: $x = \sigma(\xi, F)$. The strategic function is obtained through an equilibrium rule (e.g. Bayesian Nash equilibrium). Moreover, σ is considered to be one to one and increasing in types. Put succinctly, the player knows his own type ξ, the distribution of types F and plays x. He does not observe the type of the other players. The statistician observes only x but the equilibrium rule σ is known by both the players and the statistician as a function of ξ and F. For simplicity's sake, we consider that an iid sample of data $\{x_1, x_2, \ldots, x_n\}$ is available and that it usually comes from several games and several players. The objective of the structural analysis is to state F from the sample of x.[4] This distribution is the structural parameter in the sense that it does not depend on the rule of the game and/or the chosen strategy. The knowledge of F allows us, for example, to simulate the output of a new game rule (applying a new strategy σ).

Econometric models of games of incomplete information have been considered by many authors, in this form or in an equivalent presentation (see Guerre et al. (2000), Perrigne and Vuong (1999), Li et al. (2002), Florens and Sbaï (2010), Florens et al. (1998), Laffont et al. (1995), Perrigne and Vuong (2011)), but this paper has several specific points.

Firstly, we will focus on a particular class of strategies σ expressed as a function a of two elements:

$$\sigma(\xi, F) = a(\xi, \lambda(\xi)), \text{where } \lambda(\xi) = \frac{F(\xi)}{f(\xi)}$$

and f denotes the strictly positive probability density function corresponding to F and λ is a strictly increasing function. We call these models "hazard-rate game models",[5] even if the λ function is not strictly speaking the hazard rate. Thus, we introduce a class of well-posed problems in economics (the concept will be defined later in the paper). The characterization of a category of well-posed problems represents a novelty in the sense that many of the problems encountered in economics are ill-posed problems. Working with ill-posed problems raises many estimation issues and therefore unveiling a class of well-posed problems is "good-news". [6]

Secondly, the usual analysis is based on the relation between the distribution of x and the distribution of ξ. Compared with Guerre et al. (2000) and Perrigne and Vuong (2011), in our paper we focus on the transformation of the quantile function of ξ into the quantile function of x and not on the relation between the cumulative distribution functions. Therefore this

[4] As opposed to the reduced-form approach where the analysis is conducted mainly on the statistical properties of the sample, here the interest is focused on the data-generating process.

[5] One should note that the "hazard-rate game models" are in no way related to the theory on the "Duration models".

[6] A recent paper unveiling well-posedness in error measurement models for self-reported data is An and Hu (2012).

work contributes to developing quantile approach in the auction framework as pioneered by Marmer and Shneyerov (2012). There is no loss of information from privileging the quantile approach over the cumulative distribution function approach, while the gains are important. Moreover, sometimes the quantile methodology allows the transformation of a nonlinear inverse problem into a linear one, thus simplifying the identification analysis and providing closed-form solutions to the problem under study. The estimation stage is as well quite simplified by the use of the quantile approach. One usually obtains constructive identification and the estimator is a "plug-in" estimator. Next, the asymptotic properties can be derived using the well-established theory of order statistics. Quantile approaches in games of incomplete information have started to attract a lot of interest (see Haile et al. (2003), Marmer et al. (2013), Guerre et al. (2009), Zincenko (2013), Liu et al. (2014), Enache and Florens (2018), Gimenes et al. (2016), Gimenes (2015) and Guerre and Sabbah (2012), Enache and Florens (2020), Enache and Florens (2014)), Enache (2015).

Thirdly, we apply the above methodology to two instances of well-posed problems for which the first-order condition is characterized by the presence of the hazard-rate: the third-price auction model and the pure adverse selection model (treated also in Enache and Florens (2020), Enache and Florens (2018)). Another instance of hazard-rate game is the nonlinear pricing model which has already been treated in Luo et al. (2014). Some other possible examples where our methodology could be used are the models of war of attrition and all-pay auctions (see Krishna and Morgan (1997)), double auctions (see Wilson (1985)) and some search models (see Anderson and Renault (1999)). All these above models exhibit the property of strictly monotone hazard-rate in the first order condition.

Finally, we check the asymptotic properties of our estimator and we conclude that, although the analysis in our paper is conducted in a nonparametric fashion, we obtain a root-n speed of convergence for the estimator of the density of latent variables. This result of "super convergence" might seem surprising, but the mathematical intuition behind it is related to the fact that the model involves the presence of the "hazard-rate" of the latent variables in the equilibrium condition of the game. Thus, solving the inverse problem in this context boils down to solving a differential equation. Loosely speaking, in order to get the solution, one has to integrate and the integration will smooth the process and will lead to some nice asymptotic properties for the estimators.

As we previously mentioned, our methodology has several economic applications. In an auction model, the players know how much they value the good, but don't know how much the good is valued by other players. The auctioneer himself knows only the distribution from which the private values are drawn, but not the exact private values of the bidders. Moreover, the players might not be aware of whether the players are risk-averse or not and also the identity of bidders could be confined.[7] The econometrician can observe the bids, which are in fact the actions taken by the players accordingly to their preferences (i.e. their private values) and to a concept of equilibrium. In this situation, the latent variables or the data generating process are clearly the private values. Of course, one could suppose that both the private values and the equilibrium of the game are unknown to the econometrician, but this would make the task of recovering the structural parameter quite complicated. The direct problem of this economic instance would be to observe the private values and from there to infer what the data observed on the market should be. This is the case in experimental economics, where the designer of the game knows from the beginning what are the valuations and then, based on the players' actions, tests if they behave accordingly with the economic theory or the supposed equilibrium concept. Outside of the experimental field, these valuations are unknown to the auctioneer and in practice, contrary to the hypothesis of the economic theory, even the distribution they are drawn from is not known for the auctioneer. The role of the econometrician is to retrieve this distribution of private information, which serves in the decision-making process. For example, the theoretical results on auctions cannot be operationalized without a methodology to estimate the distribution of private values of the bidders. We know from the economic theory that the optimal format can be an auction with a reserve price, but the reserve price depends on the distribution of valuations. Therefore the theoretical recommendation is valuable only if we have a way to estimate the distribution of private values.

Another application are the regulation models. These describe economic situations in which a less informed party (called "the principal") delegates a task to another party ("the agent") that has a superior knowledge about its characteristics or about the efforts deployed to fulfill the task. The first type of asymmetric information is designated as *adverse selection* or *hidden information* problem and simply means that the agent has private information about its type and this information cannot be observed by the principal. The second informational aspect is the so-called *moral hazard* problem and it occurs because the efforts of the agent are not observable and therefore the agent has an incentive to shirk. The most common examples of principle-agent models are the contractual relationship between a local municipality that delegates the provision of a public good or service to a private contractor; between an employer who doesn't know the exact skills of the person he/she tries to hire; between the insurance company that cannot observe the diligence of the insured client; between the shareholders of a company and its manager, where, as in the previous example, the shareholders cannot observe how much effort the manager is putting in attaining the company's

[7] For self-contained works on auction theory, see Krishna (2009) and Milgrom (2004).

objectives fixed by the shareholders; or the typical example of a buyer (the principal) and the seller (the agent who provides the good and therefore has more information about its technological costs and its productivity). For a more detailed economic theory of incentives, see Laffont and Martimort (2001) or Laffont and Tirole (1993). In the particular case of a delegation model, the asymmetric information comes from the fact that the firm has more information about its productivity and technological process than the principal. The estimation of these unknowns would serve as a tool for the principal in the design of contracts of public good provision and the establishment of fares.

To summarize, this paper enriches the previous described econometric literature by mixing the quantile approach with the inverse-problem approach in the analysis of games of incomplete information.

The paper is organized as follows. Section 2 describes the inverse problem approach and presents two instances of well-posed problems, Section 3 treats the general model, Section 4 shows the local identification, Section 5 proposes an estimator and provides a numerical implementation for the estimator of the functional parameter in a third-price auction model (chosen because it is a nonlinear problem), Section 6 discusses the asymptotic properties and Section 7 concludes.

2 Two well-posed problems

Our econometric analysis is conducted using an inverse problem approach. The study of inverse problems has been for a long time a field of research reserved mainly to physicists, engineers and applied mathematicians. This situation is simply explained by the fact that a vast majority of applications belong to these areas: computer tomography, inverse scattering problem, geological prospecting, image deblurring etc. Nevertheless, there are many situations in economics where the inverse problem approach proves to be very useful. Without being exhaustive, some examples include: deconvolution, nonparametric instrumental variables estimation, finance (determining the volatility in option-pricing), game-theory models (estimation of the density of types in an auction or contract model) etc. Loosely speaking, an inverse problem is defined as the opposite of a forward problem. Any problem can be regarded either as a direct or inverse problem and therefore the choice of the direct problem/inverse problem is just a matter of convention. One can formulate the direct economic problem in the following way: given the following cause, what are the effects? Naturally, the inverse problem in this case will be: given the effects, what is the cause? Usually the inverse problem is defined as being the attempt of recovering the underlying structure from an indirect observation of it.

Finding the solution of an inverse problem involves solving a functional equation of the form $A(F) = G$, where F and G are two objects belonging to two separable Hilbert spaces, A is an operator which maps F into G and F is the parameter of interest. The inverse problems are classified into ill-posed and well-posed problems. Despite of what the terminology might suggest, the well-posedness of a problem is not related to the correct specification of a model. The study of an the well-posedness takes place in the setting of a well-specified model. The French mathematician Jacques Salomon Hadamard defined the three criteria that a well-posed problem should cumulatively meet: there exists a solution to the problem, the problem doesn't have more than one solution, a small change in the data should lead to only a small change in the solution. The uniqueness of the solution (at most one solution) corresponds to the well-known concept of identification in econometrics. The third criteria, also known as a stability issue, aims at evaluating whether the inverse mapping is continuous or not. A lack of continuity of the inverse mapping leads to an ill-posed problem. Imagine for example that the inverse mapping, A^{-1}, exists but it is not continuous. Let us suppose that F and G are two cumulative distribution functions of the latent variables and, respectively, of the observable variables. Even if one has a consistent estimator for G (for example, the empirical cumulative distribution function), the small errors related to the estimation of G will be translated in big errors in the estimation of F because of the discontinuity of A^{-1}. In this case, one cannot consistently estimate F using a consistent estimation for G because of the discontinuity in the inverse mapping (for more details, see Horowitz (2014) and Alquier et al. (2011)). Hence, in order to obtain a stable solution to the problem (which doesn't vary a lot when estimating F by using the sample distribution function and not the population distribution function), several methods of regularisation are used. The regularisation methods attempt at inverting the operator A in a manner that will allow for a consistent estimation of F. For a survey on regularisation methods see Carrasco et al. (2007).

The concept of inverse problems has been borrowed in economics mainly from the fields of physics and engineering. In non-technical terms, an inverse problem is an attempt to retrieve an unobservable input of a system from the noisy observation of its output. The output is obtained through a transformation of the input that is supposed to be known to the researcher. The method used to back-out the latent variables is the inversion of the transform operator. When the inverse of this operator exists, and is continuous, the problem is called well-posed in the Hadamard sense.

One application of the inverse problem approach in economics is the empirical treatment of data generated by games of incomplete information. Examples are given by auction models, adverse selection models or nonlinear pricing models. More details on these examples will be provided later in this paper. While many of these games lead to ill-posed problems [8], in this paper we consider a class of problems that are actually well-posed. The well-posedness of these models is directly related to the shape of the first-order condition characterized by the presence of a monotone hazard-rate. The intuition behind this remark is that a first-order condition with such a behavior gives rise to a differential equation instead of an integral equation [9] (as it is the case, for example, in the first-price auction model, where we have a Fredholm type I integral equation). Therefore, in our examples, the inverse operator used to retrieve the latent variables is an integral operator and therefore bounded, which insures the well-posedness of the problem. As mentioned previously, this paper proposes a taxonomy by introducing a class of well-posed problems that we call "hazard-rate game models". The well-posedness of the problem comes from the fact that the distribution of latent variables and the distribution of data are related through the hazard-rate.The pure adverse selection model and the third-price auction model are two models belonging to the type of games of incomplete information presented previously and that share this characteristic. Among different types of auctions and regulation models, these two games feature an equilibrium strategy that is a function of the unobserved variable and its "hazard-rate". Therefore we are going to use these two models in order to exemplify our general econometric procedure.

The Principal-Agent model. The adverse selection model is basically a case of hidden information where a principal delegates a task to an agent that has more information about its own type than the principal. The presence of asymmetric information is common in many regulatory situations and the case of a natural monopoly that provides a public service is a classical example. The equilibrium condition for this model can be expressed in terms of the hazard-rate (see Baron and Myerson (1982)):

$$P(x(\theta)) = C_x(x(\theta),\theta) + (1-\delta)\frac{F(\theta)}{f(\theta)}C_{x\theta}(x(\theta),\theta). \tag{1}$$

where P denotes the inverse demand function, x is the quantity of services/goods provided by the natural monopoly, θ is the type of the agent, C is the agent's cost function and δ is the weight that the regulator gives to profit of the firm in his/her maximization problem.

For a conveniently chosen cost function, $C(x,\theta) = c_1(\theta)x$, equation 1 simply becomes:

$$p = c_1(\theta) + (1-\delta)\frac{F(\theta)}{f(\theta)}c_1'(\theta).$$

[8]The classical example is the first-price auction model that is a mildly ill-posed problem (for more details on the degree of ill-posedness of this model see Florens et al. (1998)).

[9]Most of the ill-posed inverse problems are generated by noisy integral equations.

Using this specification, $c_1(\theta)$ is actually the marginal cost.
For the principal-agent model, the quantile equation relating the structural parameter and the reduced form parameter is:

$$G^{-1}(\alpha) = c_1(F^{-1}(\alpha)) + (1-\delta)\alpha c_1'(F^{-1}(\alpha))F^{-1'}(\alpha).$$

where $G^{-1}(\alpha)$ is the quantile function of the prices and $F^{-1}(\alpha)$ is the quantile function of the types. The solution to this problem is:

$$c_1(F^{-1}(\alpha)) = \alpha^{-\frac{1}{1-\delta}} \left[\int_0^{\alpha} \frac{G^{-1}(t)}{1-\delta} \times t^{\frac{\delta}{1-\delta}} \mathrm{d}t + constant \right]. \tag{2}$$

Intuitively, this is a well-posed problem as the solution is continuous in the data (it is an integral of the quantile function of the data). This case has been treated separately and extensive in our previous paper, see Enache and Florens (2018).

The Third-Price Auction Model. The economic model behind the third-price auction has been discussed mainly in the papers by Kagel and Levin (1993) and Monderer and Tennenholtz (2000). To summarize, the third-price auction model is a game with at least three players and a payment rule that implies the bidder with the highest bid wins the auction, but s/he will only pay the third-highest bid. This peculiar rule leads to overbidding behavior in the third-price auction. Assuming that the players are playing a Bayesian Nash Equilibrium, the bidder's maximization problem gives rise to the following first-order condition:

$$x = \xi + \frac{1}{\eta} \ln \left\{ 1 + \frac{\eta}{N-2} \frac{F(\xi)}{f(\xi)} \right\}.$$

where $N \geq 3$, f is the probability density function (hereafter p.d.f.) of F and $\eta > 0$ is the CARA (constant absolute risk aversion) parameter. Thus, in the case of the third-price auction model, if we write the FOC in terms of quantile functions we obtain the following result:

$$G^{-1}(\alpha) = F^{-1}(\alpha) + \frac{1}{\eta} \ln \left\{ 1 + \frac{\eta}{N-2} \frac{\alpha}{f\left(F^{-1}(\alpha)\right)} \right\}.$$

where $\alpha \in [0,1]$, G^{-1} denotes the quantile of the bids and F^{-1} the quantile function of the private values. This equation can be linearized by making the change of variable $\beta(\alpha) =$

$e^{\eta F^{-1}(\alpha)}$. We obtain therefore a first-order differential equation between the transformation of G^{-1} and F^{-1}:

$$\beta(\alpha) + \frac{\alpha}{N-2}\beta'(\alpha) = e^{\eta G^{-1}(\alpha)}.$$

The solution of the previous equation is simply:

$$F^{-1}(\alpha) = \frac{1}{\eta} \ln \frac{1}{\alpha^M} \int_0^\alpha M s^{M-1} e^{\eta G^{-1}(s)} \, \mathrm{d}s.$$

where $M = N-2$ in order to simplify notations.

This approach allows for a closed-form solution of the quantile of private values in terms of the quantile of bids, given the knowledge of the risk-aversion parameter and the number of bidders. We therefore obtain nonparametric global identification given the knowledge of the risk-aversion parameter. The estimation is also quite straightforward and involves only plugging-in the empirical counterpart of the distribution of bids. For a more detailed analysis of this case, see the paper Enache and Florens (2020).

3 The general model

Next we present a general setup to analyze this class of hazard-rate games, but without an explicit form of the a function.

We have the same economic theory as above, i.e. each individual has a private signal or type, ξ, generated by a probability measure characterized by its cdf F on $\mathbb{R}$. This distribution has a support $[\underline{\xi}, \overline{\xi}]$ and we assume $\underline{\xi}$ known by the statistician. We are going to consider $\underline{\xi} = 0$ given that this does not restrict the model. We will comment on this assumption later on.

The player plays a real value x which satisfies:

$$x = \sigma(\xi, F) = \sigma_F(\xi). \tag{3}$$

If $F \in \mathscr{F}$, then σ is a function defined on $[0, \overline{\xi}] \times \mathscr{F}$. This function is the equilibrium of the game and is known by the players and by the statistician (as a function of ξ and F). The players know F, but this is unknown information to the statistician.

Assumption 1. *$\sigma(\xi, F)$ is strictly increasing on $[0, \overline{\xi}]$ $\forall F \in \mathscr{F}$ and $\sigma(0, F) = 0$.*[10]

[10]Strictly speaking, it is sufficient to assume this property true for the true F. As we also need this property for the consistent estimation of F, we assume its validity in a neighborhood of the true F.

More generally the support of x is $[\underline{x},\overline{x}]$ and we assume $\sigma(\underline{\xi},F)=\underline{\xi}=\underline{x}$. Then the knowledge of $\underline{\xi}$ and of $\underline{x}$ are equivalent. In many cases $\underline{x}$ can be estimated at a rate n and the estimation of $\underline{x}$ and of $\underline{\xi}$ doesn't contaminate the rest of the estimation. This rate is reached if the density of x is strictly positive in $\underline{x}$ and we will see that our assumptions will impose this property.

Assumption 2. *We define the interval $\Xi=[\xi_1,\xi_2]$, with $0<\xi_1<\xi_2<\overline{\xi}$.*[11] *The function F is twice continuously differentiable on $[0,\overline{\xi}]$ and its derivative f is bounded from below on Ξ.*[12] *As long as the density function f does not have all its derivatives equal to 0 in 0, we can define the function λ as $\lambda:[0,\overline{\xi}]\to\mathbb{R}$, $\lambda(\xi)=\frac{F}{f}(\xi)$ and we assume that λ is strictly increasing.*

Assumption 3. *The strategic function has the form:*

$$\sigma(\xi,F)=a(\xi,\lambda(\xi))$$

where $a:[0,\overline{\xi}]\times\mathbb{R}^+\to\mathbb{R}$ is a $\mathscr{C}^1$ function.

Let us denote G the cdf of x. We have obviously:

$$G(x)=\Pr(X\leq x)=\Pr(\sigma(\xi,F)\leq x)=\Pr(\xi\leq\sigma_F^{-1}(x))=F\circ\sigma_F^{-1},$$

or

$$G\circ\sigma_F=F. \tag{4}$$

Definition 1. *The structural parameter of the model is F and the reduced parameter is G. These two functional parameters are linked by the relation $G\circ\sigma_F=F$.*

Under assumptions (2) and (3), G is strictly increasing [13] because it is a composition of F and σ_F^{-1} both strictly increasing. Then, G^{-1} and F^{-1} the quantile function of x and ξ are well defined on $[0,1]$. We obviously have :

$$G^{-1}(\alpha)=\sigma_F\circ F^{-1}(\alpha). \tag{5}$$

One can see from equation (5) that one of the main advantages of using a quantile approach as opposed to a distribution function approach (i.e. as in equation (4)) is that one does

[11] ξ_1 and ξ_2 can be as close as we want to 0 and respectively $\overline{\xi}$.

[12] Please note that the function f may be 0 in 0 or $\overline{\xi}$, but it is bounded from below on $\Xi=[\xi_1,\xi_2]$.

[13] Therefore $g>0$ and the lower bound $\underline{x}=\underline{\xi}$ can be estimated at a n rate. Indeed $g(x)=f\left(\sigma_F^{-1}(x)\right)\times\sigma_F^{-\prime 1}$ does not cancel in $\underline{x}$ and $\min x_i\Rightarrow\underline{x}$ at a rate n.

not have to invert the strategy function. This is extremely useful, as in many games of incomplete information, the strategy function is not invertible. Under assumption (3) the previous relation becomes:

$$G^{-1}(\alpha) = a\left(F^{-1}(\alpha), \frac{\alpha}{f \circ F^{-1}(\alpha)}\right) = a\left(F^{-1}(\alpha), \alpha F^{-1'}(\alpha)\right).$$

Let us now reparametrize the model. We take as the structural parameter the function, L such that $L : [0,1] \to \mathbb{R}^+$, $L = F^{-1}$, and the reduced form parameter is $Q : [0,1] \to \mathbb{R}^+$, $Q = G^{-1}$. These two parameters are related by the equation :

$$Q(\alpha) = a(L(\alpha), \alpha L'(\alpha))^{14}, \tag{6}$$

which is the core of our model. In order to avoid the boundary problems, α will be restricted to an interval $[\alpha_1, \alpha_2]$, $0 < \alpha_1 < \alpha_2 < 1$ (more details will follow in Section 6). The quantile function $Q(\alpha)$ of the data is identified and can be estimated if an iid sample is available. The identification of L is obtained by solving the nonlinear differential equation (6). The solution of (6) is constrained by the limit condition $L(0) = 0$.

4 Local analysis

At a simplified level, every model can be described by the existence of a group of latent variables, observable variables and the relationship between them. The latent variables cannot be observed or they are just not observed, while the observable variables constitute the data that is actually at the econometrician's disposal. In our case, Q denotes the quantile function of the observables and L the quantile function of the latent variables.

We know that every L leads to only one quantile of observables Q (otherwise a wouldn't be a function), but the question of identification is: under which conditions, the quantile function of observables, Q, is generated by only one quantile function of latent variables L? Otherwise said, the question is under which conditions, the mapping from L to Q is injective? One should note that the identification analysis is made in terms of populations' distributions (i.e. supposing that the econometrician knows the true parameters characterising the distributions), and that is why identification is conducted completely independent from the estimation procedure. We adopt a nonparametric identification of the games of incomplete information. Nonparametric identification is useful even for cases where para-

[14] This equation can be denoted as $Q = T(L)$ and, by using this notation we reason in terms of functions and T is an operator that maps L into Q.

metric estimation methods are employed (for example, due to the small sample size) as it is an insurance of the robustness of estimators.

Let us start with the nonlinear differential equation (6). In general this nonlinearity implies that the global identification question is difficult to solve and we will consider a local analysis of the equation (6) (see Florens et al. (1998) or Florens and Sbaï (2010)). Local identification is a weaker notion of identification because is a concept dependent on the chosen topology (in this paper we restrain to Hilbert spaces).

Let us specify the following properties of the a function.

Assumption 4. *The function* $a : [0,\overline{\xi}] \times \mathbb{R}^* \to \mathbb{R}$ *is continuously differentiable with partial derivatives denoted* $\partial_1 a$ *and* $\partial_2 a$ *and* $\partial_1 a \neq 0$.

Let us consider the following Sobolev space of functions:

$$\mathscr{W}_1 \quad = \quad \left\{ h : [0,1] \to \mathbb{R} \text{ differentiable } \middle| \sup_\alpha |h(\alpha)| \text{ and } \sup_\alpha |h'(\alpha)| \text{ are bounded} \right\}$$

provided with the norm:

$$\|h\|_1 = \sup_\alpha |h(\alpha)| + \sup_\alpha |h'(\alpha)|.$$

The parameter space of our model is the subset $\mathscr{K}$ defined by:

$$\mathscr{K} = \left\{ h \in \mathscr{W}_1 \middle| h(0) = 0 \text{ and } h'(\alpha) > 0 \right\},$$

and the quantile function L is an interior point of $\mathscr{K}$. This set has a non empty interior in $\mathscr{W}_1$. This means that there exists a neighborhood of L in $\mathscr{W}_1$ included in $\mathscr{K}$. This property implies that the tangent space of $\mathscr{K}$ in L is $\mathscr{W}_1$.

Now that we defined the tangent space, we may look at the equation (6) locally. We have that $Q = T(L)$, where Q is a function in $\mathscr{W}_0$ and $\mathscr{W}_0$ is defined as:

$$\mathscr{W}_0 = \left\{ r : [0,1] \to \mathbb{R} \middle| \sup_\alpha |r(\alpha)| \text{ is bounded} \right\},$$

provided with the norm $\|r\|_0 = \sup_\alpha |r(\alpha)|$. The operator T is continously differentiable and its Fréchet derivative in L denoted K_L verifies

$$\widetilde{Q}(\alpha) = \left(K_L\widetilde{L}\right)(\alpha) = \partial_1 a\left(L(\alpha), \alpha L'(\alpha)\right)\widetilde{L}(\alpha) + \alpha\partial_2 a\left(L(\alpha), \alpha L'(\alpha)\right)\widetilde{L}'(\alpha)$$

$$= a_1(\alpha)\widetilde{L}(\alpha) + a_2(\alpha)\widetilde{L}'(\alpha), \quad (7)$$

where

$$\begin{cases} a_1(\alpha) = \partial_1 a\left(L(\alpha), \alpha L'(\alpha)\right) \\ a_2(\alpha) = \alpha\partial_2 a\left(L(\alpha), \alpha L'(\alpha)\right). \end{cases}$$

The operator $\left(K_L\widetilde{L}\right)(\alpha)$ is continuous. The computation of the Fréchet derivative is obtained in the following way. We compute the Gâteaux derivative $\left(\frac{\partial}{\partial\varepsilon}T\left(L+\varepsilon\widetilde{L}\right)\Big|_{\varepsilon=0}\right)$ and we prove that the obtained expression (7) is actually the Fréchet derivative (i.e. double continuity in L and $\widetilde{L}$). This property requires that $K_L\left(\widetilde{L}\right)$ is a continuous linear operator, which is clear under the choice of parametric set plus some regularity conditions recalled in Florens and Sbaï (2010).

The next results show that it is also continuously invertible.

Proposition 1. *Under the limit condition $L(0) = 0$, the equation (7) has a unique solution equal to :*

$$\widetilde{L}(\alpha) = \frac{1}{A(\alpha)}\int_0^\alpha \widetilde{Q}(u)B(u)\,\mathrm{du} = K_L^{-1}\widetilde{Q}$$

where $A(\alpha) = \exp C(\alpha)$, $B(\alpha) = \frac{1}{a_2(\alpha)}\exp C(\alpha)$ for any $C(\alpha)$ such that $C'(\alpha) = \frac{a_1(\alpha)}{a_2(\alpha)}$ and K_L^{-1} is a continuous function of $\widetilde{Q}$.

Proof: Firstly, let us remark that $\widetilde{L}(\alpha)$ does not depend on an additive constant on C. Secondly, we have that $\widetilde{L}(\alpha) \to 0$ if $\alpha \to 0$ and this result will be used to show consistency. Finally, the result follows from the standard solution of the first order differential equation with no constant. □

The local analysis is useful not only in order to assess the local identification of the model, but it will also be employed in the study of the asymptotic properties of the estimator.

Corollary 1. *Under previous assumptions, the model is locally identified.*

Proof: The local linear approximation K_L is invertible and its inverse is continuous. Then the implicit function theorem (see Schwartz (1966)) applies and we have local identification. □

5 Estimation

The estimator of L is defined as the solution of:

$$\widehat{Q}(\alpha) = a\left(\widehat{L}(\alpha), \alpha\widehat{L}'(\alpha)\right), \tag{8}$$

where $\widehat{Q}(\alpha)$ is the empirical quantile function of our observations:

$$\widehat{Q}(\alpha) = \sum_{i=1}^{n} x_{in} \mathbb{1}\left[\frac{i-1}{n} < \alpha \leq \frac{i}{n}\right],$$

where x_{in} is the order statistics of the sample $x_1, x_2, \ldots, x_n$.

The resolution of the differential equation (8) is performed numerically. It may be useful to replace the function $\widehat{Q}$ by a smooth version of the empirical quantile function. This smooth version should be chosen such that its asymptotic behavior remains the same as the empirical one.

We take the nonlinear example of the third-price auction model. In this section we will treat this problem as if we had no knowledge of the closed-form solution which links the data to the unobservables and we proceed to a numerical estimation. We will imagine that we are unaware of the closed-form solution of the problem in terms of quantiles in order to see how our method works on a nonlinear example.
In the case of the third-price auction model, the a function has the following shape:

$$a(u,v) = u + \frac{1}{\eta} \ln\left\{1 + \frac{\eta}{N-2} v\right\},$$

where u corresponds here to $L(\alpha)$ and v to $\alpha L'(\alpha)$.

For the design of the simulated data, we suppose that the private values are drawn from a Beta distribution on the interval $[0,1]$, i.e: $F(\xi) = \xi^2$. In this case $\xi = \sqrt{\varepsilon}$, where ε is a uniformly distributed variable on $[0,1]$. The equilibrium bid in the third-price auction with $N = 5$ bidders and a risk-aversion parameter fixed at $\eta = 1$ is given by:

$$x = \sqrt{\varepsilon} + \ln\left(1 + \frac{1}{6}\sqrt{\varepsilon}\right).$$

Then our differential equation becomes simply:

$$L(\alpha) + \ln\left\{1 + \frac{1}{3}\alpha L'(\alpha)\right\} = Q(\alpha). \tag{9}$$

This equation can be solved using a standard ODE solver under the initial condition $L(0) = 0$. In our case, we used the "ode45" Matlab solver, which is an explicit method for nonstiff problems[15] based on Runge-Kutta 4th/5th-order (see Butcher (2016) for numerical methods for differential equations). Its solution is depicted below in Figure 1. We do not discuss the approximation errors in the numerical computation.

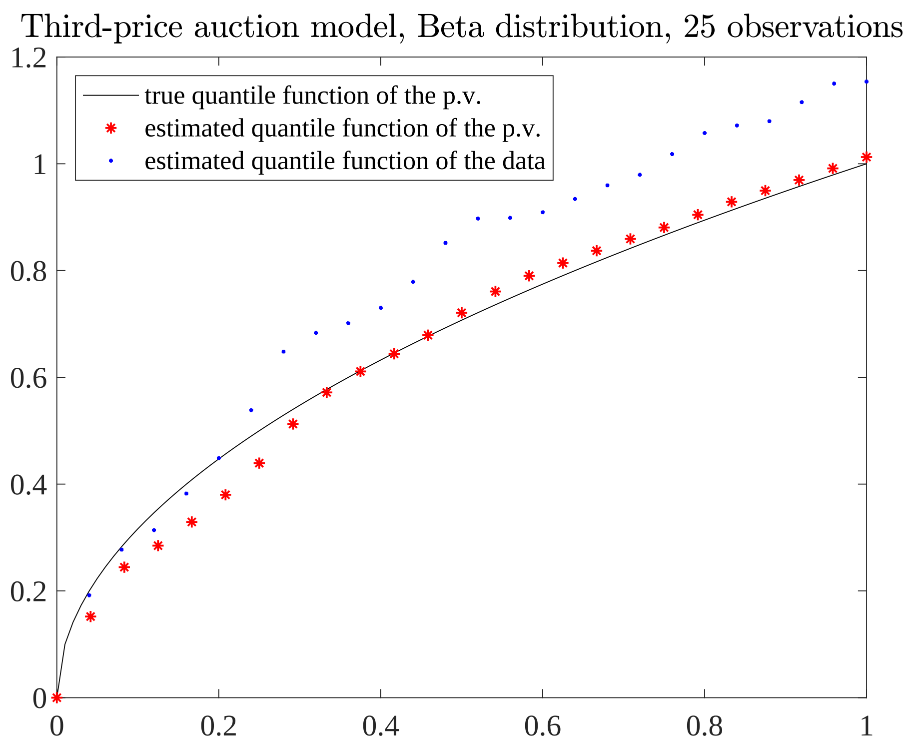

Figure 1: The solution of the differential equation for the case of a third-price auction model for a Beta distribution with 25 observations.

Below we present a second configuration where the private values are distributed Weibull,

[15] A stiff ordinary differential equation is one for which numerical errors accumulate greatly over time.

i.e.

$$F(\xi) = 1 - e^{-c_1 y^{c_2}}, \tag{10}$$

where the parameters of the Weibull distribution, c_1 and c_2 are set to be equal to 1. The equilibrium bid in this case will be: $x = \xi + \ln\left(1 + \frac{1}{3}\frac{1-e^{-\xi}}{e-\xi}\right)$. We generate the private values as: $\xi = -ln(1-\varepsilon)$, where ε is a uniformly distributed variable on $[0,1]$.

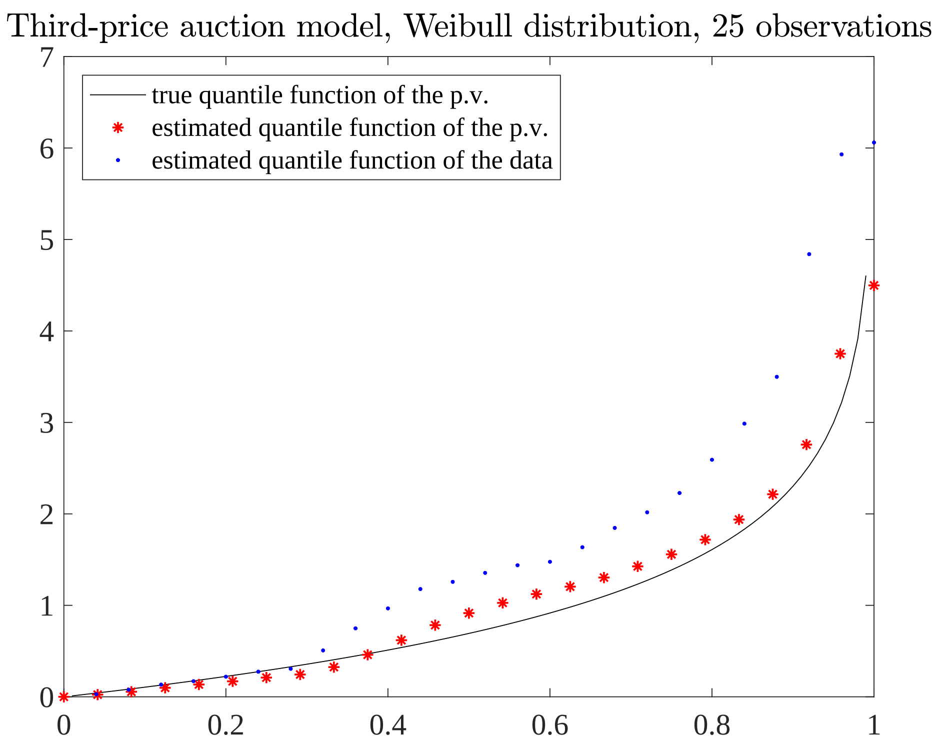

Figure 2: The solution of the differential equation for the case of a third-price auction model for a Weibull distribution with 25 observations.

Next we present the Monte Carlo simulations for the two data configurations. In Figure 3 we have the results from 50 simulations on 25 observations and where the private values are drawn from a Beta distribution.

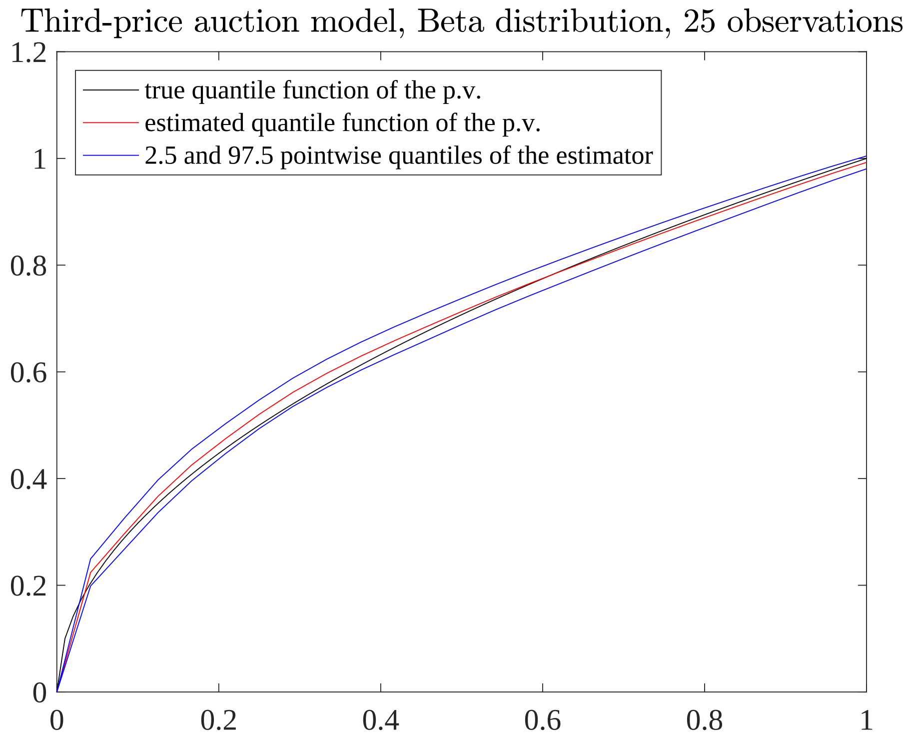

Figure 3: Monte Carlo Simulations for the case of a third-price auction model for a Beta distribution with 25 observations.

In Figure 4 we have the results from 50 simulations on 25 observations and where the private values are drawn from a Weibull distribution.

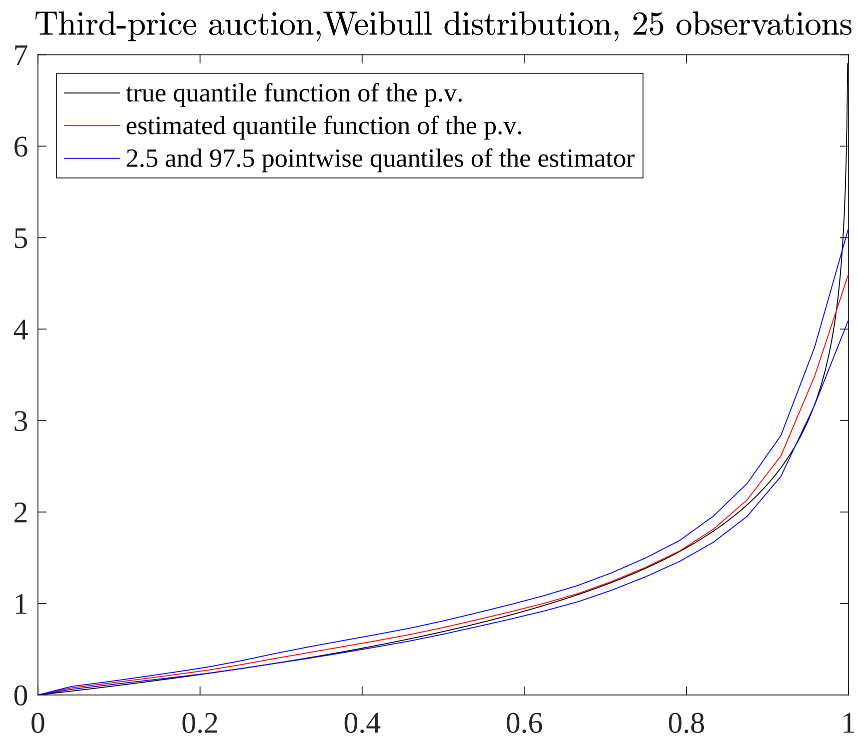

Figure 4: Monte Carlo Simulations for the case of a third-price auction model for a Weibull distribution with 25 observations.

6 Asymptotic properties

Using the Lemma 21.4 from Van der Vaart (1998) we have that $\widehat{Q}$ is asymptotically normal and converges uniformly to Q for all $\alpha \in [\alpha_1, \alpha_2]$, where $0 < \alpha_1 < \alpha_2 < 1$. Therefore all the asymptotic results in this section will be derived for $\alpha \in [\alpha_1, \alpha_2]$, $0 < \alpha_1 < \alpha_2 < 1$.

The implicit function theorem used in the corollary (1) implies that in a neighborhood of the true value Q, the estimator $\widehat{L}$ depends continuously on $\widehat{Q}$ (see th. 385 from Schwartz (1966)). Then we have the following consistency result. The asymptotic normality also follows from the implicit function theorem.

Proposition 2. *The estimator $\widehat{L}$ converges to L in $\mathscr{W}_1$, i.e. :*

$$\sup_{\alpha}|\widehat{L}(\alpha)-L(\alpha)|+\sup_{\alpha}|\widehat{L}'(\alpha)-L'(\alpha)|\xrightarrow{p}0,$$

where $\alpha\in[q_1,q_2], 0<q_1<q_2<1$.

Proposition 3. *If the true L is twice differentiable, Q is differentiable and we have:*

$$\sqrt{n}(\widehat{L}-L)\rightarrow\mathscr{G}(0,\Omega)\ \textit{in}\ \mathscr{W}_1.$$

where Ω is the variance operator characterised by the covariance function:

$$\omega(\alpha,\beta)=\frac{1}{A(\alpha)A(\beta)}\int_0^{\alpha}\int_0^{\beta}B(u)B(v)q(u)q(v)u(1-v)\,\mathrm{d}u\mathrm{d}v$$

The functions A and B are defined in Proposition 1 and $q(\alpha)$ is the derivative of $Q(\alpha)$. The proof of this proposition is given in Appendix 7.

From the estimation of L we may define an estimation of the derivative of L, denoted by l. This estimation is done numerically by computing the derivative of a smooth version of $\widehat{L}$. Using standard results of differential calculus it may be verified that:

$$\frac{\partial}{\partial\alpha}\left[\frac{1}{A(\alpha)}\int_0^{\alpha}\widetilde{Q}(u)B(u)\,\mathrm{d}u\right]=\frac{1}{A(\alpha)}\widetilde{Q}(\alpha)B(\alpha)-\frac{A'(\alpha)}{A^2(\alpha)}\int_0^{\alpha}\widetilde{Q}(u)B(u)\,\mathrm{d}(u),$$

where A' is the derivative of A wrt α. An implication of this result is that $\sqrt{n}\left(\widehat{l}(\alpha)-l(\alpha)\right)$ converges to a Gaussian process. Indeed we have that:

$$\sqrt{n}\left(\widehat{l}(\alpha)-l(\alpha)\right)=\frac{1}{A(\alpha)}B(\alpha)\left(\sqrt{n}\left(\widehat{Q}(\alpha)-Q(\alpha)\right)\right)-\frac{A'(\alpha)}{A^2(\alpha)}\int_0^{\alpha}\left(\sqrt{n}\left(\widehat{Q}(u)-Q(u).\right)\right)B(u)\,\mathrm{d}u+o_p(1)\quad(11)$$

We may also define an estimator of F by $\widehat{F}=\left(\widehat{L}\right)^{-1}$ (where $\widehat{L}$ is a smooth version of the initial estimator) and $\widehat{f}$ by $\widehat{F}'$.

The von Mises calculus implies that:

$$\sqrt{n}\left(\widehat{F(x)} - F(x)\right) = -\sqrt{n}f(x)\left(\widehat{L} - L\right) \circ F(x) + o_p(1),$$

and

$$\sqrt{n}\left(\widehat{f(x)} - f(x)\right) = -\sqrt{n}f'(x)\left(\widehat{L} - L\right) \circ F(x) - \sqrt{n}f^2(x)\left(\widehat{l} - l\right) \circ F(x) + o_p(1).$$

Then the three processes $\sqrt{n}\left(\widehat{l} - l\right)$, $\sqrt{n}\left(\widehat{F} - F\right)$, $\sqrt{n}\left(\widehat{f} - f\right)$ converge to three zero mean gaussian processes. The expression of their variances may be found in Enache and Florens (2020). In practice, confidence intervals may be computed by usual nonparametric bootstrap. The density f is estimated at the speed $\sqrt{n}$ and converges as a stochastic process. This result does not contradict the usual results on optimality of the estimation of a density because f is not the density of actual data. The convergence of $\widehat{f}$ at $\sqrt{n}$ has been remarked in particular cases (see Luo et al. (2014), Enache and Florens (2020), Enache and Florens (2018)). A central contribution of our paper is to underline the main mathematical foundations of this type of results. However, one may note that the estimation of f using the x_i gives better results than the estimation of f using the ξ_i, even if the ξ_i were observable. An interesting question in that case would be to combine the information contained by the ξ_i and the x_i.

Remark 1. *Our model is strictly speaking overidentified in the sense that the solution L may not be increasing. Then Q should be constrained such that L is a quantile function. However the model is not asymptotically overidentified, as the tangent space at the true value is the complete space or, equivalently, because the set $\mathscr{K}$ is open in $\mathscr{W}_1$. Then, a non constrained estimator will satisfy the overidentification constraint for a large (but finite) n.*

7 Conclusions

This paper proposes a general methodology to tackle the structural identification and estimation in games of incomplete information which belong to a class we call "hazard-rate" models. The examples that motivate our analysis are issued from the economics of regulation (such as pure adverse selection), auction theory (third-price auction model) or optimal pricing. Our strategy consists in rewriting the functional relation between the structural parameter (i.e. the data generating process or the latent variables) and the reduced-form parameter (i.e. the sample or the observables) in terms of quantile functions. The result is a differential equation in the quantile of primitives that we analyze locally and for which we show local identification. As the quantile functions contain the same statistical information as the cumulative distribution functions, there is no loss of information in recovering or working with the quantiles instead of the distribution functions. We therefore obtain a differential equation in terms of quantiles that can be solved numerically. Moreover we show that the estimators of the quantile function and of the density quantile function of primitives converge to a gaussian process at a root-n rate of convergence.

This model can be generalized in several directions. The general form of the model may assume that ξ is generated by a conditional distribution to some densities Z and that σ depends also on some unknown parameter or function μ and on exogenous observables, W: $x = \sigma(\xi, \lambda(\xi \mid Z), \mu, W)$ and G becomes a distribution conditional to Z and W.

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Proof of Proposition 3

The Propositions 2 and 3 are based on the properties of the estimation of the quantile function. Firstly, $\widehat{Q}$ converges to Q in $\mathscr{W}_0$ which implies the convergence of $\widehat{L}$ in $\widehat{W}_1$ as shown in Proposition 2. Secondly, if Q is continuously differentiable with a derivative q strictly positive in $(0,1)$, the process $\sqrt{n}\left(\widehat{Q}-Q\right)$ converges in $L^{\infty}[\alpha_1,\alpha_2]$ to a zero mean Gaussian process for any $0<\alpha_1<\alpha_2<1$. This limit process is equal to the usual Brownian Bridge multiplied by q or, equivalently, the covariance of this process is $\sigma(\alpha,\beta)=q(\alpha)q(\beta)\alpha(1-\beta)$. The main steps of the proof of convergence of $\sqrt{n}\left(\widehat{L}-L\right)$ are the following (more details are given in a similar proof by Enache and Florens (2020)). Firstly, the differentiability property of T implies that:

$$\sqrt{n}\left(\widehat{L}(\alpha)-L(\alpha)\right)=\frac{1}{A(\alpha)}\int_0^{\alpha}\left[\sqrt{n}\left(\widehat{Q}(u)-Q(u)\right)\right]B(u)\,\mathrm{d}u+o_p(1).$$

The control of the residual follows from Serfling (1980) (Lemma B, page 218) and from Property 5 in Enache and Florens (2020).

The second step is based on an approximation property given in Csörgö (1983):

$$\sup_{c_n<\alpha<1-c_n}|\sqrt{n}\left(\widehat{Q}-Q\right)-\delta(\alpha)|=o_p(1)$$

where δ is the Brownian Bridge multiplied by q and c_n is a sequence converging to 0 at a suitable rate (for example $c_n=\frac{1}{n}\ln n$). Then

$$\sqrt{n}\left(\widehat{L}(\alpha)-L(\alpha)\right)=\frac{1}{A(\alpha)}\int_{c_n}^{\alpha}B(u)\delta(u)\,\mathrm{d}u+o_p(1).$$

Finally, if $c_n<\alpha$ and $b<1-c_n$, we get that $\sqrt{n}\left(\widehat{L}(\alpha)-L(\alpha)\right)$ converges to $\frac{1}{A(\alpha)}\int_{c_n}^{\alpha}B(u)\delta(u)\,\mathrm{d}u$.

The variance of this process is obtained by the usual calculation:

$$\omega(\alpha,\beta)=\frac{1}{A(\alpha)A(\beta)}\int_0^{\alpha}\int_0^{\beta}B(u)B(v)q(u)q(v)u(1-v)\,\mathrm{d}u\mathrm{d}v.$$

□