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Research Article

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A fractional Kirchhoff problem involving a singular term and a critical nonlinearity

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Abstract: In this paper, we consider the following critical nonlocal problem:

$$\begin{cases} M\Big(\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)(-\Delta)^s u = \frac{\lambda}{u^\gamma} + u^{2_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with continuous boundary, dimension $N > 2s$ with parameter $s \in (0,1)$, $2_s^* = 2N/(N-2s)$ is the fractional critical Sobolev exponent, $\lambda > 0$ is a real parameter, $\gamma \in (0,1)$ and M models a Kirchhoff-type coefficient, while $(-\Delta)^s$ is the fractional Laplace operator. In particular, we cover the delicate degenerate case, that is, when the Kirchhoff function M is zero at zero. By combining variational methods with an appropriate truncation argument, we provide the existence of two solutions.

Keywords: Kirchhoff-type problems, fractional Laplacian, singularities, critical nonlinearities, perturbation methods

MSC 2010: Primary 35J75, 35R11, 49J35; secondary 35A15, 45G05, 35S15

1 Introduction

This paper is devoted to the study of a class of Kirchhoff-type problems driven by a nonlocal fractional operator and involving a singular term and a critical nonlinearity. More precisely, we consider

$$\begin{cases} \Big(\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\theta-1}(-\Delta)^s u = \frac{\lambda}{u^\gamma} + u^{2_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with continuous boundary, dimension $N > 2s$ with parameter $s \in (0,1)$, $2_s^* = 2N/(N-2s)$ is the fractional critical Sobolev exponent, $\lambda > 0$ is a real parameter, $\theta \in (1, 2_s^*/2)$, while $\gamma \in (0,1)$. Here $(-\Delta)^s$ is the fractional Laplace operator defined, up to normalization factors, by the Riesz potential as

$$(-\Delta)^s \varphi(x) = \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}}\,dy, \quad x \in \mathbb{R}^N,$$

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along any $\varphi \in C_0^\infty(\Omega)$; we refer to [11] and the recent monograph [22] for further details on the fractional Laplacian and the fractional Sobolev spaces $H^s(\mathbb{R}^N)$ and $H_0^s(\Omega)$.

As is well explained in [11, 22], problem (1.1) is the fractional version of the following nonlinear problem:

$$\begin{cases} -M\Big(\int\limits_\Omega |\nabla u(x)|^2\,dx\Big)\Delta u = \dfrac{\lambda}{u^\gamma} + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

where Δ denotes the classical Laplace operator while, just for a general discussion, $M(t) = t^{\theta-1}$ for any $t \in \mathbb{R}_0^+$. In literature, problems like (1.1) and (1.2) are called of Kirchhoff type whenever the function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ models the Kirchhoff prototype, given by

$$M(t) = a + bt^{\theta-1}, \quad a,\ b \ge 0,\ a + b > 0,\ \theta \ge 1. \tag{1.3}$$

In particular, when $M(t) \ge$ constant > 0 for any $t \in \mathbb{R}_0^+$, Kirchhoff problems are said to be *non-degenerate* and this happens for example if $a > 0$ in the model case (1.3). While if $M(0) = 0$ but $M(t) > 0$ for any $t \in \mathbb{R}^+$, Kirchhoff problems are called *degenerate*. Of course, for (1.3) this occurs when $a = 0$.

This kind of nonlocal problems has been widely studied in recent years. We refer to [17–21] for different Kirchhoff problems with M like in (1.3), driven by the Laplace operator and involving a singular term of type $u^{-\gamma}$. In [21], Liu and Sun study a Kirchhoff problem with a singular term and a Hardy potential by using the Nehari method. The same approach is used in [19] for a singular Kirchhoff problem with also a subcritical term. In [17], strongly assuming $a > 0$ in (1.3), Lei, Liao and Tang prove the existence of two solutions for a Kirchhoff problem like (1.2) by combining perturbation and variational methods. While in [18], Liao, Ke, Lei and Tang provide a uniqueness result for a singular Kirchhoff problem involving a negative critical nonlinearity by a minimization argument. By arguing similarly to [17], Liu, Tang, Liao and Wu [20] give the existence of two solutions for a critical Kirchhoff problem with a singular term of type $|x|^{-\beta}u^{-\gamma}$.

Problem (1.1) has been studied by Barrios, De Bonis, Medina and Peral [4] when $\theta = 1$, namely without a Kirchhoff coefficient. They prove the existence of two solutions by applying the sub/supersolutions and Sattinger methods. In [8], Canino, Montoro, Sciunzi and Squassina generalize the results of [4, Section 3] to the delicate case of the p-fractional Laplace operator $(-\Delta)_p^s$. While in the last section of [1], Abdellaoui, Medina, Peral and Primo provide the existence of a solution for nonlinear fractional problems with a singularity like $u^{-\gamma}$ and a fractional Hardy term by perturbation methods. Concerning fractional Kirchhoff problems involving critical nonlinearities, we refer to [2, 9, 13, 14, 16, 23] for existence results and to [5, 12, 24, 25, 29] for multiplicity results. In particular, in [9, 13, 14, 23] different singular terms appear, but are given by the fractional Hardy potential.

Inspired by the above works, we study a multiplicity result for problem (1.1). As far as we know, a fractional Kirchhoff problem involving a singular term of type $u^{-\gamma}$ has not been studied yet. We can state our result as follows.

Theorem 1.1. *Let $s \in (0,1)$, $N > 2s$, $\theta \in (1, 2_s^*/2)$, $\gamma \in (0,1)$ and let Ω be an open bounded subset of $\mathbb{R}^N$ with $\partial\Omega$ continuous. Then there exists $\overline{\lambda} > 0$ such that for any $\lambda \in (0, \overline{\lambda})$ problem (1.1) has at least two different solutions.*

The first solution of problem (1.1) is obtained by a suitable minimization argument, where we must pay attention to the nonlocal nature of the fractional Laplacian. Concerning the second solution, because of the presence of $u^{-\gamma}$, we can not apply the usual critical point theory to problem (1.1). For this, we first study a perturbed problem obtained by truncating the singular term $u^{-\gamma}$. Then by approximation we get our second solution of (1.1).

Finally, we observe that Theorem 1.1 generalizes in several directions the first part of [4, Theorem 4.1] and [17, Theorem 1.1].

The paper is organized as follows: In Section 2, we discuss the variational formulation of problem (1.1), and we introduce the perturbed problem. In Section 3, we prove the existence of the first solution of (1.1), and we give a possible generalization of this existence result at the end of the section. In Section 4, we prove the existence of a mountain pass solution for the perturbed problem. In Section 5, we prove Theorem 1.1.

2 Variational setting

Throughout this paper, we assume without further mentioning that $s \in (0,1)$, $N > 2s$, $\theta \in (1, 2_s^*/2)$, $\gamma \in (0,1)$ and Ω is an open bounded subset of $\mathbb{R}^N$ with $\partial\Omega$ continuous. As a matter of notations, we denote with $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = \max\{-\varphi, 0\}$ respectively the positive and negative part of a function φ.

Problem (1.1) has a variational structure, and the natural space where to find solutions is the homogeneous fractional Sobolev space $H_0^s(\Omega)$. In order to study (1.1) it is important to encode the "boundary condition" $u = 0$ in $\mathbb{R}^N \setminus \Omega$ in the weak formulation, by considering also that the interaction between Ω and its complementary in $\mathbb{R}^N$ gives a positive contribution in the so-called *Gagliardo norm*, given as

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \Bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \Bigg)^{1/2}. \tag{2.1}$$

The functional space that takes into account this boundary condition will be denoted by X_0 and it is defined as

$$X_0 = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

We refer to [26] for a general definition of X_0 and its properties. We also would like to point out that when $\partial\Omega$ is continuous, by [15, Theorem 6] the space $C_0^\infty(\Omega)$ is dense in X_0, with respect to the norm (2.1). This last point will be used to overcome the singularity in problem (1.1).

In X_0 we can consider the norm

$$\|u\|_{X_0} = \Bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \Bigg)^{1/2},$$

which is equivalent to the usual one defined in (2.1) (see [26, Lemma 6]). We also recall that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with the scalar product defined as

$$\langle u, v\rangle_{X_0} = \iint\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy.$$

From now on, in order to simplify the notation, we will denote $\|\cdot\|_{X_0}$ and $\langle\cdot,\cdot\rangle_{X_0}$ by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, and $\|\cdot\|_{L^q(\Omega)}$ by $\|\cdot\|_q$ for any $q \in [1, \infty]$.

In order to present the weak formulation of (1.1) and taking into account that we are looking for positive solutions, we will consider the following Kirchhoff problem:

$$\begin{cases} \Bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \Bigg)^{\theta-1} (-\Delta)^s u = \dfrac{\lambda}{(u^+)^\gamma} + (u^+)^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.2}$$

We say that $u \in X_0$ is a (weak) solution of problem (2.2) if u satisfies

$$\|u\|^{2(\theta-1)}\langle u, \varphi\rangle = \lambda \int\limits_\Omega \frac{\varphi}{(u^+)^\gamma}\,dx + \int\limits_\Omega (u^+)^{2_s^*-1}\varphi \tag{2.3}$$

for any $\varphi \in X_0$. Problem (2.2) has a variational structure and $J_\lambda : X_0 \to \mathbb{R}$, defined by

$$J_\lambda(u) = \frac{1}{2\theta}\|u\|^{2\theta} - \frac{\lambda}{1-\gamma}\int\limits_\Omega (u^+)^{1-\gamma}\,dx - \frac{1}{2_s^*}\|u^+\|_{2_s^*}^{2_s^*},$$

is the underlying functional associated to (2.2). Because of the presence of a singular term in (2.2), the functional J_λ is not differentiable on X_0. Therefore, we can not apply directly the usual critical point theory to J_λ in order to solve problem (2.2). However, it is possible to find a first solution of (2.2) by using a local minimization argument. In order to get the second solution of (2.2) we have to study an associated approximating problem. That is, for any $n \in \mathbb{N}$, we consider the following perturbed problem:

$$\begin{cases} \left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \right)^{\theta-1} (-\Delta)^s u = \frac{\lambda}{(u^+ + \frac{1}{n})^\gamma} + (u^+)^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.4}$$

For this, we say that $u \in X_0$ is a (weak) solution of problem (2.4) if u satisfies

$$\|u\|^{2(\theta-1)}\langle u, \varphi\rangle = \lambda \int\limits_\Omega \frac{\varphi}{(u^+ + \frac{1}{n})^\gamma}\,dx + \int\limits_\Omega (u^+)^{2_s^*-1}\varphi \tag{2.5}$$

for any $\varphi \in X_0$. In this case, solutions of (2.4) correspond to the critical points of the functional $J_{n,\lambda} : X_0 \to \mathbb{R}$, set as

$$J_{n,\lambda}(u) = \frac{1}{2\theta}\|u\|^{2\theta} - \frac{\lambda}{1-\gamma}\int\limits_\Omega \left[\left(u^+ + \frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma} \right] dx - \frac{1}{2_s^*}\|u^+\|_{2_s^*}^{2_s^*}. \tag{2.6}$$

It is immediate to see that $J_{n,\lambda}$ is of class $C^1(X_0)$.

We conclude this section by recalling the best constant of the fractional Sobolev embedding, which will be very useful to study the compactness property of the functional $J_{n,\lambda}$. That is, we consider

$$S = \inf_{\substack{v \in H^s(\mathbb{R}^N) \\ v \not\equiv 0}} \frac{\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\left(\int_{\mathbb{R}^N} |v(x)|^{2_s^*}\,dx \right)^{2/2_s^*}}, \tag{2.7}$$

which is well defined and strictly positive, as shown in [10, Theorem 1.1].

3 A first solution for problem (1.1)

In this section, we prove the existence of a solution for problem (1.1) by a local minimization argument. For this, we first study the geometry of the functional J_λ.

Lemma 3.1. *There exist numbers $\rho \in (0, 1]$, $\lambda_0 = \lambda_0(\rho) > 0$ and $\alpha = \alpha(\rho) > 0$ such that $J_\lambda(u) \geq \alpha$ for any $u \in X_0$, with $\|u\| = \rho$, and for any $\lambda \in (0, \lambda_0]$. Furthermore, set*

$$m_\lambda = \inf\{J_\lambda(u) : u \in \overline{B}_\rho\},$$

where $\overline{B}_\rho = \{u \in X_0 : \|u\| \leq \rho\}$. Then $m_\lambda < 0$ for any $\lambda \in (0, \lambda_0]$.

Proof. Let $\lambda > 0$. From the Hölder inequality and (2.7) for any $u \in X_0$ we have

$$\int\limits_\Omega (u^+)^{1-\gamma}\,dx \leq |\Omega|^{(2_s^*-1+\gamma)/2_s^*}\|u\|_{2_s^*}^{1-\gamma} \leq |\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\|u\|^{1-\gamma}. \tag{3.1}$$

Hence, using again (2.7) and (3.1), we get

$$J_\lambda(u) \geq \frac{1}{2\theta}\|u\|^{2\theta} - \frac{S^{-2_s^*/2}}{2_s^*}\|u\|^{2_s^*} - \frac{\lambda}{1-\gamma}|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\|u\|^{1-\gamma}.$$

Since $1 - \gamma < 1 < 2\theta < 2_s^*$, the function

$$\eta(t) = \frac{1}{2\theta}t^{2\theta-1+\gamma} - \frac{S^{-2_s^*/2}}{2_s^*}t^{2_s^*-1+\gamma}, \quad t \in [0, 1],$$

admits a maximum at some $\rho \in (0,1]$ small enough, that is, $\max_{t\in[0,1]} \eta(t) = \eta(\rho) > 0$. Thus, let

$$\lambda_0 = \frac{(1-\gamma)S^{(1-\gamma)/2}}{2|\Omega|^{(2_s^*-1+\gamma)/2_s^*}}\eta(\rho).$$

Then for any $u \in X_0$ with $\|u\| = \rho \le 1$ and for any $\lambda \le \lambda_0$, we get $J_\lambda(u) \ge \rho^{1-\gamma}\eta(\rho)/2 = \alpha > 0$.

Furthermore, fixed $v \in X_0$ with $v^+ \not\equiv 0$, for $t \in (0,1)$ sufficiently small we have

$$J_\lambda(tv) = \frac{t^{2\theta}}{2\theta}\|v\|^{2\theta} - t^{1-\gamma}\frac{\lambda}{1-\gamma}\int_\Omega (v^+)^{1-\gamma}\,dx - \frac{t^{2_s^*}}{2_s^*}\|v^+\|_{2_s^*}^{2_s^*} < 0$$

since $1-\gamma < 1 < 2\theta < 2_s^*$. □

We are now ready to prove the existence of the first solution of (1.1).

Theorem 3.2. *Let λ_0 be given as in Lemma 3.1. Then for any $\lambda \in (0,\lambda_0]$ problem* (1.1) *has a solution $u_0 \in X_0$ with $J_\lambda(u_0) < 0$.*

Proof. Fix $\lambda \in (0,\lambda_0]$ and let ρ be as given in Lemma 3.1. We first prove that there exists $u_0 \in \overline{B}_\rho$ such that $J_\lambda(u_0) = m_\lambda < 0$. Let $\{u_k\}_k \subset \overline{B}_\rho$ be a minimizing sequence for m_λ, that is, such that

$$\lim_{k\to\infty} J_\lambda(u_k) = m_\lambda. \tag{3.2}$$

Since $\{u_k\}_k$ is bounded in X_0, by applying [26, Lemma 8] and [6, Theorem 4.9], there exist a subsequence, still denoted by $\{u_k\}_k$, and a function $u_0 \in \overline{B}_\rho$ such that, as $k \to \infty$, we have

$$\begin{cases} u_k \rightharpoonup u_0 \text{ in } X_0, & u_k \rightharpoonup u_0 \text{ in } L^{2_s^*}(\Omega), \\ u_k \to u_0 \text{ in } L^p(\Omega) \text{ for any } p \in [1, 2_s^*), & u_k \to u_0 \text{ a.e. in } \Omega. \end{cases} \tag{3.3}$$

Since $\gamma \in (0,1)$, by the Hölder inequality, for any $k \in \mathbb{N}$ we have

$$\left|\int_\Omega (u_k^+)^{1-\gamma}\,dx - \int_\Omega (u_0^+)^{1-\gamma}\,dx\right| \le \int_\Omega |u_k^+ - u_0^+|^{1-\gamma}\,dx \le \|u_k^+ - u_0^+\|_2^{1-\gamma}|\Omega|^{(1+\gamma)/2},$$

which yields, by (3.3),

$$\lim_{k\to\infty}\int_\Omega (u_k^+)^{1-\gamma}\,dx = \int_\Omega (u_0^+)^{1-\gamma}\,dx. \tag{3.4}$$

Let $w_k = u_k - u_0$; by [7, Theorem 2] it holds true that

$$\|u_k\|^2 = \|w_k\|^2 + \|u_0\|^2 + o(1), \quad \|u_k\|_{2_s^*}^{2_s^*} = \|w_k\|_{2_s^*}^{2_s^*} + \|u_0\|_{2_s^*}^{2_s^*} + o(1) \tag{3.5}$$

as $k \to \infty$. Since $\{u_k\}_k \subset \overline{B}_\rho$, by (3.5) for k sufficiently large, we have $w_k \in \overline{B}_\rho$. Lemma 3.1 implies that for any $u \in X_0$, with $\|u\| = \rho$, we get

$$\frac{1}{2\theta}\|u\|^{2\theta} - \frac{1}{2_s^*}\|u^+\|_{2_s^*}^{2_s^*} \ge \alpha > 0,$$

and from this, since $\rho \le 1$, for k sufficiently large we have

$$\frac{1}{2\theta}\|w_k\|^{2\theta} - \frac{1}{2_s^*}\|w_k^+\|_{2_s^*}^{2_s^*} > 0. \tag{3.6}$$

Thus, by (3.2), (3.4)–(3.6) and considering $\theta \ge 1$, it follows that, as $k \to \infty$,

$$\begin{aligned} m_\lambda &= J_\lambda(u_k) + o(1) \\ &= \frac{1}{2\theta}(\|w_k\|^2 + \|u_0\|^2)^\theta - \frac{\lambda}{1-\gamma}\int_\Omega (u_0^+)^{1-\gamma}\,dx - \frac{1}{2_s^*}(\|w_k^+\|_{2_s^*}^{2_s^*} + \|u_0^+\|_{2_s^*}^{2_s^*}) + o(1) \\ &\ge J_\lambda(u_0) + \frac{1}{2\theta}\|w_k\|^{2\theta} - \frac{1}{2_s^*}\|w_k^+\|_{2_s^*}^{2_s^*} + o(1) \ge J_\lambda(u_0) + o(1) \ge m_\lambda \end{aligned}$$

since $u_0 \in \overline{B}_\rho$. Hence, u_0 is a local minimizer for J_λ, with $J_\lambda(u_0) = m_\lambda < 0$, which implies that u_0 is nontrivial.

Now, we prove that u_0 is a positive solution of (2.2). For any $\psi \in X_0$, with $\psi \geq 0$ a.e. in $\mathbb{R}^N$, let us consider a $t > 0$ sufficiently small so that $u_0 + t\psi \in \overline{B}_\rho$. Since u_0 is a local minimizer for J_λ, we have

$$\begin{aligned} 0 &\leq J_\lambda(u_0 + t\psi) - J_\lambda(u_0) \\ &= \frac{1}{2\theta}(\|u_0 + t\psi\|^{2\theta} - \|u_0\|^{2\theta}) - \frac{\lambda}{1-\gamma}\int_\Omega [((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}]\,dx - \frac{1}{2_s^*}(\|u_0 + t\psi\|_{2_s^*}^{2_s^*} - \|u_0^+\|_{2_s^*}^{2_s^*}). \end{aligned}$$

From this, by dividing by $t > 0$ and passing to the limit as $t \to 0^+$, it follows that

$$\liminf_{t\to 0^+} \frac{\lambda}{1-\gamma}\int_\Omega \frac{((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}}{t}\,dx \leq \|u_0\|^{2(\theta-1)}\langle u_0, \psi\rangle - \int_\Omega (u_0^+)^{2_s^*-1}\psi\,dx. \tag{3.7}$$

We observe that

$$\frac{1}{1-\gamma}\cdot\frac{((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}}{t} = ((u_0 + \xi t\psi)^+)^{-\gamma}\psi \quad \text{a.e. in } \Omega,$$

with $\xi \in (0, 1)$ and $((u_0 + \xi t\psi)^+)^{-\gamma} \to (u_0^+)^{-\gamma}\psi$ a.e. in Ω as $t \to 0^+$. Thus, by the Fatou lemma, we obtain

$$\lambda\int_\Omega (u_0^+)^{-\gamma}\psi\,dx \leq \liminf_{t\to 0^+}\frac{\lambda}{1-\gamma}\int_\Omega \frac{((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}}{t}\,dx. \tag{3.8}$$

Therefore, combining (3.7) and (3.8), we get

$$\|u_0\|^{2(\theta-1)}\langle u_0, \psi\rangle - \lambda\int_\Omega (u_0^+)^{-\gamma}\psi\,dx - \int_\Omega (u_0^+)^{2_s^*-1}\psi\,dx \geq 0 \tag{3.9}$$

for any $\psi \in X_0$ with $\psi \geq 0$ a.e. in $\mathbb{R}^N$.

Since $J_\lambda(u_0) < 0$ and by Lemma 3.1, we have $u_0 \in B_\rho$. Hence, there exists $\delta \in (0, 1)$ such that $(1 + t)u_0 \in \overline{B}_\rho$ for any $t \in [-\delta, \delta]$. Let us define $I_\lambda(t) = J_\lambda((1 + t)u_0)$. Since u_0 is a local minimizer for J_λ in $\overline{B}_\rho$, the functional I_λ has a minimum at $t = 0$, that is,

$$I_\lambda'(0) = \|u_0\|^{2\theta} - \lambda\int_\Omega (u_0^+)^{1-\gamma}\,dx - \int_\Omega (u_0^+)^{2_s^*}\,dx = 0. \tag{3.10}$$

For any $\varphi \in X_0$ and any $\varepsilon > 0$, let us define $\psi_\varepsilon = u_0^+ + \varepsilon\varphi$. Then by (3.9) we have

$$\begin{aligned} 0 &\leq \|u_0\|^{2(\theta-1)}\langle u_0, \psi_\varepsilon^+\rangle - \lambda\int_\Omega (u_0^+)^{-\gamma}\psi_\varepsilon^+\,dx - \int_\Omega (u_0^+)^{2_s^*-1}\psi_\varepsilon^+\,dx \\ &= \|u_0\|^{2(\theta-1)}\langle u_0, \psi_\varepsilon + \psi_\varepsilon^-\rangle - \lambda\int_\Omega (u_0^+)^{-\gamma}(\psi_\varepsilon + \psi_\varepsilon^-)\,dx - \int_\Omega (u_0^+)^{2_s^*-1}(\psi_\varepsilon + \psi_\varepsilon^-)\,dx. \end{aligned} \tag{3.11}$$

We observe that, for a.e. $x, y \in \mathbb{R}^N$, we obtain

$$\begin{aligned} (u_0(x) - u_0(y))(u_0^-(x) - u_0^-(y)) &= -u_0^+(x)u_0^-(y) - u_0^-(x)u_0^+(y) - [u_0^-(x) - u_0^-(y)]^2 \\ &\leq -|u_0^-(x) - u_0^-(y)|^2, \end{aligned} \tag{3.12}$$

from which we immediately get

$$(u_0(x) - u_0(y))(u_0^+(x) - u_0^+(y)) \leq |u_0(x) - u_0(y)|^2.$$

From the last inequality it follows that

$$\begin{aligned} \langle u_0, \psi_\varepsilon + \psi_\varepsilon^-\rangle &= \iint_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\psi_\varepsilon(x) + \psi_\varepsilon^-(x) - \psi_\varepsilon(y) - \psi_\varepsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \varepsilon\iint_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x-y|^{N+2s}}\,dx\,dy. \end{aligned} \tag{3.13}$$

Hence, denoting $\Omega_\varepsilon = \{x \in \mathbb{R}^N : u_0^+(x) + \varepsilon\varphi(x) \le 0\}$ and combining (3.11) with (3.13), we get

$$
\begin{aligned}
0 &\le \|u_0\|^{2\theta} + \varepsilon\|u_0\|^{2(\theta-1)}\langle u_0, \varphi\rangle + \|u_0\|^{2(\theta-1)}\langle u_0, \psi_\varepsilon^-\rangle - \lambda\int_\Omega (u_0^+)^{-\gamma}(u_0^+ + \varepsilon\varphi)\,dx - \int_\Omega (u_0^+)^{2_s^*-1}(u_0^+ + \varepsilon\varphi)\,dx \\
&\quad + \lambda\int_{\Omega_\varepsilon} (u_0^+)^{-\gamma}(u_0^+ + \varepsilon\varphi)\,dx + \int_{\Omega_\varepsilon} (u_0^+)^{2_s^*-1}(u_0^+ + \varepsilon\varphi)\,dx \\
&\le \|u_0\|^{2\theta} - \lambda\int_\Omega (u_0^+)^{1-\gamma}\,dx - \int_\Omega (u_0^+)^{2_s^*}\,dx + \|u_0\|^{2(\theta-1)}\langle u_0, \psi_\varepsilon^-\rangle \\
&\quad + \varepsilon\Big[\|u_0\|^{2(\theta-1)}\langle u_0, \varphi\rangle - \lambda\int_\Omega (u_0^+)^{-\gamma}\varphi\,dx - \int_\Omega (u_0^+)^{2_s^*-1}\varphi\,dx\Big] \\
&= \|u_0\|^{2(\theta-1)}\langle u_0, \psi_\varepsilon^-\rangle + \varepsilon\Big[\|u_0\|^{2(\theta-1)}\langle u_0, \varphi\rangle - \lambda\int_\Omega (u_0^+)^{-\gamma}\varphi\,dx - \int_\Omega (u_0^+)^{2_s^*-1}\varphi\,dx\Big],
\end{aligned} \tag{3.14}
$$

where the last equality follows from (3.10). Arguing similarly to (3.12), for a.e. $x, y \in \mathbb{R}^N$ we have

$$
(u_0(x) - u_0(y))(u_0^+(x) - u_0^+(y)) \ge |u_0^+(x) - u_0^+(y)|^2. \tag{3.15}
$$

Thus, denoting

$$
\mathcal{U}_\varepsilon(x, y) = \frac{(u_0(x) - u_0(y))(\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+2s}},
$$

by the symmetry of the fractional kernel and (3.15), we get

$$
\begin{aligned}
\langle u_0, \psi_\varepsilon^-\rangle &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \mathcal{U}_\varepsilon(x, y)\,dx\,dy + 2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)} \mathcal{U}_\varepsilon(x, y)\,dx\,dy \\
&\le -\varepsilon\Big(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \mathcal{U}(x, y)\,dx\,dy + 2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)} \mathcal{U}(x, y)\,dx\,dy\Big) \\
&\le 2\varepsilon\iint_{\Omega_\varepsilon\times\mathbb{R}^N} |\mathcal{U}(x, y)|\,dx\,dy,
\end{aligned} \tag{3.16}
$$

where we set

$$
\mathcal{U}(x, y) = \frac{(u_0(x) - u_0(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}.
$$

Clearly $\mathcal{U} \in L^1(\mathbb{R}^{2N})$, so that for any $\sigma > 0$ there exists R_σ sufficiently large such that

$$
\iint_{(\operatorname{supp}\varphi)\times(\mathbb{R}^N\setminus B_{R_\sigma})} |\mathcal{U}(x, y)|\,dx\,dy < \frac{\sigma}{2}.
$$

Also, by the definition of Ω_ε, we have $\Omega_\varepsilon \subset \operatorname{supp}\varphi$ and $|\Omega_\varepsilon \times B_{R_\sigma}| \to 0$ as $\varepsilon \to 0^+$. Thus, since $\mathcal{U} \in L^1(\mathbb{R}^{2N})$, there exist $\delta_\sigma > 0$ and $\varepsilon_\sigma > 0$ such that for any $\varepsilon \in (0, \varepsilon_\sigma]$,

$$
|\Omega_\varepsilon \times B_{R_\sigma}| < \delta_\sigma \quad\text{and}\quad \iint_{\Omega_\varepsilon\times B_{R_\sigma}} |\mathcal{U}(x, y)|\,dx\,dy < \frac{\sigma}{2}.
$$

Therefore, for any $\varepsilon \in (0, \varepsilon_\sigma]$,

$$
\iint_{\Omega_\varepsilon\times\mathbb{R}^N} |\mathcal{U}(x, y)|\,dx\,dy < \sigma,
$$

from which we get

$$
\lim_{\varepsilon\to 0^+} \iint_{\Omega_\varepsilon\times\mathbb{R}^N} |\mathcal{U}(x, y)|\,dx\,dy = 0. \tag{3.17}
$$

Combining (3.14) with (3.16), dividing by ε, letting $\varepsilon \to 0^+$ and using (3.17), we obtain

$$\|u_0\|^{2(\theta-1)}\langle u_0, \varphi\rangle - \lambda \int_\Omega (u_0^+)^{-\gamma}\varphi\, dx - \int_\Omega (u_0^+)^{2_s^*-1}\varphi\, dx \geq 0$$

for any $\varphi \in X_0$. By the arbitrariness of φ, we prove that u_0 verifies (2.3), that is, u_0 is a nontrivial solution of (2.2).

Finally, considering $\varphi = u_0^-$ in (2.3) and using (3.12), we see that $\|u_0^-\| = 0$, which implies that u_0 is nonnegative. Moreover, by the maximum principle in [28, Proposition 2.17], we can deduce that u_0 is a positive solution of (2.2), and so also solves problem (1.1). This concludes the proof. □

We end this section by observing that the result in Theorem 3.2 can be extended to more general Kirchhoff problems. That is, we can consider the problem

$$\begin{cases} M\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\Big)(-\Delta)_p^s u = \frac{\lambda}{u^\gamma} + u^{p_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.18}$$

where $p_s^* = pN/(N-ps)$, with $N > ps$ and $p > 1$, while the Kirchhoff coefficient M satisfies the following condition:

($\mathcal{M}$) $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and nondecreasing. There exist numbers $a > 0$ and ϑ such that for any $t \in \mathbb{R}_0^+$,

$$\mathscr{M}(t) := \int_0^t M(\tau)d\tau \geq at^\vartheta, \quad \text{with } \begin{cases} \vartheta \in (1, p_s^*/p) & \text{if } M(0) = 0, \\ \vartheta = 1 & \text{if } M(0) > 0. \end{cases}$$

The main operator $(-\Delta)_p^s$ is the fractional p-Laplacian which may be defined, for any function $\varphi \in C_0^\infty(\Omega)$, as

$$(-\Delta)_p^s\varphi(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. Then, arguing as in the proof of Theorem 3.2 and observing that we have not used yet the assumption that $\partial\Omega$ is continuous, we can prove the following result.

Theorem 3.3. *Let $s \in (0,1)$, $p > 1$, $N > ps$, $\gamma \in (0,1)$ and let Ω be an open bounded subset of $\mathbb{R}^N$. Let M satisfy ($\mathcal{M}$). Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$ problem (3.18) admits a solution.*

4 A mountain pass solution for problem (2.4)

In this section, we prove the existence of a positive solution for the perturbed problem (2.4) by the mountain pass theorem. For this, throughout this section we assume $n \in \mathbb{N}$ without further mentioning. Now, we first prove that the related functional $J_{n,\lambda}$ satisfies all the geometric features required by the mountain pass theorem.

Lemma 4.1. *Let $\rho \in (0,1]$, $\lambda_0 = \lambda_0(\rho) > 0$ and $\alpha = \alpha(\rho) > 0$ be given as in Lemma 3.1. Then, for any $\lambda \in (0, \lambda_0]$ and any $u \in X_0$ with $\|u\| \leq \rho$, one has $J_{n,\lambda}(u) \geq \alpha$. Furthermore, there exists $e \in X_0$, with $\|e\| > \rho$, such that $J_{n,\lambda}(e) < 0$.*

Proof. Since $\gamma \in (0,1)$, by the subadditivity of $t \mapsto t^{1-\gamma}$, we have

$$\Big(u^+ + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma} \leq (u^+)^{1-\gamma} \quad \text{a.e. in } \Omega \tag{4.1}$$

for any $u \in X_0$ and any $n \in \mathbb{N}$. Thus, we have $J_{n,\lambda}(u) \geq J_\lambda(u)$ for any $u \in X_0$ and the first part of the lemma directly follows by Lemma 3.1.

For any $v \in X_0$, with $v^+ \not\equiv 0$, and $t > 0$, we have

$$J_{n,\lambda}(tv) = \frac{t^{2\theta}}{2\theta}\|v\|^{2\theta} - \frac{\lambda}{1-\gamma}\int_\Omega \Big[\Big(tv^+ + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma}\Big]\,dx - \frac{t^{2_s^*}}{2_s^*}\|v^+\|_{2_s^*}^{2_s^*} \to -\infty \quad \text{as } t \to \infty$$

since $1-\gamma < 1 < 2\theta < 2_s^*$. Hence, we can find $e \in X_0$, with $\|e\| > \rho$ sufficiently large, such that $J_{n,\lambda}(e) < 0$. □

We discuss now the compactness property for the functional $J_{n,\lambda}$, given by the Palais–Smale condition. We recall that $\{u_k\}_k \subset X_0$ is a Palais–Smale sequence for $J_{n,\lambda}$ at level $c \in \mathbb{R}$ if

$$J_{n,\lambda}(u_k) \to c \quad \text{and} \quad J'_{n,\lambda}(u_k) \to 0 \qquad \text{in } (X_0)' \text{ as } k \to \infty. \tag{4.2}$$

We say that $J_{n,\lambda}$ satisfies the Palais–Smale condition at level c if any Palais–Smale sequence $\{u_k\}_k$ at level c admits a convergent subsequence in X_0.

Before proving this compactness condition, we introduce the following positive constants, which will help us for a better explanation:

$$D_1 = \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)S^{2_s^*\theta/(2_s^*-2\theta)}, \qquad D_2 = \frac{\big[\big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*}S^{-(1-\gamma)/2}\big]^{2\theta/(2\theta-1+\gamma)}}{\big(\frac{1}{2\theta} - \frac{1}{2_s^*}\big)^{(1-\gamma)/(2\theta-1+\gamma)}}. \tag{4.3}$$

Lemma 4.2. *Let $\lambda > 0$. Then the functional $J_{n,\lambda}$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$ verifying*

$$c < D_1 - D_2\lambda^{2\theta/(2\theta-1+\gamma)}, \tag{4.4}$$

with $D_1, D_2 > 0$ given as in (4.3).

Proof. Let $\lambda > 0$ and let $\{u_k\}_k$ be a Palais–Smale sequence in X_0 at level $c \in \mathbb{R}$, with c satisfying (4.4). We first prove the boundedness of $\{u_k\}_k$. By (4.2), there exists $\sigma > 0$ such that, as $k \to \infty$,

$$\begin{aligned}
c + \sigma\|u_k\| + o(1) &\geq J_{n,\lambda}(u_k) - \frac{1}{2_s^*}\langle J'_{n,\lambda}(u_k), u_k\rangle \\
&= \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|u_k\|^{2\theta} - \frac{\lambda}{1-\gamma}\int_\Omega \Big[\Big(u_k^+ + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma}\Big]\,dx + \frac{\lambda}{2_s^*}\int_\Omega \Big(u_k^+ + \frac{1}{n}\Big)^{-\gamma}u_k\,dx \\
&\geq \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|u_k\|^{2\theta} - \lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)\int_\Omega |u_k|^{1-\gamma}\,dx \\
&\geq \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|u_k\|^{2\theta} - \lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*}S^{-(1-\gamma)/2}\|u_k\|^{1-\gamma},
\end{aligned}$$

where the last two inequalities follow by (2.7), (4.1) and the Hölder inequality. Therefore, $\{u_k\}_k$ is bounded in X_0 since $1-\gamma < 1 < 2\theta$. Also, $\{u_k^-\}_k$ is bounded in X_0, and by (4.2) we have

$$\lim_{k\to\infty}\langle J'_{n,\lambda}(u_k), -u_k^-\rangle = \lim_{k\to\infty}\|u_k\|^{2(\theta-1)}\langle u_k, -u_k^-\rangle = 0.$$

Thus, by inequality (3.12) we deduce that $\|u_k^-\| \to 0$ as $k \to \infty$, which yields

$$J_{n,\lambda}(u_k) = J_{n,\lambda}(u_k^+) + o(1) \quad \text{and} \quad J'_{n,\lambda}(u_k) = J'_{n,\lambda}(u_k^+) + o(1) \qquad \text{as } k \to \infty.$$

Hence, we can suppose that $\{u_k\}_k$ is a sequence of nonnegative functions.

By the boundedness of $\{u_k\}_k$ and by using [26, Lemma 8] and [6, Theorem 4.9], there exist a subsequence, still denoted by $\{u_k\}_k$, and a function $u \in X_0$ such that

$$\begin{cases}
u_k \rightharpoonup u \text{ in } X_0, & \|u_k\| \to \mu, \\
u_k \rightharpoonup u \text{ in } L^{2_s^*}(\Omega), & \|u_k - u\|_{2_s^*} \to \ell, \\
u_k \to u \text{ in } L^p(\Omega) \text{ for any } p \in [1, 2_s^*), & u_k \to u \text{ a.e. in } \Omega, \quad u_k \leq h \text{ a.e. in } \Omega,
\end{cases} \tag{4.5}$$

as $k \to \infty$, with $h \in L^p(\Omega)$ for a fixed $p \in [1, 2_s^*)$. If $\mu = 0$, then immediately $u_k \to 0$ in X_0 as $k \to \infty$. Hence, let us assume that $\mu > 0$.

Since $n \in \mathbb{N}$, by (4.5) it follows that

$$\left|\frac{u_k - u}{(u_k + \frac{1}{n})^\gamma}\right| \leq n^\gamma (h + |u|) \quad \text{a.e. in } \Omega,$$

so by the dominated convergence theorem and (4.5) we have

$$\lim_{k\to\infty} \int_\Omega \frac{u_k - u}{(u_k + \frac{1}{n})^\gamma}\, dx = 0. \tag{4.6}$$

By (4.5) and [7, Theorem 2], we have

$$\|u_k\|^2 = \|u_k - u\|^2 + \|u\|^2 + o(1), \qquad \|u_k\|_{2_s^*}^{2_s^*} = \|u_k - u\|_{2_s^*}^{2_s^*} + \|u\|_{2_s^*}^{2_s^*} + o(1) \tag{4.7}$$

as $k \to \infty$. Consequently, we deduce from (4.2), (4.5), (4.6) and (4.7) that, as $k \to \infty$,

$$\begin{aligned} o(1) &= \langle J'_{n,\lambda}(u_k), u_k - u\rangle \\ &= \|u_k\|^{2(\theta-1)}\langle u_k, u_k - u\rangle - \lambda \int_\Omega \frac{u_k - u}{(u_k + \frac{1}{n})^\gamma}\, dx - \int_\Omega u_k^{2_s^*-1}(u_k - u)\, dx \\ &= \mu^{2(\theta-1)}(\mu^2 - \|u\|^2) - \|u_k\|_{2_s^*}^{2_s^*} + \|u\|_{2_s^*}^{2_s^*} + o(1) \\ &= \mu^{2(\theta-1)}\|u_k - u\|^2 - \|u_k - u\|_{2_s^*}^{2_s^*} + o(1). \end{aligned}$$

Therefore, we have proved the crucial formula

$$\mu^{2(\theta-1)} \lim_{k\to\infty} \|u_k - u\|^2 = \lim_{k\to\infty} \|u_k - u\|_{2_s^*}^{2_s^*}. \tag{4.8}$$

If $\ell = 0$, since $\mu > 0$, by (4.5) and (4.8) we have $u_k \to u$ in X_0 as $k \to \infty$, concluding the proof.

Thus, let us assume by contradiction that $\ell > 0$. By (2.7), the notation in (4.5) and (4.8), we get

$$\ell^{2_s^*} \geq S\mu^{2(\theta-1)}\ell^2. \tag{4.9}$$

Noting that (4.8) implies in particular that

$$\mu^{2(\theta-1)}(\mu^2 - \|u\|^2) = \ell^{2_s^*},$$

by using (4.9), it follows that

$$(\ell^{2_s^*})^{2s/N} = (\mu^{2(\theta-1)})^{2s/N}(\mu^2 - \|u\|^2)^{2s/N} \geq S\mu^{2(\theta-1)}.$$

From this we obtain

$$\mu^{4s/N} \geq (\mu^2 - \|u\|^2)^{2s/N} \geq S(\mu^{2(\theta-1)})^{(N-2s)/N}.$$

Considering $N < 2s\theta/(\theta - 1) = 2s\theta'$, we have

$$\mu^2 \geq S^{N/(2s\theta - N(\theta-1))}. \tag{4.10}$$

Indeed, the restriction $N/(2\theta') < s$ follows directly from the fact that $1 < \theta < 2_s^*/2 = N/(N - 2s)$. By (4.1), considering that $n \in \mathbb{N}$, for any $k \in \mathbb{N}$ we have

$$J_{n,\lambda}(u_k) - \frac{1}{2_s^*}\langle J'_{n,\lambda}(u_k), u_k\rangle \geq \left(\frac{1}{2\theta} - \frac{1}{2_s^*}\right)\|u_k\|^{2\theta} - \lambda\left(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\right)\int_\Omega u_k^{1-\gamma}\, dx.$$

From this, as $k \to \infty$, since $\theta \geq 1$, by (4.2), (4.5), (4.7), (4.10), the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned} c &\geq \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)(\mu^{2\theta} + \|u\|^{2\theta}) - \lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\|u\|^{1-\gamma} \\ &\geq \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)(\mu^{2\theta} + \|u\|^{2\theta}) - \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|u\|^{2\theta} \\ &\quad - \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)^{-(1-\gamma)/(2\theta-1+\gamma)}\Big[\lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\Big]^{2\theta/(2\theta-1+\gamma)} \\ &\geq \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)S^{2_s^*\theta/(2_s^*-2\theta)} - \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)^{-(1-\gamma)/(2\theta-1+\gamma)}\Big[\lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\Big]^{2\theta/(2\theta-1+\gamma)}, \end{aligned}$$

which contradicts (4.4) since (4.3). This concludes the proof. □

We now give a control from above for the functional $J_{n,\lambda}$ under a suitable situation. For this, we assume, without loss of generality, that $0 \in \Omega$. By [10], we know that the infimum in (2.7) is attained at the function

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2 + |x|^2)^{(N-2s)/2}} \quad \text{with } \varepsilon > 0, \tag{4.11}$$

that is, it holds true that

$$\iint_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = S\|u_\varepsilon\|^2_{L^{2_s^*}(\mathbb{R}^N)}.$$

Let us fix $r > 0$ such that $B_{4r} \subset \Omega$, where $B_{4r} = \{x \in \mathbb{R}^N : |x| < 4r\}$, and let us introduce a cut-off function $\phi \in C^\infty(\mathbb{R}^N, [0,1])$ such that

$$\phi = \begin{cases} 1 & \text{in } B_r, \\ 0 & \text{in } \mathbb{R}^N \setminus B_{2r}. \end{cases} \tag{4.12}$$

For any $\varepsilon > 0$, we set

$$\psi_\varepsilon = \frac{\phi u_\varepsilon}{\|\phi u_\varepsilon\|_{2_s^*}^2} \in X_0. \tag{4.13}$$

Then we can prove the following result.

Lemma 4.3. *There exist $\psi \in X_0$ and $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$,*

$$\sup_{t\geq 0} J_{n,\lambda}(t\psi) < D_1 - D_2\lambda^{2\theta/(2\theta-1+\gamma)},$$

with $D_1, D_2 > 0$ given as in (4.3).

Proof. Let $\lambda, \varepsilon > 0$. Let u_ε and ψ_ε be as in (4.11) and (4.13), respectively. By (2.6), we have $J_{n,\lambda}(t\psi_\varepsilon) \to -\infty$ as $t \to \infty$, so that there exists $t_\varepsilon > 0$ such that $J_{n,\lambda}(t_\varepsilon\psi_\varepsilon) = \max_{t\geq 0} J_{n,\lambda}(t\psi_\varepsilon)$. By Lemma 4.1, we know that $J_{n,\lambda}(t_\varepsilon\psi_\varepsilon) \geq \alpha > 0$. Hence, by the continuity of $J_{n,\lambda}$ there exist two numbers $t_0, t^* > 0$ such that $t_0 \leq t_\varepsilon \leq t^*$.

Now, since $\|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)}$ is independent of ε, by [27, Proposition 21] we have

$$\|\psi_\varepsilon\|^2 \leq \frac{\iint_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\|\phi u_\varepsilon\|_{2_s^*}^2} = S + O(\varepsilon^{N-2s}),$$

from which, by the elementary inequality

$$(a+b)^p \leq a^p + p(a+1)^{p-1}b, \quad \text{for any } a > 0,\ b \in [0,1],\ p \geq 1,$$

with $p = 2\theta$, it follows that, as $\varepsilon \to 0^+$,

$$\|\psi_\varepsilon\|^{2\theta} \leq S^\theta + O(\varepsilon^{N-2s}).$$

Hence, by the last inequality, (2.6) and since $t_0 \le t_\varepsilon \le t^*$, for any $\varepsilon > 0$ sufficiently small, we have

$$J_{n,\lambda}(t_\varepsilon \psi_\varepsilon) \le \frac{t_\varepsilon^{2\theta}}{2\theta} S^\theta + C_1 \varepsilon^{N-2s} - \frac{\lambda}{1-\gamma} \int_\Omega \Big[\Big(t_0\psi_\varepsilon + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma}\Big] dx - \frac{t_\varepsilon^{2_s^*}}{2_s^*}, \tag{4.14}$$

with a suitable positive constant C_1. We observe that

$$\max_{t\ge 0}\Big(\frac{t^{2\theta}}{2\theta} S^\theta - \frac{t^{2_s^*}}{2_s^*}\Big) = \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big) S^{2_s^*\theta/(2_s^*-2\theta)}.$$

Thus, by (4.14) it follows that

$$J_{n,\lambda}(t_\varepsilon \psi_\varepsilon) \le \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big) S^{2_s^*\theta/(2_s^*-2\theta)} + C_1 \varepsilon^{N-2s} - \frac{\lambda}{1-\gamma} \int_\Omega \Big[\Big(t_0\psi_\varepsilon + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma}\Big] dx. \tag{4.15}$$

Now, let us consider a positive number q, less than 1, satisfying

$$\frac{(N-2s)(1-\gamma) - 2q(N-2s)(1-\gamma) + 2_s^* qN}{2_s^*} \cdot \frac{2\theta}{2\theta - 1 + \gamma} \cdot \frac{1}{N-2s} + 1 - \frac{2\theta}{2\theta - 1 + \gamma} < 0, \tag{4.16}$$

that is, since $2 < 2\theta < 2_s^*$, $N > 2s$ and $\gamma \in (0, 1)$, such that

$$0 < q < \min\Big\{\frac{(N-2s)(2_s^* - 2\theta)(1-\gamma)}{2\theta N(2_s^* - 2) + 4\theta N\gamma + 8\theta s(1-\gamma)}, 1\Big\}.$$

By the elementary inequality

$$a^{1-\gamma} - (a+b)^{1-\gamma} \le -(1-\gamma) b^{(1-\gamma)/p} a^{(p-1)(1-\gamma)/p} \quad \text{for any } a > 0,\ b > 0 \text{ large enough},\ p > 1,$$

with $p = 2_s^*/2$, considering $\varepsilon < r^{1/q}$ sufficiently small, with r given by (4.12), and since $q < 1$, we have

$$\begin{aligned} -\frac{1}{1-\gamma} \int_{\{x\in\Omega:|x|\le\varepsilon^q\}} \Big[\Big(t_0\psi_\varepsilon + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma}\Big] dx &\le -\widetilde{C} \varepsilon^{(N-2s)(1-\gamma)/2_s^*} \int_{\{x\in\Omega:|x|\le\varepsilon^q\}} \Big[\frac{1}{(|x|^2 + \varepsilon^2)^{(N-2s)/2}}\Big]^{2(1-\gamma)/2_s^*} dx \\ &\le -C_2 \varepsilon^{((N-2s)(1-\gamma) - 2q(N-2s)(1-\gamma) + 2_s^* qN)/2_s^*}, \end{aligned} \tag{4.17}$$

with two positive constants $\widetilde{C}$ and C_2 independent of ε. By combining (4.15) with (4.17), we get

$$J_{n,\lambda}(t_\varepsilon \psi_\varepsilon) \le \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big) S^{2_s^*\theta/(2_s^*-2\theta)} + C_1 \varepsilon^{N-2s} - C_2 \lambda \varepsilon^{((N-2s)(1-\gamma) - 2q(N-2s)(1-\gamma) + 2_s^* qN)/2_s^*}. \tag{4.18}$$

Thus, let us consider $\lambda^* > 0$ such that

$$D_1 - D_2 \lambda^{2\theta/(2\theta-1+\gamma)} > 0 \quad \text{for any } \lambda \in (0, \lambda^*),$$

and let us set

$$\nu_1 = \frac{2q\theta}{(2\theta - 1 + \gamma)(N-2s)}, \quad \nu_2 = \frac{2\theta[(N-2s)(1-\gamma) - 2q(N-2s)(1-\gamma) + 2_s^* qN]}{2_s^*(2\theta - 1 + \gamma)(N-2s)} + 1,$$

$$\nu_3 = \nu_2 - \frac{2\theta}{2\theta - 1 + \gamma}, \quad \lambda_1 = \min\Big\{\lambda^*, r^{1/\nu_1}, \Big(\frac{C_2}{C_1 + D_2}\Big)^{-1/\nu_3}\Big\},$$

where r and q are given in (4.12) and (4.16), respectively, while we still consider D_1 and D_2 as defined in (4.3). Then, by considering $\varepsilon = \lambda^{\nu_1/q}$ in (4.18), since (4.16) implies that $\nu_3 < 0$, for any $\lambda \in (0, \lambda_1)$ we have

$$J_{n,\lambda}(t_\varepsilon \psi_\varepsilon) \le D_1 + C_1 \lambda^{2\theta/(2\theta-1+\gamma)} - C_2 \lambda^{\nu_2} = D_1 + \lambda^{2\theta/(2\theta-1+\gamma)}(C_1 - C_2 \lambda^{\nu_3}) < D_1 - D_2 \lambda^{2\theta/(2\theta-1+\gamma)},$$

which concludes the proof. □

We can now prove the existence result for (2.4) by applying the mountain pass theorem.

Theorem 4.4. *There exists $\overline{\lambda} > 0$ such that, for any $\lambda \in (0, \overline{\lambda})$, problem* (2.4) *has a positive solution $v_n \in X_0$ with*

$$\alpha < J_{n,\lambda}(v_n) < D_1 - D_2\lambda^{2\theta/(2\theta-1+\gamma)}, \tag{4.19}$$

where α, D_1 and D_2 are given in Lemma 3.1 and (4.3), *respectively.*

Proof. Let $\overline{\lambda} = \min\{\lambda_0, \lambda_1\}$, with λ_0 and λ_1 given in Lemmas 3.1 and 4.3, respectively. Let us consider $\lambda \in (0, \overline{\lambda})$. By Lemma 4.1, the functional $J_{n,\lambda}$ verifies the mountain pass geometry. For this, we can set the critical mountain pass level as

$$c_{n,\lambda} = \inf_{g\in\Gamma} \max_{t\in[0,1]} J_{n,\lambda}(g(t)),$$

where

$$\Gamma = \{g \in C([0,1], X_0) : g(0) = 0,\ J_{n,\lambda}(g(1)) < 0\}.$$

By Lemmas 4.1 and 4.3, we get

$$0 < \alpha < c_{n,\lambda} \le \sup_{t\ge0} J_{n,\lambda}(t\psi) < D_1 - D_2\lambda^{2\theta/(2\theta-1+\gamma)}.$$

Hence, by Lemma 4.2 the functional $J_{n,\lambda}$ satisfies the Palais–Smale condition at level $c_{n,\lambda}$. Thus, the mountain pass theorem gives the existence of a critical point $v_n \in X_0$ for $J_{n,\lambda}$ at level $c_{n,\lambda}$. Since

$$J_{n,\lambda}(v_n) = c_{n,\lambda} > \alpha > 0 = J_{n,\lambda}(0),$$

we obtain that v_n is a nontrivial solution of (2.4). Furthermore, by (2.5) with test function $\varphi = v_n^-$ and inequality (3.12), we can see that $\|v_n^-\| = 0$, which implies that v_n is nonnegative. By the maximum principle in [28, Proposition 2.17], we have that v_n is a positive solution of (2.4), concluding the proof. □

5 A second solution for problem (1.1)

In this last section, we prove the existence of a second solution for problem (1.1), as a limit of solutions of the perturbed problem (2.4). For this, here we need the assumption that $\partial\Omega$ is continuous in order to apply a density argument for the space X_0.

Proof of Theorem 1.1. Let us consider $\overline{\lambda}$ as given in Theorem 4.4, and let $\lambda \in (0, \overline{\lambda})$. Since $\overline{\lambda} \le \lambda_0$, by Theorem 3.2 we know that problem (1.1) admits a solution u_0 with $J_\lambda(u_0) < 0$.

In order to find a second solution for (1.1) let $\{v_n\}_n$ be a family of positive solutions of (2.4). By (2.7), (4.1), (4.19) and the Hölder inequality, we have

$$\begin{aligned}
D_1 - D_2\lambda^{2\theta/(2\theta-1+\gamma)} &> J_{n,\lambda}(v_n) - \frac{1}{2_s^*}\langle J'_{n,\lambda}(v_n), v_n\rangle \\
&= \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|v_n\|^{2\theta} - \frac{\lambda}{1-\gamma}\int_\Omega \Big[\Big(v_n + \frac{1}{n}\Big)^{1-\gamma} - \Big(\frac{1}{n}\Big)^{1-\gamma}\Big]\,dx + \frac{\lambda}{2_s^*}\int_\Omega \Big(v_n + \frac{1}{n}\Big)^{-\gamma} v_n\,dx \\
&\ge \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|v_n\|^{2\theta} - \frac{\lambda}{1-\gamma}\int_\Omega v_n^{1-\gamma}\,dx \\
&\ge \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)\|v_n\|^{2\theta} - \frac{\lambda}{1-\gamma}|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\|v_n\|^{1-\gamma},
\end{aligned}$$

which yields that $\{v_n\}_n$ is bounded in X_0 since $1-\gamma < 1 < 2\theta$. Hence, by using [26, Lemma 8] and [6, Theorem 4.9], there exist a subsequence, still denoted by $\{v_n\}_n$, and a function $v_0 \in X_0$ such that

$$\begin{cases}
v_n \rightharpoonup v_0 \text{ in } X_0, & \|v_n\| \to \mu, \\
v_n \rightharpoonup v_0 \text{ in } L^{2_s^*}(\Omega), & \|v_n - v_0\|_{2_s^*} \to \ell, \\
v_n \to v_0 \text{ in } L^p(\Omega) \text{ for any } p \in [1, 2_s^*), & v_n \to v_0 \text{ a.e. in } \Omega.
\end{cases} \tag{5.1}$$

We want to prove that $v_n \to v_0$ in X_0 as $n \to \infty$. When $\mu = 0$, by (5.1) we have $v_n \to 0$ in X_0 as $n \to \infty$. For this, we suppose $\mu > 0$. We observe that

$$0 \le \frac{v_n}{(v_n + \frac{1}{n})^\gamma} \le v_n^{1-\gamma} \quad \text{a.e. in } \Omega,$$

so by the Vitali convergence theorem and (5.1) it follows that

$$\lim_{n\to\infty} \int_\Omega \frac{v_n}{(v_n + \frac{1}{n})^\gamma}\, dx = \int_\Omega v_0^{1-\gamma}\, dx. \tag{5.2}$$

Using (2.5) for v_n and test function $\varphi = v_n$, by (5.1) and (5.2), as $n \to \infty$, we have

$$\mu^{2\theta} - \lambda \int_\Omega v_0^{1-\gamma}\, dx + \|v_n\|_{2_s^*}^{2_s^*} = o(1). \tag{5.3}$$

For any $n \in \mathbb{N}$, by an immediate calculation in (2.4) we see that

$$\|v_n\|^{2(\theta-1)}(-\Delta)^s v_n \ge \min\Big\{1, \frac{\lambda}{2^\gamma}\Big\} \quad \text{in } \Omega.$$

Thus, since $\{v_n\}_n$ is bounded in X_0 and by using a standard comparison argument (see [3, Lemma 2.1]) and the maximum principle in [28, Proposition 2.17], for any $\widetilde{\Omega} \Subset \Omega$ there exists a constant $c_{\widetilde{\Omega}} > 0$ such that

$$v_n \ge c_{\widetilde{\Omega}} > 0, \quad \text{a.e. in } \widetilde{\Omega} \text{ and for any } n \in \mathbb{N}. \tag{5.4}$$

Now, let $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp}\varphi = \widetilde{\Omega} \Subset \Omega$. By (5.4), we have

$$0 \le \left|\frac{\varphi}{(v_n + \frac{1}{n})^\gamma}\right| \le \frac{|\varphi|}{c_{\widetilde{\Omega}}^\gamma} \quad \text{a.e. in } \Omega,$$

so that by (5.1) and the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{\varphi}{(v_n + \frac{1}{n})^\gamma}\, dx = \int_\Omega v_0^{-\gamma}\varphi\, dx. \tag{5.5}$$

Thus, by considering (2.5) for v_n, sending $n \to \infty$ and using (5.1) and (5.5), for any $\varphi \in C_0^\infty(\Omega)$ it follows that

$$\mu^{2(\theta-1)}\langle v_0, \varphi\rangle - \lambda \int_\Omega v_0^{-\gamma}\varphi\, dx + \int_\Omega v_0^{2_s^*-1}\varphi\, dx = 0. \tag{5.6}$$

However, since $\partial\Omega$ is continuous, by [15, Theorem 6] the space $C_0^\infty(\Omega)$ is dense in X_0. Thus, by a standard density argument, (5.6) holds true for any $\varphi \in X_0$. By combining (5.3) and (5.6) with test function $\varphi = v_0$, as $n \to \infty$ we get

$$\mu^{2(\theta-1)}(\mu^2 - \|v_0\|^2) = \|v_n\|_{2_s^*}^{2_s^*} - \|v_0\|_{2_s^*}^{2_s^*} + o(1),$$

and by (5.1) and [7, Theorem 2] we have

$$\mu^{2(\theta-1)} \lim_{n\to\infty} \|v_n - v_0\|^2 = \ell^{2_s^*}. \tag{5.7}$$

If $\ell = 0$, then $v_n \to v_0$ in X_0 as $n \to \infty$ since $\mu > 0$.

Let us suppose $\ell > 0$ by contradiction. Arguing as in Lemma 4.2, by (5.1) and (5.7) we get (4.10). Therefore, since $\theta \ge 1$, by (4.1), (4.10), (4.19), (5.1), the Hölder inequality and the Young inequality we have

$$\begin{aligned}
&D_1 - D_2\lambda^{2\theta/(2\theta-1+\gamma)} \\
&\quad > J_{n,\lambda}(v_n) - \frac{1}{2_s^*}\langle J'_{n,\lambda}(v_n), v_n\rangle \\
&\quad \ge \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)(\mu^{2\theta} + \|v_0\|^{2\theta}) - \lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\|v_0\|^{1-\gamma} \\
&\quad \ge \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)S^{2_s^*\theta/(2_s^*-2\theta)} - \Big(\frac{1}{2\theta} - \frac{1}{2_s^*}\Big)^{-(1-\gamma)/(2\theta-1+\gamma)}\Big[\lambda\Big(\frac{1}{1-\gamma} + \frac{1}{2_s^*}\Big)|\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2}\Big]^{2\theta/(2\theta-1+\gamma)},
\end{aligned}$$

which is the desired contradiction, thanks to (4.3).

Therefore, $v_n \to v_0$ in X_0 as $n \to \infty$, and by (2.3) and (2.5) we immediately see that v_0 is a solution of problem (2.2). Furthermore, by (4.19) we have $J_\lambda(v_0) \geq \alpha > 0$, which also implies that v_0 is nontrivial. Reasoning as at the end of the proof of Theorem 4.4, we conclude that v_0 is a positive solution of (2.2), and so v_0 also solves problem (1.1). Finally, v_0 is different from u_0 since $J_\lambda(v_0) > 0 > J_\lambda(u_0)$. ☐

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