



UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Física *Gleb Wataghin*

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THERMODYNAMICS AND CLASSICALITY

Termodinâmica e Classicalidade

Campinas

2020

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Thermodynamics and Classicality

Termodinâmica e Classicalidade

Dissertação apresentada ao Instituto de Física *Gleb Wataghin* da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Física, na área de Física

Dissertation submitted to the *Gleb Wataghin* Institute of Physics of University of Campinas in fulfillment of the requirements for the Master's degree in Physics, in the field of Physics.

Supervisor/Orientador: Marcos César de Oliveira

ESTE EXEMPLAR CORRESPONDE À VERSÃO FINAL DA DISSERTAÇÃO DE Mestrado DEFENDIDA PELO ALUNO ALEXSSANDRE DE OLIVEIRA JUNIOR E ORIENTADA PELO PROF. DR. MARCOS CÉSAR DE OLIVEIRA.

Campinas
2020

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Física Gleb Wataghin
Lucimeire de Oliveira Silva da Rocha - CRB 8/9174

OL4t Oliveira Junior, Alexssandre de, 1994-
Thermodynamics and classicality / Alexssandre de Oliveira Junior. –
Campinas, SP : [s.n.], 2020.

Orientador: Marcos César de Oliveira.

Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Física Gleb Wataghin.

1. Termodinâmica quântica. 2. Sistemas quânticos abertos. 3. Ótica quântica. 4. Luz comprimida. 5. Máquinas quânticas. I. Oliveira, Marcos César de, 1969-. II. Universidade Estadual de Campinas. Instituto de Física Gleb Wataghin. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Termodinâmica e classicalidade

Palavras-chave em inglês:

Quantum thermodynamics

Open quantum systems

Quantum optics

Squeezed light

Quantum engines

Área de concentração: Física

Titulação: Mestre em Física

Banca examinadora:

Marcos César de Oliveira [Orientador]

Marcus Vinicius Segantini Bonança

Antonio Zelaquett Khoury

Data de defesa: 19-02-2020

Programa de Pós-Graduação: Física

Identificação e informações acadêmicas do(a) aluno(a)

- ORCID do autor: <https://orcid.org/0000-0002-5319-0835>

- Currículo Lattes do autor: <http://lattes.cnpq.br/9812004015328272>

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OBS.: Ata da defesa com as respectivas assinaturas dos membros encontra-se no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria do Programa da Unidade.

CAMPINAS
2020

*Baldo, você carecia mesmo de estudar e tirar carta-de-doutor,
porque para cuidar do trivial você jeito não tem.*

Você não é habilidoso

João Guimarães Rosa, Grande Sertão Veredas, 1956.

To my Parents

Acknowledgments/Agradecimentos

Com certeza essa é a parte mais difícil de toda a minha dissertação, pois ao longo desses anos, tive o imenso prazer de conviver com uma infinidade de pessoas maravilhosas que tiveram uma contribuição significativa, tanto na minha vida pessoal, quanto acadêmica. Primeiramente, eu gostaria de agradecer aos meus pais, Daria e Alexssandre, pela educação, ensinamentos e principalmente por todo carinho que me deram durante toda a minha jornada. Eu aprendi a ler com a minha mãe aos quatro anos e tenho memórias vivas do quanto ela se esforçou para que isso acontecesse. Enquanto, com o meu pai, aprendi a nunca desistir, ouvir e sempre respeitar opiniões divergentes das minhas. Estendo este agradecimento ao meu irmão Bruno cujo sempre me deu, e continua dar, todo o suporte que precisei e que eu ainda preciso. Você foi, e ainda é, fundamental em vários aspectos da minha vida. Além disso, também agradeço ao meu outro irmão, Gabriel, por sempre me incentivar e pelos infinitos conselhos dados — a sua ida para universidade foi uma inspiração para que eu também fosse; a minha admiração por você é imensa. Agradeço a minha avó, Rosa, e a minha tia Madalena, por sempre estarem do meu lado e me apoiarem em cada decisão. Eu amo vocês incondicionalmente e sou extremamente orgulhoso de vocês duas.

Ao longo dos últimos dois anos tive a especial oportunidade de ser orientado pelo Prof. Marcos César de Oliveira. Eu gostaria de agradecê-lo, não somente pela sua orientação ímpar, mas também por todas as conversas, sugestões e conselhos. Pelos meus poucos momentos de insegurança e teimosia, e os meus muitos momentos de confusão, agradeço também por sua paciência. Sou extremamente grato pela sua maneira tranquila e leve de ser, sem abrir mão do senso crítico que tanto me fez aprender e melhorar. Além de orientador, acabou se tornando um grande amigo — eu aprendi muito com o senhor, muito obrigado, Marcos.

Durante minha graduação e pós-graduação, tive a felicidade de ter conhecido inúmeros excelentes profissionais. Dentre eles, eu gostaria de deixar o meu muito obrigado ao meu amigo, Prof. Felipe Fanchini, que sem dúvida foi fundamental para a minha carreira. Agradeço a você por todas as nossas conversas, por ter me recomendando para o Marcos e pela enorme paciência em me aguentar ao longo desses anos. Sua amizade é muito importante para mim; ao Prof. Edson Sardella, pela orientação e por me fazer acreditar que a física foi a escolha mais feliz da minha vida; ao Prof. Ricardo Doretto, pelas aulas claras e inspiradoras e também pela sua paciência por me aturar batendo em sua porta quase que sempre, além de seus conselhos; ao Prof. Marcus Bonança pelas discussões, clareza ao me orientar em determinados pontos deste trabalho e paciência; ao Prof. Marcus Aguiar por ter lido essa dissertação em sua primeira versão e apontar pontos importantes. Ao Prof. Rafael Rabelo, pelas conversas, discussões e pelo seu companheirismo no basquete.

Não poderia deixar de mencionar aqui os meus grandes amigos. Primeiramente agradeço aos de graduação: Orisson, Wellington, Jorge, Felps e Bruno. Sinto muita saudade de vocês e também das nossas conversas; ao

grande Rafael Simões, por todas as discussões sobre mecânica quântica, vida e mestrado. Embarcamos juntos nessa aventura e acredito que estamos felizes com essa escolha; agradeço àqueles que tive a oportunidade de morar e compartilhar uma das fases mais felizes da minha vida: Bernardo, Rodrigo e Giovanni — obrigado pelos momentos de muita alegria que passamos juntos.

Em Campinas, tive a oportunidade de dividir sala e morar com o Arthur Faria, o qual agradeço sem palavras pela amizade, por sempre estar presente e, em especial, pelas inúmeras discussões, seja sobre Física ou sobre dinossauros colados na parede — muito obrigado pelo seu apoio e carinho; ao meu grande amigo Sérgio Novi, cujo tive o prazer de morar e compartilhar grandes momentos. Jamais vou me esquecer dos seus conselhos e do seu apoio incondicional; ao meu querido amigo Yago Philippe, pelas imensuráveis discussões sobre Física, suporte e companheirismo — eu te admiro, Baianinho e torço bastante pelo seu trabalho; ao Carlos Galdino, pela amizade, por ser chato em vários aspectos que me fizeram crescer e pelos vários conselhos dados. Quando cheguei em Campinas, meus primeiros amigos foram duas pessoas maravilhosas, Paulinho e Murilo. Agradeço ao Paulinho por toda paciência e ajuda, por sempre estar disposto em discutir mecânica quântica - não importando onde seja e pelas cervejas compartilhadas; ao Murilo, pela alegria e loucura, além das conversas, cervejas e por ser crítico quando necessário. Agradeço a amizade e o carinho da Débora Princepe, quem eu tive o privilégio de conhecer como monitora do FIFE e acabou se tornando uma grande amiga — você me deu conselhos que sempre foram muito frutíferos, não somente no âmbito acadêmico. Sua dedicação e competência foram, e continuam a ser, um exemplo para mim; por todo companheirismo e animação, agradeço à Gabriela Soffiati, por fazer as minhas quinta-feiras, e alguns domingos, muito mais felizes, i.e., com cerveja. Em pouco tempo, você acabou se tornando muito especial para mim, obrigado pela amizade; ao Yago Godoy, pelos momentos de felicidade e também de aflição, ou seja, como ele diz, “fritando” em mecânica quântica. Também agradeço ao meu querido amigo, Marcello Passos, o Físico experimental mais teórico que conheço. Obrigado pela companhia, discussões e colaborações e nunca mais confie em mim para planejar uma viagem; à Helena Milaré, pelo carinho e pelos nossos momentos juntos, nossas conversas tiveram um valor inestimável.

Agradeço ao meu grupo: John, por ler este trabalho e dar diversas sugestões, e também pela sua amizade ímpar e inspiradora. À Marina, à Luisa, à Tatiana e ao Pedro.

Eu gostaria de agradecer aos meus amigos de Poços de Caldas que estiveram comigo desde o ensino fundamental. Ao meu grande amigo, Carlos Eduardo (Dudu), que é uma inspiração para mim. Obrigado pelas conversas, pela companhia, momentos de felicidade e pela sólida amizade, independente de barreiras geográficas. Repito as minhas palavras para Ana Lu, obrigado pela amizade e carinho; à Isabella Faria, por todas as risadas e carinho; às irmãs Schneider, Gabriela e Larissa - eu gosto muito de vocês duas. Embora ele não seja de Poços de Caldas, agradeço à amizade do meu amigo, Henrique Kaneno, por ser tão engraçado e legal. E com certeza, à Ciça, por ser uma das pessoas que mais me apoiou quando eu decidi que seguiria este rumo, muito

obrigado pela amizade e por todo carinho e os momentos felizes que compartilhamos juntos.

Por fim, agradeço a três pessoas muito importantes que tiveram uma grande influência na minha vida acadêmica e que detém minha total admiração. Ao meu grande amigo Thiago Acconcia, por sempre me incentivar a vir para Unicamp e pelas inúmeras conversas sobre física desde os tempos de ensino fundamental; ao Daniel Dantas que, desde a 5ª série, está comigo e me encoraja em muitas ocasiões, principalmente nos meus momentos de desânimos. E também, ao meu amigo Gabriel Cozzella, um grande exemplo de pessoa e de Físico, que como os outros dois, desde o ensino fundamental, tem me dado conselhos mais que valiosos.

I also owe a very special thank you to Ahmed Tariq for helping me since the very beginning of my time in College. I have learned very much from our conversations and from your unique personality.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES)-001, que possibilitou o desenvolvimento deste projeto.

State University of Campinas, UNICAMP

Campinas, SP - Brazil

February, 2020

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Abstract

Quantum heat engines are devices comprised of one or more quantum systems operating cyclically between two heat reservoirs to produce work. Although the laws of quantum mechanics rule these engines, this does not imply that these machines function in a quantum fashion, or exhibit a quantum advantage. Indeed, quantum heat engines have shown a remarkable similarity with classical models, thus raising the question of *what is “quantum” in quantum thermodynamics*. With the help of the P-representability concept, an expression called classicality function is introduced - allowing us to quantify the degree of non-classicality of a bosonic system. As an illustration, we consider a quantum heat engine consisting of a single bosonic mode as a working substance coupled to a cold thermal and hot squeezed thermal bath. For two specific protocols, i.e., a quantum Otto and a generalized cycle, we show that the non-classicality is a resource that can be used to perform thermodynamical tasks more efficiently. Furthermore, we present a theoretical framework to study how quantum features, such as entanglement or coherence, affect the classical formulations of thermodynamic axioms.

Resumo

Máquinas térmicas que operam no regime quântico são dispositivos compostos por um ou mais sistemas quânticos que operam ciclicamente entre dois reservatórios cujo objetivo é produzir trabalho. Embora as leis da mecânica quântica governem esses dispositivos, isso não implica que essas máquinas funcionem de maneira quântica ou que exibam alguma vantagem. De fato, máquinas quânticas mostraram uma notável semelhança com os modelos clássicos, levantando assim à questão de *o que é quântico em termodinâmica quântica*. Com a ajuda do conceito de P-representabilidade, apresentamos uma simples expressão, denominada função de classicalidade, que permite quantificar o grau de não-classicalidade de um sistema bosônico. Como ilustração, consideramos uma máquina quântica que consiste em um único modo bosônico, como substância de trabalho, acoplada a um reservatório térmico frio e um reservatório térmico comprimido quente. Para dois protocolos específicos, ou seja, um ciclo de Otto quântico e um ciclo generalizado, mostramos que a não-classicalidade é um recurso para executar tarefas termodinâmicas mais eficientes. Além disso, o presente estudo apresenta um arcabouço teórico para estudar como recursos quânticos, como emaranhamento ou coerência, afetam as formulações clássicas dos axiomas da termodinâmica.

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CHAPTER 1

Quantum thermodynamics

1.1 Why quantum thermodynamics?

The science of thermodynamics has been profoundly triumphant, impacting strongly the natural sciences and allowing the development of technologies that have changed our lives, from coolers to spaceships. Until recently, it was applied to large systems composed by a vast number of particles described by the laws of classical physics. However, with modern technologies miniaturizing down to the nanoscale and into the quantum regime, the question of the applicability of thermodynamics to this new realm has become an exciting challenge.

As a result, the field of quantum thermodynamics is an emerging topic in physics aiming to understand how quantum features, such as entanglement and quantum coherence, affect classical formulations of the thermodynamics axioms. Important goals of the field are, among others, a better understanding of thermalisation in quantum systems, the characterisation of non-equilibrium fluctuations in the quantum regime, the role of quantum heat engines, and the design and realisation of new experiments exploring quantum thermodynamics using, for example, nuclear spins (Micadei et al., 2019), trapped ions (Abah et al., 2012), optomechanical setups (Klaers et al., 2017), and others.

While quantum mechanics was conceived from a consistency argument on the nature of thermal emitted light, the black-body radiation, thermodynamics stretches back to the nineteenth century with the effort to understand steam engines during the industrial revolution. For many decades, the two theories developed separately until Scovil and Schulz-DuBois (1959) showed the equivalence of the Carnot engine with the three-level maser. Thirty years later, studies on the balance of entropy in open quantum systems opened the way to

explore heat engines in the quantum realm. In conclusion, [Alicki \(1979\)](#) showed that the Carnot statement of the second law is universally valid also for quantum heat engines.

On the other hand, connections between information theory and thermodynamics started to appear tighter: one of the significant pioneer contributions is the famous maximal entropy principle introduced by [Jaynes \(1957a,b\)](#). In these seminal works, Jaynes discusses the justification of applying the methods of statistical mechanics from microscopic mechanical laws using tools from information theory. Also, the Gedanken experiment of Maxwell's demon, led to the famous statement by [Landauer \(1961\)](#) that *information is physical*. In other words, if we want to encode information in a given physical system, we must pay an energetic price, setting limitations on the processing of information by the laws of thermodynamics.

During the last twenty years, new approaches have revealed general laws applicable to non-equilibrium systems, pushing the range of validity of thermodynamic statements beyond the realm of equilibrium, and deeply into the non-equilibrium region ([Seifert, 2012](#)). While equilibrium thermodynamics deals with average quantities and is applied when the system is composed of a macroscopic number of particles, the so-called *stochastic thermodynamics* picks up where the macroscopic description starts to lose accuracy. The pillars of the stochastic thermodynamics are given by [Jarzynski \(1997\)](#) and [Crooks \(1999\)](#) relations. The fluctuation theorems link the free energy difference between states with the work done in transforming between them. These relations also hold unmodified in closed quantum systems ([Mukamel, 2003](#); [Esposito et al., 2009](#); [Campisi et al., 2011](#)), with slight adjustments when the system is open ([Venkatesh et al., 2015](#)). However, in the quantum realm, work is not an observable and has to be inferred by performing projective measurements ([Parrondo, 2007](#)). Questions such as non-linear quantum fluctuations and the potential application of fluctuation relations to quantum computing still remain open.

In general, the study of quantum thermodynamics is still in its childhood and, the broad character of the field implies that the quantum thermodynamics community should bring together people with diverse backgrounds and common interests, along with their differences about fundamental physical concepts.

1.2 The scope of this dissertation

In this dissertation, we pose the following question:

How non-classical features of light affect thermodynamic quantities?

The non-classical feature studied here is the squeezing, which corresponds to coherence in the energy eigenbasis. To answer the posed question, an expression called *classicality function* is put forward. This function allows us to discriminate which states of radiation are classical or non-classical based on its signal. Then, it is examined a quantum heat engine whose working substance is light, in particular, a single bosonic mode

prepared in a thermal state, coupled to a cold thermal and a hot squeezed thermal reservoirs. For two different cycles, it is shown that a non-classical effect of light can be used to enhance the thermodynamic efficiency of the quantum heat engine. Before digging into the world of quantum heat engines properly, some groundwork is needed. For this reason, the dissertation is divided as follows:

We start with a brief review of the fundamental concepts of thermodynamics. After a cursory look, we start chapter 2 by presenting the foundations of quantum mechanics, with the added goal of clarifying further notation and adopted conventions. Some slightly more specific notions concerning the quantum relative entropy and coherence are also included. In both chapters, a more detailed description and technical details are left to the different textbooks about these subjects Callen (1985); Breuer et al. (2002).

The basic theory and formalism are set out in chapter 4. We start with the quantization of a single-mode field and generalize it to the multiple mode case. Then, the main ideas behind the thermal, coherent, and squeezed states are presented and discussed. The connection with the phase space formalism is made by defining the characteristic function. These tools are linked to the Gaussian parametrization, which encloses this chapter.

Next, in chapter 5, the theory of open quantum systems is described and discussed. Specifically, we studied the dynamics of a single bosonic mode under the influence of a thermal and squeezed thermal bath. A master equation describing this interaction is derived and making use of the characteristic function method, it is solved. Subsequently, the problem of two-mode systems interacting with each other is discussed, and it is observed that for certain initial states of the bosonic modes, a swapping of state configuration occurs inside the recurrence time.

Finally, in chapter 6, all the tools developed in the past chapters are applied to a specific problem. We consider a quantum heat engine consisting of a single bosonic mode as the working substance undergoing a quantum Otto and a generalized cycle. Both cycles allow us to investigate the role of the non-classicality in quantum thermodynamics. Our theoretical findings show that the coherence is a resource for quantum heat engines. In particular, we conclude that non-classical states outperform classical states for some thermodynamical tasks.

It is important to stress that the symbol \bar{n} with different labels will be extensively used throughout the dissertation, and each of them has a different meaning, although they followed the same idea. Therefore, the three most important are summarized in the following table:

The mean number of photons	\bar{n}
The mean number of “thermal photons”	\bar{n}_{th}
The mean number of squeezed thermal photons	\bar{n}_{sth}

In the first row of the above table, *the mean number of photons* comprehends the general case, where this quantity is evaluated on the state other than the squeezed or thermal states. In chapter 4, the concept of P-

representability will be stated from this parameter with no label, meaning that expression depends on the basis that we are working on.

CHAPTER 2

Fundamentals of thermodynamics

Thermodynamics is a phenomenological theory that plays a prominent role in our daily lives. Its range of applicability goes from black-holes down to a rubber band. Over three centuries, the theory survived the advent of major scientific revolutions like the advent of general relativity and quantum mechanics, turning itself into one of the pillars of Physics. Its importance is, for instance, categorically stated by Einstein ([Einstein and Schilpp, 1979](#)):

“The only physical theory of universal content, which I am convinced, that within the framework of applicability of its basic concepts will never be overthrown.”

The origin of its success resides in the simplicity of its laws, and its universality comes from the fact that it deals with systems composed of a large number of particles without being concerned with the microscopic details on how these particles behave or interact with each other ([Callen, 1985](#)). Unlike the standard model of particle physics, which seeks to describe the building blocks of our physical world, the laws of thermodynamics only state what can and cannot be done with these blocks. The downside of this macroscopic description is that it necessarily deals with average quantities. While being a valid approach when the system at hand is composed of a macroscopic number of particles, it starts losing accuracy as the system size decreases, and fluctuations around these average quantities, due to thermal motion, become relevant ([Millen and Xuereb, 2016](#)).

Despite being a theory of around two hundred years old, thermodynamics has received significant updates in the 21st century. For instance, Stochastic thermodynamics picks up from where the macroscopic description starts to fail and gives a deeper insight into the fluctuations of thermodynamic quantities ([Seifert, 2012](#)). It also goes beyond the equilibrium situations typically encountered in thermodynamics and describes the behaviour

of systems that are not in thermal equilibrium. When dealing with even smaller systems, quantum effects come into play - fluctuations are no longer just thermal in their origin but also quantum. In this regime, many questions arise as we move from the macroscopic classical world to the microscopic quantum realm; it is natural to ask whether the laws of thermodynamics retain their place.

This chapter does not intend to include a detailed analysis of the foundations of thermodynamics and its refinements. We felt an introduction to the general theory was required in order to make the presentation self-sufficient and clear the following chapters from possible ambiguities. In this chapter is present an introduction to the theoretical framework of thermodynamics. The chapter starts with basic concepts, such as state variables, quasi-static process, reversible process, followed by a detailed description of the laws of thermodynamics. Then, heat engines shall be introduced, and two primary examples are discussed: the Carnot and Otto cycle. Finally, for future reference, the Helmholtz free energy is defined without carrying about its interpretation or intricate details in its derivation.

2.1 Preliminary concepts

A thermodynamic system can be a rubber band, a magnetized metal, or, as we often see in the textbooks, an ideal gas. The only restriction is that the system is defined as a macroscopic, i.e., composed by a large number of particles. For simplicity, a thermodynamic system is a part of the universe in which we are interested. Each thermodynamic system is separated from its environment by a physical or imaginary boundary. This boundary can allow different exchange processes of work, heat, or matter. We may classify it as completely closed if neither matter nor energy is exchanged; closed if it exchanges energy but not matter, and open when it exchanges both (Lemons, 2009).

A configuration of the system at any given time represents a state. A particularly fundamental one is the thermal equilibrium. In thermodynamic equilibrium, the state of the system can be described by a set of macroscopic quantities known as the state variables. The relation between the state variables is specified by the *equation of state* that describes the properties of the system. For example, in an ideal gas, these parameters can be the volume V , the pressure P , and the temperature T . The equation of state determines how these parameters are related to each other.

A change in one or more of the state variables is called a thermodynamic process. They are produced by the interaction of the system with its surroundings, and by an action of some external agent through the variation of some control parameter $\lambda(t)$. When the variation of the control parameter is infinitely slow, such that the system can be regarded as a succession of equilibrium states, we say that the process is *quasi-static*, while if it takes a negligible amount of time to occur we say it is instantaneous. Furthermore, if no external action occurs, this implies that the process is spontaneous. Only quasi-static processes are fully describable by means

of classical thermodynamics (Deffner and Campbell, 2019). A fundamental classification of thermodynamic processes is the following: we call a *reversible process* the one in which the system and its surroundings can be reversed to their original states after some process, i.e. the so-called protocol $\Lambda = \{\lambda(t) | t_{ini} \leq t \leq t_{fin}\}$, is reversed in time (Paule, 2018). Conversely, irreversible processes occur when the state of the system departs from equilibrium during the time evolution, generating a permanent change in the environment even if the system turns back to its original state by reversing the protocol Λ . However, reversible processes correspond to idealized hypothetical processes and are implemented only when the system and the surroundings are in thermal equilibrium (Callen, 1985). It is also worth stressing that, although all reversible processes are quasi-static, the converse is not always true.

Furthermore, depending on the type of contact of the system with its environment and the nature of the later, we may classify thermodynamic processes as:

- **Isothermal processes:** a process in which the temperature of the system remains constant during its interaction with the environment. Common examples of isothermal processes are melting, evaporation or other phase changes when occurring at constant pressure.
- **Isochoric process:** although the name may refer to a process in which the volume of the closed system remains constant, we define it as a process where no work is performed by the external agent (or by the system). This is the case of the heating or cooling of a liquid inside a closed container with non-zero thermal conduction.
- **Adiabatic process:** in such processes, the system cannot exchange heat nor matter with the environment, except for the work performed by the external agent. Adiabatic processes sometimes occur when a physical process takes place so rapidly that there is not enough time for heat exchange with the environment.

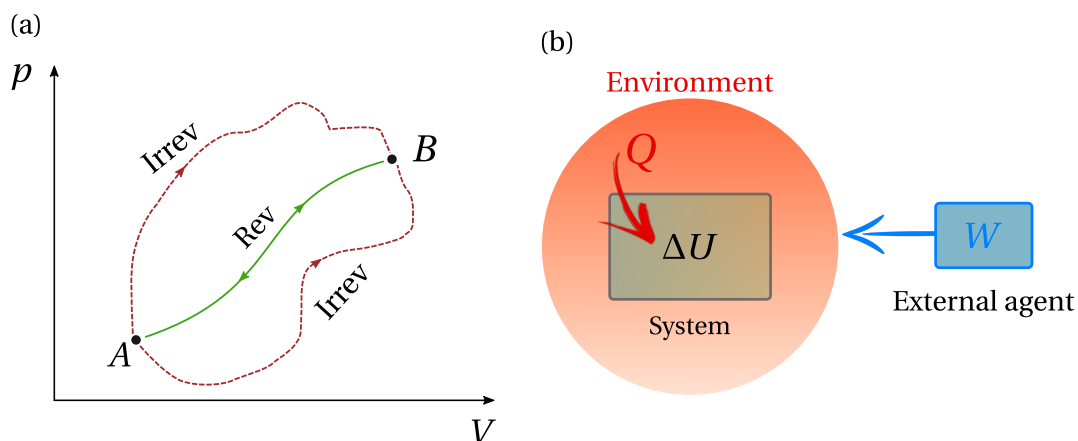


Figure 2.1: (a) p - V diagram of a reversible (green), and an irreversible process (red). (b) Schematic diagram of the first law of thermodynamics.

2.2 The laws of thermodynamics

The framework of thermodynamics is built upon four laws, which axiomatically paraphrase everyday experience and observation of nature.

Zeroth law of thermodynamics. The zeroth law of thermodynamics defines a state of equilibrium of a system relative to its environment. In its most common formulation, it can be expressed as

If two systems are in thermal equilibrium with a third system, then they are in thermal equilibrium with each other.

The zeroth law allows us to define a temperature. Any two systems in thermal equilibrium have the same temperature. Otherwise, they have different temperatures.

First law of thermodynamics. This law is a version of the law of conservation of energy, distinguishing two kinds of energy exchanges, heat, and thermodynamic work. This leads to the following decomposition:

The energy exchanged by a system can be split into two contributions called work, W , and heat Q , as

$$dU = \delta W + \delta Q, \quad (2.1)$$

where U is the internal energy, W represents a controllable (and hence useful) energy, the work, which is associated to the action of an external agent on the system via a control parameter λ , and Q denotes the energy introduced into the system in an uncontrolled way. Both quantities, W and Q , are path-dependent, and we use the symbol δ to denote that both are inexact differentials. A schematic representation of the first law is depicted in fig.(2.1).

The second law of thermodynamics. The origins of the second law of thermodynamics go back to the pioneering analysis of Sadi Carnot on the power and efficiency of heat engines in 1824, motivated by the spread of the steam engine during the industrial revolution. Carnot identified the flow of heat as a fundamental process required for the generation of work (Carnot, 2005). He introduced the condition of maximum work extraction of an idealized heat engine operating cyclically between a difference of temperatures. This condition corresponds to the case in which all the operations performed by the machine are reversible. However, it should be pointed out that Carnot's analysis was based on the old theory of heat, which considered heat as an indestructible quantity, not fulfilling the first law. Its work, while passing mostly unnoticed during decades, was essential to the later introduction of the concept of entropy and the formalization of the second law of thermodynamics by Rudolf Clausius and Lord Kelvin, whose original formulation of the second law states that (Fermi, 1956):

A transformation whose only final result is to transform into work heat extracted from a source that is at the same temperature throughout is impossible. (Postulate of Lord Kelvin.)

In other words, according to the second law, the division of energy interchanged between systems and their surroundings into heat and work is made asymmetrically: while work can be entirely transformed into heat in a cyclic process, the opposite is not valid. The second law can also be expressed as follows (Fermi, 1956):

A transformation whose only final result is to transfer heat from a body at a given temperature to a body at a higher temperature is impossible (Postulate of Clausius).

Consequently, according to the Clausius statement, if we bring two systems at different temperatures into thermal contact, heat flows spontaneously by conduction from one of these bodies to the other. The system from which heat flows is at a higher temperature than the other receiving the heat. Although both postulates seem to be different from each other, it is possible to demonstrate their equivalence. To do that, we need to prove that if the Clausius postulate were not valid, the Kelvin postulate would not be valid, and vice-versa. Finally, the Carnot statement declares that (Carnot, 2005)

No engine operating between two heat reservoirs can be more efficient than a Carnot engine operating between those same reservoirs.

As we have seen before, there are many ways to operate a thermodynamic process from some equilibrium state A to some other equilibrium state B. Each of these different paths will provide different values for the work and the heat exchange by the system during the process. Among all those paths, there always exists at least one idealized path, corresponding to a reversible process. Indeed, there exist, in general, infinitely many reversible paths connecting the two states A and B. The existence of reversible paths lead us to define the thermodynamic entropy, a concept as fundamental and universal as energy, entering into the theory as a state function. It was first defined through its change in a reversible process by Clausius and Hirst (1867)

$$\Delta S_{\text{th}} \equiv \int_A^B \frac{\delta Q_{\text{rev}}}{T} \quad \text{or} \quad \oint \frac{\delta Q_{\text{rev}}}{T} = 0. \quad (2.2)$$

where $\Delta S_{\text{th}} = S_{\text{th}}^B - S_{\text{th}}^A$ is the change in the entropy from point A to B, and Q_{rev} is the heat absorbed by the system. Notice that the thermodynamic entropy is a state function, this means that its variation only depends on its value on the initial and final states of the process, but not on the specific path followed. However, for a generic process that does not follow a reversible path, we have that $dS \geq \delta Q/T$, or

$$\Delta S_{\text{th}} \geq \int_A^B \frac{\delta Q}{T} \quad \text{or} \quad \oint \frac{\delta Q}{T} \leq 0. \quad (2.3)$$

The above inequalities are usually referred to as the Clausius inequality for arbitrary and cyclic processes, respectively.

Third law of thermodynamics. Walther Nernst developed the third law between 1906 and 1912 by noticing that the changes in thermodynamic entropy of all isothermal processes tend to zero when the temperature approaches zero (Lemons, 2009). Planck reformulated this principle in a stronger way by stating that the entropy of any system in equilibrium tends to zero as its temperature approaches zero (Callen, 1985)

$$\lim_{T \rightarrow 0} S = 0. \quad (2.4)$$

In quantum thermodynamics, the validity of eq.(2.4) has been questioned. In particular, the Nernst heat theorem has mainly been studied in Ising models and lattice systems with generalized ferromagnetic many-body interactions (Aizenman and Lieb, 2004; Leff, 1970).

2.3 Heat engines

The laws of thermodynamics were formulated in the furnaces of the XIX century with engineers and scientists motivated by the spread of the steam engine during the industrial revolution. In general, engines are machines that convert some form of energy (e.g., thermal or electrical energy) into work. The necessary ingredients of an engine are: a system, which is called the working fluid - or substance, and two reservoirs at different temperatures. The working fluid operates in a cycle: a set of thermodynamic processes that define the type of engine - extracting heat Q_A from a hot reservoir, using part of this heat as work W , and rejecting the non-useful part as heat Q_R to the colder reservoir. Their efficiency, defined as the ratio of the extracted work to the invested energy, is restricted to one at most by the energy-conservation law.

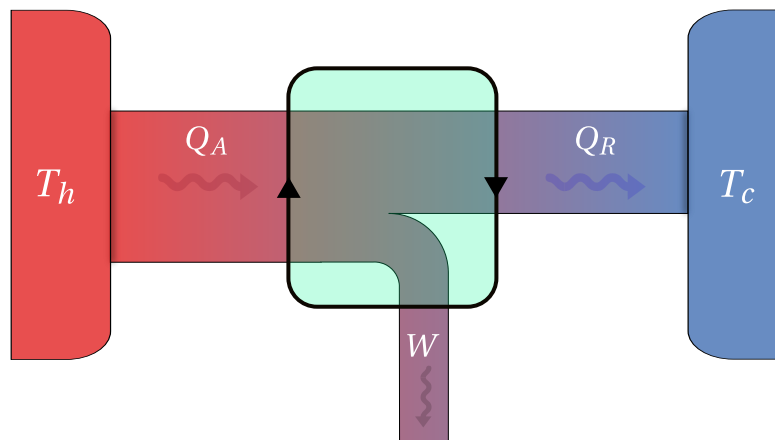


Figure 2.2: A heat engine sketched. The working substance follows a cycle where it absorbs an amount of heat Q_A from the hot reservoir at temperature T_h , use some part of this energy to perform work, and rejects an amount of heat to the cold reservoir at T_c .

2.3.1 The Carnot cycle

The Carnot cycle constitutes one of the most significant results in thermodynamics. It introduces the condition of maximum work extraction of an idealized heat engine operating cyclically between two different temperatures. This condition corresponds to the case in which all the operations performed by the machine are reversible. Under such circumstances, its efficiency establishes a fundamental limitation on the performance of ideal heat engines solely based on the temperature ratio of the reservoirs and independent of the specific model of the engine.

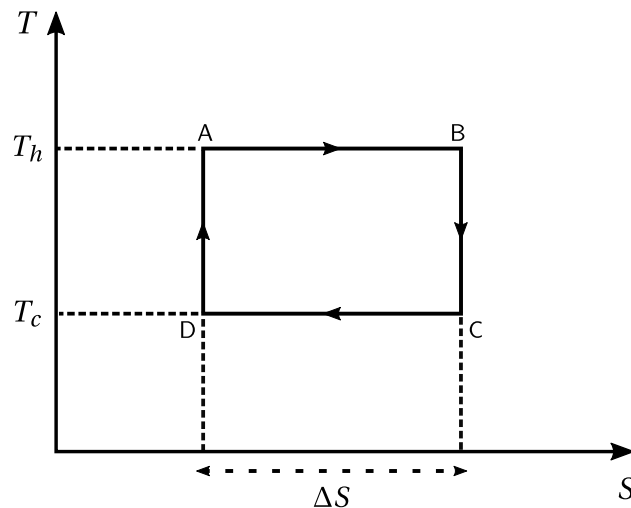


Figure 2.3: $T - S$ diagram of the Carnot cycle. During the strokes, AB and CD the entropy changes due to the coupling with the hot and cold reservoir, while in BC and DA the entropy remains constant.

The Carnot cycle is accomplished in four steps:

- (AB) Isothermal expansion at T_h

The working fluid initially at state A, and at a temperature T_h , is put in contact with a hot reservoir at the same temperature. The working fluid absorbs an amount of heat Q_A from the hot reservoir and undergoes a reversible isothermal expansion from A to B.

- (BC) Isentropic expansion

At state B, the working fluid is thermally insulated and then performs an adiabatic expansion to C. During this expansion its temperature decreases to T_c .

- (CD) Isothermal compression at T_c

In this step, the working fluid is brought into contact with a cold reservoir at temperature T_c and compressed to state D, while giving up an amount of heat Q_R to the reservoir T_c .

- (DA) Isentropic compression

Finally, the system is compressed adiabatically back to the initial state A.

The total amount of heat absorbed by the working fluid during the cycle is $Q_A - Q_R$. Let W_c be the amount of work done by the system during the transformation. Using the 1st law of thermodynamics,

$$W_c = Q_A - Q_R. \quad (2.5)$$

The efficiency is defined by the ratio of W_c and Q_A , as

$$\eta_C = \frac{W_c}{Q_A} = 1 - \frac{Q_R}{Q_A}. \quad (2.6)$$

Since both isothermal processes are reversible, we may write that

$$Q_A = T_h(S_B - S_A) \quad \text{while} \quad Q_R = T_c(S_B - S_A), \quad (2.7)$$

and therefore

$$\eta = 1 - \frac{T_c}{T_h}. \quad (2.8)$$

This result means that the efficiency of the Carnot cycle does not depend on the type of system used as a working fluid, but only on the temperatures of the hot and cold reservoirs, and the fact that all processes are reversible.

2.3.2 The Otto cycle

Another important cycle was proposed by the French engineer [de Rochas \(1862\)](#). Although Beau de Rochas proposed it first, the cycle carries the name of the German engineer, Nikolaus A. Otto, who first built it. To make it simpler, we shall address the cycle considering an ideal gas as a working substance. The cycle consists of four thermodynamic strokes:

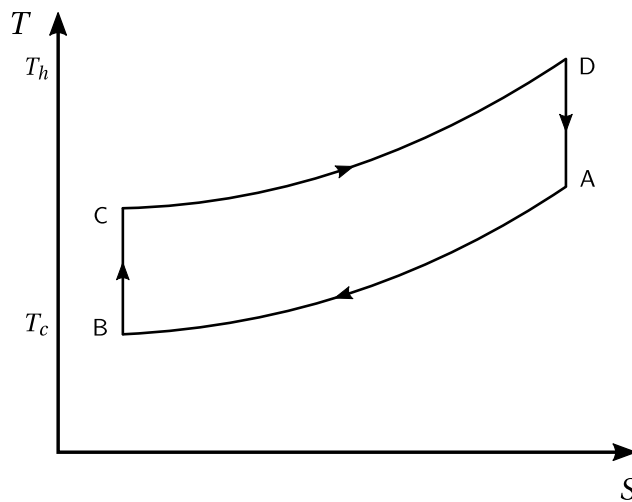


Figure 2.4: An Otto cycle heat engine sketched.

- (BC) Isentropic compression

The working fluid is thermally isolated and compressed from state A to B. Since it is an adiabatic compression, no heat is exchanged during this step.

- (CD) Hot isochoric

The working fluid at state B is put in contact with the hot reservoir and heated at constant volume C. No work is performed since the process takes place at constant volume.

- (DA) Isentropic expansion

The working fluid is again thermally insulated, and expanded adiabatically from state C to D.

- (AB) Cold isochoric

The final stroke consists in bringing the working fluid to its initial state A, isochorically cooling the system from state D to the state A.

Assuming that the working fluid obeys the ideal gas equation of state, the efficiency can be written in terms of the corresponding volumes as

$$\eta_O = 1 - \left(\frac{V_2}{V_1} \right)^{\gamma-1}, \quad (2.9)$$

where γ is the ratio of the heat capacities at constant pressure and volume, respectively. Notice that the efficiency of an Otto cycle does depend on the properties of the working fluid, as can be seen in eq.(2.9), through the heat capacity. It is worth mentioning that the Otto cycle is not reversible.

2.4 The Helmholtz free energy

The first law of thermodynamics states that

$$dU = \delta W + \delta Q = TdS - pdV, \quad (2.10)$$

ignoring electrical and or other kinds of thermodynamical work. Since $TdS = d(TS) - SdT$, eq.(2.10) can be recast as

$$dU = d(TS) - SdT - pdV \Rightarrow d(U - TS) = -SdT - pdV. \quad (2.11)$$

The thermodynamic function $U - TS$ is called the Helmholtz free energy F

$$F \equiv U - TS. \quad (2.12)$$

The Helmholtz free energy is the Legendre transformation of the internal energy U (de Oliveira, 2005), in which temperature replaces entropy as the independent variable. The Helmholtz free energy is a convenient function

when the temperature and volume are easily controlled or measured. For instance, when a fluid slowly evolves in thermal equilibrium with a constant temperature. For such a process, the variation in the Helmholtz free Energy is

$$\Delta F = - \int_{V_i}^{V_f} p dV. \quad (2.13)$$

Hence, the work performed on (by), an isothermal working fluid increases, or decreases, its Helmholtz free energy. In other words, the Helmholtz free energy is a thermodynamic potential that measures the useful work obtainable from a closed thermodynamic system.

CHAPTER 3

Fundamentals of quantum mechanics

Quantum mechanics is the most profound picture of nature we have. The development of quantum mechanics in the early years of the twentieth century, involving names like Planck, Einstein, Bohr, Heisenberg, Schrödinger, and Dirac, left us by 1927 with a sophisticated understanding that is unquestionably one of the most significant intellectual accomplishments in human history.

By the end of the nineteenth century, scientists had managed to unify particles and fields into a single entity, the wave function. The incentive to do so came from two directions: first, physicists discovered that things they thought were waves, like the electric and magnetic fields, had particle-like properties (Einstein, 1905; Compton, 1923). Then, they realized that things they thought were particles, like electrons, manifested field-like properties (Davisson and Germer, 1928). The reconciliation of these puzzles lies in the fact that the world is fundamentally field-like, it is a *quantum wave function*, but when we look at it by performing a careful measurement, it seems particle-like.

In this dissertation, the “path of the wave-function” will not be followed. Instead, a formalism based on the density operator that was introduced by von Neuman (1927) is going to be adopted. A density operator encompasses all the information about a physical system, while wave-functions are restricted to particular cases. Therefore, in this chapter, we present a cursory look at quantum mechanics. The chapter starts by introducing the formalism of the density matrix and deriving the Liouville-von Neumann Equation. Then, the interaction picture is presented to review some tools used in chapter 5. The chapter finishes with a very succinctly discussion about quantum entropies and how to quantify coherence. It is worth to emphasize that one of the central concepts discussed in this dissertation is the idea of coherence. Any state that is not diagonal carries a certain amount of coherence (with respect to the chosen basis). That refers to the principle of superposition: one of

the main building blocks of quantum physics.

3.1 The density matrix formalism

Quantum mechanics is all about states and operators. States represent the instantaneous configuration of a system, while operators are functions that map a state into another state. The quantum mechanics has presented us that all information about a given system is encompassed in an object called wave function $|\psi\rangle$ (Cohen-Tannoudji et al., 1977). However, this is not always true. The Hungarian-American mathematician and physicist von Neuman (1927) showed that the wave function is not the most general way of defining a quantum state.

To motivate the discussion, consider a quantum machine that produces states $|\psi_1\rangle$. Nevertheless, this machine is not perfect, and sometimes it can produce states $|\psi_2\rangle$ with probability $1-p$, while producing states $|\psi_1\rangle$ with probability p . How do we describe such a state? Our best guess is that the produced state will be a mixture of states $|\psi_1\rangle$ and $|\psi_2\rangle$, with probabilities p and $1-p$, respectively. Notice how some classical uncertainty is introduced in the description. Of course, there may be as many p 's as we want. We could also say that this machine now produce also states $|\psi_3\rangle$ and $|\psi_4\rangle$ with probabilities p_3 and p_4 . The general idea is all we must assume is that they behave like classical probabilities,

$$p_i \in [0, 1] \quad , \quad \sum_i p_i = 1. \quad (3.1)$$

Let us consider now an observable \hat{A} , and compute its expectation value. We must do that by taking an average over this ensemble of states,

$$\langle \hat{A} \rangle = \sum_i p_i \langle \psi_i | \hat{A} | \psi_i \rangle. \quad (3.2)$$

We weight the possible expectation values by the relative probabilities p_i that each one occurs. Notice that,

$$\langle \psi_i | \hat{A} | \psi_i \rangle = \text{tr}(\hat{A} | \psi_i \rangle \langle \psi_i |), \quad (3.3)$$

and therefore, eq.(3.2) can be recast as

$$\langle \hat{A} \rangle = \sum_i p_i \text{tr}(\hat{A} | \psi_i \rangle \langle \psi_i |) = \text{tr} \left\{ \hat{A} \sum_i p_i | \psi_i \rangle \langle \psi_i | \right\}. \quad (3.4)$$

This motivates to the following definition:

Density matrix. The most general representation of a quantum system is written in terms of an operator $\hat{\rho}$ such that

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i|. \quad (3.5)$$

This operator is called the density operator or the density matrix.

Then, eq.(3.2) can be written as

$$\langle A \rangle = \text{tr}(\hat{\rho}A). \quad (3.6)$$

The density operator satisfies some properties that can be extracted from its definition (3.5) and condition (3.1):

i. Hermiticity $\hat{\rho}$ is Hermitian

$$\hat{\rho} = \hat{\rho}^\dagger. \quad (3.7)$$

ii. Positivity $\hat{\rho}$ is positive semi-definite

$$\hat{\rho} \geq 0 \quad \text{i.e.} \quad \langle \psi | \hat{\rho} | \psi \rangle \geq 0, \quad \text{for all } |\psi\rangle \in \mathbb{H}. \quad (3.8)$$

where \mathbb{H} is the Hilbert space.

Proof: From eq.(3.5),

$$\langle \phi | \hat{\rho} | \phi \rangle = \sum_i p_i \langle \phi | \psi_i \rangle \langle \psi_i | \phi \rangle = \sum_i p_i |\langle \phi | \psi_i \rangle|^2 \geq 0. \quad (3.9)$$

This means that the average of the density matrix over any quantum state is always non-negative. Consequently, all its eigenvalues are also non-negative.

iii. Trace The density matrix has trace one

$$\text{tr}(\hat{\rho}) = 1. \quad (3.10)$$

Proof: From the definition of the trace and (3.5)

$$\text{tr}(\hat{\rho}) = \sum_j \langle j | \hat{\rho} | j \rangle = \sum_{j,i} p_i \langle j | \psi_i \rangle \langle \psi_i | j \rangle = \sum_i p_i \langle \psi_i | \psi_i \rangle = \sum_i p_i = 1. \quad (3.11)$$

This is the normalization condition for the density matrix.

iv. Purity. The density operator $\hat{\rho}$ is said to be a pure state if

$$\hat{\rho} = |\psi\rangle\langle\psi|, \quad (3.12)$$

which implies

$$\hat{\rho} = \hat{\rho}^2, \quad (3.13)$$

Conversely, states which cannot be written in the form given by eq.(3.12) are called mixed states,

$$\hat{\rho} \neq \hat{\rho}^2. \quad (3.14)$$

The purity of the system is defined by the trace of $\hat{\rho}^2$,

$$\text{tr}(\hat{\rho}^2) = \text{tr} \left\{ \sum_{i,j} p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| \right\} = \sum_i p_i^2 \leq 1. \quad (3.15)$$

For a pure ensemble, this is trivial since $\hat{\rho}^2 = \hat{\rho}$, and the purity is equal to one. However, for a mixed ensemble, the purity will be smaller than one and, consequently, the trace of $\hat{\rho}^2$ can be used as a measure of the purity of the system.

3.1.1 The Liouville-von Neumann equation

According to quantum mechanics, the state vector $|\psi(t)\rangle$ evolves in time obeying the Schrödinger equation (Cohen-Tannoudji et al., 1977),

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{\mathcal{H}}(t) |\psi(t)\rangle, \quad (3.16)$$

where $\hat{\mathcal{H}}(t)$ is the Hamiltonian of the system. The solution of the Schrödinger equation may be represented in terms of the unitary time-evolution operator $\hat{U}(t, t_0)$ which transforms the state $|\psi(t_0)\rangle$ at some initial time t_0 , to the state $|\psi(t)\rangle$, at time t ,

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle. \quad (3.17)$$

A Schrödinger equation (3.16) for the time-evolution operator is obtained by substituting (3.17) into (3.16),

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{\mathcal{H}}(t) \hat{U}(t, t_0), \quad (3.18)$$

subject to the initial condition

$$\hat{U}(t_0, t_0) = \mathbb{1}, \quad (3.19)$$

For a closed isolated physical system, the energy is a constant of motion, and eq.(3.18) yields the well-known expression

$$\hat{U}(t, t_0) = e^{-i\hat{\mathcal{H}}(t-t_0)/\hbar}. \quad (3.20)$$

In many other situations, one often encounters cases where $\hat{\mathcal{H}}$ is time-dependent. If, in such case, the dynamics of the system can still be formulated in terms of a possibly time-dependent Hamiltonian generator $\hat{\mathcal{H}}(t)$, the system is considered to be closed. For a time-dependent Hamiltonian, the evolution operator can be written as a Dyson series as

$$\hat{U}(t, t_0) = \mathcal{T} \exp \left\{ - \int_{t_0}^t dt' \hat{\mathcal{H}}(t') \right\}, \quad (3.21)$$

where \mathcal{T} denotes the chronological time-ordering operator. For instance,

$$\mathcal{T} [\hat{\mathcal{H}}(t_1) \hat{\mathcal{H}}(t_2)] = \begin{cases} \hat{\mathcal{H}}(t_1) \hat{\mathcal{H}}(t_2) & \text{if } t_2 < t_1 \\ \hat{\mathcal{H}}(t_2) \hat{\mathcal{H}}(t_1) & \text{if } t_2 > t_1. \end{cases} \quad (3.22)$$

In order to determine the time evolution of the density operator in closed quantum systems, let $\hat{\rho}(t_0)$ be the initial state of the system given by

$$\hat{\rho}(t_0) = \sum_i p_i |\psi_i(t_0)\rangle \langle \psi_i(t_0)|, \quad (3.23)$$

where $|\psi(t_0)\rangle$ are normalized state vectors which evolving in time according to the Schrödinger equation (3.16). Thus, for a given time t , the state of the system will be

$$\hat{\rho}(t) = \sum_i p_i |\psi(t)\rangle \langle \psi(t)|, \quad (3.24)$$

or in terms of the evolution operator,

$$\begin{aligned} \hat{\rho}(t) &= \sum_i p_i |\psi(t)\rangle \langle \psi(t)| = \sum_i p_i \hat{U}(t, t_0) |\psi(t_0)\rangle \langle \psi(t_0)| \hat{U}^\dagger(t, t_0) \\ &= \hat{U}(t, t_0) \left\{ \sum_i |\psi_i(t_0)\rangle \langle \psi_i(t_0)| \right\} \hat{U}^\dagger(t, t_0) \\ &= \hat{U}(t, t_0) \hat{\rho}(t_0) \hat{U}^\dagger(t, t_0), \end{aligned} \quad (3.25)$$

i.e., the time evolution is described by a unitary transformation \hat{U} . Differentiating this equation with respect to time, the equation of motion for the density operator is obtained,

$$\begin{aligned} i\hbar \frac{d\hat{\rho}}{dt} &= \hat{\mathcal{H}}(t) \hat{U}(t, t_0) \hat{\rho}(t_0) \hat{U}^\dagger(t, t_0) - \hat{U}(t, t_0) \hat{\rho}(t_0) \hat{U}^\dagger(t, t_0) \hat{\mathcal{H}}(t) \\ &= \hat{\mathcal{H}}(t) \hat{\rho}(t) - \hat{\rho}(t) \hat{\mathcal{H}}(t). \end{aligned} \quad (3.26)$$

This is known as the *Liouville-von Neumann's equation*, and may be written more concisely as

$$i\hbar \frac{d\hat{\rho}}{dt} = [\hat{\mathcal{H}}(t), \hat{\rho}(t)] = \mathcal{L} \hat{\rho}(t). \quad (3.27)$$

where \mathcal{L} is the *Liouvillian super-operator*, an operator acting on operators to yield another operator.

The formal solution of (3.27) can be written as

$$\hat{\rho}(t) = \mathcal{T} \exp \left\{ \int_{t_0}^t dt' \mathcal{L}(t') \right\} \hat{\rho}(t_0). \quad (3.28)$$

For a time-independent Hamiltonian, the Liouvillian super-operator is also time-independent, and we have

$$\hat{\rho}(t) = e^{-\mathcal{L}(t-t_0)} \hat{\rho}(t_0). \quad (3.29)$$

An interesting consequence of eq.(3.27) is that the purity (3.15) remains constant during unitary evolutions. To see that, we only need to use the cyclic property of the trace:

$$\text{tr}(\hat{\rho}(t)^2) = \text{tr}(\hat{U} \hat{\rho}_0 \hat{U}^\dagger \hat{U} \hat{\rho}_0 \hat{U}^\dagger) = \text{tr}(\hat{\rho}_0 \hat{\rho}_0 \hat{U}^\dagger \hat{U}) = \text{tr}(\hat{\rho}_0^2). \quad (3.30)$$

3.2 The interaction picture

The pictures of quantum mechanics are equivalent routes to formulate the dynamics of a quantum system mathematically. The three most important ones are the Schrödinger, Heisenberg, and Dirac (interaction) pictures. These differ simply by a basis change in respect to a time-dependency. Its use depends solely on the type of problem to be solved.

In the Schrödinger picture, quantum systems are described by state vectors that evolve in time, while the operators stay fixed (Messiah, 2014),

$$\frac{d}{dt} |\psi_S(t)\rangle = -\frac{i}{\hbar} \hat{\mathcal{H}}_S |\psi_S(t)\rangle, \quad (3.31)$$

where the label S in each expression of eq.(3.31) is used to refer to the Schrödinger picture. Conversely, in the Heisenberg picture, it is the operators that change in time while the state vectors are time-independent (Messiah, 2014),

$$\hat{O} = \hat{O}(t). \quad (3.32)$$

The differences between the Schrödinger and Heisenberg pictures of quantum mechanics revolve around how to deal with systems that evolve in time: the time-dependent nature of the system must be carried by some combination of the state vectors and the operators. What is essential about these two pictures is that they are equivalents in its description (Messiah, 2014).

For our purpose, none of these two pictures will be so useful as the *interaction picture*, or the *Dirac's picture*. The interaction picture is halfway between the Schrödinger and the Heisenberg pictures, as both the quantum states and the operators carry a time dependency. The interaction picture is useful when the type of problem in consideration include time-dependent interaction terms explicitly in the Hamiltonian - or are not precisely soluble.

The idea of the interaction picture is to split a given Hamiltonian $\hat{\mathcal{H}}(t)$, into the sum of two parts

$$\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}_0 + \hat{V}(t), \quad (3.33)$$

where $\hat{\mathcal{H}}_0$ is the Hamiltonian whose solution is known, and $\hat{\mathcal{V}}$ is the Hamiltonian describing the interaction term, and may be considered as a perturbation. Of course, the splitting in eq.(3.33) depends on the particular physical situation under study. The expectation value of a Schrödinger's operator \hat{O} may be written using eq.(3.2)

$$\langle \hat{O}(t) \rangle = \text{tr}(\hat{\rho}(t)\hat{O}(t)), \quad (3.34)$$

that can be rewritten, after inserting some identities and using the cyclic property of the trace, as

$$\langle \hat{O}(t) \rangle = \text{tr}(e^{i\hat{\mathcal{H}}_0 t/\hbar}\hat{O}(t)e^{-i\hat{\mathcal{H}}_0 t/\hbar}e^{i\hat{\mathcal{H}}_0 t/\hbar}\hat{\rho}(t)e^{-i\hat{\mathcal{H}}_0 t/\hbar}) = \text{tr}(\hat{O}_I(t)\hat{\rho}_I(t)), \quad (3.35)$$

where \hat{O}_I and $\hat{\rho}_I$ are the interaction picture operators defined by

$$\hat{O}_I(t) \equiv \hat{U}_0^\dagger(t)\hat{O}(t)\hat{U}_0(t) \quad (3.36)$$

$$\hat{\rho}_I(t) \equiv \hat{U}_0^\dagger(t)\hat{\rho}(t_0)\hat{U}_0(t), \quad (3.37)$$

and $\hat{U}_0 = e^{-i\hat{\mathcal{H}}_0 t/\hbar}$.

In order to obtain the time evolution of $\hat{\rho}_I(t)$, all we need is take the time derivative of eq.(3.37):

$$\begin{aligned} i\hbar \frac{d}{dt}\hat{\rho}_I(t) &= i\hbar \left(\frac{d\hat{U}_0^\dagger}{dt} \right) \hat{\rho}(t)\hat{U}_0 + i\hbar \hat{U}_0^\dagger \left(\frac{\partial \hat{\rho}(t)}{\partial t} \right) \hat{U}_0(t) + i\hbar \hat{U}_0^\dagger \hat{\rho}(t) \left(\frac{d\hat{U}_0}{dt} \right) \\ &= -\hat{U}_0^\dagger \hat{\mathcal{H}}_0 \hat{U}_0 \hat{U}_0^\dagger \hat{\rho} \hat{U}_0^\dagger + i\hbar \hat{U}_0^\dagger \left(\frac{\partial \hat{\rho}(t)}{\partial t} \right) \hat{U}_0(t) + \hat{U}_0^\dagger \hat{\rho} \hat{U}_0 \hat{U}_0^\dagger H_0 \hat{U}_0 \\ &= [\hat{\rho}_I(t), \hat{\mathcal{H}}_{0,I}] + i\hbar \hat{U}_0^\dagger \left(\frac{\partial \hat{\rho}(t)}{\partial t} \right) \hat{U}_0(t), \end{aligned} \quad (3.38)$$

and using the von-Neumann equation (3.27) to rewrite the second term of eq.(3.38)

$$\begin{aligned} i\hbar \hat{U}_0^\dagger \left(\frac{\partial \hat{\rho}(t)}{\partial t} \right) \hat{U}_0(t) &= \hat{U}_0^\dagger [\hat{\mathcal{H}}(t), \hat{\rho}(t)] \hat{U}_0 = i\hat{U}_0^\dagger [\hat{\mathcal{H}}_0 + \hat{\mathcal{V}}(t), \hat{\rho}] \hat{U}_0 \\ &= [H_{0,I}, \hat{\rho}_I(t)] + [\mathcal{V}_I(t), \hat{\rho}_I(t)], \end{aligned} \quad (3.39)$$

the corresponding von-Neumann equation in the interaction picture is obtained:

$$\frac{d}{dt}\hat{\rho}_I(t) = -\frac{i}{\hbar} [\hat{\mathcal{V}}_I(t), \hat{\rho}_I(t)]. \quad (3.40)$$

Furthermore, we can write this equation in integral form as

$$\hat{\rho}_I(t) = \hat{\rho}_I(t_0) - \frac{i}{\hbar} \int_{t_0}^t dt' [\hat{\mathcal{V}}_I(t'), \hat{\rho}_I(t')]. \quad (3.41)$$

This equation will be the starting point for the construction of a model describing the interaction between a system and a reservoir (chapter 5).

3.3 Quantum entropies

Given a system described by the density matrix $\hat{\rho}$, it could be asked how much it is known about $\hat{\rho}$. In other words, how much information is stored in a quantum state? The von-Neumann entropy gives a possible answer to this question. In its most straightforward interpretation, entropy is a measure of the disorder (or mixedness) of a density matrix, quite like the purity (3.15). However, with the entropy, this disorder acquires a more informational sense (Breuer et al., 2002).

The von-Neumann entropy. Given a density matrix $\hat{\rho}$, the von-Neumann entropy is defined as

$$S(\hat{\rho}) = -\text{tr}(\hat{\rho} \log \hat{\rho}) = -\sum_k p_k \log p_k, \quad (3.42)$$

where in the second equality, is used the spectral decomposition of the density matrix. The logarithm in eq.(3.42) can be either base two or base e depending on the context where it is applied.

The von Neumann entropy is a sum of functions of the negative quantity $p_k \log p_k$, defined only in its diagonal form, with $0 \leq p_k \leq 1$. The plot of this function is shown in fig.(3.1). It tends to zero for $p \rightarrow 0$ and $p \rightarrow 1$, as expected, for pure states, and has a maximum at $p = 1/e$ for a maximally mixed state.

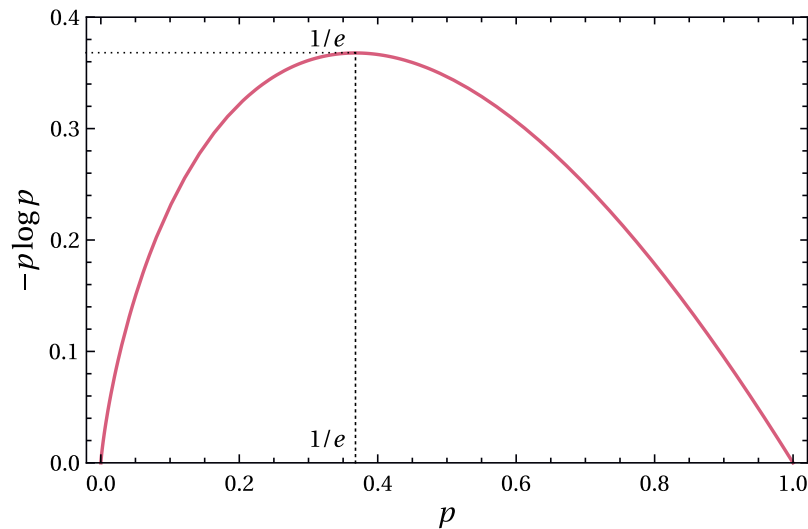


Figure 3.1: The function $-p \log p$, corresponding to each term in the von-Neumann entropy. The dimension of the Hilbert space in this example is $\log \mathcal{D}$.

Since $-p \log p$ is always non-negative, the same must be valid for $S(\rho)$:

$$S(\hat{\rho}) \geq 0. \quad (3.43)$$

As we can see in fig.(3.1), the entropy is bounded from below by 0. In the case where we are dealing with a finite Hilbert space whose dimension is \mathcal{D} , this implies that $S(\hat{\rho})$ will be bounded from above (Breuer et al.,

2002)

$$S(\rho) \leq \log \mathcal{D} \quad \text{for} \quad \hat{\rho} = \frac{\mathbb{1}}{\mathcal{D}}. \quad (3.44)$$

Equality occurs when the mixture is maximal. Therefore, the von-Neumann entropy varies between 0 for pure states and $\log(\mathcal{D})$ for maximally disordered states. Hence, it serves as a measure of how mixed a state is.

Another essential property of the entropy (3.42) is that it is invariant under unitary transformations:

$$S(U\hat{\rho}U^\dagger) = S(\hat{\rho}). \quad (3.45)$$

Throughout this dissertation, it is considered the von Neumann entropy as the entropy that appears in the second law in eq.(2.2) since the correspondence between the von Neumann entropy and the thermodynamical entropy is achieved when the system is in equilibrium (Spohn, 1978; Alicki, 1979). Hence, adiabatic processes, in analogy with classical thermodynamics, is defined as the process in which eq.(3.45) is satisfied.

3.4 The quantum relative entropy

Another important quantity is the quantum relative entropy or Kullback-Leibler divergence (Kullback, 1959). Given two density matrices $\hat{\rho}$ and σ , it is defined as

$$S(\hat{\rho}||\sigma) = \text{tr}(\hat{\rho} \log \hat{\rho} - \hat{\rho} \log \sigma). \quad (3.46)$$

A particular relevant property of the relative entropy is that it satisfies the Klein inequality:

$$S(\hat{\rho}||\sigma) \geq 0 \quad , \quad S(\hat{\rho}||\sigma) = 0 \quad \text{iff} \quad \hat{\rho} = \sigma. \quad (3.47)$$

This result gives the idea that the relative entropy can be used as a measure of the distance between two density matrices. However, this is not entirely precise since the relative entropy does not satisfy the triangle inequality,

$$d(x, z) \leq d(x, y) + d(y, z). \quad (3.48)$$

To be a true measure of distance, it must follow the triangle inequality. Nevertheless, it can still be used as a measure of distance due to its simplicity, or at least, as an information quantifier. For instance, let $\hat{\pi}$ be the maximally mixed state given by (3.44). This is the state with maximal uncertainty and minimum information. Consequently, the amount of information in a state $\hat{\rho}$ can be defined as the distance between $\hat{\rho}$ and π ,

$$I(\hat{\rho}) = S(\hat{\rho}||\pi), \quad (3.49)$$

so, the information contained in the system is given by

$$I(\hat{\rho}) = \text{tr}(\hat{\rho} \log \hat{\rho}) - \text{tr}(\hat{\rho} \log(1/\mathcal{D})) = -S(\hat{\rho}) + \log \mathcal{D}. \quad (3.50)$$

We observe that, for higher values of entropy, less is the information about the system. In the case of a maximally mixed system, $S(\hat{\rho}) = \log \mathcal{D}$, the relative entropy or information is minimum (zero). However, for a pure state, the entropy vanishes and the information is maximum and equal to $\log \mathcal{D}$.

Now, suppose that instead of wondering how much do we know about a given system, we want to quantify some property \mathcal{P} of a given quantum state. Let \mathcal{S} be the set of all states which do not have that property. Then, the distance between $\hat{\rho}$ and the closest state without that property can be used to estimate that property

$$\mathcal{P}(\hat{\rho}) = \min_{\sigma \in \mathcal{S}} S(\hat{\rho} || \sigma). \quad (3.51)$$

3.4.1 The relative entropy of coherence

As a first application of the quantum relative entropy, let us consider that the property of interest is the degree of coherence a state has relative to some basis.

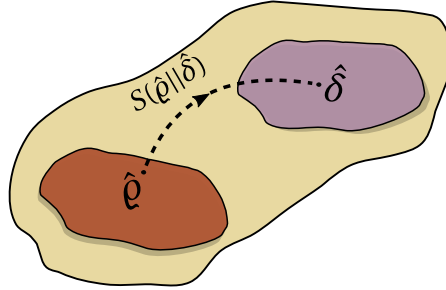


Figure 3.2: Schematic representation of the relative entropy of coherence.

Coherence is a basis dependent concept. Given a particular basis, a state is called incoherent if it is diagonal on that basis. Any state that is not diagonal carries a certain amount of coherence associated with the chosen basis. Thus, let \mathcal{S} be the set of incoherent states with respect to the basis $|i\rangle$. This will be the set of all states which are diagonal in this basis,

$$\delta = \sum_i \delta_i |i\rangle \langle i| \in \mathcal{S}. \quad (3.52)$$

Given a state $\hat{\rho}$, the amount of coherence present in it is given by

$$C(\hat{\rho}) = \min_{\delta \in \mathcal{S}} S(\hat{\rho} || \delta). \quad (3.53)$$

This is called *the relative entropy of coherence*. In this case, the minimization can be done in closed form. Given a general state

$$\hat{\rho} = \sum_{i,j} p_{i,j} |i\rangle \langle j|, \quad (3.54)$$

and defining

$$\hat{\rho}_{\text{diag}} = \sum_i |i\rangle \langle i|, \quad (3.55)$$

as its diagonal form in the $|i\rangle$ basis, it may be verified that for any $\delta \in \mathcal{S}$, $\text{tr}(\hat{\rho} \log \delta) = \text{tr}(\hat{\rho}_{\text{diag}} \log \delta)$, so

$$S(\hat{\rho}||\delta) = S(\hat{\rho}_{\text{diag}}) - S(\hat{\rho}) + S(\hat{\rho}_{\text{diag}}||\delta) \geq S(\hat{\rho}_{\text{diag}}) - S(\hat{\rho}). \quad (3.56)$$

Note that the minimum value of $S(\hat{\rho}||\delta)$ occurs precisely for $\delta = \hat{\rho}_{\text{diag}}$, so that the relative entropy of coherence can be written as

$$C(\hat{\rho}) = \min_{\delta \in \mathcal{S}} S(\hat{\rho}||\delta) = S(\hat{\rho}_{\text{diag}}) - S(\hat{\rho}). \quad (3.57)$$

A notable example of a state that has coherence in the energy eigenbasis is the set of squeezed states ([A. Serafini, 2019](#)).

CHAPTER 4

The quantum nature of light

The goal of this chapter is providing a thorough discussion about the single-mode bosonic field and its particularities. Our aim is to learn the main properties of the working substance that we shall adopt. Also, it will provide the required knowledge for the comprehension of chapter 6. We start by presenting the quantization of the electromagnetic field and discussing some of its consequences - for instance, the concept of thermal states. Given the central role played by the single-mode, we move to introduce the coherent states. This is the framework for quantum characteristic functions and quasi-probability distributions, which goes back to the seminal work by [Wigner \(1932\)](#). Also, the fundamental aspects of the squeezed states are examined. With the purpose of characterizing the states of light into classical or non-classical, the concept of P-representability is presented, and the formalism of characteristic function is also introduced. We finish the chapter with the Gaussian state parametrization ([Englert and Wódkiewicz, 2003](#)).

4.1 Quantization of a single-mode field

Let us consider the simple case of a radiation field confined to a one-dimensional cavity along the z -axis with perfectly conducting walls at $z = 0$ and $z = L$, as shown in [fig.\(4.1\)](#). The electric field must vanish at the boundaries and will take the form of a standing wave. Additionally, it is assumed that there are no other sources of radiation and that the electric field is polarized along the x -axis, $\mathbf{E}(\mathbf{r}, t) = \mathbf{e}_x E_x(z, t)$, where \mathbf{e}_x is a unit polar-

ization vector. In this case, Maxwell's equations (in SI units) are:

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{E} &= 0.\end{aligned}\tag{4.1}$$

A single-mode field satisfying Maxwell's equations and the boundary conditions is given by (Jackson, 1975):

$$E_x(z, t) = \left(\frac{2\omega^2}{V\epsilon_0} \right)^{1/2} q(t) \sin(kz),\tag{4.2}$$

where ω is the frequency of the mode, and k is the wave number related to the frequency according to $k = \omega/c$. The boundary conditions at $z = L$ determine the allowed frequencies, $\omega_m = c(m\pi/L)$, $m = 1, 2, \dots$. The term V in eq.(4.2) is the effective volume of the cavity and $q(t)$ will act as a canonical position. The magnetic field in the cavity is, from eqs.(4.1-4.2), $\mathbf{B}(\mathbf{r}, t) = \mathbf{e}_y B_y(z, t)$, with

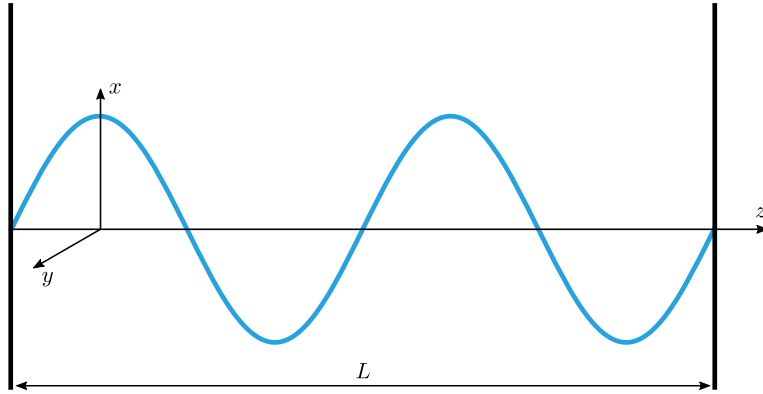


Figure 4.1: Cavity with conducting walls located at $z = 0$ and $z = L$. The electromagnetic field is polarized along the x -direction.

$$B_y(z, t) = \left(\frac{\mu_0 \epsilon_0}{k} \right) \left(\frac{2\omega^2}{V\epsilon_0} \right) \dot{q}(t) \cos(kz).\tag{4.3}$$

Here, $\dot{q}(t)$ plays the role of a canonical momentum for a "particle" of unit mass, i.e, $p(t) = \dot{q}(t)$. The classical field energy, or Hamiltonian H , of the single-mode is given by

$$\begin{aligned}H &= \frac{1}{2} \int dV \left[\epsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^2(\mathbf{r}, t) \right] \\ &= \frac{1}{2} \int dV \left[\epsilon_0 E_x^2(z, t) + \frac{1}{\mu_0} B_y^2(y, t) \right].\end{aligned}\tag{4.4}$$

From eqs.(4.2) and (4.3) the Hamiltonian can be rewritten as

$$H = \frac{1}{2} (p^2 + \omega^2 q^2),\tag{4.5}$$

from which, we see that the Hamiltonian of a single-mode field is formally equivalent to that of a harmonic oscillator of unit mass (Cohen-Tannoudji et al., 1977), where the electric and magnetic fields play the role of canonical operators.

Next, it is used the correspondence rule (Cohen-Tannoudji et al., 1977) to replace p and q by their operator equivalents \hat{q} and \hat{p} , satisfying the canonical commutation relation

$$[\hat{q}, \hat{p}] = i\hbar. \quad (4.6)$$

The electric and magnetic fields of the single-mode become the operators

$$\begin{aligned} \hat{E}_x(z, t) &= \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \hat{q}(t) \sin(kz) \\ \hat{B}_y(z, t) &= \left(\frac{\mu_0\epsilon_0}{k}\right) \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \hat{p}(t) \cos(kz), \end{aligned} \quad (4.7)$$

and the Hamiltonian becomes

$$\hat{\mathcal{H}} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2). \quad (4.8)$$

It is convenient to introduce the non-Hermitian annihilation and creation operators \hat{a} and \hat{a}^\dagger ,

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar\omega}}(\omega \hat{q} + i\hat{p}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}}(\omega \hat{q} - i\hat{p}), \end{aligned} \quad (4.9)$$

satisfying the following algebra

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (4.10)$$

Or alternatively, the operators \hat{q} and \hat{p} can be expressed in terms of the annihilation and creation operators:

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \quad , \quad \hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger). \quad (4.11)$$

In terms of these operators, the electric and magnetic field operators then become, respectively

$$\begin{aligned} \hat{E}_x(z, t) &= \mathcal{E}_0(\hat{a} + \hat{a}^\dagger) \sin(kz) \\ \hat{B}_y(z, t) &= i\mathcal{B}_0(\hat{a}^\dagger - \hat{a}) \cos(kz), \end{aligned} \quad (4.12)$$

where $\mathcal{E}_0 = (2\omega^2/V\epsilon_0)^{1/2}$ and $\mathcal{B}_0 = (\mu_0/k)(\epsilon_0\hbar\omega^3/V)^{1/2}$ represent the electric and magnetic field amplitudes.

As a result, the Hamiltonian operator takes the form

$$\hat{\mathcal{H}} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (4.13)$$

The operator $\hat{a}^\dagger \hat{a}$ has a special meaning and is called the *number operator*, denoted by \hat{n} . Let $|n\rangle$ designate an eigenstate of the number operator with eigenvalue n ,

$$\hat{n} |n\rangle = n |n\rangle, \quad (4.14)$$

thus, the “eigenproblem” for the Hamiltonian (4.13) is given by

$$\hat{\mathcal{H}}|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle = E_n|n\rangle, \quad (4.15)$$

with $E_n = \hbar\omega(n + 1/2)$ and $n \in \mathbb{N}$, where \mathbb{N} is the set of natural numbers.

The action of \hat{a} and \hat{a}^\dagger on the number eigenstate is, using the commutation relation (4.10) and eq.(4.14),

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle. \end{aligned} \quad (4.16)$$

Using these identities, it can be shown that the number eigenstate $|n\rangle$ is generated from the ground state $|0\rangle$, by the repeated action of the creation operator \hat{a}^\dagger :

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{n!}|0\rangle. \quad (4.17)$$

Also, it can be easily shown that $\{|n\rangle\}$ forms an orthonormal basis

$$\langle n|m\rangle = \delta_{n,m}, \quad (4.18)$$

and a complete set,

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1. \quad (4.19)$$

4.1.1 Multiple modes

The result for the single-mode field confined in a cavity can be generalized to multi-mode radiation fields. It is not our purpose to show that since throughout this dissertation, our focus is the single-mode. However, to generalize the idea to composite systems, it is defined a set of annihilation operators \hat{a}_i , where $i = 1, 2, \dots, n$. It is conventional to use the word *mode* to label each i . These operators are defined to satisfy

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j} \quad , \quad [\hat{a}_i, \hat{a}_j] = 0. \quad (4.20)$$

Therefore, operators of different modes commute and the Hilbert space of a multi-mode is expanded by the states

$$|\mathcal{N}\rangle = |n_1, n_2, \dots, n_N\rangle, \quad n_i = 0, 1, 2, \dots \quad (4.21)$$

These are called the *Fock states*, which are the eigenstates of the number operators $\hat{a}_i^\dagger\hat{a}_i$,

$$\hat{a}_i^\dagger\hat{a}_i|\mathcal{N}\rangle = n_i|\mathcal{N}\rangle. \quad (4.22)$$

According to eq.(4.14) each mode can have an arbitrary number n of particles.

4.2 Thermal states

So far, we presented with the quantization of the electromagnetic field and its physical properties. Now we restrict our discussion even more, and we shall see an important concept that is the *thermal state*. Some mathematical aspects related to the Fock states will be investigated, and significant results will be derived. Later on, in chapter 6, we shall address this state more carefully.

Consider a cavity filled with a single-mode bosonic field, at thermal equilibrium with its walls. The density operator for the thermal field is, in the canonical ensemble, given by

$$\hat{\rho}_{\text{th}} = \frac{e^{-\beta\hat{\mathcal{H}}}}{\text{tr}[e^{-\beta\hat{\mathcal{H}}}]}, \quad \beta = \frac{1}{k_B T}, \quad (4.23)$$

where k_B is the Boltzmann constant and $\hat{\mathcal{H}}$ is the system's Hamiltonian (4.13). Notice that,

$$\text{tr}[e^{-\beta\hat{\mathcal{H}}}] = \sum_{n=0}^{\infty} \langle n | e^{-\beta\hat{\mathcal{H}}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta E_n} \equiv Z, \quad (4.24)$$

is the partition function. We can go further and re-write the above expression as

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}, \quad (4.25)$$

since $e^{-\beta\hbar\omega} < 1$. Substituting this partition function into eq.(4.23), the density operator for the single-mode thermal state becomes

$$\hat{\rho}_{\text{th}} = \sum_n (1 - e^{-\beta\hbar\omega}) e^{-\beta\hbar\omega n} |n\rangle \langle n|. \quad (4.26)$$

Correspondingly, the mean number of photons of the thermal field is given by

$$\bar{n}_{\text{th}} = \text{tr}[\hat{n} \hat{\rho}_{\text{th}}] = \frac{1}{e^{\beta\hbar\omega} - 1}, \quad (4.27)$$

Using the last result (4.27), eq.(4.26) can be rewritten in terms of \bar{n}_{th} as

$$\hat{\rho}_{\text{th}} = \sum_n \frac{\bar{n}_{\text{th}}^n}{(1 + \bar{n}_{\text{th}})^{n+1}} |n\rangle \langle n| \quad (4.28)$$

This leads to the remarkable result that will be used extensively in chapter 6:

Single-bosonic mode energy. The mean energy of a single-bosonic mode at equilibrium with a reservoir at temperature T is

$$E = \text{tr}(\hat{\rho}_{\text{th}} \hat{\mathcal{H}}) = \frac{\hbar\omega}{2} \coth\left(\frac{\hbar\omega}{2k_B T}\right). \quad (4.29)$$

Notice that when $T \rightarrow 0$, the mean number of photons also goes to zero, thus

$$\lim_{T \rightarrow 0} E = E_0 = \frac{\hbar\omega}{2}, \quad (4.30)$$

so the equilibrium state tends to the ground state, as expected. Conversely, for high values of temperature, $\beta \rightarrow 0$, and using the approximation $\coth x \approx 1/x$, the energy tends to,

$$E \approx k_B T. \quad (4.31)$$

Thus, for higher values of temperature, the energy becomes linearly proportional to T . Whenever a system has an infinite number of states, the high-temperature result usually match its classical analog. It is possible to relate the mean energy to the partition function Z in eq.(4.25) as

$$E = -\frac{\partial}{\partial \beta} \log Z = T^2 \frac{\partial}{\partial T^2} \log Z. \quad (4.32)$$

This formula is very useful, especially in more complex problems, since finding Z can be a difficult task, and finding E would require the computation of an even more difficult sum. With this expression, this problem is avoided, and E is obtained from a simple differentiation. It also shows that Z is more than a mere normalization constant.

There is an elegant way of interpreting eq.(4.23), which says that at finite temperature, the thermal state will be the one which minimizes the free energy (Parrondo, 2015),

$$F = E - TS, \quad (4.33)$$

where E is given by (4.29), and S is the von Neumann entropy (3.42). To see this, let us write the von-Neumann entropy for the state (4.23)

$$S = -\text{tr}(\hat{\rho} \ln \hat{\rho}) = \text{tr} \left[\hat{\rho} \beta \left(\hat{\mathcal{H}} + \frac{1}{\beta} \log Z \right) \right] = \beta(E - F). \quad (4.34)$$

where

$$F = -\frac{1}{\beta} \log Z \quad \text{or} \quad Z = e^{-\beta F}, \quad (4.35)$$

This clarifies the physical meaning of Z - as being directly related to the free energy. Consequently, using (4.25), the free energy is simply

$$F = \frac{\hbar\omega}{2} + k_B T \log(1 - e^{-\beta\hbar\omega}). \quad (4.36)$$

4.3 Quantum fluctuations of a single-mode

The uncertainty principle asserts that for a pair of canonical conjugate variables, such as position and momentum (or energy and time), it is impossible to know precisely their values at the same time. While in the classical world, uncertainties arise from our ignorance, in the quantum realm, this is not true. There exist fundamental constraints on our understanding of a given system. In order to explore the role of fluctuations, let us consider two non-commuting observables, \hat{A} and \hat{B} , such that

$$[\hat{A}, \hat{B}] = i\hat{C}, \quad (4.37)$$

so it does not exist a common basis of eigenstates for \hat{A} and \hat{B} . Consequently, the two observables cannot be determined precisely. Defining the variation or fluctuation of an operator \hat{A} and \hat{B} , as

$$\delta\hat{A} \equiv \hat{A} - \langle\hat{A}\rangle \quad , \quad \delta\hat{B} \equiv \hat{B} - \langle\hat{B}\rangle \quad , \quad (4.38)$$

note that,

$$[\delta\hat{A}, \delta\hat{B}] = [\hat{A}, \hat{B}] = i\hat{C} \quad , \quad (4.39)$$

and

$$\begin{aligned} (\delta\hat{A})^2 &= \hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle - \langle\hat{A}\rangle^2 \Rightarrow \langle\delta\hat{A}^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 \equiv (\Delta A)^2 \quad , \\ (\delta\hat{B})^2 &= \hat{B}^2 - 2\hat{B}\langle\hat{B}\rangle - \langle\hat{B}\rangle^2 \Rightarrow \langle\delta\hat{B}^2\rangle = \langle\hat{B}^2\rangle - \langle\hat{B}\rangle^2 \equiv (\Delta B)^2 \quad . \end{aligned} \quad (4.40)$$

are the fluctuations of the operators.

On the other hand, the Cauchy–Schwartz inequality states that given two state vectors $|a\rangle$ and $|b\rangle$, it is satisfied that

$$\langle a|a\rangle\langle b|b\rangle \geq |\langle a|b\rangle|^2 \quad . \quad (4.41)$$

Let $|a\rangle = \delta\hat{A}|u\rangle$ and $|b\rangle = \delta\hat{B}|u\rangle$, the above inequality becomes

$$(\Delta A)^2(\Delta B)^2 \geq |\langle u|\delta\hat{A}\delta\hat{B}|u\rangle|^2 \quad . \quad (4.42)$$

The right side of eq.(4.42) can be simply written as

$$\delta\hat{A}\delta\hat{B} = \frac{1}{2}(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A}) + \frac{1}{2}(\delta\hat{A}\delta\hat{B} - \delta\hat{B}\delta\hat{A}) \equiv \hat{F} + \frac{i}{2}\hat{C} \quad (4.43)$$

where $\hat{F} \equiv \frac{1}{2}(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A})$ is a Hermitian operator.

Therefore, the “incompatibility” of two non-commuting observables can be expressed in terms of the following expression

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^2 + \langle F\rangle^2 \quad , \quad (4.44)$$

which is known as the *The Heisenberg–Robertson–Schrödinger uncertainty relation*.

Therefore, the variance can be used as a measure of the uncertainty in some observable, let say \hat{A} , we have that

$$(\Delta A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 \quad , \quad (4.45)$$

or by the standard deviation, that is the square root of eq.(4.45), which is referred to as the uncertainty of the observable \hat{A} . If we apply this equation to compute the fluctuations in the quantized electromagnetic field eq.(4.12) using the Fock state, we observe that it is not zero, even for the case where $\bar{n}_{\text{th}} = 0$ - the so-called *vacuum fluctuation*. For instance, the fluctuations in the electric field (4.7) are given by (Gerry et al., 2005):

$$\Delta E_x = \sqrt{2}\mathcal{E}_0 \sin(kz) \left(n + \frac{1}{2} \right)^{\frac{1}{2}} \quad . \quad (4.46)$$

With the single-mode prepared in the vacuum state $|0\rangle$, the fluctuation of the field strength is

$$\Delta E_x = \mathcal{E}_0 \sin(kz). \quad (4.47)$$

The vacuum fluctuation has its origin in the uncertainty principle because the number operator \hat{n} does not commute with the electric field:

$$[\hat{n}, \hat{E}_x] = \mathcal{E}_0 \sin(kz)(\hat{a}^\dagger - \hat{a}). \quad (4.48)$$

Thus, \hat{n} and \hat{E}_x are complementary quantities, i.e., operators that cannot be measured simultaneously, for which their respective uncertainties obey the inequality (4.44)

$$\Delta n \Delta E_x \geq \frac{1}{2} \mathcal{E}_0 |\sin(kz)| | \langle \hat{a}^\dagger - \hat{a} \rangle |. \quad (4.49)$$

The concept of vacuum fluctuation is fundamental to explain many situations that are not covered by the classical electromagnetic theory. For instance, spontaneous emission is a direct result of vacuum fluctuations. An atom - in free space, initially prepared in the excited state, would remain in the excited state and, therefore - never would radiate. However, the vacuum fluctuations give the essential contribution to that spontaneous emission getting started.

When we quantized the electromagnetic field, a pair of variables that played the role of the canonical operators \hat{q} and \hat{p} were identified. Then, the electric and magnetic fields of the single-mode became a combination of these operators where was lately written in terms of the non-Hermitian annihilation and creation operators. It is customary writing the electric and magnetic fields (4.12) in terms of the so-called *quadratures* \hat{X}_1 and \hat{X}_2 , that are essentially the dimensionless position and momentum operators, given by

$$\begin{aligned} \hat{X}_1 &= \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \\ \hat{X}_2 &= \frac{1}{2i}(\hat{a} - \hat{a}^\dagger). \end{aligned} \quad (4.50)$$

and satisfying the commutation relation

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \quad (4.51)$$

We could rewrite the electric and magnetic fields (4.12) in terms of these operators. However, our aim to introduce the quadrature operators is to discuss the role of the fluctuations as a motivation to present the squeezed states in section 4.5. Since $\langle n | \hat{X}_i | n \rangle = 0$, for $i = 1, 2$ the quadrature fluctuations in the thermal state are simply

$$(\Delta X_1)^2 = \text{tr}(\hat{\rho}_{\text{th}} \hat{X}_1^2) = \frac{1}{4}(2\bar{n}_{\text{th}} + 1), \quad (4.52)$$

and similarly

$$(\Delta X_2)^2 = \text{tr}(\hat{\rho}_{\text{th}} \hat{X}_2^2) = \frac{1}{4}(2\bar{n}_{\text{th}} + 1). \quad (4.53)$$

Consequently, the uncertainty in both quadratures is the same, and it increases with the temperature through \bar{n}_{th} (4.27). Furthermore, the vacuum state, $\bar{n}_{\text{th}} = 0$, minimizes the uncertainty product

$$(\Delta X_1)^2 = (\Delta X_2)^2 = \frac{1}{4}. \quad (4.54)$$

It is important to stress that the uncertainty principle does not say that “everything is uncertain”. Either quadrature (position or momentum) could be determined in an appropriate quantum state, however they cannot be precisely specified at the same time.

4.4 Coherent states

In the last years, the developments and utilization of coherent states have made tremendous progress and achieved a wide range of applications, e.g., in the semi-classical description of quantum systems, in condensed matter physics, in quantum computations, and so on (Glauber, 1963; Kapor et al., 1991; Walls and Milburn, 2008). Its history dates back to 1926, when Lorentz complained, reasonably, that Schrödinger’s wave functions did not reveal a classical motion. As a result, Schrödinger proposed the idea of coherent states in connection with the quantum states of classical motion for a harmonic oscillator. However, for 35 years, activities in this field remained almost inert. It was not until thirty-five years after Schrödinger’s seminal paper that Glauber (1963) and Sudarshan (1963a) provided a complete theoretical framework for coherent states.

The coherent states, denoted by $|\alpha\rangle$, are known as the most classical states among all pure states (Glauber, 1963). This is because its properties follow very closely to classical states. For example, the expectation value of the electric field has the form of the classical expression (Gerry et al., 2005). Also, the time evolution of a coherent state is concentrated along the classical trajectories. A coherent state is defined as the eigenstate $|\alpha\rangle$ of the annihilation operator \hat{a} , with complex eigenvalue α ,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \quad (4.55)$$

as introduced by Klauder (1960). From this definition, a coherent state $|\alpha\rangle$ can be expanded in terms in terms of the number state $|n\rangle$

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle. \quad (4.56)$$

Acting with \hat{a} on each term of the expansion (4.56),

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n} |n-1\rangle = \alpha \sum_{n=0}^{\infty} |n\rangle. \quad (4.57)$$

Multiplying on the left by $\langle m|$ and using the orthogonality of the Fock states (4.18), the recursive formula is obtained

$$C_n = \frac{\alpha}{\sqrt{n}} C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} C_{n-2} = \dots = \frac{\alpha^n}{\sqrt{n!}} C_0, \quad (4.58)$$

and then

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (4.59)$$

Using the normalization condition, $\langle\alpha|\alpha\rangle = 1$, one has:

Coherent states. A normalized coherent state can be expanded in terms of the Fock states $|n\rangle = 0, 1, 2, \dots$, as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (4.60)$$

From the result (4.60), the following propositions can be derived:

i. Orthogonality *The coherent states are not orthogonal; the inner product between two different coherent states $|\alpha\rangle$ and $|\beta\rangle$, respectively, is given by*

$$|\langle\alpha|\beta\rangle|^2 = e^{|\alpha-\beta|^2}, \quad (4.61)$$

Proof: From eq.(4.60)

$$\begin{aligned} \langle\alpha|\beta\rangle &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*n} \beta^n}{\sqrt{n!m!}} \langle n|m\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{(\alpha^* \beta)^n}{n!} \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{\alpha^* \beta}. \end{aligned} \quad (4.62)$$

Then

$$\begin{aligned} |\langle\alpha|\beta\rangle|^2 &= e^{-|\alpha|^2} e^{-|\beta|^2} e^{\alpha^* \beta} \\ &= e^{-|\alpha-\beta|^2}. \end{aligned} \quad (4.63)$$

Hence, the overlap between two coherent states decays exponentially with their distance. For larger α and β , they become approximately orthogonal.

ii. Completeness *The coherent states form a complete set*

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle\alpha| = 1, \quad (4.64)$$

Proof: From eq.(4.60)

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle \alpha| = \frac{1}{\pi} \int d^2\alpha e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^{*n} \alpha^m}{\sqrt{n!m!}} |n\rangle \langle m|, \quad (4.65)$$

in polar coordinates,

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle \alpha| = \frac{1}{\pi} \sum_{n,m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \int_0^{\infty} dr e^{-r^2} r^{n+m+1} \int_0^{2\pi} d\phi e^{-i(n-m)\phi}, \quad (4.66)$$

where $\alpha = r e^{i\phi}$. The integration over ϕ gives zero unless n is equal to m . Thus,

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle \alpha| = 2 \sum_{n,m=0}^{\infty} \frac{|n\rangle \langle n|}{n!} \int_0^{\infty} dr e^{-r^2} r^{2n+1}. \quad (4.67)$$

After integrating by parts n times,

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle \alpha| = 2 \sum_{n,m=0}^{\infty} \frac{|n\rangle \langle n|}{n!} \frac{1}{2} n! = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1. \quad (4.68)$$

Since the coherent states are not orthogonal, and yet they obey a closure relation, it is said that they form an overcomplete basis. That is, a set which is still complete even if some element is removed from it.

iii. Displacement operator *The coherent states can be generated from the vacuum state by the action of a unitary operator called the displacement operator $\hat{D}(\alpha)$*

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle, \quad (4.69)$$

where $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$.

Proof: Using (4.17), one may re-write (4.60) as

$$\begin{aligned} |\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha \hat{a}^\dagger)^n}{n!} |0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} |0\rangle \equiv \hat{D}(\alpha) |0\rangle. \end{aligned} \quad (4.70)$$

As we can see, a coherent state is generated by the application of the operator $\hat{D}(\alpha)$ on the vacuum state. However, $\hat{D}(\alpha)$ is not unitary. But notice that if,

$$\tilde{\hat{D}}(\alpha) \rightarrow e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} e^{\beta \hat{a}}, \quad (4.71)$$

this implies that

$$e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} e^{\beta \hat{a}} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} |0\rangle. \quad (4.72)$$

Also, imposing that $D(\alpha)D^\dagger(\alpha) = D^\dagger(\alpha)D(\alpha) = 1$, it is obtained that $\beta = -\alpha^*$. Therefore,

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle, \quad (4.73)$$

where

$$\hat{D}(\alpha) \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}, \quad (4.74)$$

is the displacement operator.

To understand why $\hat{D}(\alpha)$ is called a displacement operator, notice that it satisfies

$$\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(\alpha), \quad (4.75)$$

which means that if we displace by a given α and then displace back by $-\alpha$, we return to where we started. Next, computing the action of this operator on the annihilation \hat{a} and creation \hat{a}^\dagger operators, with the help of the *Baker–Campbell–Hausdorff formula* (see Appendix A.3), this reads

$$\begin{aligned} \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) &= \hat{a} + \alpha \\ \hat{D}^\dagger(\alpha) \hat{a}^\dagger \hat{D}(\alpha) &= \hat{a}^\dagger + \alpha^*, \end{aligned} \quad (4.76)$$

This is why we call \hat{D} the displacement operator, since it displaces \hat{a} by the value α . Also, the displacement operator obeys the semi-group relation:

$$\hat{D}(\alpha + \beta) = \hat{D}_{\alpha+\beta} e^{-i\text{Im}\{\alpha\beta^*\}}. \quad (4.77)$$

To finish our discussion about coherent states is worth mentioning that, coherent states are minimum uncertainty states. That is the way Schrödinger derived them. Thus,

$$\Delta q \Delta p = \frac{1}{2} \hbar, \quad (4.78)$$

this result can be checked using eq.(4.11) and (4.45). In terms of the quadratures (4.50),

$$(\Delta X_1)^2 = (\Delta X_2)^2 = \frac{1}{4}. \quad (4.79)$$

the coherent states have the same uncertainty of the vacuum state, meaning that the vacuum state is also a coherent state.

4.4.1 Phase-space picture

It is well known that the concept of phase space in quantum mechanics is not simple since the canonical variables \hat{q} and \hat{p} do not commute. Consequently, it is not possible to localize a point in phase space as it is usually done in classical mechanics. However, it is possible to establish a direct connection between operators on Hilbert space $L^2(\mathbb{R}^n)$ and functions of $2n$ variables, which stems from the fact that the displacement operators $\hat{D}(\alpha)$ form a completely orthogonal set on the space of operators on $L^2(\mathbb{R}^n)$ concerning the Hilbert-Schmidt scalar product¹. Nevertheless, we have shown that the coherent states minimize the uncertainty relation for

¹A Hilbert–Schmidt operator is a bounded operator A on a Hilbert space \mathbb{H} with a finite Hilbert–Schmidt norm.

both quadratures, and that the uncertainty of the two quadratures is the same. Observing the definition (4.50), one may argue that the complex α -plane plays the role of phase space where, up to scale factors, the real and imaginary parts of α are position and momentum variables, respectively. A coherent state $|\alpha\rangle$, with $\alpha = |\alpha|e^{i\theta}$, may be represented pictorially as shown in fig.(4.2). The shaded circle represents the “area of uncertainty” in the coherent state, the fluctuation being equal in all directions of phase space and the center of the circle is located at a distance $|\alpha| = \bar{n}^{\frac{1}{2}}$ from the origin, at an angle θ . For the vacuum, $|\alpha| = 0$, the phase-space representation is given in Fig (4.2), where it is evident that uncertainty in the phase is as large as possible. A Fock state $|n\rangle$ can also be represented in phase space as a circle of radius n , as depicted in fig.(4.3).

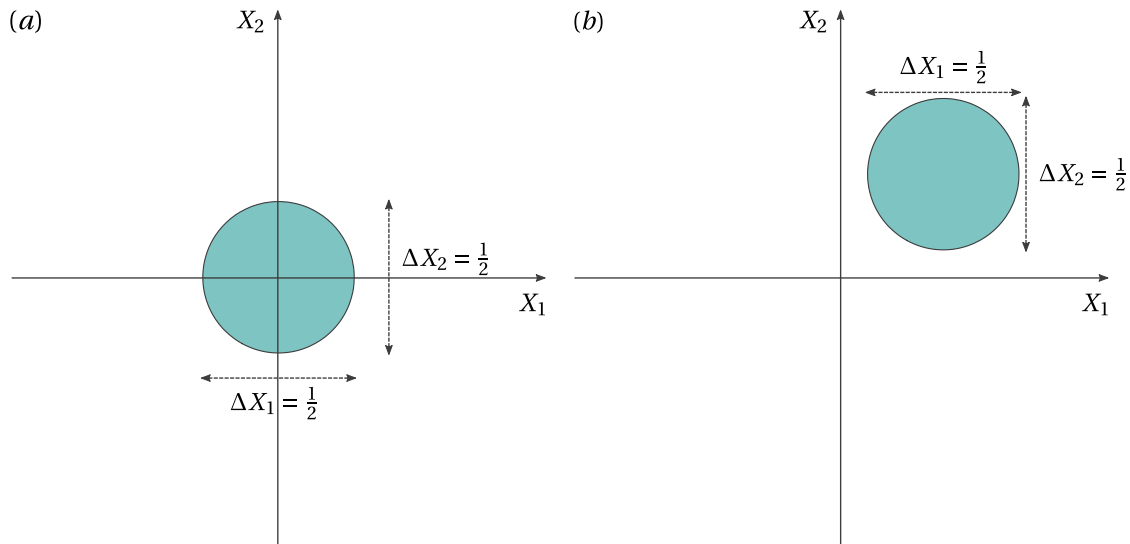


Figure 4.2: Phase-space representation for (a) the vacuum state and (b) a coherent state of amplitude $|\alpha|$.

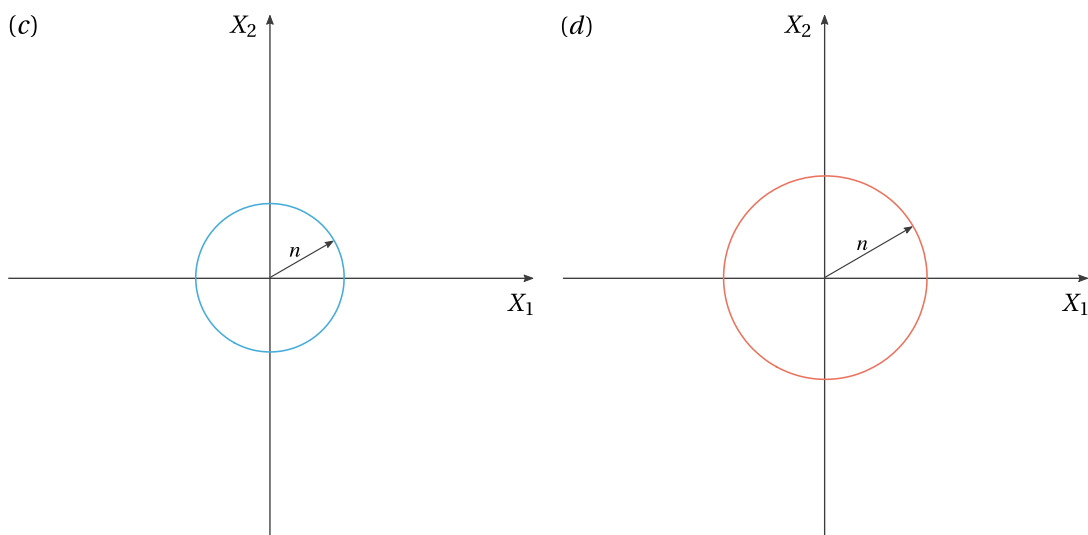


Figure 4.3: Phase-space representation for a Fock state (c) $n = 1$ and (d) $n = 3$. The uncertainty in the photon number is zero while the phase is completely random.

4.5 Squeezed states

As it was seen in the last section, the fluctuations in the quadratures decrease with the temperature (see eq. (4.53)) and reach their minimum value for the vacuum state, (4.54), having the same uncertainty as to the coherent state (4.79). In other words, even for the vacuum state, the operators that describe the state of light have an intrinsic uncertainty. This inherent noise is due to the quantum nature of light. By recognizing that these quantum fluctuations always occur in two complementary observables, physicists have been able to make an interesting trade-off. Reduced fluctuations in one variable can be realized at the expense of increased fluctuations in the other, as determined by the Heisenberg principle.

Early theoretical work in the 60s and 70s led, in the mid-80s, to the conclusion that quantum fluctuations can be reduced below the vacuum-state (shot noise) limit in many forms of nonlinear optical interactions (Slusher et al., 1985; Shelby et al., 1986; Wu et al., 1986). The *squeezed states* are these kinds of states where the quadrature operators have uncertainties below the vacuum state noise. This means that, when we turn on the squeezed light, we see less noise than no light at all! (Scully and Zubairy, 1997b). This seems apparently a paradoxical feature - at least within the classical framework. However, it is another manifestation of the non-classical behaviour of light.

4.5.1 Squeezed states and the uncertainty relation

Let us consider a quantum system described by two Hermitian operators, \hat{A} and \hat{B} , satisfying the commutation relation

$$[\hat{A}, \hat{B}] = i\hat{C}. \quad (4.80)$$

Then, according to the Heisenberg uncertainty relation, the product of its uncertainties satisfies

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \hat{C} \rangle|, \quad (4.81)$$

where the root-mean square deviation ΔA is defined as in eq.(4.45).

Squeezed state. A state of the system is said to be squeezed if the uncertainty in one of the observables (say \hat{A}) satisfies the relation

$$(\Delta A)^2 < \frac{1}{2} |\langle \hat{C} \rangle|. \quad (4.82)$$

Therefore, when the quantum fluctuations in one observable reduce below the shot noise, that is, under the limit ($\frac{1}{2}|\langle \hat{C} \rangle|$), it is said that the observable is squeezed. This can be attained only at the expense of an enhancement of fluctuations in the conjugate observable, such that the uncertainty relation is not violated. If, in

addition to the condition (4.82), the variances satisfy the minimum uncertainty relation,

$$\Delta A \Delta B = \frac{1}{2} |\langle \hat{C} \rangle|, \quad (4.83)$$

then the state is called an ideal squeezed state.

As an illustration, if we consider the quadrature operators \hat{X}_1 and \hat{X}_2 , a squeezed state of the radiation field is obtained when

$$(\Delta X_i)^2 < \frac{1}{4} \quad (i = 1 \text{ or } 2). \quad (4.84)$$

4.5.2 The squeezing operator

Squeezed states of radiation are produced in nonlinear processes where a “classical” electromagnetic field drives a nonlinear medium. In the nonlinear medium, pairs of correlated photons of the same frequency can be generated (Scully and Zubairy, 1997b). In the interaction picture, the corresponding effective Hamiltonian for such nonlinear optical process, called *parametric down-conversion*, may be written as

$$\hat{\mathcal{H}}_I = [\epsilon(\hat{a}^{\dagger 2}) + \epsilon^*(\hat{a})^2], \quad (4.85)$$

where the quantity ϵ includes the amplitude of the driving field as well as the second-order susceptibility for the down-conversion. The operator $\hat{U}(t) = \exp(-i\hat{\mathcal{H}}_I t/\hbar)$ describing the time evolution of the single-mode radiation field under consideration is

$$\hat{U}(t) \equiv \hat{S}(\xi) = \exp\left(\frac{1}{2}\xi^* \hat{a}^2 - \frac{1}{2}\xi \hat{a}^{\dagger 2}\right), \quad (4.86)$$

where $\xi = r e^{i\theta}$ is an arbitrary complex number. This quantity, $\xi = -i\epsilon t/\hbar$, determines the strength of the squeezing and depends on the interaction time (essentially the length of the nonlinear medium), the susceptibility and the amplitude of the driving field (Scully and Zubairy, 1997b). It is immediately to check that

$$\hat{S}^\dagger(\xi) = \hat{S}^{-1}(\xi) = S(\xi). \quad (4.87)$$

Using the *Baker–Campbell–Hausdorff* formula the unitary transformation of the annihilation and creation operators is written,

$$\begin{aligned} \hat{S}(\xi) \hat{a} \hat{S}^\dagger(\xi) &= \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r \\ \hat{S}(\xi) \hat{a}^\dagger \hat{S}^\dagger(\xi) &= \hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r. \end{aligned} \quad (4.88)$$

Therefore, the unitary transformation of the number operator by the squeezing operator is

$$\begin{aligned} \hat{S}(\xi) \hat{a}^\dagger \hat{a} \hat{S}^\dagger(\xi) &= \hat{S}(\xi) \hat{a}^\dagger \hat{S}^\dagger(\xi) \hat{S}(\xi) \hat{a} \hat{S}^\dagger(\xi) = (\hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r)(\hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r) \\ &= \hat{a}^\dagger \hat{a} \cosh 2r + \sinh^2 r - \frac{1}{2} \sinh 2r (\hat{a}^{\dagger 2} e^{i\theta} + \hat{a}^2 e^{-i\theta}). \end{aligned} \quad (4.89)$$

If $\theta = 0$,

$$\hat{S}(\xi) \hat{a}^\dagger \hat{a} \hat{S}^\dagger(\xi) = \hat{a}^\dagger \hat{a} \cosh 2r + \sinh^2 r - \frac{1}{2} \sinh 2r (\hat{a}^{\dagger 2} + \hat{a}^2). \quad (4.90)$$

An important result is the the unitary transformation of the annihilation and creation operators squared:

$$\hat{S}(\xi) \hat{a}^2 \hat{S}^\dagger(\xi) = \hat{a}^2 \cosh^2 r + e^{2i\theta} \sinh^2 r \hat{a}^{\dagger 2} - e^{i\theta} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \sinh 2r, \quad (4.91)$$

$$\hat{S}(\xi) \hat{a}^{\dagger 2} \hat{S}^\dagger(\xi) = \hat{a}^{\dagger 2} \cosh^2 r + e^{-2i\theta} \sinh^2 r \hat{a}^2 - e^{-i\theta} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \sinh 2r. \quad (4.92)$$

The mean value of eqs.(4.91-4.92), for $\theta = 0$, defines what is called by the *coherence parameter*,

$$m(r, \bar{n}_{\text{th}}) = \text{tr}(\hat{\rho}_{\text{sth}} \hat{a}^2) = \text{tr}(\hat{\rho}_{\text{sth}} \hat{a}^{\dagger 2}) = \left(\bar{n}_{\text{th}} + \frac{1}{2} \right) \sinh 2r, \quad (4.93)$$

A squeezed thermal state does not admit a diagonal representation on the Fock basis and, therefore, encompass coherence. As we shall see in chapter 6, the amount of coherence present in a squeezed thermal state with respect to energy eigenbasis, is proportional to the coherence parameter.

A vacuum squeezed state $|\xi\rangle$ is generated acting the squeezing operator $\hat{S}(\xi)$ on the vacuum

$$|\xi\rangle = \hat{S}(\xi) |0\rangle, \quad (4.94)$$

In the case of the vacuum squeezed state, the uncertainties in its quadratures (fig. (4.4)) are given by ($\theta=0$)

$$(\Delta X_1)^2 = \frac{1}{4} e^{-2r}, \quad (\Delta X_2)^2 = \frac{1}{4} e^{2r}, \quad (4.95)$$

assuming squeezing in the \hat{X}_1 quadrature. This result is interesting for two reasons: the first is because the fluctuation in the quadrature \hat{X}_1 is below the vacuum fluctuation for any $r > 0$. This implies the second compelling reason: sometimes, in experimental situations, we want to know only one of the canonically conjugated variables. This means that we can increase our knowledge in the desired observable, with a squeezing operation, by discarding the information about the conjugated one.

The density operator of a squeezed number state is

$$\hat{\rho}_s = \hat{S}(r) |n\rangle \langle n| \hat{S}^\dagger(r), \quad (4.96)$$

and the corresponding density operator for the squeezed thermal state is the Bose-Einstein weighted sum of the squeezed number state,

$$\hat{\rho}_{\text{sth}} = \hat{S}(\xi) \frac{e^{-\beta \hat{\mathcal{H}}}}{Z} \hat{S}^\dagger(\xi), \quad (4.97)$$

The mean number of photons for the thermal squeezed state is

$$\bar{n}_{\text{sth}} = \text{tr}[\hat{\rho}_{\text{sth}} \hat{n}] = \bar{n}_{\text{th}} \cosh 2r + \sinh^2 r, \quad (4.98)$$

which implies in the expedient result:

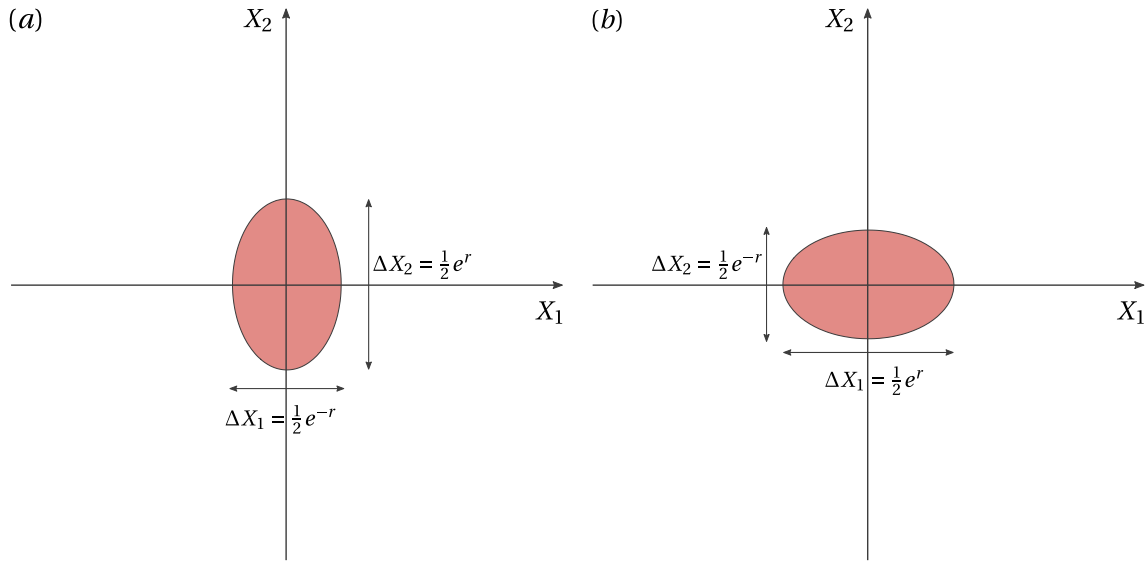


Figure 4.4: Phase-space representation for a squeezed vacuum state with the squeezing in the (a) X_1 and (b) X_2 quadratures.

Squeezed single-bosonic mode energy. The mean energy of a squeezed single-bosonic mode at equilibrium with a reservoir at temperature T and with squeezing parameter r is

$$E = \text{tr}(\hat{\rho}_{\text{sth}} \hat{\mathcal{H}}) = \frac{\hbar\omega}{2} \coth\left(\frac{\hbar\omega}{2k_B T}\right) \cosh 2r. \quad (4.99)$$

For this state, the quadrature uncertainties (see fig.(4.4)) reads

$$\begin{aligned} (\Delta X_1)^2 &= \frac{1}{4} (2\bar{n}_{\text{th}} + 1) e^{-2r}, \\ (\Delta X_2)^2 &= \frac{1}{4} (2\bar{n}_{\text{th}} + 1) e^{2r}. \end{aligned} \quad (4.100)$$

In the case that $\theta \neq 0$, it is defined the rotated quadrature operators (see fig.(4.5))

$$\begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix}. \quad (4.101)$$

We can check that eq.(4.101) also satisfies

$$(\Delta Y_1)^2 = \frac{1}{4} e^{-2r}, \quad (\Delta Y_2)^2 = \frac{1}{4} e^{2r}, \quad (4.102)$$

A more general class of squeezed state is obtained by applying the squeezing operator, followed by the action of the displacement operator, on the vacuum state (see fig.4.6), as follows

$$|\alpha, \xi\rangle = \hat{D}(\alpha) \hat{S}(\xi) |0\rangle, \quad (4.103)$$

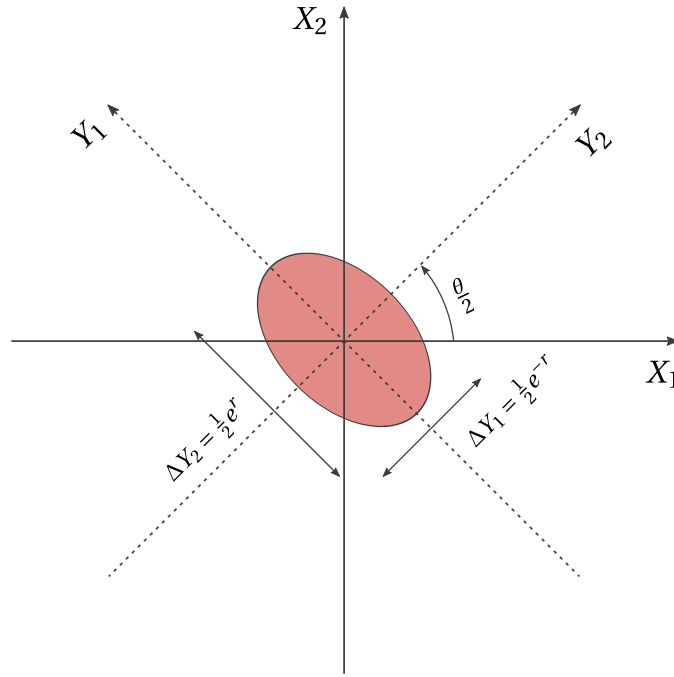


Figure 4.5: Phase-space representation of a rotated squeezed vacuum state. The squeezing is along the $\theta/2$ direction.

this application defines the squeezed coherent state. In this state, the expectation value of the annihilation operator is

$$\langle \hat{a} \rangle = \alpha, \quad (4.104)$$

independent of the squeezing parameter r , and the expectation values for the operators \hat{a}^2 and $\hat{a}^\dagger \hat{a}$ are given by

$$\begin{aligned} \langle \hat{a}^2 \rangle &= \alpha^2 - e^{i\theta} \sinh r \cosh r, \\ \langle \hat{a}^\dagger \hat{a} \rangle &= |\alpha|^2 + \sinh^2 r. \end{aligned} \quad (4.105)$$

As expected, it follows that the fluctuations in each quadrature have the same value as the non-displaced case (4.95).

To finish this section, the squeezed states can be expanded in terms of the Fock states $\{|n\rangle\}$. Consider an electromagnetic mode prepared in the vacuum state, involving under the action of the Hamiltonian (4.85), into the state

$$|\xi\rangle = S(\xi) |0\rangle. \quad (4.106)$$

Starting with

$$\hat{a} |0\rangle = 0, \quad (4.107)$$

using the fact that the squeezing operator is unitary, and acting on the last equation with $\hat{S}(\xi)$ by the right, it is

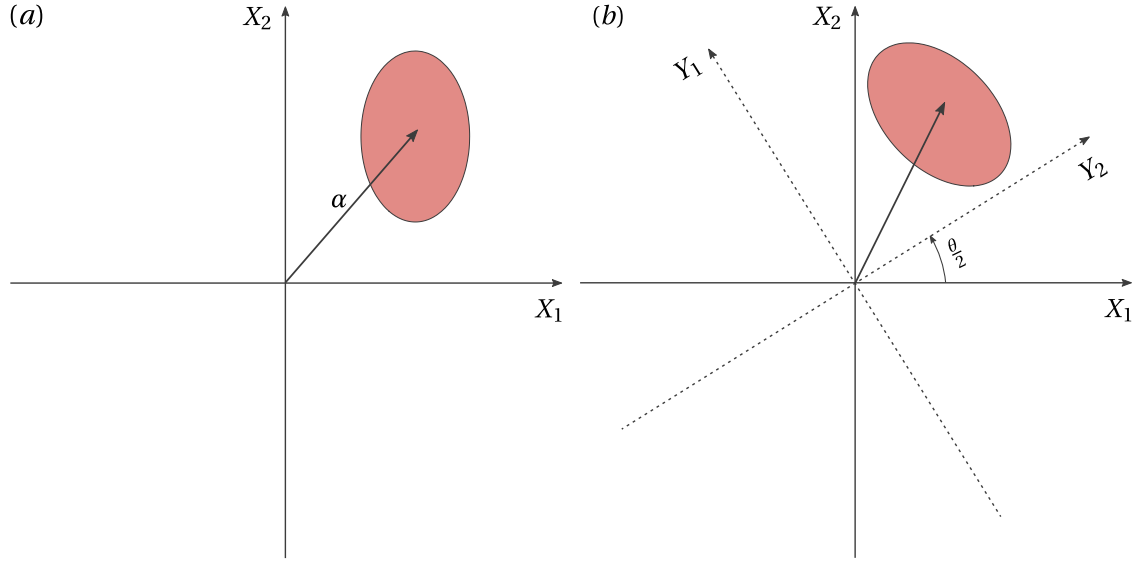


Figure 4.6: Phase-space representation of a displaced (a) squeezed vacuum state with the squeezing in the X_1 quadrature and (b) a rotated squeezed vacuum state.

obtained

$$S\hat{a}S^\dagger|0\rangle = (\hat{a}\nu - \hat{a}^\dagger\mu)|\xi\rangle = 0, \quad (4.108)$$

where we defined $\nu \equiv \cosh r$ and $\mu \equiv \sinh r$. The state $|\xi\rangle$ is an eigenstate of the operator $(\hat{a}\nu - \hat{a}^\dagger\mu)$ with eigenvalue equal to zero. Expanding $|\xi\rangle$ in terms of the Fock states, eq.(4.108) can be recast as,

$$\begin{aligned} (\hat{a}\nu - \hat{a}^\dagger\mu)|\xi\rangle &= (\hat{a}\nu - \hat{a}^\dagger\mu) \sum_{n=0}^{\infty} C_n |n\rangle \\ &= \sum_{n=0}^{\infty} \left(\nu C_n \sqrt{n} |n-1\rangle + \mu C_n \sqrt{n+1} |n+1\rangle \right) = 0. \end{aligned} \quad (4.109)$$

Projecting on $|m\rangle$ and using the orthogonality of the Fock states,

$$\nu \sum_{n=0}^{\infty} C_n \sqrt{n} \delta_{m,n-1} = \mu \sum_{n=0}^{\infty} C_n \sqrt{n+1} \delta_{m,n+1}, \quad (4.110)$$

or

$$C_{m+1} = -\frac{\nu}{\mu} \left(\frac{m}{m+1} \right)^{1/2} C_{m-1}. \quad (4.111)$$

It can be seen that there are two distinct solutions, one involving only even photon states and a second involving only odd photon states. The first solution is interesting since it contains the vacuum. Notice that the ratio

$\mu/\nu = e^{i\theta} \tanh r$, thus, the solution of the recurrence relation is

$$\begin{aligned}
C_2 &= -e^{i\theta} \tanh r \left(\frac{1}{2}\right)^{1/2} C_0, \\
C_4 &= e^{i\theta} \tanh r \left(\frac{3}{4}\right)^{1/2} C_2 = (e^{i\theta} \tanh r)^2 \left(\frac{1}{2} \times \frac{3}{4}\right)^{1/2} C_0, \\
C_6 &= -e^{i\theta} \tanh r \left(\frac{5}{6}\right)^{1/2} C_4 = (e^{i\theta} \tanh r)^3 \left(\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6}\right)^{1/2} C_0, \\
&\vdots \\
C_{2m} &= (-1)^m (e^{i\theta} \tanh r)^m \left[\frac{(2m-1)!!}{(2m)!!} \right]^{1/2} C_0.
\end{aligned} \tag{4.112}$$

where C_0 is determined by the normalization condition, $\langle \xi | \xi \rangle = 1$:

$$\begin{aligned}
\langle \xi | \xi \rangle &= |C_0|^2 \sum_{n,m=0}^{\infty} (\tanh r)^{n+m} \left[\frac{(2m-1)!!}{(2m)!!} \times \frac{(2n-1)!!}{(2n)!!} \right]^{1/2} \langle n | m \rangle \\
&= |C_0|^2 \sum_{n=0}^{\infty} (\tanh r)^{2n} \left[\frac{(2n-1)!!}{(2n)!!} \right] = |C_0|^2 (1 - \tanh^2 r)^{-1/2} \\
&= |C_0|^2 \cosh r,
\end{aligned} \tag{4.113}$$

thus

$$C_0 = \frac{1}{\sqrt{\cosh r}}. \tag{4.114}$$

In the last step, the mathematical identity was used

$$1 + \sum_{m=1}^{\infty} x^m \left(\frac{(2m-1)!!}{(2m)!!} \right) = (1-x)^{-1/2}. \tag{4.115}$$

Using the following identities

$$\begin{aligned}
(2m)!! &= 2^m m!, \\
(2m-1)!! &= \frac{1}{2^m m!} \frac{(2m)!}{m!},
\end{aligned} \tag{4.116}$$

the expansion coefficients for the squeezed vacuum state are given by

$$C_{2m} = (-1)^m \frac{\sqrt{(2m)!}}{2^m m!} \frac{(e^{i\theta} \tanh r)^m}{\sqrt{\cosh r}}. \tag{4.117}$$

Therefore, the squeezed vacuum state becomes

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{m=0}^{\infty} (-1)^m \frac{\sqrt{(2m)!}}{2^m m!} (e^{i\theta} \tanh r)^m |2m\rangle. \tag{4.118}$$

So the probability of detecting $2m$ photons in the field is oscillatory

$$P_{2m} = |\langle 2m | \xi \rangle|^2 = \frac{1}{\cosh r} \frac{(2m)!}{2^{2m} (m!)^2} (\tanh r)^{2m}, \tag{4.119}$$

while for detecting $2m+1$, an odd number of photons, is zero.

This section started with the idea that squeezed states are a manifestation of the non-classical behaviour of light. Then, it was showed that squeezing is a quantum effect rooted in Heisenberg's uncertainty principle. But, what is a non-classical effect, and how could we distinguish it from a classical one? In the next section, we will answer this question based on the concept of P-representability.

4.6 The Glauber-Sudarshan P representation

According to section 4.4, the coherent states not only form a complete set of states but an overcomplete one. Even though they are not orthogonal, they may be used as a basis set. The following representation in terms of coherent states, was introduced independently by Glauber (1963) and Sudarshan (1963b) back in the '70s. As we shall see, the P-representation establishes a correspondence between the quantum and the classical theories.

P-representation. A density operator $\hat{\rho}$ can be expanded as a diagonal sum over the coherent states

$$\hat{\rho} = \int d^2\alpha P(\alpha, \alpha^*) |\alpha\rangle \langle\alpha|, \quad (4.120)$$

where $d^2\alpha = d(\text{Re}\{\alpha\})d(\text{Im}\{\alpha\})$, and $P(\alpha, \alpha^*)$ is a quasi-probability distribution called the P -function.

Let us investigate why this representation is appealing, and see why $P(\alpha, \alpha^*)$ plays a role somewhat analogous to a classical probability distribution. First, note that

$$\int d^2\alpha P(\alpha, \alpha^*) = \int d^2\alpha \langle\alpha|\alpha\rangle P(\alpha, \alpha^*) = \text{tr} \left(\int d^2\alpha |\alpha\rangle \langle\alpha| P(\alpha, \alpha^*) \right) = \text{tr}(\hat{\rho}) = 1. \quad (4.121)$$

Thus, $P(\alpha, \alpha^*)$ is normalized like a classical probability distribution. Note also that for the expectation values of the creation \hat{a}^\dagger and annihilation \hat{a} operators, we have that

$$\begin{aligned} \langle \hat{a}^{\dagger q} \hat{a}^p \rangle &= \text{tr}(\hat{\rho} \hat{a}^{\dagger q} \hat{a}^p) = \text{tr} \left(\int d^2\alpha |\alpha\rangle \langle\alpha| P(\alpha, \alpha^*) \hat{a}^{\dagger q} \hat{a}^p \right) \\ &= \int d^2\alpha P(\alpha, \alpha^*) \langle\alpha| \hat{a}^{\dagger q} \hat{a}^p |\alpha\rangle = \int d^2\alpha P(\alpha, \alpha^*) \alpha^{*p} \alpha^q, \end{aligned} \quad (4.122)$$

these averages are calculated in the same manner as in classical statistics. However, the identification of $P(\alpha, \alpha^*)$ as a probability distribution must be made with reservation. Unlike a classical probability, $P(\alpha, \alpha^*)$ can assume negative values and can be more singular than the delta function². Thus, $P(\alpha, \alpha^*)$ is not a genuine probability distribution, and for this reason, it is often referred to as a quasi-distribution function.

P representability. States that can be written as in (4.120) with a proper P -function, $P(\alpha, \alpha^*) \geq 0$ and that are “not more singular” than a delta function, including coherent states, are said to be classical states (or P-representable). Unlike, a non-classical state is the one where $P(\alpha, \alpha^*)$ is negative or highly singular.

According to the definition of P-representability expressed above, a remarkable statement can be derived:

²The *structure theorem* for distributions states that locally, every distribution can be written as the derivative of a certain order of a continuous function. The more derivatives are needed to take more singular such a distribution will be.

i. Non-classicality *Squeezed states are non-classical states*

Proof: Expressing the expectation values of each quadrature in terms of the P-function. First, note that

$$\begin{aligned}\langle \hat{X}_1^2 \rangle &= \text{tr}(\hat{\rho} \hat{X}_1^2) = \frac{1}{4} \text{tr} \left(\int d^2 \alpha P(\alpha) |\alpha\rangle \langle \alpha| (\hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^{\dagger} \hat{a} + 1) \right) \\ &= \frac{1}{4} \left(1 + \int d^2 \alpha P(\alpha) (\alpha^* + \alpha)^2 \right),\end{aligned}\quad (4.123)$$

and that,

$$\langle \hat{X}_1 \rangle^2 = \frac{1}{4} (\langle \hat{a}^{\dagger} \rangle + \langle \alpha \rangle)^2 = \frac{1}{4} \int d^2 \alpha P(\alpha) [\alpha \langle \hat{a} \rangle + \alpha^* \langle \hat{a}^{\dagger} \rangle + \alpha \langle \hat{a} \rangle + \alpha^* \langle \hat{a} \rangle] \quad (4.124)$$

$$= \frac{1}{4} \int d^2 \alpha P(\alpha) [(\alpha + \alpha^*) (\langle \hat{a} \rangle + \langle \hat{a}^{\dagger} \rangle)]. \quad (4.125)$$

where $\langle \hat{a} \rangle = \int d^2 \alpha P(\alpha) \alpha$ and $\langle \hat{a}^{\dagger} \rangle = \int d^2 \alpha P(\alpha) \alpha^*$.

Thus,

$$(\Delta X_1)^2 = \frac{1}{4} \left\{ 1 + \int d^2 \alpha P(\alpha) [(\alpha + \alpha^*) - (\langle \hat{a}^{\dagger} \rangle + \langle \hat{a} \rangle)]^2 \right\}. \quad (4.126)$$

Equivalently,

$$(\Delta X_2)^2 = \frac{1}{4} \left\{ 1 + \int d^2 \alpha P(\alpha) [(\alpha - \alpha^*)/i - (\langle \hat{a}^{\dagger} \rangle - \langle \hat{a} \rangle)/i]^2 \right\}. \quad (4.127)$$

Since the term inside the square bracket is always positive, it is evident that the condition $(\Delta X_i)^2 < 1/4$ requires that $P(\alpha)$ must be non-positive, at least in some regions of the phase space.

Still remains, however, the question of how to calculate $P(\alpha, \alpha^*)$ explicitly. For this purpose, it is followed a general procedure introduced by Mehta (1967). Starting with eq.(4.120) and assuming that $|\beta\rangle$ and $|\beta\rangle$ are coherent states, with β and $-\beta$ being the eigenvalues of \hat{a} ,

$$\langle -\beta | \hat{\rho} | \beta \rangle = \int d^2 \alpha P(\alpha, \alpha^*) \langle -\beta | \alpha \rangle \langle \alpha | \beta \rangle, \quad (4.128)$$

using eq.(4.62),

$$\langle -\beta | \hat{\rho} | \beta \rangle = e^{-|\beta|^2} \int d^2 \alpha [P(\alpha, \alpha^*) e^{-|\alpha|^2}] e^{\beta \alpha^* - \beta^* \alpha}. \quad (4.129)$$

Now let $\alpha = x_\alpha + i y_\alpha$ and $\beta = x_\beta + i y_\beta$, then $d^2 \alpha = dx_\alpha dy_\alpha$, and $\beta \alpha^* - \beta^* \alpha = 2i(y_\beta x_\alpha - x_\beta y_\alpha)$. Thus, eq.(4.129) becomes

$$\langle -\beta | \hat{\rho} | \beta \rangle = e^{-|\beta|^2} \iint dx_\alpha dy_\alpha [P(x_\alpha, y_\alpha) e^{-(x_\alpha^2 + y_\alpha^2)}] e^{2i(y_\beta x_\alpha - x_\beta y_\alpha)}. \quad (4.130)$$

The Fourier transform in the complex plane is written:

$$\begin{aligned}g(\beta) &= \int d^2 \alpha f(\alpha) e^{\beta \alpha^* - \beta^* \alpha} \\ f(\alpha) &= \frac{1}{\pi^2} \int d^2 \beta g(\beta) e^{\beta^* \alpha - \alpha \beta^*}.\end{aligned}\quad (4.131)$$

With the identifications

$$\begin{aligned} g(\beta) &= e^{|\beta|^2} \langle -\beta | \hat{\rho} | \beta \rangle \\ f(\alpha) &= P(\alpha) e^{-|\beta|^2}, \end{aligned} \quad (4.132)$$

it is obtained,

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int d^2\beta e^{|\beta|^2} \langle -\beta | \hat{\rho} | \beta \rangle e^{\beta\alpha^* - \beta^*\alpha}. \quad (4.133)$$

P-function for the coherent and thermal states

As a first example, let us calculate the P-function $P(\alpha, \alpha^*)$ for the coherent state $\hat{\rho}_{\alpha_0} = |\alpha_0\rangle \langle \alpha_0|$:

$$\langle -\beta | \hat{\rho} | \beta \rangle = \langle -\beta | \alpha_0 \rangle \langle \alpha_0 | \beta \rangle = e^{-|\alpha_0|^2 - |\beta|^2 - \alpha_0\beta^* + \beta\alpha_0^*}. \quad (4.134)$$

It then follows from eq.(4.133)

$$P(\alpha, \alpha^*) = \frac{1}{\pi^2} e^{|\alpha| - |\alpha_0|} \int d^2\beta e^{-\beta(\alpha^* - \alpha_0^*) + \beta^*(\alpha - \alpha_0)} = \delta^{(2)}(\alpha - \alpha_0), \quad (4.135)$$

the P-representation of a coherent state is a two-dimensional delta function. Another way to see this result without following the last method is to decompose the coherent state, as in the eq.(4.120), and notice that $P(\alpha, \alpha^*)$ must to be the delta function due to the filtering property (appendix A.2).

Now for a thermal state eq.(4.26), we have that,

$$\begin{aligned} \langle -\beta | \hat{\rho} | \beta \rangle &= \sum_n \frac{\bar{n}_{\text{th}}^n}{(1 + \bar{n}_{\text{th}})^{(n+1)}} \langle -\beta | n \rangle \langle n | \beta \rangle = \frac{e^{-|\beta|^2}}{1 + \bar{n}_{\text{th}}} \sum_{n=0}^{\infty} \frac{(-|\beta|^2)^n}{n!} \left(\frac{\bar{n}_{\text{th}}}{1 + \bar{n}_{\text{th}}} \right)^n \\ &= \frac{e^{-|\beta|^2}}{1 + \bar{n}_{\text{th}}} \exp \left[\frac{-|\beta|^2}{\left(1 + \frac{1}{\bar{n}_{\text{th}}}\right)} \right], \end{aligned} \quad (4.136)$$

thus

$$P(\alpha, \alpha^*) = \frac{e^{-|\beta|^2}}{\pi^2(1 + \bar{n}_{\text{th}})} \int d^2\beta e^{-|\beta|^2/(1 + \frac{1}{\bar{n}_{\text{th}}})} e^{-\beta\alpha^* + \alpha\beta^*} \quad (4.137)$$

$$= \frac{1}{\pi \bar{n}_{\text{th}}} e^{-|\alpha|^2/\bar{n}_{\text{th}}}. \quad (4.138)$$

Notice that the P-function of the thermal distribution is given by a Gaussian distribution.

If a coherent state is a classical state, the number state $|n\rangle$ has to be the opposite. Thus, for a pure number state, $\hat{\rho}_n = |n\rangle \langle n|$,

$$\langle -\beta | n \rangle \langle n | \beta \rangle = e^{|\beta|^2} \frac{(-\beta^*\beta)^n}{n!}. \quad (4.139)$$

Thus

$$P(\alpha) = \frac{e^{|\alpha|^2}}{n!} \frac{1}{\pi^2} \int d^2\beta (\beta^*\beta)^n e^{\beta^*\alpha - \beta\alpha^*}. \quad (4.140)$$

The integral does not exist in terms of ordinary functions. Formally, it can be written

$$P(\alpha) = \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \frac{1}{\pi^2} \int d^2 \beta e^{\beta^* \alpha - \beta \alpha^*} \quad (4.141)$$

$$= \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \delta^{(2)}(\alpha). \quad (4.142)$$

The derivative of the delta function, called a tempered distribution, is more singular than a delta function, and has meaning only under the integral sign, i.e., for some function $F(\alpha, \alpha^*)$

$$\int d^2 \alpha F(\alpha, \alpha^*) \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \delta^{(2)}(\alpha) = \left[\frac{\partial^{2n} F(\alpha, \alpha^*)}{\partial \alpha^n \partial \alpha^{*n}} \right]_{\alpha=0, \alpha^*=0}. \quad (4.143)$$

From these examples, it can be easily seen that it is not a simple task to evaluate the P-function. However, if we are interested in discriminating whether a given state is classical or non-classical, we must do it. Luckily, there are methods where we could quickly check it without caring about the details of the P-function. In the following section, we shall establish the grounds to learn how we could infer, which states we are dealing with without evaluating its probability distribution function.

4.7 Characteristic functions

In probability theory and statistics, the characteristic function of a random variable completely defines its probability distribution. If a random variable admits a probability density function, then the characteristic function is the Fourier transform of the probability density function (Gerry et al., 2005). It provides an alternative description of analytical results without having to deal directly with the probability density. The description of states through their characteristic function turns out to be instrumental to several achievements. Motivated by this, we now proceed to introduce the quantum mechanical version of the characteristic function.

We shall start the discussion by introducing the three characteristic functions, and showing some mathematical properties of it without discussing the physical meaning behind which of them. After we established the precise meaning, we will motivate the definition of these function from a physical problem, and then focus on the normally-ordered characteristic function. Let us first define the characteristic function associated with a density operator $\hat{\rho}$:

Characteristic function. For a system $\hat{\rho}$, it can be introduced a real parameter $s \in [-1, 1]$, so the corresponding s-parametrized characteristic function $\chi_s(z, z^*)$ is

$$\chi_s(z, z^*) \equiv \text{tr}[\hat{\rho} \hat{D}(z)] e^{\frac{s}{2}|z|^2}. \quad (4.144)$$

where $\hat{D}(z)$ is the displacement operator (4.74).

Recalling that $\hat{D}(z) = e^{z\hat{a}^\dagger} e^{z^* \hat{a}} e^{-\frac{1}{2}|z|^2} = e^{-z^* \hat{a}} e^{z\hat{a}^\dagger} e^{\frac{1}{2}|z|^2}$, and inserting it in the definition of the s-parametrized characteristic function, the three characteristic functions can be obtained

$$\chi_{-1}(z, z^*) \equiv \chi_A(z, z^*) = \text{tr}(\hat{\rho} e^{-z^* \hat{a}} e^{z\hat{a}^\dagger}) \quad (\text{anti-normally ordered}). \quad (4.145)$$

$$\chi_0(z, z^*) \equiv \chi_W(z, z^*) = \text{tr}(\hat{\rho} e^{z\hat{a}^\dagger - z^* \hat{a}}) \quad (\text{Wigner}). \quad (4.146)$$

$$\chi_1(z, z^*) \equiv \chi_N(z, z^*) = \text{tr}(\hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) \quad (\text{normally ordered}). \quad (4.147)$$

Also, notice that

$$\chi_W(z, z^*) = \chi_N(z, z^*) e^{-\frac{1}{2}|z|^2} = \chi_A(z, z^*) e^{\frac{1}{2}|z|^2}. \quad (4.148)$$

Let us consider for the moment, the normal-ordered function ($s = 1$)

$$\chi_N(z, z^*) \equiv \text{tr}(\hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}). \quad (4.149)$$

This equation represents a two-dimensional Fourier transform of the P function. Writing $\hat{\rho}$ in terms of the P-representation (4.120) it is obtained

$$\begin{aligned} \chi_N(z, z^*) &= \int d^2\alpha P(\alpha) \langle \alpha | e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} | \alpha \rangle = \int d^2\alpha P(\alpha, \alpha^*) e^{z\alpha^* - z^* \alpha}, \\ &= \int_{-\infty}^{-\infty} dx \int_{-\infty}^{-\infty} dy P(x + iy, x - iy) e^{2i(vx - \mu y)}, \end{aligned} \quad (4.150)$$

with the inverse relation

$$\begin{aligned} P(\alpha, \alpha^*) &\equiv \frac{1}{\pi^2} \int d^2z \chi_N(z, z^*) e^{z^* \alpha - z \alpha^*} \\ &\equiv \frac{1}{\pi^2} \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\nu \chi_N(\mu + i\nu, \mu - i\nu) e^{-2i(\nu x - \mu y)}. \end{aligned} \quad (4.151)$$

Also, the definition of the characteristic function allows determining the expectation values of all normal-ordered operators via

$$\langle \hat{a}^{\dagger p} \hat{a}^q \rangle \equiv \text{tr}(\hat{\rho} \hat{a}^{\dagger p} \hat{a}^q) = \frac{\partial^{p+q}}{\partial(z)^p \partial(-z^*)^q} \chi_N(z, z^*) |_{z=z^*=0}. \quad (4.152)$$

Similarly, for the anti-normally ordered characteristic function, it can be written

$$\chi_A(z, z^*) = \text{tr}(\hat{\rho} e^{-z^* \hat{a}} e^{z\hat{a}^\dagger}) = \frac{1}{\pi} \int d^2\alpha \langle \alpha | e^{z\hat{a}^\dagger} \hat{\rho} e^{-z^* \hat{a}} | \alpha \rangle \quad (4.153)$$

$$= \int d^2\alpha Q(\alpha) e^{z\alpha^* - z^* \alpha}, \quad (4.154)$$

which is the two-dimensional Fourier transform of the so-called *Glauber Q-Function*,

$$Q = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle. \quad (4.155)$$

The inverse Fourier transform of the anti-normal characteristic function yields,

$$Q(\alpha) = \frac{1}{\pi^2} \int d^2z \chi_A(z, z^*) e^{z^* \alpha - z \alpha^*}. \quad (4.156)$$

The Q-function is, therefore, always positive and does not diverge. Because of these properties, it found ample application in the study of quantum dynamical systems (Sudarshan, 1963a; Mehta and Sudarshan, 1965). For example, the value of the Q-function at a point α represents the probability that the heterodyne measurement of the system yields the outcome α ³.

From the definition of the anti-normally ordered characteristic function, it can be obtained

$$\langle \hat{a}^p \hat{a}^{\dagger q} \rangle \equiv \text{tr}(\hat{\rho} \hat{a}^p \hat{a}^{\dagger q}) = \frac{\partial^{p+q}}{\partial(z)^p \partial(-z^*)^q} \chi_A(z, z^*)|_{z=z^*=0}. \quad (4.157)$$

Finally, it turns out that the Wigner function may be obtained as the Fourier transform of the Wigner characteristic function

$$W(\alpha) = \frac{1}{\pi^2} \int d^2 z e^{z^* \alpha - z \alpha^*} \chi_W(z, z^*) \quad (4.158)$$

$$= \frac{1}{\pi^2} \int d^2 z e^{z^* \alpha - z \alpha^*} \chi_N(z, z^*) e^{-|z|^2/2}. \quad (4.159)$$

As for the symmetrically ordered, the Wigner function, it was historically the first quasi-probability to be introduced, by Wigner (1932). The idea behind the Wigner function is that its integral over a certain phase space quadrature gives the probability of measuring its respective conjugate quadrature.

In general, there are many ways to represent continuous variables in phase space (Gerry et al., 2005). The three most important are the Glauber-Sudarshan P-function (4.156), and the Wigner function (4.158). Each has its advantages and disadvantages.

4.8 Quantum continuous variables

During this chapter, we have presented the main properties of the states of light and its features. Specifically, we studied in details a bosonic mode, a direct consequence of the quantization of the electromagnetic field. States of light are very often described/modelled by a *quantum continuous variable system*, a quantum system whose degrees of freedom are associated with operators with a continuous spectrum. They naturally obey a commutation relation and require the adoption of an infinite-dimensional Hilbert space, even when a finite number of degrees of freedom is considered.

In our case, \hat{x} and \hat{p} are the quantum counterpart of the magnetic and electric fields along one polarization direction. Besides that, it was seen that such a system can be modelled as a collection of non-interacting quantum harmonic oscillators with different frequencies. Now, we shall address the Gaussian parametrization. A Gaussian state is a ground or thermal state of a second-order Hamiltonian, which is quadratic in the creation and annihilation operators. Those states are completely determined by their first moments and covariance matrix (Serafini, 2017). The reason for this parametrization is twofold: dealing with a continuous variable system

³heterodyne measurements give complex outcomes.

in the phase-space could be easier than in the Hilbert space. Also, the criterion of P-representability (4.120) can be easily obtained by the elements of a quantity called the *covariance matrix* that can be easily calculated.

4.8.1 Parametrization of a Gaussian state

Consider a quantum system with a finite set of degrees of freedom represented by pairs of self-adjoint canonical operators \hat{q}_j and \hat{p}_j (for $j = 1, \dots, n$, and setting $\hbar = 1^4$), with canonical commutation relation

$$[\hat{q}_j, \hat{p}_k] = i\delta_{j,k}. \quad (4.160)$$

It is convenient to introduce a real and anti-symmetric form Ω , given by the direct sum of identical 2×2 block matrices:

$$\Omega = \bigoplus_{j=1}^n \Omega_1 \quad \text{with} \quad \Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.161)$$

Eq.(4.161) is known as the symplectic form and satisfies that $\Omega = \Omega^\top$ and $-\Omega^2 = \mathbb{1}$. In symplectic form, the canonical commutation relation is given by

Canonical commutation relation. By defining the vector of canonical operators $\hat{\mathbf{r}} = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)^\top$, where \top indicates the transposition operation, eq. (4.160) can be recast as the following expression

$$[\hat{\mathbf{r}}, \hat{\mathbf{r}}^\top] = i\Omega, \quad (4.162)$$

where the commutator (4.162) should be taken as an outer product (see appendix A.1).

It is also convenient to define a multi-modal vector composed of all the different bosonic mode operators, $\hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_n, \hat{a}_n^\dagger)^\top$, such that

$$\hat{a}_j = \frac{\hat{q}_j + i\hat{p}_j}{\sqrt{2}} \quad \text{and} \quad \hat{a}_j^\dagger = \frac{\hat{q}_j - i\hat{p}_j}{\sqrt{2}}, \quad (4.163)$$

where $\hat{\mathbf{r}}$ is related to $\hat{\mathbf{a}}$ by a unitary transformation \bar{U}_n , defined as

$$\bar{U} = \bigoplus_{j=1}^n \bar{u}_n \quad \text{with} \quad \bar{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad (4.164)$$

so,

$$\hat{\mathbf{r}} = \bar{U}\hat{\mathbf{a}} \quad , \quad \hat{\mathbf{a}} = \bar{U}^\dagger\hat{\mathbf{r}}. \quad (4.165)$$

In terms of \bar{U} , the canonical commutation relation for $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$ is

$$[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger] = [\bar{U}\hat{\mathbf{r}}, \hat{\mathbf{r}}^\dagger\bar{U}^\dagger] = \bar{U}[\hat{\mathbf{r}}, \hat{\mathbf{r}}^\dagger]\bar{U}^\dagger = i\bar{U}\Omega\bar{U}^\dagger = \Sigma = \bigoplus_{j=1}^n \sigma_z, \quad (4.166)$$

⁴We set $\hbar = 1$ only in this section.

where σ_z is the z Pauli matrix. To summarize, in terms of the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{r}}$ the algebra becomes

$$[\hat{a}_i, \hat{a}_j^\dagger] = \sum_{i,j} \Sigma_{i,j} \quad , \quad [\hat{r}_i, \hat{r}_j^\dagger] = i\Omega_{i,j}. \quad (4.167)$$

Evaluating the expectation value of the vector $\hat{\mathbf{r}}$ for a Gaussian state, it is obtained the first moments, or displacement vector,

$$\bar{\mathbf{r}} = \text{tr}[\hat{\rho}\hat{\mathbf{r}}]. \quad (4.168)$$

The first moment, $\bar{\mathbf{r}}$, can be determined by performing measurements of positions and momenta on systems which are described by the bosonic operators. With the connection between density matrices on $L^2(\mathbb{R}^n)$ and functions of $2n$ variables, the covariance matrix θ , and the vector of first moments $\hat{\mathbf{r}}$ are directly accessible in practice as the first and second statistical moments of measurements of the canonical operators. The covariance matrix contains all the information about the system. Our analysis can be summarised by the following characterization of a Gaussian state $\hat{\rho}$:

Parametrisation of a generic Gaussian state. A Gaussian state $\hat{\rho}$ is completely determined by its vector of first moments $\hat{\mathbf{r}}$ and the covariance matrix θ , defined as

$$\hat{\mathbf{r}} = \text{tr}[\hat{\rho}\hat{\mathbf{r}}] \quad \text{and} \quad \theta = \text{tr}[\hat{\rho}\{(\hat{\mathbf{r}} - \bar{\mathbf{r}}), (\hat{\mathbf{r}} - \bar{\mathbf{r}})^\top\}]. \quad (4.169)$$

or by \mathbf{V} , where \mathbf{V} is the covariance matrix written in terms of the bosonic operators $\theta = \bar{U}\mathbf{V}\bar{U}^\dagger$.

The components of the covariance matrix are given by

$$\theta_{i,j} = \frac{1}{2} \langle \{\hat{\mathbf{r}}_i, \hat{\mathbf{r}}_j\} \rangle - \bar{\mathbf{r}}_i \bar{\mathbf{r}}_j = \frac{1}{2} \langle \{\delta\hat{\mathbf{r}}_i, \delta\hat{\mathbf{r}}_j^\dagger\} \rangle, \quad (4.170)$$

$$\mathbf{V} = \frac{1}{2} \langle \{\mathbf{a}_i, \mathbf{a}_j\} \rangle - \bar{\mathbf{a}}_i \bar{\mathbf{a}}_j = \frac{1}{2} \langle \{\delta\hat{\mathbf{a}}_i, \delta\hat{\mathbf{a}}_j^\dagger\} \rangle, \quad (4.171)$$

where $\delta\hat{X}_i = \hat{X}_i - \langle \hat{X}_i \rangle$. For the single-mode case, $n = 1$, they are written

$$\mathbf{V} = \begin{pmatrix} \langle \delta\hat{a}^\dagger \delta\hat{a} \rangle + \frac{1}{2} & \langle \delta\hat{a} \delta\hat{a} \rangle \\ \langle \delta\hat{a}^\dagger \delta\hat{a} \rangle & \langle \delta\hat{a}^\dagger \delta\hat{a} \rangle + \frac{1}{2} \end{pmatrix}, \quad \theta = \begin{pmatrix} \langle \delta\hat{q}^2 \rangle & \frac{1}{2} \langle \{\delta\hat{q} \delta\hat{p}\} \rangle \\ \frac{1}{2} \langle \{\delta\hat{q} \delta\hat{p}\} \rangle & \langle \delta\hat{p}^2 \rangle \end{pmatrix} \quad (4.172)$$

where the curly brackets denote the anti-commutator of the operators.

As an example, the vacuum state $|0\rangle$ is a Gaussian state with zero displacement and covariance matrix

$$\mathbf{V}_{\text{va}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.173)$$

Similarly, the thermal state $\hat{\rho}_{\text{th}}$ is also a Gaussian state with zero first moments and covariance matrix

$$\mathbf{V}_{\text{th}} = \begin{pmatrix} \bar{n}_{\text{th}} + \frac{1}{2} & 0 \\ 0 & \bar{n}_{\text{th}} + \frac{1}{2} \end{pmatrix}. \quad (4.174)$$

Because θ and \mathbf{V} are linked by a unitary transformation (4.165), it can be chosen either the \mathbf{V} or θ representation. In the present case, \mathbf{V} is chosen, due to practical convenience. Therefore, the covariance matrix \mathbf{V} of $\hat{\rho}$ is given by (4.175),

$$\mathbf{V} = \begin{pmatrix} \langle \delta \hat{a}^\dagger \delta \hat{a} \rangle + \frac{1}{2} & \langle \delta \hat{a} \delta \hat{a} \rangle \\ \langle \delta \hat{a}^\dagger \delta \hat{a}^\dagger \rangle & \langle \delta \hat{a}^\dagger \delta \hat{a} \rangle + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \bar{n} + \frac{1}{2} & m \\ m^* & \bar{n} + \frac{1}{2} \end{pmatrix}, \quad (4.175)$$

where $\bar{n} = (\langle \hat{a}^\dagger a \rangle - \langle \hat{a}^\dagger \rangle \langle a \rangle)$, and $m = (\langle a^2 \rangle - \langle a \rangle^2)$. The non-diagonal terms in \mathbf{V} correspond to the non-diagonal terms in the energy eigenbasis (Fock basis) that is coherence, and m is the coherence parameter (4.93).

Finally, it can be established a direct connection between the characteristic function and the covariance matrix:

Characteristic function of a Gaussian state. The characteristic function χ_W of a Gaussian state in terms of its covariance matrix \mathbf{V} is given by:

$$\chi_W(\mathbf{z}) = e^{-\frac{1}{2} \mathbf{z}^\dagger \mathbf{V} \mathbf{z}} \quad (4.176)$$

where $\mathbf{z} = (z \ z^*)^\top$, and z is the complex eigenvalue of the annihilation operator \hat{z} , whose its eigenstate is the coherent state $|z\rangle$.

4.8.2 Generalized uncertainty relations

The Gaussian state parametrization requires that the covariance matrix satisfy some properties to describe a physical state (Englert and Wódkiewicz, 2003). In order to derive these properties, consider the following operator

$$\hat{A} = \sum_i \delta \hat{a}_i \alpha_i, \quad (4.177)$$

where α_i are arbitrary complex numbers. Since AA^\dagger is positive semi-definite, it then follows that,

$$\langle \hat{A} \hat{A}^\dagger \rangle = \sum_{i,j} \alpha_i \alpha_j^* \langle \delta \hat{a}_i \delta \hat{a}_j^\dagger \rangle \geq 0. \quad (4.178)$$

The anti-commutator $\{\hat{a}_i, \hat{a}_j^\dagger\}$ can be written, in terms of the fluctuation operators, as

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = 2\delta_{i,j} - \Sigma_{i,j}. \quad (4.179)$$

Using the algebra (4.167), which also holds for the fluctuation operators $\delta \hat{a}_i$, it is obtained

$$\langle \hat{A} \hat{A}^\dagger \rangle = \sum_{i,j} \alpha_i \alpha_j^* \left[V_{i,j} + \frac{1}{2} \Sigma_{i,j} \right] \geq 0. \quad (4.180)$$

Hence, the covariance matrix must satisfy what is called the *bona fide* relation

$$\mathbf{V} + \frac{1}{2} \Sigma \geq 0 \quad , \quad \theta + \frac{i}{2} \Omega \geq 0, \quad (4.181)$$

where the second result of eq.(4.181) can also be obtained using the algebra of $\hat{\mathbf{r}}$.

The equation (4.181) is a generalization of the Heisenberg's uncertainty relation. To see that, consider the case of a single-mode, then

$$\boldsymbol{\theta} + \frac{i}{2}\boldsymbol{\Omega} = \begin{pmatrix} \langle \delta \hat{q}^2 \rangle & \frac{1}{2} \langle \{\delta \hat{q} \delta \hat{p}\} \rangle + \frac{i}{2} \\ \frac{1}{2} \langle \{\delta \hat{q} \delta \hat{p}\} \rangle - \frac{i}{2} & \langle \delta \hat{p}^2 \rangle \end{pmatrix}. \quad (4.182)$$

For this matrix to be positive semi-definite, both of its eigenvalues must be non-negative. Alternatively, the determinant must be non-negative. Evaluating the determinant, the uncertainty relation is obtained

$$\langle \delta \hat{q}^2 \rangle \langle \delta \hat{p}^2 \rangle \geq \frac{1}{4} + \frac{1}{4} \langle \{\delta \hat{q}, \delta \hat{p}\} \rangle, \quad (4.183)$$

the eq.(4.183) is usually called the *Roberson-Schrödinger uncertainty relation*.

4.8.3 P-representable positive Gaussian operators

A positive Gaussian operator is said to be P-representable if it can be written as a mixture of coherent states with a proper probability distribution function. Otherwise, the state is not P-representable, and therefore, non-classical (MC Oliveira, 2004). The discrimination of whether a given state is classical or not is not an easy task. However, a criteria can be derived for identifying if a given state is non-classical according to its covariance matrix. For this purpose, let us rewrite the P-function in terms of the parametrization that we have been discussing,

$$P(\mathbf{z}) = \int d\mathbf{z}' e^{-\mathbf{z}^\dagger \boldsymbol{\Sigma} \mathbf{z}'} \chi(\mathbf{z}'), \quad (4.184)$$

where $\boldsymbol{\Sigma}$ is the z-Pauli matrix, and $\chi_N(\mathbf{z})$ is the normally-ordered characteristic function (4.149) that can be rewritten as

$$\chi_N(\mathbf{z}) = e^{\frac{1}{4} \mathbf{z}^\dagger \mathbf{z}} \chi_W(\mathbf{z}) = e^{\frac{1}{4} \mathbf{z}^\dagger \mathbf{z}} e^{-\frac{1}{2} \mathbf{z}^\dagger \mathbf{V} \mathbf{z}} = e^{\frac{1}{2} \mathbf{z}^\dagger (\frac{1}{2} \mathbb{1} - \mathbf{V}) \mathbf{z}}. \quad (4.185)$$

Substituting eq.(4.185) into (4.184), and using the fact that $-\mathbf{z}^\dagger \boldsymbol{\Sigma} \mathbf{z}' = -\mathbf{z}'^\dagger \boldsymbol{\Sigma} \mathbf{z}$ (Englert and Wódkiewicz, 2003), the P-function can be written as

$$P(\mathbf{z}) = \int d\mathbf{z}' e^{\mathbf{z}'^\dagger (\mathbf{V} - \frac{1}{2}) \mathbf{z}'} = \frac{1}{\sqrt{\det(\mathbf{V} - \frac{1}{2})}} e^{\frac{1}{2} (\boldsymbol{\Sigma} \mathbf{z})^\dagger \sigma_x (\mathbf{V} - \frac{1}{2})^{-1} (\boldsymbol{\Sigma} \mathbf{z})}, \quad (4.186)$$

where σ_x is the x-Pauli matrix. It can be verified that, $(\boldsymbol{\Sigma} \mathbf{z})^\dagger = \mathbf{z} \boldsymbol{\Sigma}$ and $\mathbf{z} \boldsymbol{\Sigma} \mathbf{T} = -\mathbf{z}^\dagger \boldsymbol{\Sigma}$, thus

$$P(\mathbf{z}) = \frac{1}{\sqrt{\det(\mathbf{V} - \frac{1}{2})}} e^{-\frac{1}{2} \mathbf{z}^\dagger \boldsymbol{\Sigma} (\mathbf{V} - \frac{1}{2}) \boldsymbol{\Sigma} \mathbf{z}} \equiv \sqrt{\det \mathbf{P}} e^{-\frac{1}{2} \mathbf{z}^\dagger \mathbf{P} \mathbf{z}}, \quad (4.187)$$

where

$$\mathbf{P} = \boldsymbol{\Sigma} \left(\mathbf{V} - \frac{1}{2} \right)^{-1} \boldsymbol{\Sigma}, \quad (4.188)$$

From this result, we may conclude the following statement:

P-representability of a Gaussian state A single-mode Gaussian state described by a covariance matrix \mathbf{V} is said to be P-representable if

$$\bar{n} > |m|, \quad (4.189)$$

Proof: To have a P-representable state, \mathbf{P} must be non-negative, which means that $(\mathbf{V} - 1/2) \geq 0$. This condition requires eigenvalues greater or equal to zero, or alternatively, a non-negative determinant. Thus,

$$\det\left(\mathbf{V} - \frac{\mathbb{1}}{2}\right) = \begin{vmatrix} \bar{n} & m \\ m^* & \bar{n} \end{vmatrix} = \bar{n}^2 - |m|^2 \geq 0. \quad (4.190)$$

Therefore,

$$\bar{n} \geq |m|. \quad (4.191)$$

It is important to emphasize that \bar{n} is not necessarily the mean number of thermal photons \bar{n}_{th} , but the mean number of photons correspondent to the distribution in which the system is described.

Example: single-mode squeezing

To give a non-trivial example, consider a single-mode bosonic system prepared in the squeezed thermal state

$$\hat{\rho} = \hat{S}(r) \frac{e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}}}{Z} \hat{S}^\dagger(r) \quad (4.192)$$

where $Z = (1 - e^{-\beta\hbar\omega})$, and $\hat{S}(r)$ is the squeezing operator (4.86) for $\theta = 0$. In this state, the first moments are zero, and the covariance matrix is given by

$$\mathbf{V}_{\text{th}} = \begin{pmatrix} (\bar{n}_{\text{th}} + 1/2) \cosh(2r) & (\bar{n}_{\text{th}} + 1/2) \sinh(2r) \\ (\bar{n}_{\text{th}} + 1/2) \sinh(2r) & (\bar{n}_{\text{th}} + 1/2) \cosh(2r) \end{pmatrix}, \quad (4.193)$$

where \bar{n}_{th} is the Bose-Einstein distribution (4.27).

From the condition (4.189), it is seen that not all states are classical. Increasing the squeezing parameter, a transition from classical to non-classical can be observed at some critical value of r . This value r_c is reached when $\bar{n} = |m|$:

$$r_c = \frac{1}{2} \log(2\bar{n}_{\text{th}} + 1). \quad (4.194)$$

In this regime, if we recall the results (4.100) and (4.84), we see that this transition occurs when one quadrature goes below the uncertainty bound, and therefore it is no longer P-representable. This result will be extremely significant in chapter 6.

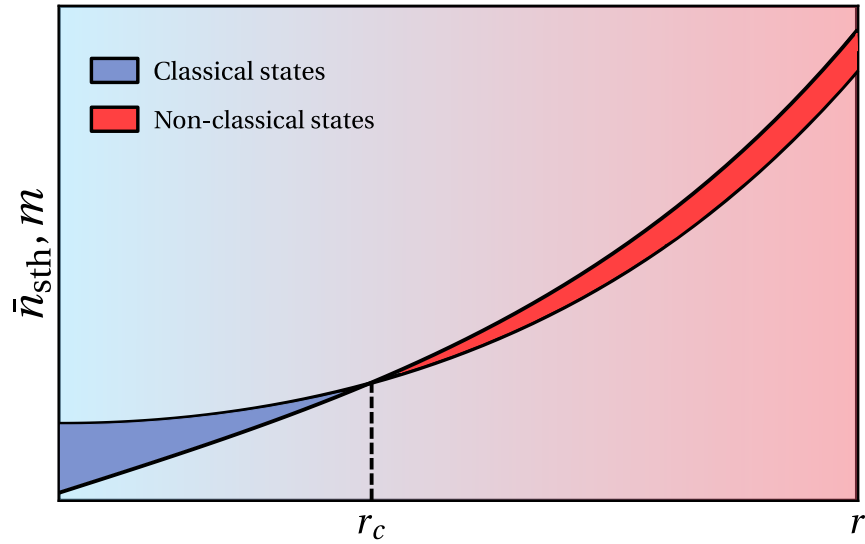


Figure 4.7: Sketch of the parameters \bar{n}_{sth} and m as a function of the squeezing parameter. At $t = t_0$, the system is prepared in a thermal state and is P-representable (blue region). By applying a squeezing operation and increasing the squeezing parameter, the system goes from a classical to a non-classical state at some specific squeezing parameter. The squeezing parameter, such that this transition occurs, when $\bar{n}_{\text{sth}} = m$, is defined as the critical squeezing parameter r_c . After r_c , the state cannot be represented as being merely a mixture of coherent states anymore since its probability distribution function is not legitimate.

In the next section, we will tackle the problem of a single-mode coupled to a reservoir. Our objective is to obtain an equation for the system under the action of environmental interaction. Many tools learned in this section will be used, both for the master equation derivation and its solution.

CHAPTER 5

Open quantum systems

No quantum system can exist in isolation — it unavoidable interacts with its environment. Over sufficiently long times, rather than undergoing the coherent evolution predicted by Schrödinger’s equation, a quantum system will exhibit a noisy, dissipative dynamics. The theory of open quantum systems is concerned precisely with formulating simple and accurate equations that predict the dynamics of open systems. This has become an essential tool, used routinely in many fields of quantum research (e.g., quantum information, quantum optics, or condensed matter physics).

Back in the late 1970s, the theory of open quantum systems was first inspected as a means to study the charming relationships between thermodynamics and quantum mechanics (Kossakowski, 1972; Spohn, 1978; Alicki, 1979). An open quantum system kept away from equilibrium by thermal contact with environments at different temperatures, is a minimal universal template for a heat engine or a refrigerator, and the analysis of such models gave rise to the quickly growing field of quantum thermodynamics (Scovil and Schulz-DuBois, 1959; Geusic et al., 1959). Perhaps, the most exciting recent developments in the field are the first proof-of-principle experiments realizing nanoscale quantum heat engines and refrigerators (Peterson et al., 2019; Klatzow et al., 2019; Roßnagel et al., 2016).

In this chapter, the essential elements of the theory of open quantum systems used in the next chapter of this dissertation are presented. The derivation given here follows the treatments by Louisell (1973) and Haake (1973). With some differences in the way approximations are introduced. Also, no attempt is made to follow the authors’ notation.

Consider a closed system described by a Hilbert space $\mathbb{H} = \mathbb{H}_S \otimes \mathbb{H}_B$, where \mathcal{H}_S and \mathcal{H}_B are the Hamiltonians

for the open system and the environment¹, respectively. The density matrix of the open system is obtained tracing out the bath from the density matrix of the global system. This is what is called a reduced density matrix, $\hat{\rho}_s(t) = \text{tr}_B[\hat{\rho}(t)]$. The formal solution for $\hat{\rho}_s(t)$ given some initial condition $\hat{\rho}(0)$ is

$$\hat{\rho}_s(t) = \text{tr}_B[e^{-i\hat{\mathcal{H}}t}\hat{\rho}(0)e^{i\hat{\mathcal{H}}t}], \quad (5.1)$$

where $\hat{\mathcal{H}}$ is the global Hamiltonian on \mathbb{H} .

5.1 The Lindblad equation

In this dissertation, the master equation method is used to model the dynamics of the open quantum systems. This is an equation of motion that describes the time evolution of the open system $\hat{\rho}_s$ under the influence of the environment. For this evolution to be physically meaningful, it is required that it be completely positive and trace-preserving (CPTP) (Rivas and Huelga, 2011). It also considered, for simplicity, that the dynamics are Markovian, that is, the time derivative of $\hat{\rho}$ at time t depends only on $\hat{\rho}(t)$. The most general Markovian master equation fulfilling these conditions is the Lindblad equation:

Lindblad's Theorem. The generator of any quantum operation satisfying the CPTP condition must have the form (Breuer et al., 2002):

$$\frac{d}{dt}\hat{\rho} = \mathcal{L}(\hat{\rho}) = -i[\hat{\mathcal{H}}, \hat{\rho}] + \sum_k \gamma_k \left[L_k \hat{\rho} L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \hat{\rho}\} \right] \quad (5.2)$$

where $\hat{\mathcal{H}}$ is a Hamiltonian describing coherent processes, L_k are the Lindblad operators describing the incoherent “jumps” and $\gamma_k \geq 0$ are the rates of these jumps.

The Lindblad equation may either be postulated on phenomenological grounds or derived from first principles. In the latter case, the derivation requires the assumption of at least two well-separated dynamical time scales, one fast and one slow Carmichael (1999).

5.2 The master equation

In this section, a master equation describing the interaction of a harmonic oscillator with frequency ω_0 and a bath of harmonic oscillators with frequencies ω_j is derived. In order to simplify the analysis, several mathematical details are ignored.

¹The terms “environment”, “reservoir” and “bath” are used interchangeably.

Let us consider an open quantum system described by the Hamiltonian $\hat{\mathcal{H}}_S, \hat{\mathcal{H}}_S \in \hat{\mathbb{H}}_S$, coupled to a reservoir described by the Hamiltonian $\hat{\mathcal{H}}_B, \hat{\mathcal{H}}_B \in \hat{\mathbb{H}}_B$. Such that, the Hamiltonian of the composite system in $\hat{\mathbb{H}}_S \otimes \hat{\mathbb{H}}_B$ is

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_S + \hat{\mathcal{H}}_B + \hat{\mathcal{V}}, \quad (5.3)$$

where $\hat{\mathcal{V}}$ is the interaction Hamiltonian. The equation of motion in the interaction picture is given by eq.(3.40)

$$\frac{d}{dt} \hat{\rho}_I(t) = -\frac{i}{\hbar} [\hat{\mathcal{V}}_I(t), \hat{\rho}_I(t)], \quad (5.4)$$

where $\hat{\rho}_I$ is the density operator of the composite system. If the interaction is turned on at $t = 0$ and no correlations exist between the system and reservoir initially, the density operator of the composite system is given by

$$\hat{\rho}_I(0) = \hat{\rho}_I(0) \otimes \hat{\rho}_{IB}, \quad (5.5)$$

where $\hat{\rho}_I(0)$ and $\hat{\rho}_{IB}$ are the initial density operator of the system and reservoir in the interaction picture, respectively. Integrating eq.(5.4), substituting the result $\hat{\rho}_I(t)$ back into the commutator of the right-side, and tracing out the environment, it is obtained:

$$\dot{\hat{\rho}}_I(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{tr}_B\{[\hat{\mathcal{V}}_I(t), [\hat{\mathcal{V}}_I(t'), \hat{\rho}_I(t')]]\}, \quad (5.6)$$

where, without loss of generality, we have eliminated the term $(-i/\hbar)\text{tr}_B\{[\hat{\mathcal{V}}_I(t), \hat{\rho}_I(0)]\}$ assuming that

$$\text{tr}_B\{\hat{\mathcal{V}}_I(t)\hat{\rho}_{IB}\} = 0, \quad (5.7)$$

meaning that the reservoir operators have zero mean in the state $\hat{\rho}_{IB}$ (Carmichael, 1999).

At later times correlations arise due to the coupling between the system and the reservoir. However, it is considered that the system is very weakly coupled to the reservoir, such that $\hat{\rho}_I(t)$ should only give terms of order $\hat{\mathcal{V}}_I$ from an uncorrelated state.

$$\hat{\rho}_I(t) = \hat{\rho}_I(t) \otimes \hat{\rho}_{IB} + O(\hat{\mathcal{V}}_I). \quad (5.8)$$

Now we can make our first approximation, a *Born approximation*: neglecting terms of second-order and higher in $\hat{\mathcal{V}}_I$, eq (5.6) can be written as

$$\dot{\hat{\rho}}_I(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{tr}_B\{[\hat{\mathcal{V}}_I(t), [\hat{\mathcal{V}}_I(t'), \hat{\rho}_I(t') \otimes \hat{\rho}_{IB}]]\}. \quad (5.9)$$

A complete, and carefully, discussion of this approximation can be found in the work of Haake (1973). This approximation implies in a weakly coupling. Hence, it is possible to write the state of the composite system as a tensorial product. Furthermore, it is assumed that the correlation time τ_c (and thus the relaxation time τ_R) of the environment is sufficiently small, such that $\hat{\rho}_{IB}(t) \approx \hat{\rho}_{IB}(0)$, if $\tau_R \gg \tau_c$.

Notice that the equation (5.9) is still complicated. In particular, the future evolution of $\hat{\rho}(t)$ depends on its past through the integration over $\hat{\rho}(t')$. Then, it is expected that the time scales involved here are shorter than

the gross time scales over which the system evolves, so that $\hat{\rho}(t') \approx \hat{\rho}(t)$. This is the second major approximation and is called the *Markov approximation* and the *master equation* (5.9) can be rewritten as

$$\dot{\hat{\rho}}_I(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{tr}_B \{ [\hat{\mathcal{V}}_I(t), [\hat{\mathcal{V}}_I(t'), \hat{\rho}_I(t) \otimes \hat{\rho}_{IB}]] \}. \quad (5.10)$$

5.2.1 Thermal bath

So far, no property about the reservoir or the system has been taken into account. It is now adopted an explicit model consisting of a single-bosonic mode, i.e., a harmonic oscillator with frequency ω_0 , and the bosonic operators \hat{a} and \hat{a}^\dagger , coupled to a reservoir modeled by a collection of harmonic oscillators with frequencies ω_j and bosonic operators \hat{b}_j and \hat{b}_j^\dagger . The single-mode couples to the reservoir via a coupling constant k_j in the rotating wave approximation². It is assumed that the reservoir is in thermal equilibrium at temperature T , and each harmonic oscillator of the bath is prepared in a thermal state (4.23):

$$\hat{\rho}_R^{(j)} = \sum_j (1 - e^{-\hbar\omega_j/k_B T}) e^{-\hbar\omega_j \hat{b}^\dagger \hat{b}}. \quad (5.11)$$

Therefore, the Hamiltonian of the composite system may be written as

$$\begin{aligned} \hat{\mathcal{H}}_S &\equiv \hbar\omega_0 \hat{a}^\dagger \hat{a} \\ \hat{\mathcal{H}}_B &\equiv \sum_j \hbar\omega_j \hat{b}_j^\dagger \hat{b}_j \\ \hat{\mathcal{V}} &\equiv \sum_j \hbar(k_j^* \hat{a} \hat{b}_j^\dagger + k_j \hat{a}^\dagger \hat{b}_j) = \hbar(\hat{a} \hat{\Gamma}^\dagger + \hat{a}^\dagger \hat{\Gamma}). \end{aligned} \quad (5.12)$$

where $\hat{H}_S \in \mathbb{H}_S$, $\hat{H}_B \in \mathbb{H}_B$, k_j is the coupling constant, $\hat{\Gamma} = \sum_j k_j \hat{b}_j$, and $\hat{\mathcal{V}}$ is the interaction Hamiltonian.

Let us first write the operators on eq.(5.12) in the interaction picture,

$$\begin{aligned} \hat{a}_I(t) &= e^{i\omega_0 \hat{a}^\dagger \hat{a}} \hat{a} e^{-i\omega_0 \hat{a}^\dagger \hat{a}} = \hat{a} e^{-i\omega_0 t} \\ \hat{a}_I^\dagger(t) &= e^{i\omega_0 \hat{a}^\dagger \hat{a}} \hat{a}^\dagger e^{-i\omega_0 \hat{a}^\dagger \hat{a}} = \hat{a}^\dagger e^{i\omega_0 t} \\ \hat{\Gamma}_I(t) &= \left(e^{i\sum_n \omega_n \hat{b}_n^\dagger \hat{b}_n} \right) \sum_j k_j \hat{b}_j \left(e^{-i\sum_n \omega_n \hat{b}_n^\dagger \hat{b}_n} \right) = \sum_j k_j \hat{b}_j e^{-i\omega_j t} \\ \hat{\Gamma}_I^\dagger(t) &= \left(e^{i\sum_n \omega_n \hat{b}_n^\dagger \hat{b}_n} \right) \sum_j k_j^* \hat{b}_j^\dagger \left(e^{-i\sum_n \omega_n \hat{b}_n^\dagger \hat{b}_n} \right) = \sum_j k_j^* \hat{b}_j^\dagger e^{i\omega_j t}, \end{aligned} \quad (5.13)$$

so, it follows that the interaction Hamiltonian in the interaction picture is given by

$$\hat{\mathcal{V}}_I(t) = \hbar(\hat{a}_I^\dagger(t) \hat{\Gamma}_I(t) e^{i\omega_0 t} + \hat{a}_I(t) \hat{\Gamma}_I^\dagger(t) e^{-i\omega_0 t}). \quad (5.14)$$

²The rotating wave is an approximation where the terms in a given Hamiltonian, which oscillate rapidly, are ignored. In this case, the interaction Hamiltonian takes the linear form.

From now on, the label I characterizing the interaction picture is dropped out so that the notation does not get confusing. Substituting eqs.(5.13) and (5.14) into eq.(5.10),

$$\begin{aligned}\dot{\hat{\rho}} = & - \int_0^t dt' \{ [\hat{a}\hat{a}\hat{\rho}(t') - \hat{a}\hat{\rho}(t')\hat{a}] e^{-i\omega_0(t+t')} \langle \hat{\Gamma}^\dagger(t)\hat{\Gamma}^\dagger(t') \rangle_B + \text{h.c.} \\ & + [\hat{a}^\dagger\hat{a}^\dagger\hat{\rho}(t') - \hat{a}^\dagger\hat{\rho}(t')\hat{a}^\dagger] e^{i\omega_0(t+t')} \langle \hat{\Gamma}(t)\hat{\Gamma}(t') \rangle_B + \text{h.c.} \\ & + [\hat{a}\hat{a}^\dagger\hat{\rho}(t') - \hat{a}^\dagger\hat{\rho}(t')\hat{a}] e^{-i\omega_0(t-t')} \langle \hat{\Gamma}^\dagger(t)\hat{\Gamma}(t') \rangle_B + \text{h.c.} \\ & + [\hat{a}^\dagger\hat{a}\hat{\rho}(t') - \hat{a}\hat{\rho}(t')\hat{a}^\dagger] e^{i\omega_0(t-t')} \langle \hat{\Gamma}(t)\hat{\Gamma}^\dagger(t') \rangle_B + \text{h.c.},\end{aligned}\quad (5.15)$$

where h.c means the Hermitian conjugate, and the notation $\langle \cdot \rangle$ is called the reservoir correlation function and is given explicitly by

$$\begin{aligned}\langle \hat{\Gamma}(t)^\dagger \hat{\Gamma}(t')^\dagger \rangle_B &= \sum_{j,k} k_j^* k_k^* e^{i\omega_j t} e^{i\omega_k t'} \langle \hat{b}_j^\dagger \hat{b}_k^\dagger \rangle = 0, \\ \langle \hat{\Gamma}(t) \hat{\Gamma}(t') \rangle_B &= \sum_{j,k} k_j k_k e^{-i\omega_j t} e^{-i\omega_k t'} \langle \hat{b}_j \hat{b}_k \rangle = 0, \\ \langle \hat{\Gamma}(t)^\dagger \hat{\Gamma}(t') \rangle_B &= \sum_{j,k} k_j^* k_k e^{i\omega_j t} e^{-i\omega_k t'} \langle \hat{b}_j^\dagger \hat{b}_k \rangle = \sum_j |k_j|^2 e^{i\omega_j(t-t')} \bar{n}_{\text{th}}(\omega_j, T), \\ \langle \hat{\Gamma}(t) \hat{\Gamma}(t')^\dagger \rangle_B &= \sum_{j,k} k_j k_k^* e^{-i\omega_j t} e^{i\omega_k t'} \langle \hat{b}_j \hat{b}_k^\dagger \rangle = \sum_j |k_j|^2 e^{-i\omega_j(t-t')} [\bar{n}_{\text{th}}(\omega_j, T) + 1],\end{aligned}\quad (5.16)$$

with

$$\bar{n}_{\text{th}}(\omega_j, T) = \text{tr}_B [\hat{\rho}_B \hat{b}_j^\dagger \hat{b}_j] = \frac{e^{-\hbar\omega_j/k_B T}}{1 - e^{-\hbar\omega_j/k_B T}}. \quad (5.17)$$

The non-vanishing terms in eqs.(5.16) involve a summation over the reservoir oscillators. This summation can be changed to an integration by introducing the density of states $g(\omega)$, such that $g(\omega)d\omega$ gives the number of oscillators with frequencies in the interval $\omega + d\omega$. Also, making the change of variable

$$\tau = t - t', \quad (5.18)$$

the equation (5.15) can be restated as

$$\begin{aligned}\dot{\hat{\rho}} = & - \int_0^t d\tau \{ [\hat{a}\hat{a}^\dagger\hat{\rho}(t-\tau) - \hat{a}^\dagger\hat{\rho}(t-\tau)\hat{a}] e^{-i\omega_0\tau} \langle \Gamma^\dagger(t)\Gamma(t-\tau) \rangle_B + \text{h.c.} \\ & + [\hat{a}^\dagger\hat{a}\hat{\rho}(t-\tau) - \hat{a}\hat{\rho}(t-\tau)\hat{a}^\dagger] e^{i\omega_0\tau} \langle \Gamma(t)\Gamma^\dagger(t-\tau) \rangle_B + \text{h.c.},\end{aligned}\quad (5.19)$$

where now the nonzero correlation functions takes the form

$$\begin{aligned}\langle \hat{\Gamma}(t)^\dagger \hat{\Gamma}(t') \rangle_B &= \int_0^\infty d\omega e^{i\omega\tau} g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega_j, T), \\ \langle \hat{\Gamma}(t) \hat{\Gamma}(t')^\dagger \rangle_B &= \int_0^\infty d\omega e^{-i\omega\tau} g(\omega) |k(\omega)|^2 [\bar{n}_{\text{th}}(\omega, T) + 1].\end{aligned}\quad (5.20)$$

It is convenient to evaluate the time integral in eq.(5.19) first, without performing the frequency integrals, to obtain an explicit form for the reservoir correlation functions. This is possible because the τ integration is

dominated by times that are much shorter than the time scale for the evolution of $\hat{\rho}$ (Carmichael, 1999). With $\hat{\rho}(t - \tau)$ replaced by $\hat{\rho}(t)$ (Markov Approximation), and writing $\hat{\rho}(t) = \hat{\rho}$ just to make it easier, (5.18) becomes

$$\begin{aligned} \dot{\hat{\rho}} &= - \int_0^t d\tau \{ [\hat{a}\hat{a}^\dagger \hat{\rho} - \hat{a}^\dagger \hat{\rho} \hat{a}] e^{-i\omega_0 \tau} \langle \hat{\Gamma}^\dagger(t) \hat{\Gamma}(t - \tau) \rangle + [\hat{a}^\dagger \hat{a} \hat{\rho} - \hat{a} \hat{\rho} \hat{a}^\dagger] e^{i\omega_0 \tau} \langle \hat{\Gamma}(t) \hat{\Gamma}^\dagger(t - \tau) \rangle_B + \text{h.c.} \}_B \\ &= - \int_0^t d\tau \{ [\hat{a}\hat{a}^\dagger \hat{\rho} - \hat{a}^\dagger \hat{\rho} \hat{a}] e^{-i\omega_0 \tau} \} \int_0^\infty d\omega e^{i\omega \tau} g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega, T) \\ &\quad - \int_0^t d\tau \{ [\hat{a}^\dagger \hat{a} \hat{\rho} - \hat{a} \hat{\rho} \hat{a}^\dagger] e^{i\omega_0 \tau} \} \int_0^\infty d\omega e^{-i\omega \tau} g(\omega) |k(\omega)|^2 [\bar{n}_{\text{th}}(\omega, T) + 1] + \text{h.c.}, \end{aligned} \quad (5.21)$$

and then,

$$\begin{aligned} \dot{\hat{\rho}} &= \int_0^t d\tau \int_0^\infty d\omega e^{i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega, T) [\hat{a}^\dagger \hat{\rho} \hat{a} - \hat{a} \hat{a}^\dagger \hat{\rho}] \\ &\quad + \int_0^t d\tau \int_0^\infty d\omega e^{-i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega, T) [\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}] \\ &\quad + \int_0^t d\tau \int_0^\infty d\omega e^{-i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2 [\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}] + \text{h.c.} \end{aligned} \quad (5.22)$$

Note that the second term of this equation can be combined with the Hermitian conjugate of the first term and vice-versa, obtaining

$$\begin{aligned} \dot{\hat{\rho}} &= \int_0^t d\tau \int_0^\infty d\omega e^{-i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega, T) [\hat{a}^\dagger \hat{\rho} \hat{a} + \hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a} \hat{a}^\dagger] \\ &\quad + \int_0^t d\tau \int_0^\infty d\omega e^{-i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2 [\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}] + \text{h.c.}, \end{aligned} \quad (5.23)$$

or,

$$\dot{\hat{\rho}} = \alpha (\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}) + \beta (\hat{a} \hat{\rho} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a} \hat{a}^\dagger) + \text{h.c.}, \quad (5.24)$$

with the definitions

$$\begin{aligned} \alpha &= \int_0^t d\tau \int_0^\infty d\omega e^{-i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2, \\ \Lambda &= \int_0^t d\tau \int_0^\infty d\omega e^{-i(\omega - \omega_0)\tau} g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega, T). \end{aligned} \quad (5.25)$$

Since τ dominates by much shorter times $\approx \tau_R$, the τ integration can be extended to infinity and the quantities α and β can be evaluated (appendix A.5) using the Sokhotski–Plemelj formula:

$$\lim_{t \rightarrow \infty} \int_0^t d\tau e^{-i(\omega - \omega_0)\tau} = \pi \delta(\omega - \omega_0) + i \frac{P}{\omega - \omega_0}, \quad (5.26)$$

where P indicates the Cauchy principal value. Hence,

$$\begin{aligned} \alpha &= \pi g(\omega_0) |k(\omega_0)|^2 + i \Delta, \\ \Lambda &= \pi g(\omega_0) |k(\omega_0)|^2 \bar{n}_{\text{th}}(\omega_0) + i \Delta', \end{aligned} \quad (5.27)$$

with

$$\begin{aligned} \Delta &= P \int_0^\infty d\omega \frac{g(\omega) |k(\omega)|^2}{\omega - \omega_0} \\ \Delta' &= P \int_0^\infty d\omega \frac{g(\omega) |k(\omega)|^2 \bar{n}_{\text{th}}(\omega, T)}{\omega - \omega_0}. \end{aligned} \quad (5.28)$$

Finally, the master equation for the harmonic oscillator coupled to a thermal bath is obtained - from (5.25) and (5.27-5.30), as

$$\dot{\hat{\rho}} = \frac{\gamma}{2}(\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) - i\Delta[\hat{a}^\dagger\hat{a}, \hat{\rho}] + \gamma\bar{n}_{\text{th}}(\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho} + \hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \quad (5.29)$$

where

$$\begin{aligned} \gamma &\equiv 2\pi g(\omega_0)|k(\omega_0)|^2, \\ \bar{n}_{\text{th}} &\equiv \bar{n}_{\text{th}}(\omega_0, T). \end{aligned} \quad (5.30)$$

The density operator of the system $\hat{\rho}$ is still in the interaction picture. To transform it back to the Schrödinger picture, it is only needed the following transformation

$$\hat{\rho} = -\frac{i}{\hbar}[\mathcal{H}_s, \hat{\rho}_s] + U\hat{\rho}U^\dagger, \quad (5.31)$$

with $\mathcal{H}_s = \hbar\omega_0\hat{a}^\dagger\hat{a}$. Thus, the master equation in the Schrödinger picture is written

$$\dot{\hat{\rho}} = -i\omega'_0[\hat{a}^\dagger\hat{a}, \hat{\rho}] + \frac{\gamma}{2}(\bar{n}_{\text{th}} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}_{\text{th}}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \quad (5.32)$$

where

$$\omega'_0 \equiv \omega_0 + \Delta. \quad (5.33)$$

5.2.2 Squeezed thermal bath

In the last section, we derived an equation that describes the time evolution of an open system when it is coupled with a bosonic bath prepared in a thermal state (4.23). In this section, the derivation of the master equation is extended to the case where the open system is interacting with a non-thermal bath. In other words, the single mode is coupled to a collection of harmonic oscillators, each of them prepared in a squeezed thermal state, that is, its states are given by

$$\hat{\rho}_B^{(j)} = \hat{S}(\xi) \frac{e^{-\beta\hat{\mathcal{H}}_B^{(j)}}}{Z_j} \hat{S}^\dagger(\xi) \quad (5.34)$$

where \hat{S} is the squeezing operator (4.86). For this case, it follows that the correlation function (5.16) now satisfies,

$$\begin{aligned} \langle \hat{b}_j^\dagger(\omega)\hat{b}_j^\dagger(\omega') \rangle &= 2\pi m(\omega)\delta(2\omega' - \omega - \omega_0), \\ \langle \hat{b}_j^\dagger(\omega)\hat{b}_j(\omega') \rangle &= 2\pi\bar{n}_{\text{sth}}(\omega, T)\delta(\omega - \omega_0), \\ \langle \hat{b}_j(\omega)\hat{b}_j^\dagger(\omega_0) \rangle &= 2\pi[\bar{n}_{\text{sth}}(\omega, T) + 1]\delta(\omega - \omega_0), \end{aligned} \quad (5.35)$$

where $\bar{n}_{\text{sth}}(r, T) = (\bar{n}_{\text{th}} + 1/2)\cosh 2r$ is the mean photon number of the i th squeezed reservoir, and $m(r, T) = (\bar{n}_{\text{th}} + 1/2)\sinh 2r$ is coherence parameter.

The same procedure as in the case of a thermal bath can be followed. In this case, the master equation in the interaction picture is given by (Scully and Zubairy, 1997b)

$$\begin{aligned} \dot{\hat{\rho}} = & \frac{\gamma}{2}(\bar{n}_{\text{sth}} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}_{\text{sth}}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{\gamma}{2}m(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}^\dagger) + \frac{\gamma}{2}m^*(2\hat{a}\hat{\rho}\hat{a} - \hat{a}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}) \end{aligned} \quad (5.36)$$

The difference between eq.(5.36) and eq.(5.29) are the two additional terms corresponding to the non-diagonal terms on the density operator when it is represented in a Fock basis. To recovery eq.(5.29), we only need to make $r = 0$.

5.3 Solution of the master equation

In the last section, we discussed the master equation for the single-mode coupled to a squeezed thermal bath,

$$\begin{aligned} \dot{\hat{\rho}} = & \frac{\gamma}{2}(\bar{n}_{\text{sth}} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}_{\text{sth}}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{\gamma}{2}m(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}^\dagger) + \frac{\gamma}{2}m^*(2\hat{a}\hat{\rho}\hat{a} - \hat{a}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}) \end{aligned} \quad (5.37)$$

The time evolution of the density matrix can be obtained by solving this equation. For this, it is convenient to use the characteristic function formalism, so rewriting eq.(5.37) in terms of the characteristic function (4.144), the equation of motion is now

$$\frac{\partial \chi_W}{\partial t} = e^{-\frac{1}{2}|z|^2} \text{tr}(\dot{\hat{\rho}} e^{za^\dagger} e^{-z^*a}). \quad (5.38)$$

where it was used the correspondence (see. eq(4.185)) between the Wigner and the normally-ordered characteristic functions. Substituting eq.(5.37) into eq.(5.38),

$$\begin{aligned} \frac{\partial \chi_W}{\partial t} = & e^{-\frac{1}{2}|z|^2} \text{tr} \left\{ \left[\frac{\gamma}{2}(\bar{n}_{\text{sth}} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}_{\text{sth}}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \right. \right. \\ & \left. \left. + \frac{\gamma}{2}m(2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}^\dagger) + \frac{\gamma}{2}m^*(2\hat{a}\hat{\rho}\hat{a} - \hat{a}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}) \right] e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} \right\}. \end{aligned} \quad (5.39)$$

Re-expressing each of the twelve terms on the right-hand side of equation (5.39) in terms of χ_N and its derivative with respect to z and z^* . Some of them can be achieved directly. For instance,

$$\text{tr}(\hat{a}\hat{\rho}\hat{a}^\dagger e^{z\hat{a}^\dagger} e^{-z^*\hat{a}}) = \text{tr}(\hat{\rho}\hat{a}^\dagger e^{z\hat{a}^\dagger} e^{-z^*\hat{a}}\hat{a}) = -\frac{\partial^2}{\partial z^* \partial z} \chi_N, \quad (5.40)$$

where we have simply used the cyclic property of the trace. For the remaining terms, we use the cyclic property of the trace and a little more algebraic manipulation to rearrange the terms inside the trace. Generally, the rearrangement may require us to pass \hat{a}^\dagger through the exponential $e^{iz^*\hat{a}}$. For this purpose, it is used the relation:

$$\begin{aligned} e^{-z^*\hat{a}}\hat{a}^\dagger e^{z^*\hat{a}} &= \hat{a}^\dagger - z^* \\ e^{z\hat{a}^\dagger}\hat{a} e^{-z\hat{a}^\dagger} &= \hat{a} - z. \end{aligned} \quad (5.41)$$

A more detailed discussion about this passage can be found on Appendix (A.5). The equation of motion for χ_N becomes

$$\frac{\partial \chi_W}{\partial t} = \left\{ -\frac{\gamma}{2} \left[(1 + 2\bar{n}_{\text{sth}})|z|^2 + m^* z^2 + m z^{*2} + z^* \frac{\partial}{\partial z^*} + z \frac{\partial}{\partial z} \right] \right\} e^{-\frac{1}{2}|z|^2} \chi_N. \quad (5.42)$$

Defining the quantities

$$\begin{aligned} A &\equiv -\frac{\gamma}{2} [(1 + 2\bar{n}_{\text{sth}})|z|^2 + m^* z^2 + m z^{*2}] \\ B &\equiv -\frac{\gamma}{2} \left(z^* \frac{\partial}{\partial z^*} + z \frac{\partial}{\partial z} \right). \end{aligned} \quad (5.43)$$

The choice in defining A and B as in eq.(5.45) is because it satisfies the following commutation relation

$$[A, B] = \gamma A, \quad (5.44)$$

this result will be useful for the next steps. Then, the equation of motion for the characteristic function can be written as:

$$\frac{\partial \chi_W}{\partial t} = (A + B) \chi_W(z, t), \quad (5.45)$$

whose the formal solution is

$$\chi_W(t) = e^{(A+B)t} \chi_W(0). \quad (5.46)$$

Using the *The Zassenhaus formula* and other identities (on Appendix A.3), the solution (5.46) can be recast as

$$\chi_W(z, t) = \exp \left[\frac{A}{\gamma} (1 - e^{-\gamma t}) \right] e^{tB} \chi_W(z, 0), \quad (5.47)$$

where $\chi(z, 0)$ is the initial normally-ordered characteristic function of the single bosonic mode.

Considering the following relation:

$$\exp(\epsilon x \partial_x + y \epsilon \partial_y) f(x, y) = f(x \exp(\epsilon), y \exp(\epsilon)), \quad (5.48)$$

we may use it by recalling that $B = -\frac{\gamma}{2} \left(z^* \frac{\partial}{\partial z^*} + z \frac{\partial}{\partial z} \right)$. Thus,

$$e^{tB} \chi_W(z, 0) = \chi_W(e^{-\frac{\gamma}{2}t} z, 0), \quad (5.49)$$

defining $z(t) \equiv e^{-\frac{\gamma}{2}t} z$, therefore, the characteristic function can be written as

$$\chi_W(z, t) = \exp \left[\frac{A}{\gamma} (1 - e^{-\gamma t}) \right] \chi_W(z(t), 0). \quad (5.50)$$

The initial characteristic function of a Gaussian state is given by (4.176):

$$\chi_W(\mathbf{z}, 0) = e^{-\frac{1}{2} \mathbf{z}^\dagger \mathbf{v} \mathbf{z}}. \quad (5.51)$$

Introducing the following matrices

$$\mathbf{N} = \begin{pmatrix} 2\bar{n}_{\text{sth}} + 1 & 0 \\ 0 & 2\bar{n}_{\text{sth}} + 1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}, \quad (5.52)$$

and substituting eq.(5.51) into eq.(5.50), the characteristic function of the single bosonic mode weakly dissipating into a squeezed thermal reservoir is

$$\begin{aligned} \chi(z, t) &= \exp \left[\frac{A}{\gamma} (1 - e^{-\gamma t}) \right] \chi(z(t), 0) \\ &= \exp \left\{ -\frac{1}{2} [(1 + 2\bar{n}_{\text{sth}})|z|^2 + m^* z_i^2 + m z^{*2}] (1 - e^{-\gamma t}) \right\} \chi(z(t), 0) \\ &= \exp \left\{ -\frac{1}{2} \mathbf{z}^\dagger \left(\frac{\mathbf{N}}{2} + \mathbf{M} \right) \mathbf{z} + \frac{e^{-\gamma t}}{2} \mathbf{z}^\dagger \left(\frac{\mathbf{N}}{2} + \mathbf{M} \right) \mathbf{z} \right\} \exp \left[-\frac{1}{2} e^{-\gamma t} \mathbf{z}^\dagger \mathbf{V} \mathbf{z} \right] \\ &= e^{-\frac{1}{2} \mathbf{z}^\dagger \mathbf{C} \mathbf{z}}, \end{aligned}$$

where

$$\mathbf{C} \equiv e^{-\gamma t} (\mathbf{V} - \boldsymbol{\sigma}) + \boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\sigma} = \frac{\mathbf{N}}{2} + \mathbf{M}. \quad (5.53)$$

Explicitly,

$$\boldsymbol{\sigma} = \begin{pmatrix} \bar{n}_{\text{sth}} + \frac{1}{2} & m \\ m^* & \bar{n}_{\text{sth}} + \frac{1}{2} \end{pmatrix} \quad (5.54)$$

Therefore, from eqs.(5.53) and (5.54), we observe that when t goes to infinity, the open system is described by a squeezed thermal state. However, setting $r = 0$, i.e., when the open system is interacting with a thermal bath, the system evolves to the thermal state. In order to confirm that, let us determine the P-function for the case where $r = 0$. Using eq.(4.186),

$$\begin{aligned} P(\mathbf{z}) &= \int d\mathbf{z}' e^{-\mathbf{z}^\dagger \Sigma \mathbf{z}'} e^{-\frac{1}{2} \mathbf{z}^\dagger (\mathbf{V} - \frac{1}{2} \mathbb{1}) \mathbf{z}} \\ &= \frac{1}{\sqrt{\det(\mathbf{V} - \frac{1}{2} \mathbb{1})}} e^{-\frac{1}{2} \mathbf{z}^\dagger \Sigma (\mathbf{V} - \frac{1}{2} \mathbb{1})^{-1} \Sigma \mathbf{z}} \\ &= \frac{1}{\pi \bar{n}_{\text{th}}} \exp \left\{ -\frac{1}{2} \begin{pmatrix} z^* & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\bar{n}_{\text{th}}} & 0 \\ 0 & \frac{1}{\bar{n}_{\text{th}}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \right\} \\ &= \frac{1}{\pi \bar{n}_{\text{th}}} e^{-\frac{|z|^2}{\bar{n}_{\text{th}}}}, \end{aligned} \quad (5.55)$$

which is the same result derived in (4.137).

We can go further and verify that $\mathcal{L}(e^{-\beta \omega \hat{a}^\dagger \hat{a}}) = 0$, meaning that the steady-state is a thermal state with the same temperature as that imposed by the bath.

5.3.1 Some properties of the squeezed thermal bath

Until now, non-thermal reservoirs such as the squeezed thermal bath, has been essentially conceptualized as stationary non-equilibrium reservoirs, constructed, i.e., from reservoir engineering techniques. We have

seen so far that a squeezed thermal bath is modeled as a collection of harmonic oscillator where each of them is in a thermal state at some inverse temperature β to which the squeezing operator has been applied (5.34) :

$$\hat{\rho}_R^{(j)} = \hat{S}(\xi) \frac{e^{-\beta \hat{\mathcal{H}}_R^{(j)}}}{Z_j} \hat{S}^\dagger(\xi), \quad (5.56)$$

Although a squeezed thermal bath has been called as nonequilibrium reservoirs, [Manzano \(2018\)](#) interpreted it, as a generalized Gibbs ensemble. In order to see that, let us write (5.34) in a more convenient way. Using (4.90), the density operator of the reservoir can be written as,

$$\begin{aligned} \hat{\rho}_R^{(j)} &= \frac{1}{Z_s} \exp \left\{ -\beta \hbar \omega_j \left(\hat{b}_j^\dagger \hat{b}_j \cosh 2r - \frac{1}{2} \sinh 2r (\hat{b}_j^{\dagger 2} + \hat{b}_j^2) + \sinh^2 r + \frac{1}{2} \right) \right\} \\ &= \frac{1}{Z_s} \exp \left\{ -\beta \cosh 2r \left[\hbar \omega_j \left(\hat{b}_j^\dagger \hat{b}_j + \frac{1}{2} \right) - \frac{\hbar \omega_j}{2} (\hat{b}_j^2 + \hat{b}_j^{\dagger 2}) \tanh 2r \right] \right\} \\ &= \frac{1}{Z_s} \exp \{ -\beta_s (H_R^{(j)} - \mu \hat{m}_R^{(j)}) \}. \end{aligned} \quad (5.57)$$

where the following quantities were defined:

$$\beta_s \equiv \beta \cosh 2r \quad \text{Generalized inverse temperature} \quad (5.58)$$

$$\mu \equiv \tanh 2r \quad \text{Chemical-like potential of squeezing,} \quad (5.59)$$

together with the operator

$$\hat{m}_R^{(j)} \equiv -\frac{\hbar \omega_j}{2} (\hat{b}_j^2 + \hat{b}_j^{\dagger 2}). \quad (5.60)$$

The operator (5.60) is called as the second-order moment asymmetry, or just the asymmetry for the j th reservoir mode, and it receives this name because if we rewrite it in terms of the dimensionless position and momentum quadratures (4.50), the operator takes the form

$$\hat{m}_R^{(j)} = \frac{\hbar \omega_j}{2} (\hat{X}_1^2 - \hat{X}_2^2). \quad (5.61)$$

For the cases that we are dealing here, a single bosonic mode, i.e, a Gaussian state with first moments equal to zero, it corresponds to the difference between the uncertainties in the mode quadratures \hat{X}_1 and \hat{X}_2 ,

$$\langle \hat{m}_R^{(j)} \rangle = \frac{\hbar \omega_j}{2} (\Delta X_2^2 - \Delta X_1^2), \quad (5.62)$$

and is a measure of squeezing. For instance, in the case of a thermal state (4.52) and (4.53), the uncertainties are symmetric, and therefore the asymmetry is zero. However, for squeezed states, the asymmetry depends on the squeezing parameter and increases with it.

Moreover, this reservoir behaves in a very similar way to a particle reservoir³ at thermal equilibrium, but replacing the number of particles by the asymmetry, and consequently the traditional chemical potential by the chemical-like potential of squeezing ([Manzano, 2018](#)).

³Grand canonical ensemble

5.4 Dynamics of a two-mode system and swapping time

Along the last section we discussed the behaviour of a single bosonic mode interacting with a bath that is modeled as a collection of harmonic oscillators. We have seen that at the asymptotic limit, when $t \rightarrow \infty$, the steady-state of the open system tends to become the state in which the reservoir was prepared. Now let us consider the problem where instead of coupling the single-mode with an infinite number of modes, we coupled to another single mode bosonic field, using the same interaction Hamiltonian (5.14). Then, we will analyze the dynamical evolution of a two-mode harmonic system and conclude that for certain values of time, namely the *swapping time*, the two-modes exchange their identities. The aim of this section is to shortly discuss the result obtained by Oliveira et al. (1999) in order to use it as a future reference in chapter 6.

The Hamiltonian of the composite system can be written as

$$\hat{\mathcal{H}} = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_c \hat{b}^\dagger \hat{b} + \hbar(e^{i(\phi-\nu t)} \hat{a} \hat{b}^\dagger + e^{i(\phi-\nu t)} \hat{a}^\dagger \hat{b}), \quad (5.63)$$

where

$$\hat{\mathcal{V}} = \hbar(e^{i(\phi-\nu t)} \hat{a} \hat{b}^\dagger + e^{i(\phi-\nu t)} \hat{a}^\dagger \hat{b}), \quad (5.64)$$

so the equations of motion for the bosonic operators in the interaction picture, $\hat{a}_I(t) = \hat{a}(t)e^{i\omega_a t}$ and $\hat{b}_I(t) = \hat{b}(t)e^{i\omega_b t}$, reads

$$\frac{d\hat{a}_I}{dt} = \frac{1}{i\hbar} [\hat{a}_I, \hat{\mathcal{V}}_I] = -i\lambda e^{i(\Omega t + \phi)} \hat{b}_I \quad (5.65)$$

$$\frac{d\hat{b}_I}{dt} = \frac{1}{i\hbar} [\hat{b}_I, \hat{\mathcal{V}}_I] = -i\lambda e^{-i(\Omega t + \phi)} \hat{a}_I, \quad (5.66)$$

Deriving eq.(5.65) with respect to time and substituting it in eq.(5.66), one obtains a second order differential equation

$$\frac{d^2 \hat{a}_I}{dt^2} - i\Omega \frac{d\hat{a}_I}{dt} + \lambda^2 \hat{a}_I = 0. \quad (5.67)$$

The same can be done for \hat{b}_I . The general solution of eq.(5.67) is given by

$$\hat{a}_I(t) = \hat{c}_1 e^{\mu_+ t} + \hat{c}_2 e^{\mu_- t}, \quad (5.68)$$

with

$$\mu_{\pm} = \pm \frac{i}{2} \left(\sqrt{\Omega^2 + 4\lambda^2} \pm \Omega \right), \quad (5.69)$$

the amplitude operators \hat{c}_1 and \hat{c}_2 are determined by the initial conditions, $\hat{a}(0)$ and $\hat{b}(0)$:

$$\begin{aligned} \hat{c}_1 &= -\frac{[\mu_- \hat{a}(0) + i\lambda \hat{b}(0) e^{-i\phi}]}{\mu_+ - \mu_-} \\ \hat{c}_2 &= \frac{[\mu_- \hat{a}(0) + i\lambda \hat{b}(0) e^{-i\phi}]}{\mu_+ - \mu_-}. \end{aligned} \quad (5.70)$$

After some manipulation, the solution of eq.(5.67) are

$$\begin{aligned}\hat{a}_1(t) &= u_1(t)\hat{a}_1(0) + v_1(t)\hat{b}_1(0), \\ \hat{b}_1(t) &= u_2(t)\hat{b}_1(0) + v_2(t)\hat{a}_1(0),\end{aligned}\tag{5.71}$$

where

$$\begin{aligned}u_1(t) &= \frac{\mu_+ e^{\mu_- t} - \mu_- e^{\mu_+ t}}{\mu_+ - \mu_-}, \\ u_2(t) &= \frac{\mu_+ e^{\mu_+ t} - \mu_- e^{\mu_- t}}{\mu_+ - \mu_-} e^{-i\Omega t}, \\ v_1(t) &= i\lambda \frac{(e^{\mu_- t} - e^{\mu_+ t})}{\mu_+ - \mu_-} e^{-i\phi}, \\ v_2(t) &= i\lambda \frac{(e^{\mu_- t} - e^{\mu_+ t})}{\mu_+ - \mu_-} e^{-i(\Omega t - \phi)},\end{aligned}\tag{5.72}$$

or more explicitly,

$$\begin{aligned}u_1(t) &= \frac{e^{i\Omega t/2}}{\sqrt{\Omega^2 + 4\lambda^2}} \left[\sqrt{\Omega^2 + 4\lambda^2} \cos\left(\frac{\sqrt{\Omega^2 + 4\lambda^2}}{2} t\right) - i \sin\left(\frac{\sqrt{\Omega^2 + 4\lambda^2}}{2} t\right) \right], \\ u_2(t) &= \frac{e^{-i\Omega t/2}}{\sqrt{\Omega^2 + 4\lambda^2}} \left[\sqrt{\Omega^2 + 4\lambda^2} \cos\left(\frac{\sqrt{\Omega^2 + 4\lambda^2}}{2} t\right) + i \sin\left(\frac{\sqrt{\Omega^2 + 4\lambda^2}}{2} t\right) \right],\end{aligned}\tag{5.73}$$

$$\begin{aligned}v_1(t) &= -\frac{2i\lambda e^{i\Omega t/2} e^{-i\phi}}{\sqrt{\Omega^2 + 4\lambda^2}} \sin\left(\frac{\sqrt{\Omega^2 + 4\lambda^2}}{2} t\right), \\ v_2(t) &= -\frac{2i\lambda e^{-i\Omega t/2} e^{-i\phi}}{\sqrt{\Omega^2 + 4\lambda^2}} \sin\left(\frac{\sqrt{\Omega^2 + 4\lambda^2}}{2} t\right).\end{aligned}\tag{5.74}$$

Observing eq.(5.74), we see that for certain values of time the coefficients go to zero, and the operators $\hat{a}_1(t)$ and $\hat{b}_1(t)$ assume their initial values, up to a unitary phase factors. Another characteristic time of the two-mode system is the swapping time τ_{sw} when an exchange (up to a phase factor) of the two-modes operators takes place, namely

$$\hat{a}_1(t + \tau_{sw}) = \hat{b}_1(t),\tag{5.75}$$

$$\hat{b}_1(t + \tau_{sw}) = \hat{a}_1(t),\tag{5.76}$$

such that,

$$u_1(t + \tau_R) = v_2(t),\tag{5.77}$$

$$v_1(t + \tau_R) = u_2(t).\tag{5.78}$$

Therefore, for eq.(5.77) we have that,

$$\begin{aligned}e^{i\Omega(2t + \tau_R)/2} \left\{ \cos\left(\frac{\lambda' t}{2}\right) \left[\lambda' \cos\left(\frac{\lambda' t}{2}\right) - i\Omega \sin\left(\frac{\lambda' \tau_R}{2}\right) \right] - \sin\left(\frac{\lambda' t}{2}\right) \left[\cos\left(\frac{\lambda' t}{2}\right) + i\Omega \sin\left(\frac{\lambda' \tau_R}{2}\right) \right] \right\} \\ = -2i\lambda' e^{i\phi} \sin\left(\frac{\lambda' t}{2}\right),\end{aligned}\tag{5.79}$$

where $\lambda' = \sqrt{\Omega^2 + 4\lambda^2}$. eq.(5.79) is satisfied only if $\Omega = 0$, implying that $\lambda' = \lambda$. Thus, eq.(5.79) becomes

$$\cos \lambda t \cos \lambda \tau_R - \sin \lambda t \sin \lambda \tau_R = -i e^{i\phi} \sin \lambda t. \quad (5.80)$$

If we choose τ_R such that

$$\tau_R^n = \frac{(n + 1/2)\pi}{\lambda}, \quad n = 1, 2, 3, \dots \quad (5.81)$$

thus

$$e^{i\phi} = i(-1)^n = (i)^{2n+1} = e^{i\pi(2n+1)/2}, \quad (5.82)$$

if and only if

$$\phi_n = \left(n + \frac{1}{2}\right)\pi = \lambda \tau_R + \pi. \quad (5.83)$$

The same result is valid for (5.78), resulting in the identity exchange of the operators \hat{a} and \hat{b} . Therefore, we have seen that when we consider the problem of coupling by a linear interaction Hamiltonian, the dynamics followed by the joint system is one where the two-modes exchange their identities, i.e, the mode A assumes the initial value of mode B and vice-versa. The identity exchange occurs periodically, with a period given by τ_R .

CHAPTER 6

Thermodynamics and classicality

Over the last decades, it has been investigated whether quantum features could be controlled and whether quantum devices could operate with efficiencies larger than the Carnot's efficiency – therefore requiring a reformulation of Carnot's statement of the second law of thermodynamics. In this scenario, quantum engines operating between a thermal and a squeezed-thermal bath have been shown to overcome this bound (Roßnagel et al., 2014; Abah and Lutz, 2014; Niedenzu et al., 2017). Nevertheless, the use of squeezed thermal baths in quantum thermodynamics has been mostly unexplored. In this dissertation, we study a quantum heat engine based on a single bosonic mode as a working substance. The bosonic mode operates between a cold thermal bath and a hot squeezed thermal bath, and performs two different cycles. We address the problem using the concept of *P-representability*, which asserts that a given state is said to be classical (or P-representable) if it is written as a mixture of coherent states with a proper probability distribution function (Gerry et al., 2005; Lee, 1991). Conversely, a non-classical state does not have a legitimate probability distribution function over an ensemble of coherent states. Additionally, squeezed states encompass states with non-zero coherence in the energy eigenbasis, presenting an ideal platform to understand how generic quantum features affect classical formulations of the thermodynamics axioms.

We start by looking at the most straightforward Gaussian resource theory of all, namely squeezing or non-classicality. The free states of this theory, also called classical states, are merely convex mixtures of coherent states (Lami et al., 2018). Within this framework, a function called *classicality* is defined. This function is constructed based on the definition of P-representability and only depends on the temperature and the squeezing parameter. Since its negativity indicates a non-classical behavior, for higher values of temperature, the system is more classical, and a considerable amount of squeezing is needed to achieve a non-classical character. This is

also related to the uncertainty in position or momentum quadratures of the bosonic mode. A non-classical state is attained when this uncertainty falls below shot noise (Scully and Zubairy, 1997a). Afterwards, we develop a microscopic model that provides us with a master equation.

We then move to consider the implementation of a traditional Otto and a generalized cycle. The bosonic mode performs a thermodynamic four-stroke cycle. The first and the third strokes are the thermalization (isochoric) processes where the work parameter, which is the squeezing, is fixed, and the system is allowed to relax in contact with the heat baths. The second and fourth strokes, on the other hand, are the unitary (isentropic) work strokes, where the mode is squeezed and unsqueezed. In both cases, we start with a classical state. By applying a squeezing operation to the mode, a transition to a non-classical state occurs. The second cycle follows the same idea as the first. However, during the system's thermalization with the hot squeezed thermal bath, instead of keeping fixed the squeezing parameter, the classicality function describing the system is maintained constant, unlike in the traditional Otto cycle, where classicality increases in this step. A general scheme of the two protocols are sketched below (see fig.(6.1))

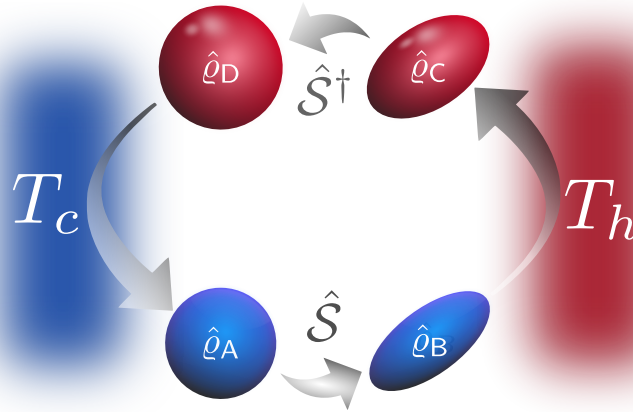


Figure 6.1: Schematic diagram of the four-step protocol in which the bosonic mode is coupled with two different reservoirs. The unitary S represents the adiabatic squeezing modulation and S^\dagger the unitary unsqueezing.

6.1 System plus reservoir approach

We address the problem assuming a weak coupling regime between the single-mode and the heat baths. The dynamics can be modelled adopting a time-independent Lindblad master equation of the form, $\dot{\hat{\rho}} = \mathcal{L}(\hat{\rho})$. The working fluid is a harmonic oscillator with natural frequency ω_0 , and creation and annihilation operators, a^\dagger and \hat{a} , respectively; the reservoir is modeled as a collection of harmonic oscillators with frequencies ω_j , and corresponding creation and annihilation operators b_j^\dagger and b_j . The oscillator couples to the j th reservoir oscillator via a coupling constant k_j in the rotating-wave approximation.

The Hamiltonian of the composite system is written as

$$\begin{aligned}\hat{\mathcal{H}}_s &\equiv \hbar\omega_0\hat{a}^\dagger a, \\ \hat{\mathcal{H}}_B &\equiv \sum_j \hbar\omega_j\hat{b}_j^\dagger\hat{b}_j, \\ \hat{\mathcal{V}} &\equiv \sum_j \hbar(k_j^*\hat{a}\hat{b}_j^\dagger + k_j\hat{a}^\dagger\hat{b}_j) = \hbar(\hat{a}\hat{\Gamma}^\dagger + \hat{a}^\dagger\hat{\Gamma}).\end{aligned}\tag{6.1}$$

6.1.1 Thermal bath

First, it is assumed that the reservoir is prepared in a thermal state, such that it is parameterized only by its temperature T . In this case, the density operator is given by

$$\hat{\rho}_R^{(i)} = \frac{e^{-\beta\hat{\mathcal{H}}_R^{(i)}}}{Z_i}\tag{6.2}$$

where Z_i is the partition function, $\hat{\mathcal{H}}_R^{(i)}$ is the Hamiltonian of each reservoir oscillator, and β is the inverse temperature. Under the *Markov-Born approximation*, the master equation for the single-modes field can be written as

$$\dot{\hat{\rho}} = \frac{\gamma}{2}(\bar{n}_{\text{th}} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}_{\text{th}}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger),\tag{6.3}$$

where γ is the damping rate of the i th mode, and

$$\bar{n}_{\text{th}}(\omega_i, T) = \text{tr}(\hat{\rho}_R\hat{b}_i^\dagger\hat{b}_i) = \frac{e^{-\hbar\omega_i/k_B T}}{1 - e^{-\hbar\omega_i/k_B T}},\tag{6.4}$$

is the mean number of thermal photons.

6.1.2 Squeezed thermal bath

Next, it is also considered the situation where the bosonic mode is coupled to a collection of harmonic oscillators, each of them prepared in a squeezed thermal state, that is,

$$\hat{\rho}_R^{(i)} = \hat{S}(\xi) \frac{e^{-\beta\hat{\mathcal{H}}_R^{(i)}}}{Z_i} \hat{S}^\dagger(\xi),\tag{6.5}$$

where \hat{S} is the squeezing operator (4.86).

The correlation function of the squeezed thermal reservoir is given by

$$\begin{aligned}\langle \hat{b}_i^\dagger(\omega)\hat{b}_i^\dagger(\omega') \rangle &= 2\pi m(\omega)\delta(2\omega' - \omega - \omega_0), \\ \langle \hat{b}_i^\dagger(\omega)\hat{b}(\omega') \rangle &= 2\pi\bar{n}_{\text{sth}}(\omega, T)\delta(\omega - \omega_0), \\ \langle \hat{b}(\omega)\hat{b}_i^\dagger(\omega_0) \rangle &= 2\pi[\bar{n}_{\text{sth}}(\omega, T) + 1]\delta(\omega - \omega_0),\end{aligned}\tag{6.6}$$

where $\bar{n}_{\text{sth}}(\omega, T)$ is the mean photon number of the i th squeezed reservoir, and $m(\omega)$ is the coherence parameter related to the phase correlations of the i th squeezed reservoir. They are related to the squeezing parameter, r_i ,

by $\bar{n}_{\text{sth}}(\omega, T) = (\bar{n}_{\text{th}} + 1/2) \cosh 2r_i$ and $m(\omega) = (\bar{n}_{\text{th}} + 1/2) \sinh 2r_i$. In this case, the evolution of the density operator in the interaction picture is described by the master equation

$$\begin{aligned} \dot{\hat{\rho}} = & \frac{\gamma}{2} (\bar{n}_{\text{sth}} + 1) (2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2} \bar{n}_{\text{sth}} (2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{\gamma}{2} m (2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}^\dagger) + \frac{\gamma}{2} m^* (2\hat{a}\hat{\rho}\hat{a} - \hat{a}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}). \end{aligned} \quad (6.7)$$

In chapter 5 we obtained the solution of the master equation (6.7) by applying the formalism developed in chapter 4. Summarizing:

Diffusive master equation. The dynamics of a single-mode system subject to a linearly coupled Markovian environment of n modes (6.7) has the following steady-state state solution:

$$\hat{\rho} = \frac{e^{\beta_s(\hat{\mathcal{H}}_R - \mu\hat{m}_R)}}{Z_{\text{sq}}} \quad , \quad \text{if the bath is squeezed,} \quad (6.8)$$

where $Z_{\text{sq}} = \text{tr}[e^{\beta_s(\hat{\mathcal{H}}_R - \mu\hat{m}_R)}]$. For the case of a thermal bath, i.e $m = 0$, we have

$$\hat{\rho} = \frac{e^{\beta\hat{\mathcal{H}}_R}}{Z_{\text{th}}} \quad , \quad Z_{\text{th}} = \text{tr}(e^{-\beta_0\hat{\mathcal{H}}}). \quad (6.9)$$

In eq.(6.8) we defined the a generalized inverse temperature β_s , and a chemical-like potential of squeezing:

$$\beta_s \equiv \beta \cosh 2r \quad , \quad \mu \equiv \tanh 2r. \quad (6.10)$$

where β is the inverse temperature. Also, the *coherence operator* was defined as

$$\hat{m}_R \equiv -\frac{\hbar\omega}{2} (\hat{b}^{\dagger 2} + \hat{b}^2). \quad (6.11)$$

6.2 Classicality and coherence

Our primary motivation for introducing the classicality function is to define a measure of how non-classical quantum states are. With this in hand, we may ask how the degree of non-classicality influences thermodynamic quantities, such as heat, work, and efficiency.

It is well known in the quantum optics community that the origin of non-classical effects relies on the P-function be non-positive or highly singular (Scully and Zubairy, 1997b; Gerry et al., 2005). However, we have shown in chapter 4 that evaluating this function is not an easy task, and sometimes we end up with high order derivatives of the delta function. Nevertheless, for a Gaussian state, we have a precise condition to determine whether a given state is P-representable or not. The P-representability condition (4.189) is stated in terms of the parameters \bar{n} and m - covariance matrix elements of the system (see eq.(4.175)). When $\bar{n} > m$, the system

can be written as a mixture of coherent states with a proper probability distribution function. Otherwise, the system is said to be non-classical.

Following this definition, but in a slightly different way, for squeezed states, the *classicality function*, \mathcal{C} , is defined as the difference between \bar{n}_{sth} and m ,

$$\begin{aligned}\mathcal{C} &\equiv \bar{n}_{\text{sth}} - m = \left(\bar{n}_{\text{sth}} + \frac{1}{2} \right) - \left(m + \frac{1}{2} \right) \\ &= \frac{1}{2} \coth\left(\frac{\hbar\omega}{2k_B T} \right) e^{-2r} - \frac{1}{2},\end{aligned}\tag{6.12}$$

From eq.(6.12), we observe that the classicality only depends on the temperature and the squeezing parameter. Its negativity indicates the non-classical behavior, which means that for higher values of temperature, more classical is the system, and a more considerable value of r is needed to achieve the non-classical character (see fig.6.2). Therefore, by defining the classicality function, it is possible to distinguish whether a given state is

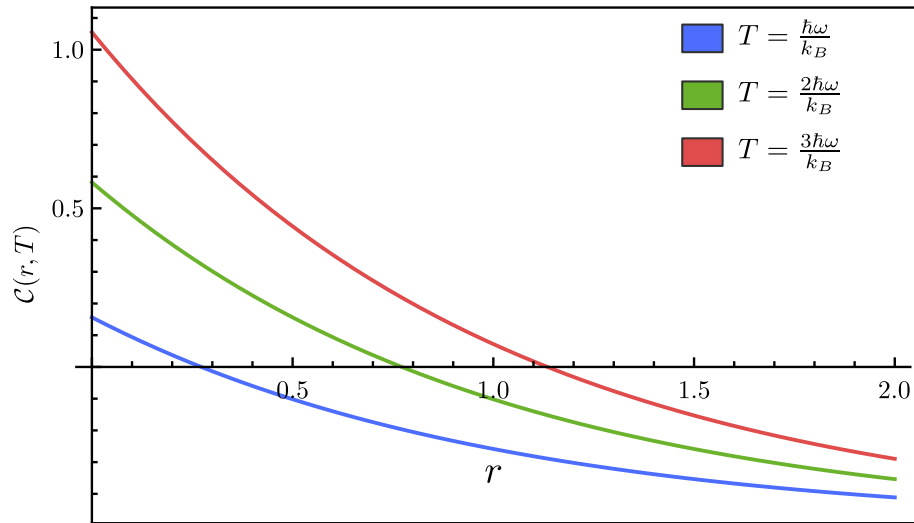


Figure 6.2: Classicality function (6.12) as a function of the squeezing parameter for three different temperature. There is a competition between T and r . For higher values of temperature, a more significant amount of squeezing is required to change the state of the system from classical to non-classical.

classical or non-classical by its signal.

6.2.1 Coherence in squeezed states

In section (4.8), we termed the non-diagonal elements m of the covariance matrix by coherence parameter. Besides, we discussed in chapter 3, that the relative entropy of coherence could be used to quantify the amount of coherence in a given basis. Therefore, let us estimate how much coherence squeezed states have in the energy eigenbasis

$$C(\hat{\rho}) = S(\hat{\rho}_{\text{diag}}) - S(\hat{\rho}).\tag{6.13}$$

Then, considering that $\hat{\rho}$ is described by a squeezed thermal state (6.9) and determining the von-Neumann entropy,

$$\begin{aligned} S(\hat{\rho}) &= -\text{tr}(\hat{\rho} \ln \hat{\rho}) = \text{tr} [\hat{\rho} \beta_s (\hat{\mathcal{H}} - \mu \hat{m}) - \beta \log Z_{\text{sq}}] \\ &\approx \beta_s (E - m), \end{aligned} \quad (6.14)$$

where $E = \text{tr}(\hat{\rho} \hat{\mathcal{H}})$ and $m = \text{tr}(\hat{\rho} \hat{m})$. Note that operator \hat{m} in eq.(6.8) is the term responsible for adding the non-diagonal elements in the energy eigenbasis. Consequently, we have that

$$S(\hat{\rho}_{\text{diag}}) \approx \beta_s E, \quad (6.15)$$

and therefore,

$$C(\hat{\rho}) = \beta_s m + \mathcal{O} \approx \beta_s m, \quad (6.16)$$

where the term \mathcal{O} comes from the contribution of the partition function.

The result (6.16) indicates squeezed states encompass states with nonzero coherence in the energy eigenbasis, and the amount of coherence is proportional to the parameter m , that is defined by the eq.(4.193). Also, the amount of coherence (6.16) is proportional to the squeezing parameter but inversely proportional to the temperature.

6.3 Quasi-static condition, work and heat

Before we tackle the proposed problem in this dissertation, let us establish a necessary concept first. Consider a quantum system with Hamiltonian $\hat{\mathcal{H}}(\lambda)$, where λ is an external control parameter. Then the dynamics of the system are governed by a Liouville type equation, $\hat{\rho} = \mathcal{L}(\hat{\rho})$, where the super-operator \mathcal{L} reflects both the unitary dynamics generated by $\hat{\mathcal{H}}$ and the non-unitary contribution induced by the interaction with the environment. Also, we further assume that the equation for the steady-state, $\mathcal{L}(\hat{\rho}) = 0$, has a unique solution and that both cycles are formulated for quasi-static processes. Hence, for the system plus reservoir approach, we will take the state when $t \rightarrow \infty$.

Upon moving from the macroscopic description to the microscopic quantum realm, a clear picture of what is work and heat becomes blurred, and it is not easy to distinguish what is the contribution of each quantity to the internal energy. However, in analogy with classical thermodynamics, we will define work as a controllable (coherent) energy exchange related to a parameter of the system, and heat as an (incoherent) flux of energy that cannot be controlled (or be useful for some process). Essentially, the definition says that work is related to a change on the system's Hamiltonian (Alicki, 1979),

$$\delta W = \text{tr}[(\delta H) \hat{\rho}], \quad (6.17)$$

while heat is an incoherent energy change determined by exclusion, that is

$$\delta Q = dU - \delta W. \quad (6.18)$$

We are interested in a single bosonic mode prepared in a squeezed thermal state. Its mean energy is given by (4.99),

$$E(r, \bar{n}_{\text{th}}) = \hbar\omega \left(\bar{n}_{\text{th}} + \frac{1}{2} \right) \cosh 2r, \quad (6.19)$$

differentiating this equation results in

$$\begin{aligned} dE &= \left(\frac{\partial E}{\partial r} \right)_{\bar{n}_{\text{th}}} dr + \left(\frac{\partial E}{\partial \bar{n}_{\text{th}}} \right)_r d\bar{n}_{\text{th}} \\ &= \left[2\hbar\omega \left(\bar{n}_{\text{th}} + \frac{1}{2} \right) \sinh 2r \right] dr + (\hbar\omega \cosh 2r) d\bar{n}_{\text{th}}, \end{aligned} \quad (6.20)$$

where we identified the first term as work and the second as heat:

$$\begin{aligned} \delta W &= -2\hbar\omega \left(\bar{n}_{\text{th}} + \frac{1}{2} \right) \sinh 2r dr, \\ \delta Q &= \hbar\omega \cosh 2r d\bar{n}_{\text{th}}. \end{aligned} \quad (6.21)$$

The minus signal in the work expression is adopted because, during the preparation of the squeezed state, external energy is needed to perform it. We must pay an energetic price to create a squeezing state. While the work performed to unsqueeze the mode is positive. In other words, the work performed on the system is negative, and the work performed by the system is positive.

6.4 Quantum Otto cycle

As a first application of the theoretical framework derived in the past chapters, it is studied a cycle inspired in the classical Otto cycle (Section.(2.3.2)). To understand the role of non-classicality in quantum thermodynamics, we first analyze a simple protocol that consists of giving an amount of non-classicality (coherence) to a single-mode and use it to operate a thermodynamical cycle. In each stroke, quantities as work and heat will be computed, and then the thermodynamic efficiency is determined.

The quantum Otto cycle consists of four consecutive strokes, namely, two isochoric processes and two isentropic processes, as shown in fig.(6.4). During the isentropic stroke, the squeezing parameter of the harmonic oscillator is modulated between 0 and r by an external agent. Heating and cooling result from the coupling to two heat baths at temperatures $T_1 = \hbar\omega/k_B$ and $T_2 = 2\hbar\omega/k_B$. We denote the state of the working fluid performing the cycle at the fourth points as $\hat{\rho}_A$, $\hat{\rho}_B$, $\hat{\rho}_C$, and $\hat{\rho}_D$, respectively. Also, the thermal distribution is indicated by \bar{n}_{T_i} where $i = 1, 2$.

The cycle starts with the single bosonic mode at point A, in equilibrium with the cold thermal reservoir at $T_1 = \hbar\omega/k_B$,

$$\hat{\rho}_A = \frac{\exp(-\beta_1 \hat{\mathcal{H}}_1)}{Z_A} \quad \text{with} \quad Z_A = \text{Tr}\{e^{-\beta_1 \hat{\mathcal{H}}_1}\}, \quad (6.22)$$

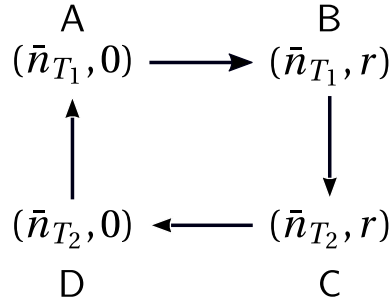


Figure 6.3: Schematic representation of the thermodynamic cycle obtained by varying the squeezing parameter r and the temperature of the working fluid. Each one of the strokes are obtained by keeping one of the variables constant while quasi-statically varying the other one.

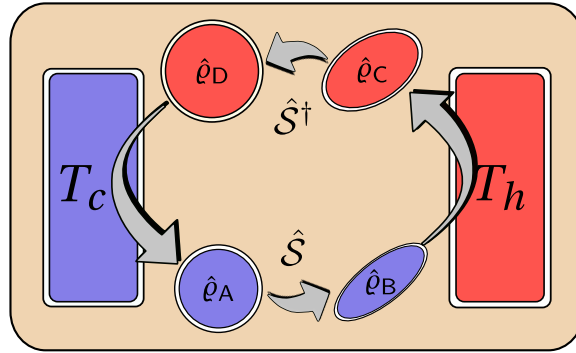


Figure 6.4: Diagram of the thermodynamic cycle. In each stroke, one of the variables is kept constant while the other one is changing.

where the initial Hamiltonian is given by $\hat{\mathcal{H}}_1 = \hbar\omega(\hat{a}_1^\dagger\hat{a}_1 + 1/2)$, Z_A is the partition function, and $\beta_1 = 1/k_B T_1$. The mean energy at point A is simply

$$\langle \hat{\mathcal{H}} \rangle_A = \hbar\omega \left(\bar{n}_{T_1} + \frac{1}{2} \right), \quad (6.23)$$

During the first stroke, the working fluid is decoupled from the reservoirs, and a unitary squeezing operation is applied to the system. The density operator at point B is $\hat{\rho}_B = \hat{S}_1 \hat{\rho}_A \hat{S}_1^\dagger$. The work done on the single-mode in the isentropic compression is negative, yielding

$$\begin{aligned}
 W_{AB} &= \langle \hat{\mathcal{H}}_A \rangle - \langle \hat{\mathcal{H}}_B \rangle = \hbar\omega \left(\bar{n}_{T_1} + \frac{1}{2} \right) - \hbar\omega \left(\bar{n}_{T_1} + \frac{1}{2} \right) \cosh 2r \\
 &= -2\hbar\omega \left(\bar{n}_{T_1} + \frac{1}{2} \right) \sinh^2 r,
 \end{aligned} \quad (6.24)$$

In the second stroke, the bosonic mode is put into contact with the squeezed thermal bath, at $T_2 = 2\hbar\omega/k_B$, while the squeezing parameter stays constant, resulting in an ‘‘isochoric’’ process where the mode relaxes to

the steady-state $\hat{\rho}_C$. The heat entering the system from the squeezed thermal bath in the relaxation is

$$\begin{aligned} Q_A &= \langle \hat{\mathcal{H}}_C \rangle - \langle \hat{\mathcal{H}}_B \rangle = \hbar\omega \left(\bar{n}_{T_2} + \frac{1}{2} \right) \cosh 2r - \hbar\omega \left(\bar{n}_{T_1} + \frac{1}{2} \right) \cosh 2r \\ &= \hbar\omega (\bar{n}_{T_2} - \bar{n}_{T_1}) \cosh 2r. \end{aligned} \quad (6.25)$$

During the third stroke, the working fluid is again detached from the reservoirs, and a unitary unsqueezing operation is applied to the mode, S^\dagger , bringing its squeezing parameter back to 0. Therefore, the system is taken to the state at point D, and then $\hat{\rho}_D = \exp(-\beta_2 \hat{\mathcal{H}}_2) / Z_D$ with $Z_D = \text{Tr}\{e^{-\beta_2 \hat{\mathcal{H}}_2}\}$. Consequently, the work extracted is given by

$$W_{CD} = 2\hbar\omega \left(\bar{n}_{T_2} + \frac{1}{2} \right) \sinh^2 r, \quad (6.26)$$

The cycle is closed by bringing the bosonic mode into contact with the cold thermal bath at temperature T_1 and hence relaxing back to $\hat{\rho}_A$. During the last isochoric process, the heat transferred from the cold reservoir to the system is

$$Q_R = \langle \hat{\mathcal{H}} \rangle_A - \langle \hat{\mathcal{H}} \rangle_D = \hbar\omega (\bar{n}_{T_1} - \bar{n}_{T_2}). \quad (6.27)$$

The work over the cycle is given by the contributions of the two isentropic strokes, as

$$W_{\text{cycle}} = 2\hbar\omega (\bar{n}_{T_2} - \bar{n}_{T_1}) \sinh^2 r. \quad (6.28)$$

Also, the efficiency of the engine, defined as the ratio of the work over the cycle to the energy absorbed from the hot reservoir, can be obtained from the previous results, eqs.(6.25) and (6.28),

$$\eta_o = 1 - \frac{1}{\cosh 2r}. \quad (6.29)$$

In fig. (6.5), we notice the efficiency increases with the squeezing parameter. For higher values of squeezing, the efficiency will approach the unity, but never surpasses it. It is interesting to mention that, as in the classical case, the thermodynamic efficiency does not depend explicitly on the temperature of the reservoirs. The reason is in the way that the protocol was designed: the cycle consists of modulating the squeezing parameter, while the system is operating between two different temperatures. During the interaction systems plus reservoir, the squeezing parameter stays constant. When the systems interact with the hot squeezed bath, the squeezing parameter is fixed at r . Thus the absorbed heat is a function of r . However, when the system is interacting with the cold thermal bath, the squeezing parameter is fixed at zero - the system must return to its initial state. Because the engine is operated between two temperatures, these quantities are cancelled, resulting in a thermodynamic efficiency that solely depend on the squeezing parameter.

The behaviour of the classicality function over the cycle is plotted in fig.(6.6). Observe that, in the first stroke (A-B), a non-classicality is added to the single-mode by the action of the squeezing operator. In other words, we took a classical state, without any coherence, and end up with a non-classical state which contains

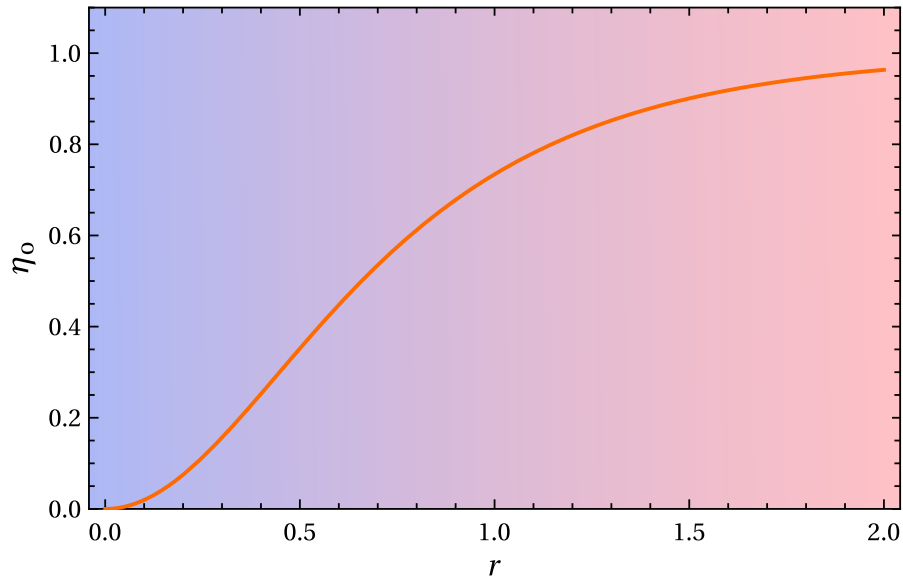


Figure 6.5: The efficiency η_o of the heat engine as a function of the squeezing parameter r . The efficiency does not depend on temperatures, but only of the squeezing parameter. The difference in the color indicates the degree of non-classicality.

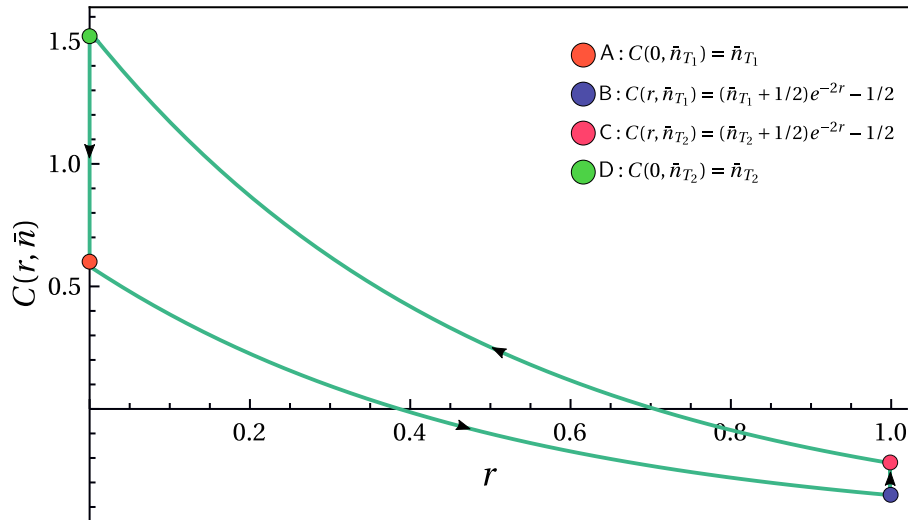


Figure 6.6: Classicality function (6.12) over the cycle as a function of the squeezing parameter for $T_1 = \hbar\omega/k_B$ and $T_2 = 2\hbar\omega/k_B$.

coherence in the energy eigenbasis. In the next stroke (B-C), the squeezing parameter stays constant, while the temperature of the bosonic mode increases, which means an increase of the classicality function. We may interpret this result as if the bath has consumed part of the coherence added in the first stroke. The last strokes (C-D) and (D-A) correspond to the unsqueezing and the lowering of the temperature of the bosonic mode. That is, the classicality function increases until its maximum, and goes to its initial value. From fig. (6.5), we see that non-classicality is a resource since the efficiency increases with the degree of squeezing. However, only a part

of the non-classicality is used over the cycle. The interaction between the single-mode and the reservoir causes an increase in the system's classicality, meaning that the bath consume part of the coherence that is added to the system.

6.5 Generalized cycle

In order to explore non-classicality in our favour, we present a cycle where the classicality function stays constants when the single-mode interacts with the squeezed thermal bath. The idea is to preserve the non-classicality added and use it for the work extraction.

As a second application, we consider the same setup: a quantum heat engine operating between a cold thermal bath at temperature $T_1 = \hbar\omega/k_B$, and a hot squeezed thermal reservoir at $T_2 = \hbar\omega/k_B$ with squeezing parameter r_R . The main difference between the two protocols is that we want to preserve the non-classical character of the system during its interaction with the hot reservoir, this implies that the squeezed thermal bath must be prepared with a different squeezing parameter than the system after the first step. Because, the system will absorb heat from bath and its temperature will increase, meaning an increase in its classicality. Consequently, to compensate for this increase, the system must have its squeezing parameter modulated from r_t to r_R , where r_t is the squeezing parameter related to the first stroke. The condition that must be fulfilled is:

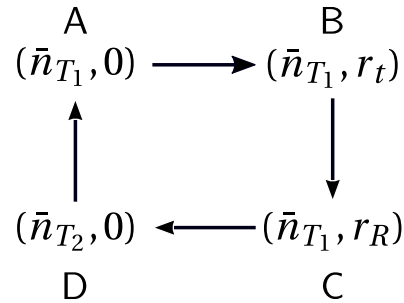


Figure 6.7: A sketch of the thermodynamic cycle obtained by varying the squeezing parameter r and the temperature of the working fluid while keeping constant its classicality.

$$\mathcal{C}(r_t, \bar{n}_{T_1}) = \mathcal{C}(r_R, \bar{n}_{T_2}), \quad (6.30)$$

where r_t is the squeezing parameter provided by the unitary transformation in the first stroke, and r_R is the squeezing parameter of the hot squeezed bath.

The cycle starts with the working fluid at point A, and the procedure taken in the first stroke is the same as in the Otto cycle - the single-mode has its squeezing parameter modulated from 0 to r_t . In the second stroke, the bosonic mode is put into contact with the squeezed thermal bath. In order to keep the classicality constant, the stationary state of the system is slowly varied, such that the condition (6.30) is satisfied. According to the

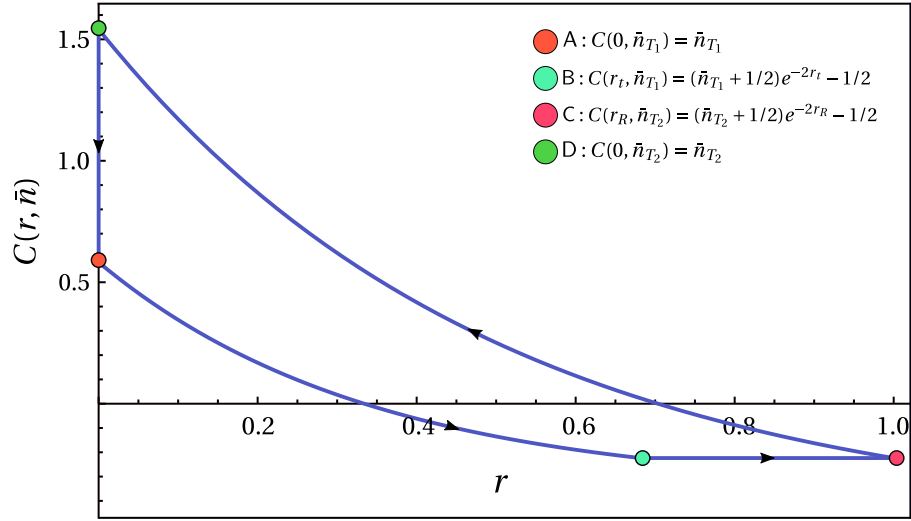


Figure 6.8: Classicality function (6.12) over the cycle as a function of the squeezing parameter for $T_1 = \hbar\omega/k_B$ and $T_2 = 2\hbar\omega/k_B$. The squeezing parameter of the transformation is such that $r_t > r_{c_2}$.

definition given in Sec.(6.3 - eqs.(6.21)), during the second stroke, work and heat can be obtained as,

$$W_{BC} = 2\hbar\omega \int_{r_t}^{r_R} \left(\bar{n}_T + \frac{1}{2} \right) \sinh 2r \, dr \quad , \quad Q_{BC} = \hbar\omega \int_{r_t}^{r_R} \cosh 2r \, d\bar{n}_T, \quad (6.31)$$

Therefore, to satisfy the condition (6.30), work is performed on the system during its interaction with the squeezed thermal bath; at the same time, heat is exchanged, so

$$\begin{aligned} W_{BC} &= \frac{\hbar\omega}{4} \left(\bar{n}_{T_2} + \frac{1}{2} \right) [e^{2r_R} + 4(r_T - r_R)e^{-2r_R}] - \frac{\hbar\omega}{4} \left(\bar{n}_{T_1} + \frac{1}{2} \right) e^{2r_T}, \\ Q_{BC} &= -\hbar\omega \left(\bar{n}_{T_2} + \frac{1}{2} \right) e^{-2r_R}(r_T - r_R) + \frac{\hbar\omega}{4} \left(\bar{n}_{T_2} + \frac{1}{2} \right) e^{2r_R} - \frac{\hbar\omega}{4} \left(\bar{n}_{T_1} + \frac{1}{2} \right) e^{2r_T}. \end{aligned} \quad (6.32)$$

The relation between r_T and r_R , can be obtained by using the condition (6.30),

$$r_R = r_T + \frac{1}{2} \ln \left(\frac{\bar{n}_{T_2} + \frac{1}{2}}{\bar{n}_{T_1} + \frac{1}{2}} \right). \quad (6.33)$$

In the third stroke, the system is again detached from the reservoirs, and an unsqueezing operation, \hat{S}^\dagger , is applied to the mode, changing its squeezing parameter adiabatically back to 0. The cycle is closed by putting the bosonic mode in contact with the cold thermal reservoir and relaxing back to $\hat{\rho}_A$.

Once the work and the heat are determined in each stroke, we may calculate the efficiency of the cycle. Different from the first protocol, the efficiency now depends on the temperature and does not have a straightforward expression, as in eq.(6.29)

$$\eta_G = 1 - \frac{4(1+e)e^{2r_T + \frac{1}{2}}}{2e^{4r_T + 1} + 2e^{4r_T + \frac{1}{2}} + 2e^{4r_T + \frac{3}{2}} + e^2 \log \left(\frac{(\sqrt{e}+1)^2}{1+e} \right) + \log \left(\frac{(\sqrt{e}+1)^2}{1+e} \right) - 2e \log(1+e) + 4e \log(\sqrt{e}+1)}. \quad (6.34)$$

Figure 6.9 shows the engine's efficiency as a function of the squeezing parameter for both protocols. The efficiency of the generalized cycle is always higher than the Otto cycle, because now the classicality is kept constant along the third stroke. Consequently, work is performed on the system to fulfill the established condition (6.30), which means a higher extraction of work on the third stroke, in comparison with the Otto cycle. It is important to stress that in the first treated cycle, the bath partially consumed the non-classicality introduced in the first stroke. Here, we prevent the dissipation of non-classicality by imposing restrictions to keep it.

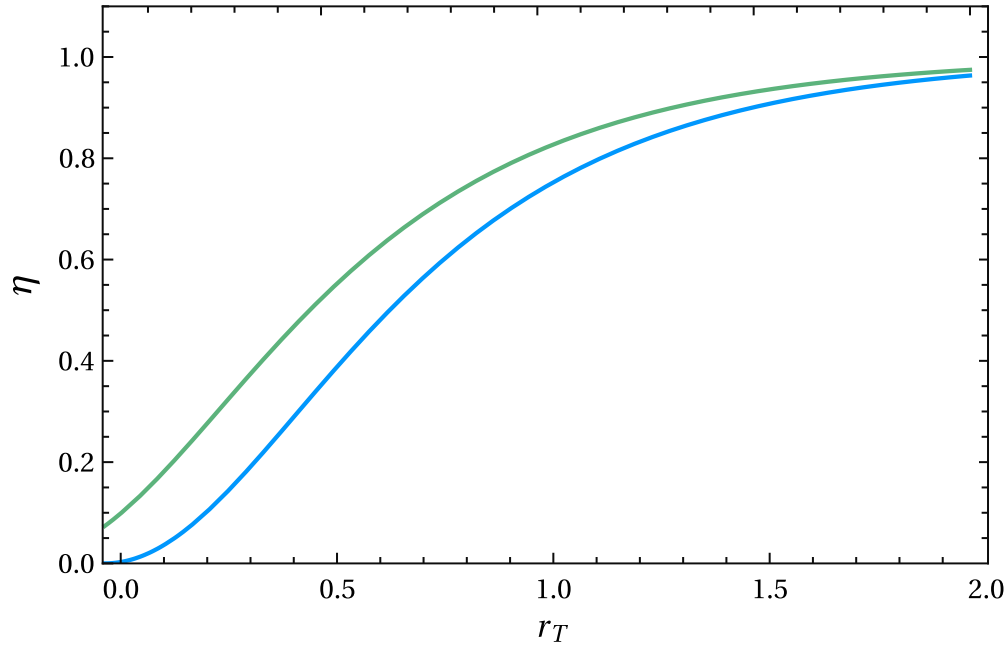


Figure 6.9: Efficiency of the heat engine as a function of the squeezing parameter of the transformation r_T for protocol I: the Otto cycle (blue line), and protocol II: the generalized cycle (green line), with bath temperatures equal to $T_1 = \hbar\omega/k_B$ and $T_2 = 2\hbar\omega/k_B$.

As we notice in section 6.2, the non-classical character of the system depends on its temperature and squeezing parameter. For higher values of temperature, higher is the required value of the critical squeezing parameter, to take the system from a classical state to a non-classical one. Consequently, for a single-mode operating between two different temperatures, it is related two different critical squeezing parameters, denoted by r_{c_1} and r_{c_2} . This suggests that both protocols can be investigated in three different situations. That is, when the state of the single-mode is:

- i classical for both temperatures
- ii non-classical for T_1 , but classical for T_2
- iii non-classical for both temperatures.

These cycles present different regimes of operation depending on the squeezing r and temperature T , as is summarized in the phase diagram of fig.(6.10).

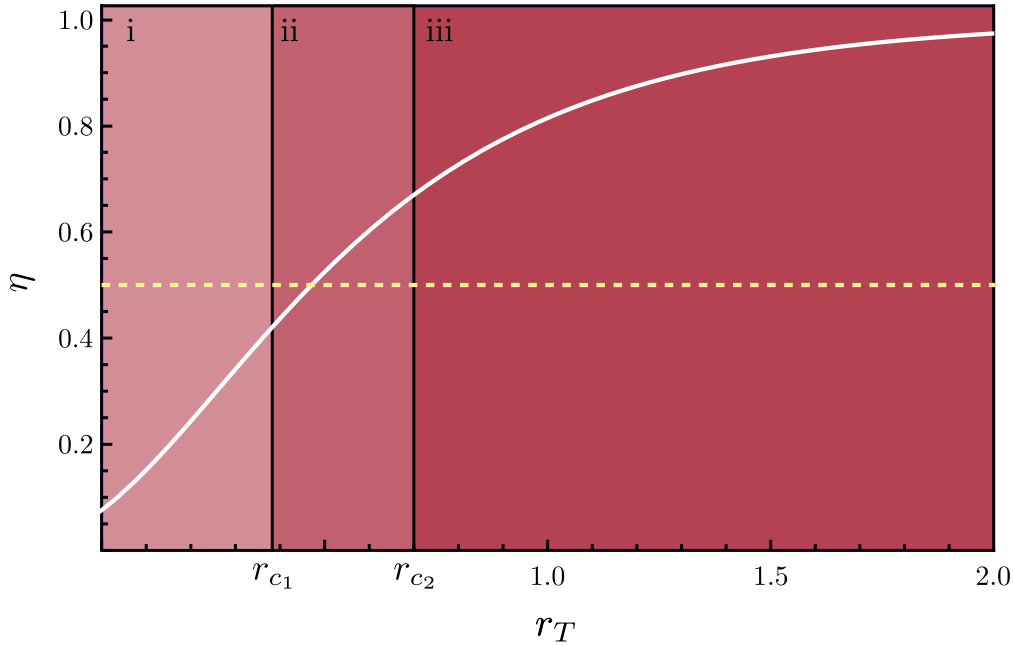


Figure 6.10: The thermodynamic efficiency of the generalized (in white) and Carnot cycle as a function of the squeezing parameter. Also illustrated, a phase diagram with the three regimes of the cycle (i, ii, and iii). The color scale corresponds to the degree of non-classicality.

Region i corresponds to the case where the state of the system is characterized as classical. Although there exists squeezing, the condition (4.84) is not satisfied due to the temperature of the system, that compensates for the bosonic mode quadratures to be higher than the shot noise. This operation can be seen as modulation of the single-mode frequency from ω_1 to ω_2 , where $\omega_2 > \omega_1$.

Region ii corresponds to the case where the state of the system is characterized as non-classical for T_1 , but classical for T_2 . That is, the bath consumed the non-classicality added to the system during the first step. Consequently, the classicality function of the single-mode undergoes two transitions over the cycle: classical to non-classical (first stroke) and non-classical to classical (second stroke) due to the temperature increasing.

Finally, in the *region iii*, the most efficient of all three, where the state of the system is non-classical for both temperatures, and it is possible to verify that the non-classicality is a resource for quantum heat engines. Just for comparison, the classical Carnot efficiency is also plotted. It is observed that the Carnot's efficiency is surpassed in region *ii*, where the working fluid starts the cycle with the non-classical character. We may understand that the coherence added is used to perform the cycle more efficiently, and helped to beat Carnot's limit. However, this comparison is not totally appropriate since the problem in consideration is formulated

in a different context from which the Carnot's cycle was proposed. For comparison with the first protocol, it is also plotted the relative efficiency, defined by the ratio of the efficiency difference by the efficiency of the first protocol (the ‘‘Otto cycle’’). As we can see in fig.(6.11), with the increase in the squeezing parameter, the efficiencies get closer to each other. This is because the absorbed heat increases with the squeezing parameter, while the rejected heat does not. Therefore, for higher values of squeezing ($r_T \gg r_{c_2}$), the difference between the two protocols becomes tight. In other words, the degree of non-classicality is so high that the type of cycle does not affect the performance of the quantum heat engine.

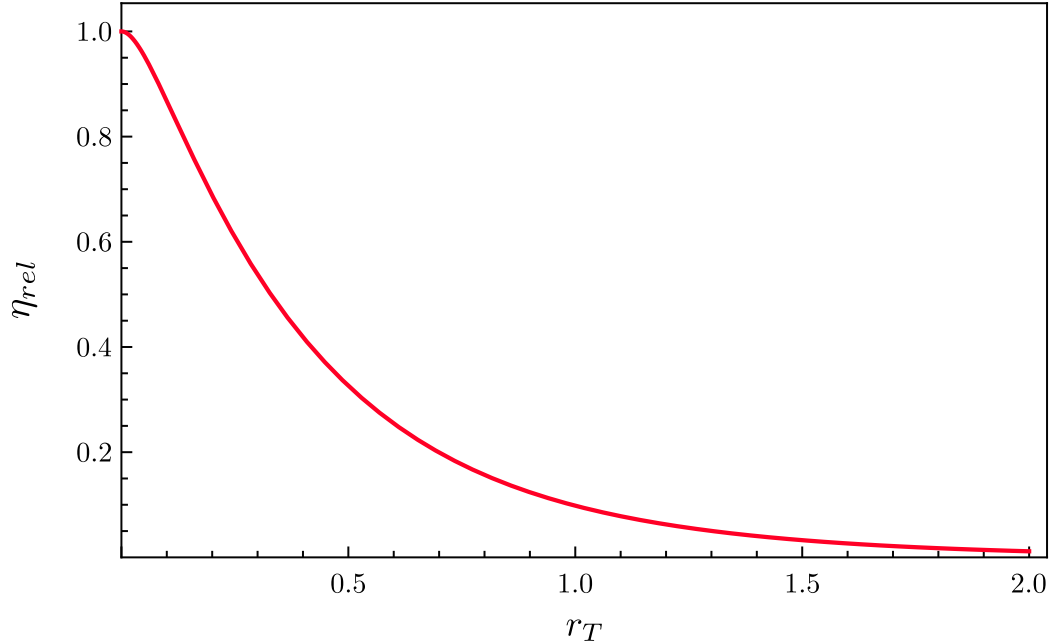


Figure 6.11: The relative efficiency: $\eta_R = \frac{\eta_G - \eta_o}{\eta_G}$, where η_G is the efficiency of the generalized cycle; The reservoirs temperature are: $T_1 = \hbar\omega/k_B$ and $T_2 = 2\hbar\omega/k_B$.

6.6 Work extraction

For both cycles, the two initial strokes are essential, where the working fluid is prepared in a squeezed thermal state by varying its squeezing parameter, and when it is put into contact with the squeezed thermal reservoir. In the first stroke, there is a price to pay to create a squeezed state. The energy involved in this process is counted as work since we are interacting externally with the system in a controlled way. Conversely, in the second stroke, the system interacts with the resonant modes of the reservoir through an interaction Hamiltonian $\hat{\mathcal{V}}$. In terms of energy exchange, this interaction is uncontrollable. Although in some cases, the interaction Hamiltonian $\hat{\mathcal{V}}$ may also depend on a controllable parameter λ , this is not our case.

It is in the third stroke, that part of the invested energy is converted into work, so it can be seen as the process where the work extraction occurs. This step can alternatively be done by coupling the bosonic mode to

an auxiliary system, an ancilla, via a linear interaction Hamiltonian \hat{V} . The only condition is that the auxiliary system is prepared in a thermal state with a temperature T_2 . In chapter 5, we showed that when two modes interact via a linear interaction Hamiltonian, their identities (states) are exchanged for some multiple values of a time, τ_R , called swapping time. In the course of the swapping time, the work produced by the engine can be delivered to the auxiliary system. This interaction mimics the application of a unitary unsqueezing operation on the bosonic mode. Therefore, if the interaction between the two modes is cut in the first swapping time, the system will end up in a thermal state $\hat{\rho}_D = \exp(-\beta_2 \hat{\mathcal{H}}_2) / Z_D$. The motivation to implement this model is because we could avoid another interaction with the system during the third step. The only disadvantage is that this analysis characterizes a single-shot thermodynamic operation. For each round, a new ancilla is required. However, the implementation where the system interacts sequentially with a series of independent ancillae, all identically prepared in a thermal state $\hat{\rho}_D$ is a way to solve this issue and is called as a *collisional model* (Rodrigues et al., 2019).

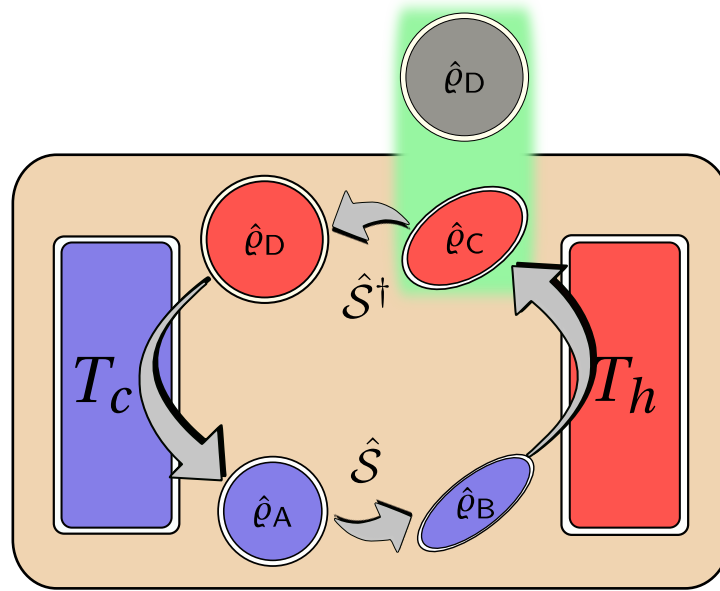


Figure 6.12: Schematic diagram of the four-stroke cycle operating between reservoirs at different temperatures. The ancilla (in grey) extracts work from the cycle, and represents the unsqueezing operation on the single mode.

6.7 Free energy and chemical-like potential of squeezing

To finish this chapter, we shall establish the concept of *nonequilibrium free energy* that was introduced by Esposito and den Broeck (2011) to investigate how coherence is linked to thermodynamical work. In chapter 3, it was discussed that the von-Neumann entropy coincides with the thermodynamic entropy when a thermal state describes the system. In this case, the thermodynamic relation between the Helmholtz free energy,

internal energy and entropy is recovered:

$$F = E - TS. \quad (6.35)$$

One can go further and define a nonequilibrium free energy (\mathcal{F}) for the system, with state $\hat{\rho}$, in contact with a thermal bath at temperature T and described by the Hamiltonian $\hat{\mathcal{H}}$

$$\mathcal{F}(\hat{\rho}, \hat{\mathcal{H}}) \equiv \langle \hat{\mathcal{H}} \rangle_{\hat{\rho}} - TS(\hat{\rho}). \quad (6.36)$$

It can be shown that the von-Neumann entropy and the associated nonequilibrium free energy are similar to their equilibrium counterparts in nonequilibrium isothermal processes. Here, isothermal means that the system is in contact with a thermal reservoir at temperature T , although the system itself may not have a well-defined temperature. Additionally, [Esposito et al. \(2009\)](#) showed that the nonequilibrium free energy characterizes the optimal amount of work extractable from a system with Hamiltonian $\hat{\mathcal{H}}$, in a state $\hat{\rho}$, with the help of a thermal reservoir at temperature T . Besides that, this quantity is limited by the Helmholtz free energy F , such that $\mathcal{F}(\hat{\rho}, \hat{\mathcal{H}}) \geq F(\hat{\rho})$

Let us use this idea to analyze the relationship between the coherence and the extracted work. For a squeezed thermal state, the density operator is given by

$$\hat{\rho}_{\text{sth}} = \frac{e^{\beta_s(\hat{\mathcal{H}}_R - \mu\hat{m}_R)}}{Z_{\text{sq}}}. \quad (6.37)$$

Considering an infinitesimal change in the state of the system, $\hat{\rho}' = \hat{\rho} + \epsilon\delta\hat{\rho}$, where for simplicity $\delta\hat{\rho}$ is a traceless operator accounting for this change, and $\epsilon \ll 1 \in \mathbb{R}^+$, it follows that up to first order in ϵ , the entropy variation is given by (see Appendix A.6)

$$\Delta S = S(\hat{\rho}') - S(\hat{\rho}) = \beta(\Delta E - \hbar\omega\mu\Delta m), \quad (6.38)$$

where $S(\hat{\rho})$ is the von-Neumann entropy, $\Delta E = \text{etr}(\hat{\mathcal{H}}\delta\hat{\rho})$ and $\Delta m = \text{etr}(m\delta\hat{\rho})$ are the corresponding changes in the internal energy and coherence parameter. Therefore, using eqs.(6.36-6.37), the infinitesimal change in the nonequilibrium free energy is

$$\Delta\mathcal{F} = \Delta E - k_B T \Delta S = \hbar\omega\mu\Delta m. \quad (6.39)$$

Remarkably, eq.(6.39) leads to the interpretation that the maximum work that can be extracted from the nonequilibrium state $\hat{\rho}_{\text{sth}}$ is directly proportional do the coherence parameter m . Now, looking closely to the definitions made in section 6.3, it was derived that,

$$\begin{aligned} dE &= \left(\frac{\partial E}{\partial r}\right)_{\bar{n}_{\text{th}}} dr + \left(\frac{\partial E}{\partial \bar{n}_{\text{th}}}\right)_r d\bar{n}_{\text{th}} \\ &= \left[2\hbar\omega\left(\bar{n}_{\text{th}} + \frac{1}{2}\right)\sinh 2r\right] dr + (\hbar\omega\cosh 2r)d\bar{n}_{\text{th}}. \end{aligned} \quad (6.40)$$

The coherence parameter can be written as

$$m = \left(\bar{n}_{\text{th}} + \frac{1}{2}\right)\sinh 2r. \quad (6.41)$$

Differentiating eq.(6.41),

$$dm = \left[2 \left(\bar{n}_{\text{th}} + \frac{1}{2} \right) \cosh 2r \right] dr + (\sinh 2r) d\bar{n}_{\text{th}}, \quad (6.42)$$

solving for dr and inserting the result in eq.(6.40), it is obtained

$$dE = \hbar\omega\mu dm + \frac{\hbar\omega}{\cosh 2r} d\bar{n}_{\text{th}} \quad (6.43)$$

$$= \delta W + \delta Q, \quad (6.44)$$

Therefore, for an isothermal process, $d\bar{n}_{\text{th}} = 0$, it is obtained that the work is directly related to the amount of coherence, which is directly connected to the difference between the nonequilibrium free energy

$$\Delta\mathcal{F} = W = \hbar\omega\mu\Delta m. \quad (6.45)$$

The link between work and coherence is made by defining the nonequilibrium free-energy, although it is not necessary to define this quantity since the nonequilibrium free energy \mathcal{F} reaches its minimum precisely for the equilibrium state $\hat{\rho}_{eq}$, we felt the necessity to show that this result is also valid for the general case.

CHAPTER 7

Conclusions and outlook

7.1 Conclusions

At the heart of quantum mechanics lies a peculiar phenomenon called *the principle of superposition*. Ideas such as Schrödinger's cat, entanglement, and quantum teleportation all stem from the same concept, which also is responsible for all the “weirdness” of this theory. In chapter 3, it was seen that the most general way of defining a quantum state is by an operator called the density matrix, whose off-diagonal elements, *coherence*, corresponds to superposition concerning to a given basis. A proper understanding of the interplay between coherence and thermodynamics constituted one of the main aims of this dissertations. My contributions to the work presented in each chapter of this dissertation can be distinguished as follows:

Chapter 2 and 3 mainly reviews introductory material in a manner suitable for use later in the dissertation and reproduces the work by (Streltsov et al., 2017).

Chapter 4 essentially summarises the main ideas about quantum optics for later use. Some calculations in section 4.8 , includes part of my own work, regarding the interpretation of classical and non-classical state according to the sketch (4.7).

Chapter 5 is based on work published by Haake (1973) and by Louisell (1973), which I have adapted to focus on my personal contributions. All calculations presented in section 5.3 represent part of my own work, although closely related ideas may be found throughout the literature in one form or another.

Chapter 6 consisted in the main findings of the present research, regarding coherence and work extraction of quantum thermodynamical systems.

This dissertation covers, in detail and generality, the study of the dynamics of a single bosonic mode, as an open system interacting with certain specific kinds of reservoirs. A theoretical framework to treat the problem and its main properties is discussed. The formalism, developed in the first chapters, is applied to an open question in quantum thermodynamics, which has been unfolding over the last years (Lostaglio et al., 2015b; Uzdin, 2016; Abah et al., 2012), that is how coherence is related to quantum thermodynamics. There are many ways to tackle this problem, which makes the field extremely broad. However, here, a quantum optics perspective/approach is used since the thermodynamic working substance of interest is a single-mode of a bosonic field in contact with thermal reservoirs, and under the action of an external agent, which is able to introduce squeezing in the quadratures of the field. The main question addressed is:

How non-classical features, such as coherence and entanglement, affect thermodynamic quantities?

Motivated by the fact that squeezed states do not admit a diagonal representation in the Fock basis and, therefore, encompass an intrinsic coherence, plus the fact that squeezed thermal baths have been mostly unexplored in the field of quantum thermodynamics, the setup showed in chapter 6 is proposed.

The concept of classicality presented in chapter 6 is used to propose more efficient thermodynamic cycles. Emphasis is put on the physical aspect of its definition, based on how to distinguish classical from non-classical states, and this information is applied to design a thermodynamic protocol. Therefore, a general framework to explore the thermodynamics of squeezed states is developed.

Our main result shows that the efficiency of a quantum heat engine is enhanced by increasing the squeezing parameter. Identifying that squeezing is directly related to coherence, it is concluded that coherence can be a resource for quantum heat engines, meaning that it can be used to perform thermodynamical tasks more efficiently. In particular, two different cycles are studied showing that the efficiency of a quantum engine may be enhanced by keeping the classicality constant, suggesting that more work can be extracted in the non-classical regime. It is worth to mention that the thermodynamic cost of preparing squeezed states is explicitly considered. Therefore, both studied cases provide valuable information to further elucidate the main mechanisms behind squeezed states in quantum thermodynamics. Finally, a direct connection between the work extracted and the coherence parameter is made noting that the work extracted is proportional to the coherence added to the system. The results presented in this dissertation details our original contribution to this new field of quantum thermodynamics.

An exciting subject for future work regards the possibility of using two bosonic systems as a working sub-

stance. This may have many applications under the problem considered in this dissertation; for example, the role of the entanglement between the two modes and the work extraction could be examined since the non-classicality presented in this scenario will not be only coherence. Also, recently [Lostaglio et al. \(2015a\)](#) showed that in a multipartite setting, i.e., a collection of harmonic oscillators or a multi-mode bosonic system, only what is called by *relational coherence* contributes to thermodynamic work. In other words, there are specific off-diagonal blocks on the density matrix representation, with respect to some basis, that that does not help to the work extraction. We finished chapter 6 designating the details of the work extraction mechanism by coupling the bosonic mode with an ancilla. This idea can be adequately refined by introducing a collisional model describing the interaction of a system with a series of independent ancillae prepared in states state.

Bibliography

- S. L. U. S.-B. H. G. A. A. Serafini, M. Lostaglio. *Gaussian Thermal Operations and the Limits of Algorithmic Cooling*. arxiv <https://arxiv.org/abs/1909.06123v2>, Imperial College London, 2019.
- O. Abah and E. Lutz. Efficiency of heat engines coupled to nonequilibrium reservoirs. *EPL (Europhysics Letters)*, 106(2):20001, apr 2014. doi: 10.1209/0295-5075/106/20001. URL <https://doi.org/10.1209/0295-5075/106/20001>.
- O. Abah, J. Roßnagel, G. Jacob, S. Deffner, F. Schmidt-Kaler, K. Singer, and E. Lutz. Single-ion heat engine at maximum power. *Phys. Rev. Lett.*, 109:203006, Nov 2012. doi: 10.1103/PhysRevLett.109.203006. URL <https://link.aps.org/doi/10.1103/PhysRevLett.109.203006>.
- M. Aizenman and E. H. Lieb. *The Third Law of Thermodynamics and the Degeneracy of the Ground State for Lattice Systems*, pages 333–351. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004. ISBN 978-3-662-10018-9. doi: 10.1007/978-3-662-10018-9_18. URL https://doi.org/10.1007/978-3-662-10018-9_18.
- R. Alicki. The quantum open system as a model of the heat engine. *Journal of Physics A: Mathematical and General*, 12(5):L103, 1979. URL <http://stacks.iop.org/0305-4470/12/i=5/a=007>.
- H. Breuer, F. Petruccione, and S. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, 2002. ISBN 9780198520634. URL <https://books.google.com.br/books?id=w2U0nwEACAAJ>.
- Callen. *Thermodynamics and an Introduction to Thermostatistics*. Wiley, 1985. ISBN 9780471862567. URL <https://books.google.com.br/books?id=XJORAQAIAAJ>.
- M. Campisi, P. Hänggi, and P. Talkner. Colloquium: Quantum fluctuation relations: Foundations and applica-

- tions. *Rev. Mod. Phys.*, 83:771–791, Jul 2011. doi: 10.1103/RevModPhys.83.771. URL <https://link.aps.org/doi/10.1103/RevModPhys.83.771>.
- S.-V. Carmichael. *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations*. Physics and astronomy online library. Springer, 1999. ISBN 9783540548829. URL <https://books.google.com.br/books?id=ocgRgM-yJacC>.
- S. Carnot. *Reflections on the motive power of fire: and other papers on the second law of thermodynamics*. Dover Books on Physics. Dover, Mineola, NY, 2005. URL <http://cds.cern.ch/record/1986977>.
- R. Clausius and T. Hirst. *The Mechanical Theory of Heat: With Its Applications to the Steam-engine and to the Physical Properties of Bodies*. J. Van Voorst, 1867. URL <https://books.google.com.br/books?id=8LIEAAAAYAAJ>.
- C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Quantum mechanics*. Quantum Mechanics. Wiley, 1977. ISBN 9782705658342. URL <https://books.google.com.br/books?id=CnkfAQAAMAAJ>.
- A. H. Compton. A quantum theory of the scattering of x-rays by light elements. *Phys. Rev.*, 21:483–502, May 1923. doi: 10.1103/PhysRev.21.483. URL <https://link.aps.org/doi/10.1103/PhysRev.21.483>.
- G. E. Crooks. Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences. *Phys. Rev. E*, 60:2721–2726, Sep 1999. doi: 10.1103/PhysRevE.60.2721. URL <https://link.aps.org/doi/10.1103/PhysRevE.60.2721>.
- C. J. Davisson and L. H. Germer. Reflection of electrons by a crystal of nickel. *Proceedings of the National Academy of Sciences*, 14(4):317–322, 1928. ISSN 0027-8424. doi: 10.1073/pnas.14.4.317. URL <https://www.pnas.org/content/14/4/317>.
- M. de Oliveira. *Termodinâmica*. Livraria da Física, 2005. ISBN 9788588325470. URL <https://books.google.com.br/books?id=sneRJSuC-QC>.
- A. B. de Rochas. *Alphonse Beau de Rochas*. Editions de Haute-Provence, 1862. ISBN 9782909800295. URL <https://books.google.com.br/books?id=JqohAQAIAAJ>.
- S. Deffner and S. Campbell. *Quantum Thermodynamics: An Introduction to the Thermodynamics of Quantum Information*. Iop Concise Physics. Morgan & Claypool Publishers, 2019. ISBN 9781643276557. URL <https://books.google.com.br/books?id=YdAuxwEACAAJ>.
- A. Einstein. Über einen die erzeugung und verwandlung des lichtetes betreffenden heuristischen gesichtspunkt. *Annalen der Physik*, 322(6):132–148, 1905. doi: 10.1002/andp.19053220607. URL <https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19053220607>.

- A. Einstein and P. Schilpp. *Autobiographical Notes*. Library of living philosophers. Open Court, 1979. ISBN 9780875483528. URL <https://books.google.com.br/books?id=SpYuAAAAIAAJ>.
- B.-G. Englert and K. Wódkiewicz. Tutorial notes on one-party and two-party gaussian states. *International Journal of Quantum Information*, 01(02):153–188, 2003. doi: 10.1142/S0219749903000206. URL <https://doi.org/10.1142/S0219749903000206>.
- M. Esposito and C. V. den Broeck. Second law and landauer principle far from equilibrium. *EPL (Europhysics Letters)*, 95(4):40004, aug 2011. doi: 10.1209/0295-5075/95/40004. URL <https://doi.org/10.1209/0295-5075/95/40004>.
- M. Esposito, U. Harbola, and S. Mukamel. Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems. *Rev. Mod. Phys.*, 81:1665–1702, Dec 2009. doi: 10.1103/RevModPhys.81.1665. URL <https://link.aps.org/doi/10.1103/RevModPhys.81.1665>.
- E. Fermi. *Thermodynamics*. Dover books in physics and mathematical physics. Dover Publications, 1956. ISBN 9780486603612. URL <https://books.google.com.br/books?id=VEZ11jsT3IwC>.
- C. Gerry, P. Knight, and P. Knight. *Introductory Quantum Optics*. Cambridge University Press, 2005. ISBN 9780521527354. URL <https://books.google.com.br/books?id=CgByyoBJJwC>.
- J. E. Geusic, E. O. S.-D. Bois, R. W. De Grasse, and H. E. D. Scovil. Three Level Spin Refrigeration and Maser Action at 1500 mc/sec. *Journal of Applied Physics*, 30(7):1113–1114, Jul 1959. doi: 10.1063/1.1776991.
- R. J. Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev.*, 131:2766–2788, Sep 1963. doi: 10.1103/PhysRev.131.2766. URL <https://link.aps.org/doi/10.1103/PhysRev.131.2766>.
- F. Haake. *Statistical treatment of open systems by generalized master equations*, pages 98–168. Springer Berlin Heidelberg, Berlin, Heidelberg, 1973. ISBN 978-3-540-38458-8. doi: 10.1007/BFb0044957. URL <https://doi.org/10.1007/BFb0044957>.
- J. Jackson. *Classical electrodynamics*. Wiley, 1975. ISBN 9780471431329. URL https://books.google.com.br/books?id=_7rvAAAAMAAJ.
- C. Jarzynski. Nonequilibrium equality for free energy differences. *Phys. Rev. Lett.*, 78:2690–2693, Apr 1997. doi: 10.1103/PhysRevLett.78.2690. URL <https://link.aps.org/doi/10.1103/PhysRevLett.78.2690>.
- E. T. Jaynes. Information theory and statistical mechanics. *Phys. Rev.*, 106:620–630, May 1957a. doi: 10.1103/PhysRev.106.620. URL <https://link.aps.org/doi/10.1103/PhysRev.106.620>.

- E. T. Jaynes. Information theory and statistical mechanics. ii. *Phys. Rev.*, 108:171–190, Oct 1957b. doi: 10.1103/PhysRev.108.171. URL <https://link.aps.org/doi/10.1103/PhysRev.108.171>.
- D. V. Kapor, M. J. Škrinjar, and S. D. Stojanović. Relation between spin-coherent states and boson-coherent states in the theory of magnetism. *Phys. Rev. B*, 44:2227–2230, Aug 1991. doi: 10.1103/PhysRevB.44.2227. URL <https://link.aps.org/doi/10.1103/PhysRevB.44.2227>.
- J. Klaers, S. Faelt, A. Imamoglu, and E. Togan. Squeezed thermal reservoirs as a resource for a nanomechanical engine beyond the carnot limit. *Phys. Rev. X*, 7:031044, Sep 2017. doi: 10.1103/PhysRevX.7.031044. URL <https://link.aps.org/doi/10.1103/PhysRevX.7.031044>.
- J. Klatzow, J. N. Becker, P. M. Ledingham, C. Weinzetl, K. T. Kaczmarek, D. J. Saunders, J. Nunn, I. A. Walmsley, R. Uzdin, and E. Poem. Experimental demonstration of quantum effects in the operation of microscopic heat engines. *Phys. Rev. Lett.*, 122:110601, Mar 2019. doi: 10.1103/PhysRevLett.122.110601. URL <https://link.aps.org/doi/10.1103/PhysRevLett.122.110601>.
- J. R. Klauder. The action option and a feynman quantization of spinor fields in terms of ordinary c-numbers. *Annals of Physics*, 11(2):123–168, 1960. ISSN 0003-4916. doi: [https://doi.org/10.1016/0003-4916\(60\)90131-7](https://doi.org/10.1016/0003-4916(60)90131-7). URL <http://www.sciencedirect.com/science/article/pii/0003491660901317>.
- A. Kossakowski. On quantum statistical mechanics of non-hamiltonian systems. *Reports on Mathematical Physics*, 3(4):247–274, 1972. ISSN 0034-4877. doi: [https://doi.org/10.1016/0034-4877\(72\)90010-9](https://doi.org/10.1016/0034-4877(72)90010-9). URL <http://www.sciencedirect.com/science/article/pii/0034487772900109>.
- S. Kullback. *Information Theory and Statistics*. A Wiley publication in mathematical statistics. Dover Publications, 1959. ISBN 9780486696843. URL <https://books.google.com.br/books?id=05LwShwkhFYC>.
- L. Lami, B. Regula, X. Wang, R. Nichols, A. Winter, and G. Adesso. Gaussian quantum resource theories. *Phys. Rev. A*, 98:022335, Aug 2018. doi: 10.1103/PhysRevA.98.022335. URL <https://link.aps.org/doi/10.1103/PhysRevA.98.022335>.
- R. Landauer. Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development*, 5(3):183–191, July 1961. ISSN 0018-8646. doi: 10.1147/rd.53.0183.
- C. T. Lee. Measure of the nonclassicality of nonclassical states. *Phys. Rev. A*, 44:R2775–R2778, Sep 1991. doi: 10.1103/PhysRevA.44.R2775. URL <https://link.aps.org/doi/10.1103/PhysRevA.44.R2775>.
- H. S. Leff. Proof of the third law of thermodynamics for ising ferromagnets. *Phys. Rev. A*, 2:2368–2370, Dec 1970. doi: 10.1103/PhysRevA.2.2368. URL <https://link.aps.org/doi/10.1103/PhysRevA.2.2368>.

- D. Lemons. *Mere Thermodynamics*. Johns Hopkins University Press, 2009. ISBN 9780801890147. URL <https://books.google.com.br/books?id=WFZf2G80tNcC>.
- M. Lostaglio, D. Jennings, and T. Rudolph. Description of quantum coherence in thermodynamic processes requires constraints beyond free energy. *Nature Communications*, 6(1):6383, 2015a. ISSN 2041-1723. doi: 10.1038/ncomms7383. URL <https://doi.org/10.1038/ncomms7383>.
- M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph. Quantum coherence, time-translation symmetry, and thermodynamics. *Phys. Rev. X*, 5:021001, Apr 2015b. doi: 10.1103/PhysRevX.5.021001. URL <https://link.aps.org/doi/10.1103/PhysRevX.5.021001>.
- W. Louisell. *Quantum statistical properties of radiation*. Wiley Series in Pure and Applied Optics Series. John Wiley & Sons Canada, Limited, 1973. ISBN 9780471547853. URL <https://books.google.com.br/books?id=NR1RAAAAMAAJ>.
- G. Manzano. Squeezed thermal reservoir as a generalized equilibrium reservoir. *Phys. Rev. E*, 98:042123, Oct 2018. doi: 10.1103/PhysRevE.98.042123. URL <https://link.aps.org/doi/10.1103/PhysRevE.98.042123>.
- G. Manzano, J. M. Horowitz, and J. M. R. Parrondo. Quantum fluctuation theorems for arbitrary environments: Adiabatic and nonadiabatic entropy production. *Phys. Rev. X*, 8:031037, Aug 2018. doi: 10.1103/PhysRevX.8.031037. URL <https://link.aps.org/doi/10.1103/PhysRevX.8.031037>.
- M. C. MC Oliveira. p -representable subset of all bipartite gaussian separable states. *Phys. Rev. A*, 70:034303, Sep 2004. doi: 10.1103/PhysRevA.70.034303. URL <https://link.aps.org/doi/10.1103/PhysRevA.70.034303>.
- C. L. Mehta. Diagonal coherent-state representation of quantum operators. *Phys. Rev. Lett.*, 18:752–754, May 1967. doi: 10.1103/PhysRevLett.18.752. URL <https://link.aps.org/doi/10.1103/PhysRevLett.18.752>.
- C. L. Mehta and E. C. G. Sudarshan. Relation between quantum and semiclassical description of optical coherence. *Phys. Rev.*, 138:B274–B280, Apr 1965. doi: 10.1103/PhysRev.138.B274. URL <https://link.aps.org/doi/10.1103/PhysRev.138.B274>.
- A. Messiah. *Quantum Mechanics*. Dover Books on Physics. Dover Publications, 2014. ISBN 9780486784557. URL <https://books.google.com.br/books?id=8FvLAgAAQBAJ>.

- K. Micadei, J. P. Peterson, A. M. d. Souza, R. S. Sarthour, I. S. Oliveira, G. T. Landi, T. B. Batalhão, R. M. Serra, and E. Lutz. Reversing the direction of heat flow using quantum correlations. *Nature Communications*, 2019. doi: 10.1038/s41467-019-10333-7.
- J. Millen and A. Xuereb. Perspective on quantum thermodynamics. *New Journal of Physics*, 18(1):011002, jan 2016. doi: 10.1088/1367-2630/18/1/011002. URL <https://doi.org/10.1088%2F1367-2630%2F18%2F1%2F011002>.
- S. Mukamel. Quantum extension of the jarzynski relation: Analogy with stochastic dephasing. *Phys. Rev. Lett.*, 90:170604, May 2003. doi: 10.1103/PhysRevLett.90.170604. URL <https://link.aps.org/doi/10.1103/PhysRevLett.90.170604>.
- W. Niedenzu, V. Mukherjee, A. J. Ghosh, A. G. Kofman, and G. Kurizki. Quantum engine efficiency bound beyond the second law of thermodynamics. In *Nature Communications*, 2017.
- M. Oliveira, S. S. Mizrahi, and V. V. Dodonov. Information transfer in the course of a quantum interaction. *Journal of Optics B: Quantum and Semiclassical Optics*, 1(5):610–617, sep 1999. doi: 10.1088/1464-4266/1/5/319. URL <https://doi.org/10.1088%2F1464-4266%2F1%2F5%2F319>.
- J. M. S.-T. Parrondo, Juan M. R.; Horowitz. Fluctuation theorems: Work is not an observable. *Phys. Rev. E*, 75:050102, May 2007. doi: 10.1103/PhysRevE.75.050102. URL <https://link.aps.org/doi/10.1103/PhysRevE.75.050102>.
- J. M. S.-T. Parrondo, Juan M. R.; Horowitz. Thermodynamics of information. *Nature Physics*, 11:131–139, 2015. ISSN 1745-2473,1745-2481. doi: 10.1038/nphys3230. URL <http://doi.org/10.1038/nphys3230>.
- G. Paule. *Thermodynamics and Synchronization in Open Quantum Systems*. Springer Theses. Springer International Publishing, 2018. ISBN 9783319939636. URL <https://books.google.com.br/books?id=ZLMatwEACAAJ>.
- J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra. Experimental characterization of a spin quantum heat engine. *Phys. Rev. Lett.*, 123:240601, Dec 2019. doi: 10.1103/PhysRevLett.123.240601. URL <https://link.aps.org/doi/10.1103/PhysRevLett.123.240601>.
- Á. Rivas and S. Huelga. *Open Quantum Systems: An Introduction*. SpringerBriefs in Physics. Springer Berlin Heidelberg, 2011. ISBN 9783642233548. URL <https://books.google.com.br/books?id=FGCuYsIZAA0C>.
- F. L. S. Rodrigues, G. De Chiara, M. Paternostro, and G. T. Landi. Thermodynamics of weakly coherent collisional models. *Phys. Rev. Lett.*, 123:140601, Oct 2019. doi: 10.1103/PhysRevLett.123.140601. URL <https://link.aps.org/doi/10.1103/PhysRevLett.123.140601>.

- J. Roßnagel, O. Abah, F. Schmidt-Kaler, K. Singer, and E. Lutz. Nanoscale heat engine beyond the carnot limit. *Phys. Rev. Lett.*, 112:030602, Jan 2014. doi: 10.1103/PhysRevLett.112.030602. URL <https://link.aps.org/doi/10.1103/PhysRevLett.112.030602>.
- J. Roßnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer. A single-atom heat engine. *Science*, 352(6283):325–329, 2016. ISSN 0036-8075. doi: 10.1126/science.aad6320. URL <https://science.sciencemag.org/content/352/6283/325>.
- H. E. D. Scovil and E. O. Schulz-DuBois. Three-level masers as heat engines. *Phys. Rev. Lett.*, 2:262–263, Mar 1959. doi: 10.1103/PhysRevLett.2.262. URL <https://link.aps.org/doi/10.1103/PhysRevLett.2.262>.
- M. Scully and M. Zubairy. *Quantum Optics*. Cambridge University Press, 1997a. ISBN 9780521435956. URL <https://books.google.com.br/books?id=20ISsQCKKmQC>.
- M. Scully and M. Zubairy. *Quantum Optics*. Cambridge University Press, 1997b. ISBN 9780521435956. URL <https://books.google.com.br/books?id=20ISsQCKKmQC>.
- U. Seifert. Stochastic thermodynamics, fluctuation theorems and molecular machines. *Reports on Progress in Physics*, 75(12):126001, nov 2012. doi: 10.1088/0034-4885/75/12/126001. URL <https://doi.org/10.1088/0034-4885/75/12/126001>.
- A. Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods*. CRC Press, Taylor & Francis Group, 2017. ISBN 9781482246346. URL <https://books.google.com.br/books?id=zHtgvgAACAAJ>.
- R. M. Shelby, M. D. Levenson, S. H. Perlmutter, R. G. DeVoe, and D. F. Walls. Broad-band parametric deamplification of quantum noise in an optical fiber. *Phys. Rev. Lett.*, 57:691–694, Aug 1986. doi: 10.1103/PhysRevLett.57.691. URL <https://link.aps.org/doi/10.1103/PhysRevLett.57.691>.
- R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley. Observation of squeezed states generated by four-wave mixing in an optical cavity. *Phys. Rev. Lett.*, 55:2409–2412, Nov 1985. doi: 10.1103/PhysRevLett.55.2409. URL <https://link.aps.org/doi/10.1103/PhysRevLett.55.2409>.
- H. Spohn. Entropy production for quantum dynamical semigroups. *Journal of Mathematical Physics*, 19(5): 1227–1230, 1978. doi: 10.1063/1.523789. URL <https://doi.org/10.1063/1.523789>.
- A. Streltsov, G. Adesso, and M. B. Plenio. Colloquium: Quantum coherence as a resource. *Rev. Mod. Phys.*, 89:041003, Oct 2017. doi: 10.1103/RevModPhys.89.041003. URL <https://link.aps.org/doi/10.1103/RevModPhys.89.041003>.

- E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Phys. Rev. Lett.*, 10:277–279, Apr 1963a. doi: 10.1103/PhysRevLett.10.277. URL <https://link.aps.org/doi/10.1103/PhysRevLett.10.277>.
- E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Phys. Rev. Lett.*, 10:277–279, 1963b. doi: 10.1103/PhysRevLett.10.277.
- R. Uzdin. Coherence-induced reversibility and collective operation of quantum heat machines via coherence recycling. *Phys. Rev. Applied*, 6:024004, Aug 2016. doi: 10.1103/PhysRevApplied.6.024004. URL <https://link.aps.org/doi/10.1103/PhysRevApplied.6.024004>.
- B. P. Venkatesh, G. Watanabe, and P. Talkner. Quantum fluctuation theorems and power measurements. *New Journal of Physics*, 17(7):075018, aug 2015. doi: 10.1088/1367-2630/17/7/075018. URL <https://doi.org/10.1088%2F1367-2630%2F17%2F7%2F075018>.
- von Neuman. *Mathematical Foundations of Quantum Mechanics*. Goldstine Printed Materials. Princeton University Press, 1927. ISBN 9780691028934. URL <https://books.google.com.br/books?id=JLyCo3R04qUC>.
- D. Walls and G. Milburn. *Quantum Optics*. Springer Berlin Heidelberg, 2008. ISBN 9783540285731. URL <https://books.google.com.br/books?id=LiWsc3Nlf0kC>.
- E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–759, Jun 1932. doi: 10.1103/PhysRev.40.749. URL <https://link.aps.org/doi/10.1103/PhysRev.40.749>.
- L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu. Generation of squeezed states by parametric down conversion. *Phys. Rev. Lett.*, 57:2520–2523, Nov 1986. doi: 10.1103/PhysRevLett.57.2520. URL <https://link.aps.org/doi/10.1103/PhysRevLett.57.2520>.

APPENDIX A

Appendix

In this appendix, it is briefly discussed some properties of the delta function, Gaussian integrals, some particularities of the used notation, some details from chapter 5, and the variation of the von-Neumann entropy during infinitesimal changes in the state of the system discussed in chapter 6.

A.1 Compact outer product notation

Let $\hat{\mathbf{a}}$ be a vector of Hermitian operators, where the general case is the one with $\hat{\mathbf{a}}\hat{\mathbf{a}}^T \neq (\hat{\mathbf{a}}\hat{\mathbf{a}}^T)^T$. That is, the operators in the vector might not commute with each other. It is hence useful to define the symmetrized and anti-symmetrized forms of such products, as

$$\begin{aligned} \{\hat{\mathbf{a}}, \hat{\mathbf{a}}^T\} &= \hat{\mathbf{a}}\hat{\mathbf{a}}^T + (\hat{\mathbf{a}}\hat{\mathbf{a}}^T)^T \\ [\hat{\mathbf{a}}, \hat{\mathbf{a}}^T] &= \hat{\mathbf{a}}\hat{\mathbf{a}}^T - (\hat{\mathbf{a}}\hat{\mathbf{a}}^T)^T. \end{aligned} \tag{A.1}$$

In components, one has

$$\begin{aligned} \{\hat{\mathbf{a}}, \hat{\mathbf{a}}^T\}_{j,k} &= \hat{a}_j \hat{a}_k - \hat{a}_k \hat{a}_j \\ [\hat{\mathbf{a}}, \hat{\mathbf{a}}^T]_{j,k} &= \hat{a}_j \hat{a}_k + \hat{a}_k \hat{a}_j \end{aligned} \tag{A.2}$$

To make the idea clearer, consider $\mathbf{a} = (A_i, B_i)^\top$. Then

$$\begin{aligned} [\hat{\mathbf{a}}, \hat{\mathbf{a}}^\top] &= \begin{pmatrix} A_i \\ B_i \end{pmatrix} (A_j \ B_j) - \left[\begin{pmatrix} A_i \\ B_i \end{pmatrix} (A_j \ B_j) \right]^\top \\ &= \begin{pmatrix} A_i A_j - A_j A_i & A_i B_j - B_i A_j \\ B_i A_j - A_i B_j & B_i B_j - B_j B_i \end{pmatrix}. \end{aligned}$$

A.1.1 The direct sum

The direct sum \oplus of any pair of matrices \mathbf{A} of size $m \times n$ and \mathbf{B} of size $p \times q$ is a matrix of size $(m+p) \times (n+q)$ defined as

$$\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{p1} & \cdots & b_{pq} \end{pmatrix}. \quad (\text{A.3})$$

For instance

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \oplus \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 \end{pmatrix}.$$

Therefore,

$$\bigoplus_{i=1}^n \mathbf{A}_i = \text{diag}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n) = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_n \end{pmatrix}. \quad (\text{A.4})$$

A.2 Delta function

The Dirac delta function $\delta(x)$ admits the following representations

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ipx}, \quad (\text{A.5})$$

$$\delta(x) = \lim_{t \rightarrow \infty} \frac{1}{\pi} \frac{\sin(tx)}{x}. \quad (\text{A.6})$$

which satisfies the fundamental property:

$$\int_{-\infty}^{\infty} dx f(x) \delta(x-a) = f(a). \quad (\text{A.7})$$

The delta function can be converted into the following relation for complex variables:

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dp' e^{ipx'} e^{-ip'x} = \frac{1}{\pi^2} \int_{\mathbb{C}} d\beta^2 e^{\alpha\beta^* - \alpha^*\beta}, \quad (\text{A.8})$$

where it was applied the changes of variable $\sqrt{2}\alpha = (x + ip)$ and $\sqrt{2}\beta = (x' + ip')$. Also, $\delta^2(\alpha) = \delta(\text{Re}(\alpha))\delta(\text{Im}(\alpha))$.

A.3 Important relations

Very often, the canonical commutation relations are expressed by exponentiating the canonical operators. So the *Baker-Campbell-Hausdorff* formula for the product of the exponentials of two operators \hat{A} and \hat{B} is

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + 1/2! [\hat{A}, \hat{B}] + \dots}. \quad (\text{A.9})$$

A related expansion that is useful is the called the *Zassenhaus formula*:

$$e^{t(\hat{A} + \hat{B})} = e^{t\hat{A}} e^{t\hat{B}} e^{-\frac{t^2}{2} [\hat{A}, \hat{B}]} e^{\frac{t^3}{6} (2[\hat{B}, [\hat{A}, \hat{B}]] + [\hat{A}, [\hat{A}, \hat{B}]])} e^{-\frac{t^4}{24} ([[[\hat{A}, \hat{B}], \hat{A}], \hat{A}] + 3[[[\hat{A}, \hat{B}], \hat{A}], \hat{B}] + 3[[[\hat{A}, \hat{B}], \hat{B}], \hat{B}])} \dots \quad (\text{A.10})$$

A.4 Gaussian integrals

Gaussian integrals with many variables occurred in Chapter 4. Given a symmetric, real, positive definite $2n \times 2n$ matrix B , and a $2n$ -dimensional vector \mathbf{b} , one has the equality:

$$\int_{\mathbb{R}^{2n}} d\mathbf{r} e^{-\mathbf{r}A\mathbf{r}^T \mathbf{b}} = \frac{\pi^n}{\sqrt{\text{Det}A}} e^{\frac{1}{4} \mathbf{b}^T A^{-1} \mathbf{b}}, \quad (\text{A.11})$$

where the shorthand notation $d\mathbf{r}$ corresponds the product of differentials of the $2n$ integration variables that compose the vector \mathbf{r} : $d\mathbf{r} = dr_1 \dots dr_{2n}$.

A.5 Chapter 5 procedures

A.5.1 Expressing each term of the master equation as a function of the characteristic function

Using (5.41) and the cyclic property of the trace, the remaining terms in (5.39) are:

$$\begin{aligned}
 \text{tr}(\hat{a}^\dagger \hat{a} \hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) &= \text{tr}(\hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} \hat{a}^\dagger \hat{a}) \\
 &= \text{tr}[\hat{\rho} e^{z\hat{a}^\dagger} (e^{-z^* \hat{a}} \hat{a}^\dagger e^{z^* \hat{a}}) e^{-z^* \hat{a}} \hat{a}] \\
 &= \text{tr}[\hat{\rho} (\hat{a}^\dagger - z^*) e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} \hat{a}] \\
 &= \left(\frac{\partial}{\partial z} - z^* \right) \text{tr}(\hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} \hat{a}) \\
 &= - \left(\frac{\partial}{\partial z} - z^* \right) \frac{\partial}{\partial z^*} \chi_N,
 \end{aligned}$$

$$\begin{aligned}
 \text{tr}(\hat{\rho} \hat{a}^\dagger \hat{a} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) &= \text{tr}[\hat{\rho} \hat{a}^\dagger e^{z\hat{a}^\dagger} (e^{-z^* \hat{a}} \hat{a} e^{z\hat{a}^\dagger}) e^{-z^* \hat{a}}] \\
 &= \text{tr}[\hat{\rho} \hat{a}^\dagger e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} (\hat{a} + z)] \\
 &= \left(-\frac{\partial}{\partial z^*} + z \right) \text{tr}[\hat{\rho} \hat{a}^\dagger e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} \hat{a}] \\
 &= \left(-\frac{\partial}{\partial z^*} + z \right) \frac{\partial}{\partial z} \chi_N,
 \end{aligned}$$

$$\begin{aligned}
 \text{tr}(\hat{\rho} \hat{a} \hat{a}^\dagger e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) &= \text{tr}[\hat{\rho} (\hat{a}^\dagger \hat{a} + 1) e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}] \\
 &= \left[\left(-\frac{\partial}{\partial z^*} + z \right) \frac{\partial}{\partial z} + 1 \right] \chi_N,
 \end{aligned}$$

$$\begin{aligned}
 \text{tr}(\hat{a} \hat{a}^\dagger \hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) &= \text{tr}[(\hat{a}^\dagger \hat{a} + 1) \hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}] \\
 &= \left[- \left(\frac{\partial}{\partial z} - z^* \right) \frac{\partial}{\partial z^*} + 1 \right] \chi_N,
 \end{aligned}$$

$$\text{tr}(\hat{a}^\dagger \hat{\rho} \hat{a} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) = \left(1 - |z|^2 + z^* \frac{\partial}{\partial z^*} + z \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^* \partial z} \right) \chi_N,$$

$$\begin{aligned}
 \text{tr}(\hat{a} \hat{a} \hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) &= \text{tr}(\hat{\rho} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} \hat{a} \hat{a}) \\
 &= \frac{\partial^2}{\partial z^{*2}} \chi_N,
 \end{aligned}$$

$$\begin{aligned}
 \text{tr}(\hat{\rho} \hat{a} \hat{a} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) &= \text{tr}(\hat{\rho} \hat{a} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}} \hat{a} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) \\
 &= \text{tr}[\hat{\rho} \hat{a} e^{z\hat{a}^\dagger} (\hat{a} + z) e^{-z^* \hat{a}}] \\
 &= \left(-\frac{\partial}{\partial z^*} + z \right) \text{tr}(\hat{\rho} \hat{a} e^{z\hat{a}^\dagger} e^{-z^* \hat{a}}) \\
 &= \left(-\frac{\partial}{\partial z^*} + z \right) \left(-\frac{\partial}{\partial z^*} + z \right) \chi_N,
 \end{aligned}$$

$$\begin{aligned}
\text{tr}(\hat{a}\hat{\rho}\hat{a}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}) &= \text{tr}(\hat{\rho}\hat{a}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}\hat{a}) \\
&= -\frac{\partial}{\partial z^*}\text{tr}[\hat{\rho}\hat{a}e^{z\hat{a}^\dagger}e^{-z^*\hat{a}}] \\
&= -\frac{\partial}{\partial z^*}\left(-\frac{\partial}{\partial z^*}+z\right)\chi_N,
\end{aligned}$$

For the last term of eq.(5.39) the procedure is similar. Collecting all the terms the equation of motion for the characteristic function can be written as,

$$\begin{aligned}
\frac{\partial\chi_W}{\partial t} &= e^{-\frac{1}{2}|z|^2}\left\{\frac{\gamma}{2}(\bar{n}_{\text{sth}}+1)\left[-z^*\frac{\partial}{\partial z^*}-z\frac{\partial}{\partial z}\right]+\frac{\gamma}{2}\bar{n}_{\text{sth}}\left[-2|z|^2+z^*\frac{\partial}{\partial z^*}+z\frac{\partial}{\partial z}\right]\right. \\
&\quad \left.-\frac{\gamma}{2}M^*z^2-\frac{\gamma}{2}Mz^{*2}\right\}\chi_N \\
&= \left\{\frac{\gamma}{2}(\bar{n}_{\text{sth}}+1)\left[-z^*\left(\frac{\partial}{\partial z^*}+\frac{z}{2}\right)-z\left(\frac{\partial}{\partial z}+\frac{z^*}{2}\right)\right]\right. \\
&\quad \left.+\frac{\gamma}{2}\bar{n}_{\text{sth}}\left[-2|z|^2+z^*\left(\frac{\partial}{\partial z^*}+\frac{z}{2}\right)+z\left(\frac{\partial}{\partial z}+\frac{z^*}{2}\right)\right]-\frac{\gamma}{2}M^*z^2-\frac{\gamma}{2}Mz^{*2}\right\}e^{-\frac{1}{2}|z|^2}\chi_N.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial\chi_W}{\partial t} &= \left\{-\frac{\gamma}{2}\left[z^*\frac{\partial}{\partial z^*}+z\frac{\partial}{\partial z}+M^*z^2+Mz^{*2}+|z|^2\right]-2\gamma\bar{n}_{\text{sth}}|z|^2\right\}\chi \\
&= \left\{-\frac{\gamma}{2}\left[(1+2\bar{n}_{\text{sth}})|z|^2+M^*z^2+Mz^{*2}+z^*\frac{\partial}{\partial z^*}+z\frac{\partial}{\partial z}\right]\right\}\chi.
\end{aligned}$$

A.5.2 Frequency integration

Let us look to the equation (5.25). We have to evaluate a double integral with respect to τ and ω . The integration concerning to the variable τ can be immediately solved. Note that,

$$\int_0^t d\tau e^{-i(\omega-\omega_0)\tau} = \frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} - i\frac{1-\cos(\omega-\omega_0)t}{\omega-\omega_0}.$$

The limit as t tends to infinity in eq.(5.26) is defined anticipating the role of the right-side inside an integration over ω . Then, it left us to solve the following integral:

$$\lim_{t\rightarrow\infty}\int_{-\infty}^{\infty} d\omega f(\omega)\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} - i\lim_{t\rightarrow\infty}\int_{-\infty}^{\infty} d\omega f(\omega)\left[\frac{1-\cos(\omega-\omega_0)t}{\omega-\omega_0}\right]. \quad (\text{A.12})$$

The first integration is straightforward and can be solved directly by using the delta function representation A.6:

$$\lim_{t\rightarrow\infty}\int_{-\infty}^{\infty} d\omega f(\omega)\frac{\sin(\omega-\omega_0)t}{\omega-\omega_0} = \int_{-\infty}^{\infty} d\omega\pi\delta(\omega-\omega_0)f(\omega) = \pi f(\omega_0),$$

while the second integration is not trivial as the first one. Because of that, let I be the integral

$$I \equiv \lim_{t\rightarrow\infty}\int_{-\infty}^{\infty} d\omega f(\omega)\left[\frac{1-\cos(\omega-\omega_0)t}{\omega-\omega_0}\right] = \int_{-\infty}^{\infty} d\omega\frac{f(\omega)}{\omega-\omega_0} - \lim_{t\rightarrow\infty}\int_{-\infty}^{\infty} d\omega f(\omega)\left[\frac{\cos(\omega-\omega_0)t}{\omega-\omega_0}\right]. \quad (\text{A.13})$$

Defining

$$A = \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0} \quad \text{and} \quad B = \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} d\omega f(\omega) \left[\frac{\cos(\omega - \omega_0)t}{\omega - \omega_0} \right]. \quad (\text{A.14})$$

It is assumed that $f(\omega)$ is a well-behaved function. The integrands in eq.(A.14) have singularities at ω_0 . To evaluate this integral, we shall use the residue theorem, and the integration will be performed in the complex plane. The integral is evaluated via the loop contour $C = C_1 \cup C_2$ shown in the figure (A.1): Note that,

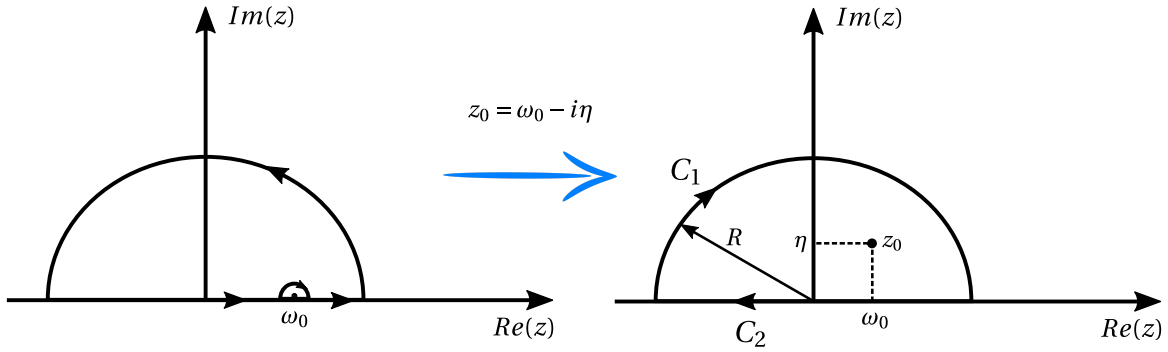


Figure A.1: Complex plane sketched. The function has a simple pole on the real axis at ω_0 . Instead of deforming the contour to avoid the singularity, one can displace the singularity introducing z_0 , such that $z_0 = \omega_0 - i\eta$.

$$\oint_C dz \frac{f(z)}{z - z_0} = \int_{C_1} dz \frac{f(z)}{z - z_0} + \int_{C_2} dz \frac{f(z)}{z - z_0} = 2\pi i \text{Res}(f(z)). \quad (\text{A.15})$$

Furthermore, observe that the first term in eq.(A.15) goes to zero when $R \rightarrow \infty$. While,

$$\lim_{\eta \rightarrow 0^+} \oint_C dz \frac{f(z)}{z - z_0} = \lim_{\eta \rightarrow 0^+} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - z_0} = \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0}, \quad (\text{A.16})$$

and

$$\lim_{\eta \rightarrow 0^+} \oint_C f(z) \frac{\cos(\omega - \omega_0)t}{z - z_0} = \int_{-\infty}^{\infty} d\omega f(\omega) \frac{\cos(\omega - \omega_0)t}{\omega - \omega_0}. \quad (\text{A.17})$$

The integration,

$$\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - z_0}, \quad (\text{A.18})$$

can be evaluated using the property

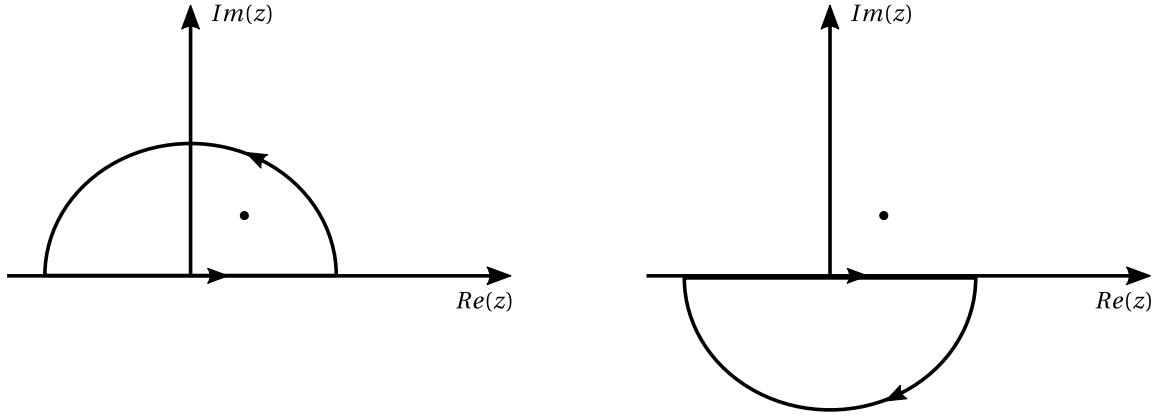
$$\frac{1}{x \pm i\delta} = \text{P} \left(\frac{1}{x} \right) \mp i\pi\delta(x) \quad \text{where} \quad \delta \rightarrow 0^+. \quad (\text{A.19})$$

Thus,

$$\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - z_0} = \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0 - i\eta} = \text{P} \left(\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0} \right) + i\pi \int_{-\infty}^{\infty} d\omega f(\omega) \delta(\omega - \omega_0). \quad (\text{A.20})$$

Taking the limit,

$$A = \lim_{\eta \rightarrow 0^+} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - z_0} = \text{P} \left(\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0} \right) + i\pi f(\omega_0). \quad (\text{A.21})$$

Figure A.2: The closed contours C used in the evaluation of the integral eq.(A.22).

The integral B is solved explicitly in the complex plane

$$\begin{aligned}
 \int_C dz f(z) \frac{\cos(z - z_0)t}{z - z_0} &= \int_C dz \frac{f(z)}{z - z_0} \frac{e^{i(z-z_0)t} + e^{-i(z-z_0)t}}{2} \\
 &= \frac{e^{-iz_0t}}{2} \oint_C dz \frac{e^{izt}}{z - z_0} + \frac{f(z)e^{iz_0t}}{2} \oint_C dz \frac{f(z)e^{-izt}}{z - z_0} \\
 &= \frac{e^{-iz_0t}}{2} 2\pi i \lim_{z \rightarrow z_0} (z - z_0) \frac{f(z)e^{izt}}{z - z_0} = i\pi f(z_0)
 \end{aligned} \tag{A.22}$$

Therefore,

$$I = \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} d\omega f(\omega) \left[\frac{1 - \cos(\omega - \omega_0)t}{\omega - \omega_0} \right] = P \left(\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_0} \right) \tag{A.23}$$

A.6 Infinitesimal changes in the state of the system and its entropy.

In the following we discuss the proof of eq.(6.38). The proof of this equation follows from the work by [Manzano et al. \(2018\)](#). Consider a change in the state of the system

$$\hat{\rho}' = \hat{\rho} + \epsilon \Delta \hat{\rho}, \tag{A.24}$$

where $\hat{\rho}'$ is the state of the system after some interaction, $\hat{\rho}$ is the original state, and $\delta \hat{\rho}$ is a traceless operator accounting for the change. In order to proceed, it is necessary to evaluate the eigenvalues and eigenvectors of the state $\hat{\rho}'$. This can be done by using perturbation theory when $\epsilon \ll 1$. Up to second order in ϵ , we have that

$$\begin{aligned}
 \lambda'_n &\approx \lambda_n + \epsilon \lambda_n^{(1)} + \epsilon^2 \lambda_n^{(2)}, \\
 |\lambda'\rangle &\approx |\lambda_n\rangle + \epsilon |\lambda_n^{(1)}\rangle + \epsilon^2 |\lambda_n^{(2)}\rangle.
 \end{aligned} \tag{A.25}$$

where $\{\lambda_n, |\lambda_n\rangle\}$ are the eigenvalues and eigenvectors of $\hat{\rho}$. In particular, the first order terms read

$$\lambda_n^{(1)} = \langle \lambda_n | \Delta \hat{\rho} | \lambda_n \rangle, \tag{A.26}$$

$$|\lambda_n^{(1)}\rangle = \sum_{m \neq n} \frac{\langle \lambda_m | \Delta \hat{\rho} | \lambda_m \rangle}{\lambda_n - \lambda_m}. \tag{A.27}$$

Then, the change in the von-Neumann entropy (3.42) can be written as

$$\Delta S = S(\hat{\rho}') - S(\hat{\rho}) = -\sum_n \lambda'_n \log \lambda'_n + \sum_j \lambda_j \log \lambda_j, \quad (\text{A.28})$$

and using eqs.(A.25) we obtain up to second order in ϵ :

$$\Delta S \approx -\epsilon \sum_n \lambda_n^{(1)} \log \lambda_n - \epsilon^2 \left(\lambda_n^{(2)} \log \lambda_n + \sum_j \frac{1}{2} \frac{\lambda_j^{(1)2}}{\lambda_j} \right). \quad (\text{A.29})$$

Dropping out the second order terms, and using eq.(A.27), it is obtained

$$\Delta S \approx -\epsilon \sum_n \lambda_n^{(1)} \log \lambda_n = -\epsilon \sum_n \langle \lambda_n | \Delta \hat{\rho} | \lambda_n \rangle \log \lambda_n \quad (\text{A.30})$$

$$= -\epsilon \text{tr}(\Delta \hat{\rho} \log \hat{\rho}). \quad (\text{A.31})$$