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Statistical thermodynamics of the Fröhlich-Bose-Einstein condensation of magnons out of equilibrium

Fabio Stucchi Vannucchi *
São Paulo State University (UNESP), Institute of Biosciences, 11380-972 São Vicente, SP, Brazil

Roberto Luzzi
Condensed Matter Physics Department, Institute of Physics "Gleb Wataghin," State University of Campinas (UNICAMP), 13083-859 Campinas, SP, Brazil

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This work presents a nonequilibrium statistical-thermodynamic approach to the study of a Fröhlich-Bose-Einstein condensation of magnons under radio-frequency radiation pumping. Fröhlich-Bose-Einstein condensates display a complex behavior consisting in steady-state conditions to the emergence of a synergetic dissipative structure resembling the Bose-Einstein condensation of systems in equilibrium. Then a kind of "two-fluid model" arises: the "normal" nonequilibrium structure and the Fröhlich condensate, which is shown to be an attractor to the system. In this study we analyze some aspects of the irreversible thermodynamics of this dissipative complex system. We obtained the expression for the informational entropy of the two-fluid condensate and introduced an order parameter to characterize the role of the Fröhlich interaction in ordering the system. The analysis highlights the order increase due to the Fröhlich interaction. We also study the informational entropy production of the system, considering its internal and external parts. Finally, the Glansdorff-Prigogine criteria for evolution and (in)stability are verified.

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I. INTRODUCTION

Several formulations of thermodynamics have been recently proposed and used to deal with nonequilibrium open quantum systems. Among these approaches are Markovian dynamics with the Lindblad-Gorini-Kossakowski-Sudarshan semigroup generator [1,2] or the quantum Fokker-Planck-Kramers equation [3,4]. An alternative possible approach, similar to other open quantum systems treatments [5,6], is to follow the path of Gibbs and others to consider a formalism of ensembles [nonequilibrium statistical ensemble formalism (NESEF)] [7–13]. In this case, evolution equations of the macroscopic state of the system are the quantum mechanical equations of motion averaged over the nonequilibrium ensemble, with the NESEF-kinetic theory providing a practical way of calculation [14–17]. Informational thermodynamic aspects of the system are then obtained in terms of the NESEF-based nonequilibrium-statistical irreversible thermodynamics [18,19] (also chap. 7 in Ref. [7]).

In this work we use the above formalism to describe the irreversible thermodynamic of a system of spins in thin films of yttrium-iron-garnets in the presence of a constant magnetic field and being excited by a source of radio-frequency (rf) radiation. This magnetic system, driven toward far-removed equilibrium conditions, has been reported in detailed experiments performed by Demokritov *et al.* [20,21] forming a Bose-Einstein condensate (BEC) of magnons. These experimental results have evidenced an unexpected increase of the magnons population in the lowest-energy state in their energy dispersion relation. That is, instead of the energy pumped into the system being redistributed among the magnons in such nonthermal conditions it is transferred to the lowest-frequency mode (with a fraction being dissipated to the surrounding media). Some theoretical studies along certain approaches have been presented by several authors (see, for example, Refs. [22–26]). We proceed here to describe the thermodynamic aspects of the system within the framework of a nonequilibrium ensemble formalism.

For that purpose, we consider a system of N-localized spins in the presence of a constant magnetic field being pumped by a rf source of radiation that drives them out of equilibrium. The spin system is embedded in a thermal bath consisting of the phonon system (the lattice vibrations) supposedly in equilibrium with an external reservoir at temperature T_0. The quantum state of the system is characterized by the full Hamiltonian of spins and lattice vibrations after going through Holstein-Primakov and Bogoliubov transformations [27–29] and NESEF.

The evolution of the nonequilibrium state of magnons under rf-radiation excitation is summarized in Sec. II where a two-fluid approximation is proposed and the nonequilibrium Fröhlich-Bose-Einstein condensate (NEFBEC) is obtained (full description may be obtained in Refs. [25,26]). In Sec. III we present an extended study of the nonequilibrium irreversible thermodynamics of the NEFBEC of such "hot" magnons. We calculate the informational entropy of the system and define an order parameter to analyze the role of

*fs.vannucchi@unesp.br

Fröhlich contribution. Entropy production is then obtained and finally two criteria for thermodynamic evolution and stability are verified.

II. FRÖHLICH-BOSE-EINSTEIN CONDENSATION OF HOT MAGNONS IN BRIEF

The system we are considering consists of a subsystem of spins being pumped by a microwave source and interacting nonlinearly with a thermal bath (black-body radiation and crystalline lattice) that is in contact with a thermal reservoir in equilibrium at temperature T_0. This system is well described by the Hamiltonian

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_S + \hat{\mathscr{H}}_Z + \hat{\mathscr{H}}_{\rm SR} + \hat{\mathscr{H}}_R + \hat{\mathscr{H}}_{\rm SL} + \hat{\mathscr{H}}_L, \quad (1)$$

where $\hat{\mathscr{H}}_S$ accounts for the internal (exchange and magnetic dipole) interactions between spins, and $\hat{\mathscr{H}}_Z$ is associated with the effect of the constant magnetic field (Zeeman effect). $\hat{\mathscr{H}}_L$ and $\hat{\mathscr{H}}_R$ are the Hamiltonian of the thermal bath (lattice and radiation, respectively), $\hat{\mathscr{H}}_{\rm SL}$ and $\hat{\mathscr{H}}_{\rm SR}$ are their interaction with the spin subsystem ($\hat{\mathscr{H}}_{\rm SR}$ includes also the effect of the source). By introducing the quasiparticles related to the spin, lattice, and radiation variables (respectively the magnons, phonons, and photons) and their creation and annihilation operators ($\hat{c}^\dagger_{\mathbf{q}}$, $\hat{c}_{\mathbf{q}}$, $\hat{b}^\dagger_{\mathbf{q}}$, $\hat{b}_{\mathbf{q}}$, $\hat{d}^\dagger_{\mathbf{q}}$, and $\hat{d}_{\mathbf{q}}$), we may write the Hamiltonian of Eq. (1) as

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_0 + \hat{\mathscr{H}}' \quad (2)$$

with

$$\hat{\mathscr{H}}_0 = \hat{\mathscr{H}}_S^{(2)} + \hat{\mathscr{H}}_L + \hat{\mathscr{H}}_R \quad (3)$$

being the noninteracting term formed by the Hamiltonians of free magnons, phonons, and photons (with energies $\hbar\omega_{\mathbf{q}}$, $\hbar\Omega_{\mathbf{k}}$, and $\hbar\zeta_{\mathbf{p}}$, respectively). The other term,

$$\hat{\mathscr{H}}' = \hat{\mathscr{H}}_{\rm MM} + \hat{\mathscr{H}}_{\rm SL} + \hat{\mathscr{H}}_{\rm SR}, \quad (4)$$

includes the interactions between quasiparticles: $\hat{\mathscr{H}}_{\rm MM}$ is the magnon-magnon scattering term, $\hat{\mathscr{H}}_{\rm SL}$ accounts for the relevant magnon-phonon interaction, and $\hat{\mathscr{H}}_{\rm SR}$ is the interaction between magnons and photons (details in Appendix A).

The thermodynamic description of the system follows the mechanical one. The thermodynamic state can be defined in terms of the time-dependent thermodynamic variables, which are average values of the so-called basic microdynamical variables. We are then led to choose a basic set of variables that should characterize the macroscopic state of the system (the appropriate nonequilibrium thermodynamic state of the system [18,19]).

For a general magnetic system one should include in the description all experimental-related variables (e.g., magnetic moment operator). In second quantization form this choice would impose inclusion of populations of magnons, coherent states, and Gorkov pairs. The kinetic equations of these thermodynamic variables form a system of nonlinear coupled integrodifferential equations which is discussed in Ref. [25] and in a detailed form in Ref. [26]. As stated in these references, for the specific spin systems considered here, it is enough to follow the evolution of populations of magnons $\mathcal{N}_{\mathbf{q}}(t)$ and, to complete the thermodynamic description of the entire system, also include the energy evolution of the thermal bath $E_B(t)$ (lattice and black-body radiation). Thus the set of relevant thermodynamic variables is

$$\{\{\mathcal{N}_{\mathbf{q}}(t)\};\ E_B\}, \quad (5)$$

average values of the so-called basic microdynamical variables

$$\{\{\hat{\mathcal{N}}_{\mathbf{q}} = \hat{c}^\dagger_{\mathbf{q}}\hat{c}_{\mathbf{q}}\};\ \hat{\mathscr{H}}_B\}, \quad (6)$$

with $\hat{\mathcal{N}}_{\mathbf{q}}$ being the population operator of magnons in mode $\mathbf{q}$ and $\hat{\mathscr{H}}_B = \hat{\mathscr{H}}_L + \hat{\mathscr{H}}_R$ is the Hamiltonian of the thermal bath (phonons and photons).

These averages are weighted through a nonequilibrium statistical operator $\hat{\mathscr{R}}_\varepsilon(t)$,

$$\mathcal{N}_{\mathbf{q}}(t) = \mathrm{Tr}\{\hat{\mathcal{N}}_{\mathbf{q}}\,\hat{\mathscr{R}}_\varepsilon(t)\} \quad (7)$$

and

$$E_B = \mathrm{Tr}\{\hat{\mathscr{H}}_B\,\hat{\mathscr{R}}_\varepsilon(t)\}. \quad (8)$$

We introduce a factorization between the thermal bath in equilibrium and the magnetic subsystem

$$\hat{\mathscr{R}}_\varepsilon(t) = \hat{\varrho}_B \times \hat{\varrho}_\varepsilon(t), \quad (9)$$

where

$$\hat{\varrho}_B = \frac{1}{Z_B}\exp\{-\beta_B(\hat{\mathscr{H}}_B)\} \quad (10)$$

is the canonical distribution function of the phonons and photons in stationary condition near equilibrium at temperature $T_B = (k_B\beta_B)^{-1}$ (with Z_B its partition function) and $\hat{\varrho}_\varepsilon(t)$ the nonequilibrium statistical operator of the magnon system.

The last term of Eq. (9) may be obtained solving a modified Liouville-Dirac equation for $\hat{\varrho}_\varepsilon(t)$,

$$\frac{\partial}{\partial t}\hat{\varrho}_\varepsilon(t) + \frac{1}{i\hbar}[\hat{\varrho}_\varepsilon(t), \hat{\mathscr{H}}] = -\varepsilon\{\hat{\varrho}_\varepsilon(t) - \hat{\bar{\varrho}}(t,0)\}, \quad (11)$$

where the right term (with $\varepsilon \to 0$) introduces the "Bogoliubov's symmetry-breaking procedure" in time and $\hat{\bar{\varrho}}(t,0)$ is the auxiliary statistical operator. Equation (11) ensures on the one hand that the nonequilibrium statistical operator $\hat{\varrho}_\varepsilon(t)$ incorporates the dynamical evolution while, on the other hand, includes irreversibility [7–10].

The auxiliary statistical operator $\hat{\bar{\varrho}}(t,0)$ is written in terms of the chosen microdynamical variables taken the form

$$\hat{\bar{\varrho}}(t,0) = \frac{1}{\bar{Z}(t)}\exp\left\{-\sum_{\mathbf{q}}[F_{\mathbf{q}}(t)\hat{\mathcal{N}}_{\mathbf{q}}]\right\}, \quad (12)$$

where $F_{\mathbf{q}}(t)$ is the nonequilibrium thermodynamic variable conjugated to the populations of magnons in the sense of the Eqs. (14) below. The normalization of $\hat{\bar{\varrho}}(t,0)$ introduces the nonequilibrium partition function

$$\bar{Z}(t) \equiv \mathrm{Tr}\left\{\exp\left\{-\sum_{\mathbf{q}}[F_{\mathbf{q}}(t)\hat{\mathcal{N}}_{\mathbf{q}}]\right\}\right\}. \quad (13)$$

It is important to stress that since the intensive nonequilibrium thermodynamic variable $F_{\mathbf{q}}(t)$ equivalently describes the macro state of the system and that

$$-\frac{\delta\ln\bar{Z}(t)}{\delta F_{\mathbf{q}}(t)} = \mathcal{N}_{\mathbf{q}}(t) \quad (14)$$

may be considered nonequilibrium equations of state, there is a close analogy with the intensive thermodynamic variables in equilibrium. Moreover, for the equation of state it follows that

$$\langle \hat{c}^\dagger_{\mathbf{q}} \hat{c}_{\mathbf{q}} | t \rangle = \mathcal{N}_{\mathbf{q}}(t) = \frac{1}{e^{F_{\mathbf{q}}(t)} - 1}, \tag{15}$$

or, alternatively,

$$F_{\mathbf{q}}(t) = \ln\left\{1 + \frac{1}{\mathcal{N}_{\mathbf{q}}(t)}\right\} = -\ln\left\{\frac{\mathcal{N}_{\mathbf{q}}(t)}{\mathcal{N}_{\mathbf{q}}(t)+1}\right\}. \tag{16}$$

We recall that the equations of evolution for the populations are the quantum mechanical equations of motion for the dynamical quantities $\hat{\mathcal{N}}_{\mathbf{q}}$ averaged over the nonequilibrium ensemble. They are handled resorting to the NESEF-based nonlinear quantum kinetic theory, with the calculations performed in the approximation that incorporates only quadratic terms in the interaction strength—with memory and vertex renormalization neglected, that is, we keep what in kinetic theory is called the irreducible part of the two-particle collisions. In a compact form (details on Appendix B) we may write

$$\frac{d}{dt}\mathcal{N}_{\mathbf{q}}(t) = \mathfrak{S}_{\mathbf{q}}(t) + \mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t) + \mathfrak{M}_{\mathbf{q}}(t), \tag{17}$$

where $\mathfrak{S}_{\mathbf{q}}$ is the source term that accounts for the pumping of energy to the system, $\mathfrak{R}_{\mathbf{q}}(t)$ is a nonlinear term of interaction between the spin subsystem and the black-body radiation, and $L_{\mathbf{q}}(t)$ is the linear relaxation to the lattice with characteristic time $\tau_{\mathbf{q}}$. The last two terms are nonlinear contributions: $\mathfrak{F}_{\mathbf{q}}(t)$, the so-called Fröhlich term, which is a nonlinear interaction between magnons mediated by the lattice, and $\mathfrak{M}_{\mathbf{q}}(t)$ accounts for the magnon-magnon scattering interaction term. In a similar form of Eq. (17) we have that

$$\frac{d}{dt}E_B(t) = J^{(2)}_{\mathrm{TD}}(t) - \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)], \tag{18}$$

the first term,

$$J^{(2)}_{\mathrm{TD}}(t) = -\frac{E_B(t) - E_B^{(0)}}{\tau_{\mathrm{TD}}}, \tag{19}$$

is the contribution which accounts for the thermal diffusion to the reservoir with a thermal diffusion time τ_{TD} and tends to lead the thermal bath to equilibrium [characterized by the equilibrium energy $E_B^{(0)}$]. The other contribution is related to the energy received from the subsystem of magnons.

Our system has its thermodynamic evolution described by the kinetic equations (17) and (18) and they must be solved. Since we have affirmed that the thermal bath is in a stationary state near the equilibrium condition defined by the external reservoir we have that

$$\frac{d}{dt}E_B(t) = 0, \tag{20}$$

and the thermal diffusion effect is rapid enough to keep this configuration. In this case $E_B(t) \simeq E_B^{(0)}$, $T_B \simeq T_0$, and $\beta_B \simeq \beta_0$.

Considering again the evolution of the population of magnons, we emphasize that Eq. (17) constitutes a nonlinear system of coupled integrodifferential equations. Its resolution in an approximate form called "two-fluid model" is discussed on Refs. [25,26], where the mean populations $\mathcal{N}_1(t)$ and $\mathcal{N}_2(t)$ were defined respectively as representations of the populations of magnons around the minimum frequency and those being fed by the external source,

$$\mathcal{N}_{1,2}(t) = \frac{\sum_{\mathbf{q}\in R_{1,2}} \mathcal{N}_{\mathbf{q}}(t)}{\sum_{\mathbf{q}\in R_{1,2}} 1} = \frac{\sum_{\mathbf{q}\in R_{1,2}} \mathcal{N}_{\mathbf{q}}(t)}{n_{1,2}}, \tag{21}$$

where R_1 and R_2 are the correspondent regions in the reciprocal space. Their evolution equations were obtained from Eq. (17),

$$f_1\frac{d}{d\bar{t}}\mathcal{N}_1(\bar{t}) = -D\mathcal{N}_1\big(\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big) - f_1\big[\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big] \tag{22a}$$
$$+ F\{\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1\} \tag{22b}$$
$$- M\{\mathcal{N}_1(\mathcal{N}_1+1) + \mathcal{N}_2(\mathcal{N}_2+1)\}$$
$$\times\left(\mathcal{N}_1\frac{\mathcal{N}_2^{(0)}}{\mathcal{N}_1^{(0)}} - \mathcal{N}_2\right), \tag{22c}$$

and

$$f_2\frac{d}{d\bar{t}}\mathcal{N}_2(\bar{t}) = I\,(1 + 2\mathcal{N}_2) \tag{23a}$$
$$- D\mathcal{N}_2\big(\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big) - f_2\big[\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big] \tag{23b}$$
$$- F\{\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1\} \tag{23c}$$
$$+ M\{\mathcal{N}_1(\mathcal{N}_1+1) + \mathcal{N}_2(\mathcal{N}_2+1)\}$$
$$\times\left(\mathcal{N}_1\frac{\mathcal{N}_2^{(0)}}{\mathcal{N}_1^{(0)}} - \mathcal{N}_2\right), \tag{23d}$$

where $\bar{t}$ is the scaled time t/τ, taking the relaxation time $\tau_{\mathbf{q}}$ as having a unique constant value ($\mathbf{q}$ independent), $\mathcal{N}_{1,2}^{(0)}$ are the populations in equilibrium, and f_1 and f_2 are the fractions of the Brillouin zone corresponding to the two regions in the two-fluid model. Moreover, the coefficients M and F are respectively the coupling strengths associated to magnon-magnon interaction and to the Fröhlich contribution, while D is associated to decay with emission of photons, and $\bar{\nu}$ is an average phonon population. Finally, the parameter I is related to the rf-radiation field rate transferred to the spin system, whose absorption is reinforced by a positive feedback effect. All these coefficients are dimensionless when multiplied by the relaxation time τ.

In a similar fashion, in the two fluid model, the energy of the thermal bath has an evolution given by

$$\frac{d}{d\bar{t}}E_B(t) = \tau J^{(2)}_{\mathrm{TD}}(t) + n\hbar\omega_1\big\{D\mathcal{N}_1\big(\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big) + f_1\big[\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big]$$
$$-\; F\{\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1\}\big\}$$
$$+\; n\hbar\omega_2\big\{D\mathcal{N}_2\big(\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big) + f_2\big[\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big]$$
$$+\; F\{\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1\}\big\}, \tag{24}$$

being $\hbar\omega_1$ and $\hbar\omega_2$ the energy of the magnons in the regions R_1 and R_2, and $n = \sum_{\mathbf{q}} 1$.

In Fig. 1 we show the evolution of the populations $\mathcal{N}_1$ and $\mathcal{N}_2$, starting from equilibrium, under the pumping source action (we adopted $\tau = 1\,\mu$s to compare with experimental data [20]), solving numerically Eqs. (22) and (23). As stated in

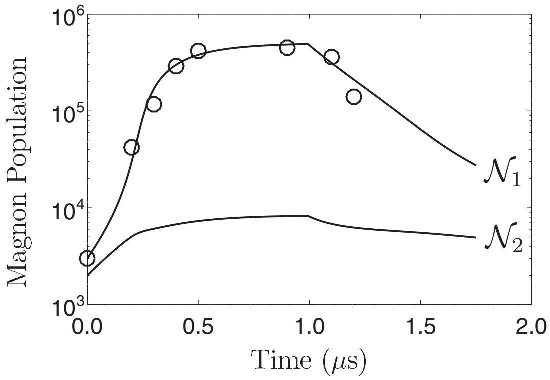

FIG. 1. Evolution of the magnon population. Circles represent Demokritov's *et al.* data for the low-energy magnon population [20], with the pumping being switched off after 1 μs. Solid lines show low- and high-energy magnon populations, obtained after numerical integration of Eqs. (22) and (23) using the following parameters: $\mathcal{N}_1^{(0)} = 3 \times 10^3$, $\mathcal{N}_2^{(0)} = 2 \times 10^3$, $f_1 = 3 \times 10^{-6}$, $f_2 = 3 \times 10^{-4}$, $F = 2 \times 10^{-6}$, $M = 3 \times 10^{-14}$, $D = 4 \times 10^{-11}$, and $I = 8 \times 10^{-4}$. After Ref. [25].

Refs. [25,26], besides the fine agreement with the experimental data, this result clearly shows the accumulation of magnons on the minimum-frequency mode ($\mathcal{N}_1$).

In addition, the analysis of the steady state of the system, i.e., the solutions of Eqs. (22) and (23) such that $\frac{d}{d\bar{t}}\mathcal{N}_1(\bar{t})$ and $\frac{d}{d\bar{t}}\mathcal{N}_2(\bar{t})$ are null, evidence the role of the Fröhlich term to the condensation of magnons. In Fig. 2 we show the values of the steady-state populations $\mathcal{N}_1^S$ and $\mathcal{N}_2^S$ as a function of the scaled rate of pumping I, and the existence of two pumping scaled rate thresholds can be noticed. The first, followed by a steep increase in the population of the lowest-frequency modes, corresponds to the emergence of BEC, while the second, for higher values of I, accounts for the internal thermalization of the magnons which acquire a common quasitemperature, implying that the magnon-magnon interaction overcomes the Fröhlich contribution and BEC is impaired.

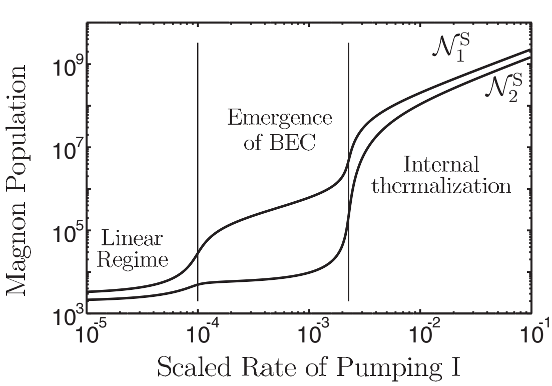

FIG. 2. Steady-state magnon populations as a function of the pumping source intensity which are solutions for Eqs. (22) and (23) using the same parameters as in Fig. 1. After Ref. [25].

III. INFORMATIONAL STATISTICAL THERMODYNAMICS OF NEFBEC

We proceed to the description of the NESEF-based informational irreversible thermodynamics (IST) [18,19] of the NEFBEC of magnons [20,21] with the description given in Refs. [25,26] and summarized in the previous section.

A. IST entropy and order parameter for magnons

Here we introduce the informational entropy in the framework of IST,

$$\bar{S}(t) = -\mathrm{Tr}\{\hat{\mathscr{R}}_\varepsilon(t)\mathcal{P}_\varepsilon(t)\ln\hat{\mathscr{R}}_\varepsilon(t)\}, \tag{25}$$

where, we recall, $\hat{\mathscr{R}}_\varepsilon(t)$ is the nonequilibrium statistical operator of Eq. (9) and $\mathcal{P}_\varepsilon(t)$ is a time-dependent projection operator (it is characterized by the nonequilibrium state of the system at any time t) such that [7,30–32]

$$\mathcal{P}_\varepsilon(t)\ln\hat{\varrho}_\varepsilon(t) = \ln\hat{\bar{\varrho}}(t,0) \tag{26}$$

and

$$\mathcal{P}_\varepsilon(t)\ln\hat{\varrho}_B = \ln\hat{\varrho}_B, \tag{27}$$

where $\hat{\bar{\varrho}}(t,0)$ and $\hat{\varrho}_B$ are those of Eqs. (12) and (10).

Hence we have that

$$\begin{aligned}\bar{S}(t) &= -\mathrm{Tr}\{\hat{\mathscr{R}}_\varepsilon(t)\ln\{\hat{\bar{\varrho}}(t,0)\times\hat{\varrho}_B\}\}\\ &= \phi(t) + \beta_B E_B + \sum_{\mathbf{q}} F_{\mathbf{q}}(t)\mathcal{N}_{\mathbf{q}}(t),\end{aligned} \tag{28}$$

$$\phi(t) = \ln Z_B + \ln\bar{Z}(t), \tag{29}$$

where Z_B and $\bar{Z}(t)$ are the canonical and nonequilibrium partition functions [see Eqs. (10) and (13)]. The last one depends on time and must be explicitly written in terms of the nonequilibrium thermodynamic variables. In a analogous way to the equilibrium Bose statistics we obtain that

$$\bar{Z}(t) = \mathrm{Tr}\exp\left\{-\sum_{\mathbf{q}} F_{\mathbf{q}}(t)\hat{\mathcal{N}}_{\mathbf{q}}\right\} = \prod_{\mathbf{q}}\frac{1}{1-e^{-F_{\mathbf{q}}(t)}}. \tag{30}$$

Thus

$$\ln\bar{Z}(t) = \sum_{\mathbf{q}}\ln\frac{1}{1-e^{-F_{\mathbf{q}}(t)}} \tag{31}$$

and using the relation between $F_{\mathbf{q}}(t)$ and $\mathcal{N}_{\mathbf{q}}(t)$, Eq. (16), one obtains the expression for the informational entropy,

$$\begin{aligned}\bar{S}(t) = \ln Z_B + \beta_B E_B &- \sum_{\mathbf{q}}\mathcal{N}_{\mathbf{q}}(t)\ln[\mathcal{N}_{\mathbf{q}}(t)]\\ &- \sum_{\mathbf{q}}[\mathcal{N}_{\mathbf{q}}(t)+1]\ln[\mathcal{N}_{\mathbf{q}}(t)+1],\end{aligned} \tag{32}$$

whereas Z_B, β_B, and E_B are constants [see Eq. (20) and subsequent discussion].

The informational entropy of the above-cited "two-fluid model" is obtained partitioning the $\mathbf{q}$ sums of Eq. (32) in the regions R_1 and R_2 of Eq. (21). Neglecting the constant part related to the bath, the informational entropy may be

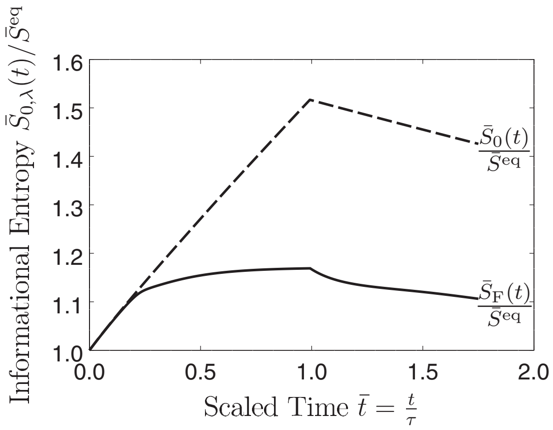

FIG. 3. Informational entropy as function of time associated with magnon populations from Eqs. (22) and (23) displayed in Fig. 1. Solid line represents the system with Fröhlich contribution, while the dashed line refers to a system in which the Fröhlich contribution is absent [$F = 0$ on Eqs. (22) and (23)]. Radiation pumping switched off at $\bar{t} = 1$. We recall that the values of the different parameters are indicated in the caption of Fig. 1.

written as

$$\bar{S}(t) = -n_1\{\mathcal{N}_1 \ln(\mathcal{N}_1) - (\mathcal{N}_1 + 1)\ln(\mathcal{N}_1 + 1)\} - n_2\{\mathcal{N}_2 \ln(\mathcal{N}_2) - (\mathcal{N}_2 + 1)\ln(\mathcal{N}_2 + 1)\}, \quad (33)$$

where we omitted time dependence on the right for practical convenience.

It is possible to study the role of the Fröhlich contribution using expression (33): Changing the value of F (the coupling strength associated to Fröhlich contribution) in Eqs. (22), (23), and (24), we may virtually compare the informational entropy in systems with different Fröhlich coupling strengths.

The pumped system of magnons presented in Fig. 1—where the magnon populations were numerically obtained from Eqs. (22) and (23)—has the informational entropy, obtained with the aid of Eq. (33), displayed as function of time in Fig. 3. In this figure we also show the time evolution of the informational entropy for the magnon populations obtained from Eqs. (22) and (23) with $F = 0$, i.e., a pumped magnon system with negligible Fröhlich contributions.

It can be noticed that the informational entropy values are lower when Fröhlich contribution is present, as it should as a result of having increasing ordering, that is, information increase. The same behavior occurs in the case of the informational entropy of the steady states as function of the scaled rate of pumping. The presence of the Fröhlich contribution leads to a decrease of the informational entropy precisely in the region of the condensate as shown in Fig. 4.

This decrease of informational entropy due to the Fröhlich contribution may be understood as some kind of increase in order and we introduce the order parameter to characterize this point:

$$\Delta(F, I) = \frac{\bar{S}_0^S(I) - \bar{S}_F^S(I)}{\bar{S}_0^S(I)} = 1 - \frac{\bar{S}_F^S(I)}{\bar{S}_0^S(I)}, \quad (34)$$

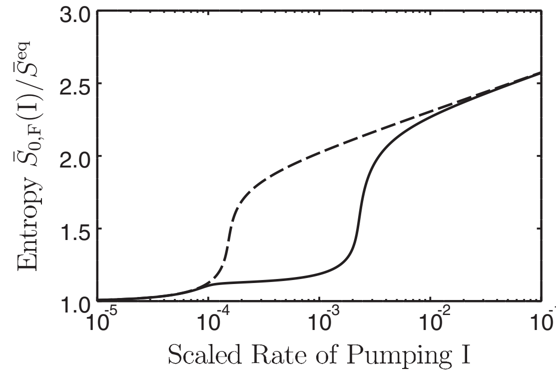

FIG. 4. Informational entropy of the steady states as function of the scaled rate of pumping I for systems with (solid) and without (dashed) Fröhlich contribution.

where $\bar{S}_F^S(I)$ and $\bar{S}_0^S(I)$ are the steady-state informational entropies with and without Fröhlich contribution, that is,

$$\bar{S}_F^S(I) = f_1\{(\mathcal{N}_1^S + 1)\ln(\mathcal{N}_1^S + 1) - \mathcal{N}_1^S \ln(\mathcal{N}_1^S)\} + f_2\{(\mathcal{N}_2^S + 1)\ln(\mathcal{N}_2^S + 1) - \mathcal{N}_2^S \ln(\mathcal{N}_2^S)\} \quad (35)$$

and

$$\bar{S}_0^S(I) = f_1\{(\mathcal{N}_1^{S,F=0} + 1)\ln(\mathcal{N}_1^{S,F=0} + 1) - \mathcal{N}_1^{S,F=0}\ln(\mathcal{N}_1^{S,F=0})\} + f_2\{(\mathcal{N}_2^{S,F=0} + 1)\ln(\mathcal{N}_2^{S,F=0} + 1) - \mathcal{N}_2^{S,F=0}\ln(\mathcal{N}_2^{S,F=0})\}, \quad (36)$$

where the dependence of the steady-state populations on I has not been explicitly indicated. Figure 5 presents the order parameter as function of the scaled rate of pumping which highlights this kind of complex order.

The role of the Fröhlich contribution may be evidenced through the numerical analysis of the order parameter as function of the Fröhlich contribution coupling strength when the rate of pumping is fixed. In Fig. 6 we present the mean steady-state populations $\mathcal{N}_{1,2}^S$ as function of F for fixed scaled rate of pumping $I = 8 \times 10^{-4}$ and the corresponding informational entropy order parameter.

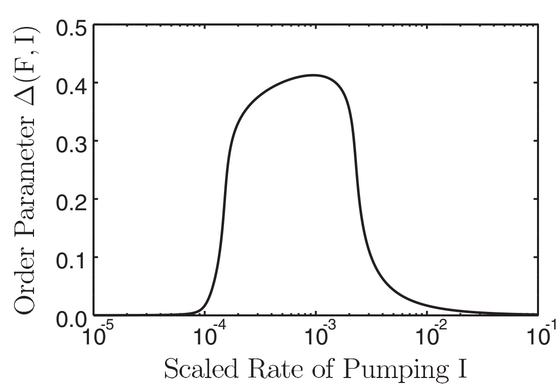

FIG. 5. Order parameter of Eq. (34) as function of the scaled rate of pumping I.

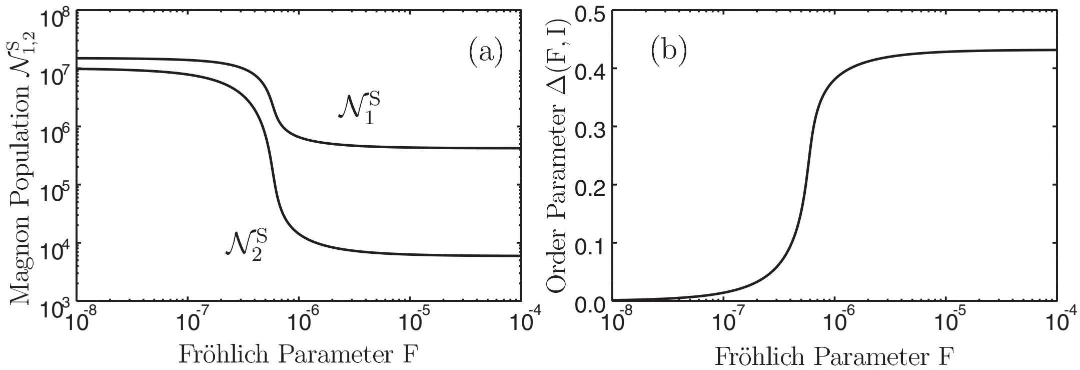

FIG. 6. (a) Steady-state magnon populations as function of F. (b) The related order parameter.

As can be seen in the Fig. 6(a), the magnon steady-state populations $\mathcal{N}^S_{1,2}$ decrease as the nonlinear Fröhlich contribution coupling strength increases (notably the mean population associated with high-frequency magnons $\mathcal{N}^S_2$). This complex behavior of the steady-state populations may be understood considering that (i) the Fröhlich contribution leads to the formation of the condensate in which the magnons with lower frequency are overpopulated at the expense of the higher in frequency populations and (ii) since the substantial decrease of $\mathcal{N}^S_2$ relative to $\mathcal{N}^S_1$ diminishes the absorbance of the material [because of the positive feedback effect of the parallel pumping, see Eq. (23a)] the net flux of absorbed energy is lower than the case without Fröhlich interaction, justifying the global fall of the mean population. The order parameter behavior, shown in Fig. 6(b), corroborates the idea that Fröhlich contribution enhances the complex order mentioned before.

B. IST entropy production

We analyze the informational-entropy production and, using Eq. (28) [paying attention to the logarithm of the partition functions in Eq. (29)], it can be shown that it is given by

$$\bar{\sigma}(t) = \frac{d}{dt}\bar{S}(t) = \beta_0 \frac{dE_B(t)}{dt} + \sum_{\mathbf{q}} F_{\mathbf{q}}(t)\frac{d\mathcal{N}_{\mathbf{q}}(t)}{dt}. \tag{37}$$

Taking into account Eqs. (17) and (18), it can be rewritten in terms of two contributions,

$$\bar{\sigma}(t) = \bar{\sigma}_i(t) + \bar{\sigma}_e(t), \tag{38}$$

consisting of the so-called internal one, $\bar{\sigma}_i(t)$, which results from internal interactions in the system, and the external one, $\bar{\sigma}_e(t)$, related to the interactions with the surroundings, in this case with the source and the thermal reservoir. These informational-entropy production terms are sometimes also called Π and $-\Phi$, respectively (cf. Ref. [4]) and, in our case, are given by

$$\bar{\sigma}_i(t) = \sum_{\mathbf{q}} \{F_{\mathbf{q}}(t)\,[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t) + \mathfrak{M}_{\mathbf{q}}(t)] - \beta_0\hbar\omega_{\mathbf{q}}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)]\}, \tag{39}$$

$$\bar{\sigma}_e(t) = \sum_{\mathbf{q}} \left\{F_{\mathbf{q}}(t)\,\mathfrak{S}_{\mathbf{q}}(t) + \beta_0\, J^{(2)}_{\mathrm{TD}}(t)\right\}, \tag{40}$$

or, using Eq. (20),

$$\bar{\sigma}_e(t) = \sum_{\mathbf{q}} \{F_{\mathbf{q}}(t)\,\mathfrak{S}_{\mathbf{q}}(t) + \beta_0\hbar\omega_{\mathbf{q}}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)]\}. \tag{41}$$

One may calculate the informational-entropy production on a two-fluid approach and complete expressions are given on Appendix C. We show the informational-entropy production of the system of magnons evolving in time (i.e., the entropy production associated with the evolution described in Fig. 1) in Fig. 7. We can observe that the total entropy production may have positive and negative values, although the internal entropy production is strictly non-negative—as it should.

Another important result, shown in Fig. 8, is the informational-entropy production for the steady states (see Fig. 2).

C. The evolution criterion

The change in time of IST-entropy production can be separated into two parts, namely

$$\frac{d}{dt}\bar{\sigma}(t) = \frac{d_Q}{dt}\bar{\sigma}(t) + \frac{d_F}{dt}\bar{\sigma}(t), \tag{42}$$

where

$$\frac{d_Q}{dt}\bar{\sigma}(t) = \sum_{\mathbf{q}} F_{\mathbf{q}}(t)\frac{d^2\mathcal{N}_{\mathbf{q}}(t)}{dt^2}, \tag{43}$$

that is, the part that accounts for the change in time of $\mathcal{N}_{\mathbf{q}}(t)$, and

$$\frac{d_F}{dt}\bar{\sigma}(t) = \sum_{\mathbf{q}} \frac{dF_{\mathbf{q}}(t)}{dt}\,\frac{d\mathcal{N}_{\mathbf{q}}(t)}{dt}, \tag{44}$$

accounting for the part of change in time of the nonequilibrium thermodynamics variables $F_{\mathbf{q}}(t)$. Recalling that $F_{\mathbf{q}}(t)$ may be expressed in terms of the populations [cf. Eq. (16)], we have that

$$\frac{dF_{\mathbf{q}}(t)}{dt} = \frac{d}{dt}\ln\left\{\frac{\mathcal{N}_{\mathbf{q}}(t)+1}{\mathcal{N}_{\mathbf{q}}(t)}\right\} = \left[\frac{1}{\mathcal{N}_{\mathbf{q}}(t)+1} - \frac{1}{\mathcal{N}_{\mathbf{q}}(t)}\right]\frac{d\mathcal{N}_{\mathbf{q}}(t)}{dt} = -\frac{1}{(\mathcal{N}_{\mathbf{q}}+1)\mathcal{N}_{\mathbf{q}}}\,\frac{d\mathcal{N}_{\mathbf{q}}(t)}{dt}, \tag{45}$$

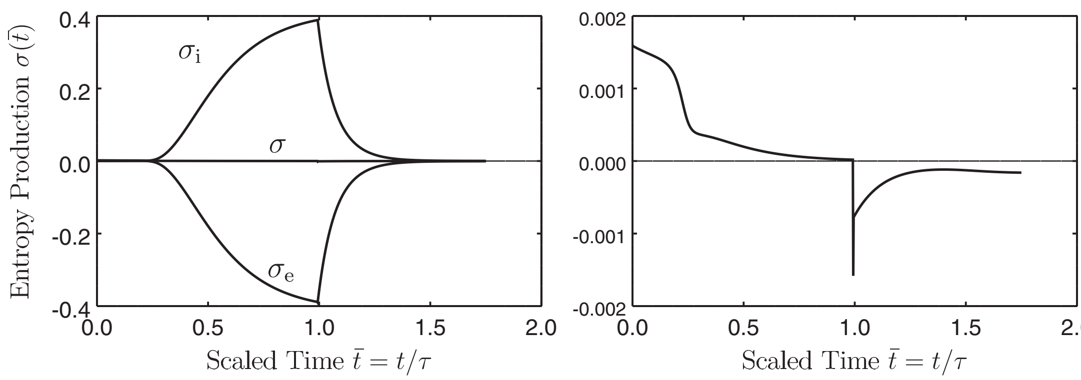

FIG. 7. Informational-entropy production of the system of magnons as function of time (associated with Figs. 1 and 3). On the left the internal, external and total entropy production (and we call attention to the expected non-negative values of the internal entropy production) can be observed; on the right only the total entropy production is shown. We recall that the pumping source is switched off at $\bar{t} = 1$.

and therefore

$$\frac{d_F}{dt}\bar{\sigma}(t) = -\sum_{\mathbf{q}} \frac{1}{(\mathcal{N}_{\mathbf{q}}+1)\mathcal{N}_{\mathbf{q}}}\left(\frac{d\mathcal{N}_{\mathbf{q}}}{dt}\right)^2 \leqslant 0. \tag{46}$$

This inequality verifies for this system the generalization of Glansdorff-Prigogine's thermodynamic criterion of evolution [18,19,33]. That is, along the trajectory of the macrostate of the system in the thermodynamic (or Gibbs) space of states, the quantity of Eq. (44) is always nonpositive, a quantity which in classical Onsagerian thermodynamics is the product of the change in time of the thermodynamic forces times the fluxes of matter and energy.

D. The (in)stability criterion

Within the above discussed framework of a nonequilibrium thermodynamics of the Fröhlich-Bose-Einstein condensation of magnons, we may analyze the stability of the steady-state populations $\mathcal{N}^S_{\mathbf{q}}$. Considering arbitrary small deviations, say, $\epsilon\,\eta_{\mathbf{q}}(t)$, from the steady state, we may expand the informational entropy in the form

$$\bar{S}(\{\mathcal{N}_{\mathbf{q}}(t)\}) = \bar{S}(\{\mathcal{N}^S_{\mathbf{q}} + \epsilon\eta_{\mathbf{q}}(t)\}) = \bar{S}(\{\mathcal{N}^S_{\mathbf{q}}\}) + \delta\bar{S} + \delta^2\bar{S} + \dots, \tag{47}$$

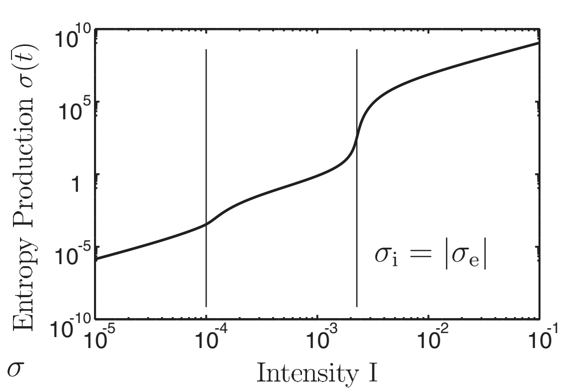

FIG. 8. Internal and external informational-entropy production absolute values of the system in steady state as function of the scaled rate of pumping I (related to Fig. 2); $\sigma_i = -\sigma_e$.

with

$$\delta^n\bar{S} = \left.\frac{\partial^n\bar{S}}{\partial\epsilon^n}\right|_{\epsilon=0}\frac{\epsilon^n}{n!}. \tag{48}$$

Since

$$\frac{\partial^2\bar{S}}{\partial\epsilon^2} = -\sum_{\mathbf{q}}\frac{\eta^2_{\mathbf{q}}(t)}{\big[\mathcal{N}^S_{\mathbf{q}} + \epsilon\eta_{\mathbf{q}}(t) + 1\big]\big[\mathcal{N}^S_{\mathbf{q}} + \epsilon\eta_{\mathbf{q}}(t)\big]}, \tag{49}$$

we have that the second variation of the entropy is

$$\delta^2\bar{S} = -\sum_{\mathbf{q}}\frac{\epsilon^2\eta^2_{\mathbf{q}}(t)}{(\mathcal{N}^S_{\mathbf{q}}+1)\mathcal{N}^S_{\mathbf{q}}} = -\sum_{\mathbf{q}}\frac{[\Delta\mathcal{N}_{\mathbf{q}}(t)]^2}{(\mathcal{N}^S_{\mathbf{q}}+1)\mathcal{N}^S_{\mathbf{q}}} \leqslant 0, \tag{50}$$

where $\Delta\mathcal{N}_{\mathbf{q}}(t)$ represents the value of the imposed arbitrary deviation from the steady state and the nonpositiveness of Eq. (50) is a manifestation of the convexity of the maximized informational entropy. Differentiation in time of Eq. (50) introduces the quantity called *excess of entropy production function*, namely

$$\delta^2\bar{\sigma}(t) = \frac{1}{2}\frac{d}{dt}\delta^2\bar{S}(t) = -\sum_{\mathbf{q}}\frac{\Delta\mathcal{N}_{\mathbf{q}}(t)}{(\mathcal{N}^S_{\mathbf{q}}+1)\mathcal{N}^S_{\mathbf{q}}}\frac{d}{dt}\Delta\mathcal{N}_{\mathbf{q}}(t), \tag{51}$$

which, in the two-fluid model has the following form:

$$\delta^2\bar{\sigma}(I,t) = -\frac{n_1\,\Delta\mathcal{N}_1(I,t)}{\big[\mathcal{N}^S_1(I)+1\big]\mathcal{N}^S_1(I)}\frac{d}{dt}\Delta\mathcal{N}_1(I,t) - \frac{n_2\,\Delta\mathcal{N}_2(I,t)}{\big[\mathcal{N}^S_2(I)+1\big]\mathcal{N}^S_2(I)}\frac{d}{dt}\Delta\mathcal{N}_2(I,t). \tag{52}$$

According to Glansdorff-Prigogine (in)stability criterion [18,19,33], if

$$\tfrac{1}{2}\,\delta^2\bar{S}(t)\,\delta^2\bar{\sigma}(t) \leqslant 0, \tag{53}$$

then the steady state is stable.

We have already proved above [cf. Eq. (50)] that $\delta^2\bar{S} \leqslant 0$ in the studied case. $\delta^2\bar{\sigma} \geqslant 0$ and this is so once $\Delta\mathcal{N}_{1,2}\frac{d\mathcal{N}_{1,2}}{dt} \leqslant 0$, as it follows from solving the evolution equations Eqs. (22) and (23). Therefore Eq. (53) is verified and hence, for the given constraints, the reference steady state is stable for all fluctuations compatible with the equations of evolution.

IV. CONCLUSION

We have considered the nonequilibrium statistical thermodynamics of the Fröhlich-Bose-Einstein condensation of magnons excited under the action of radio-frequency-radiation pumping. It constitutes an example of complexity in which, after a certain threshold has been attained in the value of the pumping source intensity, the energy pumped onto the system is transferred from higher- to lower-frequency modes in a cascading process.

Several important characteristics have been analyzed. First, we derived the so-called informational entropy for the system of magnons (Sec. III A) and then we have specified the magnon informational entropy for the "two-fluid model." Later, we introduced an order parameter in terms of informational entropy. This order parameter shows that the informational entropy is smaller when the nonlinear interaction responsible for the onset of the NEFBEC predominates, thus evidencing the order increase due to the Fröhlich interaction.

In Sec. III B we calculated the informational-entropy production function to characterize the contributions of the internal and external informational entropy production. The former has, as expected, non-negative values that characterizes dissipation, while the latter is negative as a result of the pumping on the system.

In informational nonequilibrium statistical thermodynamics, this is related with the generalized $\mathscr{H}$ theorem in the sense of Jancel [34], which we have called *weak principle of informational entropy increasing*. Namely, given the informational statistical entropy $\bar{S}(t)$ of Eq. (28) and the informational entropy production $\sigma(t)$ of Eq. (37), the principle tells us that

$$\Delta\bar{S}(t) = \bar{S}(t) - \bar{S}(t_0) = \int_{-\infty}^{t} dt'\,\sigma(t') \geqslant 0. \tag{54}$$

The informational entropy with the evolution property of Eq. (54) is the coarse-grained entropy of Eq. (25), with the coarse-graining being performed by the action of the projection operator $\mathcal{P}_\varepsilon(t)$ of Eqs. (26) and (27). This operator projects at any time the logarithm of the NSD over the logarithm of the auxiliary (coarse-grained) distribution embedded in the subspace spanned by the basic dynamic quantities (cf. set (5), see also Ref. [32]). The consequence of this projection is the information loss that is reflected in the increase of informational entropy expressed in Eq. (54). This result is a consequence of the presence of the irreversible part of $\hat{\varrho}_\varepsilon(t)$ not contained in $\hat{\bar{\varrho}}(t)$, which is the part that accounts for the processes which generate dissipation in the description of the macroscopic state of the system.

We restate that the information lost in each particular problem as a result of truncating the set of basic variables must be carefully evaluated [35,36].

Finally, it has been verified that the Glansdorff-Prigogine evolution criterion is satisfied (Sec. III C), and from the generalization of the Glansdorff-Prigogine (in)stability principle we have shown that the nonequilibrium thermodynamic state of the system is stable under any condition (Sec. III D). We stress that such instability has been derived in relation to other possible homogeneous states, but instability against the onset of a spatially ordered state cannot be ruled out and is still under consideration.

ACKNOWLEDGMENT

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APPENDIX A: HAMILTONIAN OF THE SYSTEM

The Hamiltonian of the spin system in quasiparticle formalism may be written as

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_0 + \hat{\mathscr{H}}', \tag{A1}$$

with

$$\hat{\mathscr{H}}_0 = \hat{\mathscr{H}}_S^{(2)} + \hat{\mathscr{H}}_L + \hat{\mathscr{H}}_R = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{q}}\hat{c}_{\mathbf{q}} + \sum_{\mathbf{k}} \hbar\Omega_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \sum_{\mathbf{p}} \hbar\zeta_{\mathbf{p}}\hat{d}^\dagger_{\mathbf{q}}\hat{d}_{\mathbf{q}} \tag{A2}$$

being the noninteracting term formed by the Hamiltonians of free magnons, phonons, and photons and $\hbar\omega_{\mathbf{q}}$, $\hbar\Omega_{\mathbf{k}}$, and $\hbar\zeta_{\mathbf{p}}$ their energies. The other term,

$$\hat{\mathscr{H}}' = \hat{\mathscr{H}}_{\rm MM} + \hat{\mathscr{H}}_{\rm SL} + \hat{\mathscr{H}}_{\rm SR}, \tag{A3}$$

includes the interactions between quasiparticles:

$$\hat{\mathscr{H}}_{\rm MM} = \sum_{\mathbf{q},\mathbf{q}_1,\mathbf{q}_2} \mathcal{V}_{\mathbf{q},\mathbf{q}_1,\mathbf{q}_2}\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{q}_1}\hat{c}_{\mathbf{q}_2}\hat{c}_{\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2} \tag{A4}$$

is the magnon-magnon scattering term,

$$\hat{\mathscr{H}}_{\rm SL} = \sum_{\mathbf{q},\mathbf{k}\neq 0}(\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}})\{\mathcal{F}_{\mathbf{q},\mathbf{k}}\hat{c}^\dagger_{\mathbf{q}}\hat{c}_{\mathbf{q}-\mathbf{k}} + \mathcal{L}_{\mathbf{q},\mathbf{k}}\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{k}-\mathbf{q}} + \mathcal{L}^*_{\mathbf{q},-\mathbf{k}}\hat{c}_{\mathbf{q}}\hat{c}_{-\mathbf{k}-\mathbf{q}}\} + \sum_{\mathbf{q},\mathbf{k}\neq 0}\{\mathcal{R}_{\mathbf{q},\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}-\mathbf{q}} + \mathcal{R}^+_{\mathbf{q},\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{q}-\mathbf{k}} + \mathcal{R}^{+*}_{-\mathbf{q},-\mathbf{k}}\hat{b}_{-\mathbf{k}}\hat{b}_{\mathbf{k}-\mathbf{q}}\}(\hat{c}_{\mathbf{q}} + \hat{c}^\dagger_{-\mathbf{q}}) \tag{A5}$$

accounts for the relevant magnon-phonon interaction, and

$$\hat{\mathscr{H}}_{\rm SR} = \sum_{\mathbf{p}}(\hat{d}_{\mathbf{p}} + \hat{d}^\dagger_{-\mathbf{p}})(\mathcal{S}^{\perp *}_{\mathbf{p}}\hat{c}^\dagger_{\mathbf{p}} + \mathcal{S}^{\perp}_{-\mathbf{p}}\hat{c}_{-\mathbf{p}}) + \sum_{\mathbf{p},\mathbf{q}}(\hat{d}_{\mathbf{p}} + \hat{d}^\dagger_{-\mathbf{p}})\{\mathcal{S}^{\|\rm a}_{\mathbf{q},\mathbf{p}}\hat{c}^\dagger_{\mathbf{q}}\hat{c}_{\mathbf{q}-\mathbf{p}} + \mathcal{S}^{\|\rm b}_{\mathbf{q},\mathbf{p}}\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}-\mathbf{q}} + \mathcal{S}^{\|\rm b*}_{\mathbf{q},-\mathbf{p}}\hat{c}_{-\mathbf{q}}\hat{c}_{\mathbf{q}-\mathbf{p}}\} \tag{A6}$$

is the interaction between magnons and photons (source and black-body radiation).

APPENDIX B: KINETIC EQUATIONS

$$\frac{d}{dt}\mathcal{N}_{\mathbf{q}}(t) = \frac{1}{i\hbar}\mathrm{Tr}\{[\hat{\mathcal{N}}_{\mathbf{q}}, \hat{\mathscr{H}}]\,\hat{\varrho}_\varepsilon(t)\times\hat{\varrho}_B\} = J^{(0)}_{\mathcal{N}_{\mathbf{q}}}(t) + J^{(1)}_{\mathcal{N}_{\mathbf{q}}}(t) + \mathscr{J}^{(2)}_{\mathcal{N}_{\mathbf{q}}}(t), \tag{B1}$$

$$J^{(0)}_{\mathcal{N}_{\mathbf{q}}}(t) = \frac{1}{i\hbar}\mathrm{Tr}\{[\hat{\mathcal{N}}_{\mathbf{q}}, \hat{\mathscr{H}}_0]\,\bar{\hat{\varrho}}(t,0)\times\hat{\varrho}_B\} = 0, \tag{B2}$$

$$J^{(1)}_{\mathcal{N}_{\mathbf{q}}}(t) = \frac{1}{i\hbar}\mathrm{Tr}\{[\hat{\mathcal{N}}_{\mathbf{q}}, \hat{\mathscr{H}}']\,\bar{\hat{\varrho}}(t,0)\times\hat{\varrho}_B\} = 0, \tag{B3}$$

$$\begin{aligned}\mathscr{J}^{(2)}_{\mathcal{N}_{\mathbf{q}}}(t) \simeq J^{(2)}_{\mathcal{N}_{\mathbf{q}}}(t) = {}& \frac{1}{(i\hbar)^2}\int_{-\infty}^{t} dt'\, \mathrm{e}^{\varepsilon(t'-t)}\,\mathrm{Tr}\{[\hat{\mathscr{H}}'(t'-t)_0, [\hat{\mathscr{H}}', \hat{\mathcal{N}}_{\mathbf{q}}]]\,\bar{\hat{\varrho}}(t,0)\times\hat{\varrho}_B\} \\ & + \frac{1}{i\hbar}\sum_{\ell}\int_{-\infty}^{t} dt'\, \mathrm{e}^{\varepsilon(t'-t)}\mathrm{Tr}\{[\hat{\mathscr{H}}'(t'-t)_0, \hat{P}_\ell]\,\bar{\hat{\varrho}}(t,0)\times\hat{\varrho}_B\}\frac{\delta J^{(1)}_{\mathcal{N}_{\mathbf{q}}}(t)}{\delta Q_\ell(t)},\end{aligned} \tag{B4}$$

with $\hat{P}_\ell$ and $\hat{Q}_\ell$ being the variables of sets (6) and (5), respectively, and

$$\hat{O}(t)_0 = e^{-\frac{t}{i\hbar}\hat{\mathscr{H}}_0}\hat{O}e^{\frac{t}{i\hbar}\hat{\mathscr{H}}_0}, \tag{B5}$$

δ stands for functional differentiation.

In a compact form we may write

$$\frac{d}{dt}\mathcal{N}_{\mathbf{q}}(t) = \mathfrak{S}_{\mathbf{q}}(t) + \mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t) + \mathfrak{M}_{\mathbf{q}}(t), \tag{B6}$$

where

$$\mathfrak{S}_{\mathbf{q}}(t) = \frac{8\pi}{\hbar^2}\sum_{\mathbf{q}'\neq-\mathbf{q}}\left|\mathcal{S}^{\parallel \mathrm{b}}_{\mathbf{q},\mathbf{q}+\mathbf{q}'}\right|^2\left\{(1+\mathcal{N}_{\mathbf{q}}+\mathcal{N}_{\mathbf{q}'})f^{S}_{\mathbf{q}'+\mathbf{q}}\right\}\delta(\omega_{\mathbf{q}}+\omega_{\mathbf{q}'}-\zeta_{\mathbf{q}+\mathbf{q}'}) \tag{B7}$$

is the source term that accounts for the pumping of energy to the system, $f^{S}_{\mathbf{q}'+\mathbf{q}}$ stands for the population of photons of the source;

$$\mathfrak{R}_{\mathbf{q}}(t) = \frac{8\pi}{\hbar^2}\sum_{\mathbf{q}'\neq-\mathbf{q}}\left|\mathcal{S}^{\parallel \mathrm{b}}_{\mathbf{q},\mathbf{q}+\mathbf{q}'}\right|^2\left\{(\mathcal{N}_{\mathbf{q}'}+1)(\mathcal{N}_{\mathbf{q}}+1)f^{\mathrm{T}}_{\mathbf{q}'+\mathbf{q}} - \mathcal{N}_{\mathbf{q}'}\mathcal{N}_{\mathbf{q}}\left(f^{\mathrm{T}}_{\mathbf{q}'+\mathbf{q}}+1\right)\right\}\delta(\omega_{\mathbf{q}}+\omega_{\mathbf{q}'}-\zeta_{\mathbf{q}+\mathbf{q}'}) \tag{B8}$$

is a nonlinear term of interaction between the spin subsystem and the black-body radiation ($f^{\mathrm{T}}_{\mathbf{q}'+\mathbf{q}}$ being its photon's population);

$$L_{\mathbf{q}}(t) = -\frac{1}{\tau_{\mathbf{q}}}\left[\mathcal{N}_{\mathbf{q}} - \mathcal{N}^{(0)}_{\mathbf{q}}\right] \tag{B9}$$

is the linear relaxation to the lattice with characteristic time $\tau_{\mathbf{q}}$. The last two terms are nonlinear contributions;

$$\begin{aligned}\mathfrak{F}_{\mathbf{q}}(t) = {}& \frac{2\pi}{\hbar^2}\sum_{\mathbf{q}'\neq\mathbf{q}}|\mathcal{F}_{\mathbf{q},\mathbf{q}-\mathbf{q}'}|^2\{\mathcal{N}_{\mathbf{q}'}(\mathcal{N}_{\mathbf{q}}+1)(\nu_{\mathbf{q}'-\mathbf{q}}+1) - (\mathcal{N}_{\mathbf{q}'}+1)\mathcal{N}_{\mathbf{q}}\nu_{\mathbf{q}'-\mathbf{q}}\}\delta(\omega_{\mathbf{q}'}-\omega_{\mathbf{q}}-\Omega_{\mathbf{q}'-\mathbf{q}}) \\ & + \frac{2\pi}{\hbar^2}\sum_{\mathbf{q}'\neq\mathbf{q}}|\mathcal{F}_{\mathbf{q},\mathbf{q}-\mathbf{q}'}|^2\{(\mathcal{N}_{\mathbf{q}}+1)\mathcal{N}_{\mathbf{q}'}\nu_{\mathbf{q}-\mathbf{q}'} - \mathcal{N}_{\mathbf{q}}(\mathcal{N}_{\mathbf{q}'}+1)(\nu_{\mathbf{q}-\mathbf{q}'}+1)\}\delta(\omega_{\mathbf{q}'}-\omega_{\mathbf{q}}+\Omega_{\mathbf{q}-\mathbf{q}'}),\end{aligned} \tag{B10}$$

the so-called Fröhlich term, a nonlinear interaction between magnons mediated by the lattice, and

$$\begin{aligned}\mathfrak{M}_{\mathbf{q}}(t) = {}& \frac{16\pi}{\hbar^2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}|\mathcal{V}_{\mathbf{q},\mathbf{q}_1,\mathbf{q}_2}|^2\{(\mathcal{N}_{\mathbf{q}}+1)(\mathcal{N}_{\mathbf{q}_1}+1)\mathcal{N}_{\mathbf{q}_2}\mathcal{N}_{\mathbf{q}_3} - \mathcal{N}_{\mathbf{q}}\mathcal{N}_{\mathbf{q}_1}(\mathcal{N}_{\mathbf{q}_2}+1)(\mathcal{N}_{\mathbf{q}_3}+1)\} \\ & \times\ \delta(\omega_{\mathbf{q}}+\omega_{\mathbf{q}_1}-\omega_{\mathbf{q}_2}-\omega_{\mathbf{q}_3})\delta_{\mathbf{q}_3,\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2}\end{aligned} \tag{B11}$$

accounts for the magnon-magnon scattering interaction term.

In a similar form of Eq. (B1) we have that

$$\frac{d}{dt}E_B(t) = \frac{1}{i\hbar}\mathrm{Tr}\{[\hat{\mathscr{H}}_B, \hat{\mathscr{H}}]\,\hat{\varrho}_\varepsilon(t)\times\hat{\varrho}_B\} \simeq J^{(0)}_{E_B}(t) + J^{(1)}_{E_B}(t) + J^{(2)}_{E_B}(t). \tag{B12}$$

It is simple to show that $J^{(0)}_{E_B}(t)$ and $J^{(1)}_{E_B}(t)$ are null. The last term is composed of two contributions:

$$J^{(2)}_{E_B}(t) = J^{(2)}_{\mathrm{TD}}(t) - \sum_{\mathbf{q}}\hbar\omega_{\mathbf{q}}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)], \tag{B13}$$

the first,

$$J_{\mathrm{TD}}^{(2)}(t) = -\frac{E_B(t) - E_B^{(0)}}{\tau_{\mathrm{TD}}}, \tag{B14}$$

is the contribution which accounts for the thermal diffusion to the reservoir with a thermal diffusion time τ_{TD} and tends to lead the thermal bath to equilibrium (characterized by the equilibrium energy $E_B^{(0)}$). The other contribution is related to the energy received from the subsystem of magnons.

APPENDIX C: TWO-FLUID INFORMATIONAL-ENTROPY PRODUCTION

In the two-fluid model the informational-entropy production is thus given by

$$\bar{\sigma}_i(t) = \sum_{\mathbf{q}}\{[F_{\mathbf{q}}(t) - \beta_0\hbar\omega_{\mathbf{q}}]\,[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)] + F_{\mathbf{q}}(t)\,\mathfrak{M}_{\mathbf{q}}(t)\} \tag{C1}$$

$$\begin{aligned}
&\approx \left\{\ln\left(\frac{\mathcal{N}_1+1}{\mathcal{N}_1}\right) - \beta_0\hbar\omega_1\right\}\sum_{\mathbf{q}\in R_1}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)] + \left\{\ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right) - \beta_0\hbar\omega_2\right\}\sum_{\mathbf{q}\in R_2}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)]\\
&\quad + \ln\left(\frac{\mathcal{N}_1+1}{\mathcal{N}_1}\right)\sum_{\mathbf{q}\in R_1}\mathfrak{M}_{\mathbf{q}}(t) + \ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right)\sum_{\mathbf{q}\in R_2}\mathfrak{M}_{\mathbf{q}}(t)\\
&= -\frac{n}{\tau}\left\{\ln\left(\frac{\mathcal{N}_1+1}{\mathcal{N}_1}\right) - \beta_0\hbar\omega_1\right\}\left\{D\,\mathcal{N}_1\big(\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big) + f_1\big[\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big]\right\}\\
&\quad + \frac{n}{\tau}\left\{\ln\left(\frac{\mathcal{N}_1+1}{\mathcal{N}_1}\right) - \beta_0\hbar\omega_1\right\}F\{\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1\}\\
&\quad - \frac{n}{\tau}\left\{\ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right) - \beta_0\hbar\omega_2\right\}\left\{D\,\mathcal{N}_2\big(\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big) + f_2\big[\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big]\right\}\\
&\quad - \frac{n}{\tau}\left\{\ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right) - \beta_0\hbar\omega_2\right\}F\{\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1\}\\
&\quad + \frac{n}{\tau}\left\{\ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right) - \ln\left(\frac{\mathcal{N}_1+1}{\mathcal{N}_1}\right)\right\}M\{\mathcal{N}_1(\mathcal{N}_1+1) + \mathcal{N}_2(\mathcal{N}_2+1)\}\left(\mathcal{N}_1\frac{\mathcal{N}_2^{(0)}}{\mathcal{N}_1^{(0)}} - \mathcal{N}_2\right),
\end{aligned} \tag{C2}$$

and

$$\begin{aligned}
\bar{\sigma}_e(t) &= \sum_{\mathbf{q}}\{F_{\mathbf{q}}(t)\,\mathfrak{S}_{\mathbf{q}}(t) + \beta_0\hbar\omega_{\mathbf{q}}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)]\}\\
&\approx \ln\left(\frac{\mathcal{N}_1+1}{\mathcal{N}_1}\right)\sum_{\mathbf{q}\in R_1}\mathfrak{S}_{\mathbf{q}}(t) + \ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right)\sum_{\mathbf{q}\in R_2}\mathfrak{S}_{\mathbf{q}}(t)\\
&\quad + \beta_0\hbar\omega_1\sum_{\mathbf{q}\in R_1}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)] + \beta_0\hbar\omega_2\sum_{\mathbf{q}\in R_2}[\mathfrak{R}_{\mathbf{q}}(t) + L_{\mathbf{q}}(t) + \mathfrak{F}_{\mathbf{q}}(t)]\\
&= \frac{n}{\tau}\ln\left(\frac{\mathcal{N}_2+1}{\mathcal{N}_2}\right)I\,(1 + 2\mathcal{N}_2)\\
&\quad + \frac{n}{\tau}\,\beta_0\hbar\omega_1\big\{-D\,\mathcal{N}_1\big(\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big) - f_1\big[\mathcal{N}_1 - \mathcal{N}_1^{(0)}\big] + F[\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1]\big\}\\
&\quad + \frac{n}{\tau}\,\beta_0\hbar\omega_2\big\{-D\,\mathcal{N}_2\big(\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big) - f_2\big[\mathcal{N}_2 - \mathcal{N}_2^{(0)}\big] - F[\mathcal{N}_1\mathcal{N}_2 + (\bar{\nu}+1)\mathcal{N}_2 - \bar{\nu}\mathcal{N}_1]\big\},
\end{aligned} \tag{C3}$$

where we used that

$$\ln\left(\frac{\mathcal{N}_{1,2}^{(0)}+1}{\mathcal{N}_{1,2}^{(0)}}\right) = \beta_0\hbar\omega_{1,2}, \tag{C4}$$

with $\mathcal{N}_{1,2}^{(0)}$ being the distribution in equilibrium.

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