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DANIEL GOMES FADEL

ON THE BEHAVIOR OF SEQUENCES OF ARBITRARILY LARGE MASS MONOPOLES IN DIMENSIONS 3 AND 7

SOBRE O COMPORTAMENTO DE SEQUÊNCIAS DE MONOPOLOS COM MASSAS ARBITRARIAMENTE GRANDES EM DIMENSÕES 3 E 7

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Resumo

Estudamos aspectos analíticos de soluções suaves das equaçõs de Yang–Mills–Higgs em variedades riemannianas não-compactas com geometria limitada, focando no problema de compacidade para soluções do tipo monopolo sob um certo regime assintótico de massas arbitrariamente grandes em variedades assintóticamente cônicas (AC) de dimensões 3 e 7. No caso de dimensão 3, as componentes conexas do espaço de módulos de monopolos são indexadas por um inteiro chamado de carga, e consideramos o problema do comportamento limite de sequências de monopolos com carga fixa k e massas arbitrariamente grandes, cf. Fadel–Oliveira [FO19]. Provamos que a comportamento limite de tais monopolos é caracterizado pela concentração de energia ao longo de um conjunto finito Z, consistindo de no máximo k pontos nos quais os zeros dos campos de Higgs se acumulam e um monopolo em $\mathbb{R}^3$ de massa 1 e carga 1 borbulha. Também apresentamos alguns resultados na direção do problema de convergência da sequência fora de Z. Finalmente, seguindo uma sugestão do artigo seminal de Donaldson–Segal [DS09], desenvolvemos o mesmo tipo de análise para monopolos de dimensões mais altas em G_2-variedades AC de dimensão 7 e, sob certas hipóteses brandas, provamos, entre outras coisas, que o conjunto de acumulação dos zeros dos campos de Higgs nesse caso é de medida $\mathcal{H}^4$ finita e está contido em um conjunto $\mathcal{H}^4-$retificável onde a energia intermediária da sequência se concentra.

Palavras-chave: Gauge, Teorias de; Monopolos magnéticos; Blow-up locus; Geometria calibrada.

Abstract

We study analytical aspects of smooth solutions of the Yang–Mills–Higgs equations on noncompact Riemannian manifolds of bounded geometry, focusing on the compactness problem for monopole solutions under a certain asymptotic regime of arbitrarily large mass on asymptotically conical (AC) $3-$ and $7-$ dimensional manifolds. In the $3-$dimensional case, the connected components of the moduli space of monopoles are labeled by an integer called the charge, and we consider the problem of the limiting behavior of sequences of monopoles with fixed charge k and arbitrarily large masses, cf. Fadel–Oliveira [FO19]. We prove that the limiting behavior of such monopoles is characterized by energy concentration along a finite set Z, consisting of at most k points at which the zeros of the Higgs fields accumulate and a mass 1 and charge 1 $\mathbb{R}^3-$monopole bubbles off. We also give some results on the direction of the convergence problem of the sequence outside Z. Finally, following a suggestion of the seminal paper of Donaldson–Segal [DS09], we develop the same sort of analysis for higher dimensional monopoles on $7-$dimensional AC G_2-manifolds and under certain mild assumptions we prove, among other things, that the accumulation set of the Higgs fields zeros in this case is of finite $\mathcal{H}^4-$measure and is included in a $\mathcal{H}^4-$rectifiable set where the intermediate energy of the sequence concentrates.

Keywords: Gauge theory; Magnetic monopoles; Blow-up locus; Calibrated geometry.

Contents

Introduction

The study of gauge theory, particularly Yang–Mills–Higgs theory, has been a powerful tool in geometry and topology. The pioneering work[1] of Atiyah, Donaldson, Taubes, Uhlenbeck et al has led to revolutionary advances in low dimensional topology, differential and algebraic geometry. The basic concept in the theory is the *Yang–Mills–Higgs functional*, defined for *configurations* (A,Φ) consisting of a connection A on a $\mathrm{SU}(2)-$bundle over a given Riemannian manifold X and a section Φ of the associated adjoint bundle, as the average square L^2-norm of the curvature of A and the exterior covariant derivative of Φ:

$$\mathcal{E}(A,\Phi) := \frac{1}{2}\left(\|F_A\|^2_{L^2} + \|\mathrm{d}_A\Phi\|^2_{L^2}\right).$$

Its critical points are characterized by a second-order partial differential equation on (A,Φ) called the *Yang–Mills–Higgs equations*: $\mathrm{d}_A^*F_A = [\mathrm{d}_A\Phi,\Phi]$ and $\Delta_A\Phi = 0$, which one can think of as a nonlinear analogue of Maxwell's equations. In the 'pure' Yang–Mills case where $\Phi \equiv 0$, there is a very important type of manifest solutions in four-dimensions consisting of so-called *(anti-)selfdual* connections or *instantons*, satisfying

$$*F_A = \pm F_A,$$

under the Hodge star operator. This is a first-order condition, which implies the Yang–Mills equation: $\mathrm{d}_A^*F_A = 0$. Moreover, on closed manifolds, instantons are absolute minimizers of the Yang–Mills energy. The space of equivalence classes of such solutions, modulo symmetries, is called the *(A)SD instanton moduli space* (respectively). In particular, outstanding results on topology of $4-$manifolds derive from the study of moduli spaces of anti-selfdual (ASD) instantons.

From the dimensional reduction of the instanton equation one can also obtain very interesting gauge theoretical equations in lower dimensions; in particular, in $3-$dimensions. In

[1] Just to cite some of them: [Ati78, AHDM78, AB83, Don83, Don85, Don90, DK90, Tau82b, Tau83, Tau84, Tau87, Tau88, Tau89, Tau90, Uhl82a, Uhl82b, UY86].

fact, given a noncompact, oriented, Riemannian $3-$manifold X^3, one can study the so-called *Bogomolnyi monopole equation*,

$$F_A = *\mathrm{d}_A\Phi,$$

for configurations (A, Φ) on a $\mathrm{SU}(2)-$bundle over X^3. Solutions are critical points of the Yang–Mills–Higgs energy functional and are translation-invariant instantons on $X^3 \times \mathbb{R}$. One usually also imposes suitable asymptotic conditions ensuring finite energy $\mathcal{E}(A, \Phi) < \infty$, the most important being that $|\Phi|$ is asymptotic to a constant m at infinity, for some $m > 0$ called the *mass* of the monopole. A special case is when $X = \mathbb{R}^3$, where the scaling invariance of the Euclidean metric allows one to reduce to the case $m = 1$. But in general m is a genuine parameter and one may study the behavior of solutions as m varies, in particular as $m \to \infty$.

Extensions of the monopole and instanton equations to higher dimensions exist under the presence of additional geometric structure [DT98], which is available in certain special holonomy manifolds: the monopole equation find parallels in $6-$dimensional Calai–Yau manifolds and $7-$dimensional G_2-manifolds, and they arise as dimensional reductions of the parallel instanton equation in $8-$dimensional $\mathrm{Spin}(7)-$manifolds. Donaldson and Segal [DS09, §6.3] proposed to study these higher dimensional monopoles under the finite mass asymptotic condition and suggested that under the regime where the mass of the monopoles gets arbitrarily large, these concentrate along codimension 3 calibrated cycles in these contexts much in the same way as higher dimensional instantons on closed special holonomy manifolds experience energy concentration and bubbling phenomena along codimension 4 calibrated cycles, cf. [Tia00]. In fact, they suggest that it may be possible that a detailed study of the moduli spaces of finite mass monopoles, in particular the compactness problem of large mass monopoles and its relation with codimension 3 calibrated geometry, potentially could lead to the definition of invariants of certain noncompact special holonomy manifolds.

In this thesis we investigate the problem of the behavior of arbitrarily large mass monopoles on asymptotically conical $3-$ and G_2-manifolds. In the $3-$dimensional case, the connected components of the moduli space of monopoles are labeled by an integer called the charge, and we consider the problem of the limiting behavior of sequences of monopoles with fixed charge k and arbitrarily large masses, cf. [FO19]. In Chapter 3 we prove that the limiting behavior of such monopoles is characterized by energy concentration along a finite subset Z of X, consisting of at most k points at which the zeros of the Higgs fields accumulate and a mass one and charge one $\mathbb{R}^3-$monopole bubbles off, cf. Theorem 3.1.1. We also give some results on the direction of the convergence problem of the sequence outside Z, cf. Section 3.8.

Under certain mild assumptions, in Chapter 4 we develop the same sort of analysis for the G_2-case and prove that the accumulation set of the Higgs fields zeros in this case is of finite $\mathcal{H}^4-$measure and is included in a $\mathcal{H}^4-$rectifiable subset where the intermediate energy of the sequence concentrates (cf. Theorem 4.5.9, Proposition 4.6.12 and Theorem 4.6.13). Moreover, we give some results concerning the bubbling analysis (cf. Proposition 4.7.4) and finish stating some conjectures of its possible outcomes.

The first two chapters set the stage and provide important tools for the analysis performed in the chapters described above. Chapter 1 introduces the reader to the central objects and equations of this thesis. We start with the general setup of Yang–Mills–Higgs theory, the action functional and its Euler–Lagrange equations, then review the unified description of pure Yang–Mills instantons in dimensions $n \geq 4$, mentioning the failure of compactness of their moduli spaces on closed manifolds via energy concentration along codimension 4 calibrated integer rectifiable currents. Next, we introduce monopoles, also in a unified setup, in dimensions $n \geq 3$, explaining the particular cases of interest, giving some of their basic properties and explaining the notions of finite mass configurations as well as the notions of charge and monopole class in the context of asymptotically conical $3-$ and G_2- manifolds respectively. We finish the chapter presenting the conjectural picture on their relation with codimension 3 calibrated geometry.

In Chapter 2 we collect some important analytical properties of YMH configurations in general Riemannian $n-$manifolds and prove some new results that does not appear in the literature. We start with a basic scaling property of the YMH/monopole equations and a known important monotonicity formula for the YMH energy on small geodesic balls in dimensions $n \geq 4$. Then we give simple Bochner–Weitzenböck formulae that imply a known important estimate on the Laplacian of the energy density. By using these results and a nonlinear mean value inequality, we prove a $n-$dimensional $\varepsilon-$regularity result (Theorem 2.3.1) which has some strong analytical consequences and whose $3-$dimensional instance is key to the analysis in Chapter 3 (cf. Theorem 3.3.1). We then prove that on a noncompact manifold of bounded geometry the energy density of a YMH configuration with finite energy decays uniformly to zero at infinity and attains its maximum (Corollary 2.3.3). Moreover, we prove that for an irreducible YMH configuration (A, Φ) on an AC $n-$manifold with $n \geq 3$, finite energy forces $|\Phi|$ to uniformly converge to a constant at infinity and, conversely, that if this holds then $|\mathrm{d}_A\Phi| \in L^2(X)$ (Proposition 2.3.5). In particular, in an AC $3-$dimensional manifold, an irreducible monopole has finite energy if and only if its Higgs field norm uniformly converges to a constant at infinity (Corollary 2.3.8).

More detailed summaries of the contents of each chapter can be found in their own introductions.

Conventions and notations throughout this thesis. X always denotes a connected smooth manifold *without boundary*. If g is a Riemannian metric on X then given any $x \in X$ we denote by $i_x(g)$ the *injectivity radius at* x, and then we let $i_X(g) := \inf_{x\in X} i_x(g)$ be the *injectivity radius of* X. Whenever (X, g) is of bounded geometry, we let $0 < r_0(g) <_g i_X(g)$ be a small enough constant satisfying the scaling property $r_0(\lambda^2 g) = \lambda r_0(g)$ for all $\lambda > 0$, and for which the ball $B_{r_0}(x) \subset (X, g)$ is geometrically uniformly controlled for any $x \in X$. We use the geometers' convention for the Laplacian: $\Delta := \mathrm{d}^*\mathrm{d}$. The gauge theory setup will always be as in §1.1; in particular, G will always denote a compact semi-simple Lie group and a $G-$bundle means a vector bundle associated to a principal $G-$bundle via a faithful representation. We denote by $c > 0$ (sometimes c_0, c_1 and c') a generic constant, which depends only on n, the geometry of the base manifold (X, g) and possibly on the structure group G of the bundle over X in consideration (it should be clear in each context). Its value might change from one occurrence to the next. Should c depend on further data we indicate this by a non-numerical subscript. Finally, we write $x \lesssim y$ for $x \le cy$.

Chapter 1

Gauge theories

In this introductory chapter we briefly review the language of Yang–Mills–Higgs theory, introducing the relevant objects and equations that will be studied in more detail in the rest of this thesis. We start in Section 1.1 with a general description of the gauge theory setup, the action functional and its Euler–Lagrange equations, together with basic observations on its solutions. We then continue in Section 1.2 with the special case of Yang–Mills instantons in a generalized context of dimensions $n \geq 4$ and mention the failure of compactness of their moduli spaces on closed manifolds via energy concentration along codimension 4 calibrated integer rectifiable currents, cf. [Tia00]. Next, in Section 1.3, we introduce the central notion of monopoles, also in a unified setup, in dimensions $n \geq 3$, explaining the particular cases of interest in 3, 6 and 7 dimensions. We give some of their basic properties, introduce the notions of finite mass and (generalized) charge in the context of asymptotically conical $3-$ and G_2-manifolds and cite important energy formulas (cf. [Oli14a]). We finish this chapter presenting a conjectural picture on the relation of large mass monopoles with codimension 3 calibrated geometry. The main references for this chapter are Jaffe-Taubes' book [JT80] and Oliveira's thesis [Oli14a].

1.1 Yang–Mills–Higgs theory

Let (X^n, g) be an oriented Riemannian $n-$manifold and let E be a $G-$bundle over X, where the structure group (or *gauge group*) G is a compact semi-simple Lie group. More concretely, we shall mostly work with $G = \mathrm{SU}(2)$. We write $\mathfrak{g}_E$ for the associated adjoint bundle, equipped with the metric induced by a suitable normalization of the negative of the

Killing form on the Lie algebra $\mathfrak{g}$ of G; e.g., when $G = \mathrm{SU}(2)$ we take the metric induced by the inner product $(a, b) \mapsto -\mathrm{tr}(ab)$ for $a, b \in \mathfrak{su}(2)$. We refer to (smooth) sections of $\mathfrak{g}_E$ as *Higgs fields*. We write $\mathcal{A}(E)$ for the affine space modelled on $\Omega^1(X, \mathfrak{g}_E)$ of (smooth) $G-$connections on E, and we let $\mathcal{C}(E) := \mathcal{A}(E) \times \Gamma(\mathfrak{g}_E)$ denote the space of (smooth) *configurations* on E.

The **Yang–Mills–Higgs (YMH) energy** of a configuration $(A, \Phi) \in \mathcal{C}(E)$ over an open subset $U \subseteq X$ is given by

$$\mathcal{E}_U(A, \Phi) := \int_U e(A, \Phi),$$

where

$$e(A, \Phi) := \frac{1}{2} \left(|F_A|^2 + |\mathrm{d}_A \Phi|^2 \right)$$

denotes the YMH *energy density* function. Here the integral is taken with respect to the usual Riemannian measure (which we shall mostly omit), $F_A \in \Omega^2(X, \mathfrak{g}_E)$ denotes the curvature of A, while $\mathrm{d}_A \Phi \in \Omega^1(X, \mathfrak{g}_E)$ is the exterior covariant derivative of Φ induced by A, and $|\cdot|$ stands for the norm induced on $\mathfrak{g}_E-$valued forms by the tensor product metric. YMH theory is the variational theory of the functional $\mathcal{E}_X$, which is usually considered to be defined on the set of configurations (A, Φ) with $|F_A|, |\mathrm{d}_A \Phi| \in L^2(X)$, also known as *finite energy* configurations. As a particular instance, the **Yang–Mills (YM) energy** of a connection $A \in \mathcal{A}(E)$ over U is given by

$$\mathcal{E}_U(A) := \mathcal{E}_U(A, 0) = \frac{1}{2} \|F_A\|^2_{L^2(U)}.$$

The Euler–Lagrange equations[1] for the YMH energy functional are

$$\mathrm{d}_A^* F_A = [\mathrm{d}_A \Phi, \Phi], \quad \Delta_A \Phi = 0, \tag{1.1.1}$$

and these are called the **YMH equations**. Here d_A^* stands for the L^2-adjoint operator of d_A, which acting on $\mathfrak{g}_E-$valued $k-$forms has the form $\mathrm{d}_A^* = (-1)^{n(k+1)+1} * \mathrm{d}_A *$, where $*$ is the Hodge star operator associated to g; $\Delta_A := \mathrm{d}_A^* \mathrm{d}_A$ denotes the Laplacian induced by A. In particular, a smooth critical point $A \in \mathcal{A}(E)$ of the YM energy satisfies the **YM equation**:

$$\mathrm{d}_A^* F_A = 0. \tag{1.1.2}$$

Definition 1.1.3. A configuration $(A, \Phi) \in \mathcal{C}(E)$ is called **YMH** if it satisfies (1.1.1). A connection $A \in \mathcal{A}(E)$ is called **YM** if it satisfies (1.1.2).

Remark 1.1.4. We do not ask for finite energy in any of the definitions.

[1] with respect to compactly supported variations.

If $(A,\Phi)\in\mathcal{C}(E)$ is YMH then, in particular, $\Delta_A\Phi=0$ and this implies that[2]

$$\Delta\frac{|\Phi|^2}{2}=\langle\Phi,\Delta_A\Phi\rangle-|\mathrm{d}_A\Phi|^2=-|\mathrm{d}_A\Phi|^2\leq 0. \tag{1.1.5}$$

As a consequence, the function $|\Phi|^2$ is subharmonic, and so, has no local maxima by the Maximum Principle (cf. [JT80, Proposition 3.3, Chapter VI]). Thus, if X was to be a compact manifold (without boundary), then $|\Phi|^2$ would be constant and $\mathrm{d}_A\Phi=0=\mathrm{d}_A^*F_A$, in which case A would be a YM connection. Therefore, if one is to study smooth *irreducible* YMH configurations, meaning those with $\mathrm{d}_A\Phi\neq 0$, on a manifold without boundary X, then X must be noncompact.

Remark 1.1.6. Of course one can also consider the theory of weak solutions of the YMH equations, in some appropriate functional analytic setup depending on X. In this thesis, however, we shall mostly restrict ourselves to smooth solutions (unless otherwise stated).

We denote by $\mathcal{G}(E)$ the group of (smooth) *gauge transformations* on E. There is a natural action of $\mathcal{G}(E)$ on $\mathcal{C}(E)$, acting on connections via pullback and on Higgs fields by the adjoint action. By the Ad-invariance of the metric on $\mathfrak{g}_E$ it follows that the YMH energy (density) is invariant under the action of $\mathcal{G}(E)$, and so are the YMH equations (one can also check this directly).

Now that we introduced the general framework of YMH theory, we turn to special solutions of the YMH equations.

1.2 Instantons

In pure YM theory, a rather important type of first-order solutions of the YM equation (1.1.2) appear in four-dimensions, the so-called *instantons*. Let us briefly recall this classical notion before giving a more general definition. Let (X^4,g) be an oriented Riemannian $4-$manifold. Then it is well-known that the space of $2-$forms splits orthogonally into so-called selfdual and anti-selfdual parts, corresponding respectively to $\pm$-eigenspaces of the Hodge $*$ operator. Given a $G-$bundle E over X, this splitting immediately extends to $\mathfrak{g}_E-$valued $2-$forms and a connection $A\in\mathcal{A}(E)$ is said to be a anti-selfdual (ASD) instanton if its curvature satisfies $*F_A=-F_A$. These are automatically Yang–Mills connections by the Bianchi identity and for $\mathrm{SU(r)}-$bundles of positive topological charge over closed $4-$manifolds these are in fact

[2] Δ denotes the geometers' Laplacian on functions; $\Delta:=\mathrm{d}^*\mathrm{d}$.

absolute minimizers of the Yang–Mills functional. If (X^n, g) is an $n-$dimensional oriented Riemannian manifold, with $n > 4$, the Hodge $*$ operator is not an endomorphism on $2-$forms anymore but rather maps $2-$forms into $(n-2)-$forms. But if, rather naively, we suppose that X^n is endowed with a $(n-4)-$form Ξ, then the operator $*(\cdot \wedge \Xi)$ maps $2-$forms into $2-$forms and we could make precise a notion of instanton depending on Ξ. This turns out to be very interesting in special cases and in this section we shall introduce this higher dimensional notion of instanton and the particularly important cases it includes.

1.2.1 The instanton equation

In the context of §1.1, now suppose that $n \geq 4$ and furthermore that X admits a closed $(n-4)-$form $\Xi \in \Omega^{n-4}(X, \mathbb{R})$.

Definition 1.2.1 (Instantons). A connection $A \in \mathcal{A}(E)$ is called a $\Xi-$**instanton** on X if it satisfies

$$*(F_A \wedge \Xi) = -F_A. \tag{1.2.2}$$

Since Ξ is closed, it follows from the Bianchi identity that every $\Xi-$instanton is a Yang–Mills connection.

However, even when Ξ is a parallel form, the algebraic equation (1.2.2) is in general overdetermined and may admit no solutions at all. When $n > 4$, this definition behaves best when (X^n, g) has special holonomy so that it admits a natural geometric structure defining Ξ. Indeed, the interest in studying this equation in general lies on the following particular types of settings (X^n, g, Ξ):

(i) (X^4, g) an oriented Riemannian $4-$manifold and $\Xi \equiv 1$. Then equation (1.2.2) is of course the classical **anti-self-dual (ASD) instanton** equation: $*F_A = -F_A$.

(ii) (X^{2m}, g) Kähler $m-$fold with $m \geq 2$ and $\Xi = \frac{\omega^{m-2}}{(m-2)!}$, where ω is the associated fundamental $(1,1)-$form. Then equation (1.2.2) is the **Hermitian–Yang–Mills** (HYM) equation: $F_A^{0,2} = 0$ and $\Lambda_\omega F_A = 0$, where Λ_ω is the dual Lefschetz operator.

(iii) (X^7, g) a G_2-manifold and $\Xi = \phi$ the G_2-structure $3-$form. Then equation (1.2.2) is the G_2-**instanton** equation which can be written as $F_A \wedge \psi = 0$, where $\psi = *\phi$.

(iv) (X^8, g) a $\mathrm{Spin}(7)-$manifold and $\Xi = \Omega$ the $\mathrm{Spin}(7)-$structure $4-$form. Then equation (1.2.2) is the $\mathrm{Spin}(7)-$**instanton** equation: $*(F_A \wedge \Omega) = -F_A$.

In all the above cases, it turns out that Ξ is in fact a *calibration*. Recall that a $k-$form $\alpha \in \Omega^k(X)$ is called a *calibration* if it is closed and has comass$(\alpha) \leq 1$, i.e., $\mathrm{d}\alpha = 0$ and for each $x \in X$ and each oriented $k-$plane $V^k \leq T_x M$ we have $\alpha|_V \leq \mathrm{vol}_V$. Then a closed oriented $k-$dimensional submanifold $Q^k \subset X^n$ is called calibrated with respect to α if $\alpha|_{T_x Q} = \mathrm{vol}_{T_x Q}$ for all $x \in Y$. We note that this notion also extends to rectifiable $k-$currents, which are some sort of generalized (possibly very singular) submanifolds in the sense of geometric measure theory. The key property of calibrated submanifolds (and currents) is that they are volume minimizers in their homology class, attaining a topological volume bound.

It turns out that $\Xi-$instantons in all the above cases, when they exist, are also energy minimizers for the $\mathrm{SU(r)}$ YM energy attaining a topological energy bound. The energy bound is given by the following simple result.

Lemma 1.2.3 (Topological energy bound)**.** *Let (X^n, g, Ξ) be a closed, oriented, Riemannian $n-$manifold, $n \geq 4$, together with a calibration $(n-4)-$form $\Xi \in \Omega^{n-4}(X)$. Let E be a $\mathrm{SU(r)}-$bundle over X. If $\langle c_2(E) \cup [\Xi], [X] \rangle \geq 0$ and $A \in \mathcal{A}(E)$ is a $\Xi-$instanton then*

$$\mathcal{E}(A) = 4\pi^2 \langle c_2(E) \cup [\Xi], [X] \rangle.$$

1.2.2 Non-compactness phenomena and codimension 4 calibrated geometry

The $\Xi-$instanton equation (1.2.2) (and, more generally, the Yang–Mills equation (1.1.2)) is invariant under the group $\mathcal{G}(E)$ of gauge transformations of E. A major difficulty in the study of $\Xi-$instantons is that their moduli spaces need not be compact. This non-compactness phenomenon has two causes: the formation of non-removable singularities and bubbling in codimension four. In fact, Tian [Tia00] discovered that there is an interesting relation between gauge theory in higher dimension and calibrated geometry via the bubbling process. In particular, his general foundational compactness result—extending work of Uhlenbeck [Uhl82a], Price [Pri83] and Nakajima [Nak88] —, together with the removable singularity theorem of Tao–Tian [TT04], can be summarized as follows.

Theorem 1.2.4 (Uhlenbeck, Price, Nakajima, Tian, Tao)**.** *Let (X^n, g, Ξ) be a closed, oriented, Riemannian $n-$manifold, $n \geq 4$, together with a calibration $(n-4)-$form $\Xi \in \Omega^{n-4}(X)$. Let E be a $\mathrm{SU(r)}-$bundle over X and let $\{A_i\} \subset \mathcal{A}(E)$ be a sequence of $\Xi-$instantons over (X, g). Then there exist constants $c \geq 0$ and $\varepsilon_{\mathrm{inst}} > 0$, where c depends only on n and the* *geometry of* (X^n, g)*, and* $\varepsilon_{\text{inst}}$ *depends furthermore on the structure constants of* $\mathfrak{su}(r)$*, such that the following holds.*

- *The subset* $S \subseteq X$ *defined by*

$$S = S(\{A_i\}) := \bigcap_{0<r<r_0} \left\{ x \in X : \liminf_{i\to\infty} e^{cr^2} r^{4-n} \int_{B_r(x)} |F_{A_i}|^2 \mathcal{H}^n \geq \varepsilon_{\text{inst}} \right\}$$

is closed and satisfies $\mathcal{H}^{n-4}(S) < \infty$*. Moreover, after passing to a subsequence, there are gauge transformations* $\sigma_i \in \mathcal{G}(E|_{X\setminus S})$ *such that* $\sigma_i^* A_i$ *converges to a* $\Xi-$*instanton* A *on* $E|_{X\setminus S}$ *in* C^∞_{loc} *outside* S*.*

- *There exists a bounded upper semicontinuous function* $\Theta : S \to [\varepsilon_{\text{inst}}, \infty)$ *such that, as Radon measures,*

$$|F_{A_i}|^2 \mathcal{H}^n \rightharpoonup |F_A|^2 \mathcal{H}^n + \Theta \mathcal{H}^{n-4} \lfloor S.$$

- S *decomposes as* $S = \Gamma \cup \text{sing}(A)$*, where*

$$\Gamma := \text{supp}(\Theta \mathcal{H}^{n-4} \lfloor S) \quad \text{and}$$
$$\text{sing}(A) := \left\{ x \in X : \limsup_{r \downarrow 0} r^{4-n} \int_{B_r(x)} |F_A|^2 \mathcal{H}^n > 0 \right\};$$

Γ *is countably* $\mathcal{H}^{n-4}-$*rectifiable and, for* $\mathcal{H}^{n-4}-$*a.e.* $x \in \Gamma$*,* $\Xi|_{T_x\Gamma}$ *is a volume form on* $T_x\Gamma$*; moreover,* $\mathcal{H}^{n-4}(\text{sing}(A)) = 0$*.*

- *The connection* A *extends to a* $\Xi-$*instanton on a* $\mathrm{SU(r)}-$*bundle* $\tilde{E}$ *over* $X \setminus \text{sing}(A)$ *which is isomorphic to* E *over* $X \setminus S$*.*

- *For* $\mathcal{H}^{n-4}-$*a.e.* $x \in \Gamma$*, there exists a non-trivial ASD instanton* $\mathfrak{I}(x)$ *on* $T_x\Gamma^\perp$ *satisfying*

$$\Theta(x) \geq \mathcal{E}(\mathfrak{I}(x)) \tag{1.2.5}$$

and whose pullback to T_xX *is gauge-equivalent to the limit of a blowing-up of the sequence* $\{A_i\}$ *around the point* x*.*

- *Finally, the* $(n-4)-$*current* $C(\Gamma, \Theta)$ *given by*

$$C(\Gamma, \Theta)(\varphi) := \frac{1}{8\pi^2} \int_\Gamma \langle \Xi|_\Gamma, \varphi \rangle \Theta \mathrm{d}\left(\mathcal{H}^{n-4} \lfloor \Gamma\right), \quad \forall \varphi \in \Omega^{n-4}(X),$$

is a $\Xi-$*calibrated integer rectifiable current satisfying conservation of instanton charge density, in the following sense: for every* $\varphi \in \Omega^{n-4}(X)$*,*

$$\lim_{i\to\infty} \int_X \text{tr}(F_{A_i} \wedge F_{A_i}) \wedge \varphi = \int_X \text{tr}(F_A \wedge F_A) \wedge \varphi + 8\pi^2 C(\Gamma, \Theta)(\varphi). \tag{1.2.6}$$

In particular, the L^2-energy is conserved:

$$8\pi^2\langle c_2(E)\cup[\Xi],[X]\rangle = \|F_A\|^2_{L^2(X\setminus S)} + \int_\Gamma \Theta \mathrm{d}\left(\mathcal{H}^{n-4}\lfloor\Gamma\right).$$

Remark 1.2.7. In the situation of Theorem 1.2.4:

- The function Θ measures the energy density lost by the sequence around a point $x\in\Gamma$. If, instead of a single ASD instanton bubbling off transversely at $x\in\Gamma$, there is actually a whole bubbling tree of ASD instantons, then the inequality (1.2.5) is necessarily strict.

- In the simplest case, the singularities of A are removable, Γ is a smooth $\Xi-$calibrated submanifold, and the bubbling trees of ASD instantons consist of single ASD instantons forming a smooth section $\mathfrak{I}$ of an instanton bundle associated to the normal bundle of Γ and the restriction $E|_\Gamma$ of the ambient bundle. Conjecturally, in case (X,g,Ξ) is a G_2- or $\mathrm{Spin}(7)-$manifold, $\mathfrak{I}$ should satisfy a certain non-linear Dirac equation, associated to Ξ and the restriction $A|_\Gamma$, called the *Fueter equation*, see [Wal17b, Wal17c, Hay17].

Remark 1.2.8. Two of the most important analytical results underlying the blow-up analysis and proof of Theorem 1.2.4 are the pure YM instances of the monotonicity formula of Theorem 2.1.2 and the $\varepsilon-$regularity of Theorem 2.3.1. A self-contained comprehensive treatment on the proof of Tian's bubbling theorem (accounting for all the results in Theorem 1.2.4 but the singularity removal result of Tao–Tian) can be found in the book by the author and his supervisor Sá Earp [FSE19].

We finish this section with a simple linear algebra result which is the reason why ASD instantons bubbles off transversally.

Lemma 1.2.9 ([Fad16, Proposition 2.3.6])**.** *Suppose $n>4$, let $\Xi\in\Omega^{n-4}(\mathbb{R}^n)$ be a calibration, let $\mathbb{R}^n=\mathbb{R}^{n-4}\oplus\mathbb{R}^4$ be an orthogonal decomposition and let E be a $G-$bundle over $\mathbb{R}^4$ where G is a compact semi-simple Lie group. If $I\in\mathcal{A}(E)$ is a* non-flat *connection then the following are equivalent:*

(i) The lifted connection $\underline{I}$ is a $\Xi-$ASD instanton.

(ii) There exist orientations on $\mathbb{R}^{n-4}$ and $\mathbb{R}^4$ with respect to which Ξ calibrates $\mathbb{R}^{n-4}$ and I is an ASD instanton on $\mathbb{R}^4$.

1.3 Monopoles

1.3.1 The monopole equation

In the context of §1.1, now suppose that $n \geq 3$ and furthermore that X admits a closed $(n-3)-$form $\Theta \in \Omega^{n-3}(X, \mathbb{R})$. As some sort of a dimensional reduction of the previous section notion, we introduce the following.

Definition 1.3.1 (Monopoles)**.** A configuration (A, Φ) on E is called a $\Theta-$**monopole** on (X, g) if it satisfies

$$F_A \wedge \Theta = *\mathrm{d}_A \Phi. \tag{1.3.2}$$

Since Θ is closed, it follows from the Bianchi identity that a solution (A, Φ) to (1.3.2) satisfies $\Delta_A \Phi = 0$.

In this thesis we are particularly interested in monopoles (rather than instantons), and our interest in introducing equation (1.3.2) in general lies mainly on two of the following three particular types of settings (X^n, g, Θ):

(a) When (X^3, g) is an oriented Riemannian $3-$manifold we take $\Theta = 1$, so that (1.3.2) is the classical **Bogomolnyi equation**:

$$F_A = *\mathrm{d}_A \Phi, \tag{1.3.3}$$

and its solutions are simply called (Bogomolnyi) **monopoles**. This equation can be obtained from the dimensional reduction of the ASD instanton equation on $M^4 := X^3 \times \mathbb{R}_t$ with the standard product structure. Indeed, let $\underline{E} \to M^4$ be the pullback bundle. Then an $\mathbb{R}-$invariant connection $\mathbb{A}$ on $\underline{E}$ is always of the form $\mathbb{A} = \underline{A} + \underline{\Phi} \otimes \mathrm{d}t$, where (A, Φ) is a configuration on $E \to X^3$, and $\underline{A}$ and $\underline{\Phi}$ denote the corresponding pullbacks. Now one may easily check that $*_4 F_{\mathbb{A}} = -F_{\mathbb{A}}$ if and only if (A, Φ) satisfy (1.3.3).

Monopoles are easily seen to be YMH configurations and, in fact, in suitable cases (e.g. $G = \mathrm{SU}(2)$ and $X^3 = \mathbb{R}^3$) they are absolute minimizers of the YMH energy functional. They have been focus of intense study in conformally flat manifolds such as $\mathbb{R}^3$ (some of the earlier references in the mathematics literature are [Tau82a, JT80, AH88]) and $\mathbb{R}^2 \times S^1$ (see for example [CK01, CK02, Fos16]), as in these cases the moduli spaces of monopoles are (noncompact) Hyperkähler manifolds. In more general geometries, Braam [Bra89] considered monopoles on asymptotically hyperbolic manifolds, while

Floer [Flo95, Flo95] and Ernst [Ern95] studied monopoles on asymptotically Euclidean (AE) ones, which are natural generalizations of the $\mathbb{R}^3$ situation. A further generalization of the $\mathbb{R}^3$ situation, which contains the AE case as a subcase, is that of *asymptotically conical* (AC) manifolds [Kot15], [Oli16c], [FO19], which we shall focus in this thesis. We shall study these on Chapter 3.

(b) If (X^6, g) is a Calabi–Yau $3-$fold, we take $\Theta = \Omega_1$ the real part of the holomorphic volume form $\Omega \in \Omega^{3,0}(X, \mathbb{C})$. Then (1.3.2) gives the **Calabi–Yau monopoles** equation $F_A \wedge \Omega_1 = *\mathrm{d}_A\Phi$, which comes with the condition $\Lambda_\omega F_A = 0$. These can be obtained from the dimensional reduction of the G_2-instanton equation. More generally, one may study complex Calabi–Yau monopoles, i.e. pairs (A, Φ), where $\Phi \in \Omega^0(X, \mathfrak{g}_E^{\mathbb{C}})$ is a complex Higgs field, such that $\frac{1}{2}F_A \wedge \Omega = *\partial_A\Phi$ and $\Lambda F_A = \frac{i}{2}[\Phi, \overline{\Phi}]$. Besides its mention in [DS09, §6.3], the only mathematical reference studying these monopoles that the author knows is Oliveira's work [Oli14a, Oli16a], where he constructs the first nontrivial examples under a symmetry assumption, on the cotangent bundle of the $3-$sphere endowed with the (AC) Stenzel metric. His main results constructs the moduli space of such symmetric monopoles, show that they are parametrized by their mass (i.e. the asymptotic value of $|\Phi|$) and then describes their behavior in the large mass limit.

(c) When (X^7, g) is a G_2-manifold we take $\Theta = \psi$ the G_2-structure $4-$form. Then equation (1.3.2) is the G_2-**monopole** equation:

$$F_A \wedge \psi = *\mathrm{d}_A\Phi. \tag{1.3.4}$$

This can be obtained from dimensional reduction of the $\mathrm{Spin}(7)-$instanton equation, and includes G_2-instantons as pure Yang–Mills solutions (these are the only solutions in the compact setting, cf. Proposition 1.3.11). Oliveira [Oli14a, Oli14b] studied these under a symmetry assumption on the three Bryant–Salamon examples of AC G_2-manifolds. He gives the first nontrivial examples of G_2-monopoles constructing the moduli space of symmetric monopoles on two of these manifolds: the total spaces of the bundles of ASD $2-$forms over the $4-$sphere and $\mathbb{CP}^2$, and showed analogous results as those described above in (b), as well as proves a vanishing result for monopoles on the total space of the spinor bundle over the $3-$sphere. In [Oli16b] he also studies G_2-monopoles with singularities. We shall study more closely G_2-monopoles on AC manifolds in Chapter 4.

As it should be clear from the above, in this thesis we shall focus in cases (a) and (c), i.e. the 3−dimensional Bogomolnyi equation (1.3.3) and the higher dimensional G_2−monopole equation (1.3.4). Even so, it is convenient to use the Θ−monopole notion to unify some common properties that we shall need.

There is an important energy functional associated to the notion of Θ−monopoles and directly related to the YMH energy functional.

Definition 1.3.5. Let $U \subset X$ be a precompact open subset. The Θ−**energy** of $\mathcal{E}_U^\Theta$ of a configuration (A, Φ) on U is defined by

$$\mathcal{E}_U^\Theta(A, \Phi) := \int_U e^\Theta(A, \Phi),$$

where

$$e^\Theta(A, \Phi) := \frac{1}{2}\left(|F_A \wedge \Theta|^2 + |\mathrm{d}_A\Phi|^2\right)$$

is the Θ−*energy density*.

A standard computation of the first variation gives the following.

Proposition 1.3.6 (cf. [Oli14a, Proposition 1.3.2])**.** *The Euler–Lagrange equations for the Θ−energy functional are*

$$\mathrm{d}_A^*\pi_\Theta(F_A) = [\mathrm{d}_A\Phi, \Phi], \quad \Delta_A\Phi = 0, \tag{1.3.7}$$

where $\pi_\Theta(F_A) := *(*(F_A \wedge \Theta) \wedge \Theta)$.

Some comments are in order:

- The YMH energy and the Θ−energy agree in the classical 3−dimensional case (a), and so does their Euler–Lagrange equations.

- As for the Calabi–Yau case (b), the complex structure gives the bi-degree splitting $\Omega^2_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$, then $\pi_\Theta(F_A) = -2(F_A^{2,0} + F_A^{0,2})$. So the Θ−energy just measures the L^2−norm of $F_A^{0,2}$ (for unitary A) and its Euler–Lagrange equations are $\Delta_A\Phi = 0$ and $\partial_A^* F_A^{2,0} = -\frac{1}{2}[\partial_A\Phi, \Phi]$.

- In the G_2−case (c), the G_2−structure gives the splitting $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$ and $\pi_\Theta(F_A) = 3\pi_7(F_A)$. So the Θ−energy just measures the L^2−norm of $\pi_7(F_A)$ and the Euler–Lagrange equations are the YMH equations since by G_2−linear algebra one has

$$\mathrm{d}_A^* F_A = 3\mathrm{d}_A^*\pi_7(F_A) = \mathrm{d}_A^*\pi_\Theta(F_A).$$

Using integration by parts, one can easily prove the following energy identities:

Proposition 1.3.8 (cf. [Oli14a, Proposition 1.3.4])**.** *Let $U \subset X$ be a precompact subset with smooth boundary ∂U, and let (A, Φ) be a configuration on E. Then:*

$$\mathcal{E}_U^{\Theta}(A, \Phi) = \int_{\partial U} \langle \Phi, F_A \rangle \wedge \Theta + \frac{1}{2} \| F_A \wedge \Theta - *\mathrm{d}_A \Phi \|^2_{L^2(U)}. \tag{1.3.9}$$

Moreover, in case (X^7, g) is a G_2-*manifold one has*

$$\mathcal{E}_U(A, \Phi) = -\frac{1}{2} \int_U \langle F_A \wedge F_A \rangle \wedge \phi + \int_{\partial U} \langle \Phi, F_A \rangle \wedge \psi + \frac{1}{2} \| F_A \wedge \psi - *\mathrm{d}_A \Phi \|^2_{L^2(U)}. \tag{1.3.10}$$

For noncompact manifolds X with an end of a suitable type, it can happen that the first term in the energy identity (1.3.9) will give rise to a quantity that is fixed in terms of the topology of the bundle E and Θ (e.g. $X = \mathbb{R}^3$ or, more generally, on asymptotically conical manifolds under mild assumptions, cf. Propositions 1.3.18 and 1.3.23). Since the second term is always greater than or equal to zero with equality if and only if (A, Φ) is a $\Theta-$monopole, it follows that in such cases the $\Theta-$monopoles minimize the $\Theta-$energy.

In accordance with what we already remarked for YMH configurations in §1.1 (recall that Bogomolnyi and G_2-monopoles are YMH configurations), Proposition 1.3.8 gives:

Corollary 1.3.11. *If X is closed and (A, Φ) is a $\Theta-$monopole then $\mathcal{E}_X^{\Theta} = 0$ and $\mathrm{d}_A \Phi = 0 = F_A \wedge \Theta$. In particular, if $\Phi \neq 0$, the connection A is reducible.*

It follows from the above result that in order to study solutions of equation (1.3.2) with $\mathrm{d}_A \Phi \neq 0$ one must either let X be compact with *nonempty* boundary, noncompact or allow the monopoles to have singularities. In this thesis we shall focus on the noncompact manifolds of bounded geometry, specially on asymptotically conical manifolds.

1.3.2 Monopoles on asymptotically conical (AC) manifolds

In this section we introduce the main class of noncompact Riemannian manifolds of bounded geometry with which we shall work in this thesis.

Definition 1.3.12. Let (X^n, g) be a complete, oriented, Riemannian $n-$manifold. Then (X^n, g) is called **asymptotically conical** (AC) with rate $\nu < 0$ if there exist a compact subset $K \subset X$, an oriented, closed (compact and without boundary) Riemannian $(n-1)-$manifold (N^{n-1}, g_N), and an orientation preserving diffeomorphism

$$\varphi : C(N) := (1, \infty)_r \times N \to X \setminus K$$

such that the cone metric $g_C := dr^2 + r^2 g_N$ on $C(N)$ satisfies

$$|\nabla^j (\varphi^* g - g_C)|_C = O(r^{\nu - j}) \quad \text{as } r \to \infty \text{ for all } j \in \mathbb{N}_0.$$

Here ∇ is the Levi–Civita connection of g_C. We say furthermore that X has *one end* if N is connected, and we refer to $X \setminus K$ as the *end* of X. N is called the asymptotic link of X. A *distance function* on X will be any positive smooth function $\rho : X \to \mathbb{R}^+$ such that $\rho|_{X \setminus K} = r \circ \varphi^{-1}$.

Finite mass configurations. On AC manifolds, we shall be interested in the following particular class of configurations.

Definition 1.3.13. Let (X^n, g) be an AC oriented Riemannian $n-$manifold with one end, as in Definition 1.3.12. Let E be a $G-$bundle over X and suppose there exists a $G-$bundle E_∞ over the asymptotic link N of X together with an isomorphism of bundles $\varphi^* \left(E|_{X \setminus K} \right) \cong \pi^* E_\infty$, where $\pi : (1, \infty) \times N \to N$ is the projection onto the second factor. A configuration (A, Φ) on E is said to have **finite mass** if there exists a connection A_∞ on E_∞ such that A is asymptotic to A_∞ with some rate $\eta < 0$, i.e.

$$|\nabla^j_\infty (\varphi^* \nabla_A - \pi^* \nabla_\infty)| = O(\rho^{\eta - 1 - j}) \quad \text{as } \rho \to \infty \text{ for all } j \in \mathbb{N}_0 \text{ and for some } \eta < 0, \tag{1.3.14}$$

and there is a positive constant $m \in \mathbb{R}^+$ with

$$\lim_{\rho \to \infty} |\Phi| = m. \tag{1.3.15}$$

We call the constant m the **mass** of (A, Φ).

Remark 1.3.16. Definition 1.3.13 may not be the most general one, but the analysis of the next chapters will motivate it better in the case of YMH configurations and monopoles. For instance, Proposition 2.3.5 and Corollary 2.3.8 will relate the asymptotic condition (1.3.15) to finite energy assumption. Below we shall also see that one can prove nice energy formulas with it for YMH configurations under reasonable assumptions. Moreover one should have in mind that all nontrivial examples of monopoles on AC $3-$dimensional, Calabi–Yau and G_2-manifolds constructed in [Oli14a] satisfy these assumptions.

Remark 1.3.17. If (A, Φ) is a YMH configuration with finite mass m then, by subharmonicity, as $|\Phi|$ converges to m along the end of X, the maximum principle yields that either $|\Phi| < m$ on X or $|\Phi|$ is constant equal to m.

Charge and energy formula in the $3-$dimensional case. Let (X^3, g) be an AC oriented Riemannian $3-$manifold with asymptotic link N, fix a smooth radius function ρ on X, and let E be an $\mathrm{SU}(2)-$bundle over X. We note that E is necessarily trivializable, since $\mathrm{SU}(2)$ is $2-$connected. Now let (A, Φ) be a configuration with finite mass $m \neq 0$. Then there is $r \gg 1$ so that Φ does not vanish in $\rho^{-1}[r, \infty)$. It follows that the various $\Phi|_{\rho^{-1}(r)}$ yield a well defined homotopy class of maps $\rho^{-1}(r) \cong N^2 \to \mathfrak{su}(2)\backslash\{0\} \cong S^2$. The degree of such maps is therefore a well defined integer k called the **charge** of (A, Φ). Equivalently, the eigenspaces of Φ split the bundle in this region as $E|_{\rho^{-1}(r)} \cong L \oplus L^{-1}$, for some complex line bundle L over $N \cong \rho^{-1}(r)$. Moreover, the degree of any such L does not depend on r and equals to the charge k of (A, Φ).

In particular, in this $3-$dimensional AC case, using the energy identity of Proposition 1.3.8 one can prove the following energy formula.

Proposition 1.3.18 ([Oli14a, Corollary 1.4.11])**.** *Let (X^3, g) be an AC oriented Riemannian $3-$manifold and E be an $\mathrm{SU}(2)-$bundle over X. If $(A, \Phi) \in \mathcal{C}(E)$ is a finite mass monopole of mass $m \neq 0$ and charge k, then*

$$\mathcal{E}_X(A, \Phi) = 4\pi m k. \tag{1.3.19}$$

In fact, more generally, one has the following. Suppose that $(A, \Phi) \in \mathcal{C}(E)$ has finite mass $m \neq 0$, charge k and satisfy the following conditions:

- *(i) (A, Φ) is asymptotic to a configuration (A_∞, Φ_∞) on E_∞, with rates $\eta - 1$ for the connection and -1 for the Higgs field, with all derivatives, for some $\eta < 0$;*
- *(ii) A_∞ is YM, $d_{A_\infty}\Phi_\infty = 0$ and $\langle \Phi_\infty, F_{A_\infty} \rangle \neq 0$.*
- *(iii) $\mathcal{E}_X(A, \Phi) < \infty$.*

Then A_∞ is reducible and

$$\mathcal{E}_X(A, \Phi) = 4\pi m k + \frac{1}{2}\| * F_A - \mathrm{d}_A \Phi\|^2_{L^2(X)}. \tag{1.3.20}$$

Notice that the first term of (1.3.20) is fixed by the charge and mass while the second is nonnegative, and vanishes if and only if (A, Φ) is a monopole. Thus showing, in particular, that monopoles minimize the YMH energy amongst such finite mass m and charge k configurations.

AC G_2-structures and energy formula in the $7-$dimensional case.

Definition 1.3.21. Let N^6 be a $6-$manifold. A pair of forms $(\omega, \Omega_1) \in \Omega^2 \oplus \Omega^3(N, \mathbb{R})$ determine an $\mathrm{SU}(3)-$**structure** on N if:

- The $\mathrm{GL}(6, \mathbb{R})$ orbit of Ω_1 is open, with stabilizer a covering of $\mathrm{SL}(3, \mathbb{C})$;
- The following compatibility relations hold
$$\omega \wedge \Omega_1 = \omega \wedge \Omega_2 = 0, \quad \frac{\omega^3}{3!} = \frac{1}{4}\Omega_1 \wedge \Omega_2,$$
where $\Omega_2 := J\Omega_1$ and J denotes the almost complex structure determined by Ω_1;
- $g_N := \omega(\cdot, J\,\cdot\,)$ determines a Riemannian metric on N.

If, furthermore, the forms $(\omega, \Omega_1, \Omega_2)$ satisfy
$$\mathrm{d}\Omega_2 = -2\omega^2 \quad \text{and} \quad \mathrm{d}\omega = 3\Omega_1,$$
then (N, g_N) is said to be **nearly Kähler**.

Lemma 1.3.22. *Suppose that N^6 is endowed with an $\mathrm{SU}(3)-$structure determined by (ω, Ω_1). Then the Riemannian cone $(C(N) := (1, \infty)_r \times N, g_C := dr^2 + r^2 g_N)$ with the G_2-structure*
$$\phi_C := r^2 dr \wedge \omega + r^3\Omega_1, \quad \psi_C := r^4\frac{\omega^2}{2} - r^3 dr \wedge \Omega_2,$$
is a G_2-manifold if and only if (N^6, g_N) is nearly Kähler.

Thus if X is an AC oriented Riemannian $7-$manifold endowed with a compatible torsion-free G_2-structure (ψ, g), then its asymptotic link (N^6, g_N) is necessarily nearly Kähler. Moreover, it follows that ψ has the same asymptotic rate of the metric:
$$|\nabla^j\left(\varphi^*\psi - \psi_C\right)|_C = O(r^{\nu-j}) \quad \text{as } r \to \infty \text{ for all } j \in \mathbb{N}_0.$$

In this context, one can prove the following energy formula.

Proposition 1.3.23 ([Oli14b, Proposition 4])**.** *Let (X^7, ψ, g) be an AC G_2-manifold with one end, with asymptotic link N, and E be an $\mathrm{SU}(2)-$bundle over X. Suppose that $(A, \Phi) \in \mathcal{C}(E)$ is a finite mass G_2-monopole of mass $m \neq 0$ and $|A - A_\infty| = O(\rho^{\eta-5})$ as $\rho \to \infty$ for some $\eta < 0$. Then (E_∞, A_∞) is reducible to a complex line bundle $L \to N$ such that $E|_{\rho^{-1}(r) \cong N} \cong L \oplus L^{-1}$ for $r \gg 1$, and*
$$\mathcal{E}_X^{\psi}(A, \Phi) = 4\pi m\langle c_1(L) \cup [\iota^*\psi], [N]\rangle, \tag{1.3.24}$$

where $[\iota^*\psi] \in H^4(N, \mathbb{R})$ is the cohomology class obtained by restricting $[\psi] \in H^4(X, \mathbb{R})$ to any cross section $\rho^{-1}(r) \cong N$ for $r \gg 1$. In fact, more generally, one has the following. Suppose that $(A, \Phi) \in \mathcal{C}(E)$ is of finite mass $m \neq 0$ and satisfy the following properties:

(i) *(A, Φ) is asymptotic to a configuration (A_∞, Φ_∞) on E_∞, with rates $\eta - 5$ for the connection and -5 for the Higgs field, with all derivatives, for some $\eta < 0$;*

(ii) *A_∞ is Hermitian–Yang–Mills with respect to the nearly Kähler structure of N, i.e. $F_{A_\infty} \wedge \Omega_2 = 0$ and $F_{A_\infty} \wedge \omega^2 = 0$;*

(iii) *$\mathcal{E}^\psi_X(A, \Phi) < \infty$.*

Then A_∞ is reducible and

$$\mathcal{E}^\psi_X(A, \Phi) = 4\pi m \langle c_1(L) \cup [\iota^*\psi], [N] \rangle + \frac{1}{2} \| * F_A \wedge \psi - \mathrm{d}_A \Phi \|^2_{L^2(X)}. \tag{1.3.25}$$

Under the hypothesis of Proposition 1.3.23, keeping the class $c_1(L) \cup [\iota^*\psi]$ fixed, it follows that G_2-*monopoles minimize the* $\psi-$*energy*. The class $[\iota^*\psi]$ is determined by the calibration and $c_1(L) \in H^2(N, \mathbb{R})$ by the asymptotic behavior of the G_2-monopole. We call such $c_1(L)$ a **monopole class** and this is the higher dimensional analog of what in 3 dimensions is known as the monopole charge.

1.3.3 Conjectural picture: large mass monopoles and codimension 3 calibrated geometry

By the invariance of equation (1.3.2) under the action of the gauge group $\mathcal{G} = \mathcal{G}(E)$, one is interested in the *moduli space* of irreducible monopoles on E:

$$\mathcal{M}_\Theta(X, E) := \{(A, \Phi \neq 0) : (A, \Phi) \text{ is a solution of (1.3.2) and } A \text{ is irreducible}\}/\mathcal{G}.$$

Let us retrict attention to monopoles on a G_2-manifold X. Then, in parallel with the instanton case (cf. Tian's theorem 1.2.4), Donaldson–Segal [DS09] suggested that these might be related to coassociative submanifolds of X, i.e. $4-$dimensional $\psi-$calibrated submanifolds (in the Calabi–Yau case these would be the special Lagrangian submanifolds). Joyce [Joy16] gives some conjectures which attempt to define an invariant of a G_2-manifold by counting rigid, compact coassociative submanifolds. In fact, it follows from McLean's work that a closed coassociative submanifold $Q \subset X$ deforms in a smooth moduli space of dimension $b_2^-(Q)$.

Hence these are rigid when $b_2^-(Q) = 0$ and one could hope to count these. Another (possibly complementary) attempt to define an enumerative invariant of G_2-manifolds is by an appropriate count of monopoles and the idea is that this may be related to a count of coassociative submanifolds via the (non)compactness problem for $\mathcal{M}_\Theta(X, E)$: the general expectation is that sequences of monopoles under the regime where the asymptotic values of the Higgs fields norms at infinity (their *mass*) becomes arbitrarily large, should concentrate along coassociative cycles whose homology class is determined by the topological type of the bundle E. Such a concentration phenomena is then expected to be modelled on $\mathbb{R}^3-$monopoles along the fibers of the normal bundle to the coassociatives; in fact, there is an analogue of Lemma 1.2.9 for G_2-monopoles. (One can write down a similar result for the Calabi-Yau case.)

Lemma 1.3.26. *Let $\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3$ be an orthogonal decomposition and let E be a $G-$bundle over $\mathbb{R}^3$. If (A, Φ) is a configuration on E with A non-flat and $\mathrm{d}_A\Phi \neq 0$, then the following are equivalent:*

(i) The lifted configuration $(\underline{A}, \underline{\Phi})$ is a G_2-monopole.

(ii) There exist orientations on $\mathbb{R}^4$ and $\mathbb{R}^3$ with respect to which ψ calibrates $\mathbb{R}^4$ and (A, Φ) is a monopole on $\mathbb{R}^3$.

The works of Oliveira [Oli14b, Oli16a] provides concrete instances of these general expectations on the codimension 3 concentration phenomena for large mass monopoles under symmetry assumptions on certain examples of AC Calabi–Yau and G_2-manifolds.

In Chapters 3 and 4 of this thesis we start the investigation of the general behavior of large mass monopoles on AC $3-$dimensional and G_2- manifolds respectively.

Chapter 2

General properties of YMH configurations

In this chapter we gather some important analytical properties of YMH configurations in general. These will be useful for the analysis of sequences of large mass YMH/monopoles configurations on the next chapters.

We start with a basic scaling property (Proposition 2.1.1) of the YMH/monopole equations and a key monotonicity formula (Theorem 2.1.2) enjoyed by YMH configurations in dimensions $n \geq 4$. Then we move to Bochner–Weitzenböck formulas that imply an important estimate on the Laplacian of the energy density (Corollary 2.2.4). By using these results and a nonlinear mean value inequality, we then derive an $\varepsilon-$regularity result (Theorem 2.3.1) that has some powerful analytic consequences. For instance, we prove that on a noncompact manifold of bounded geometry the energy density of a YMH configuration with finite energy decays uniformly to zero at infinity and attains its maximum (Corollary 2.3.3). Moreover, we prove that for an irreducible YMH configuration (A, Φ) on an AC $n-$manifold with $n \geq 3$, finite energy forces $|\Phi|$ to uniformly converge to a constant at infinity and, conversely, that if this holds then $|\mathrm{d}_A\Phi| \in L^2(X)$ (Proposition 2.3.5). In particular, in an AC $3-$dimensional manifold, an irreducible monopole has finite energy if and only if its Higgs field norm uniformly converges to a constant at infinity (Corollary 2.3.8).

In the last section, we generalize a result of Taubes for monopoles on the Euclidean space $\mathbb{R}^3$ to finite mass YMH configurations on AC $n-$dimensional manifolds with $n \geq 3$ (Theorem 2.4.1). Our result controls how big the radius of a ball should be so that it contains points where the Higgs field norm of a finite mass YMH configuration is bigger than a given portion of its mass.

2.1 Scaling properties and a monotonicity formula

Notations. Consider the setup of §1.1. Henceforth, it will be convenient to use the following notations. If we scale the metric g by λ^2, for some $\lambda > 0$, then for the new metric $g_\lambda := \lambda^2 g$ we write:

- $B_r^\lambda(x) :=$ open $g_\lambda-$ball of center x and radius r;
- $e_\lambda(A, \lambda^{-1}\Phi) := g_\lambda-$YMH energy density of $(A, \lambda^{-1}\Phi)$;
- $\mathrm{vol}_\lambda := g_\lambda-$volume form;
- $\mathcal{E}_U^\lambda := g_\lambda-$YMH functional over U.

With these notations, note that the following identities holds:

- $B_{\lambda r}^\lambda(x) = B_r(x)$;
- $e_\lambda(A, \lambda^{-1}\Phi) = \lambda^{-4} e(A, \Phi)$;
- $\mathrm{vol}_\lambda = \lambda^n \mathrm{vol}$;
- $\mathcal{E}_U^\lambda(A, \lambda^{-1}\Phi) = \lambda^{n-4}\mathcal{E}_U(A, \Phi)$.

We shall use analogous notations for the $\Theta-$energy density and functional.

We start with a basic scaling property of the YMH and $\Theta-$monopole equations.

Proposition 2.1.1. *Let (X^n, g) be an oriented Riemannian n-manifold, E a G-bundle over X, and (A, Φ) a configuration on E. For any real $\lambda > 0$, the following holds:*

- *If (A, Φ) is YMH on (X^n, g), then $(A, \lambda^{-1}\Phi)$ is YMH on (X^n, g_λ).*
- *Suppose $\Theta \in \Omega^{n-3}(X, \mathbb{R})$ is a calibration. If (A, Φ) is a $\Theta-$monopole on (X^n, g), then $(A, \lambda^{-1}\Phi)$ is a $(\lambda^{n-3}\Theta)-$monopole on (X^n, g_λ).*

Proof. Acting on $k-$forms, the Hodge-$*$ operators associated to g_λ and g are related by $*_\lambda = \lambda^{n-2k}*$. Therefore, we have $\mathrm{d}_A^{*_\lambda} F_A = \lambda^{-2}\mathrm{d}_A^* F_A = \lambda^{-2}[\mathrm{d}_A\Phi, \Phi]$ in the YMH case, and $*_\lambda(\mathrm{d}_A(\lambda^{-1}\Phi)) = \lambda^{n-3} * \mathrm{d}_A\Phi = F_A \wedge (\lambda^{n-3}\Theta)$ in the $\Theta-$monopole case. ■

We now cite a key monotonicity property enjoyed by the renormalized scale-invariant YMH energy of YMH configurations on geodesic balls in dimensions $n \geq 4$. For YM connections this was proved by Price [Pri83]; the proof in the YMH case is analogous.

Theorem 2.1.2 ([Afu19, Theorem 2.1]). *Let (X^n, g) be an oriented Riemannian $n-$manifold, where $n \geq 4$, and let E be a $G-$bundle over X. Let $x \in X$, write $\rho := d^g(x, \cdot)$ for the distance function from x, and suppose the Hessian estimate*

$$c_- \rho^2 g_\rho \leq g - \nabla^2 \left(\frac{1}{2} \rho^2 \right) \leq c_+ \rho^2 g_\rho$$

holds on $B_{r_x}(x)$ for some $r_x \in (0, i_x(g)]$ and $c_\pm \geq 0$, where $g_\rho := g - \mathrm{d}\rho \otimes \mathrm{d}\rho$. Then there is a constant $c = c(n, c_\pm) \geq 0$ such that if $(A, \Phi) \in \mathcal{C}(E)$ is YMH then the inequality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} \left(e^{cr^2} r^{4-n} \mathcal{E}_{B_r(x)}(A, \Phi) \right) &\geq e^{cr^2} r^{4-n} \int_{\partial B_r(x)} |\partial_\rho \lrcorner F_A|^2 + |\mathrm{d}_{A,\partial_\rho} \Phi|^2 \\ &+ e^{cr^2} r^{3-n} \int_{B_r(x)} |\mathrm{d}_A \Phi|^2 \geq 0 \end{aligned} \tag{2.1.3}$$

holds for all $r < r_x$. In particular, for all $0 < s < r \leq r_x$ one has

$$\begin{aligned} &e^{cr^2} r^{4-n} \mathcal{E}_{B_r(x)}(A, \Phi) - e^{cs^2} s^{4-n} \mathcal{E}_{B_s(x)}(A, \Phi) \\ &\qquad \geq \int_{A_{r,s}(x)} e^{c\rho^2} \rho^{4-n} \left(|\partial_\rho \lrcorner F_A|^2 + |\mathrm{d}_{A,\partial_\rho} \Phi|^2 \right), \end{aligned} \tag{2.1.4}$$

where $A_{r,s}(x) := B_r(x) \setminus B_s(x)$. Furthermore:

- *If $X^n = \mathbb{R}^n$ or[1] $S^n(-\kappa^2)$, then $c = 0$ and the inequalities (2.1.3) and (2.1.4) holds for all $x \in X$ and $r > 0$.*
- *If (X^n, g) is of bounded geometry (e.g. AC or compact), then there are uniform constants $r_0 \in (0, i_X(g)]$ and $c = c(n, g) \geq 0$ such that the inequalities (2.1.3) and (2.1.4) holds for all $x \in X$ and $r \in (0, r_0]$.*

One consequence of the monotonicity formula is the following vanishing result (compare with [Pri83, Corollary 2, p. 148] and [JT80, Corollary 2.3]).

Corollary 2.1.5 ([Afu19, Theorem A]). *Let (A, Φ) be a YMH configuration on $X^n = \mathbb{R}^n$ or $S^n(-\kappa^2)$. If there is some $x \in \mathbb{R}^n$ such that[2]*

$$\mathcal{E}_{B_R(x)}(A, \Phi) = o(R^{n-4}) \quad \textit{as } R \to \infty, \tag{2.1.6}$$

[1] $S^n(-\kappa^2)$ denotes the hyperbolic $n-$space of constant sectional curvature $-\kappa^2$.

[2] Here we use the standard little-o notation.

then (A,Φ) *is gauge equivalent to the canonical flat connection and a constant function* $\mathbb{R}^n \to \mathfrak{g}$. *In particular, for* $n \geq 5$ *there is no non-trivial finite energy YMH configuration on* $\mathbb{R}^n$ *or* $S^n(-\kappa^2)$.

Proof. Suppose, by contradiction, that F_A or $\mathrm{d}_A\Phi$ is nonzero. Then there exists some $R_0 > 0$ large enough so that

$$\Delta := R_0^{4-n}\mathcal{E}_{B_{R_0}(x)}(A,\Phi) > 0.$$

On the other hand, for each $R \geq R_0$, Theorem 2.1.2 implies that

$$\Delta \leq R^{4-n}\mathcal{E}_{B_R(x)}(A,\Phi).$$

Thus, making $R \to \infty$ and using the hypothesis (2.1.6) we conclude $\Delta \leq 0$ ($\Rightarrow\Leftarrow$). This proves the main statement. ∎

2.2 Bochner–Weitzenböck formulas and estimate

The following are simple but very useful Bochner–Weitzenböck formulae for general YMH configurations that immediately gives an important estimate on the Laplacian of their YMH energy density and later we will used in Chapters 3 and 4 to derive improved estimates for the energy density of monopoles in dimensions 3 and 7 (cf. Corollary 3.8.2, Proposition 4.2.1 and Corollary 4.2.6).

Lemma 2.2.1. *Let* (X^n, g) *be an oriented Riemannian* $n-$*manifold and* E *be a* $G-$*bundle over* X *where* G *is a compact Lie group. Equip* $\mathfrak{g}_E$ *with an Ad-invariant inner product. Suppose that* (A,Φ) *is a smooth YMH configuration on* E. *Then*

$$\begin{aligned}\frac{1}{2}\Delta|\mathrm{d}_A\Phi|^2 = &-\langle(\mathrm{d}_A\Phi)\circ \mathrm{Ric}^g, \mathrm{d}_A\Phi\rangle - 2\langle *[*F_A, \mathrm{d}_A\Phi], \mathrm{d}_A\Phi\rangle \\ &- |[\mathrm{d}_A\Phi,\Phi]|^2 - |\nabla_A(\mathrm{d}_A\Phi)|^2 \end{aligned}\tag{2.2.2}$$

and[3]

$$\begin{aligned}\frac{1}{2}\Delta|F_A|^2 = &-\langle F_A\circ(\mathrm{Ric}^g\wedge I + 2R^g), F_A\rangle - \langle[\mathrm{d}_A\Phi, \mathrm{d}_A\Phi], F_A\rangle \\ &- \sum_{i,j,k}\langle[F_{ij},F_{jk}],F_{ki}\rangle - |[F_A,\Phi]|^2 - |\nabla_A F_A|^2. \end{aligned}\tag{2.2.3}$$

Here $\Delta := \mathrm{d}^*\mathrm{d}$ *denotes the (positive definite) Laplacian and* Ric^g *the Ricci curvature tensor of* g.

[3] $F_{ij} := F_A(e_i, e_j)$ denotes the components of the curvature on a local orthonormal frame $\{e_i\}$.

Proof. The formulas follows by straightforward computations using the YMH equations and the standard Bochner–Weitzenböck formulas. We shall do the proof of (2.2.2). First recall the Bochner–Weitzenböck formula (cf. [BLJ81]) for the Laplacian of $\mathfrak{g}_E-$valued $1-$forms:

$$\Delta_A(\mathrm{d}_A\Phi) = \nabla_A^*\nabla_A(\mathrm{d}_A\Phi) + (\mathrm{d}_A\Phi)\circ \mathrm{Ric}^g + *[*F_A, \mathrm{d}_A\Phi].$$

Then,

$$\begin{aligned}\frac{1}{2}\Delta|\mathrm{d}_A\Phi|^2 &= \langle \nabla_A^*\nabla_A \mathrm{d}_A\Phi, \mathrm{d}_A\Phi\rangle - |\nabla_A(\mathrm{d}_A\Phi)|^2\\ &= \langle \Delta_A(\mathrm{d}_A\Phi), \mathrm{d}_A\Phi\rangle - \langle (\mathrm{d}_A\Phi)\circ\mathrm{Ric}^g, \mathrm{d}_A\Phi\rangle - \langle *[*F_A, \mathrm{d}_A\Phi], \mathrm{d}_A\Phi\rangle - |\nabla_A(\mathrm{d}_A\Phi)|^2.\end{aligned}$$

But since (A,Φ) is YMH, one has

$$\begin{aligned}\Delta_A(\mathrm{d}_A\Phi) &= \mathrm{d}_A^*\mathrm{d}_A(\mathrm{d}_A\Phi)\\ &= \mathrm{d}_A^*[F_A,\Phi]\\ &= *\mathrm{d}_A[*F_A,\Phi]\\ &= *\left([\mathrm{d}_A * F_A, \Phi] - [*F_A, \mathrm{d}_A\Phi]\right)\\ &= [\mathrm{d}_A^* F_A, \Phi] - *[*F_A, \mathrm{d}_A\Phi]\\ &= [[\mathrm{d}_A\Phi,\Phi],\Phi] - *[*F_A,\mathrm{d}_A\Phi].\end{aligned}$$

Putting the above together using the Ad_G-invariance of the inner product $\langle\cdot,\cdot\rangle$ gives the result.

■

Corollary 2.2.4 (Bochner type estimate)**.** *Let (X^n, g) be an oriented Riemannian $n-$manifold, let E be a $G-$bundle over X, and let (A,Φ) be a YMH configuration on E. Then the following estimate holds pointwise on X:*

$$\Delta e(A,\Phi) \lesssim |\mathrm{R}^g| e(A,\Phi) + e(A,\Phi)^{3/2}. \tag{2.2.5}$$

2.3 $\varepsilon-$regularity and some consequences

Combining the Bochner estimate and the monotonicity formula of the last sections, we can use the nonlinear mean value inequality of Theorem A.3 to prove the following.

Theorem 2.3.1 ($\varepsilon-$regularity for the total YMH energy density)**.** *Let (X^n, g) be an oriented Riemannian $n-$manifold of bounded geometry and let E be a $G-$bundle over X where G is* *a compact Lie group. Then there are (scale invariant) constants $\varepsilon_0 > 0$ and $C_0 > 0$ with the following significance. Let (A, Φ) be a YMH configuration on E. If $x \in X$ and $0 < r \leq r_0$ are such that*

$$\varepsilon := r^{4-n} \mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_0,$$

then

$$\sup_{B_{\frac{r}{2}}(x)} e(A, \Phi) \leq C_0 r^{-4} \varepsilon. \tag{2.3.2}$$

Proof. We apply the mean value inequality given by Theorem A.3 for $f := e(A, \Phi)$, $d = 4$, $\tau = 0$, $a_0 = 0$ and $0 < a_1, a \lesssim 1$. Indeed, with this setup, the monotonicity formula of Theorem 2.1.2 implies (A.4) in case $n \geq 4$ (in case $n < 4$ it is trivially satisfied), and the Bochner estimate of Corollary 2.2.4 implies (A.5) with critical exponent $\alpha = 3/2 = (d+2)/2$. Therefore, noting that $1 \lesssim r^{-4}$, it follows from (A.7) that there is $\varepsilon_0 := \hbar a^{-2} > 0$ and $C_0 > 0$ (depending only on n, the geometry of (X^n, g) and the structure constants of $\mathfrak{g}$) such that (2.3.2) holds. (Note that the statement is scaling invariant, and so are the desired constants.) ■

This result is of key importance in the compactness theory of YMH configurations and in this thesis we shall use it most notably in Chapter 3 in a 3−dimensional version (cf. Theorem 3.3.1). We now state some of its general consequences.

Corollary 2.3.3. *Let (X^n, g) be a noncompact, oriented, Riemannian $n-$manifold of bounded geometry and let E be a $G-$bundle over X where G is a compact Lie group. Suppose that (A, Φ) is a YMH configuration of finite energy, i.e. with $e := e(A, \Phi) \in L^1(X)$. Then $e \in L^p(X)$ for all $p \in [1, \infty]$ and decays uniformly to zero at infinity.*

Proof. Since (X^n, g) is of bounded geometry, we can cover X with a countable collection of geodesic balls $\{B_s(x_i)\}_{i=1}^{\infty}$ of radius $s := \frac{1}{4}\min\{1, r_0\}$, with a uniform bound on the number of balls containing any point of X and the half-radius balls pairwise disjoint (cf. [Heb00, Lemma 1.1]). Then since $e \in L^1(X)$ it follows that for each $\delta \in (0, \varepsilon_0]$ there exists $N_\delta \in \mathbb{N}$ so that up to removing a finite number of balls one has

$$C_0 s^{-n} \mathcal{E}_{B_{2s}(x_i)}(A, \Phi) < \delta, \quad \forall i > N_\delta.$$

Thus, by Theorem 2.3.1, we conclude that for any $\delta \in (0, \varepsilon_0]$,

$$\sup_{B_s(x_i)} e < \delta \quad \forall i \gg_\delta 1.$$

This implies that e decays uniformly to zero at infinity and its supremum is achieved at some $x_0 \in X$. Since $L^1(X) \cap L^\infty(X) \subseteq L^p(X)$ for all $p \geq 1$ this already imply the result. Furthermore, applying again Theorem 2.3.1 also gives the estimate

$$\|e\|_{L^\infty(X)} = e(x_0) \leq C\|e\|_{L^1(X)},$$

for some $C > 0$; in particular, for $p \geq 1$ we have

$$\|e\|^p_{L^p(X)} \leq \|e\|^{p-1}_{L^\infty(X)}\|e\|_{L^1(X)} \leq C^{p-1}\|e\|^p_{L^1(X)} < \infty.$$

■

In fact, by combining Theorem 2.3.1, Corollary 2.3.3 and invoking Uhlenbeck's Coulomb gauge theorem [Uhl82a, Theorem 1.3] (also see [Weh04, Theorem 6.1]), standard elliptic techniques proves (see [Ste10, §2]):

Corollary 2.3.4. *Under the same hypothesis of Corollary 2.3.3, one has* $|\nabla^k_A F_A|, |\nabla^k_A(\mathrm{d}_A\Phi)| \in L^2(X) \cap L^\infty(X)$ *for all* k.

Moreover, in the AC case we can prove the following:

Proposition 2.3.5. *Let* (X^n, g) *be an AC oriented Riemannian* $n-$*manifold with one end, where* $n \geq 3$*, let* E *be a* $G-$*bundle over* X *and* $(A, \Phi) \in \mathcal{C}(E)$ *be an irreducible YMH configuration. If* (A, Φ) *has finite energy* $\mathcal{E}_X(A, \Phi) < \infty$ *then there exists a constant* $m \in \mathbb{R}^+$ *with*

$$\lim_{\rho\to\infty} |\Phi| = m. \tag{2.3.6}$$

Conversely, if (2.3.6) *holds then* $|\mathrm{d}_A\Phi| \in L^2(X)$.

Proof. In one direction, suppose that (A, Φ) is an irreducible YMH configuration with finite energy. Then, by Corollary 2.3.3, we know that $e(A, \Phi)$ decays uniformly to zero at infinity; in particular,

$$\lim_{\rho\to\infty} |\mathrm{d}_A\Phi| = 0.$$

Since A is irreducible, (2.3.6) follows using Kato's inequality.

For the converse we follow [Oli14a, proof of Proposition 1.4.4]. Fix a smooth radius function ρ on (X^n, g). Consider the function

$$w := \frac{1}{2}\left(m^2 - |\Phi|^2\right).$$

Since (A, Φ) is YMH (and so $\Delta_A \Phi = 0$), we have that $\Delta|\Phi|^2 = -2|\mathrm{d}_A\Phi|^2 \leq 0$ so that the hypothesis (2.3.6) together with the Maximum Principle implies that w is a smooth nonnegative superharmonic function satisfying

$$\Delta w = |\mathrm{d}_A\Phi|^2 \tag{2.3.7}$$

and $\lim_{\rho\to\infty} w = 0$. Since (X^n, g) is AC and $n > 2$, all Green's functions are $\sim \rho^{2-n}$ at infinity (cf. [LT95]), so it follows by the Maximum Principle that there is a constant $c_w > 0$ such that $w \leq c_w \rho^{2-n}$ along the end. Now, for $R \gg 1$, let χ_R be a smooth bump function which is identically 1 in $B_R := \rho^{-1}([0, R])$ and has support contained in B_{2R}. From the fact that (X^n, g) is AC, the derivatives of ρ are uniformly bounded and we can assume that $|\nabla^2 \chi_r| \leq cR^{-2}$. Thus, multiplying the identity (2.3.7) by χ_R and integrating by parts yields

$$\begin{aligned}
\|\mathrm{d}_A\Phi\|^2_{L^2(B_R)} &\leq \int_X \chi_R |\mathrm{d}_A\Phi| = \int_X \chi_R \Delta w \\
&= \int_X w \Delta \chi_R \\
&\leq c_w R^{-2} \int_{B_{2R}\setminus B_R} \rho^{2-n} \\
&\leq c_w R^{-2} \int_R^{2R} \rho \mathrm{d}\rho \leq c_w.
\end{aligned}$$

(Of course, in this computation the value of c_w changes from one occurrence to the next, but the important point is that it is independent of R.) This gives a uniform bound on $\|\mathrm{d}_A\Phi\|^2_{L^2(B_R)}$ for any $R \gg 1$ and so, by dominated convergence, we get $|\mathrm{d}_A\Phi| \in L^2(X)$ as we wanted. ∎

Corollary 2.3.8. *Let (X^3, g) be an AC oriented Riemannian $3-$manifold with one end, E a $G-$bundle over X and $(A, \Phi) \in \mathcal{C}(E)$ be an irreducible monopole. Then (A, Φ) has finite energy $\mathcal{E}_X(A, \Phi) < \infty$ if and only if there exists a constant $m \in \mathbb{R}^+$ with*

$$\lim_{\rho\to\infty} |\Phi| = m.$$

Proof. Immediate from Proposition 2.3.5 using that for a monopole $\mathcal{E}_X(A, \Phi) = \|\mathrm{d}_A\Phi\|^2_{L^2(X)}$. ∎

2.4 AC and mass dependent $n-$dimensional version of Taubes' small Higgs field estimates

Let (X^3, g) be an AC oriented Riemannian $3-$manifold with one end, let E be an SU(2)-bundle over X (necessarily trivializable) and let (A, Φ) be a finite mass monopole on E with charge $k \neq 0$ and mass $m \neq 0$. Then for $r \gg 1$ we know that Φ does not vanish in $\rho^{-1}[r, \infty)$ and $\frac{\Phi}{|\Phi|}$ defines a degree k map from the various $\rho^{-1}(r) \cong N$ to the unit sphere in $\mathfrak{su}(2)$. Therefore, since $k \neq 0$, the Higgs field Φ must vanish at points in X. With this observation in mind, Taubes [Tau14] poses and addresses the following question, in the case where (X^3, g) is the Euclidean $3-$dimensional space $(\mathbb{R}^3, g_E)$:

Question. What is the *largest* radius of a ball in X that contains *only* points where $|\Phi| \ll m$?

The goal of this section is to prove an analogue of Taubes' result [Tau14, Theorem 1.2] in a much more general context. The following generalizes the $3-$dimensional case proved in [FO19, Theorem 4.1].

Theorem 2.4.1. *Let (X^n, g) be an oriented AC Riemannian $n-$manifold with one end, $n \geq 3$; E be a $G-$bundle over X; $\delta \in (0,1)$ and $\Lambda \in \mathbb{R}_+$. Then there is a constant $m_* > 0$, depending only on g, Λ and δ, with the following significance. If (A, Φ) is a finite mass YMH configuration on E with mass $m > m_*$ and $m^{-1}\|\mathrm{d}_A\Phi\|^2_{L^2(X)} \leq \Lambda$, then*

$$r_\delta(x) := \sup \left\{ r \in [0, \infty) : \sup_{B_r(x)} |\Phi| < m\delta \right\}$$

satisfy the upper bound[4]

$$r_\delta(x) \leq \left(\frac{4\Lambda c_1}{m(1-\delta)(n-2)c_2} \right)^{\frac{1}{n-2}}, \tag{2.4.2}$$

where $c_1, c_2 > 0$ are the constants of Theorem 2.4.4.

Remark 2.4.3. Recall from Proposition 2.3.5 that for any finite mass YMH configuration (A, Φ) on an AC $n-$manifold there is some $\Lambda > 0$ such that $m^{-1}\|\mathrm{d}_A\Phi\|^2_{L^2(X)} \leq \Lambda$.

The essential ingredient in the proof of this result concerns Green's functions (see [Oli16c, Proposition 2] for the $3-$dimensional case):

Theorem 2.4.4. *Let $n \geq 3$ and (X^n, g) be an oriented AC Riemannian $n-$manifold. Then, there are constants $c_1 > 0$ and $c_2 > 0$ such that for any given point $x \in X$, there exists a distribution $H_x \in (C_c^\infty(X))'$ such that $\Delta H_x = \delta_x$. Moreover, H_x is represented by an integral operator*

$$H_x(f) := \int_X f\phi_x,$$

[4] It will be clear from the proof that one can sharpen this upper bound, but for our purposes here the important point is that $r_\delta(x) \lesssim \left(\frac{\Lambda}{m(1-\delta)} \right)^{\frac{1}{n-2}}$.

where ϕ_x is a harmonic *function on $X \setminus \{x\}$ such that*

$$\phi_x|_{V(x)} = -\frac{c_1}{r^{n-2}} + O(1) \quad \textit{and} \tag{2.4.5}$$

$$\phi_x|_{V(N)} = -\frac{c_2}{\mathrm{vol}(N)}\frac{1}{r^{n-2}} + O(r^{1-n}), \tag{2.4.6}$$

where $r := \mathrm{dist}_g(\cdot, x)$, and $V(x)$, $V(N)$ denote a neighborhood of x and the end of X respectively.

The rest of this section is dedicated to the proof of Theorem 2.4.4 and then Theorem 2.4.1. Henceforth, we let $n \geq 3$ and (X^n, g) be a connected AC oriented Riemannian $n-$manifold with one end, so that there is a compact set $K \subset X$ outside of which X is asymptotic to the Riemannian cone $(C(N) := (1,\infty)_r \times N, g_C := dr^2 + r^2 g_N)$ over a closed and connected $(n-1)-$dimensional Riemannian manifold (N, g_N). Choose a smooth radius function $\rho \sim 1 + \mathrm{dist}_g(o, \cdot)$ on the cone factor, and let (E, A) be a *metric pair* over X, i.e. E is a Hermitian vector bundle over X endowed with a metric connection A. Suppose also that we have fixed asymptotics, i.e. there is a metric pair (E_∞, A_∞) over N such that $\varphi^* \left(E|_{X\setminus K}\right) \cong \pi^* E_\infty$, where $\pi\colon (1,\infty) \times N \to N$ is the projection onto the second factor, and A is asymptotic to A_∞ with some rate $\eta < 0$, i.e. (1.3.14) holds.

We now state a key estimate we shall need in this setup.

Proposition 2.4.7 ([Hei11, Corollary 1.3])**.** *For all $s \in C_c^\infty(X)$ and $\alpha \in [1, \frac{n}{n-2}]$, one has*

$$\|s\|^2_{L^{\frac{2n}{n-2}}} \leq C_K \|\nabla_A s\|^2_{L^2}.$$

Let H be the Hilbert space given by the completion of $C_c^\infty(X)$ with respect to the inner product

$$\langle \xi_1, \xi_2 \rangle_H := \langle \mathrm{d}\xi_1, \mathrm{d}\xi_2 \rangle_{L^2} = \int_X \mathrm{d}\xi_1 \wedge * \mathrm{d}\xi_2.$$

In particular, if $\xi \in H$ then $\xi \in L^1_{\mathrm{loc}}(X)$, $\nabla\xi \in L^2(X)$ and $\xi|_N := \lim_{\rho\to\infty} \xi = 0$.

Definition 2.4.8. Given $\eta \in L^1_{\mathrm{loc}}(X)$, a function $\phi \in L^1_{\mathrm{loc}}(X)$ with $\nabla\phi \in L^2(X)$ is called a **weak solution** to $\Delta\phi = \eta$ if

$$\langle \mathrm{d}\phi, \mathrm{d}\psi \rangle_{L^2} = \int_X \eta\psi, \quad \forall \psi \in C_c^\infty(X).$$

Lemma 2.4.9. *Let $\eta \in L^{\frac{2n}{n+2}}(X)$; then there is a unique weak solution $\phi_\eta \in H$ of*

$$\Delta\phi_\eta = \eta$$

Proof. To prove existence, we prove that the linear functional $\varphi_\eta : \xi \mapsto \int_X \eta\xi$ is bounded on H. To show this, recall from Proposition 2.4.7 that any $\xi \in H$ satisfies

$$\|\xi\|_{L^{2n/n-2}} \leq c\|\xi\|_H.$$

Hence, using Hölder's inequality and the hypothesis $\eta \in L^{\frac{2n}{n+2}}(X)$, we get

$$\left|\int_X \eta\xi\right| \leq c\|\eta\|_{L^{2n/n+2}}\|\xi\|_H,$$

as we wanted. Then, the Riesz representation theorem gives an element $\phi_\eta \in H$ such that $\langle\phi_\eta, \xi\rangle_H = \int_X \eta\xi$ for all $\xi \in H \supseteq C_c^\infty(X)$; therefore, ϕ_η is a weak solution to the problem. Moreover, one has $\|\phi_\eta\|_H = \|\varphi_\eta\|_{H'} \leq c\|\eta\|_{L^{2n/n+2}}$.

To prove uniqueness, let $\phi, \tilde{\phi} \in H$ be weak solutions of the problem. Then $h := \phi - \tilde{\phi} \in H$ satisfy weakly $\Delta h = 0$. By elliptic regularity, it follows that h is in fact a smooth harmonic function on X.[5] In particular, since (X^n, g) is AC and $n > 2$, $h \sim \rho^{2-n}$ and $\mathrm{d}h \sim \rho^{1-n}$ along the end. Therefore

$$\begin{aligned}
0 = \int_X h\Delta h &= -\int_X (\mathrm{d}(h \wedge *\mathrm{d}h) - \mathrm{d}h \wedge *\mathrm{d}h) \\
&= -\lim_{r\to\infty}\int_{\rho^{-1}(r)} h \wedge *\mathrm{d}h + \|h\|_H^2 \\
&= \|h\|_H^2,
\end{aligned}$$

Hence, $\phi = \tilde{\phi}$ as we wanted. ■

Consider the transpose of the Laplacian, still denoted by Δ, acting on $(C_c^\infty(X))'$ in the usual fashion: given $H \in (C_c^\infty(X))'$, then $\Delta H(\psi) := H(\Delta\psi)$ for all $\psi \in C_c^\infty(X)$. In what follows, given a point $x \in X$, we denote by $\delta_x \in C_c^\infty(X)$ the Dirac delta distribution supported on x, i.e.

$$\delta_x(\psi) := \psi(x), \quad \forall\psi \in C_c^\infty(X).$$

We are now in position to give the

Proof of Theorem 2.4.4. There is a $1-$parameter family $\{\delta_x^\varepsilon\}_{\varepsilon>0} \subset C_c^\infty(X)$ of smoothings of the current δ_x, i.e. such that

$$\delta_x(\psi) = \lim_{\varepsilon\to 0}\int_X \psi\delta_x^\varepsilon, \quad \forall\psi \in C_c^\infty(X),$$

[5] If $h \in L^2$ it would readily follow by a result of Yau [Yau76, see e.g. Proposition 1 or Theorem 3] that h must be constant. Since $h|_N = 0$ it follows that $h \equiv 0$ (or because $\mathrm{vol}(X) = \infty$).

where each δ_x^ε vanishes outside an $\varepsilon-$neighborhood of x and satisfy the uniform bound $\|\delta_x^\varepsilon\|_{L^1} = 1$.

For each $\varepsilon > 0$, the $L^{\frac{2n}{n+2}}-$norm of δ_x^ε is still bounded, however not independently of ε. Thus Lemma 2.4.9 gives a family of functions $\phi_x^\varepsilon \in H$, weakly solving

$$\Delta\phi_x^\varepsilon = \delta_x^\varepsilon$$

and with ϕ_x^ε unique for each ε. Since the δ_x^ε are smooth, elliptic regularity implies that so are the ϕ_x^ε. However, one must note that the norm $\|\phi_x^\varepsilon\|_H$ is not uniformly bounded independently of ε.

For $\psi \in C_c^\infty(X)$,

$$\delta_x(\psi) = \lim_{\varepsilon\to 0}\int_X \psi\delta_x^\varepsilon = \lim_{\varepsilon\to 0}\langle\phi_x^\varepsilon, \psi\rangle_H,$$

and since, for all ε, we have $\langle\phi_x^\varepsilon, \psi\rangle_H = \int_X \psi\delta_x^\varepsilon \leq \|\psi\|_{L^\infty}$, the weak limit as $\varepsilon \to 0$ of the ϕ_x^ε exists and gives a distribution H_x weakly solving $\Delta H_x = \delta_x$. This distribution is represented by an unbounded function which we denote by ϕ_x, such that the integral

$$\int_X \psi\phi_x = \lim_{\varepsilon\to 0}\int_X \psi\phi_x^\varepsilon$$

is well-defined for all $\psi \in C_c^\infty(X)$. As the L^1-norm of the δ_x^ε is bounded independently of ε and $\Delta\phi_x^\varepsilon = 0$ outside an $\varepsilon-$neighborhood of x, it follows that $\phi_x \in L^1_{\mathrm{loc}}(X)$ and $\phi_x \in C^\infty(X \setminus \{x\})$.

Finally, as the metric is AC it readily follows from the expression of the Laplacian on the cone, namely

$$\Delta_g = -(\partial_r)^2 - \frac{n-1}{r}\partial_r + r^{-2}\Delta_N,$$

that ϕ_x behaves as (2.4.6) along the end. Locally on a small ball $U(x)$ around x, the metric is approximately Euclidean, so that on these (2.4.5) holds. ■

Now the last ingredient we need for the proof of Theorem 2.4.1 is the following (see [Oli14a, Section 1.4.1] and, for the 3−dimensional version, [FO19, Proposition 2.2])

Proposition 2.4.10. *Let (X^n, g) be an AC oriented Riemannian $n-$manifold with one end, E a $G-$bundle over X and $(A, \Phi) \in \mathcal{C}(E)$ a finite mass YMH configuration on (X^n, g), with mass $m \neq 0$ and such that $|\mathrm{d}_A\Phi| \in L^2(X)$. Then in a neighborhood $V(N)$ of the end, we have*

$$|\Phi| = m - \frac{\|\mathrm{d}_A\Phi\|^2_{L^2(X)}}{m(n-2)\mathrm{vol}(N)\rho^{n-2}} + o(\rho^{2-n}) \quad \textit{as } \rho \to \infty. \tag{2.4.11}$$

Sketch of proof of Proposition 2.4.10. By the arguments in [Oli14a, proof of Proposition 1.4] (also see [JT80, Theorem 10.5 in Chap. IV]), one can write

$$|\Phi| = m - c\rho^{2-n} + o(\rho^{2-n}) \quad \text{on } V(N),$$

for some constant $c \in \mathbb{R}_+$ that we will now compute. Since $|\mathrm{d}_A\Phi| \in L^2(X)$, by dominate convergence we can write

$$\int_X |\mathrm{d}_A\Phi|^2 = \lim_{r\to\infty} \int_{\rho^{-1}(0,r]} |\mathrm{d}_A\Phi|^2.$$

Now, since $\Delta_A\Phi = 0$, we know that $\Delta|\Phi|^2 = -2|\mathrm{d}_A\Phi|^2$. Hence, by Stokes' theorem,

$$\int_{\rho^{-1}(0,r]} |\mathrm{d}_A\Phi|^2 = \frac{1}{2}\int_{\rho^{-1}(r)} *\mathrm{d}|\Phi|^2.$$

Therefore, we can compute:

$$\begin{aligned}
\int_X |\mathrm{d}_A\Phi|^2 &= \lim_{r\to\infty} \int_{\rho^{-1}(r)} *|\Phi|\mathrm{d}|\Phi| \\
&= \lim_{r\to\infty} \int_{\rho^{-1}(r)} |\Phi|\partial_\rho|\Phi| * \mathrm{d}\rho \\
&= \lim_{r\to\infty} \int_{\rho^{-1}(r)} |\Phi|\partial_\rho(m - c\rho^{2-n} + o(\rho^{2-n}))\rho^{n-1} \\
&= \lim_{r\to\infty} \int_{\rho^{-1}(r)} |\Phi|((n-2)c + o(1)) \\
&= cm(n-2)\mathrm{vol}(N),
\end{aligned}$$

where in the last equality we used that (A, Φ) has finite mass equal to m. ■

Proof of Theorem 2.4.1. Suppose, by contradiction, that for all $m_* > 0$ depending only on the indicated data, there exists a finite mass YMH configuration (A, Φ) with mass $m > m_*$, satisfying $m^{-1}\|\mathrm{d}_A\Phi\|^2_{L^2(X)} \leq \Lambda$, and such that

$$s := \left(\frac{4\Lambda c_1}{m(1-\delta)(n-2)c_2}\right)^{\frac{1}{n-2}} < r_\delta(x).$$

Let ϕ_x be the harmonic function on $X \setminus \{x\}$ obtained from applying Theorem 2.4.4, and let $\phi_0 := ((n-2)c_2)^{-1}2\Lambda\phi_x$. Then, for small enough $r = \mathrm{dist}(x, \cdot)$, equation (2.4.5) yields

$$\phi_0|_{U(x)} \geq -\frac{m(1-\delta)s^{n-2}}{2r^{n-2}} + c_\Lambda, \tag{2.4.12}$$

for some constant $c_\Lambda \in \mathbb{R}$, depending only on g and Λ.

Now, as s is inversely proportional to $m^{1/(n-2)}$, there is $m_* > 0$, depending only on g, Λ and δ, so that the expansion (2.4.12) is valid for $r = s$. At this point, it is convenient to further define the harmonic function on $X\backslash\{x\}$ given by $\phi := \phi_0 + m$. Then, by possibly increasing m_* so that $m_* > -2c_\Lambda(1-\delta)^{-1}$, we have

$$\phi|_{\partial B_s(x)} \geq -\frac{m(1-\delta)}{2} + c_\Lambda + m > m\delta \geq |\Phi|_{\partial B_s(x)}|,$$

where in the last inequality we used the assumption that our $s < r_\delta$. Then, the previous inequality, and the fact that both the harmonic function ϕ and the subharmonic function $|\Phi|$ converge to m along the end show that

$$|\Phi| < \phi \quad \text{in } X \setminus B_s(x).$$

On the other hand, recall from equations (2.4.11) and (2.4.6) that

$$|\Phi| \geq m - \Lambda(n-2)^{-1}\mathrm{vol}(N)^{-1}\rho^{2-n} + o(\rho^{2-n}), \quad \text{and}$$
$$\phi = m - 2\Lambda(n-2)^{-1}\mathrm{vol}(N)^{-1}\rho^{2-n} + o(\rho^{2-n}),$$

as $\rho \to \infty$. Putting these together, we conclude that $\Lambda\mathrm{vol}(N)^{-1} \geq 2\Lambda\mathrm{vol}(N)^{-1}$, hence a contradiction. ■

Remark 2.4.13. Under its hypothesis, Theorem 2.4.1 implies that for any finite mass YMH configuration with mass $m > m_*$ and $m^{-1}\|\mathrm{d}_A\Phi\|^2_{L^2(X)} \leq \Lambda$, whenever $r > c(\Lambda m^{-1}(1-\delta)^{-1})^{1/(n-2)}$ for some suitable constant $c > 0$ (depending only on g) then

$$\sup_{\partial B_r(x)} |\Phi| \geq \sup_{B_r(x)} |\Phi| \geq m\delta,$$

where in the first inequality we applied the maximum principle.

Chapter 3

Large mass monopoles on AC $3-$manifolds

In this chapter we describe the work of the author and his co-supervisor Gonçalo Oliveira [FO19], together with some minor improvements, corrections, new observations and a new section (cf. Section 3.8). We consider $\mathrm{SU}(2)$ monopoles on an asymptotically conical, oriented, Riemannian 3-manifold with one end and analyze the limiting behavior of sequences of finite mass monopoles with fixed charge, and whose sequence of masses (or, equivalently, YMH energies) is unbounded. We prove that the limiting behavior of such monopoles is characterized by energy concentration along a certain set, which we call the blow-up set. Our work shows that this set is finite, and using a bubbling analysis obtain effective bounds on its cardinality, with such bounds depending solely on the charge of the monopole. Moreover, for such sequences of monopoles there is another naturally associated set, the zero set, which consists on the set at which the zeros of the Higgs fields accumulate. In this regard, our results show that for such sequences of monopoles, the zero set and the blow-up set coincide. In particular, proving that in this "large mass" limit, the zero set is a finite set of points. We end this chapter with a brief discussion on the problem of convergence of the monopole sequence outside the zero set.

Some of our work extends for sequences of finite mass critical points of the YMH functional for which the YMH energies are $O(m_i)$ as $i \to \infty$, where m_i are the masses of the configurations.

3.1 Main results

The virtual dimension of the moduli space of finite mass $\mathrm{SU}(2)$ monopoles of fixed charge over an AC oriented Riemannian $3-$manifold X was computed in [Kot15], and a smooth open set constructed by a gluing theorem in [Oli16c]. Such gluing is an AC version of Taubes' original gluing of well separated multi-monopoles in the $\mathbb{R}^3$ case, [JT80]. In the case of [Oli16c], the mass plays the role of a parameter controlling the concentration of the resulting multi-monopole around its centers. Indeed, allowing the mass to vary gives the freedom of bringing these centers as close as one wants. In order to motivate our main results we shall now summarize this construction of large mass, charge k monopoles on X. This goes as follows: Start with k points in X; Insert charge one and mass one monopoles in $\mathbb{R}^3$ scaled down to fit in small disjoint balls around these points; As a byproduct of having been scaled down the monopoles must have mass larger than $O(d^{-2})$, where d is the minimum separation between the k-points; Then, by making use of a partition of unity these can be glued with a certain mass $O(d^{-2})$ monopole in the complement of these balls; The resulting configuration does not solve the monopole equations, but by a version of the contraction mapping principle it can be deformed to a nearby one which does. Moreover, we further remark that this configuration produces monopoles with any mass $m \geq O(d^{-2})$, for more details and the precise statements see Theorem 1 in [Oli16c] or Theorem 3.2.11 later in this chapter. [1]

The goal of our paper [FO19] is to take the inverse point of view and consider a sequence of finite mass monopoles $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}}$ with unbounded masses, $\limsup m_i = \infty$, but fixed charge k, over an AC manifold (X^3, g). In this case, the natural expectation would be an inverse construction to that of [Oli16c], with the monopoles either "escaping" through the end, or getting concentrated around at most k points $x_1, \ldots, x_k$ in X, where a monopole in the Euclidean $\mathbb{R}^3 \cong T_{x_i}X$ bubbles off.[2] See Section 3.2 for a plethora of examples motivating this expectation.

From the analytic point of view, the case when the energies $\mathcal{E}_X(A_i, \Phi_i)$ of the sequence $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}}$ are uniformly bounded has a well known limiting behavior, which is easily understood. In this case, the monopoles are either converging smoothly everywhere on X, or "escaping" to infinity through the end, see for example [AH88] for the more general statement in the $\mathbb{R}^3$ case. In fact, independently of whether they are escaping through the end or not, the restriction of such a sequence of monopoles to any compact subset $K \subset X$ smoothly con-

[1] We further point out that it should be possible to start this construction by using higher charge monopoles in $\mathbb{R}^3$ (monopole clusters). A metric version of this gluing have been carried out in [KS15] for the case of $\mathbb{R}^3$.

[2] Even though in this introduction, and for motivation purposes, we restrict to the case when the $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}}$ are monopoles, many of our results hold in the more general case of families of YMH configurations on a fixed G-bundle.

verges to a monopole. Therefore, the most interesting case is when these energies do not remain bounded. Indeed, the energy formula (1.3.19) for monopoles $\mathcal{E}_X(A_i,\Phi_i) = 4\pi k m_i$, shows that this is precisely the case under consideration, where the sequence of masses m_i is unbounded.

We now introduce some preparation needed in order to state our main results. Let $\{(A_i,\Phi_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{C}(E)$ be a sequence of finite mass YMH configurations on (X^3,g) whose masses satisfy $\limsup m_i = \infty$. Define the *blow-up set* S of $\{(A_i,\Phi_i)\}$ by

$$S := \bigcup_{\varepsilon>0} \bigcap_{0<r\leq r_0} \left\{ x \in X : \liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x)}(A_i,\Phi_i) \geq \varepsilon \right\}.$$

This may be interpreted as the set $S \subset X$ where the energy of the sequence is concentrating. On the other hand, we have the *zero set*

$$Z := \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)},$$

which consists of the accumulation points of the Higgs fields zeros, i.e. the limit set of the zeros. Under suitable assumptions, our main results show that these two sets are equal and the failure of compactness is entirely due to monopole bubbling at its points. In what follows, we shall use $\mathcal{H}^0$ to denote the counting measure on X and $\mathcal{H}^3$ to denote the standard Riemannian measure on (X^3,g).

Theorem 3.1.1 (cf. [FO19, Theorem 1.1])**.** *Let (X^3,g) be an AC oriented Riemannian $3-$manifold with one end, E an $\mathrm{SU}(2)-$bundle over X and $\{(A_i,\Phi_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{A}(E)\times\Gamma(\mathfrak{su}(2)_E)$ a sequence of finite mass monopoles on (X^3,g) with fixed charge $k\neq 0$ and masses m_i satisfying $\limsup m_i = \infty$. Then, after passing to a subsequence, the following hold:*

(a) For each $x \in S$, the sequence (A_i,Φ_i) bubbles off a mass 1 monopole (A_x,Φ_x) on $\mathbb{R}^3 \cong T_xX$. Moreover, $\mathcal{E}_{\mathbb{R}^3}(A_x,\Phi_x) = 4\pi k_x$, where $k_x \in \mathbb{Z}_{>0}$, $k_x \leq k$, is its charge.

(b) The blow-up set S can be written as

$$S = \bigcap_{0<r\leq r_0} \left\{ x \in X : \liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x)}(A_i,\Phi_i) \geq 4\pi \right\}.$$

Moreover, it coincides with the zero set Z:

$$S = Z.$$

(c) $S = Z$ is a finite set of at most k points. In fact, we actually have

$$\mathcal{H}^0(S) \leq \frac{k}{\min\limits_{x\in S} k_x}.$$

(d) *The following weak convergence of Radon measures holds:*

$$m_i^{-1} e(A_i, \Phi_i)\mathcal{H}^3 \rightharpoonup 4\pi \sum_{x \in S} k_x \delta_x,$$

where $e(A_i, \Phi_i) := |F_{A_i}|^2 + |\mathrm{d}_{A_i}\Phi_i|^2$*, and* δ_x *denotes the Dirac delta measure supported on* $\{x\}$.

(e) *For each* $x \in X \setminus S$*, we can find* $r > 0$ *and a subsequence* $i'(x) \to \infty$ *such that*

$$\sup_{B_r(x)} m_{i'}^{-1} e(A_{i'}, \Phi_{i'}) \to 0 \quad \text{as} \quad i' \to \infty.$$

In the more general case where we have a sequence of YMH configurations on a $G-$bundle, we can still guarantee some of the above results under the assumption that $\mathcal{E}_X(A_i, \Phi_i) = O(m_i)$ as $i \to \infty$, which amounts to the fixed charge assumption in the case of $\mathrm{SU}(2)-$monopoles.

Theorem 3.1.2. [FO19, Theorem 1.2] *Let* (X^3, g) *be an AC oriented Riemannian* $3-$*manifold with one end,* E *a* $G-$*bundle over* X *where* G *is a compact semi-simple Lie group, and* $\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{C}(E)$ *a sequence of finite mass YMH configurations on* (X^3, g) *whose masses* m_i *satisfy* $\limsup m_i = \infty$*. Suppose that*

$$m_i^{-1} \mathcal{E}_X(A_i, \Phi_i) \leq C,$$

for some uniform constant $C > 0$*. Then, after passing to a subsequence, the following holds:*

(a') *For each* $x \in S$*, the sequence* (A_i, Φ_i) *bubbles off a mass 1 YMH configuration* (A_x, Φ_x) *in* $\mathbb{R}^3 \cong T_x X$*. Moreover, each bubble* (A_x, Φ_x) *has strictly positive energy* $\mathcal{E}_{\mathbb{R}^3}(A_x, \Phi_x) > 0$.

(b') *The blow-up set* S *can be written as*

$$S = \bigcap_{0 < r \leq r_0} \left\{ x \in X : \liminf_{i \to \infty} m_i^{-1} \mathcal{E}_{B_r(x)}(A_i, \Phi_i) > 0 \right\}.$$

Moreover, it contains the zero set Z*:*

$$Z \subset S.$$

(c') S *(therefore* Z*) is countable.*

We shall now explain how this chapter is organized. Section 3.2 gives several examples of families of monopoles whose masses converges to infinity. The results are very illustrative and allow for the realization of all cases in our Theorem 3.1.1, and give a good intuition for the behavior of large mass monopoles. The proof of Theorems 3.1.1–3.1.2 takes up every section from Section 3.3 to 3.8, and their content is summarized below.

Having in mind the aim of relating the zero set Z and the blow-up set S of such sequences of large mass monopoles, a key result is given by the $3-$dimensional instance of our AC version of Taubes' small Higgs field radius estimate in Theorem 2.4.1. This provides a way to control how big, in terms of the mass $m \neq 0$ and the charge $k \neq 0$ of the monopole, one needs the radius of a ball in X to be so that the value of the Higgs field outside such a ball is a sufficiently large portion of m.

In Section 3.3 we prove an appropriate $3-$dimensional version of the ε-regularity theorem (Theorem 3.3.1) for YMH configurations, which is an important ingredient to relate the zero set with the blow-up set. Indeed, in Section 3.4, using a simple argument involving the fundamental theorem of calculus, together with the $\varepsilon-$regularity, we prove that a large mass YMH configuration with (locally) small energy has an interior lower bound on its Higgs field, provided it is bounded from below in some boundary ball. Together with our analogue of Taubes small Higgs field estimate, this is used in Section 3.5 to prove the inclusion $Z \subseteq S$, i.e. the last part of (b') in Theorem 3.1.2; here we also prove (c').

Section 3.6 uses the scaling properties of the YMH/monopole equations and their ellipticity on a fixed Coulomb gauge, to perform the bubbling analysis. In particular, we are able to show that at each point $x \in S$, a mass 1 (YMH/)monopole in $\mathbb{R}^3$ with strictly positive energy bubbles off. This gives part (a) and half of part (b) in Theorem 3.1.1 and (a') and half of (b') in Theorem 3.1.2. In the case of monopoles, the energy formula and a degree argument then yield the reverse inclusion $S \subseteq Z$, and thus the equality in part (b) of Theorem 3.1.1. This part of the proof somewhat resembles Taubes' proof of the Weinstein conjecture where a degree argument and the energy identity for the vortex equations is used to prove that a certain component of the spinor involved in the Seiberg–Witten equations vanishes, see [Tau07, Section 6.4].

Using all this and some simple measure theory, Section 3.7 is dedicated to describe the convergence of the relevant measures as in statement (d) of Theorem 3.1.1 and an estimate on the maximum number of elements in $S = Z$ follows, depending on the fixed charge k of the sequence and the minimum of the charges k_x of each bubble at $x \in S$; this corresponds to part (c) of Theorem 3.1.1. Finally, in Section 3.8 after stating a conjecture on the problem of convergence of the (translated) sequence outside Z, we provide some ideas on how to tackle it, proving an improved (linear) Bochner estimate for the Laplacian of the energy densities outside Z and deriving part (e) of Theorem 3.1.1. All these together gives a full proof of the main Theorems 3.1.1 and 3.1.2.

Conventions for this chapter. Unless otherwise stated, (X^3, g) will denote an oriented Riemannian $3-$manifold of bounded geometry, and E will be a $G-$bundle over X, where G is a compact semi-simple Lie group. In fact, for the most part (X^3, g) will be an AC manifold with one end and whenever we restrict attention to *monopoles* configurations on E we will always consider $G = \mathrm{SU}(2)$ for simplicity.

3.2 Motivating examples

In this section we collect a few examples which motivate the current work. The first of these consists of exploring the explicit Prasad–Sommerfield monopole in the limit when its mass is sent off to infinity. The second examples uses Taubes' construction of multi-monopoles on $\mathbb{R}^3$ to produce sequences of charge $k \geq 1$ monopoles with unbounded masses, such that the corresponding zero set Z is any a priori prescribed set of l pairwise distinct points in X, for any given $1 \leq l \leq k$. Next, we include a simple general way to produce, from a given charge $k > 1$ monopole, examples of sequences of charge k monopoles in $\mathbb{R}^3$ with unbounded masses and for which the zero set $Z = \{0\}$ and the charge k_0 of the bubble at the origin equals $k > 1$. Finally, using the multi-monopole construction of [Oli16c] in the more general setting of an AC $3-$manifold (X^3, g) with $b^2(X) = 0$, we construct sequences of charge k monopoles with unbounded masses whose zero set is any a priori prescribed set of k pairwise distinct points in X.

3.2.1 The BPS Monopole

In this section we shall write down the standard mass m BPS monopole (A_m, Φ_m) on $\mathbb{R}^3$, constructed by Prasad and Sommerfield in [PS75]. For any $m \in \mathbb{R}^+$, this has a unique zero $\Phi_m^{-1}(0) = \{0\}$ and is spherically symmetric. Obviously, by considering the sequence letting $m \to \infty$ we will have $Z = \{0\}$, however, the interesting thing of considering this specific example is that we shall be able to check the convergence to the delta function on Z explicitly.

Write $\mathbb{R}^3\backslash\{0\} \cong \mathbb{R}_+ \times \mathbb{S}^2$, and pullback from $\mathbb{S}^2 \cong \mathrm{SU}(2)/\mathrm{U}(1)$ the homogeneous bundle

$$P = \mathrm{SU}(2) \times_\chi \mathrm{SU}(2),$$

with $\chi : \mathrm{U}(1) \to \mathrm{SU}(2)$ the group homomorphism given by $\chi(e^{i\theta}) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. In this polar form, and actually working on the pullback to the total space of the radially extended Hopf bundle $\mathbb{R}^+ \times \mathrm{SU}(2)$, the Euclidean metric can be written as

$$g_E = dr^2 + 4r^2(\omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3),$$

where r is the radial direction, i.e. the distance to the origin. Now fix the standard basis $\{S_i\}$ of $\mathfrak{su}(2)$ given by the Pauli matrices, and let $\omega_1, \omega_2, \omega_3$ be the dual coframe. The 1-form $S_1 \otimes \omega^1 \in \Omega^1(\mathrm{SU}(2), \mathfrak{su}(2))$ equips the Hopf bundle $\mathrm{SU}(2) \to \mathbb{S}^2$ with an $\mathrm{SU}(2)$-invariant connection, which in turn, induces a connection in P. Making use of Wang's theorem [Wan58], one can write any other spherically symmetric connection on $\mathbb{R}^3\backslash\{0\}$ as

$$A = S_1 \otimes \omega_1 + a(r)(S_2 \otimes \omega^2 + S_3 \otimes \omega^3),$$

for some function $a : \mathbb{R}^+ \to \mathbb{R}$. Similarly, seeing an Higgs field $\Phi(r)$ as a function in the total space with values in $\mathfrak{su}(2)$ one can show, see the Appendix in [Oli14b], that any spherically symmetric Higgs field must be of the form $\Phi = \phi(r)\ S_1$, with $\phi : \mathbb{R}^+ \to \mathbb{R}$ some function. A computation yields that

$$\begin{aligned} F_A &= 2(a^2-1)S_1 \otimes \omega^{23} + \dot{a}(S_2 \otimes dr \wedge \omega^2 + S_3 \otimes dr \wedge \omega^3), \\ \nabla_A \Phi &= \dot{\phi}\ S_1 \otimes dr + 2a\phi\ (S_2 \otimes \omega^3 - S_3 \otimes \omega^2), \end{aligned}$$

with the dot denoting differentiation with respect to r. The energy density, as a function of r, is then

$$e = \frac{(a^2-1)^2}{4r^4} + \frac{\dot{a}^2}{2r^2} + \dot{\phi}^2 + \frac{2a^2\phi^2}{r^2}. \tag{3.2.1}$$

In this spherically symmetric setting, the monopole equations turn into the following system of ODE

$$\dot{\phi} = \frac{1}{2r^2}(a^2-1), \quad \dot{a} = 2a\phi,$$

Some particular solutions are given by the flat connection $(a, \phi) = (\pm 1, 0)$, and the Dirac monopole $(a, \phi) = (0, m - 1/2r)$, where $m \in \mathbb{R}$. However, the regularity conditions so that the configuration (A, Φ) smoothly extends over the origin yield that $\phi(0) = 0$ and $a(0) = 1$. One can then show, see the Appendix in [Oli14b], that any such solution is given by

$$\phi_m = \frac{1}{2}\left(\frac{1}{r} - \frac{2m}{\tanh(2mr)}\right), \quad a_m = \frac{2mr}{\sinh(2mr)}, \tag{3.2.2}$$

for some $m \in \mathbb{R}^+$, which is the mass of the resulting monopole. The resulting formula for the energy density in (3.2.1) is

$$e_m := \frac{\cosh^4(2mr) + (32\, m^4 r^4 - 2)\cosh^2(2mr) - 32\sinh(2mr)\cosh(2mr) m^3 r^3 + 16 m^4 r^4 + 1}{2\sinh^4(2\, mr) r^4}. \tag{3.2.3}$$

Recall that in this case we have $Z = \{0\}$. Given the formula above it is easy to see that in $\mathbb{R}^3 \backslash Z$ we have

$$m^{-1} e_m \leq \frac{1}{2m}\coth^4(2mr) + O(m^{-2}) \to 0, \text{ as } m \to \infty.$$

On the other hand, using this fact together with the dominated convergence theorem, we have

$$\begin{aligned} I &:= \lim_{m\to\infty} \int_{\mathbb{R}^3} m^{-1} e_m \\ &= \lim_{m\to\infty}\lim_{s\to\infty} \int_{B_s(0)} m^{-1} e_m = 4\pi \lim_{m\to\infty}\lim_{s\to\infty}\int_0^s r^2\, m^{-1} e_m(r) \mathrm{dr} \\ &= 4\pi \lim_{m\to\infty}\int_0^\infty r^2\, m^{-1} e_m(r)\mathrm{dr} \\ &= 4\pi \lim_{m\to\infty}\left(\lim_{r\to\infty} f_m(r) - \lim_{r\to 0^+} f_m(r)\right), \end{aligned} \tag{3.2.4}$$

where

$$f_m(r) = -\frac{1}{2mr} - \frac{2}{(e^{4mr}-1)^3}\left((8m^2r^2 - 4mr - 1)e^{8mr} + (8m^2r^2 + 4mr + 2)e^{4mr} - 1\right).$$

Thus, inserting this into equation (3.2.4) shows that $I = 4\pi$ and thus

$$m^{-1} e_m \mathcal{H}^3 \rightharpoonup 4\pi\delta_0, \text{ as } m \to \infty.$$

3.2.2 Sequences of Taubes' multi-monopoles on $\mathbb{R}^3$ with prescribed Z

We start by recalling the following Theorem of Taubes, see [JT80].

Theorem 3.2.5 (Theorems 1.1 and 1.2 in [JT80])**.** *Let $k \in \mathbb{N}$. Then, there is $d_0 > 0$ and $c > 0$ such that for any $y_1, \ldots, y_k \in \mathbb{R}^3$ with $d = \min_{j,l} \operatorname{dist}(y_i, y_j) > d_0$, there is a charge k, mass 1, monopole (A, Φ) in $\mathbb{R}^3$. Furthermore, for $R = cd^{-1/2}$ we have that $\Phi^{-1}(0) \subset \cup_{i=1}^k B_R(y_i)$ and $\Phi|_{\partial B_R(y_i)}$ has degree 1. In particular, Φ does have zeros inside each of the ball's $B_R(y_i)$, for $i = 1, \ldots, k$.*

We shall now use this construction to give a number of different examples of sequences of monopoles as those we consider in this chapter.

Proposition 3.2.6. *Let* $1 \leq l \leq k$ *be integers,* $\{x_1, \ldots, x_l\} \subseteq \mathbb{R}^3$ *a subset of pairwise distinct points, and* $\{m_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ *an unbounded increasing sequence, i.e.* $m_i \uparrow \infty$*. Then, there is a sequence* $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}}$ *of charge* k*, mass* m_i *monopoles on* $\mathbb{R}^3$ *with zero set*

$$Z = \{x_1, \ldots, x_l\}.$$

Remark 3.2.7. In this construction, as will be evident during the proof, we have $k_{x_j} = 1$ for all $j = 1, \ldots, l$. The case $l < k$ is precisely the case where there are $k - l$ monopoles "escaping through the end", or "run of to infinity". In the construction below we shall see that the monopole (A_i, Φ_i) has a zero

$$z_j^i \in B_{cm_i^{-1/2}}(m_i x_j),$$

for $j = l + 1, \ldots, k$. And the centers of these balls leave any compact set as $i \to \infty$. Thus, the sequence of zeros $z_j^i \to \infty$ has no convergent subsequence, and so does not contribute to Z.

In the rest of this subsection we prove this result by using Theorem 3.2.5 to construct the monopoles. Let $\lambda > 0$ and consider the scaling map $s_\lambda(x) = \lambda^{-1}x$ for $x \in \mathbb{R}^3$. Recall that the Euclidean metric g_E is invariant by scaling, i.e. $g_E = \lambda^2 s_\lambda^* g_E$ for any such positive λ. Therefore, by Proposition 2.1.1, if (A, Φ) is a charge k mass 1 monopole, we have that

$$(A_\lambda, \Phi_\lambda) = (s_\lambda^* A, \lambda^{-1} s_\lambda^* \Phi) \tag{3.2.8}$$

is a charge k, mass λ^{-1} monopole.

It is instructive to split the proof in two different cases:

Case $l = k$*.* We now construct a sequence of charge k, large mass monopoles on $\mathbb{R}^3$ with prescribed $Z = \{x_1, \ldots, x_k\}$ being k distinct points in $\mathbb{R}^3$. After choosing such k points, we fix a sequence of masses $m_i \to \infty$ and suppose, with no loss of generality, that $m_1 \gg 1$ so as to $m_1 \mathrm{dist}(x_j, x_l) > d_0$, for all $j, l \in \{1, \ldots, k\}$. Then, we can use Taubes' Theorem 3.2.5 to construct a sequence, labeled by i, of charge k, mass 1 monopoles using in the construction the points $y_j^i = m_i x_j$ for $j = 1, \ldots, k$. Rescaling these as in equation (3.2.8) with $\lambda = m_i^{-1}$ we obtain a sequence of monopoles (A_i, Φ_i) with charge k, mass m_i and

$$\Phi_i^{-1}(0) \subset \bigcup_{j=1}^{k} B_{cm_i^{-1/2}}(x_j), \quad \deg(\Phi_i\big|_{\partial B_{cm_i^{-1/2}}(x_j)}) = 1. \tag{3.2.9}$$

Now, recall from the definition of the zero set that

$$Z = \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)}.$$

Hence, as the sequence $\{m_i\}_i$ is increasing, it follows that

$$\overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)} \subseteq \bigcup_{j=1}^{k} \overline{B_{cm_n^{-1/2}}(x_j)},$$

and thus

$$Z \subseteq \bigcap_{n\geq 1} \bigcup_{j=1}^{k} \overline{B_{cm_n^{-1/2}}(x_j)} = \bigcup_{j=1}^{k} \bigcap_{n\geq 1} \overline{B_{cm_n^{-1/2}}(x_j)} = \bigcup_{j=1}^{k} \{x_j\}.$$

On the other hand, for every fixed $j \in \{1, \ldots, k\}$, the degree of the map Φ restricted to a normal sphere of radius $cm_i^{-1/2}$ equals 1 and thus Φ_i has a zero

$$z_i \in B_{cm_i^{-1/2}}(x_j),$$

for each $i \gg 1$. Since $m_i \uparrow \infty$, it follows that $z_i \to x_j$ as $i \to \infty$. Thus we get the reverse inclusion $\{x_1, \ldots, x_k\} \subseteq Z$, proving that indeed

$$Z = \{x_1, \ldots, x_k\}.$$

■

Case $l < k$. We can modify the above construction in order to make $l < k$ of the monopoles “escape to infinity”. We shall proceed as before and fix k distinct points $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^3$. Then, we consider the charge k, mass 1 monopole obtained through Taubes' Theorem 3.2.5 using the points $y_j^i = m_i x_j$ for $j = 1, \ldots, l$ and the points $y_j^i = m_i^2 x_j$ for $j = l+1, \ldots, k$.[3] Then, rescaling this monopole as before, i.e. using equation (3.2.8) with $\lambda = m_i^{-1}$, we obtain a mass m_i, charge k, monopole on $\mathbb{R}^3$. This has the property that

$$\Phi_i^{-1}(0) \subset \left(\bigcup_{j=1}^{l} B_{cm_i^{-1/2}}(x_j)\right) \cup \left(\bigcup_{j=l+1}^{k} B_{cm_i^{-1/2}}(m_i x_j)\right), \quad \deg\left(\Phi_i\big|_{\partial B_{cm_i^{-1/2}}(x_j)}\right) = 1. \quad (3.2.10)$$

Similarly to before we now have

$$\overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)} \subseteq \left(\bigcup_{j=1}^{l} \overline{B_{cm_n^{-1/2}}(x_j)}\right) \cup \left(\bigcup_{i\geq n} \bigcup_{j=l+1}^{k} \overline{B_{cm_n^{-1/2}}(m_i x_j)}\right),$$

and for n sufficiently large the second sets inside parenthesis above are disjoint for n sufficiently large. It then follows again from the same degree argument as before that

$$Z = \{x_1, \ldots, x_l\}.$$

■

[3] By slight modification of this we can also let the points y_j for $j > l$ go off to infinity at different rates.

3.2.3 An example with $k_x > 1$

In the examples above we have already seen that it is possible to have $\mathcal{H}^0(Z) < k$ by letting the monopoles "escape through the end". One other possibility would be to have points $x \in Z$ with $k_x > 1$, in this Subsection we give the simplest of such examples.

Let (A, Φ) be a finite mass SU(2)−monopole in $(\mathbb{R}^3, g_E)$ with mass $m \neq 0$ and charge $k > 1$. Since g_E is scale-invariant, taking any null-sequence $\lambda_i \downarrow 0$ we get a corresponding sequence

$$(A_{\lambda_i}, \Phi_{\lambda_i}) := (s_{\lambda_i}^* A, \lambda_i{}^{-1} s_{\lambda_i}^* \Phi)$$

of monopoles in $(\mathbb{R}^3, g_E)$ with masses $m_i := m\lambda_i^{-1} \to \infty$. Note that for such sequence $Z = \{0\}$ and $k_0 = k > 1$.

3.2.4 Sequences of monopoles with prescribed Z on any AC $3-$manifold with $b^2(X) = 0$

Let (X, g) be an AC oriented Riemannian 3−manifold with $b^2(X) = 0$ (i.e., the second Betti number of X vanishes), and let $k \in \mathbb{Z}_{>0}$. For a real number $m > 0$, denote by $\mathcal{M}_{k,m}$ the moduli space of mass m and charge k monopoles on (X, g). In this setting, the main result of [Oli16c] yields

Theorem 3.2.11 ([Oli16c, Theorem 1])**.** *There is $\mu \in \mathbb{R}$, so that for all $m \geq \mu$ and $X^k(m) \subset X^k$ defined by*

$$X^k(m) = \left\{ (x_1, ..., x_k) \in X^k \,\middle|\, \mathrm{dist}(x_i, x_j) > 4m^{-1/2} \text{ , for } i \neq j \right\},$$

while $\check{\mathbb{T}}^{k-1} = \{(e^{i\theta_1}, ..., e^{i\theta_k}) \in \mathbb{T}^k \mid e^{i(\theta_1 + ... + \theta_k)} = 1\}$, there is a local diffeomorphism onto its image

$$h_m : X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1} \to \mathcal{M}_{k,m}. \tag{3.2.12}$$

In order to use this theorem we shall fix once and for all $\alpha \in H^1(X, S^1)$ and $\theta = (e^{i\theta_1}, \ldots, e^{i\theta_k}) \in \mathbb{T}^k$ satisfying $e^{i(\theta_1 + ... + \theta_k)} = 1$. Then, we chose any k disjoint points $(x_1, ..., x_k) \in X^k$ and take an increasing sequence of positive real numbers $m_i \uparrow \infty$ with $m_1 > \max\{16 \min_{j,l} \mathrm{dist}(x_j, x_l)^2, \mu\}$ and consider the monopoles

$$(A_i, \Phi_i) = h_{m_i}((x_1, \ldots, x_k), \alpha, \theta).$$

Then, using the results of [Oli16c] we have the following

Proposition 3.2.13. *The zero set*

$$Z = \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)},$$

of the family of monopoles (A_i, Φ_i) *defined above is precisely the set of points* $\{x_1, \ldots, x_k\}$.

Proof. The result follows from proving that the zeros of the monopole (A_i, Φ_i) are contained in balls of radius $O(m_i^{-1/2})$ around the points $\{x_1, \ldots, x_k\}$, i.e. for sufficiently large i we have

$$\Phi_i^{-1}(0) \subset \bigcup_{j=1}^{k} B_{10m_i^{-1/2}}(x_j) \quad \text{and} \quad \Phi_i^{-1}(0) \cap B_{10m_i^{-1/2}}(x_j) \neq \emptyset, \quad \forall j \in \{1, \ldots, k\}. \tag{3.2.14}$$

Indeed, if we prove this assertion, then, as the sequence $\{m_i\}$ is increasing, on the one hand note that

$$\overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)} \subseteq \bigcup_{j=1}^{k} \overline{B_{10m_n^{-1/2}}(x_j)},$$

and thus

$$Z \subseteq \bigcap_{n\geq 1} \bigcup_{j=1}^{k} \overline{B_{10m_n^{-1/2}}(x_j)} = \bigcup_{j=1}^{k} \bigcap_{n\geq 1} \overline{B_{10m_n^{-1/2}}(x_j)} = \bigcup_{j=1}^{k} \{x_j\}.$$

On the other hand, for every fixed $j_0 \in \{1, \ldots, k\}$, we can find $z_i \in \Phi_i^{-1}(0) \cap B_{10m_i^{-1/2}}(x_{j_0})$ for each $i \gg 1$. Since $m_i \uparrow \infty$, it follows that $z_i \to x_{j_0}$ as $i \to \infty$. Thus we get the reverse inclusion $\{x_1, \ldots, x_k\} \subseteq Z$, and equality follows as claimed.

We are thus left with proving the assertion (3.2.14), which is the content of Appendix B. ∎

3.3 ε-regularity estimate

In this section we give an appropriate restatement of the 3−dimensional instance of Theorem 2.3.1 that will be useful in the analysis of sequences of large mass monopoles in the rest of this chapter. The following is essentially [FO19, Theorem 5.1] but stated in a stronger form and derived in a way that provides a more general perspective on the result.

Theorem 3.3.1 (ε−regularity estimate in 3−dimensions)**.** *Let* (X^3, g) *be an oriented Riemannian* 3−*manifold of bounded geometry and* E *a* G−*bundle over* X*, where* G *is a compact Lie group. Then there are scaling invariant constants* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *with the following significance. Let* $(A, \Phi) \in \mathcal{C}(E)$ *be a YMH configuration on* (X^3, g)*. For any* $m > 0$ *and* $R > 0$*, if* $x \in X$ *and* $0 < r \leq \min\{Rm^{-1}, r_0\}$ *are such that*

$$\varepsilon := m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_0,$$

then

$$\sup_{B_{\frac{r}{2}}(x)} m^{-1}e(A,\Phi) \leq C_R r^{-3}\varepsilon, \tag{3.3.2}$$

where $C_R := C_0 \max\{1, R^3\}$.

Proof. First of all, we note that, by scaling, we may assume $m = 1$. Indeed, assume the result is true for $m = 1$. Then given a YMH configuration (A,Φ) with respect to a metric g, it follows from Proposition 2.1.1 that $(A, m^{-1}\Phi)$ is a YMH configuration with respect to the scaled metric $g_m := m^2 g$. Now, by hypothesis, for $r \in (0, \min\{Rm^{-1}, r_0(g)\}]$,

$$\mathcal{E}^m_{B^m_{rm}(x)}(A, m^{-1}\Phi) = m^{-1}\mathcal{E}_{B_r(x)}(A,\Phi) < \varepsilon_0.$$

Noting that $r_0(g_m) = mr_0(g)$, the result for $m = 1$ implies that

$$\sup_{B^m_{\frac{rm}{4}}(x)} e_m(A, m^{-1}\Phi) \leq C_R(rm)^{-3}\mathcal{E}^m_{B^m_{rm}(x)}(A, m^{-1}\Phi).$$

Thus, rescaling back to g we get precisely (3.3.2). This proves our claim.

Given the above observation, in order to prove the theorem we are left to prove that there are $\varepsilon_0 > 0$ and $C_0 > 0$ such that for any YMH configuration (A,Φ) and any $R > 0$, if $x \in X$ and $0 < r \leq \min\{R, r_0\}$ are such that $\varepsilon := \mathcal{E}_{B_r(x)}(A,\Phi) < \varepsilon_0$, then

$$\sup_{B_{\frac{r}{4}}(x)} e(A,\Phi) \leq C_0 \max\{1, R^3\} r^{-3}\varepsilon. \tag{3.3.3}$$

We take ε_0 and C_0 as in the case $n = 3$ of Theorem 2.3.1. Now note that if $r \leq 1$, then the hypothesis imply

$$r\mathcal{E}_{B_r(x)}(A,\Phi) \leq \mathcal{E}_{B_r(x)}(A,\Phi) < \varepsilon_0,$$

so that by (2.3.2) we have[4]

$$\sup_{B_{\frac{r}{2}}(x)} e(A,\Phi) \leq C_0 r^{-3}\varepsilon,$$

which proves the assertion. Otherwise, i.e. in case $r > 1$, then for all $y \in B_{\frac{r}{2}}(x)$ one has $B_{\frac{1}{2}}(y) \subseteq B_r(x)$ and thus the hypothesis imply

$$\frac{1}{2}\mathcal{E}_{B_{\frac{1}{2}}(y)}(A,\Phi) \leq \mathcal{E}_{B_r(x)}(A,\Phi) < \varepsilon_0.$$

Hence, using Theorem 2.3.1 and noting that $1 \leq R^3 r^{-3}$, we get

$$e(A,\Phi)(y) \leq \sup_{B_{\frac{1}{4}}(y)} e(A,\Phi) \leq 8C_0\varepsilon \leq 8C_0 R^3 r^{-3}\varepsilon.$$

[4]Note that the ε of (2.3.2) differs from this one by a factor of r, that's why here we get r^{-3} in the right-hand side of the inequality instead of r^{-4}.

Since $y \in B_{\frac{r}{2}}(x)$ is arbitrary, this proves the assertion with C_0 replaced by $8C_0$ and finishes the proof. ■

Note that taking $r \in (0, r_0]$, $m = r^{-1}$ and $R = 1$ on the statement of Theorem 3.3.1 gives back the $n = 3$ case of Theorem 2.3.1, so that these are indeed equivalent (the above proof gives the other implication). In fact, playing around with the parameters m and R in Theorem 3.3.1 one obtains the following easy consequences.

Corollary 3.3.4. *Let (X^3, g) be an oriented Riemannian $3-$manifold of bounded geometry and E a $G-$bundle over X, where G is a compact Lie group. Let $\varepsilon_0, C_0 > 0$ be the constants given by Theorem 3.3.1 and $(A, \Phi) \in \mathcal{C}(E)$ a YMH configuration on (X^3, g). Then:*

(i) For any $m > 0$ such that $2m^{-1} < r_0$, if $x \in X$ and $r \in (2m^{-1}, r_0]$ are such that

$$\varepsilon := m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_0,$$

then

$$\sup_{B_{\frac{r}{2}}(x)} m^{-1}e(A, \Phi) \leq m^3 C_0 \varepsilon. \tag{3.3.5}$$

(ii) If $x \in X$ and $0 < r \leq \min\{1, r_0\}$ are such that

$$\varepsilon := \mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_0,$$

then

$$\sup_{B_{\frac{r}{2}}(x)} e(A, \Phi) \leq C_0 r^{-3} \varepsilon. \tag{3.3.6}$$

Proof. (i) Let $y \in B_{\frac{r}{2}}(x)$. Since $0 < 2m^{-1} < r$, one has $B_{m^{-1}}(y) \subseteq B_r(x)$ and thus by hypothesis

$$m^{-1}\mathcal{E}_{B_{m^{-1}}(y)}(A, \Phi) \leq m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_0.$$

Then we can apply Theorem 3.3.1 to conclude that

$$\sup_{B_{(2m)^{-1}}(y)} m^{-1}e(A, \Phi) \leq m^3 C_0 \varepsilon.$$

Since $y \in B_{\frac{r}{2}}(x)$ is arbitrary and the bound is uniform, the result follows.

(ii) This is just a direct consequence of Theorem 3.3.1 using $m = R = 1$. ■

3.4 An interior lower bound on the Higgs field

The next result is a consequence of the previous $\varepsilon-$regularity estimate and will prove to be useful in analyzing large mass YMH configurations. It is an improved version of [FO19, Theorem 6.1].

Theorem 3.4.1. *Let (X^3, g) be an oriented Riemannian $3-$manifold of bounded geometry, let E be a $G-$bundle over X, and let (A, Φ) be a YMH configuration on E. Given $\delta \in (0,1)$ and $R > 0$, set*

$$\varepsilon_{\delta,R} := \min\left\{C_R^{-1} R\delta^2, \varepsilon_0\right\}.$$

Let $x \in X$ and $m > 0$. If $r := Rm^{-1} \leq r_0$ and $\sup_{\partial \overline{B}_{\frac{r}{4}}(x)} |\Phi| \geq m\delta$, then

$$m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_{\delta,R} \quad \Longrightarrow \quad |\Phi| > \frac{m\delta}{2} \quad \text{on} \quad B_{\frac{r}{4}}(x).$$

Here C_R and ε_0 are the constants given by Theorem 3.3.1.

Proof. Fix $q \in \partial \overline{B}_{\frac{r}{4}}(x)$ such that the restriction of $|\Phi|$ to $\partial \overline{B}_{\frac{r}{4}}(x)$ attains its maximum at q. For any $p \in B_{\frac{r}{4}}(x)$ we can choose a (smooth by parts) path γ_p in $B_{\frac{r}{4}}(x)$ with length $L(\gamma_p) \leq \frac{r}{2}$ joining p to q. Thus, using the fundamental theorem of calculus and Kato's inequality we get:

$$|\Phi|(q) - |\Phi|(p) \leq \left| \int_{\gamma_p} \mathrm{d}|\Phi| \right| \leq \int_{\gamma_p} |\mathrm{d}_A\Phi| \leq \frac{r}{2} \sup_{B_{\frac{r}{4}}(x)} |\mathrm{d}_A\Phi|. \tag{3.4.2}$$

On the other hand, by the $\varepsilon-$regularity estimate (Theorem 3.3.1), the hypothesis $m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_{\delta,R} \leq \varepsilon_0$ implies that

$$\sup_{B_{\frac{r}{4}}(x)} |\mathrm{d}_A\Phi| \leq \sup_{B_{\frac{r}{4}}(x)} e(A, \Phi)^{\frac{1}{2}} < C_R^{\frac{1}{2}} r^{-\frac{3}{2}} m^{\frac{1}{2}} \varepsilon_{\delta,R}^{\frac{1}{2}}. \tag{3.4.3}$$

Putting (3.4.2) and (3.4.3) together and using the definitions of r and $\varepsilon_{\delta,R}$ along with the lower bound on $|\Phi|(q) = \sup_{\partial \overline{B}_{\frac{r}{4}}(x)} |\Phi|$ gives the statement. ∎

Combining the above result with Theorem 2.4.1, we get (cf. [FO19, Corollary 6.1]):

Corollary 3.4.4. *Let (X^3, g) be an oriented AC Riemannian $3-$manifold with one end, E be a $G-$bundle over X and $\Lambda \in (0, \infty)$. Then there are constants $R_\Lambda > 0$ and $\varepsilon_\Lambda > 0$, $\varepsilon_\Lambda \leq \varepsilon_0$, such that the following holds. Let (A, Φ) be a finite mass YMH configuration on E with mass $m > m_*$ and $m^{-1}\|\mathrm{d}_A\Phi\|^2_{L^2(X)} \leq \Lambda$. If $r := R_\Lambda m^{-1} \leq r_0$ then*

$$m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_\Lambda \quad \Longrightarrow \quad |\Phi| > \frac{m}{4} \quad \text{on} \quad B_{\frac{r}{4}}(x). \tag{3.4.5}$$

Proof. Let

$$R_\Lambda := c\Lambda \quad \text{and} \quad \varepsilon_\Lambda := \varepsilon_{1/2,R_\Lambda} = \min\{4^{-1}C_{R_\Lambda}^{-1}R_\Lambda, \varepsilon_0\}.$$

Here we choose $c > 0$ depending only on g big enough (e.g. $c := 64c_1c_2^{-1}$) so that by letting $\delta := 1/2$ it follows from Theorem 2.4.1 that

$$\sup_{\partial \overline{B}_{\frac{r}{4}}(x)} |\Phi| \geq \frac{m}{2},$$

therefore by hypothesis we can apply Theorem 3.4.1 with $R = R_\Lambda$ to get the desired result. ■

This has the following immediate corollary which is very important in the large mass limit.

Corollary 3.4.6. *Let (X^3, g) be an oriented AC Riemannian 3−manifold with one end, E be a* $\mathrm{SU}(2)$*−bundle over X and $k \in \mathbb{N}$. Let $R_k > 0$ and $\varepsilon_k > 0$, $\varepsilon_k \leq \varepsilon_0$, be the constants given by Corollary 3.4.4 when $\Lambda = 4\pi k$. Let (A, Φ) be a monopole of charge k and mass $m > \max\{m_*, 2R_k r_0^{-1}\}$. Then for all $r \in (2R_k m^{-1}, r_0]$ one has*

$$m^{-1}\mathcal{E}_{B_r(x)}(A, \Phi) < \varepsilon_k \quad \Longrightarrow \quad |\Phi| > \frac{m}{4} \quad \text{on} \quad B_{\frac{r}{2}}(x). \tag{3.4.7}$$

Proof. Let $y \in B_{\frac{r}{2}}(x)$. Since $0 < 2R_k m^{-1} < r$, one has $B_{R_k m^{-1}}(y) \subseteq B_r(x)$ and thus using the smallness hypothesis on the energy we can apply Corollary 3.4.4 to conclude that

$$|\Phi|(y) > \frac{m}{4}.$$

Since $y \in B_{\frac{r}{2}}(x)$ is arbitrary and the bound is uniform, the result follows. ■

3.5 The blow-up set and the zero set

From now on we will be dealing with a sequence $\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{C}(E)$ of finite mass YMH configurations on (X^3, g) satisfying the uniform bound

$$m_i^{-1}\mathcal{E}_X(A_i, \Phi_i) \leq C, \tag{3.5.1}$$

for some constant $C > 0$, and whose masses m_i satisfy $\limsup m_i = \infty$. In fact, for convenience, we may assume (after passing to a subsequence) that $m_i \uparrow \infty$. We note that in case the (A_i, Φ_i) are $\mathrm{SU}(2)$−monopoles of fixed charge $k \neq 0$, the energy formula (1.3.19) guarantee an a priori uniform bound of the form (3.5.1) with equality for $C = 4\pi k$.

In order to study the behavior of such sequence of infinitely large mass YMH configurations, it is convenient to consider the corresponding sequence of Radon measures

$$\mu_i := m_i^{-1} e(A_i, \Phi_i)\mathcal{H}^3. \tag{3.5.2}$$

By (3.5.1) this sequence is of bounded mass. Thus, after passing to a subsequence which we do not relabel, it converges weakly to a Radon measure μ. By Fatou's lemma and Riesz representation theorem, we can write

$$\mu = e_\infty \mathcal{H}^3 + \nu, \tag{3.5.3}$$

where $e_\infty : X \to [0, \infty]$ is the L^1-function

$$e_\infty := \liminf_{i\to\infty} m_i^{-1} e(A_i, \Phi_i),$$

and ν is some nonnegative Radon measure, singular with respect to $\mathcal{H}^3$, called the *defect measure*.

Let Θ be the $0-$dimensional density function of μ, i.e.

$$\Theta(x) := \lim_{r\downarrow 0} \mu(B_r(x)), \quad \forall x \in X. \tag{3.5.4}$$

Note that Θ is well-defined and bounded by C. In this section, we will be considering the *blow-up set* S of $\{(A_i, \Phi_i)\}$, which is defined to be

$$S := \{x \in X : \Theta(x) > 0\}.$$

The fact that $\{\mu_i\}$ weakly converges to μ a priori only implies that $\mu(B_r(x)) \leq \liminf_{i\to\infty} \mu_i(B_r(x))$ and $\mu(\overline{B}_r(x)) \geq \limsup_{i\to\infty} \mu_i(\overline{B}_r(x))$. Thus, for each $x \in X$, it is convenient to set

$$\mathcal{R}_x := \{r \in (0, r_0] : \mu(\partial B_r(x)) > 0\}.$$

For all $r \in (0, r_0] \setminus \mathcal{R}_x$ one has

$$\mu(B_r(x)) = \lim_{i\to\infty} \mu_i(B_r(x)).$$

Since μ is locally finite, the set $\mathcal{R}_x$ is at most countable. In particular, for each point $x \in X$ we can find a null-sequence $\{r_i\} \subseteq (0, r_0] \setminus \mathcal{R}_x$ so that

$$\Theta(x) = \lim_{i\to\infty} \mu_i(B_{r_i}(x)).$$

From these facts, the following is immediate.

Lemma 3.5.5. *The blow-up set S can be written as*

$$S = \bigcup_{j \in \mathbb{N}} S_j,$$

where

$$S_j := \bigcap_{0<r\leq r_0} \left\{ x \in X : \liminf_{i\to\infty} m_i^{-1} \mathcal{E}_{B_r(x)}(A_i, \Phi_i) \geq j^{-1} \right\}.$$

Our first result relates the blow-up set S with the accumulation points of the Higgs fields zeros, called the *zero set* Z, and defined by

$$Z := \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)}.$$

In the following statement, we shall use $\mathcal{H}^0$ to denote the counting measure.

Theorem 3.5.6. *Let (X^3, g) be an AC oriented Riemannian $3-$manifold with one end, E be an $G-$bundle over X, and $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{C}(E)$ be a sequence of finite mass YMH configurations on (X^3, g) satisfying the uniform bound (3.5.1) and whose masses m_i satisfy $\limsup_{i\to\infty} m_i = \infty$. Then, after passing to a subsequence for which $m_i \uparrow \infty$ and $\mu_i \rightharpoonup \mu$, where μ_i are the Radon measures given by (3.5.2), the following holds:*

(i) $\mathcal{H}^0(S_j) \leq jC$, for all $j \in \mathbb{N}$; in particular, each S_j is finite and S is countable.

(ii) The blow-up set contains the zero set:

$$Z \subseteq S.$$

Proof. (i) Given $0 < r \leq r_0$, we can find a countable open covering $\{B_{5r_l}(x_l)\}$ of S_j with $x_l \in S_j$, $10r_l < r$ and $B_{r_l}(x_l)$ pairwise disjoint. Then

$$\begin{aligned}
\sum_l (5r_l)^0 &\leq j \sum_l \liminf_{i\to\infty} m_i^{-1} \mathcal{E}_{B_{r_l}(x_l)}(A_i, \Phi_i) \quad (x_l \in S_j) \\
&\leq j \liminf_{i\to\infty} m_i^{-1} \sum_l \mathcal{E}_{B_{r_l}(x_l)}(A_i, \Phi_i) \quad \text{(by Fatou's lemma)} \\
&\leq j \liminf_{i\to\infty} m_i^{-1} \mathcal{E}_X(A_i, \Phi_i) \quad \text{(the } B_{r_l}(x_l)\text{'s are disjoint)} \\
&= jC. \quad \text{(by (3.5.1))}
\end{aligned}$$

Since this bound is uniform in $r \in (0, r_0]$, it follows that $\mathcal{H}^0(S_j) \leq jC$.

(ii) We shall apply Corollary 3.4.4 with $\Lambda := 2C$. Let $x \in X \setminus S$. Then $x \in X \setminus S_j$ for all $j \in \mathbb{N}$. In particular, there is $r \in (0, r_0]$ such that

$$\liminf_{i\to\infty} m_i^{-1} \mathcal{E}_{B_r(x)}(A_i, \Phi_i) < \varepsilon_\Lambda.$$

We may assume $r \notin \mathcal{R}_x$, otherwise we just work in a smaller ball for which we still have the above energy bound. In particular, it follows that there is $i_0 \in \mathbb{N}$ such that

$$m_i^{-1} \mathcal{E}_{B_r(x)}(A_i, \Phi_i) < \varepsilon_\Lambda, \quad \forall i \geq i_0.$$

Since $m_i \uparrow \infty$, by increasing i_0 if necessary we may also assume that

$$m_i > m_* \quad \text{and} \quad r_i := R_\Lambda m_i^{-1} < \frac{r}{2}, \quad \forall i \geq i_0.$$

Hence, given any $y \in B_{r/2}(x)$, it follows that

$$m_i^{-1} \mathcal{E}_{B_{r_i}(y)}(A_i, \Phi_i) < \varepsilon_\Lambda, \quad \forall i \geq i_0,$$

so that applying Corollary 3.4.4 we get that

$$|\Phi_i|(y) > \frac{m_i}{4}, \quad \forall i \geq i_0.$$

Therefore, by possibly increasing i_0, we have

$$\inf_{B_{\frac{r}{2}}(x)} |\Phi_i| \geq 100 > 0, \quad \forall i \geq i_0.$$

In particular, it follows that $x \in X \setminus Z$. By the arbitrariness of $x \in X \setminus S$, this shows that $Z \subset S$. ∎

3.6 Bubbling

Let (A, Φ) be a finite mass configuration on E with mass $m \neq 0$, and pick a point $x \in X$. For each $r \in (0, r_0]$, consider the geodesic ball $B_r(x) \subset X$. Then, identify $\mathbb{R}^3 \cong T_x X$ and use the exponential map $s_m(\cdot) = \exp(m^{-1} \cdot)$ to define

$$(A_m, \Phi_m) = (s_m^* A, m^{-1} s_m^* \Phi), \quad g_m = m^2 s_m^* g. \tag{3.6.1}$$

It follows from Proposition 2.1.1 that if (A, Φ) is a YMH configuration on $B_r(x) \subset X$, then (A_m, Φ_m) is a YMH configuration in $B_{rm}(0) \subset \mathbb{R}^3$ with respect to the metric g_m.[5] Moreover,

[5]Here $B_{rm}(0) \subset \mathbb{R}^3$ is a radius r ball with respect to both the metrics $g_m = m^2 s_m^* g$ and $m^2 \exp^* g$, by the Gauss Lemma.

we note that as $m \to \infty$ the metric g_m geometrically converges to the Euclidean one, g_E, on compact subsets of $\mathbb{R}^3$. The main result of this section is

Theorem 3.6.2. *Let (X^3, g) be an AC oriented Riemannian $3-$manifold with one end, let E be a $G-$bundle over X, and let $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{C}(E)$ be a sequence of finite mass YMH configurations on (X^3, g) with masses m_i satisfying $\limsup m_i = \infty$. Denote by S and Z the corresponding blow-up and zero sets. Then, for each $x \in S$, after passing to a subsequence and changing gauge, the rescaled sequence (A_{m_i}, Φ_{m_i}) converges uniformly with derivatives, in compact subsets of $\mathbb{R}^3 \cong T_xX$, to a YMH configuration (A_x, Φ_x) of mass $m_x \leq 1$ and strictly positive energy[6] $\mathcal{E}_{\mathbb{R}^3}(A_x, \Phi_x) > 0$. If, moreover, the sequence satisfy (3.5.1), then $m_x = 1$. Furthermore, in case $G = \mathrm{SU}(2)$ and the (A_i, Φ_i) are monopoles with fixed charge $k \neq 0$, then the limit (A_x, Φ_x) is a monopole of mass $m_x = 1$ and charge $k_x > 0$, $k_x \leq k$, and we have $S = Z$.*

The rest of this section is devoted to proving Theorem 3.6.2. We shall start with two auxiliary results.

Lemma 3.6.3. *Let $r \in (0, r_0]$ and $K \subset \mathbb{R}^3$ be a compact set. Then, there are constants $c > 0$ and $m_* \gg 1$ such that: If (A, Φ) is a mass $m \geq m_* \gg 1$ YMH configuration on X, then there is a gauge such that on $K \subset \mathbb{R}^3$:*

$$|\Phi_m| < 1$$
$$|\nabla_{A_m}\Phi_m|_{g_E} + |\nabla^2_{A_m}\Phi_m|_{g_E} \leq c$$
$$|A_m|_{g_E} + |\nabla_{A_m}A_m|_{g_E} + |\nabla^2_{A_m}A_m|_{g_E} \leq c.$$

Furthermore, the following inequalities holds on K

$$|F_m|_{g_E} + |\nabla_{A_m}\Phi_m|_{g_E} \leq c$$
$$|\nabla_{A_m}F_m|_{g_E} + |\nabla^2_{A_m}\Phi_m|_{g_E} \leq c.$$

Proof. We start recalling Remark 1.3.17, which in terms of Φ_m reads $|\Phi_m| < 1$.

Now, let (x^1, x^2, x^3) be geodesic normal coordinates on $B_r(x) \subset X$ and (y^1, y^2, y^3) coordinates in $B_{rm}(0) \subset \mathbb{R}^3$ so that $s_m(y^1, y^2, y^3) = (mx^1, mx^2, mx^3)$. In these coordinates we can write the metric g as

$$g = \left(\delta_{ij} + \frac{1}{3}\mathrm{R}_{iklj}x^k x^l + O(|x|^2)\right) dx^i \otimes dx^j.$$

[6] Possibly ∞.

Thus, by defining the symmetric 2-tensor $\gamma = \frac{1}{3}\mathrm{R}_{iklj} y^k y^l dy^i \otimes dy^j$, with R_{iklj} is the Riemann curvature tensor of g, we can write the metric g_m in $B_{rm}(0) \subset \mathbb{R}^3$, as

$$g_m = g_E + m^{-2}\gamma + O(m^{-3}). \tag{3.6.4}$$

It is at this point that we choose m_* to be large enough so that $K \subset B_{rm^{1/2}}(0) \subset B_{rm}$. Hence, given that $|\gamma| \leq |y|^2$, in K we have $|\nabla^j (g_m - g_E)|_{g_E} \leq O(m^{-1})$, for all $j \in \mathbb{N}_0$. In particular, these metrics are quasi-isometric in K.

Now, the YMH equations (1.1.1), in Coulomb gauge, give an elliptic system for Φ_m and the components of the connection A_m. Furthermore, for $m_* \gg 1$ and $m \geq m_*$ all the components of such system, written in the coordinates y on K, are uniformly bounded in m. Thus, elliptic regularity supplied by the m independent bound $|\Phi_m| < 1$, gives m independent bounds on the first and second y-derivatives of Φ_m and A_m. These bounds can be further iterated to yield bounds on higher y-derivatives these fields. Moreover, given that the metrics g_m and g_E on K are quasi-isometric it is irrelevant with respect to which of these metric such bounds are written. ∎

Next, we prove the following result which states that as $m \to \infty$ the (A_m, Φ_m) is not only a YMH configuration with respect to g_m as it approaches one for g_E in compact subsets of $\mathbb{R}^3$. This is a consequence of the geometric convergence of g_m to g_E but a complete proof is given below.

Lemma 3.6.5. *Let $r \in (0, r_0]$ and $K \subset \mathbb{R}^3$, then there is $m_* \gg 1$ and a constant $c > 0$ with the following significance. If (A, Φ) is a mass m YMH configuration on X, then the inequality*

$$|\Delta^E_{A_m} \Phi_m|_{g_E} + |\mathrm{d}^{*_E}_{A_m} F_{A_m} - [\mathrm{d}_{A_m}\Phi_m, \Phi_m]|_{g_E} \leq cm^{-1},$$

holds in K. Moreover, in the particular case when (A, Φ) is actually a monopole, we further have

$$| *_E F_{A_m} - \mathrm{d}_{A_m}\Phi_m|_{g_E} \leq cm^{-1}.$$

Proof. We shall prove only the case of a YMH configuration, for monopoles the result follows from similar, but somewhat easier computations. We continue to work with the coordinates y introduced during the proof of Lemma 3.6.3. Start by using equation (3.6.4) to relate the action of the Hodge-$*$ operators of both g_m and g_E. Let ω be a k-form and Ric_{ij} the Ricci curvature of

g, a computation gives

$$\begin{aligned} *_{g_m}\omega &= \left(1 + m^{-2}\frac{\gamma(\omega,\omega)}{|\omega|^2_{g_E}} - m^{-2}\frac{1}{6}\mathrm{Ric}_{ij}y^i y^j\right) *_E \omega + O(m^{-3}) && (3.6.6) \\ &= *_E\omega - m^{-2}\gamma_m(\omega), && (3.6.7) \end{aligned}$$

where in the last equality γ_m denotes an algebraic operator. This has the property of being uniformly bounded with all derivatives, i.e. there are a m-independent constants $c_j > 0$ so that for all $j \in \mathbb{N}_0$ and $m > m_* \gg 1$, we have $|(\nabla_E^j \gamma_m)(\omega)|_{g_E} \leq c_j(1+|y|^2)|\omega|_{g_E}$, where ∇_E denotes the Levi–Civita connection of the Euclidean metric in $B_{rm}(0) \subset \mathbb{R}^3$. By possibly further increasing m_* so that $K \subset B_{rm^{1/2}}(0)$, as in the proof of Lemma 3.6.3, we have that as a consequence of this we have that $|(\nabla_E^j \gamma_m)(\omega)|_{g_E} \leq c_j m|\omega|_{g_E}$ on K. Then, we compute

$$\begin{aligned} \Delta^E_{A_m}\Phi_m &= *_E \mathrm{d}_{A_m} *_E \mathrm{d}_{A_m}\Phi_m = *_E\mathrm{d}_{A_m}\left(*_m \mathrm{d}_{A_m}\Phi_m + m^{-2}\gamma_m(\mathrm{d}_{A_m}\Phi_m)\right) \\ &= *_E\left(*_m\Delta^m_{A_m}\Phi_m + m^{-2}(\nabla_E\gamma_m)(\mathrm{d}_{A_m}\Phi_m) + m^{-2}\gamma_m(\nabla_{A_m}\mathrm{d}_{A_m}\Phi_m)\right) \\ &= \Delta^m_{A_m}\Phi_m + *_m m^{-2}\left(\gamma_m(\nabla_{A_m}\mathrm{d}_{A_m}\Phi_m) + (\nabla_E\gamma_m)(\mathrm{d}_{A_m}\Phi_m)\right) \\ &\quad + m^{-2}\gamma_m\left(*_m\Delta^m_{A_m}\Phi_m + m^{-2}(\nabla_E\gamma_m)(\mathrm{d}_{A_m}\Phi_m) + m^{-2}\gamma_m(\nabla_{A_m}\mathrm{d}_{A_m}\Phi_m)\right). \end{aligned}$$

Recall that (A_m,Φ_m) is a YMH configuration for g_m, $\Delta^m_{A_m}\Phi_m = 0$, and so, on K we have

$$|\Delta^E_{A_m}\Phi_m|_{g_E} \leq cm^{-1}\left(|\mathrm{d}_{A_m}\Phi_m|_{g_E} + |\nabla^2_{A_m}\Phi_m|_{g_E}\right) \tag{3.6.8}$$

Similarly, we consider

$$\begin{aligned} \mathrm{d}^{*_E}_{A_m}F_{A_m} &= *_E\mathrm{d}_{A_m} *_E F_{A_m} = *_E\mathrm{d}_{A_m}\left(*_m F_{A_m} + m^{-2}\gamma_m(F_{A_m})\right) \\ &= *_E\left(*_m\mathrm{d}^{*_m}_{A_m}F_{A_m} + m^{-2}(\nabla_E\gamma_m)(F_{A_m}) + m^{-2}\gamma_m(\nabla_{A_m}F_{A_m})\right) \\ &= \mathrm{d}^{*_m}_{A_m}F_{A_m} + m^{-2} *_m \left((\nabla_E\gamma_m)(F_{A_m}) + \gamma_m(\nabla_{A_m}F_{A_m})\right) \\ &\quad + m^{-2}\gamma_m\left(\left(*_m\mathrm{d}^{*_m}_{A_m}F_{A_m} + m^{-2}(\nabla_E\gamma_m)(F_{A_m}) + m^{-2}\gamma_m(\nabla_{A_m}F_{A_m})\right)\right) \end{aligned}$$

Again, the fact that (A_m,Φ_m) is a YMH configuration for g_m, implies that $\mathrm{d}^{*_m}_{A_m}F_{A_m} = [\mathrm{d}_{A_m}\Phi_m,\Phi_m]$, which together with the previous computation yields that on K

$$|\mathrm{d}^{*_E}_{A_m}F_{A_m} - [\mathrm{d}_{A_m}\Phi_m,\Phi_m]|_{g_E} \leq cm^{-1}\left(|F_{A_m}|_{g_E} + |\nabla_{A_m}F_{A_m}|_{g_E}\right) \tag{3.6.9}$$

Then, putting together equations (3.6.8)–(3.6.9) with the result of Lemma 3.6.3 we conclude that on K

$$|\Delta^E_{A_m}\Phi_m|_{g_E} + |\mathrm{d}^{*_E}_{A_m}F_{A_m} - [\mathrm{d}_{A_m}\Phi_m,\Phi_m]|_{g_E} \leq cm^{-1}, \tag{3.6.10}$$

for some $c > 0$ independent of m. ■

This lemmata has the following consequence:

Corollary 3.6.11. *Let* $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{C}(E)$ *be a sequence of finite mass YMH configurations on* (X, g) *with masses* m_i *satisfying* $\limsup m_i = \infty$*, and let* $x \in X$*. Then, after passing to a subsequence and changing gauge, the rescaled sequence* $\{(A_{m_i}, \Phi_{m_i})\}_{i\in\mathbb{N}}$ *defined in equation* (3.6.1) *converges uniformly with derivatives, in compact subsets of* $\mathbb{R}^3 \cong T_xX$*, to a YMH configuration* (A_x, Φ_x) *of mass* $m_x \leq 1$*. Moreover, if the* (A_i, Φ_i) *satisfy the uniform bound* (3.5.1) *then* $m_x = 1$*. In particular, if the* (A_i, Φ_i) *are monopoles, then so is* (A_x, Φ_x) *and* $m_x = 1$.

Proof. Lemma 3.6.3, together with a standard patching argument (see e.g. [DK90, Section 4.4.2]) and the Arzelà–Ascoli theorem, imply immediately that, after passing to a subsequence and changing gauge, the sequence (A_{m_i}, Φ_{m_i}) converges uniformly with derivatives on compact subsets of $\mathbb{R}^3$ to a configuration (A_x, Φ_x) with mass $m_x \leq 1$. The fact that (A_x, Φ_x) is a YMH configuration/monopole is then immediate from Lemma 3.6.5. Finally, to see that we have $m_x = 1$ in the case (3.5.1) is satisfied, fix $r \in (0, r_0]$ and note that, since $\limsup m_i = \infty$, given a sequence $\{\delta_i\} \subset (0, 1)$ with $\delta_i \uparrow 1$, then a diagonal argument shows that up to taking a subsequence we can assume that $m_i > m_*$ and $r > 8Cc_1c_2^{-1}m_i^{-1}(1-\delta_i)^{-1}$, so that by Theorem 2.4.1 we get

$$1 \geq \sup_{\partial B_{rm_i}(0)} |\Phi_{m_i}| = m_i^{-1} \sup_{\partial B_r(x)} |\Phi_i| > \delta_i.$$

Taking the limit $i \to \infty$, we get the desired conclusion. ■

Remark 3.6.12. Since g_m converges to g_E in C^∞_{loc} (cf. proof of Lemma 3.6.3), the first part of Corollary 3.6.11 could be directly deduced as a consequence of the $\varepsilon-$regularity (Theorem 3.3.1).

Corollary 3.6.13. *Let* $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{C}(E)$ *be a sequence of YMH configurations on* (X^3, g) *satisfying the uniform bound* (3.5.1) *and whose masses satisfy* $\limsup m_i = \infty$*. Then, after passing to a subsequence,*

$$S = \bigcap_{0<r\leq r_0} \left\{ x \in X : \liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x)}(A_i, \Phi_i) > 0 \right\}. \tag{3.6.14}$$

Moreover, in case $G = \mathrm{SU}(2)$ *and the* (A_i, Φ_i) *are monopoles of fixed charge* $k \neq 0$*, then*

$$S = \bigcap_{0<r\leq r_0} \left\{ x \in X : \liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x)}(A_i, \Phi_i) \geq 4\pi \right\}. \tag{3.6.15}$$

Proof. We start proving (3.6.14). One inclusion ($\subseteq$) is clear from Lemma 3.5.5. On the other hand, if $x \in X$ is such that for all $r \in (0, r_0]$ one has

$$\varepsilon := \liminf_{i\to\infty} m_i^{-1} \mathcal{E}_{B_r(x)}(A_i, \Phi_i) > 0,$$

then by Corollary 3.6.11 one has that $\varepsilon > 0$ is the energy of a YMH configuration (A_x, Φ_x) in $\mathbb{R}^3$; hence, for $j = j(x) \in \mathbb{N}$ such that $\mathcal{E}_{\mathbb{R}^3}(A_x, \Phi_x) \geq j^{-1}$, we get that $x \in S_j \subset S$, thereby proving the other inclusion.

In the case of (3.6.15), the trivial inclusion is ($\supseteq$) and it suffices to note that the above (A_x, Φ_x) is a positive energy monopole in this case, so that $\varepsilon = 4\pi k_x$ for some positive integer k_x; hence, $\varepsilon \geq 4\pi$, as we wanted. ■

We are now in position to prove the main result of this section.

Proof of Theorem 3.6.2. Let $\{(A_i, \Phi_i)\}$ be a sequence of YMH configurations with masses m_i satisfying $\limsup m_i = \infty$. For $x \in S$, we consider the rescaled sequence $\{(A_{m_i}, \Phi_{m_i})\}$ obtained from the construction in equation (3.6.1). It follows from the definition of S that there is $j = j(x) \in \mathbb{N}$ such that for all $r \in (0, r_0]$ we have

$$\liminf_{i\to\infty} \int_{B_{rm_i}(0)} e(A_{m_i}, \Phi_{m_i}) \geq j^{-1}. \tag{3.6.16}$$

Moreover, it follows from Corollary 3.6.11 that, after passing to a subsequence and changing gauge, the (A_{m_i}, Φ_{m_i}) converges uniformly with derivatives on compact subsets of $\mathbb{R}^3$ to a YMH configuration (A_x, Φ_x) with mass $m_x \leq 1$ (and $m_x = 1$ in case (3.5.1) holds). Furthermore, equation (3.6.16) implies that

$$\int_{\mathbb{R}^3} e(A_x, \Phi_x) \geq j^{-1} > 0. \tag{3.6.17}$$

This condition gives that (A_x, Φ_x) has strictly positive energy. Now, in case $G = \mathrm{SU}(2)$ and the (A_i, Φ_i) are monopoles with fixed charge $k \neq 0$, then (A_x, Φ_x) is a monopole of mass $m_x = 1$ and the energy formula (1.3.19) shows that

$$4\pi k \geq \int_{\mathbb{R}^3} e(A_x, \Phi_x) = 4\pi k_x, \tag{3.6.18}$$

where $k_x \in \mathbb{Z}_{>0}$ is the charge of (A_x, Φ_x); in particular, $k_x \leq k$. Recalling that $k_x > 0$ is the degree of Φ_x restricted to a large sphere, we conclude that Φ_x must have zeros. Thus, by Lemma 3.6.5, for all sufficiently large i so does (A_{m_i}, Φ_{m_i}) in $B_{rm_i}(0) \subset \mathbb{R}^3$ (since as $i \to \infty$ the (A_{m_i}, Φ_{m_i}) becomes as close as one wants of being a positive energy monopole with respect to g_E). Rescaling back, we have that (A_i, Φ_i) must have zeros in $B_r(x) \subset X$ for $i \gg 1$. However, given that the value of $r \in (0, r_0)$ is arbitrary, as $i \to \infty$ such zeros becomes as close as one wants to x yielding that $x \in Z$. This together with Theorem 3.5.6 shows that $Z = S$. This finishes the proof of the theorem. ■

Remark 3.6.19. Notice that, by the equality (3.6.14) of Corollary 3.6.13, if $x \in X \backslash S$ then rather than equation (3.6.17) we have

$$\int_{\mathbb{R}^3} e(A_x, \Phi_x) = 0,$$

which means that (A_x, Φ_x) is gauge equivalent to a flat connection and a constant function $\mathbb{R}^3 \to \mathfrak{g}$. This means that S is indeed precisely the set where the bubbling occurs.

3.7 Convergence as measures

Our aim in this section is to prove the following:

Theorem 3.7.1. *Let (X^3, g) be an AC oriented Riemannian $3-$manifold with one end, let E be a $\mathrm{SU}(2)$-bundle over X and let $\{(A_i, \Phi_i)\}_{i \in \mathbb{N}} \subseteq \mathcal{C}(E)$ be a sequence of finite mass monopoles on (X^3, g) with fixed charge $k \neq 0$ and masses m_i satisfying $\limsup m_i = \infty$. Then, up to taking a subsequence, the corresponding blow-up set S is finite with at most k points and we have the following weak convergence of Radon measures:*

$$m_i^{-1} e(A_i, \Phi_i) \mathcal{H}^3 \rightharpoonup 4\pi \sum_{x \in S} k_x \delta_x,$$

where δ_x denotes the Dirac delta measure supported on $\{x\}$.

In what follows we fix a sequence of monopoles (A_i, Φ_i) as in the hypothesis of Theorem 3.7.1. Recall the sequence $\{\mu_i\}$ of Radon measures (3.5.2) which, by the energy formula (1.3.19), is of bounded mass $4\pi k$ and hence, after passing to a subsequence which we do not relabel, converges weakly to a Radon measure μ, where μ decomposes as in (3.5.3).

The first observation we make is that on the one hand $e_\infty(x) = 0$ for every $x \in X \setminus S$ by equation (3.6.14) of Corollary 3.6.13. On the other hand, it follows from assertion (i) of Theorem 3.5.6 that $\mathcal{H}^3(S) = 0$. Therefore, we conclude that $\mu = \nu$.

Next, we provide some properties of the $0-$dimensional density function Θ of μ.

Proposition 3.7.2. *The function $\Theta : X \to [0, 4\pi k]$ satisfy the following properties:*

(i) $\Theta(x) = 4\pi k_x \geq 4\pi$ *for all* $x \in S$*, where* k_x *is the charge of the bubble* (A_x, Φ_x) *at* x *obtained as in Corollary 3.6.11.*

(ii) Θ *is upper semicontinuous.*

Proof. (i) Let $x \in S$. Then, by Corollary 3.6.11, after passing to a subsequence and changing gauge, the rescaled sequence $\{(A_{m_i}, \Phi_{m_i})\}_{i\in\mathbb{N}}$ defined in equation (3.6.1) converges uniformly with derivatives, in compact subsets of $\mathbb{R}^3 \cong T_xX$, to a mass 1 monopole (A_x, Φ_x) of charge k_x. Then, for any $r \in (0, r_0]$,

$$4\pi k_x = \mathcal{E}_{\mathbb{R}^3}(A_x, \Phi_x) = \liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x)}(A_i, \Phi_i).$$

Now recall that for $r \in (0, r_0] \setminus \mathcal{R}_x$ the weak convergence of measures implies

$$\mu(B_r(x)) = \liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x)}(A_i, \Phi_i) = 4\pi k_x.$$

Moreover, we can find a null-sequence $(r_j) \subset (0, r_0] \setminus \mathcal{R}_x$ so that

$$\Theta(x) = \lim_{j\to\infty} \mu(B_{r_j}(x)) = 4\pi k_x,$$

as we wanted.

(ii) Suppose $\{x_n\}$ is a sequence of points in X with $x_n \to x \in X$ as $n \to \infty$. Let $\delta > 0$ and $r > 0$. Thus, for $n \gg 1$ we have

$$\Theta(x_n) \leq \mu(B_r(x_n)) \leq \mu(B_{r+\delta}(x)),$$

and so $\limsup_{n\to\infty} \Theta(x_n) \leq \mu(B_r(x))$. The result then follows from taking the limit as $r \downarrow 0$. ■

Corollary 3.7.3. $S = Z$ *is a finite set with at most* k *points; in fact,*

$$\mathcal{H}^0(S) \leq \frac{k}{\min_{x\in S} k_x}.$$

Proof. Proposition 3.7.2 immediately implies that S is closed: if $x_n \to x$, with $x_n \in S$, then the upper semicontinuity of Θ implies that $\Theta(x) \geq \limsup \Theta(x_n) \geq 4\pi$, thereby showing that $x \in S$. Now if $K \subset X$ is a compact subset, then $S\cap K$ is also compact. Then, given $0 < r \leq r_0$, we can find a finite open covering $\{B_{2r_j}(x_j)\}_{1\leq j\leq l}$ of $S \cap K$ with $x_j \in S \cap K$, $2r_j < r$ and

$B_{r_j}(x_j)$ pairwise disjoint. Hence,

$$\begin{aligned}\sum_{j=1}^{l} r_j^0 &\leq \frac{1}{4\pi}\sum_{j=1}^{l}\liminf_{i\to\infty} m_i^{-1}\mathcal{E}_{B_{r_j}(x_j)}(A_i,\Phi_i) \quad (x_j\in S)\\ &\leq \frac{1}{4\pi}\liminf_{i\to\infty} m_i^{-1}\sum_{j=1}^{l}\mathcal{E}_{B_{r_j}(x_j)}(A_i,\Phi_i)\\ &\leq \frac{1}{4\pi}\liminf_{i\to\infty} m_i^{-1}\mathcal{E}_X(A_i,\Phi_i) \quad \text{(the } B_{r_j}(x_j)\text{'s are disjoint)}\\ &= k. \quad \text{(by the energy formula (1.3.19))}\end{aligned}$$

Finally, since S is finite, we can write $S=\{x_1,\dots,x_l\}$ (for some $l\leq k$) and choose $r\in(0,r_0]$ such that the balls $B_r(x_j)$, $j=1,\dots,l$, are pairwise disjoint. Then

$$\begin{aligned}\mathcal{H}^0(S) &= \sum_{j=1}^{l} r^0 = \sum_{j=1}^{l}\frac{1}{4\pi k_{x_j}}\mathcal{E}_{\mathbb{R}^3}(A_{x_j},\Phi_{x_j})\\ &= \sum_{j=1}^{l}\frac{1}{4\pi k_{x_j}}\lim_{i\to\infty} m_i^{-1}\mathcal{E}_{B_r(x_j)}(A_i,\Phi_i)\\ &\leq \frac{1}{4\pi\min_j k_{x_j}}\lim_{i\to\infty} m_i^{-1}\sum_{j=1}^{l}\mathcal{E}_{B_r(x_j)}(A_i,\Phi_i)\\ &\leq \frac{1}{4\pi\min_j k_{x_j}}\lim_{i\to\infty} m_i^{-1}\mathcal{E}_X(A_i,\Phi_i) \quad \text{(the } B_r(x_j) \text{ are pairwise disjoint)}\\ &= \frac{k}{\min_j k_{x_j}}. \quad \text{(by (1.3.19))}\end{aligned}$$

■

Finally, writting $S=\{x_1,\dots,x_l\}$, for some $l\leq k$ (by Corollary 3.7.3), we have

Proposition 3.7.4. $\mu = 4\pi\sum_{j=1}^{l} k_{x_j}\delta_{x_j}$.

Proof. Firstly, we show that $\mathrm{spt}(\mu)=S$. Indeed, in one hand, by Proposition 3.7.2, note that $x\in S$ implies $\Theta(x)\geq 4\pi>0$ and therefore $x\in\mathrm{spt}(\mu)$. On the other hand, if $x\in X\setminus S$ then $\mu(B_r(x))\leq\liminf\mu_i(B_r(x))=0$ for all $r\in(0,r_0]$. Therefore, $x\in X\setminus\mathrm{spt}(\mu)$.

By the energy formula, we know that $\Theta(x)\leq 4\pi k$ for all $x\in X$. In particular, given $A\subseteq\mathrm{spt}(\mu)=S$, it follows that

$$\mu(A)=\sum_{j=1}^{l}\lim_{r\downarrow 0}\mu(A\cap B_r(x_j))\leq 4\pi k\mathcal{H}^0(S).$$

Hence, μ is absolutely continuous with respect to $\mathcal{H}^0$. Putting these facts together, the Radon–Nikodym theorem implies that we can write $\mu=\theta\mathcal{H}^0\lfloor S$, for some L^1−function $\theta: S\to\mathbb{R}^+$.

Since S is finite, by the definition of the density function Θ it immediately follows that $\theta = \Theta|_S$, thereby proving the desired statement. ■

This completes the proof of Theorem 3.7.1.

3.8 Questions on convergence outside Z

In order to improve the description of the large mass limit behavior of monopoles obtained in [FO19] (and described in the last sections), one question that arises is about a possible convergence result for the monopoles outside the zero set. More specifically, we hope that the following result is true

Conjecture 3.8.1. *Under the hypothesis of Theorem 3.1.1, there is a subsequence of* $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}}$, *which we do not relabel, and a sequence of gauge transformations* $g_i \in \mathcal{G}(E|_{X\setminus Z})$ *such that the gauge transformed translated sequence*

$$g_i^*\left(A_i, \Phi_i - m_i\frac{\Phi_i}{|\Phi_i|}\right)$$

converges in C^∞_{loc}*−topology to a configuration of zero mass on* $X \setminus Z$.

In order to prove Conjecture 3.8.1, we need to show that for each point $x \in X \setminus Z$ we can find a subsequence $i'(x) \to \infty$ such that $e(A_{i'}, \Phi_{i'})$ is uniformly bounded in a neighborhood of x. Then it follows by Uhlenbeck's Coulomb gauge theorem, elliptic regularity, Arzelà–Ascoli, and a standard patching argument, that we can choose a single subsequence $i'' \to \infty$ such that

$$\left(A_{i''}, \Phi_{i''} - m_{i''}\frac{\Phi_{i''}}{|\Phi_{i''}|}\right)$$

is C^∞_{loc}−convergent modulo gauge to a configuration of zero mass on $X \setminus Z$.

We note that, a priori, given a sequence $\{(A_i, \Phi_i)\}$ as in Theorem 3.1.1, Corollary 3.3.4 doesn't even give us a local uniform bound on (a subsequence of) $m_i^{-1}e(A_i, \Phi_i)$; in fact, the bound it produces goes to infinity like $O(m_i^3)$ as $i \to \infty$. One possible way to get around this problem is to explore Corollary 3.4.6 together with the Lie algebra structure of $\mathfrak{su}(2)$ to get an improved estimate on the Laplacian of the energy density outside Z. In what follows we give some results in this direction.

Suppose that E is an $\mathrm{SU}(2)$−bundle over (X^3, g). For each $p \in \mathbb{N}_0$, consider the bundle $\mathcal{E}^p := \Lambda^p T^*X \otimes \mathfrak{su}(2)_E$ of $\mathfrak{su}(2)_E$−valued p−forms on X with the natural metric induced by

g and the Ad-invariant metric on $\mathfrak{su}(2)_E$. Let $\Phi \in \Omega^0(X, \mathfrak{su}(2)_E)$ be a Higgs field on E. Then there is an orthogonal decomposition of $\mathcal{E}^p$ over the open subset $U := X \setminus \Phi^{-1}(0)$,

$$\mathcal{E}^p|_U = (\mathcal{E}^p)^{\parallel} \oplus (\mathcal{E}^p)^{\perp},$$

into parallel ($^{\parallel}$) and orthogonal ($^{\perp}$) components with respect to Φ as follows: if ξ is a $\mathfrak{su}(2)_E$−valued form then

$$\xi^{\parallel} := |\Phi|^{-2}\langle \xi, \Phi\rangle_{\mathfrak{su}(2)_E}\Phi,$$

and

$$\xi^{\perp} := \xi - \xi^{\parallel} = 4^{-1}|\Phi|^{-2}[[\xi, \Phi], \Phi].$$

Note that, for any $p, q \in \mathbb{N}_0$,

$$[(\mathcal{E}^p)^{\parallel}, (\mathcal{E}^q)^{\parallel}] = 0, \quad [(\mathcal{E}^p)^{\parallel}, (\mathcal{E}^q)^{\perp}] \subseteq \left(\mathcal{E}^{p+q}\right)^{\perp} \text{ and } \quad [(\mathcal{E}^p)^{\perp}, (\mathcal{E}^q)^{\perp}] \subseteq \left(\mathcal{E}^{p+q}\right)^{\parallel}.$$

Moreover, given $\xi \in \Omega^p(X, \mathfrak{su}(2)_E)$, $\eta \in \Omega^q(X, \mathfrak{su}(2)_E)$ and $\zeta \in \Omega^r(X, \mathfrak{su}(2)_E)$, one has

$$\langle [\xi, \eta], \zeta\rangle = \langle [\xi^{\perp}, \eta^{\perp}], \zeta^{\parallel}\rangle + \langle [\xi^{\perp}, \eta^{\parallel}], \zeta^{\perp}\rangle + \langle [\xi^{\parallel}, \eta^{\perp}], \zeta^{\perp}\rangle.$$

For each $p \in \mathbb{N}_0$, the operator $\mathrm{ad}(\Phi) = [\Phi, \cdot]$ restricted to $\mathfrak{su}(2)_E$−valued p−forms vanishes precisely on the forms parallel to Φ and its eigenvalues restricted to the purely orthogonal ones are $\pm 2i|\Phi|$; in particular, for all $\xi \in \Omega^p(X, \mathfrak{su}(2)_E)$,

$$|[\Phi, \xi]| \geq 2|\Phi||\xi^{\perp}|.$$

With the above observations in mind and using Lemma 2.2.1, we get the following Bochner estimate on the energy density of a monopole outside the Higgs field zero.

Corollary 3.8.2. *Let (X^3, g) be an oriented Riemannian 3−manifold and E be a $\mathrm{SU}(2)$−bundle over X. Suppose that (A, Φ) is a monopole on E and let $U := X \setminus \Phi^{-1}(0)$ be the complement of the zero set of Φ. Then, on U one has*

$$\Delta|\mathrm{d}_A\Phi|^2 \leq c|\mathrm{Ric}^g||\mathrm{d}_A\Phi|^2 + (c'|(\mathrm{d}_A\Phi)^{\parallel}| - 4|\Phi|^2)|(\mathrm{d}_A\Phi)^{\perp}|^2, \tag{3.8.3}$$

where $c, c' > 0$ are constants depending only on g and the structure constants of $\mathfrak{su}(2)$.

Equipped with Corollary 3.8.2, we can prove the following.

Proposition 3.8.4. *Under the hypothesis of Theorem 3.1.1, the following holds.*

(i) For each $x \in X \setminus Z$ we can find $r > 0$ and a subsequence $i'(x) \to \infty$ such that

$$\sup_{B_r(x)} m_{i'}^{-1} e(A_{i'}, \Phi_{i'}) \to 0 \quad \text{as} \quad i' \to \infty.$$

(ii) Write $Z = \{x_1, \ldots, x_l\}$, with $l \leq k$, and define the functions

$$w_i := \frac{1}{2}\left(m_i^2 - |\Phi_i|^2\right).$$

For each $r \in (0, r_0]$ such that the $B_r(x_\alpha)$ are pairwise disjoint, there is a subsequence $i_j(r) \to \infty$ such that

$$e_{i_j} \lesssim w_{i_j} \quad \text{on} \quad X \setminus \overline{B_r(Z)}. \tag{3.8.5}$$

Proof. (i) Suppose $x \in X \setminus Z$. Then there exists $r \in (0, r_0]$ and a subsequence $i' \to \infty$ such that

$$m_{i'}^{-1} \mathcal{E}_{B_r(x)}(A_{i'}, \Phi_{i'}) < \varepsilon_k.$$

By taking a further subsequence if necessary, we may assume that $m_{i'} > \max\{m_*, 2R_k r^{-1}, 2r^{-1}\}$ for all i'. In particular, by Corollaries 3.3.4 and 3.4.6, for all i' one has

$$\sup_{B_{\frac{r}{2}}(x)} |\mathrm{d}_{A_{i'}} \Phi_{i'}| \leq m_{i'}^2 C_0^{1/2} \varepsilon_k^{1/2} \quad \text{and} \tag{3.8.6}$$

$$\inf_{B_{\frac{r}{2}}(x)} |\Phi_{i'}| \geq \frac{m_{i'}}{4}. \tag{3.8.7}$$

Thus, on $B_{\frac{r}{2}}(x)$, the term $c'|\mathrm{d}_{A_{i'}} \Phi_{i'}| - 4|\Phi_{i'}|^2$ in the Bochner estimate (3.8.2) is nonpositive provided we suppose ε_k is sufficiently small. Thus, we get

$$\Delta e(A_{i'}, \Phi_{i'}) \leq c e(A_{i'}, \Phi_{i'}) \quad \text{on} \quad B_{\frac{r}{2}}(x). \tag{3.8.8}$$

With this improved (linear) bound on the Laplacian of the energy densities it follows from a standard mean value inequality (cf. Theorem A.3) that

$$\sup_{B_{\frac{r}{4}}(x)} m_{i'}^{-1} e(A_{i'}, \Phi_{i'}) \lesssim r^{-3} m_{i'}^{-1} \int_{B_r(x)} e(A_{i'}, \Phi_{i'}).$$

Now the convergence of measures given by Theorem 3.7.1 implies that

$$m_{i'}^{-1} \int_{B_r(x)} e(A_{i'}, \Phi_{i'}) \to 0 \quad \text{as} \quad i' \to \infty,$$

thus proving the desired result.

(ii) Using the above part (i) and a standard diagonal argument, given $r > 0$ as in the statement, there is a subsequence $i_j(r) \to \infty$ such that

$$\Delta e(A_{i_j}, \Phi_{i_j}) \leq ce(A_{i_j}, \Phi_{i_j}) \quad \text{on} \quad X \setminus \overline{B_r(Z)}. \tag{3.8.9}$$

Since $\Delta_{A_{i_j}} \Phi_{i_j} = 0$ and (A_{i_j}, Φ_{i_j}) has finite mass m_{i_j}, it follows that for each i_j we have

$$\Delta w_{i_j} = e(A_{i_j}, \Phi_{i_j}) =: e_{i_j} \tag{3.8.10}$$

and

$$\lim_{\rho \to \infty} w_{i_j} = 0. \tag{3.8.11}$$

Putting together (3.8.9) and (3.8.10) gives

$$\Delta(cw_{i_j} - e_{i_j}) \geq 0. \tag{3.8.12}$$

Now since both e_{i_j} (cf. Corollary 2.3.3) and w_{i_j} (cf. (3.8.11)) goes to zero as $\rho \to \infty$, inequality (3.8.5) follows from the Maximum Principle.

■

These facts may be helpful in establishing Conjecture 3.8.1, but at the time of writing it is not clear to the author how to prove it completely.

Chapter 4

Partial results on large mass monopoles on AC G_2-manifolds

We turn to the problem of analyzing sequences of large mass G_2-monopoles using analogous techniques of those employed in Chapter 3 (cf. [FO19]). In this higher dimensional case, new technicalities and difficulties arises. More specifically, a priori it is not so clear what should be the precise definition of the blow-up set and, in fact, a suitable $\varepsilon-$regularity result for the more relevant $\psi-$energy density requires a monotonicity formula for the renormalized $\psi-$energy and a Bochner estimate that in general are not available without further hypothesis. Moreover, one expects that the relevant blow-up set(s), including the zero set, have Hausdorff dimension 4 (codimension 3) and to treat the bubbling analysis on these one needs regularity results and techniques from geometric measure theory.

We start by proving the desired versions of the above mentioned results under a certain mild assumption, satisfied by, e.g., the examples in [Oli14b]. In Section 4.1 we give codimension 3 and 4 monotonicity formulas for the renormalized $\psi-$energy on small geodesic balls as a consequence of the general formula in §2.1. Next, in Section 4.2 we give the G_2-monopoles version of the Bochner–Weitzenböck formula of §2.2 and its consequent estimates, including a version of the estimate outside the zero set of the Higgs field for $\mathrm{SU}(2)-$bundles analogous to the one given by Corollary 3.8.2 in $3-$dimensions. As a consequence of these results, in Section 4.3 we derive appropriate $\varepsilon-$regularities for the $\psi-$energy density.

With the previous results at hand, we then proceed to prove a number of properties concerning the behavior of sequences (A_i, Φ_i) of G_2-monopoles of arbitrarily large mass and fixed monopole class, satisfying mild assumptions. We begin in Section 4.5 defining a notion of blow-up set S in this context, which corresponds to the set of points where the $\psi-$energy of the sequence concentrates, and as in the $3-$dimensional case we define the zero set Z as the limiting set of the Higgs fields zeros. We prove that $Z \subseteq S$ and that Z has finite $\mathcal{H}^4-$measure. After passing to a subsequence if necessary, the Radon measures $\mu_i^\psi := m_i^{-1} e^\psi(A_i, \Phi_i)\mathcal{H}^7$ have a weak* limit $\mu^\psi = e_\infty^\psi \mathcal{H}^7 + \nu$, where $e_\infty^\psi := \liminf_{i\to\infty} m_i^{-1} e^\psi(A_i, \Phi_i)$ and $\nu \ll \mathcal{H}^4 \lfloor S$ is some (nonnegative) Radon measure singular with respect to $\mathcal{H}^7$. We begin Section 4.6 showing that the $4-$dimensional density function Θ^ψ of μ^ψ exists, is bounded, upper semi-continuous and its support equals S. We then proceed to show that S decomposes as:

$$S = Q \cup \operatorname{sing}(e_\infty^\psi),$$

where $Q := \operatorname{supp}(\nu)$ and $\operatorname{sing}(e_\infty^\psi)$ is the support of the $4-$dimensional upper density of $e_\infty^\psi \mathcal{H}^7$. Further, $\operatorname{sing}(e_\infty^\psi)$ is shown to have zero $\mathcal{H}^4-$measure. We end this section showing a first regularity result for the blow-up set: Q is a $\mathcal{H}^4-$*rectifiable* set, i.e. at $\mathcal{H}^4-$a.e. $x \in Q$ the approximate $4-$dimensional tangent space T_xQ exists, and ν can be written as $\nu = \Theta^\psi \mathcal{H}^4 \lfloor Q$.

We end this chapter with Section 4.7, where we start the analysis of the behavior of blow-up configurations of (A_i, Φ_i) for $i \gg 1$ at a *smooth point* $x \in Q$, i.e. T_xQ exists and $x \notin \operatorname{sing}(e_\infty^\psi)$. We show an asymptotic translation invariance result and finish stating some conjectures on the possible outcomes of a still lacking complete analysis.

4.1 Monotonicity for the renormalized $\psi-$energy

Let (X^7, g) be a G_2-manifold of bounded geometry and let E be a $G-$bundle over X. Fixing a G_2-monopole (A, Φ) on E, a point $x \in X$ and $k \in \mathbb{N}_0$, define functions $\theta_k, \theta_k^\psi : (0, r_0] \to [0, \infty)$ and $\theta_k^\phi : (0, r_0] \to \mathbb{R}$ by

$$\begin{aligned}
\theta_k(r) &:= e^{cr^2} r^{-k} \mathcal{E}_{B_r(x)}(A, \Phi) \\
\theta_k^\psi(r) &:= e^{cr^2} r^{-k} \mathcal{E}^\psi_{B_r(x)}(A, \Phi) \\
\theta_k^\phi(r) &:= \frac{e^{cr^2} r^{-k}}{2} \int_{B_r(x)} (|\pi_{14}(F_A)|^2 - 2|\pi_7(F_A)|^2) \\
&= -\frac{e^{cr^2} r^{-k}}{2} \int_{B_r(x)} \langle F_A \wedge F_A \rangle \wedge \phi.
\end{aligned}$$

Note that, by the energy identity (1.3.10), we have

$$\theta_k = \theta_k^\phi + \theta_k^\psi. \tag{4.1.1}$$

In what follows, we want to prove monotonicity formulas for θ_3^ψ and θ_4^ψ. We shall consider the following additional assumption:

$$\left\| |\pi_{14}(F_A)|^2 - 2|\pi_7(F_A)|^2 \right\|_{L^\infty(X)} \lesssim 1. \tag{4.1.2}$$

Then we have:

Proposition 4.1.3 (Monotonicity for the renormalized $\psi-$energy under assumption (4.1.2))**.** *Let (A,Φ) be a G_2-monopole on E satisfying (4.1.2). Then for all $x \in X$ and $0 < s \leq r \leq r_0$ one has*

$$e^{cr^2} r^{-3} \mathcal{E}^\psi_{B_r(x)}(A,\Phi) - e^{cs^2} s^{-3} \mathcal{E}^\psi_{B_s(x)}(A,\Phi) \geq \int_{A_{r,s}(x)} e^{c\rho^2} \rho^{-3} \left(|\iota_{\partial_\rho} F_A|^2 + |\iota_{\partial_\rho} \mathrm{d}_A \Phi| \right) - c(r^4 - s^4). \tag{4.1.4}$$

and

$$e^{cr^2} r^{-4} \mathcal{E}^\psi_{B_r(x)}(A,\Phi) - e^{cs^2} s^{-4} \mathcal{E}^\psi_{B_s(x)}(A,\Phi) \geq \int_{A_{r,s}(x)} e^{c\rho^2} \rho^{-4} \left(|\iota_{\partial_\rho} F_A|^2 + |\iota_{\partial_\rho} \mathrm{d}_A \Phi|^2 \right) - c(r^3 - s^3). \tag{4.1.5}$$

Proof. Note that assumption (4.1.2) implies $\theta_k^\phi(r) = O(r^{7-k})$. The result then readily follows by combining the general monotonicity formula (2.1.3) of Theorem 2.1.2 for $n = 7$ with the identity (4.1.1). ∎

4.2 Bochner–Weitzenböck formula and estimates (G_2-case)

The following results are direct consequences of the formulas in Lemma 2.2.1.

Lemma 4.2.1 (Bochner-Weitzenböck for G_2-monopoles)**.** *If (X^7, ψ^4, g) is a G_2-manifold and (A,Φ) is a G_2-monopole then*

$$\begin{aligned} \frac{1}{2}\Delta|\mathrm{d}_A\Phi|^2 = &- \frac{2}{3}\langle * \left([\mathrm{d}_A\Phi, \mathrm{d}_A\Phi] \wedge \psi \right), \mathrm{d}_A\Phi \rangle - 2\langle *[*\pi_{14}(F_A), \mathrm{d}_A\Phi], \mathrm{d}_A\Phi\rangle \\ &- |[\mathrm{d}_A\Phi, \Phi]|^2 - |\nabla_A(\mathrm{d}_A\Phi)|^2. \end{aligned} \tag{4.2.2}$$

In particular,

$$\Delta e^\psi(A,\Phi) \lesssim |\pi_{14}(F_A)| e^\psi(A,\Phi) + e^\psi(A,\Phi)^{3/2}. \tag{4.2.3}$$

Proof. This follows from Lemma 2.2.1 observing that $\mathrm{Ric}^g \equiv 0$ for a G_2-manifold and that $*3\pi_7(F_A) = \mathrm{d}_A\Phi \wedge \psi$ for a G_2-monopole. ∎

Corollary 4.2.4 (Bochner type estimate under assumption (4.1.2))**.** *Let (X^7, ψ, g) be a G_2−manifold and E be a G−bundle over X. If (A, Φ) is a G_2−monopole on E satisfying assumption (4.1.2), then*

$$\Delta e^{\psi}(A, \Phi) \lesssim e^{\psi}(A, \Phi) + e^{\psi}(A, \Phi)^{3/2}. \tag{4.2.5}$$

Proof. Direct consequence of Lemma 4.2.1 noting that (4.1.2) implies $|\pi_{14}(F_A)| \lesssim 1 + e^{\psi}(A, \Phi)$. ∎

Restricting attention to $\mathrm{SU}(2)$−bundles and recalling the discussion on §3.8, by Lemma 4.2.1 have also have the following Bochner estimate on the energy density of a G_2−monopole outside the Higgs field zero.

Corollary 4.2.6. *Let (X^7, g) be G_2−manifold and E be a $\mathrm{SU}(2)$−bundle over X. Suppose that (A, Φ) is a G_2−monopole on E and let $U := X \setminus \Phi^{-1}(0)$ be the complement of the zero set of Φ. Then on U one has*

$$\begin{aligned}\Delta|\mathrm{d}_A\Phi|^2 \leq \left(c_0|(\mathrm{d}_A\Phi)^{\parallel}| + c_1|(\pi_{14}(F_A))^{\parallel}| - 4|\Phi|^2\right)|(\mathrm{d}_A\Phi)^{\perp}|^2 \\ + c_2|(\pi_{14}(F_A))^{\perp}||(\mathrm{d}_A\Phi)^{\perp}||(\mathrm{d}_A\Phi)^{\parallel}|,\end{aligned} \tag{4.2.7}$$

where the $c_i > 0$, $i = 0, 1, 2$, are constants depending only on g and the structure constants of $\mathfrak{su}(2)$.

4.3 ε−regularity for the ψ−energy density

We are now able to prove ε−regularity results for the ψ−energy density of G_2−monopoles satisfying assumption (4.1.2) using the monotonicity of Proposition 4.1.3 and the Bochner estimate of Corollary 4.2.4.

Theorem 4.3.1. *Let (X^7, ψ, g) be an oriented G_2−manifold of bounded geometry and let E be a G−bundle over X. Then there are constants $\varepsilon^{\psi} > 0$ and $C^{\psi} > 0$ such that the following holds. Let (A, Φ) be a G_2−monopole satisfying assumption (4.1.2). If $x \in X$ and $0 < r \leq r_0$ are such that*

$$\varepsilon := r^{-3}\mathcal{E}^{\psi}_{B_r(x)}(A, \Phi) < \varepsilon^{\psi},$$

then

$$\sup_{B_{\frac{r}{2}}(x)} e^{\psi}(A, \Phi) \leq C^{\psi}(r^{-4}\varepsilon + 1). \tag{4.3.2}$$

Proof. We apply the mean value inequality of Theorem A.3 for $f := e^{\psi}(A,\Phi)$, $d = 4$, $\tau(r) = cr^4$, $a_0 = 0$ and $0 < a_1, a \lesssim 1$. Indeed, with this setup, the monotonicity formula of Proposition 4.1.3 implies (A.4), and the Bochner estimate of Corollary 4.2.4 implies (A.5) with critical exponent $\alpha = 3/2 = (d+2)/2$. Therefore, noting that $1 \lesssim r^{-4}$, it follows from (A.7) that there is $\varepsilon^{\psi} := \hbar a^{-2} > 0$ and $C^{\psi} > 0$ (depending only on the geometry of (X^7, g) and the structure constants of $\mathfrak{g}$) such that if $\varepsilon + \tau < \varepsilon^{\psi}$ then (4.3.2) holds. Now since $\tau(r) = cr^4$ and $r \leq r_0$, by choosing r_0 smaller (depending only on the geometry) we may assume that $\tau(r_0) < 4^{-1}\varepsilon^{\psi}$, so that replacing ε^{ψ} by $2^{-1}\varepsilon^{\psi}$ we get the desired result. ∎

As in the 3−dimensional case, in order to analyze sequences of large mass G_2−monopoles the following version of the ε−regularity will come in handy.

Theorem 4.3.3 (ε−regularity for the ψ−energy density under assumption (4.1.2))**.** *Let (X^7, ψ, g) be an oriented G_2−manifold of bounded geometry and let E be a G−bundle over X. Then there are constants $\varepsilon^{\psi} > 0$ and $C^{\psi} > 0$ such that the following holds. Let (A, Φ) be a G_2−monopole satisfying assumption (4.1.2). For any $m \geq 1$ and $R > 0$, if $x \in X$ and $0 < r \leq \min\{Rm^{-1}, r_0\}$ are such that*

$$\varepsilon := m^{-1}r^{-4}\mathcal{E}^{\psi}_{B_r(x)}(A,\Phi) < \varepsilon^{\psi},$$

then

$$\sup_{B_{\frac{r}{4}}(x)} m^{-1}e^{\psi}(A,\Phi) \leq C^{\psi}_R \left(r^{-3}\varepsilon + m^{-1}\right), \tag{4.3.4}$$

where $C^{\psi}_R := C^{\psi}\max\{1, R^3\}$.

Proof. We give a direct proof, also based on the Heinz trick (like the proof of Theorem A.3) but more on the spirit of [Wal17a, Appendix A]. Consider the function $\theta \colon \overline{B}_{r/2}(x) \to [0,\infty)$ given by

$$\theta(y) := \left(\frac{r}{2} - d(x,y)\right)^3 m^{-1}e^{\psi}(A,\Phi)(y).$$

By continuity, θ attains a maximum. Since θ is non-negative and vanishes on the boundary $\partial B_{r/2}(x)$, it achieves its maximum

$$M := \max_{\overline{B}_{r/2}(x)} \theta$$

in the interior of $B_{r/2}(x)$. We will derive a bound for M of the form $M \lesssim \max\{1, R^3\}(\varepsilon + m^{-1}r^3)$, from which the assertion of the theorem follows. Let $y_0 \in B_{r/2}(x)$ be a point with $\theta(y_0) = M$, set

$$e_0 := m^{-1}e^{\psi}(A,\Phi)(y_0)$$

and

$$s_0 := \frac{1}{2}\left(\frac{r}{2} - d(x, y_0)\right),$$

Note that

$$y \in B_{s_0}(y_0) \quad \Rightarrow \quad \left(\frac{r}{2} - d(x, y)\right) \geq s_0.$$

Therefore,

$$y \in B_{s_0}(y_0) \quad \Rightarrow \quad m^{-1}e^{\psi}(A, \Phi)(y) \leq s_0^{-3}\theta(y) \leq s_0^{-3}\theta(y_0) \lesssim e_0.$$

In particular, from the Bochner estimate (4.2.5) that

$$\Delta(m^{-1}e^{\psi}) \lesssim (m^{-1}e^{\psi}) + m^{1/2}(m^{-1}e^{\psi})^{3/2} \lesssim e_0 + m^{1/2}e_0^{3/2} \quad \text{on} \quad B_{s_0}(y_0).$$

Now by Lemma A.1 we have

$$e_0 \lesssim s^{-7}\int_{B_s(y_0)} m^{-1}e^{\psi}(A, \Phi)\mathrm{vol} + s^2(e_0 + m^{1/2}e_0^{3/2}), \quad \forall s \leq s_0.$$

Then the almost monotonicity of Proposition 4.1.3 implies

$$e_0 \lesssim s^{-3}\varepsilon + s^{-3}m^{-1}r^3 + s^2(e_0 + m^{1/2}e_0^{3/2}),$$

which we rewrite as

$$s^3 e_0 \lesssim \varepsilon + m^{-1}r^3 + s^5(e_0 + m^{1/2}e_0^{3/2}). \tag{4.3.5}$$

Since $m \geq 1$, (4.3.5) implies

$$s^3 e_0 \lesssim \varepsilon + m^{-1}r^3 + s^5(m^2 e_0 + m^{1/2}e_0^{3/2}). \tag{4.3.6}$$

We split into two cases.

<u>Case (i).</u> $m^2 e_0 \geq m^{1/2}e_0^{3/2}$ ($\iff$ $m^3 \geq e_0$): in this case, for each $s \leq s_0$, it follows from (4.3.6) that

$$s^3 e_0 \leq \frac{c(\varepsilon + m^{-1}r^3)}{1 - cm^2 s^2}. \tag{4.3.7}$$

Thus, if $cm^2 s_0^2 \leq 1/2$ we get

$$M = \theta(y_0) \lesssim s_0^3 e_0 \lesssim \varepsilon + m^{-1}r^3.$$

Otherwise, setting $s := (2c)^{-1/2}m^{-1} < s_0$ and plugging into (4.3.7) yields

$$e_0 \lesssim m^3(\varepsilon + m^{-1}r^3).$$

Since $s_0 \leq r$ and by hypothesis $r \leq Rm^{-1}$, we conclude that

$$M \lesssim s_0^3 e_0 \leq R^3(\varepsilon + m^{-1}r^3).$$

Case (ii). $m^{1/2}e_0^{3/2} > m^2 e_0$ ($\iff$ $m^3 < e_0$): in this case, $m^{1/2} < e^{1/6}$ and thus $e_0^{5/3} > m^{1/2}e_0^{3/2} > m^2 e_0$, so that from (4.3.6) we derive

$$s^3 e_0 \leq c(\varepsilon + m^{-1}r^3) + cs^5 e_0^{5/3} = c(\varepsilon + m^{-1}r^3) + s^3 e_0 c(s^2 e_0^{2/3}), \quad \forall s \leq s_0. \tag{4.3.8}$$

Thus, setting $t = t(s) := se_0^{1/3}$, the inequality (4.3.8) can be expressed as

$$t^3(1 - ct^2) \leq c(\varepsilon + m^{-1}r^3).$$

Now we can choose $\varepsilon_\psi > 0$ sufficiently small, where the smallness depends only on c, which in turn depends only on g and G, so that for $\varepsilon \leq \varepsilon_\psi$ the corresponding equation $t^3(1 - ct^2) = c(\varepsilon + m^{-1}r^3)$ has three small (real) roots t_1, t_2, t_3, which are approximately $\pm(c\varepsilon + cm^{-1}r^3)^{1/3}$, and two large (complex) roots. Since $t(0) = 0$ and t is continuous, for each $s \in [0, s_0]$, $t(s)$ must be less than the smallest positive (real) root. Therefore, $t(s) \lesssim (\varepsilon + m^{-1}r^3)^{1/3}$ for all $s \in [0, s_0]$; in particular, $M \lesssim s_0^3 e_0 \lesssim \varepsilon + m^{-1}r^3$. This finishes the proof. ∎

Remark 4.3.9. We conjecture that Theorem 4.3.3 should be a consequence of Theorem 4.3.1. Also note that the first imply the case $r \in (0, \min\{1, r_0\}]$ of the later by setting $m = r^{-1}$ and $R = 1$.

4.4 An interior lower bound on the Higgs field (G_2 case)

Let (X^7, ψ, g) be a G_2-manifold of bounded geometry and E be a $G-$bundle over X. Given real parameters $\delta \in (0, 1)$ and $R > 0$, we define

$$\varepsilon^\psi_{\delta,R} := \min\{(C^\psi_R)^{-1}R\delta^2, \varepsilon^\psi\}, \tag{4.4.1}$$

where $\varepsilon^\psi > 0$ and $C^\psi_R > 0$ are the constants given by Theorem 4.3.3.

A simple argument using the fundamental theorem of calculus together with Theorem 4.3.3 proves the following.

Proposition 4.4.2. *Let $\delta \in (0, 1)$, $m \geq 1$, $R > 0$ be real parameters and $(A, \Phi) \in \mathcal{C}(E)$ be a G_2-monopole satisfying assumption (4.1.2). If $r := Rm^{-1} \leq r_0$ and $x \in X$ are such that*

$$\sup_{\overline{B}_{\frac{r}{4}}(x)} |\Phi| \geq m\delta, \tag{4.4.3}$$

then

$$m^{-1}r^{-4}\mathcal{E}^{\psi}_{B_r(x)}(A,\Phi) < \varepsilon^{\psi}_{\delta,R} \quad \Longrightarrow \quad |\Phi| > \frac{m\delta}{2} - \frac{R(C^{\psi}_R)^{\frac{1}{2}}}{2m} \quad \textit{on} \quad B_{\frac{r}{4}}(x). \tag{4.4.4}$$

Proof. Fix $q \in \partial\overline{B}_{\frac{r}{4}}(x)$ such that the restriction of $|\Phi|$ to $\partial\overline{B}_{\frac{r}{4}}(x)$ attains its maximum at q. For any $p \in B_{\frac{r}{4}}(x)$ we can choose a (smooth by parts) path γ_p in $B_{\frac{r}{4}}(x)$ with length $L(\gamma_p) \leq \frac{r}{2}$ joining p to q. Thus, using the fundamental theorem of calculus and Kato's inequality we get:

$$|\Phi|(q) - |\Phi|(p) \leq \left| \int_{\gamma_p} \mathrm{d}|\Phi| \right| \leq \int_{\gamma_p} |\mathrm{d}_A\Phi| \leq \frac{r}{2} \sup_{B_{\frac{r}{4}}(x)} |\mathrm{d}_A\Phi|.$$

Using the assumption (4.4.3), it follows that

$$|\Phi|(p) \geq m\delta - \frac{r}{2} \sup_{B_{\frac{r}{4}}(x)} |\mathrm{d}_A\Phi|, \quad \forall p \in B_{\frac{r}{4}}(x). \tag{4.4.5}$$

On the other hand, using the hypothesis, Theorem 4.3.3 gives the following:

$$\frac{r}{2} \sup_{B_{\frac{r}{4}}(x)} |\mathrm{d}_A\Phi| \leq \frac{1}{2}(C^{\psi}_R)^{1/2} r^{-1/2} m^{1/2} (\varepsilon^{\psi}_{\delta,R})^{1/2} + \frac{r}{2}(C^{\psi}_R)^{1/2} = \frac{m\delta}{2} + \frac{R(C^{\psi}_R)^{1/2}}{2m}. \tag{4.4.6}$$

Putting (4.4.5) and (4.4.6) together completes the proof. ∎

Combining the above with Theorem 2.4.1, we get:

Theorem 4.4.7. *Let (X^7, ψ, g) be an AC G_2-manifold with one end, E be an $\mathrm{SU}(2)-$bundle over X and $k \in \mathbb{N}$. Then there are positive constants $R^{\psi}_k, \varepsilon^{\psi}_k \leq \varepsilon^{\psi}$ and c^{ψ}_k such that the following holds. Let $(A,\Phi) \in \mathcal{C}(E)$ be a finite mass G_2-monopole over (X^7,ψ,g) with fixed monopole class $c_1(L)$ such that $k := \langle c_1(L) \cup [\iota^*\psi], [N] \rangle \neq 0$, mass $m \gg_{g,k} 1$ and satisfying both the energy formula (1.3.24) and assumption (4.1.2). Then for $r := R^{\psi}_k m^{-1}$ one has*

$$m^{-1}r^{-4}\mathcal{E}^{\psi}_{B_r(x)}(A,\Phi) < \varepsilon^{\psi}_k \quad \Longrightarrow \quad |\Phi| > \frac{m}{4} - m^{-1}c^{\psi}_k \quad \textit{on} \quad B_{\frac{r}{4}}(x). \tag{4.4.8}$$

Proof. Let

$$\begin{aligned} R^{\psi}_k &:= ck, \\ \varepsilon^{\psi}_k &:= \varepsilon^{\psi}_{1/2,R^{\psi}_k} = \min\{4^{-1}(C^{\psi}_{R^{\psi}_k})^{-1}R^{\psi}_k, \varepsilon^{\psi}\}, \quad \text{and} \\ c^{\psi}_k &:= 2^{-1}R^{\psi}_k(C^{\psi}_{R^{\psi}_k})^{1/2}. \end{aligned}$$

Here we choose $c > 0$ depending only on g big enough so that it follows from Theorem 2.4.1 with $\delta := 1/2$ and $\Lambda = 4\pi k$ that

$$\sup_{\partial\overline{B}_{\frac{r}{4}}(x)} |\Phi| \geq \frac{m}{2}.$$

Therefore, by hypothesis, we can apply Proposition 4.4.2 with $\delta = 1/2$ and $R = R^{\psi}_k$ to get the desired result. ∎

4.5 The blow-up set and the zero set (G_2 case)

Let E be an SU(2)−bundle over an AC G$_2$−manifold (X^7, ψ, g). From now on we shall consider a sequence $\{(A_i, \Phi_i)\}_{i\in\mathbb{N}}$ of finite mass G$_2$−monopoles on E with fixed monopole class $c_1(L)$ such that $k := \langle c_1(L) \cup [\iota^*\psi], [N]\rangle \neq 0$ and masses m_i satisfying $\limsup m_i = \infty$; in fact, by taking a subsequence if necessary, we may assume that $m_i \uparrow \infty$. Finally, we shall suppose that the (A_i, Φ_i) satisfy the energy formula (1.3.24) (e.g., they satisfy the conditions of Proposition 1.3.23), so that

$$\mathcal{E}_X^{\psi}(A_i, \Phi_i) = 4\pi k m_i, \quad \forall i \in \mathbb{N}. \tag{4.5.1}$$

In order to study the behavior of such sequence of infinitely large mass G$_2$−monopoles, we shall consider the corresponding sequence of Radon measures

$$\mu_i^{\psi} := m_i^{-1} e^{\psi}(A_i, \Phi_i)\mathcal{H}^7 \tag{4.5.2}$$

These are finite and their total mass are uniformly bounded by the energy formula (4.5.1), which yields $\mu_i^{\psi}(X) = 4\pi k$. Thus, after passing to a subsequence which we do not relabel, they weakly converge to a positive Radon measure μ^{ψ}. By Fatou's lemma and Riesz representation theorem, we can write

$$\mu_i^{\psi} \rightharpoonup \mu^{\psi} = e_{\infty}^{\psi}\mathcal{H}^7 + \nu, \tag{4.5.3}$$

where $e_{\infty}^{\psi} : X \to [0, \infty]$ is the L^1−function

$$e_{\infty}^{\psi} := \liminf_{i\to\infty} m_i^{-1} e^{\psi}(A_i, \Phi_i)$$

and ν is singular with respect to $\mathcal{H}^7$; ν is called the *defect measure*.

We set

$$\mathcal{R}_x^{\psi} := \{r \in (0, r_0] : \mu^{\psi}(\partial B_r(x)) > 0\}.$$

For all $r \in (0, r_0] \setminus \mathcal{R}_x^{\psi}$ one has

$$\mu^{\psi}(B_r(x)) = \lim_{i\to\infty} \mu_i^{\psi}(B_r(x)).$$

Since μ^{ψ} is locally finite, the set $\mathcal{R}_x^{\psi}$ is at most countable.

Fixed the above context, we now make the following definitions.

Definition 4.5.4 (Blow-up and zero sets)**.** The **blow-up set** is defined by

$$S := \bigcup_{\varepsilon>0} S_{\varepsilon},$$

where

$$S_\varepsilon := \bigcap_{0<r\leq r_0} \left\{ x \in X : \liminf_{i\to\infty} m_i^{-1} e^{cr^2} r^{-4} \int_{B_r(x)} |F_{A_i} \wedge \psi|^2 \geq \varepsilon \right\}.$$

The **zero set** is the accumulation points of the Higgs fields zeros:

$$Z := \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} \Phi_i^{-1}(0)}.$$

Note that $Z \subseteq X$ is a closed subset.

For the rest of this chapter, we shall work under the following:

Assumption 1: The (A_i, Φ_i) satisfy assumption (4.1.2) uniformly, i.e.

$$\left\| |\pi_{14}(F_{A_i})|^2 - 2|\pi_7(F_{A_i})|^2 \right\|_{L^\infty(X)} \lesssim 1, \text{ for all } i \geq 1. \tag{4.5.5}$$

Example 4.5.6. We note that the only known nontrivial finite mass G_2-monopoles on AC manifolds, namely the invariant G_2-monopoles on the Bryant–Salamon manifolds $\Lambda^2_-(\mathbb{CP}^2)$ and $\Lambda^2_-(\mathbb{S}^4)$, were constructed in [Oli14b] and they satisfy assumption (4.5.5). For instance, recall the following:

Theorem 4.5.7 ([Oli14b])**.** *The moduli space $\mathcal{M}_{inv}(\Lambda^2_-(\mathbb{CP}^2), P) = \mathcal{M}_{inv}$ of $\mathrm{SU}(3)-$invariant irreducible G_2-monopoles on a homogeneous principal $\mathrm{SU}(2)-$bundle P over $\Lambda^2_-(\mathbb{CP}^2)$ is non empty and the following hold:*

1. *For all $(A, \Phi) \in \mathcal{M}_{inv}$, $\Phi^{-1}(0) = \mathbb{CP}^2$ is the zero section and the mass gives a bijection*

$$m \colon \mathcal{M}_{inv} \to \mathbb{R}^+.$$

Let $(A_m, \Phi_m) \in \mathcal{M}_{inv}$ be a monopole with mass $m \in \mathbb{R}^+$, there is a gauge in which

$$(A_m, \Phi_m) = \left(A^c_{1,-1} + f^2 b_m \left(\nu_1 \otimes T_2 + \nu_2 \otimes T_3\right), \phi_m T_1\right),$$

with $b_m(0) = 1$ and $\phi_m(0) = 0$. In this gauge, the curvature of the connection A_m is

$$\begin{aligned} F_{A_m} = {} & \frac{d}{d\rho}\left(f^2 b_m\right)\left(T_2 \otimes d\rho \wedge \nu_1 + T_3 \otimes d\rho \wedge \nu_3\right) + f^2 b_m \left(T_2 \otimes \Omega_2 + T_3 \otimes \Omega_3\right) \\ & + \left(2\left(f^4 b_m^2 - 1\right)\nu_{12} + \Omega_1\right) \otimes T_1. \end{aligned} \tag{4.5.8}$$

2. *Let $R > 0$, and $\{(A_\lambda, \Phi_\lambda)\}_{\lambda\in[\Lambda,+\infty)} \subset \mathcal{M}_{inv}$ be a sequence of monopoles with masses $\lambda \to \infty$. Then there is a sequence $\eta(\lambda, R)$ converging to 0 as $\lambda \to \infty$ such that for all $x \in \mathbb{CP}^2$*

$$\exp^*_\eta(A_\lambda, \eta\Phi_\lambda)|_{\Lambda^2_-(\mathbb{CP}^2)_x}$$

converges uniformly to the BPS monopole (A^{BPS}, Φ^{BPS}) *in the ball of radius* R *in* $(\mathbb{R}^3, g_E)$*. Here* $\exp_\eta$ *denotes the exponential map along the fibre* $\Lambda^2_-(\mathbb{CP}^2)_x \cong \mathbb{R}^3$*.*

3. *Let* $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)}$ *be the sequence above. Then the translated sequence*

$$\left(A_\lambda, \Phi_\lambda - \lambda \frac{\Phi_\lambda}{|\Phi_\lambda|}\right),$$

converges uniformly with all derivatives to a reducible, singular monopole on $\Lambda^2_-(\mathbb{CP}^2)$ *with zero mass and which is smooth on* $\Lambda^2_-(\mathbb{CP}^2) \setminus \mathbb{CP}^2$*.*

Write the curvature of (A_m, Φ_m) as $F_m = F_1 \otimes T_1 + \ldots$, then

$$\begin{aligned}\langle F_m \wedge F_m \rangle \wedge \phi &= \langle F_1 \wedge F_1 \rangle \wedge \phi + \ldots \\ &= \left(8 f^4 b_m^2 (s^2 f^2 + 2g^2) + 4s^2 f^2 - 16 g^2\right) d\rho \wedge \nu_{12} \wedge \Omega_1^2.\end{aligned}$$

Now, recall that $b_m(0) = 1$ and b_m is decreasing so the above is uniformly bounded independently of m, i.e.

$$|\langle F_m \wedge F_m \rangle \wedge \phi| \lesssim 1.$$

Our first result relates the blow-up set S with the zero set Z. In the following statement, we shall use $\mathcal{H}^4$ to denote the Hausdorff $4-$dimensional measure.

Theorem 4.5.9. *Under the above conditions, the following holds:*

(i) $\mathcal{H}^4(S_\varepsilon) \leq 5^4 e^c 4\pi k \varepsilon^{-1}$*, for all* $\varepsilon > 0$*; in particular,* S *has Hausdorff dimension at most* 4*.*

(ii) Let $0 < \varepsilon_k^\psi \leq \varepsilon^\psi$ *be as in Theorem 4.4.7. Then:*

$$Z \subseteq S_{\varepsilon_k^\psi}.$$

In particular, the zero set in contained in the blow-up set *and* $\mathcal{H}^4(Z) < \infty$.

Proof. We prove each item separately:

(i) Fix $\varepsilon > 0$. Given $0 < r \leq \min\{1, r_0\}$, we can find a countable open covering $\{B_{5r_l}(x_l)\}$ of S_ε with $x_l \in S_j$, $10 r_l < r$ and $B_{r_l}(x_l)$ pairwise disjoint. Then

$$\begin{aligned}\sum_l (5r_l)^4 &\leq 5^4 e^c \varepsilon^{-1} \sum_l \liminf_{i\to\infty} m_i^{-1} \mathcal{E}^\psi_{B_{r_l}(x_l)}(A_i, \Phi_i) \quad (x_l \in S_j) \\ &\leq 5^4 e^c \varepsilon^{-1} \liminf_{i\to\infty} m_i^{-1} \sum_l \mathcal{E}^\psi_{B_{r_l}(x_l)}(A_i, \Phi_i) \quad \text{(by Fatou's lemma)} \\ &\leq 5^4 e^c \varepsilon^{-1} \liminf_{i\to\infty} m_i^{-1} \mathcal{E}^\psi_X(A_i, \Phi_i) \quad \text{(the } B_{r_l}(x_l)\text{'s are disjoint)} \\ &= 5^4 e^c 4\pi k \varepsilon^{-1}. \quad \text{(by the energy formula (1.3.24))}\end{aligned}$$

Since this bound is uniform in $r \in (0, r_0]$, the result follows.

(ii) We shall apply Theorem 4.4.7. Let $x \in X \setminus S_{\varepsilon_k^\psi}$. Then there is $r \in (0, r_0]$ such that

$$\liminf_{i\to\infty} m_i^{-1} r^{-4} \mathcal{E}^\psi_{B_r(x)}(A_i, \Phi_i) < \varepsilon_k^\psi.$$

Now note that we may assume $r \notin \mathcal{R}_x$; otherwise, by an approximation argument of taking an increasing sequence of radii $r_j \uparrow r$ with $r_j \notin \mathcal{R}_x$ we can replace r by some r_j for large j and work in a smaller ball for which the above energy bound still holds. In particular, it follows that there is $i_0 \in \mathbb{N}$ such that

$$m_i^{-1} r^{-4} \mathcal{E}^\psi_{B_r(x)}(A_i, \Phi_i) < \varepsilon_k^\psi, \quad \forall i \geq i_0.$$

Since $m_i \uparrow \infty$, by increasing i_0 if necessary we may also assume that

$$m_i \gg_{g,k} 1 \quad \text{and} \quad r_i := R_k^\psi m_i^{-1} < \frac{r}{2}, \quad \forall i \geq i_0.$$

Hence, given any $y \in B_{r/2}(x)$, it follows that

$$m_i^{-1} r_i^{-4} \mathcal{E}^\psi_{B_{r_i}(y)}(A_i, \Phi_i) < \varepsilon_k^\psi, \quad \forall i \geq i_0,$$

so that applying Theorem 4.4.7 we get that

$$|\Phi_i|(y) > \frac{m_i}{4} - m_i^{-1} c_k^\psi, \quad \forall i \geq i_0.$$

Therefore, by possibly increasing i_0, we have

$$\inf_{B_{\frac{r}{2}}(x)} |\Phi_i| \geq 100 > 0, \quad \forall i \geq i_0.$$

In particular, it follows that $x \in X \setminus Z$. We are done.

■

4.6 Decomposition and rectifiability of the blow-up set

We continue the analysis of the last section under the same hypothesis.

Lemma 4.6.1. *For every $x \in X$ and $0 < s \leq r \leq r_0$,*

$$e^{cs^2} s^{-4} \mu^\psi(B_s(x)) \leq e^{cr^2} r^{-4} \mu^\psi(B_r(x)).$$

Proof. Fix $x \in X$. If $r \notin \mathcal{R}_x^\psi$, then the result immediately follows from Proposition 4.1.3. As for the case where $r \in \mathcal{R}_x$, we proceed by an approximation argument. Since μ^ψ is a locally finite measure, the set $\mathcal{R}_x^\psi$ is countable. Thus, we can pick a sequence $\{r_j\} \subset (s, r) \setminus \mathcal{R}_x^\psi$ with $r_j \uparrow r$. Then, by dominated convergence

$$\mu^\psi(B_r(x)) = \lim_{j\to\infty} \mu^\psi(B_{r_j}(x)),$$

and the result follows from the first part. ∎

Proposition 4.6.2 (The $4-$dimensional density Θ^ψ)**.** *The limit*

$$\Theta^\psi(x) := \lim_{r\downarrow 0} r^{-4}\mu^\psi(B_r(x))$$

exists for all $x \in X$ and is bounded by $e^{cr_0^2}4\pi k r_0^{-4}$. The function $\Theta^\psi : X \to [0, \infty)$ is upper semicontinuous and $\mathrm{supp}(\Theta^\psi) = S$.

Proof. The first part is a direct consequence of Lemma 4.6.1 and the energy formula (1.3.24).

To show upper semicontinuity, let $\{x_n\} \subseteq X$ be a sequence of points converging to $x \in X$. Let $r \notin \mathcal{R}_x^\psi$ and $\varepsilon > 0$. Then, for $n \gg 1$, one has $B_r(x_n) \subseteq B_{r+\varepsilon}(x)$ and thus

$$\Theta^\psi(x_n) \le e^{cr^2} r^{-4}\mu^\psi(B_r(x_n)) + cr^3 \le e^{cr^2} r^{-4}\mu^\psi(B_{r+\varepsilon}(x)) + cr^3.$$

Thus $\limsup \Theta^\psi(x_n) \le e^{cr^2} r^{-4}\mu^\psi(B_r(x)) + cr^3$. Taking the limit $r \downarrow 0$ shows what we desired.

To show that $S \subseteq \mathrm{supp}(\Theta^\psi)$, one should note that since $\Theta(x)$ exists for every $x \in X$, then by letting $\{r_j\} \subset (0, r_0) \setminus \mathcal{R}_x^\psi$ be a null-sequence one can write (see the proof of Lemma 4.6.1)

$$\Theta^\psi(x) = \lim_{j\to\infty} \liminf_{i\to\infty} r^{-4}\mu_i^\psi(B_{r_j}(x)).$$

Thus if $x \in S$, then there exists $\varepsilon > 0$ such that $\Theta^\psi(x) \ge \varepsilon > 0$, so that $x \in \mathrm{supp}(\Theta^\psi)$.

Finally, the reverse inclusion $\mathrm{supp}(\Theta^\psi) \subseteq S$ follows by monotonicity. ∎

Corollary 4.6.3. *The measure μ^ψ is absolutely continuous with respect to $\mathcal{H}^4$.*

Proof. This follows from the fact that $\Theta^\psi(x)$ exists and is uniformly bounded from above. ∎

Corollary 4.6.4. *$S_{\varepsilon_k^\psi} \subseteq X$ is a closed subset.*

Proof. This follows from the upper semi-continuity of Θ^ψ and the monotonicity for μ^ψ given by Lemma 4.6.1. Indeed, if $\{x_n\}$ is a sequence of points in $S_{\varepsilon_k^\psi}$ such that $x_n \to x$, then

$$\Theta^\psi(x) \ge \limsup_{n\to\infty} \Theta^\psi(x_n) \ge \varepsilon_k^\psi, \tag{4.6.5}$$

where in the first and second inequalities we used that Θ^ψ is upper semi-continuous and $x_n \in S_{\varepsilon_k^\psi}$ respectively. Now the monotonicity for μ^ψ gives

$$\Theta^\psi(x) \leq e^{cr^2} r^{-4} \mu^\psi(B_r(x)) \tag{4.6.6}$$

for all $r \in (0, r_0]$. Putting together (4.6.5) and (4.6.6) and recalling that $\mu^\psi(B_r(x)) \leq \liminf_{i\to\infty} \mu_i^\psi(B_r(x))$ completes the proof. ■

Definition 4.6.7.

$$\text{sing}(e_\infty^\psi) := \left\{ x \in X : (\Theta_\infty^\psi)^*(x) := \limsup_{r\downarrow 0} r^{-4} \int_{B_r(x)} e_\infty^\psi > 0 \right\}.$$

Lemma 4.6.8. $\mathcal{H}^4(\text{sing}(e_\infty^\psi)) = 0$.

Proof. For each $\varepsilon > 0$, define

$$\Delta_\varepsilon = \{x \in X : (\Theta_\infty^\psi)^*(x) > \varepsilon\}.$$

Given $0 < \delta \ll 1$, appealing to Vitali's covering lemma we can find countably many points $\{x_j\} \subset \Delta_\varepsilon$ and radii $r_j \in (0, \delta]$ so that the balls $B_{r_j}(x_j)$ are pairwise disjoint and the balls $B_{5r_j}(x_j)$ cover Δ_ε. Moreover, we can suppose that

$$r_j^{-4} \int_{B_{r_j}(x_j)} e_\infty^\psi > \varepsilon.$$

By definition of Δ_ε, it is clear that $x \in \Delta_\varepsilon$ implies $\Theta(x) > 0$, so that as $\text{supp}(\Theta) \subset S$, it follows that $\Delta_\varepsilon \subseteq S$. Hence,

$$\mathcal{H}^4(\Delta_\varepsilon) \leq 5^4 \sum_j r_j^4 \leq 5^4 \varepsilon^{-1} \sum_j \int_{B_{r_j}(x_j)} e_\infty^\psi \leq 5^4 \varepsilon^{-1} \int_{B_\delta(S)} e_\infty^\psi,$$

which goes to zero as $\delta \downarrow 0$ since $\mathcal{H}^7(S) = 0$. Thus $\mathcal{H}^4(\Delta_\varepsilon) = 0$ for all $\varepsilon > 0$, thereby proving the lemma. ■

As a consequence of Proposition 4.6.2 and Lemma 4.6.8, we get:

Corollary 4.6.9. *For $\mathcal{H}^4$−a.e. point $x \in X$ the density*

$$\Theta_\nu(x) := \lim_{r\downarrow 0} r^{-4} \nu(B_r(x))$$

exists and agrees with $\Theta^\psi(x)$; in fact, the assertion is true for all $x \in X \setminus \text{sing}(e_\infty^\psi)$.

In order to continue the analysis of the decomposition of S, and in fact for it to be considered a blow-up set in the usual sense, a key property that we need is that the defect measure ν should have its support contained in S. In oder to show this, we actually shall need another assumption on the sequence (A_i, Φ_i). In what follows, recall that $Z \subseteq S$ (by Theorem 4.5.9) and that each Φ_i decomposes $\mathfrak{su}(2)_E-$valued forms orthogonally into parallel and orthogonal components outside $\Phi_i^{-1}(0)$ (cf. §3.8). From now on we make the following:

Assumption 2: At each $x \in X \setminus S$, the (A_i, Φ_i) satisfy

$$|\pi_{14}(F_{A_i})^{\perp}| \lesssim |\pi_7(F_{A_i})^{\perp}|, \text{ for } i \gg 1. \tag{4.6.10}$$

Using (4.6.10) together with Corollary 4.2.6, we are able to improve the Bochner estimate on the Laplacian of the $\psi-$energy densities of the sequence (A_i, Φ_i) to a linear estimate outside S, in the same way as we did in Section 3.8, and then a mean value inequality together with Arzelà–Ascoli proves that $m_i^{-1}e^{\psi}(A_i, \Phi_i)$ is (up to taking a subsequence) C^0-convergent outside S, so that the desired inclusion $\text{supp}(\nu) \subseteq S$ follows.

Proposition 4.6.11. *Under the above assumptions, the support of the defect measure ν is contained in the blow-up set S:*

$$\text{supp}(\nu) \subseteq S.$$

Now we are able to easily prove the following:

Proposition 4.6.12 (Decomposition of the blow-up set)**.** *The blow-up locus decomposes as follows:*

$$S = \text{supp}(\nu) \cup \text{sing}(e^{\psi}_{\infty}).$$

Proof. We divide the proof as follows:

- $\supseteq$: We know that $\text{supp}(\nu) \subseteq S$ from Proposition 4.6.11. Moreover, by definition, $\text{sing}(e^{\psi}_{\infty}) \subseteq \text{supp}(\Theta^{\psi})$ and, by Proposition 4.6.2, $\text{supp}(\Theta^{\psi}) = S$; thus, $\text{sing}(e^{\psi}_{\infty}) \subseteq S$.

- $\subseteq$: Let $x \in S$. Then there is $\varepsilon > 0$ such that $\Theta^{\psi}(x) \geq \varepsilon > 0$. Now if $x \notin \text{sing}(e^{\psi}_{\infty})$, then $\Theta_{\nu}(x) = \Theta^{\psi}(x) > 0$, so that $x \in \text{supp}(\Theta_{\nu}) \subset \text{supp}(\nu)$. Since $\text{sing}(e^{\psi}_{\infty}) \subseteq S$, the result follows.

∎

We now prove the main result of this section.

Theorem 4.6.13 (Rectifiability of the blow-up set)**.** *The measure ν is $\mathcal{H}^4-$rectifiable, i.e. $\nu = \Theta^{\psi}\mathcal{H}^4\lfloor_{\mathrm{supp}(\nu)}$ and $\mathrm{supp}(\nu)$ is a $\mathcal{H}^4-$rectifiable set.*

Proof. Since $\mathrm{supp}(\nu) \subseteq S$, Corollary 4.6.9 and Proposition 4.6.2 imply that $\Theta_\nu(x) = \Theta^{\psi}(x) > 0$ for $\mathcal{H}^4-$a.e. $x \in \mathrm{supp}(\nu)$. Furthermore, in general $\Theta^*_\nu \leq \Theta^{\psi} < \infty$, which implies that $\nu \ll \mathcal{H}^4$. Thus the rectifiability follows from Preiss' criteria [Pre87]. ■

Corollary 4.6.14. *The blow-up set S is $\mathcal{H}^4-$rectifiable. In particular, at $\mathcal{H}^4-$a.e. point $x \in S$, the tangent space T_xS exists.*

Proof. This follows from Theorem 4.6.13 by using Lemma 4.6.8 and the decomposition of S given by Proposition 4.6.12. ■

4.7 Partial progress on bubbling analysis and a few conjectures

In this final section we give some progress on the development of the bubbling analysis of the problem under the assumptions of the last sections. In the end, we state some conjectures on what we believe to be some of the outcomes of such an analysis.

Definition 4.7.1 (Bubbling set and smooth point)**.** We call $Q := \mathrm{supp}(\nu)$ the **bubbling set** and a point $x \in Q$ is said to be **smooth** if the tangent space T_xQ exists and $x \notin \mathrm{sing}(e^{\psi}_{\infty})$.

By Theorem 4.6.13, at a smooth point $x \in Q$ the measure ν has a unique *tangent measure*, i.e., the limit

$$T_x\nu := \lim_{\delta \downarrow 0} \delta^{-4} s^*_{x,\delta}\nu$$

exists and is given by

$$T_x\nu = \Theta^{\psi}(x)\mathcal{H}^4\lfloor T_xQ.$$

Here $s_{x,\delta}(\cdot) := \exp_x(\delta\cdot)$. An alternative way of obtaining this is to construct $T_x\nu$ explicitly from a blow up sequence as we shall now do. Let (A,Φ) be a mass m configuration, $x \in X$, $\delta > 0$ and $s_{x,\delta}(\cdot) = \exp_x(\delta\cdot)$. Define the *blow up configuration at* x to be

$$(A_{x,\delta}, \Phi_{x,\delta}) = (s^*_{x,\delta}A, \delta s^*_{x,\delta}\Phi), \quad g_{x,\delta} = \delta^{-2}s^*_{x,\delta}g, \quad \psi_{x,\delta} = \delta^{-4}s^*_{x,\delta}\psi, \tag{4.7.2}$$

and respectively denote by $F_{x,\delta}$, $*_\delta$, $\mathrm{vol}_{x,\delta}$ the curvature of $A_{x,\delta}$, the Hodge-$*$, and volume form of $g_{x,\delta}$. These are being regarded as defined in a ball of radius at most $\delta^{-1}i_x$ in T_xM, where

i_x is the injectivity radius at x. A simple computation gives $|F_{x,\delta}|^2_{g_{x,\delta}} = \delta^4|F|^2_g \circ s_{x,\delta}$ and $|\mathrm{d}_{A_{x,\delta}}\Phi_{x,\delta}|^2_{g_{x,\delta}} = \delta^4|\mathrm{d}_A\Phi|^2_g \circ s_{x,\delta}$, so that

$$s^*_{x,\delta}(e(A,\Phi)\mathrm{vol}) = \delta^{7-4}e(A_{x,\delta},\Phi_{x,\delta})\mathrm{vol}_{x,\delta}.$$

We further point out that as $\delta \downarrow 0$, the metric $g_{x,\delta}$ and 4-form $\psi_{x,\delta}$ geometrically converge to those of their flat counterparts in $\mathbb{R}^7 \cong T_xX$.

We shall begin by proving the following result.

Lemma 4.7.3 (Scale fixing)**.** *Let $x \in Q$ be a smooth point. Then there is a null-sequence $\{\delta_i\} \subset (0,1)$ so that*

$$(\delta_i m_i)^{-1}|F_{x,\delta_i} \wedge \psi_{x,\delta_i}|^2_{g_{x,\delta_i}}\mathrm{vol}_{x,\delta_i} \rightharpoonup T_x\nu = \Theta^\psi(x)\mathcal{H}^4\lfloor T_xQ,$$

with F_{x,δ_i} denoting the curvature of $(A_i)_{x,\delta_i}$.

Proof. Since $x \notin \mathrm{sing}(e^\psi_\infty)$, we have

$$T_x\nu = \lim_{\delta\downarrow 0}\delta^{-4}s^*_{x,\delta}\nu = \lim_{\delta\downarrow 0}\delta^{-4}s^*_{x,\delta}\mu^\psi$$

Thus

$$T_x\nu = \lim_{\delta\downarrow 0}\lim_{i\to\infty}\delta^{-4}s^*_{x,\delta}\mu^\psi_i.$$

Therefore we can find a null-sequence $\{\delta_i\} \subset (0,1)$ such that

$$T_x\nu = \lim_{i\to\infty}\delta_i^{-4}s^*_{x,\delta_i}\mu^\psi_i.$$

This gives the result since

$$\delta_i^{-4}s^*_{x,\delta_i}\mu^\psi_i = (\delta_i m_i)^{-1}|F_{x,\delta_i} \wedge \psi_{x,\delta_i}|^2_{g_{x,\delta_i}}\mathrm{vol}_{x,\delta_i}.$$

■

Henceforth, we let $N_xQ := (T_xQ)^\perp \subset T_xX$, write (z,w) to denote points in $T_xQ \times N_xQ = T_xX$ and work with generalized cubes of the form

$$Q_{r,s}(z_0,w_0) := B_r(z_0) \times B_s(w_0) \subset T_xQ \times N_xQ = T_xX.$$

Proposition 4.7.4 (Asymptotic translation invariance)**.** *Let $x \in Q$ be a smooth point and $\{\delta_i\}$ be the null-sequence in Lemma 4.7.3. Then (after passing to a subsequence) there is a null-sequence $\{z_i\} \subset T_xQ$ so that*

$$\lim_{i\to\infty}\sup_{r\le 1} r^{-4}\int_{Q_{r,1}(z_i,0)}(\delta_i m_i)^{-1}\left(|\iota_v F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_v \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right) = 0,$$

for all $v \in T_xQ$.

We split the proof of Proposition 4.7.4 into the following lemmas.

Lemma 4.7.5. *Under the hypothesis of Proposition 4.7.4, for all $v \in T_xQ$ we have*

$$\lim_{i\to\infty}\int_{Q_{2,1}(0)} (\delta_i m_i)^{-1}\left(|\iota_v F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_v \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right) = 0.$$

Proof. Fix $g|_{T_xX}-$orthogonal coordinates $\{y_l\}_{l=1}^4$ on T_xQ, with ∂_{y_l} having length 4. Let $\partial_{\rho_{\delta_i}}$ denote the radial vector field emanating from the point ∂_{y_l} associated with the metric g_{x,δ_i}. Then the monotonicity of Proposition 4.1.3 and $|F_{A_{x,\delta_i}} \wedge F_{A_{x,\delta_i}} \wedge \phi_{x,\delta_i}| \lesssim \delta_i^4$ (by assumption 4.1.2), applied to the blow up sequence $(A_{x,\delta_i}, \Phi_{x,\delta_i})$ centered at ∂_{y_l} implies that for $0 < s \leq r$

$$\begin{aligned}
&\int_{B_r(\partial_{y_l})\backslash B_s(\partial_{y_l})} e^{c(\delta_i\tau)^2}\tau^{-4}(\delta_i m_i)^{-1}\left(|\iota_{\partial_{\rho_{\delta_i}}} F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_{\partial_{\rho_{\delta_i}}} \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right)\\
&\leq e^{c(\delta_i r)^2}r^{-4}\int_{B_r(\partial_{y_l})}(\delta_i m_i)^{-1}|F_{x,\delta_i}\wedge\psi_{x,\delta_i}|^2_{g_{x,\delta_i}} - e^{c(\delta_i s)^2}s^{-4}\int_{B_s(\partial_{y_l})}(\delta_i m_i)^{-1}|F_{x,\delta_i}\wedge\psi_{x,\delta_i}|^2_{g_{x,\delta_i}}\\
&\quad + cm_i^{-1}\delta_i^4 r^3.
\end{aligned}$$

Taking the limit as $i \to \infty$, the first two terms of the right-hand side both converge to $\Theta^\psi(\exp_x(\partial_{y_l}))$ (since $T_x\nu = \Theta^\psi(x)\mathcal{H}^4\lfloor T_xQ$) and the last term tends to zero. Since $Q_{2,1}(0) \subset B_8(\partial_{y_l})\backslash B_1(\partial_{y_l})$, it follows that

$$\lim_{i\to\infty}\int_{Q_{2,1}(0)} (\delta_i m_i)^{-1}\left(|\iota_{\partial_{\rho_{\delta_i}}} F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_{\partial_{\rho_{\delta_i}}} \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right) = 0.$$

Furthermore, at the origin $\partial_{\rho_{\delta_i}}$ generate T_xQ and as the metrics g_{x,δ_i} converge to $g|_{T_xX}$ we may state the result in terms of it. ■

Lemma 4.7.6. *Under the hypothesis of Proposition 4.7.4, for $\mathcal{H}^4-$a.e. $z \in B_1(0) \subset T_xQ$ one has*

$$\lim_{i\to\infty}\sup_{r\leq 1} r^{-4}\int_{Q_{r,1}(z,0)} (\delta_i m_i)^{-1}\left(|\iota_v F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_v \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right) = 0,$$

for all $v \in T_xQ$.

Proof. Define $f_i \colon B_2(0) \subset T_xQ \to [0,\infty)$ by

$$f_i(z) := \int_{B_1(0)\subset N_xQ} (\delta_i m_i)^{-1}\left(|\iota_v F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_v \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right)(z,\cdot)$$

and denote by $Mf_i \colon B_1(0) \subset T_xQ \to [0,\infty)$ the Hardy-Littewood maximal function associated with f_i:

$$Mf_i(z) := \sup_{r\leq 1} r^{-4}\int_{B_r(z)\subset T_xQ} f_i.$$

We then want to show that the set

$$A := \{z \in B_1(0) : \liminf_{i\to\infty} Mf_i(z) > 0\}$$

is such that $\mathcal{H}^4(A) = 0$. For each $j \in \mathbb{N}$, define

$$A_{i,j} := \{z \in B_1(0) : Mf_i(z) \geq j^{-1}\}.$$

Then we can write

$$A = \bigcup_{j\geq 1}\bigcup_{n\geq 1}\bigcap_{i\geq n} A_{i,j}.$$

For each j, on the one hand, by the weak-type L^1 estimate for the maximal operator, we have

$$\mathcal{H}^4(A_{i,j}) \lesssim j\|f_i\|_{L^1}.$$

On the other hand, by Lemma 4.7.5, it follows that $\|f_i\|_{L^1} \to 0$ as $i \to \infty$. Therefore, for each j and n we get

$$\mathcal{H}^4\left(\bigcap_{i\geq n} A_{i,j}\right) = 0,$$

which in turn implies that $\mathcal{H}^4(A) = 0$ by monotone convergence. ∎

Proof of Proposition 4.7.4. By Lemma 4.7.6, for each $j \in \mathbb{N}$ we can find $z_j \in B_{1/j}(0) \subset T_xQ$ such that

$$\lim_{i\to\infty}\sup_{r\leq 1} r^{-4}\int_{Q_{r,1}(z_j,0)} (\delta_i m_i)^{-1}\left(|\iota_v F_{A_{x,\delta_i}}|^2_{\delta_i} + |\iota_v \mathrm{d}_{A_{x,\delta_i}}\Phi_{x,\delta_i}|^2_{\delta_i}\right) = 0.$$

The conclusion then follows by applying a standard diagonal sequence argument. ∎

The natural next step would be a bubble detection result, but we now stop the development of the bubbling analysis since this is as far as the author could go until the time of writing this thesis. A more complete account on this should appear in an upcoming future work in collaboration with Gonçalo Oliveira.

Nonetheless, we now mention some conjectures. The following is what we believe to be a first main result regarding the bubbling analysis.

Conjecture 4.7.7 (Bubbling)**.** *Let $x \in Q$ be a smooth point. Then:*

(a) There are null-sequences $x_i = (z_i, w_i) \in T_xQ \times N_xQ = T_xX$ and $\{\delta_i\}, \{\eta_i\} \subset (0,1)$ so that the "inner bubble"

$$(A_i(y), \Phi_i(y)) = \left(A_{x,\delta_i}(\eta_i^{-1}x_i + y), \eta_i^{-1}\Phi_{x,\delta_i}(\eta_i^{-1}x_i + y)\right), \tag{4.7.8}$$

converges, up to gauge, to the pullback of a mass 1 Bogomolnyi monopole (A_x, Φ_x) *on* $N_xQ \cong \mathbb{R}^3$ *such that*

$$0 < \mathcal{E}_{\mathbb{R}^3}(A_x, \Phi_x) \leq \Theta^{\psi}(x).$$

(b) The tangent space T_xQ *is coassociative.*

We note that proven part (a) of the above, part (b) can be proved as follows. Decompose[1] $\psi_x = \alpha \mathrm{vol}_{T_xQ} + \psi'$, where $\alpha \in [0,1]$ and $\psi'|_{T_xQ} = 0$. Then the monopole equation becomes

$$\mathrm{d}_{A_x}\Phi_x = *(F_x \wedge \psi_x) = \alpha *_{N_xQ} F_x + *(F_x \wedge \psi'). \tag{4.7.9}$$

On the other hand we may equally write the monopole equation as

$$F_x + *(F_x \wedge \phi) = *(\mathrm{d}_{A_x}\Phi_x \wedge \psi_x) = \alpha *_{N_xQ} \mathrm{d}_{A_x}\Phi_x + *(\mathrm{d}_{A_x}\Phi_x \wedge \psi'). \tag{4.7.10}$$

Then, by combining equations (4.7.9) and (4.7.10) using the fact that $*^2_{N_xQ} = 1$ yields

$$F_x = \alpha^2 F_x,$$

and so $\alpha = 1$ since A_x is not flat. In particular, this shows that T_xQ is calibrated by ψ.

Finally, we mention that if Conjecture 4.7.7 is indeed true, then we believe that by a degree argument as in the proof of Theorem 3.6.2 one should be able to prove:

Conjecture 4.7.11. *The zero set* Z *contains all the smooth points in* Q*.*

Since $Z \subseteq S$ (by Theorem 4.5.9), if Conjecture 4.7.11 is true then by the decomposition of S of Proposition 4.6.12, and Lemma 4.6.8, it would follow that the blow-up set and the zero set coincide up to a set of zero $\mathcal{H}^4$−measure. Thus by the rectifiability of Q (Theorem 4.6.13) and the finiteness $\mathcal{H}^4(Z) < \infty$, it would follow that S and Z are $\mathcal{H}^4$−rectifiable sets of finite $\mathcal{H}^4$−measure.

The establishment of Conjectures 4.7.7 and 4.7.11 would be very important steps in the direction of a general concrete result on the concentration phenomena of large mass G_2−monopoles with fixed monopole class on AC G_2−manifolds: under suitable assumptions, one would be able to say that these concentrate along their zero set Z, that Z define a $\mathcal{H}^4$−rectifiable coassociative current of finite mass and that at each point $x \in Z$ a mass one $\mathbb{R}^3$−monopole bubbles off transversally carrying part of the lost energy at x.

[1] As ψ is a calibration this can certainly be done, by choosing an orientation on T_xQ.

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Appendix A

Mean value inequalities

This appendix intends to summarize in a unified statement (cf. Theorem A.3) some well-known (nonlinear) mean value inequalities for the Laplacian. This is an important analytical tool for this thesis and have been successfully applied in various geometric PDE problems, e.g. minimal submanifolds, harmonic maps, pseudo-holomorphic curves, Yang–Mills connections and so on. The common feature of these problems is an energy functional associated to the PDE, say scale-invariant in dimension d; in general the PDE are nonlinear, elliptic, of second order and, in fact, are the associated Euler–Lagrange equations of such functional. The solutions of the PDE then satisfy a nonlinear bound on the Laplacian of its energy density, with the nonlinear term being of order $(d+2)/d$, and they also enjoy a monotonicity property on their renormalized scale-invariant energy on small geodesic balls. One then derives a phenomena of 'energy quantization' for sequences of solutions with uniformly bounded energy as a consequence of a mean value inequality applied to the energy density of the functional; in fact, one derives the existence of a 'quanta' of energy $\hbar > 0$ and a codimension d subset of points at which such a sequence experiences energy concentration of at least $\hbar$.

The results here are basically a personal organization and mixing of (some of) the ones appearing in the very nice approaches to the subject in [Weh05], [HNS09, Appendix B] and [Wal17a, Appendix A].

We say that a Riemannian manifold (X^n, g) is of *bounded geometry* if the following conditions holds:

- The global injectivity radius is positive: $i_X(g) > 0$ (in particular, (X^n, g) is complete);
- The Riemannian curvature tensor R^g and its covariant derivative ∇R^g are uniformly

bounded:

$$|R^g| \lesssim 1 \quad \text{and} \quad |\nabla R^g| \lesssim 1 \quad \text{on } X.$$

We shall need the following standard result which follows e.g. from [GT01, Theorem 9.20, p.244] and [HNS09, Step 2 in the Proof of Theorem B.1].

Lemma A.1. *Let (X^n, g) be a Riemannian $n-$manifold of bounded geometry. Then there is $0 < \delta < i(g)$ with the following significance. If $x \in X$, $r \in (0, \delta]$ and $A \geq 0$, then every $f \in C^2(B_r(x), [0,\infty))$ satisfies:*

$$\Delta f \leq A \quad \Longrightarrow \quad f(x) \lesssim Ar^2 + r^{-n} \int_{B_r(x)} f. \tag{A.2}$$

Theorem A.3 (Mean value inequalities)**.** *Under the hypothesis of Lemma A.1, let $x \in X$, $r \in (0, \delta]$ and suppose $f \in C^2(B_r(x), [0,\infty))$ satisfy the following conditions:*

(C1) There is $d \in \mathbb{N}$ such that if $B_s(y) \subseteq B_{r/2}(x)$ then

$$s^{d-n} \int_{B_s(y)} f \lesssim r^{d-n} \int_{B_r(x)} f + \tau, \tag{A.4}$$

for some $\tau = \tau(r) \geq 0$.

(C2) There are constants $a_0, a_1, a \geq 0$ and $\alpha \in [1, (d+2)/d]$ such that

$$\Delta f \leq a_0 + a_1 f + a f^\alpha. \tag{A.5}$$

Setting

$$\varepsilon := r^{d-n} \int_{B_r(x)} f,$$

we have:

(i) If $\alpha < (d+2)/d$ then

$$\sup_{B_{\frac{r}{2}}(x)} f \lesssim_d a_0 r^2 + (a_1^{d/2} + r^{-d})(\varepsilon + \tau) + \left(a^{d/2}(\varepsilon + \tau)\right)^\beta, \tag{A.6}$$

where $\beta := 2/(2 + d - \alpha d)$.

(ii) If $\alpha = (d+2)/d$, then there is a constant $\hbar > 0$ (depending only on the geometry and d) such that

$$a^{d/2}(\varepsilon + \tau) < \hbar \quad \Longrightarrow \quad \sup_{B_{\frac{r}{2}}(x)} f \lesssim_d a_0 r^2 + (a_1^{d/2} + r^{-d})(\varepsilon + \tau) \tag{A.7}$$

Proof of Theorem A.3. The proof is based on the so-called 'Heinz trick'. In order for the reader to see precisely where each relevant constant appears in the estimates, in this proof we shall mostly avoid the notation '$\lesssim$' and show the constants explicitly. Thus, choose $\delta > 0$ as in Lemma A.1, let $c_1 > 0$ be the hidden constant in (A.2) and let $c_2 > 0$ be the hidden constant in (A.4). We shall see that both $\hbar$ and the final estimates depend only on c_1, c_2 and d.

Fix any point $y \in B_{r/2}(x)$. Define $h : [0, r/2] \to [0, \infty)$ by

$$h(s) := \left(\frac{r/2 - s}{r/2}\right)^d \max_{B_s(y)} f.$$

Note that $h(0) = f(y)$ and $h(r/2) = 0$. Since h is nonnegative there is an $s^* \in [0, r/2)$ and a $y^* \in B_{s^*}(y)$ such that

$$h(s^*) = \max_{0 \le s \le r/2} h(s), \quad F := f(y^*) = \max_{B_{s^*}(y)} f.$$

Set

$$s_0 := \frac{r/2 - s^*}{2} > 0.$$

Then

$$\max_{B_{s_0}(y^*)} f \le \max_{B_{s^*+s_0}(y)} f = \frac{(r/2)^d}{(r/2 - s^* - s_0)^d} h(s^* + s_0) \le \frac{2^d (r/2)^d}{(r/2 - s^*)^d} h(s^*) = 2^d \max_{B_{s^*}(y)} f = 2^d F.$$

Hence, by assumption (A.5), in $B_{s_0}(y^*)$ we have the inequality

$$\mathcal{L} f \le a_0 + a_1 f + a f^\alpha \le a_0 + a_1 (2^d F) + a (2^d F)^\alpha.$$

By Lemma A.1 this implies

$$F = f(y^*) \le c_1 \left((a_0 + a_1 (2^d F) + a (2^d F)^\alpha) s^2 + s^{-n} \int_{B_s(y^*)} f \right), \tag{A.8}$$

for all $s \in [0, s_0]$. Now using assumption (A.4), and observing that $s_0 \le r/2$, we get

$$F \le c_1 a_0 r^2/4 + c_1 \left(a_1 (2^d F) + a (2^d F)^\alpha \right) s^2 + c_1 c_2 s^{-d} (\varepsilon + \tau), \quad \forall s \in [0, s_0]. \tag{A.9}$$

We now make a case by case distinction.

Case 1. $F \le c_1 a_0 r^2$

In this case $f(y) \le F \le c_1 a_0 r^2 \lesssim a_0 r^2$, which already proves the assertion.

Case 2. $F \ge c_1 a_0 r^2$ and $a_1(2^d F) \ge a(2^d F)^\alpha$.

From (A.9) we derive

$$F \le F/4 + 2c_1 a_1 2^d F s^2 + c_1 c_2 s^{-d}(\varepsilon + \tau), \quad \forall s \in [0, s_0]. \tag{A.10}$$

so that we have two possibilities:

- If $c_1 a_1 2^d s_0^2 < 1/8$, then by (A.10) we get

  $$F \le 2c_1 c_2 s_0^{-d}(\varepsilon + \tau)$$

  Since $r/2 - s^* = 2s_0$ this gives

  $$f(y) = h(0) \le h(s^*) = \left(\frac{r/2 - s^*}{r/2}\right)^d F = \frac{2^{2d} s_0^d}{r^d} F \le 2^{2d+1} c_1 c_2 r^{-d}(\varepsilon+\tau) \lesssim_d r^{-d}(\varepsilon+\tau),$$

  and so the desired estimate holds.

- Otherwise, we choose $s \le s_0$ such that $c_1 a_1 2^d s^2 = 1/8$. Then by (A.10) we have

  $$f(y) \le F \le 2c_1 c_2 s^{-d}(\varepsilon + \tau) = 2c_1 c_2 (8c_1 a_1 2^d)^{d/2}(\varepsilon + \tau) \lesssim_d a_1^{d/2}(\varepsilon + \tau),$$

  as we wanted.

Case 3. $F \ge c_1 a_0 r^2$ and $a_1(2^d F) \le a(2^d F)^\alpha$.

Here from (A.9) we get

$$F \le F/4 + 2c_1 a (2^d F)^\alpha s^2 + c_1 c_2 s^{-d}(\varepsilon + \tau), \quad \forall s \in [0, s_0]. \tag{A.11}$$

Hence we have the following possibilities:

- If $c_1 a 2^{\alpha d} F^{\alpha-1} s_0^2 < 1/8$ then by (A.11),

  $$F \le 2c_1 c_2 s_0^{-d}(\varepsilon + \tau),$$

  and again by the same argument as above we get

  $$f(y) \le h(s^*) \le 2^{2d+1} c_1 c_2 r^{-d}(\varepsilon + \tau) \lesssim_d r^{-d}(\varepsilon + \tau),$$

  as we wanted.

- Otherwise we choose $s \leq s_0$ such that $c_1 a 2^{\alpha d} F^{\alpha-1} s^2 = 1/8$ so that

$$F \leq 2c_1c_2 s^{-d}(\varepsilon+\tau) = 2c_1c_2(c_1 2^{\alpha d+3})^{d/2} a^{d/2} F^{(\alpha d-d)/2}(\varepsilon+\tau). \tag{A.12}$$

Now if $\alpha < (d+2)/d$ then $2+d-\alpha d > 0$ and hence

$$f(y) \leq F \leq \left(2c_1c_2(c_1 2^{\alpha d+3})^{d/2} a^{d/2}(\varepsilon+\tau)\right)^{\beta} \lesssim_d \left(a^{d/2}(\varepsilon+\tau)\right)^{\beta},$$

as we wanted. For the critical exponent $\alpha = (d+2)/d$ we have that $(\alpha d - d)/2 = 1$ and thus it follows from (A.12) that $1 \leq c a^{d/2}(\varepsilon+\tau)$ for some $c > 0$ depending only on c_1, c_2 and d. Thus if $a^{d/2}(\varepsilon+\tau) < \hbar := c^{-1}$ this case can be excluded. We are done.

■

Appendix B

The proof of assertion (3.2.14).

This appendix is basically the same as the one appearing in our paper [FO19]. Here we shall prove assertion (3.2.14), which says that the zeros of the monopoles constructed via Theorem 3.2.11 are contained in balls of radius $10m^{-1/2}$ around the k-points in X used in the construction. This requires a number of technical ingredients from [Oli16c] and so we decided to include this section as an Appendix.

It follows from [Oli16c, Proposition 6] that the monopole (A_i, Φ_i) can be written as $(A_i, \Phi_i) = (A_i^0, \Phi_i^0) + (a_i, \phi_i)$, where

A.a (A_i^0, Φ_i^0) is an approximate monopole constructed in [Oli16c, Proposition 4]. Moreover, by its own construction, we have that the restriction

$$\Phi_i^0 : X \backslash \cup_{j=1}^k B_{10m_i^{-1/2}}(x_j) \to \mathfrak{su}(2),$$

satisfies $|\Phi_i^0| \geq m_i/2$ has no zeros and yields a splitting of the trivial rank-2 complex vector bundle $\underline{\mathbb{C}^2} \cong L \oplus L^{-1}$, where the complex line bundle L is such that

$$\deg(L|_{\partial B_{10m_i^{-1/2}(x_j)}}) = 1,$$

for all $j = 1, \ldots, k$. In other words, the restricted map

$$\Phi_i^0 : \partial B_{10m_i^{-1/2}}(x_j) \to \mathfrak{su}(2) \backslash \{0\},$$

has degree 1.

A.b $(a_i, \phi_i) \in \Gamma((\Lambda^1 \oplus \Lambda^0) \otimes \mathfrak{su}(2))$ satisfies an elliptic equation, when in a certain Coulomb gauge (see [Oli16c, Lemma 13]). Moreover, from [Oli16c, Proposition 6], it satisfies

$$\|(a_i, \phi_i)\|_{H_{1,-\frac{1}{2}}} \lesssim m_i^{-7/4}, \tag{B.1}$$

where $H_{1,-1/2}$ is a certain Sobolev space.

The Sobolev space $H_{1,\nu+1}$, with $\nu = -3/2$ here, is one of several $H_{n,\nu+n}$ constructed using the approximate monopole (A_i^0, Φ_i^0). These are well adapted to solving the monopole equation, and have the property that, in a certain gauge (see [Oli16c, Section 5]), one can iterate estimate (B.1) to obtain that

$$\|(a_i, \phi_i)\|_{H_{n,\nu+n}} \lesssim m_i^{-7/4},$$

for all $n \in \mathbb{N}$. Moreover, once restricted to certain subsets of X, these spaces satisfy a number of interesting properties. Some of these can be easily read from the definition in [Oli16c, Section 4.1], and we summarize them below

B.a Restricted to compact set $K \subset X$, the norm $H_{1,1+\nu}(K)$ is equivalent to the usual $L^{2,1}(K)$. However, not in an m_i-independent way. In fact, there is a constant c_n, only depending on g and K, not m_i, so that

$$\|(a_i, \phi_i)\|_{L^{2,n}(K)} \leq c_n(K)\, m_i^2\, \|(a_i, \phi_i)\|_{H_{n,\nu+n}(K)} \lesssim m_i^{1/4}. \tag{B.2}$$

B.b For $\epsilon > 0$ we consider

$$C_\epsilon = X \backslash \cup_{j=1}^k B_\epsilon(x_j).$$

If we let $4d = \min_{j,l} \mathrm{dist}(x_j, x_l)$, then the balls of radius d around the points x_i are disjoint. Using d, we shall consider C_d. Then, certain weight functions W_n, on which the spaces $H_{n,\nu+n}$ depend, can be arranged so that

$$\|(a_i, \phi_i)\|_{L^{2,n}(K)} \leq c_n(K)\, \|(a_i, \phi_i)\|_{H_{n,\nu+n}(K)} \lesssim m_i^{-7/4}. \tag{B.3}$$

for any $K \subset C_d$.

B.c On C we can use the fact that $\Phi_i^0 > 0$, as mentioned in A.a, to write any $\mathfrak{su}(2)$-valued tensor f as $f = f^{\parallel} + f^{\perp}$, with the $f^{\parallel}$ denoting the component parallel to Φ_i^0 and $f^{\perp}$ the orthogonal one. On C, and for large m_i, we can write

$$\|(a_i^{\parallel}, \phi_i^{\parallel})\|_{L^{2,n}_{\nu+n}(C)} + \|(a_i^{\perp}, \phi_i^{\perp})\|_{L^{2,n}(C)} = \|(a_i, \phi_i)\|_{H_{n,\nu+n}(C)} \lesssim m_i^{-7/4}, \tag{B.4}$$

where the spaces $L^{2,n}_{\nu+n}$ are the more standard Lockhart–McOwen conically weighted spaces.

Combining item B.a. above and the Sobolev embedding $L^{2,n}(K) \hookrightarrow C^{n-2}(K)$ one obtains that

$$\|(a_i, \phi_i)\|_{C^{n-2}(K)} \lesssim m_i^{1/4} \Rightarrow \|\Phi_i - \Phi_i^0\|_{C^0(K)} \lesssim m_i^{1/4}, \tag{B.5}$$

for any compact set $K \subset X$. In particular (a_i, ϕ) is smooth. Moreover, as mentioned in A.a, $|\Phi_i^0| \geq m_i/2$ in $C_{10m_i^{-1/2}}$ and thus

$$|\Phi_i| \geq |\Phi_i^0| - \|\phi_i\|_{C^0} \geq \frac{m_i}{2} - cm_i^{1/4}, \text{ in any compact } K \subset C_{10m_i^{-1/2}}$$

and so, for $m_i \gg 1$, does not vanish in $\partial \overline{B_{10m_i^{-1/2}}(x_j)}$ for any $j \in \{1, \ldots, k\}$. In particular, putting this together with the estimate (B.1) in A.b, which shows that ϕ_i is decaying, we conclude that Φ_i does not vanish in C, and so any of its zeros must be inside one of the balls of radius $10m_i^{-1/2}$ around the points x_i. Furthermore, this estimate shows that 1-parameter family of maps

$$\Phi_i^t = \Phi_i^0 + t\phi_i : C_{10m_i^{-1/2}} \to \mathfrak{su}(2) \backslash \{0\},$$

gives an homotopy between Φ_i^0 and Φ_i. Combining this with the discussion in A.a we conclude that Φ_i

$$\deg(\Phi_i|_{\partial B_{10m_i^{-1/2}}(x_j)}) = \deg(\Phi_i^0|_{\partial B_{10m_i^{-1/2}}(x_j)}) = 1.$$

Thus, Φ_i does have zeros inside $B_{10m_i^{-1/2}}(x_j)$.