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Homological finiteness properties

Propriedades homológicas de finitude

Campinas

2020

Luis Augusto de Mendonça

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Resumo

Consideramos problemas nas teorias de grupos discretos, álgebras de Lie e grupos pro-p. Apresentamos resultados relacionados sobretudo a propriedades homológicas de finitude de tais estruturas algébricas.

Primeiramente, discutimos Σ-invariantes de produtos entrelaçados de grupos discretos. Descrevemos completamente o invariante Σ^1, relacionado à herança por subgrupos da propriedade de ser finitamente gerado, e descrevemos parcialmente o invariante Σ^2, relacionado à herança por subgrupos da propriedade de admitir uma apresentação finita. Aplicamos tais resultados ao estudo de números de Reidemeister de isomorfismos de certos produtos entrelaçados.

Na sequência definimos e estudamos uma versão da construção de comutatividade fraca de Sidki na categoria de álgebras de Lie sobre um corpo de característica diferente de dois. Tal construção pode ser vista como um funtor que recebe uma álgebra de Lie $\mathfrak{g}$ e retorna um certo quociente $\chi(\mathfrak{g})$ da soma livre de duas cópias isomorfas de $\mathfrak{g}$. Demonstramos resultados sobre a preservação de certas propriedades algébricas por tal funtor e mostramos que o multiplicador de Schur de $\mathfrak{g}$ é um subquociente de $\chi(\mathfrak{g})$. Mostramos em particular que, para uma álgebra de Lie livre $\mathfrak{g}$ de posto ao menos três, $\chi(\mathfrak{g})$ é finitamente apresentável mas não é de tipo FP_3, e tem dimensão cohomológica infinita.

Por fim, consideramos também uma versão da construção de comutatividade fraca na categoria de grupos pro-p para um número primo fixado p. Mostramos que tal construção também preserva diversas propriedades algébricas, como ocorre nos casos de grupos discretos e álgebras de Lie. Para tanto estudamos também produtos subdiretos de grupos pro-p; em particular demonstramos uma versão do Teorema $(n-1)-n-(n+1)$.

Palavras-chave: teoria dos grupos, álgebra de Lie, grupos profinitos.

Abstract

We consider problems in the theories of discrete groups, Lie algebras, and pro-p groups. We present results related mainly to homological finiteness properties of such algebraic structures.

First, we discuss Σ-invariants of wreath products of discrete groups. We give a complete description of the Σ^1-invariant, which is related to the inheritance of the property of being finitely generated by subgroups. We also describe partially the invariant Σ^2, which is related to the inheritance of finite presentability by subgroups. We apply such results in the study of Reidemeister numbers of isomorphisms of certain wreath products.

Then we define and study a version of Sidki's weak commutativity construction in the category of Lie algebras over a field whose characteristic is not two. Such construction can be seen as a functor that receives a Lie algebra $\mathfrak{g}$ and returns a certain quotient $\chi(\mathfrak{g})$ of the free sum of two isomorphic copies of $\mathfrak{g}$. We prove some results on the preservation of certain algebraic properties by this functor, and we show that the Schur multiplier of $\mathfrak{g}$ is a subquotient of $\chi(\mathfrak{g})$. We show in particular that, for a free Lie algebra $\mathfrak{g}$ with at least three free generators, $\chi(\mathfrak{g})$ is finitely presentable but not of type FP_3, and has infinite cohomological dimension.

Finally, we also consider a version of the weak commutativity construction in the category of pro-p groups for a fixed prime number p. We show that such construction also preserves several algebraic properties, as occurs in the cases of discrete groups and Lie algebras. To this end, we also study subdirect products of pro-p groups. In particular we prove a version of the $(n-1)-n-(n+1)$ Theorem.

Keywords: group theory, Lie algebras, profinite groups.

Contents

Introduction

In this thesis we deal with problems in the theories of discrete groups, Lie algebras over a field and pro-p groups, for a prime integer p. The unifying principle is that we study (mostly homological) finiteness properties of these algebraic structures, which will usually be given by a presentation, that is, by generators and relators.

Some of the finiteness properties that we are interested are well known: we will discuss finitely generated and finitely presented groups (Lie algebras, pro-p groups). Other finiteness properties are of homological nature. Concretely, a module A over a unitary associative ring R is *of type* FP_m if there is a projective resolution

$$\mathcal{P} : \ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to A \to 0$$

with P_j finitely generated for all $j \leq m$. This defines a sequence of finiteness properties *for the module* A. We can then specialize it to the algebraic structures that we want to study by taking R and A to be:

1. $R = \mathbb{Z}G$ the group ring, $A = \mathbb{Z}$ the trivial module for any discrete group G;
2. $R = \mathcal{U}(\mathfrak{g})$ the universal enveloping algebra, $A = K$ the trivial module for any Lie algebra $\mathfrak{g}$ over the field K;
3. $R = \mathbb{Z}_p[[H]]$ the completed group algebra, $A = \mathbb{Z}_p$ the trivial module, where H is any pro-p group and $\mathbb{Z}_p$ is the ring of p-adic integers.

Thus a discrete group G is of type FP_m if $\mathbb{Z}$ is of type FP_m over $\mathbb{Z}G$, and similarly in the other cases.

In the part of this thesis in which we deal with discrete groups, we actually study the *inheritance* of finiteness properties by subgroups. This falls into the world of Σ-invariants. These are some geometric invariants of groups, containing information on these inheritance problems, whose definitions and most general results appeared in a series of papers by Bieri, Neumann, Strebel, Renz ([13, 14, 15]) and others. In what follows we describe briefly the theory.

Let Γ be a finitely generated group. The *character sphere* $S(\Gamma)$ is the set of non-zero homomorphisms $\chi : \Gamma \to \mathbb{R}$ (these homomorphisms are called *characters*) modulo the equivalence relation given by $\chi_1 \sim \chi_2$ if there is some $r \in \mathbb{R}_{>0}$ such that $\chi_2 = r\chi_1$. The class of χ will be denoted by $[\chi]$. The character sphere may be seen as the $(n-1)$-sphere in the vector space $Hom(\Gamma, \mathbb{R}) \simeq \mathbb{R}^n$, where n is the torsion-free rank of the abelianization of Γ.

The Σ-invariants are defined as certain subsets of $S(\Gamma)$. The first of them is denoted by $\Sigma^1(\Gamma)$ and can be defined as follows. For any finite generating set $X \subset \Gamma$ we can associate the Cayley graph $Cay(\Gamma; X)$. Recall that its vertex set is Γ and two vertices $\gamma_1, \gamma_2 \in \Gamma$ are connected by an edge if $\gamma_2 = \gamma_1 x$ for some $x \in X$. Any non-trivial homomorphism $\chi : \Gamma \to \mathbb{R}$ defines a submonoid

$$\Gamma_\chi = \{\gamma \in \Gamma \mid \chi(\gamma) \geq 0\}.$$

We may consider then the full subgraph $Cay(\Gamma; X)_\chi \subset Cay(\Gamma; X)$ determined by the vertex set Γ_χ, as defined above. We put:

$$[\chi] \in \Sigma^1(\Gamma) \Leftrightarrow Cay(\Gamma; X)_\chi \text{ is connected.}$$

This is the invariant that appears (with a different language) in [13], which is also called the *BNS-invariant* or *Bieri-Neumann-Strebel invariant* of Γ.

The invariant Σ^2 is defined similarly. If Γ is finitely presented and $\langle \mathcal{X} | \mathcal{R} \rangle$ is a finite presentation, we consider the Cayley complex $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)$. This complex is obtained from the Cayley graph by gluing 2-dimension cells with boundary determined by the loops defined by the relators $r \in R$, for each base point in Γ. The resulting complex is always 1-connected. Again we define $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ to be the full subcomplex spanned by Γ_χ. The 1-connectedness of this complex depends on the choice of the presentation. We define $\Sigma^2(\Gamma)$ as the subset of $S(\Gamma)$ containing exactly all the classes $[\chi]$ of characters such that $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ is 1-connected for some finite presentation $\langle \mathcal{X} | \mathcal{R} \rangle$, depending on χ, of Γ. More details on these definitions may be found in [60].

The main feature of these invariants is that they classify the related finiteness properties for subgroups containing the derived subgroup: for a finitely generated group Γ, a subgroup $N \subseteq \Gamma$ such that $[\Gamma, \Gamma] \subseteq N$ is finitely generated if and only if

$$\Sigma^1(\Gamma) \supseteq \{[\chi] \in S(\Gamma) \mid \chi|_N = 0\} =: S(\Gamma; N).$$

Similarly, if we assume further that Γ is finitely presented, then N is finitely presented if and only if $\Sigma^2(\Gamma) \supseteq S(\Gamma; N)$ ([13, 67]).

There are also some homological invariants that can be defined in terms of the monoid ring $\mathbb{Z}\Gamma_\chi$. This is of course the subring of $\mathbb{Z}\Gamma$ containing exactly all elements $\sum a_\gamma \gamma \in \mathbb{Z}\Gamma$ such that $a_\gamma \neq 0$ only if $\gamma \in \Gamma_\chi$. We put

$$\Sigma^m(\Gamma; \mathbb{Z}) = \{[\chi] \in S(\Gamma) \mid \mathbb{Z} \text{ is of type } FP_m \text{ over } \mathbb{Z}\Gamma_\chi\}.$$

A version of the result connecting the inheritance of the type FP_m for subgroups above the derived subgroup and the invariant $\Sigma^m(\Gamma; \mathbb{Z})$ also holds ([14]).

All these invariants are in general hard to describe for specific groups, and this has been done only for a few classes of groups. For right-angled Artin groups (RAAGs), they are completely described in the work of Meier, Meinert and VanWyk [56, 58]. By definition the right-angled Artin group defined by a finite graph Δ is the group A_Δ with presentation

$$A_\Delta = \langle x \in V(\Delta) \mid [x, y] = 1 \text{ if } x \text{ and } y \text{ are connected by an edge in } \Delta\rangle,$$

where $V(\Delta)$ is the vertex set of Δ. These groups have attracted great attention in modern group theory, see for instance [25]. The description of their Σ-invariants is connected with the existence of subgroups of these groups having a wide variety of finiteness properties, as shown by Bestvina and Brady [9].

There is also substancial work on the more general class of Artin groups (not-necessarily right-angled) by the same Meier, Meinert and VanWyk [57] and by Almeida and Kochloukova [1, 2, 3, 4]. Another line of generalization was followed by Meinert, who computed the invariants Σ^1 of graph products [59].

Another interesting group for which the invariants are known is Thompson's group F. Both homological and homotopical invariants have been computed in all dimensions by Bieri, Geoghegan and Kochloukova [12]. The Σ^2-invariants of the generalized Thompson groups $F_{n,\infty}$ were then computed by Kochloukova [44] and recently Zaremsky extended it to higher dimensions [82].

The Σ^1-invariants of groups of basis conjugating automorphisms of free groups were computed by Orlandi-Korner [66], and then generalized by Koban and Piggott [41] for the class of groups of pure symmetric automorphisms of RAAGs.

This last result shows that the Σ-theory has applications outside the world of finiteness properties. The groups that Koban and Piggott studied are the groups of all automorphisms φ of a fixed RAAG A_Γ such that $\varphi(v) = v^g$ for some $g \in A_\Gamma$ and for all v a vertex of Γ (recall that A_Γ is generated by the set of vertices of the graph Γ). It turns out that the group of automorphisms with this property can be another RAAG under some conditions on the graph Γ and, by looking at the Σ^1-invariant of these groups, Koban and Piggott were able to tell exactly when this is the case. Day and Wade then did the same with respect to the group of pure symmetric outer automorphisms of a RAAG [30].

Our work is about the invariants for the class of permutational wreath products of groups. Recall that given H and G groups and a G-set X, the wreath product $H \wr_X G$ is defined as the semi-direct product $M \rtimes G$, where $M = \oplus_{x\in X} H_x$ is the direct sum (that is, the restricted direct product) of copies of H indexed by X and G acts by permuting these copies according to its action on X. The finiteness properties of these groups were studied by Cornulier [26] (finite generation and finite presentability) and more recently by Bartholdi, Cornulier and Kochloukova [5] (properties FP_m).

Our first result is the full description of Σ^1.

Theorem A1. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-trivial character. Set $M = \oplus_{x \in X} H_x \subseteq \Gamma$.*

1. *If $\chi|_M = 0$, then $[\chi] \in \Sigma^1(\Gamma)$ if and only if $[\chi|_G] \in \Sigma^1(G)$ and $\chi|_{stab_G(x)} \neq 0$ for all $x \in X$.*
2. *If $\chi|_M \neq 0$, then $[\chi] \in \Sigma^1(\Gamma)$ if and only if at least one of the following conditions holds:*
   (a) There exist $x, y \in X$ with $x \neq y$, $\chi|_{H_x} \neq 0$ and $\chi|_{H_y} \neq 0$;
   (b) There exists $x \in X$ with $\chi|_{H_x} \neq 0$ and $[\chi|_{H_x}] \in \Sigma^1(H)$ or
   (c) $\chi|_G \neq 0$.

Part 1 of the theorem above generalizes [5, Thm. 8.1] in dimension 1, where H has infinite abelianization by hypothesis. For regular wreath products, that is, $\Gamma = H \wr_G G$, the action being by multiplication on the left, the Σ^1-invariant was already computed by Strebel in [76, Prop. C1.18].

For the invariant Σ^2 we consider two cases, the same as in the theorem above. For characters $\chi : H \wr_X G \to \mathbb{R}$ such that $\chi|_M \neq 0$ the criteria developed by Renz [68] are especially powerful, and have allowed us to prove part 2 of Theorem A1 and a similar result for Σ^2.

Theorem A2. *Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-trivial character. If the set*

$$T = \{x \in X \mid \chi|_{H_x} \neq 0\}$$

has at least 3 elements, then $[\chi] \in \Sigma^2(\Gamma)$.

The cases where T is non-empty but has less than 3 elements can be dealt with using the direct product formula (see Theorem 2.8) and the results on the Σ^1-invariant (see Theorem 2.28 and the comment right before it).

For the characters $\chi : \Gamma \to \mathbb{R}$ with $\chi|_M = 0$ we were not able to obtain a complete result, by lack of a general method to study necessary conditions for $[\chi] \in \Sigma^2(\Gamma)$. By the results of Bartholdi, Cornulier and Kochloukova on homological invariants, the most general theorem we can prove is the following, where $stab_G(x, y)$ denotes the stabilizer subgroup associated to an element (x, y) of X^2, which is equipped with the diagonal G-action.

Theorem A3. *Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. Then $[\chi] \in \Sigma^2(\Gamma)$ if all three conditions below hold*

1. $[\chi|_G] \in \Sigma^2(G)$;

2. $[\chi|_{stab_G(x)}] \in \Sigma^1(stab_G(x))$ *for all* $x \in X$ *and*

3. $\chi|_{stab_G(x,y)} \neq 0$ *for all* $(x, y) \in X^2$.

In general, conditions 1 and 3 are necessary for $[\chi] \in \Sigma^2(\Gamma)$. *If we assume further that the abelianization of* H *is infinite, then condition 2 is necessary as well.*

Restrictions on the abelianization of the basis group H have been recurrent in the study of finiteness properties of wreath products and related constructions. Besides appearing in the work of Bartholdi, Cornulier and Kochloukova [5], they also pop up in the paper by Kropholler and Martino [50], which deals with the wider class of *graph-wreath products* (see Section 2.4) from a more homotopical point of view.

The results that we obtained also say something about the homological invariants $\Sigma^1(\Gamma;\mathbb{Z})$ and $\Sigma^2(\Gamma;\mathbb{Z})$. First, $\Sigma^1(\Gamma;\mathbb{Z}) = \Sigma^1(\Gamma)$ for any finitely generated groups. Also, Theorem A3 holds for $\Sigma^2(\Gamma;\mathbb{Z})$ by [5]. Finally, $\Sigma^2(\Gamma) \subseteq \Sigma^2(\Gamma;\mathbb{Z})$ for any *finitely presentable* group, which can be applied under the conditions of Theorem A2. We conjecture that such theorem holds for $\Sigma^2(\Gamma;\mathbb{Z})$ if we assume only that Γ is of homological type FP_2, but we cannot give a proof at the moment.

We also considered some applications to twisted conjugacy. Recall that given an automorphism φ of a group G, the *Reidemeister number* $R(\varphi)$ is defined as the number of orbits of the twisted conjugacy action, which is given by $g \cdot h := gh\varphi(g^{-1})$, for $g, h \in G$.

Exploring the connections between Σ-theory and Reidemeister numbers, as found out by Koban and Wong [42] and Gonçalves and Kochloukova [34], we obtain some results about the Reidemeister numbers of automorphisms contained in some subgroups of finite index of $Aut(H \wr_X G)$, under some relatively strong restrictions. For precise statements, see Corollaries 2.39 and 2.41.

Theorems A1, A2 and A3 are the main results of Chapter 2 in this thesis, and have already been published in [61]. Chapter 3 is about Lie algebras. Its motivation, however, comes from some work on discrete groups that we summarize below.

The weak commutativity construction was first defined for groups by Sidki [75] and goes as follows: for a group G, we define $\mathfrak{X}(G)$ as the quotient of the free product $G * G^{\psi}$ of two isomorphic copies of G by the normal subgroup generated by the elements $[g, g^{\psi}]$ for all $g \in G$. We think of this as a functor that receives the group G and returns the group with weak commutativity $\mathfrak{X}(G)$.

In a series of papers, many group theoretic properties were shown to be preserved by this functor. For instance, it preserves finiteness and solvability [75] and finite presentability [24]. Moreover, if G is finitely generated nilpotent, polycyclic-by-finite or solvable of type FP_∞, then $\mathfrak{X}(G)$ has the same property [37, 49, 53].

The group $\mathfrak{X}(G)$ has a chain of normal subgroups with some nice properties, which allows some of the proofs of the results cited above to be carried on. We write this series as $R(G) \subseteq W(G) \subseteq L(G)$ (or $R \subseteq W \subseteq L$ if G is understood) and we observe the following: W is always abelian and $\mathfrak{X}(G)/W$ is isomorphic to a subdirect product living inside $G \times G \times G$; the subquotient W/R is isomorphic to the Schur multiplier of G [70]; and $\mathfrak{X}(G)$ is a split extension of L by G.

Chapter 3 is an analysis from the scratch of an analogue of this construction in the category of Lie algebras over a field. We fix once and for all a field K with $char(K) \neq 2$, and we only consider Lie algebras over K. For any Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^\psi$ be an isomorphic copy, with isomorphism written as $x \mapsto x^\psi$. We define

$$\chi(\mathfrak{g}) = \langle \mathfrak{g}, \mathfrak{g}^\psi \mid [x, x^\psi] = 0 \text{ for all } x \in \mathfrak{g} \rangle.$$

This must be understood as the quotient of the free sum of $\mathfrak{g}$ and $\mathfrak{g}^\psi$ by the ideal generated by the relators $[x, x^\psi]$ for all $x \in \mathfrak{g}$.

We show that $\chi(\mathfrak{g})$ has a chain of ideals

$$R(\mathfrak{g}) \subseteq W(\mathfrak{g}) \subseteq L(\mathfrak{g}) \subseteq \chi(\mathfrak{g})$$

satisfying the analogous properties as the chain of normal subgroups in the group case. We can define $L(\mathfrak{g})$ as the kernel of the homomorphism $\alpha : \chi(\mathfrak{g}) \to \mathfrak{g}$ such that $\alpha(x) = \alpha(x^\psi) = x$ for all $x \in \mathfrak{g}$. Similarly, $W(\mathfrak{g})$ is defined as the kernel of $\rho : \chi(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ such that

$$\rho(x) = (x, x, 0) \text{ and } \rho(x^\psi) = (0, x, x)$$

for all $x \in \mathfrak{g}$. Finally, $R(\mathfrak{g})$ is defined as $[\mathfrak{g}, [L(\mathfrak{g}), \mathfrak{g}^\psi]]$.

Again, we write only R, W and L if there is no risk of confusion. As in the case of groups, it turns out that W is an abelian ideal, $\chi(\mathfrak{g})/W$ is a subdirect sum living inside $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$, and $\chi(\mathfrak{g})$ is a split extension of L by $\mathfrak{g}$.

We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Recall that $\mathfrak{g}$ is of homological type FP_m if the trivial $\mathcal{U}(\mathfrak{g})$-module K if of type FP_m. A Lie algebra is finitely generated if and only if it is of type FP_1, and it is of type FP_2 if it is finitely presentable. It is plausible that there are Lie algebras of type FP_2 that are not finitely presentable, but the problem of finding an example remains open.

In the case of groups, much is known about finiteness properties of subdirect products. For finite presentability and related homotopical finiteness properties, see [8, 20, 21, 22, 55] to cite a few. Some homological counterparts of the results in these papers were treated in [51] and [45].

For Lie algebras, the only tools available come from some work of Kochloukova and Martínez-Pérez [46], which contains versions of the group-theoretic results from the articles cited above. Using this, we showed that $\chi(\mathfrak{g})/W$ is of type FP_2 or finitely presentable if and only if $\mathfrak{g}$ has the same property. This, together with the exactness of $W \rightarrowtail \chi(\mathfrak{g}) \twoheadrightarrow \chi(\mathfrak{g})/W$, could be used to deduce finiteness properties for $\chi(\mathfrak{g})$ when W is finite dimensional. In the same spirit of Theorem B in [49], we give a sufficient condition for that.

Theorem B1. *If $\mathfrak{g}$ is of type FP_2 and $\mathfrak{g}'/\mathfrak{g}''$ is finite dimensional, then $W(\mathfrak{g})$ is finite dimensional.*

In [49] the authors show that the group-theoretic weak commutativity construction preserves the property of being solvable of type FP_∞. The analogous result also holds for Lie algebras, but for a much simpler reason. By [36, Thm. 1], if $\mathfrak{g}$ is solvable of type FP_∞, then it is finite dimensional. In this case of course $\mathfrak{g}'/\mathfrak{g}''$ is also finite dimensional, and then so is W, by the theorem above. Moreover, $\chi(\mathfrak{g})/W$ is clearly finite dimensional and solvable, being a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. Thus we have the following corollary.

Corollary B2. *If $\mathfrak{g}$ is solvable of type FP_∞, then so is $\chi(\mathfrak{g})$.*

The same reasoning above can be used if we assume at first that $\mathfrak{g}$ is finite dimensional.

Corollary B3. *If $\mathfrak{g}$ is finite dimensional, then so is $\chi(\mathfrak{g})$.*

The condition $\mathfrak{g}'/\mathfrak{g}''$ *is finite dimensional* is strong and does not apply, for instance, to free non-abelian Lie algebras. The following theorem shows that we actually do not need that restriction to study finite presentability.

Theorem B4. *Let $\mathfrak{g}$ be a Lie algebra. Then $\chi(\mathfrak{g})$ is finitely presentable (resp. of type FP_2) if and only if $\mathfrak{g}$ is finitely presentable (resp. of type FP_2). If $\mathfrak{f}$ is a free non-abelian Lie algebra, then $\chi(\mathfrak{f})$ is not of type FP_3.*

The proof of the analogous results for discrete groups in [24] is of geometric nature, and it uses concepts that only make sense for groups. We had to find completely algebraic proofs, and we implicitly used the construction of HNN-extensions for Lie algebras, as defined by Wasserman [79].

We have also proved the analogous result on the Schur multiplier. To prove our version, we used the description of the Schur multiplier of Lie algebras given by Ellis [32].

Theorem B5. *For all $\mathfrak{g}$ we have $W(\mathfrak{g})/R(\mathfrak{g}) \simeq H_2(\mathfrak{g}; K)$ as vector spaces over K.*

Theorem B1 raises the question of whether W can be of infinite dimension. From the theorem above we see that this can be the case for some even finitely generated Lie algebras. If we assume that $\mathfrak{g}$ is finitely presentable, which implies that $H_2(\mathfrak{g}; K)$ is finite dimensional, we fall into the same question with respect to $R(\mathfrak{g})$.

We have analyzed $R(\mathfrak{g})$ for some specific Lie algebras. It turns out that $R(\mathfrak{g})$ is trivial whenever $\mathfrak{g}$ is abelian, in contrast with the case of groups: $R(G) \neq 1$ when G is any elementary abelian 2-group of order at least 8 ([53, Prop. 4.5]). A naive explanation for this is that those groups are 2-torsion, and here we exclude the characteristic 2 case. We also showed that $R(\mathfrak{g})$ is zero if $\mathfrak{g}$ is perfect or 2-generated.

In any of these cases (that is, when $R(\mathfrak{g}) = 0$) and for $m \geq 2$, we have that $\chi(\mathfrak{g})$ is of type FP_m if and only if $\chi(\mathfrak{g})/W(\mathfrak{g})$ is of type FP_m. This is because under each of these hypotheses $\mathfrak{g}$ is of type FP_2 (being a retract of both $\chi(\mathfrak{g})$ and $\chi(\mathfrak{g})/W(\mathfrak{g})$), thus the Schur multiplier $H_2(\mathfrak{g}; K)$ is finite dimensional. This is especially interesting because $\chi(\mathfrak{g})/W(\mathfrak{g})$ has a more concrete description as a subdirect sum living inside $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$.

In order to obtain some information about $R(\mathfrak{g})$ in other cases, we investigated the structure of $L(\mathfrak{g})$. By definition $L(\mathfrak{g})$ is the kernel of the homomorphism $\alpha : \chi(\mathfrak{g}) \to \mathfrak{g}$ defined by

$$\alpha(x) = x, \;\; \alpha(x^\psi) = x$$

for all $x \in \mathfrak{g}$. It can be shown that $L(\mathfrak{g})$ is generated as a Lie subalgebra of $\chi(\mathfrak{g})$ by the elements $x - x^\psi$, for all $x \in \mathfrak{g}$.

We started by describing a finite generating set for it as a Lie algebra, for any finitely generated Lie algebra $\mathfrak{g}$. We wrote then a presentation for $L(\mathfrak{g})$ in terms of these generators. This is Theorem 3.32 and the remarks following it. We do not state it here completely because that would require introducing a lot of notation. The set of generators is

$$\{x - x^\psi, [x, y] - [x, y]^\psi\}_{x,y},$$

where x and y run through a generating set for $\mathfrak{g}$, and all defining relations come from some manipulation of the identity

$$[x - x^\psi, [y - y^\psi, z - z^\psi]] = [x, [y, z]] - [x^\psi, [y^\psi, z^\psi]], \tag{1}$$

which holds for all $x, y, z \in \mathfrak{g}$.

There is no counterpart of this in the case of discrete groups, that is, no presentation for $L(G)$ is known in that case. This lead to slightly different results in what follows.

Even if $\mathfrak{g}$ is finitely presented, our presentation will be in general infinite. This is expected: $L(\mathfrak{g})$ is not finitely presented if $\mathfrak{g}$ is free of rank at least 2 (see Proposition 3.33).

What makes this presentation interesting is that it helps us to perform computations on $L(\mathfrak{g})$, and since $\chi(\mathfrak{g}) \simeq L \rtimes \mathfrak{g}$, this allows us to obtain results on the structure of $\chi(\mathfrak{g})$. We used this to study the question of nilpotency.

Theorem B6. *Suppose that $\mathfrak{g}$ is nilpotent of class c.*

1. *If c is odd, then $\chi(\mathfrak{g})$ is nilpotent of class at most $c+1$.*
2. *If c is even, then $\chi(\mathfrak{g})$ is nilpotent of class at most $c+2$.*

This should be compared with the article of Gupta, Rocco and Sidki [37], where the nilpotency of the group-theoretic construction is studied. There it is shown that if a group G is nilpotent of class c, then $\mathfrak{X}(G)$ is nilpotent of class at most $max\{c+2, d(G)\}$, where $d(G)$ is the minimal number of generators of G. Here the number of generators play no role.

Another consequence is that we could obtain concrete descriptions of $\chi(\mathfrak{n}_{m,c})$, where $\mathfrak{n}_{m,c}$ is the free nilpotent Lie algebra of class c and rank m, for $c = 2$ or 3. In particular, we obtained

$$dim R(\mathfrak{n}_{m,2}) = \frac{1}{24}(3m^4 - 2m^3 - 15m^2 + 14m),$$

which grows with m and is non-zero for $m \geq 3$. Interestingly enough, we have

$$dim R(\mathfrak{n}_{m,2}) = dim R(\mathfrak{n}_{m,3})$$

for all m.

This should not be interpreted as a clue that $dim R(\mathfrak{n}_{m,c})$ is constant on $c \geq 2$. In fact, the same principle that gives better bound for the nilpotency class of $\chi(\mathfrak{g})$ when $\mathfrak{g}$ is nilpotent of *odd* class (which is basically the fact that (1) is an identity involving brackets of odd length) makes it not that surprising that $R(\mathfrak{n}_{m,3})$ does not have "new" non-trivial elements with respect to $R(\mathfrak{n}_{m,2})$.

The presentation we wrote for $L(\mathfrak{g})$ is also suitable for the application of methods of Gröbner-Shirshov bases. These are some techniques that allow us to decide in some cases if a given element w of a free Lie algebra $\mathfrak{f}$ lies or not in some ideal $I \subseteq \mathfrak{f}$. Equivalently, we may decide if the image of such element w is non-trivial in the quotient $\mathfrak{f}/I$. See for instance [16, 17, 73].

We did not solve such problem for $L(\mathfrak{g})$ in the generality mentioned above, but we argued enough to obtain the following.

Theorem B7. *If $\mathfrak{g}$ is free non-abelian of rank at least 3, then $R(\mathfrak{g})$ is infinite dimensional.*

We can obtain from the theorem above the first example of a finitely presentable Lie algebra $\mathfrak{g}$ such that $W(\mathfrak{g})$ is of infinite dimension. A proof that the analogue of Theorem B7 also holds for groups has been announced by Bridson and Kochloukova (see [24, Question 5.3]).

Recall that the *cohomological dimension* $cd(\mathfrak{h})$ of a Lie algebra $\mathfrak{h}$ is the smallest positive integer n such that K admits a projective resolution

$$\mathcal{P} : 0 \to P_n \to \ldots \to P_1 \to P_0 \to K \to 0$$

over $\mathcal{U}(\mathfrak{h})$. If there is no such resolution, we say that $\mathfrak{h}$ is of infinite cohomological dimension. An infinite dimensional abelian Lie algebra, for instance, has infinite cohomological dimension. Thus we can obtain the following corollary of Theorem B7.

Corollary B8. *If $\mathfrak{g}$ is free non-abelian of rank at least 3, then $\chi(\mathfrak{g})$ is of infinite cohomological dimension.*

This ends the contents of Chapter 3, which also appear in [62] and [63]. In Chapter 4 we talk again about weak commutativity, but now in the category of pro-p groups for a fixed prime p. The results are similar to those we obtained for Lie algebras. The methods, however, are a little bit different: the main strategy to prove the results is to reduce somehow the problem to the same problem in the discrete case, using the inverse limit properties or pro-p completions. The material in Chapter 4 was developed and submitted for publication in joint work with Kochloukova [47].

The theory of homological finiteness properties of pro-p groups is in some sense simpler when compared to the other categories that we have discussed. For instance a pro-p group G is of type FP_m if and only if $H_i(G; \mathbb{Z}_p)$ is a finitely generated pro-p group for all $i \leq m$, which in turn is equivalent with $H_i(G; \mathbb{F}_p)$ being a finite p-group for all $i \leq m$ [39]. In the cases of discrete groups and Lie algebras, the analogous assertion about homologies would be only necessary in general. Furthermore, finitely presentability is equivalent to the property FP_2 for pro-p groups. We know that this is not true for discrete groups by the work and Bestvina and Brady [9], and for Lie algebras the question is open.

Let G be a pro-p group. We define $\mathfrak{X}_p(G)$ by the pro-p presentation

$$\mathfrak{X}_p(G) = \langle G, G^{\psi} \mid [g, g^{\psi}] = 1 \text{ for all } g \in G \rangle_p,$$

where G^{ψ} is an isomorphic copy of G via $g \mapsto g^{\psi}$ and $\langle - \mid - \rangle_p$ denotes presentation by generators and relators in the category of pro-p groups.

The structural theory of $\mathfrak{X}_p(G)$ is deduced similarly to the cases of discrete groups and Lie algebras. Thus we defined by analogy the normal subgroups $L_p(G)$, $W_p(G)$ and $R_p(G)$ of $\mathfrak{X}_p(G)$.

Again $\mathfrak{X}_p(G)/W_p(G)$ is a subdirect product of $G \times G \times G$, but the theory for finiteness properties of subdirect products that we needed was not developed in this case, though some results for subdirect products of free pro-p or Demushkin groups were obtained by Kochloukova and Short [48].

First, we obtained a version of the $(n-1)-n-(n+1)$ Theorem for pro-p groups.

Theorem C1. *Let $p_1 : G_1 \to Q$ and $p_2 : G_2 \to Q$ be surjective homomorphisms of pro-p groups. Suppose that $ker(p_1)$ is of type FP_{n-1}, both G_1 and G_2 are of type FP_n and Q is of type FP_{n+1}. Then the fiber product*

$$G = \{(g_1, g_2) \in G_1 \times G_2 \mid p_1(g_1) = p_2(g_2)\}$$

is of type FP_n.

Versions of this were considered for discrete groups in [7, 20, 45, 51], but it was not proved in the general case. Here we build on the work of Kuckuck [51], which leads to stronger results in the category of pro-p groups.

As a corollary we deduced the following result.

Corollary C2. *Let $G_1, \ldots, G_n$ be pro-p groups of type FP_k for some $n \geq 1$. Denote by $p_{i_1,\ldots,i_k}$ the projection $G_1 \times \ldots \times G_n \twoheadrightarrow G_{i_1} \times \ldots \times G_{i_k}$ for $1 \leq i_1 < \ldots < i_k \leq n$. Let $H \subseteq G_1 \times \ldots \times G_n$ be a closed subgroup such that $p_{i_1,\ldots,i_k}(H)$ is of finite index in $G_{i_1} \times \ldots \times G_{i_k}$ for all $1 \leq i_1 < \ldots < i_k \leq n$. Then H is of type FP_k.*

We used the corollary above, for instance, to show that $\mathfrak{X}_p(G)/W_p(G)$ is finitely presented if and only if G has the same property. In the following theorem we compile some of the main results that we obtained for $\mathfrak{X}_p(G)$. We denote by $\mathfrak{X}(H)$ the original weak commutativity construction for discrete groups.

Theorem C3. *Let G be a pro-p group. Then*

1. *If G is a finite p-group, then $\mathfrak{X}(G) \simeq \mathfrak{X}_p(G)$;*

2. *If $\mathcal{P}$ is one of the following classes of pro-p groups: solvable; finitely generated nilpotent; finitely presented; poly-procyclic; analytic pro-p and $G \in \mathcal{P}$, then $\mathfrak{X}_p(G) \in \mathcal{P}$;*

3. *If G is a free non-procyclic finitely generated pro-p group, then $\mathfrak{X}_p(G)$ is not of homological type FP_3.*

In some of the proofs above we used two important connections between the weak commutativity construction for pro-p groups and the discrete one. The first is that if a given pro-p group G is the inverse limit $G = \varprojlim G_i$, where each G_i is a finite p-group, then $\mathfrak{X}_p(G)$ is the inverse limit of the system $\{\mathfrak{X}(G_i)\}_i$ of finite p-groups. The second is the completion: if $\widehat{H}$ is the pro-p completion of the discrete group H, then $\mathfrak{X}_p(\widehat{H})$ is the pro-p completion of $\mathfrak{X}(H)$. Both these facts allow us to use the results that we already know that hold for discrete groups in order to deduce the analogues in the pro-p case.

Theorem C4. *$H_2(G; \mathbb{Z}_p) \simeq W_p(G)/R_p(G)$ for all pro-p groups G.*

Here we used the notion of non-abelian tensor product of pro-p groups developed by Moravec in [65].

In item 3 of Theorem C3, we considered the case of *p-adic analytic pro-p groups*. These are the pro-p groups that admit an analytic structure of Lie group over the ring $\mathbb{Q}_p$ of p-adic numbers [52]. We consider here however a pure group theoretic approach developed by Lubotzky, Mann, du Sautoy and Segal [31]. Concretely, a pro-p group is p-adic analytic if there is a universal bound on the number of (topological) generators of its closed subgroups.

The thesis is structured as follows: in Chapter 1 we introduce the notation and discuss some background material. Chapters 2, 3 and 4 contains our results about discrete groups, Lie algebras and pro-p groups, respectively. These three chapters can be read independently, though Chapter 4 makes reference from time to time to Chapter 3 for motivation or comparison.

1 Preliminaries

In this chapter we review some of the theory that we will be using in the thesis. First, we recall some facts about modules of type FP_m over any associative unitary ring, and then we specialize it to discrete groups, Lie algebras and pro-p groups in separate sections. We also introduce the theory of Gröbner-Shirshov bases of Lie algebras, which we apply in Section 3.9. Most of the content here is based on [10] and [80].

1.1 Generalities

Let R be an associative ring with unity. Let A be a (left or right) R-module. For an integer $m \geq 0$, we say that A is of *homological type* FP_m if there is a free resolution

$$\mathcal{P} : \ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to A \to 0$$

with P_j finitely generated for all $0 \leq j \leq m$.

It is clear that the properties FP_0 and FP_1 are equivalent to finite generation and finite presentability of modules, respectively. We think of the type FP_m for $m \geq 2$ as successive generalizations of these well known finiteness properties. We say that A is *of type* FP_∞ if it is of type FP_m for all $m \geq 0$.

Suppose for instance that R is a (left) noetherian ring and let A be a finitely generated (left) R-module. Then there is an epimorphism $d_0 : F_0 \to A$, where F_0 is a free R-module of finite rank. By noetherianity the kernel $ker(d_0)$ is finitely generated as an R-module, so we can again find an epimorphism $\tilde{d}_1 : F_1 \to ker(d_0)$, with F_1 free of finite rank. By composing $\tilde{d}_1$ with the inclusion $inc : ker(d_0) \to F_0$ we obtain $d_1 : F_1 \to F_0$, and the following sequence is exact:

$$F_1 \to F_0 \to A \to 0.$$

We can continue with this reasoning: $ker(d_1)$ is finitely generated, so we can find an epimorphism $\tilde{d}_2 : F_2 \to ker(d_1)$ with F_2 free of finite rank and so on. This procedure results in a free resolution for A as an R-module with finitely generated free modules in each step, so A is of type FP_∞.

The FP_m properties can be characterized in terms of the derived functors Tor and Ext and how they relate with direct products and direct limits of R-modules, as follows.

Theorem 1.1. *[10, Theorem 1.3] Let B be a left R-module and let $m \geq 1$ be a positive integer. The following assertions are equivalent:*

1. *B is of type FP_m;*

2. *For any direct system $\{A_i\}_{i\in I}$ of left R-modules over the directed set I with $\varinjlim A_i = 0$, one has $\varinjlim Ext^k_R(B, A_i) = 0$ for all $k \leq m$;*

3. *B is finitely presented and for any direct product $\prod_\lambda R$ of copies of R, we have $Tor^R_k(\prod_\lambda R, B) = 0$ for all $1 \leq k < m$.*

With this in hand, one may prove some useful criteria that make these finiteness properties more treatable.

Lemma 1.2. *[10, Prop. 1.4] Let $A \rightarrowtail B \twoheadrightarrow C$ be an exact sequence of R-modules. Then:*

1. *If A and C are of type FP_m, then so is B;*

2. *If A is of type FP_m and B is of type FP_{m+1}, then C is of type FP_{m+1};*

3. *If B is of type FP_m and C is of type FP_{m+1}, then A is of type FP_m.*

1.2 Discrete groups

For a discrete group G, we denote by $\mathbb{Z}G$ its group ring. By definition this is the free abelian group with basis G, and the multiplication is induced by the operation of G:

$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{g,h\in G} a_g b_h gh,$$

where $a_g, b_g \in \mathbb{Z}$ for all $g \in G$. The *augmentation map* $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the ring homomorphism defined by $\epsilon(g) = 1$ for all $g \in G$. Its kernel, denoted by $Aug(\mathbb{Z}G)$, is the *augmentation ideal* of $\mathbb{Z}G$.

We say that G is *of type FP_m* if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is of type FP_m. It is clear that any group is of type FP_0. For $m = 1$ and $m = 2$ we have the following.

Proposition 1.3. *For any group G we have:*

1. *G is type FP_1 if and only if it is finitely generated;*

2. *If G is finitely presentable, then it is of type FP_2.*

Proof. See [10], Propositions 2.1 and 2.2. □

The question of whether all groups of type FP_2 were finitely presentable remained open for a long time, but it was answered in the negative by Bestvina and Brady [9].

Some other important facts about the property FP_m for groups are collected below.

Proposition 1.4. *Let G be any group and $H \subseteq G$ a subgroup.*

1. *If H is of finite index in G, then G is of type FP_m if and only if H is of type FP_m;*
2. *If H is a retract of G and G is of type FP_m, then H is of type FP_m too;*
3. *If H is normal in G and H is of type FP_∞, then G is of type FP_m if and only if G/H is of type FP_m.*

Proof. See [10], Section 2 of Chapter I. □

Example 1.5. *Many well known groups are of type FP_∞, that is, of type FP_m for all m. This includes all finite groups, all right-angled Artin groups (including free groups of finite rank), polycyclic groups and Thompson's groups F, T and V.*

Example 1.6. *We can distinguish the sequence $\{FP_m\}_m$ of finiteness properties of groups by looking at subgroups of direct products of free groups. Concretely, if F is free of finite rank and $G = F \times \cdots \times F = F^m$, then the subgroup $S \subseteq G$ containing exactly the tuples $(g_1, \ldots, g_m)$ such that $g_1 \cdots g_m \in [F, F]$ is of type FP_{m-1}, but it is not of type FP_m. This can be proved, for instance, with methods of Σ-theory.*

The example above illustrates that the property FP_m is not in general inherited by subgroups. The study of this situation is mostly done via Σ-invariants, which we treat in Chapter 2.

1.2.1 Wreath products

Recall that the (permutational restricted) *wreath product* $H \wr_X G$ is the semi-direct product

$$\Gamma = H \wr_X G = (\oplus_{x \in X} H_x) \rtimes G,$$

where each H_x is a copy of H and G acts by permuting the copies of H with respect some fixed G-action on X. To avoid trivialities, we assume that $X \neq \emptyset$ and $H \neq 1$.

A famous case is when $H = C_2$ is the cyclic group of order two and $G = \mathbb{Z}$, which acts on itself by left multiplication. The wreath product $C_2 \wr_{\mathbb{Z}} \mathbb{Z}$ is known as the *Lamplighter group*. More generally, if $X = G$ and G acts on X be left multiplication, we say that $H \wr_G G =: H \wr G$ is a *regular* wreath product.

It was shown by G. Baumslag [6] that regular wreath products are rarely finitely presentable. Namely, he showed that if this is the case, then the base group H is trivial or the top group G is finite.

In the more general case of permutational wreath products, Cornulier showed in [26] that the situation is more complicated. Recall that given an action of a group G on a set X, the *diagonal action* of G on X^2 is defined by $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Theorem 1.7. *[26] Let $\Gamma = H \wr_X G$ be a wreath product. Then:*

1. *Γ is finitely generated if and only if G and H are finitely generated and G acts with finitely many orbits on X;*
2. *Γ is finitely presentable if and only if G and H are finitely presentable, G acts (diagonally) with finitely many orbits on X^2 and the stabilizers $stab_G(x)$ are finitely generated for all $x \in X$.*

It will be useful to see how a finite presentation for $\Gamma = H \wr_X G$ (when possible) was obtained by Cornulier. By definition Γ is generated by one copy of H_x of H for each $x \in X$, together with G, and has relations of the following types:

$$[H_x, H_y] = 1$$

if $x \neq y$ and

$$^g h_x = h_{g \cdot x}$$

for all $g \in G$ and $h \in H$, where h_x denotes the image of the element $h \in H$ in H_x. Of course we must also add the relations of G and H_x, for all x.

Now suppose that X is the disjoint union of the orbits of $G \cdot x_i$, for $1 \leq i \leq n$. By writing each H_x in terms of some H_{x_i}, we obtain

$$\Gamma = \langle H_{x_1}, \ldots, H_{x_n}, G | [^g H_{x_i}, H_{x_j}] = 1, [stab_G(x_i), H_{x_i}] = 1 \text{ for all } 1 \leq i, j \leq n, g \in J_{i,j} \rangle,$$

where $J_{i,j} \subset G$ is a set of representatives for the non-trivial double cosets of the pair (H_{x_i}, H_{x_j}) in G. It can be shown that all $J_{i,j}$ are finite exactly when G acts with finitely many orbits on X^2. By choosing finite presentations for G and H and finite generating sets for $stab_G(x_i)$ for all i, we can finally write down a presentation of Γ.

A version of Theorem 1.7 for the FP_m properties was obtained by Bartholdi, Cornulier and Kochloukova. Recall that if G acts on X, then the diagonal action of G on X^n is defined by $g \cdot (x_1, \ldots, x_n) = (g \cdot x_1, \ldots, g \cdot x_n)$.

Theorem 1.8. *[5] Let $\Gamma = H \wr_X G$ be a wreath product. Suppose that $H/[H, H]$ is infinite. Then Γ is of type FP_m if and only if the following conditions hold:*

1. *G and H are of type FP_m;*
2. *G acts (diagonally) with finitely many orbits on X^m;*

3. All the stabilizers of the action G on X^k are of type FP_{m-k} for all $1 \leq k \leq m$.

Remark 1.9. *For $m \leq 2$ the hypothesis on $H/[H,H]$ is not necessary by Theorem 1.7 and [5, Lemma 5.1].*

1.3 Lie algebras

Let K be a field and let $\mathfrak{g}$ be a Lie algebra over K. For any subset $S \subset \mathfrak{g}$, we denote by $\langle S \rangle$ and by $\langle\langle S \rangle\rangle$ the subalgebra and the ideal, respectively, of $\mathfrak{g}$ generated by S.

We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. This is the quotient of the tensor algebra $T(\mathfrak{g})$ defined on the vector space $\mathfrak{g}$ by the ideal generated by the elements $x \otimes y - y \otimes x - [x,y]$, for all $x, y \in \mathfrak{g}$. The Poincaré-Birkhoff-Witt Theorem tells us that the canonical map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.

Let $\mathfrak{g}$ be a free Lie algebra of rank n. Denote by $\gamma_k(\mathfrak{g})$ the k-th term of the lower central series, that is, $\gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\gamma_{k+1}(\mathfrak{g}) = [\mathfrak{g}, \gamma_k(\mathfrak{g})]$ for all k. *Witt's dimension formula* ([18], Théorème 3 of II.3.3) tells us the dimension of the successive quotients of this series:

$$dim(\gamma_k(\mathfrak{g})/\gamma_{k+1}(\mathfrak{g})) = \frac{1}{k}(\sum_{d|k} \mu(d) n^{k/d}), \tag{1.1}$$

where μ is the Möbius function. Recall that $\mu : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n \text{ is square-free and has an even number prime divisors} \\ -1, & \text{if } n \text{ is square-free and has an odd number prime divisors} \\ 0, & \text{otherwise.} \end{cases}$$

Notice that we can use formula (1.1) to compute the dimension of the free nilpotent Lie algebra of class c on n generators.

The field K has a structure of $\mathcal{U}(\mathfrak{g})$-module defined by $x \cdot \lambda = 0$ for all $x \in \mathfrak{g}$ and $\lambda \in K$ (here $x \in \mathfrak{g}$ is identified with its image in $\mathcal{U}(\mathfrak{g})$). In this case we say that K is the *trivial* $\mathcal{U}(\mathfrak{g})$-module. The module structure of K can also be seen as the one that comes from the K-algebra homomorphism $\epsilon : \mathcal{U}(\mathfrak{g}) \to K$ defined by $\epsilon(x) = 0$ for all $x \in \mathfrak{g}$. We call this homomorphism the *augmentation map*, and its kernel $ker(\epsilon) = Aug(\mathcal{U}(\mathfrak{g}))$ is the *augmentation ideal* of $\mathcal{U}(\mathfrak{g})$.

The homology of $\mathfrak{g}$ is defined in terms of its universal enveloping algebra. Namely, for A a $\mathcal{U}(\mathfrak{g})$-module, we put

$$H_i(\mathfrak{g}; A) := Tor_i^{\mathcal{U}(\mathfrak{g})}(K, A)$$

for any $i \geq 0$. In low homological degree we have:

$$H_0(\mathfrak{g}; A) = A_{\mathfrak{g}} := A/\langle x \cdot a \mid x \in \mathfrak{g}, a \in A \rangle$$

and

$$H_1(\mathfrak{g}; K) = \mathfrak{g}^{ab} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

Example 1.10. *If $\mathfrak{g} = K^n$ is an abelian Lie algebra of dimension n, then $\mathcal{U}(\mathfrak{g})$ is a polynomial K-algebra on n variables. In particular, $H_i(\mathfrak{g}; K) \simeq \bigwedge^i \mathfrak{g} \simeq K^{\binom{n}{i}}$.*

For any extension $\mathfrak{h} \rightarrowtail \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ of Lie algebras and a $\mathcal{U}(\mathfrak{g})$-module A, one can consider the Lyndon-Hochschild-Serre (LHS) spectral sequence, which has the following term on its second page:

$$E^2_{p,q} = H_p(\mathfrak{q}; H_q(\mathfrak{h}; A)) \Rightarrow H_{p+q}(\mathfrak{g}; A).$$

The arrow above indicates the convergence. The differential d^r of the r-th page has bidegree $(-r, r+1)$. Furthermore, the associated 5-term exact sequence is written as

$$H_2(\mathfrak{g}; A) \to H_2(\mathfrak{q}; H_0(\mathfrak{h}; A)) \to H_0(\mathfrak{q}; H_1(\mathfrak{h}; A)) \to H_1(\mathfrak{g}; A) \to H_1(\mathfrak{q}; H_0(\mathfrak{h}; A)) \to 0.$$

This can be applied for instance to an extension $\mathfrak{r} \to \mathfrak{f} \to \mathfrak{g}$ with $\mathfrak{f}$ free, which gives the familiar Hopf formula:

Proposition 1.11. *Let $\mathfrak{g} = \mathfrak{f}/\mathfrak{r}$, where $\mathfrak{f}$ is a free Lie algebra. Then*

$$H_2(\mathfrak{g}; K) \simeq \frac{[\mathfrak{f}, \mathfrak{f}] \cap \mathfrak{r}}{[\mathfrak{f}, \mathfrak{r}]}.$$

Another particular case that we will use often is when $A = K$ is the trivial $\mathcal{U}(\mathfrak{g})$-module and $\mathfrak{h}$ is a central ideal of $\mathfrak{g}$. Then the sequence is:

$$H_2(\mathfrak{g}; K) \to H_2(\mathfrak{q}; K) \to \mathfrak{h} \to \mathfrak{g}^{ab} \to \mathfrak{q}^{ab} \to 0,$$

where $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$.

We say that a Lie algebra $\mathfrak{g}$ is *of type FP_m* if K, with the trivial module structure, is of type FP_m over $\mathcal{U}(\mathfrak{g})$. If $\mathfrak{g}$ is of type FP_m for all m, we say that $\mathfrak{g}$ is *of type FP_∞*. It is clear that all Lie algebras are of type FP_0.

Example 1.12. *If $\mathfrak{g}$ is a free Lie algebra with free basis $\{x_1, \dots, x_n\}$, then $Aug(\mathcal{U}(\mathfrak{g}))$ is a free $\mathcal{U}(\mathfrak{g})$-module with basis $\{x_1, \dots, x_n\}$. In particular, K admits the free resolution*

$$0 \to \oplus_{j=1}^n \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \to K \to 0,$$

so $\mathfrak{g}$ is of type FP_∞.

Example 1.13. *If $\mathfrak{g}$ is finite dimensional, then $\mathcal{U}(\mathfrak{g})$ is both left and right noetherian (see [38], Theorem 6 in Section 3, Chapter V). The trivial $\mathcal{U}(\mathfrak{g})$-module K is finitely generated, so it admits a free resolution that is finitely generated in each step, that is, $\mathfrak{g}$ is of type FP_∞.*

The same argument used in the case of groups gives the following proposition.

Proposition 1.14. *Let $\mathfrak{g}$ be a Lie algebra over K. Then*

1. *$\mathfrak{g}$ is of type FP_1 if and only if it is finitely generated;*
2. *If $\mathfrak{g}$ is finitely presentable, then it is of type FP_2.*

We note that, unlike in the case of groups, there are no known examples of Lie algebras of type FP_2 that are not finitely presentable. These two finiteness properties can however be related by the following result.

Proposition 1.15. *A Lie algebra $\mathfrak{g}$ is of type FP_2 if and only if it is the quotient of a finitely presented Lie algebra $\mathfrak{h}$ by an ideal I that is a perfect Lie algebra, that is, $I = [I, I]$.*

Proof. First we note that a finitely generated Lie algebra $\mathfrak{g}$ is of type FP_2 if and only if its relation module is finitely generated, that is, if $\mathfrak{g} = F/R$ with F free, then $R^{ab} = R/[R, R]$ is finitely generated as a $\mathcal{U}(F)$-module via the adjoint action. This can be proved exactly as in the group case [10, Prop. 2.2].

Suppose that $\mathfrak{g}$ is of type FP_2. Write $\mathfrak{g} = F/R$, where F is finitely generated free and R^{ab} is finitely generated over $\mathcal{U}(F)$. Choose a finite set $S = \{s_1, \dots, s_n\} \subset R$ whose image in R^{ab} is a generating set. Let N be the ideal of F generated by S. Then the Lie algebra $\mathfrak{h} = F/N$ is finitely presented and the natural map $\pi : \mathfrak{h} \to \mathfrak{g}$ has kernel R/N. The choice of S implies that any element of R is congruent modulo N to some element of $[R, R]$, that is, R/N is perfect.

Conversely, if $\mathfrak{g} = F/R$ is the quotient of the finitely presented Lie algebra $\mathfrak{h} = F/N$ and the kernel R/N is perfect, then the image in R^{ab} of any finite generating set of N is a generating set for R^{ab} over $\mathcal{U}(F)$. Thus $\mathfrak{g}$ is of type FP_2. □

Recall that $\mathfrak{h}$ is a *split quotient* of a Lie algebra $\mathfrak{g}$ if there are homomorphisms $\pi : \mathfrak{g} \to \mathfrak{h}$ and $\sigma : \mathfrak{h} \to \mathfrak{g}$ such that $\pi \circ \sigma = id_{\mathfrak{h}}$.

Proposition 1.16. *If $\mathfrak{g}$ is finitely presentable (resp. of type FP_m), then any split quotient of $\mathfrak{g}$ is also finitely presentable (resp. of type FP_m).*

Proof. This is well-known for finite presentability. For the FP_m properties we consider split homomorphisms $\pi : \mathfrak{g} \to \mathfrak{h}$ and $\sigma : \mathfrak{h} \to \mathfrak{g}$ (with $\pi \circ \sigma = id$). If A is any $\mathcal{U}(\mathfrak{g})$-module, then by functoriality we obtain homomorphisms

$$\pi_* : H_k(\mathfrak{g}; A) \to H_k(\mathfrak{h}; A) \text{ and } \sigma_* : H_k(\mathfrak{h}; A) \to H_k(\mathfrak{g}; A)$$

with $\pi_* \circ \sigma_* = (\pi \circ \sigma)_* = 0$ for any k. By taking A to be any product of copies of $\mathcal{U}(\mathfrak{g})$, we see that if the condition 3 of Theorem 1.1 holds for $\mathfrak{g}$, then it also holds for $\mathfrak{h}$. □

The following technical result from the work of Kochloukova and Martínez-Pérez [46] concerns the homology of free Lie algebras. We will need it in Section 3.3.3.

Lemma 1.17. *[46, Lemma 3.1] Let $\mathfrak{f}$ be a free Lie algebra and let A be a $\mathcal{U}(\mathfrak{f})$-module. Suppose that $H_1(\mathfrak{f}; A)$ is finite dimensional over K. Suppose further that $c \cdot A$ is also finite dimensional over K for some $c \in \mathcal{U}(\mathfrak{f}) \smallsetminus \{0\}$. Then A is itself finite dimensional over K.*

The article we mentioned above is actually devoted to subdirect sums of Lie algebras. Recall that a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ of a direct sum is a *subdirect sum* if $p_i(\mathfrak{h}) = \mathfrak{g}_i$ for all i, where $p_i : \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \twoheadrightarrow \mathfrak{g}_i$ is the canonical projection. We can also consider the image of $\mathfrak{h}$ by the projections $p_{i,j} : \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \twoheadrightarrow \mathfrak{g}_i \oplus \mathfrak{g}_j$ for $i \neq j$. These homomorphisms play a role in the following criterion for finiteness properties of subdirect sums.

Theorem 1.18. *[46, Cor. D1, Cor. F1] Let $\mathfrak{h} \subseteq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a subdirect sum with $\mathfrak{h} \cap \mathfrak{g}_i \neq 0$ for all i. Suppose that $\mathfrak{g}_i$ is finitely presentable (resp. of type FP_2) for all i. Suppose further that $p_{i,j}(\mathfrak{h}) = \mathfrak{g}_i \oplus \mathfrak{g}_j$ for all $i \neq j$. Then $\mathfrak{h}$ is finitely presentable (resp. of type FP_2).*

1.3.1 Gröbner-Shirshov bases

We recall here briefly the theory of Gröbner-Shirshov bases, following the exposition in [17]. The original arguments are due to Shirshov [73], and the modern approach was initiated by Bokut [16].

Let $\mathfrak{g}$ be the free Lie algebra with free basis $X = \{x_1, \ldots, x_n\}$. Consider in the set of associative words with letters in X the lexicographic order, with $x_1 > \cdots > x_n$ and $u > v$ if u is a initial subword of v. One of such words w is *regular* (or a *Lyndon-Shirshov associative word*) if $w = uv$ implies $uv >_{lex} vu$, for non-trivial subwords u, v.

Example 1.19. *If $X = \{x_1, x_2, x_3\}$, then $w = x_1x_3x_2$ is regular, but $u = x_1x_3x_1x_2$ is not, since $u <_{lex} x_1x_2x_1x_3$.*

A *non-associative word* $[w]$ is an associative word w endowed with some bracketing. A non-associative word $[w]$ is *regular* (or a *Lyndon-Shirshov non-associative word*) if the associative word w obtained by removing all brackets is associative regular and:

1. If $[w] = [u][v]$, then both $[u]$ and $[v]$ are non-associative regular, and
2. If $[w] = [[u_1][u_2]][v]$, then $u_2 \leq_{lex} v$.

For any regular associative word w there is a unique bracketing (w) that is non-associative regular, and the set of all non-associative regular words is a basis of the free Lie algebra [72]. For short we will also call these words *monomials*.

Remark 1.20. *The unique bracketing* (w) *for a regular word* w *can be obtained inductively as follows: if* v *is the longest proper regular suffix of* w*, with* $w = uv$*, then* $(w) = (u)(v)$*.*

Example 1.21. *If* $w = x_1x_3x_2$*, then* x_2 *is the longest proper regular suffix, thus* $(w) = (x_1x_3)(x_2)$*. In the usual bracketing notation, this is* $(w) = [[x_1, x_3], x_2]$*.*

Example 1.22. *If* $w = x_1x_2x_1x_3$*, the word* x_1x_3 *is the longest proper regular suffix. Then* $(w) = (x_1x_2)(x_1x_3)$*, or simply* $(w) = [[x_1, x_2], [x_1, x_3]]$*.*

Let $d : X \to \mathbb{N} = \{1, 2, \ldots\}$ be any function. For any regular associative word $w = x_{i_1} \cdots x_{i_m}$, we put:

$$d(w) = d(x_{i_1} \cdots x_{i_m}) := d(x_{i_1}) + \ldots + d(x_{i_m}).$$

We say that $d(w)$ is the *degree* of w. We consider the *weight lexicographic* ordering, with respect to d, on the set of associative regular words: $w_1 \preceq w_2$ if $d(w_1) < d(w_2)$, or if $d(w_1) = d(w_2)$ but $w_1 \leq_{lex} w_2$. The degree function and the associated ordering can be considered also for non-associative regular words via the bijection given by the unique bracketing.

Any $f \in \mathfrak{g} \setminus \{0\}$ may be written as a linear combination of regular non-associative words. We denote by $\overline{f}$ the highest (with respect to the weight lexicographic ordering) corresponding regular associative word appearing with non-zero coefficient. We say that $\overline{f}$ is the *associative carrier* of f. If the coefficient of $\overline{f}$ in f is $1 \in K$, we say that f is *monic*.

Lemma 1.23. *[17, Lemma 2.11.15] Let* w *be a regular associative word and suppose that* u *is subword of* w *that is also regular. Let* $[u]$ *be an arbitrary bracketing of* u*. Then there is a bracketing* $[w]$ *of* w *that extends the bracketing* $[u]$ *and such that* $\overline{[w]} = w$*.*

Example 1.24. *Let* $w = x_1x_2x_3x_2$ *and* $u = x_2x_3$*. The only bracketing we can consider for* u *is* $[u] = (u) = [x_2, x_3]$*. Then the bracketing* $[[x_1, [x_2, x_3]], x_2]$ *of* w *satisfies what is required in the lemma. The bracketing* $[w]' = [x_1, [[x_2, x_3], x_2]]$ *also extends* $[u]$*, but* $\overline{[w]'} = x_1x_2x_2x_3 \neq w$*.*

Suppose that $f, g \in \mathfrak{g}$ are monic and satisfy $\overline{f} = ab$ and $\overline{g} = bc$ for some associative words a, b, c. Let $u = abc$. Notice that u is regular. By Lemma 1.23, we can consider two bracketings $[u]_1$ and $[u]_2$ for the word u such that $[u]_1$ extends the regular bracketing $(\overline{f})$ of $\overline{f}$ and $[u]_2$ extends the regular bracketing $(\overline{g})$ of $\overline{g}$, and such that $\overline{[u]_1} = \overline{[u]_2} = u$.

Let u_1 and u_2 be the elements of $\mathfrak{g}$ obtained from $[u]_1$ and $[u]_2$ by substituting $(\overline{f})$ and $(\overline{g})$ with f and g, respectively. The *first-order composition* $(f,g)^I_u$ of f and g with respect to u is defined as

$$(f,g)^I_u = u_1 - u_2.$$

Remark 1.25. *The "substitution" above can be formalized by writing $[u]_1$, say, as the image of some right-normed bracket $\delta = [y_1, \ldots, y_m]$ of a free Lie algebra $\mathfrak{h}$ with free basis $\{y_1, \ldots, y_m\}$ by some homomorphism*

$$\varphi : \mathfrak{h} \to \mathfrak{g}$$

with $\varphi(y_m) = (\overline{f})$. Then u_1 is $\varphi'(\delta)$, where $\varphi' : \mathfrak{h} \to \mathfrak{g}$ is defined by $\varphi'(y_m) = f$ and $\varphi'(y_i) = \varphi(y_i)$ for $i < m$.

Remark 1.26. *The composition depends on the choice of the bracketings of u. This will not be a problem: we will only need to use any composition for a given pair of elements $f, g \in \mathfrak{g}$ such that $\overline{f}$ and $\overline{g}$ share a subword as above.*

Example 1.27. *Let $f = [x_1, x_2] + x_4$ and $g = [[x_2, x_3], x_3] + 2[x_1, x_2]$. Then $\overline{f} = x_1x_2$ and $\overline{g} = x_2x_3x_3$. The composition of f and g is defined with respect to*

$$u = x_1x_2x_3x_3 = \overline{f}x_3x_3 = x_1\overline{g}.$$

We can take $[u]_1 = [[[x_1, x_2], x_3], x_3]$ and $[u]_2 = [x_1, [[x_2, x_3], x_3]]$. Thus

$$u_1 = [[f, x_3], x_3] = [x_1, [[x_2, x_3], x_3]] + [[[x_1, x_3], x_3], x_2] + 2[[x_1, x_3], [x_2, x_3]] + [[x_4, x_3], x_3]$$

and

$$u_2 = [x_1, g] = [x_1, [[x_2, x_3], x_3]] + 2[x_1, [x_1, x_2]].$$

Finally, subtracting u_2 from u_1 we get:

$$(f,g)^I_u = [[[x_1, x_3], x_3], x_2] + 2[[x_1, x_3], [x_2, x_3]] + [x_3, [x_3, x_4]] - 2[x_1, [x_1, x_2]].$$

Similarly, suppose that $\overline{g}$ is a subword of $\overline{f}$ for some monic elements $f, g \in \mathfrak{g}$. Again by Lemma 1.23 we can find a bracketing $[u]$ of $u = \overline{f}$ that extends the regular bracketing $(\overline{g})$ of $\overline{g}$, and such that $\overline{[u]} = u$. By substituting $(\overline{g})$ with g in $[u]$ we obtain an element f_* of $\mathfrak{g}$. The *second-order composition* $(f,g)^{II}_u$ of f and g is

$$(f,g)^{II}_u = f - f_*.$$

A subset $S \subset \mathfrak{g}$ is *reduced* if all its elements are monic and if no second-order composition can be formed between two of its elements (that is, for any $s_1, s_2 \in S$, $\overline{s}_1$ is not a subword of $\overline{s}_2$).

Finally, a *Gröbner-Shirshov basis* of an ideal $I \subset \mathfrak{g}$ is a reduced set $S \subset \mathfrak{g}$ such that $I = \langle\langle S \rangle\rangle$ and such that if $f, g \in S$ define a first-order composition with respect to some word u, then

$$(f, g)_u^I = \sum_{i=1}^{m} f_i,$$

where each f_i lies in $\langle\langle s_i \rangle\rangle$ for some $s_i \in S$ and $\overline{f}_i \prec u$ for $1 \leq i \leq m$.

Remark 1.28. *As we remarked before, for each f, g and u, we only need to write one choice of composition $(f, g)_u^I$ as a sum as above.*

Theorem 1.29. *[17, Lemma 3.2.7] Let S be a Gröbner-Shirshov basis of the ideal $I \subset \mathfrak{g}$ and let $f \in \mathfrak{g}$. If $f \in I$, then $\overline{f}$ contains $\overline{s}$ as a subword for some $s \in S$.*

Remark 1.30. *In [17] (and in Shirshov's article [73]) it is assumed that the monomial ordering is the usual deg-lex, that is, $d(x_i) = 1$ for all i. The proofs, however, carry to our setting because the weight lexicographic ordering is admissible, i.e. $u \preceq v$ implies $aub \preceq avb$ for all a, b, and has the descending chain condition.*

If $S \subset \mathfrak{g}$ is a finite homogeneous set (in the sense of the degree function d defined above), then we can decide if a certain fixed element $f \in \mathfrak{g}$ belongs to the ideal generated by S as follows. First, multiplying by elements of K we can assume that S is monic. Then we *reduce* S, that is, if $s_1, s_2 \in S$ and $\overline{s}_1$ is a subword of $\overline{s}_2$, then we can find some $s_0 \in \langle\langle s_1 \rangle\rangle$ with $\overline{s}_0 = \overline{s}_2$ and then substitute s_2 with $s_2 - s_0$ in S. Now, $s_2 - s_0$ is smaller than s_2 in the weight lexicographic ordering, so by repeating this process finitely many times, we reach a reduced set S' that generates the same ideal as S.

Now we *complete* S' in the following sense: we take all compositions between elements of S', add them to the generating set, and then reduce again as in the previous paragraph. We repeat this process until we reach a reduced set $S^{(n)}$ such that all compositions between elements of $S^{(n)}$ are either an element of $S^{(n)}$, or of degree greater than the degree of $\overline{f}$. It is essencial here that S is homogeneous, so that compositions between two elements $f, g \in S$ are either trivial, or have degree strictly greater than the degrees of f and g.

If $\overline{f}$ does not contain $\overline{s}$ as a subword for any $s \in S^{(n)}$, then $f \notin \langle\langle S^{(n)} \rangle\rangle$. Otherwise we can find some $f_2 \in \langle\langle S^{(n)} \rangle\rangle$ such that $\overline{f}_2 = \overline{f}$, and the problem is reduced to deciding if $f - f_2$ (which is smaller with respect to the weight lexicographic ordering) lies in this ideal.

1.4 Profinite groups

Most of this section can be found in [69].

A *profinite group* is a topological group G given by the inverse limit of a system of finite groups $\{G_i\}_i$, each of those equipped with the discrete topology. The topology of G is the subspace topology inherited by G in its realization as a subgroup of $\prod_i G_i$. If each of the G_i is a finite p-group for a fixed prime p, we say that G is a *pro-p group*.

A *homomorphism* of profinite (pro-p) groups is a group homomorphism that is also continuous. We will mostly talk about *closed subgroups* of a profinite group, but we will use the notation $\overline{H}$ to denote the closure of a subset (subgroup) of a profinite group.

Any finite group G is profinite: it is the inverse limit of the constant system $\{G_i\}_i$ where $G_i = G$ for all i. A less trivial example is the group of *p-adic integers*:

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z},$$

where the homomorphisms $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ are the natural maps for $n > m$. This is a pro-p group.

Completions give another way of building profinite groups. The *pro-p completion* $\widehat{G}$ of a discrete group G is defined as

$$\widehat{G} = \varprojlim G/U$$

where U runs through all normal subgroups $U \lhd G$ such that G/U is a finite p-group. It has the following universal property: there is a group homomorphism $i : G \to \widehat{G}$ such that $i(G)$ is dense in $\widehat{G}$ and for every homomorphism $\varphi : G \to H$, where H is a finite p-group, there is a continuous homomorphism of pro-p groups $\widehat{\varphi} : \widehat{G} \to H$ such that $\widehat{\varphi} \circ i = \varphi$.

Let us define free pro-p groups. Suppose that X is a *profinite space*, that is, X is the inverse limit of an inverse system $\{X_i\}_i$ of finite sets endowed with the discrete topology. Of course, X has the topology of a subspace of the cartesian product $\prod X_i$. Equivalently, X is a topological space that is Hausdorff, compact and totally disconnected (see [69, Thm. 1.1.12])

Let F be a pro-p group and suppose that $i : X \to F$ is a continuous map such that $F = \overline{\langle i(X)\rangle}$. We say that the pair (F, i) is a *free pro-p group* on X if the following universal property holds:

($\star$) For any pro-p group G and $\varphi : X \to G$ a continuous map, there exists a unique continuous homomorphism $\phi : F \to G$ such that $\phi \circ i = \varphi$.

In particular, we can take X to be any finite set. In this case the free pro-p group with basis X can be realized as the pro-p completion $\widehat{F(X)}$, where $F(X)$ is the free discrete group with free basis X. In general, it can be shown that for any profinite space X there is a (unique) free pro-p group on X ([69, Prop. 3.3.2]).

Let $\varphi : F \to G$ be a surjective homomorphism of pro-p groups, where F is a free pro-p group on the profinite set X. Let $R \subset F$ be a set of topological generators of

$ker(\varphi)$ as a normal subgroup of F. In this case, we write $G = \langle X|R\rangle_p$, and we say that this is a *presentation* of G in the category of pro-p groups. The usual notions of *finite generation* and *finite presentability* now make sense also for pro-p groups.

A *profinite ring* is a topological ring given by the inverse limit of finite rings equipped with the discrete topology. For instance, using the ring structure of $\mathbb{Z}/p^n\mathbb{Z}$ we see that $\mathbb{Z}_p$ is actually a profinite ring.

Fix R a profinite ring. A *left profinite R-module* is an abelian profinite group A together with a continuous map $\varphi : R \times A \to A$ satisfying the usual properties of the action of a ring on an abelian group. As usual, we will denote $\varphi(r, a)$ by $r \cdot a$ or simply ra. The definition of a *right profinite R-module* is analogous.

The familiar notions of free and projective modules make sense here too. In particular, for any profinite space X there exists a unique free profinite module $R[[X]]$ with free basis X [69, Prop. 5.2.2].

In order to talk about homological properties of pro-p groups, we need to define the appropriate ring over which we can apply the concepts of homological algebra. This is the completed group algebra, which is defined as follows. Let $G = \varprojlim G/U$ be a pro-p group, where each U is a normal subgroup of G and G/U is a finite p-group. The *completed group algebra* of G is defined as

$$\mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G/U],$$

where the homomorphisms $\mathbb{Z}_p[G/U_1] \to \mathbb{Z}_p[G/U_2]$ are induced by the corresponding homomorphisms $G/U_1 \to G/U_2$.

If A is right pro-p $\mathbb{Z}_p[[G]]$-module and B is a left pro-p $\mathbb{Z}_p[[G]]$-module, the *complete tensor product* A $\widehat{\otimes}_{\mathbb{Z}_p[[G]]} B$ is defined as

$$A \widehat{\otimes}_{\mathbb{Z}_p[[G]]} B = \varprojlim A_i \otimes_{\mathbb{Z}_p[[G]]} B_j$$

where $\{A_i\}$ and $\{B_j\}$ are the systems of (right or left) finite $\mathbb{Z}_p[[G]]$-modules such that $A = \varprojlim A_i$ and $B = \varprojlim B_j$.

We now have a homology theory for pro-p groups. By definition for a left pro-p $\mathbb{Z}_p[[G]]$-module A the (continuous) homology group $H_n(G; A)$ is the n-th homology of the complex $\mathcal{P} \widehat{\otimes}_{\mathbb{Z}_p[[G]]} A$, where $\mathcal{P}$ is a projective resolution of $\mathbb{Z}_p$ as a $\mathbb{Z}_p[[G]]$-module. As usual, we have $H_1(G; \mathbb{Z}_p) \simeq G/\overline{[G, G]}$.

As in the other cases, the tool we can use to study homology of groups given by extensions is a spectral sequence. For G be pro-p group, $N \subseteq G$ a closed normal subgroup and A a pro-p $\mathbb{Z}_p[[G]]$-module, the associated Lyndon-Hochschild-Serre (LHS) spectral sequence is

$$E^2_{i,j} = H_i(G/N; H_j(N; A)) \Rightarrow H_{i+j}(G; A).$$

The facts about convergence, bidegree and 5-term exact sequence in the Lie algebra case hold here as well.

Remark 1.31. *The cohomology theory that makes sense here has coefficients in discrete modules (rather than profinite). We will not describe it here, since we only make use of homology. Details may be found in [69, Chapter 6].*

A pro-p group G has *homological type* FP_m if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a free resolution with all free modules finitely generated up to degree m. Of course we assume that the differentials between the free modules in such resolution are continuous maps. The theory of pro-p groups of type FP_m is simpler than the cases of discrete groups and Lie algebras.

Lemma 1.32. *[39] Let G be a pro-p group. The following conditions are equivalent:*

1. *G is of type FP_m;*
2. *$H_i(G;\mathbb{F}_p)$ is finite for all $i \leq m$;*
3. *$H_i(G;\mathbb{Z}_p)$ is finitely generated as a pro-p group for all $i \leq m$.*

In particular, G is finitely presentable if and only it is of type FP_2.

Other resuls of niceness of the FP_m properties hold similarly to the other categories that we considered.

Lemma 1.33. *Let G be a pro-p group and let m be a positive integer. Then:*

1. *If G is of type FP_m and $H \leq G$ is a retract, then H is of type FP_m;*
2. *If $H \leq G$ is a closed subgroup of finite index, then G is of type FP_m if and only if H is of type FP_m.*

Proof. Item 1 can be proved exactly as in Lemma 1.16 for Lie algebras. Suppose that $H \leq G$ is of finite index. By Shapiro's Lemma ([69, Thm. 6.10.8]), there is a a natural isomorphism

$$H_n(G;\mathbb{F}_p) \simeq H_n(H;\mathbb{Z}_p[[G]]\widehat{\otimes}_{\mathbb{Z}_p[[H]]}\mathbb{F}_p) \simeq H_n(H;\mathbb{F}_p[G/H]),$$

where G/H is the set of cosets. Notice that $\mathbb{F}_p[G/H]$ is a finite abelian p-group. In particular, $H_n(H;\mathbb{F}_p[G/H])$ is finite if and only if $H_n(H;\mathbb{F}_p)$ is finite, thus G is of type FP_m if and only if H is of type FP_m by Lemma 1.32. □

Example 1.34. *Finitely generated free and abelian pro-p group are of type FP_m for all m.*

A nice class of pro-p groups with good homological finiteness properties are the analytic groups, defined as follows. For a pro-p group G, we denote by $d(G)$ the cardinality of a minimal (topological) generating set of G. The *rank* of G is defined as

$$rk(G) = sup\{d(H) \mid H \text{ is a closed subgroup of } G\}.$$

A pro-p group is *p-adic analytic* if $rk(G) < \infty$. Equivalently, G is p-adic analytic if it admits a strucure of analytic manifold over $\mathbb{Q}_p$ (the field of p-adic numbers) where the group product and inversion are analytic maps [54].

If G is p-adic analytic, then the completed group algebra $\mathbb{Z}_p[[G]]$ is noetherian [52]. In particular, G is of type FP_m for all m.

2 Σ-invariants of wreath products

In this chapter we deal with discrete groups. More specifically, we describe our work on the Σ-invariants of permutational wreath products of groups. We develop the theory, prove Theorems A1, A2 and A3, and discuss their consequences. Most of what is found here was published in [61].

2.1 Background on the Σ-invariants

Let us start by recalling the definition of the invariant Σ^1. For a finitely generated group Γ and a finite generating set $\mathcal{X} \subset \Gamma$, we consider the Cayley graph $Cay(\Gamma; \mathcal{X})$. Its vertex set is Γ and two vertices γ_1 and γ_2 are connected by an edge if and only if there is some $x \in \mathcal{X}^{\pm 1}$ such that $\gamma_2 = \gamma_1 x$. With this definition, $Cay(\Gamma; \mathcal{X})$ admits a left action by Γ which is induced by left multiplication in the group. An important fact here is that this graph is always connected: any $\gamma \in \Gamma$ can be written as a word on the elements of $\mathcal{X}$, and this defines a label for a path connecting 1 and γ in $Cay(\Gamma; \mathcal{X})$.

Recall that the character sphere $S(\Gamma)$ is the set of classes non-zero homomorphisms $\chi : \Gamma \to \mathbb{R}$ modulo the equivalence relation $\chi_1 \sim \chi_2$ if $\chi_1 = r\chi_2$ for some $r \in \mathbb{R}_{>0}$.

For a fixed non-zero homomorphism (character) $\chi : \Gamma \to \mathbb{R}$ we can define the submonoid

$$\Gamma_\chi = \{\gamma \in \Gamma \mid \chi(\gamma) \geq 0\}.$$

Notice that $\Gamma_{\chi_1} = \Gamma_{\chi_2}$ if and only if χ_1 and χ_2 represent the same class in the character sphere $S(\Gamma)$. The full subgraph of $Cay(\Gamma; \mathcal{X})$ spanned by Γ_χ, which we denote by $Cay(\Gamma; \mathcal{X})_\chi$, may not be connected. We put:

$$\Sigma^1(\Gamma) = \{[\chi] \in S(\Gamma) \mid Cay(\Gamma; \mathcal{X})_\chi \text{ is connected}\}.$$

It can be shown that this definition does not depend on the (finite) generating set $\mathcal{X}$. This invariant is known as the *Bieri-Neumann-Strebel* invariant (or simply BNS-invariant) of G, in reference to the authors who studied it first [13].

The invariant Σ^2 is defined similarly. If Γ is finitely presented and $\langle \mathcal{X} | \mathcal{R} \rangle$ is a finite presentation, we consider the Cayley complex $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)$. This complex is obtained from the Cayley graph by gluing 2-dimension cells with boundary determined by the loops defined by the relators $r \in \mathcal{R}$, for each base point in Γ. The resulting complex is always 1-connected. Again for $\chi : \Gamma \to \mathbb{R}$ a non-zero homomorphism, we define $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ to be the full subcomplex spanned by Γ_χ. The 1-connectedness of this complex depends on the choice of the presentation. We define $\Sigma^2(\Gamma)$ as the subset of $S(\Gamma)$ containing exactly all the classes $[\chi]$ of characters such that $Cay(\Gamma; \langle\mathcal{X}|\mathcal{R}\rangle)_\chi$ is 1-connected for some finite presentation $\langle\mathcal{X}|\mathcal{R}\rangle$ of Γ. More details on these definitions may be found in [60].

The following theorem is the main reason why we study these invariants.

Theorem 2.1 ([13], [67]). *Suppose that Γ is finitely generated and let $N \subseteq \Gamma$ be a subgroup such that $[\Gamma, \Gamma] \subseteq N$. Then N is finitely generated if and only if*

$$\Sigma^1(\Gamma) \supseteq \{[\chi] \in S(\Gamma) \mid \chi|_N = 0\}.$$

Similarly, if Γ is further finitely presented, then N is finitely presented if and only if

$$\Sigma^2(\Gamma) \supseteq \{[\chi] \in S(\Gamma) \mid \chi|_N = 0\}.$$

Remark 2.2. *Notice that any subgroup N as above is automatically normal in Γ and Γ/N is a finitely generated abelian group. The proof of the theorem involves reducing somehow to the case where $\Gamma/N \simeq \mathbb{Z}^n \subseteq \mathbb{R}^n$ and then studying the connectedness (resp. 1-connectedness) of pre-images in Cayley graphs (resp. Cayley complexes) of open balls in $\mathbb{R}^n$ via the maps $\Gamma \to \mathbb{R}^n$.*

The invariants that we have defined so far will be referred to as *homotopical invariants*, in contrast with the *homological* ones that we will define now. For a finitely generated group Γ and $\chi : \Gamma \to \mathbb{R}$ a non-zero homomorphism, consider the monoid ring $\mathbb{Z}\Gamma_\chi$. This is by definition the subring of $\mathbb{Z}\Gamma$ containing exactly all elements $\sum a_\gamma \gamma \in \mathbb{Z}\Gamma$ such that $a_\gamma \neq 0$ only if $\gamma \in \Gamma_\chi$. We put

$$\Sigma^m(\Gamma; \mathbb{Z}) = \{[\chi] \in S(\Gamma) \mid \mathbb{Z} \text{ is of type } FP_m \text{ over } \mathbb{Z}\Gamma_\chi\}.$$

As observed by Bieri and Renz in [14], if $\Sigma^m(\Gamma; \mathbb{Z}) \neq \emptyset$, then Γ is of type FP_m. It can be verified that invariants $\Sigma^1(\Gamma)$ and $\Sigma^1(\Gamma; \mathbb{Z})$ coincide [14, Prop. 6.4].

Remark 2.3. *Proposition 6.4 in [14] actually establishes that $\Sigma^1(\Gamma)$ (as defined in [13]) and $\Sigma^1(\Gamma; \mathbb{Z})$ are antipodal sets in $S(\Gamma)$, but they do coincide if the actions of Γ (on Cayley graphs and on modules) are chosen consistently on the left (or on the right).*

It is clear by the definition that we have a chain

$$S(G) \supset \Sigma^1(\Gamma) = \Sigma^1(\Gamma; \mathbb{Z}) \supset \Sigma^2(\Gamma; \mathbb{Z}) \supset \Sigma^3(\Gamma; \mathbb{Z}) \supset \dots$$

Moreover, $\Sigma^2(\Gamma) \subseteq \Sigma^2(\Gamma; \mathbb{Z})$ whenever Γ is finitely presented. Details may be found in [14] and [67]. Of course, the analogous version of Theorem 2.1 holds.

Theorem 2.4 ([14]). *Suppose that Γ is of type FP_m and let $N \subseteq \Gamma$ be a subgroup such that $[\Gamma, \Gamma] \subseteq N$. Then N is of type FP_m if and only if*

$$\Sigma^m(\Gamma; \mathbb{Z}) \supseteq \{[\chi] \in S(\Gamma) \mid \chi|_N = 0\}.$$

Example 2.5. *It can be deduced from Theorems 2.1 and 2.4 that $\Sigma^1(\Gamma) = \Sigma^2(\Gamma) = \Sigma^m(\Gamma; \mathbb{Z}) = S(\Gamma)$ for all m if Γ is a finitely generated abelian group (or, more generally, polycyclic-by-finite).*

Example 2.6. *If F is a free non-abelian group of finite rank, then $\Sigma^1(F) = \emptyset$. Indeed, for any non-zero character $\chi : F \to \mathbb{R}$ we can assume that $\chi(x) > 0$ and $\chi(y) < 0$ for some distinct elements x and y of a free basis X of F. Then $\chi(yx^n) \geq 0$ for some $n \in \mathbb{N}$. Now, $Cay(F; X)$ is a tree, thus the unique path joining 1 and yx^n passes through y. Thus $Cay(F; X)_\chi$ is not connected. From this also follows that $\Sigma^2(F) = \Sigma^m(F; \mathbb{Z}) = \emptyset$ for all m.*

We will give more complex examples in the sequence, after we establish some general results.

Lemma 2.7. *Let G be a finitely generated group and let $N \subseteq G$ be a normal subgroup. Let $\chi : G \to \mathbb{R}$ be a non-zero character such that $\chi(N) = 0$. If $[\chi] \in \Sigma^1(G)$, then $[\bar{\chi}] \in \Sigma^1(G/N)$, where $\bar{\chi} : G/N \to \mathbb{R}$ is the induced character.*

Proof. Let $\mathcal{X} \subset G$ be a finite generating set. Denote $Q = G/N$ and let $\pi : G \to Q$ be the canonical projection. If $q \in Q_{\bar{\chi}}$, then $q = \pi(g)$ for some $g \in G_\chi$. By hypothesis the elements 1 and g are connected in $Cay(G; \mathcal{X})_\chi$. By applying π we obtain a path connecting 1 and q in $Cay(Q; \pi(\mathcal{X}))_{\bar{\chi}}$, so $Cay(Q; \pi(\mathcal{X}))_{\bar{\chi}}$ is connected. □

The lemma above does not hold for the higher Σ-invariants. Some of the other results that we will need about these invariants concern direct products of groups.

Theorem 2.8 (Direct product formulas, [33]). *Let G_1 and G_2 be finitely generated groups and let $\chi = (\chi_1, \chi_2) : G_1 \times G_2 \to \mathbb{R}$ be a non-zero character. Then $[\chi] \in \Sigma^1(G_1 \times G_2)$ if and only if at least one of the following conditions holds:*

1. *$\chi_i \neq 0$ for $i = 1, 2$ or*
2. *$[\chi_i] \in \Sigma^1(G_i)$ for some $i \in \{1, 2\}$.*

Similarly, if G_1 and G_2 are finitely presented (resp. of type FP_2), then $[\chi] \in \Sigma^2(G_1 \times G_2)$ (resp. $[\chi] \in \Sigma^2(G_1 \times G_2; \mathbb{Z})$) if and only if at least one of the following conditions holds:

1. *$[\chi_1] \in \Sigma^1(G_1)$ and $\chi_2 \neq 0$;*

2. $[\chi_2] \in \Sigma^1(G_2)$ *and* $\chi_1 \neq 0$ *or*

3. $[\chi_i] \in \Sigma^2(G_i)$ *(resp.* $[\chi_i] \in \Sigma^2(G_i;\mathbb{Z})$*) for some* $i \in \{1,2\}$.

Theorem 2.8 has versions for $\Sigma^n(G_1 \times G_2;\mathbb{Z})$ with $n \leq 3$. There was a conjecture suggesting how to compute the higher Σ-invariants of direct products, but it turned out to be false. Counterexamples were found by Meier, Meinert and VanWyk [57] and Schütz [71]. For precise statements see [11], which also brings a proof of the conjecture for the homological invariants if coefficients are taken in a field (rather than $\mathbb{Z}$).

Example 2.9. *If F is free non-abelian of finite rank, then $\Sigma^1(F \times F)$ is the set of classes characters $[\chi]$, where $\chi = (\chi_1, \chi_2)$ and both χ_1 and χ_2 are non-zero characters of F. Furthermore, $\Sigma^2(F \times F) = \Sigma^m(F \times F;\mathbb{Z}) = \emptyset$ for all m.*

Similarly to what happens in the study of finiteness properties of groups,the Σ-invariants behave well with respect to the notions of subgroups of finite index and retracts.

Theorem 2.10 ([56, 60])**.** *Let G be a finitely presentable group and let $H \leqslant G$ be a subgroup of finite index. Let $\chi : G \to \mathbb{R}$ be a non-zero character. Then $[\chi] \in \Sigma^2(G)$ if and only if $[\chi|_H] \in \Sigma^2(H)$. The same holds for G a group of type FP_m with respect to the invariant $\Sigma^m(-;\mathbb{Z})$.*

Theorem 2.11 (Retracts, [60])**.** *Let G be a finitely presented group and suppose that H is a retract, that is, there are homomorphisms $p : G \to H$ and $j : H \to G$ such that $p \circ j = id_H$. Suppose that $\chi : H \to \mathbb{R}$ is a non-zero character. Then*

$$[\chi \circ p] \in \Sigma^2(G) \Rightarrow [\chi] \in \Sigma^2(H).$$

The same holds for G a group of type FP_m with respect to the invariant $\Sigma^m(-;\mathbb{Z})$.

The following theorem is more specific, but it will be essential for our work on wreath products.

Theorem 2.12. *[43, Thm. C] Let G be a finitely generated group and suppose that N is a normal subgroup of G that is locally nilpotent-by-finite.*

1. *If G is of type FP_m, then*

$$\{[\chi] \in S(G) \mid \chi(N) \neq 0\} \subseteq \Sigma^m(G;\mathbb{Z}).$$

2. *If G is finitely presentable, then*

$$\{[\chi] \in S(G) \mid \chi(N) \neq 0\} \subseteq \Sigma^2(G).$$

The result in [43] is stated for N locally polycyclic-by-finite, but actually the proof works for nilpotent-by-finite. We will use it with N being abelian. The case $m = 1$, with N abelian, can also be found as Lemma C1.20 in Strebel's notes [76].

2.2 The Σ^1-invariant of wreath products

We will now compute the Σ^1-invariant of an arbitrary finitely generated wreath product.

Fix $\Gamma = H \wr_X G$ and denote $M = \oplus_{x\in X} H_x \subseteq \Gamma$. We will start working with the characters $\chi : \Gamma \to \mathbb{R}$ such that $\chi|_M = 0$, for which there are some partial results by Bartholdi, Cornulier and Kochloukova. We quote their result in its most general form, which deals with the higher homological invariants.

Theorem 2.13 ([5], Theorem 8.1)**.** *Let $\Gamma = H \wr_X G$ be a wreath product of type FP_m and let $M = \oplus_{x\in X} H_x \subseteq \Gamma$. Let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. The following conditions are sufficient for $[\chi] \in \Sigma^m(\Gamma; \mathbb{Z})$:*

1. *$[\chi|_G] \in \Sigma^m(G; \mathbb{Z})$;*
2. *$[\chi|_{stab_G(\bar{x})}] \in \Sigma^{m-i}(stab_G(\bar{x}); \mathbb{Z})$ for all stabilizers $stab_G(\bar{x})$ of the diagonal action of G on X^i and for all $1 \le i \le m$.*

Moreover, if the abelianization of H is infinite, then such conditions are also necessary.

Notice that item 2 contains a statement about invariants in degree 0. By convention we put $\Sigma^0(V; \mathbb{Z}) = S(V)$ for any finitely generated group V. In other words, the condition $[\chi] \in \Sigma^0(V; \mathbb{Z})$ means that $\chi : V \to \mathbb{R}$ is a non-zero homomorphism.

Recall that the homological and homotopical invariants coincide in degree 1, that is, $\Sigma^1(V; \mathbb{Z}) = \Sigma^1(V)$ whenever V is a finitely generated group . We can now extract from Theorem 2.13 a set of sufficient conditions for $[\chi] \in \Sigma^1(\Gamma)$.

Proposition 2.14. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. If $[\chi|_G] \in \Sigma^1(G)$ and if $\chi|_{stab_G(x)} \neq 0$ for all stabilizers $stab_G(x)$ of the action of G on X, then $[\chi] \in \Sigma^1(\Gamma)$.*

Remark 2.15. *This result could also be obtained by considering an action of Γ on a nice complex. We shall apply this reasoning in the study of the invariant $\Sigma^2(H \wr_X G)$.*

This set of conditions is in fact necessary. First, if $\chi : \Gamma \to \mathbb{R}$ and $M \subseteq ker(\chi)$, then

$$[\chi] \in \Sigma^1(\Gamma) \Rightarrow [\chi|_G] \in \Sigma^1(G)$$

by Lemma 2.7, since $\chi|_G$ coincides with the character $\bar{\chi}$ induced on the quotient $\Gamma/M \simeq G$. It suffices then to analyze the restriction of χ to the stabilizer subgroups under the hypothesis that $[\chi] \in \Sigma^1(\Gamma)$.

Proposition 2.16. *If* $[\chi] \in \Sigma^1(\Gamma)$ *and* $\chi|_M = 0$, *then* $\chi|_{stab_G(x)} \neq 0$ *for all* $x \in X$.

Proof. Recall that by Theorem 1.7, G acts with finitely many orbits on X and both G and H are finitely generated. Let $X = G \cdot x_1 \sqcup \ldots \sqcup G \cdot x_n$. We only need to show that $\chi|_{stab_G(x_i)} \neq 0$ for all i. By taking the quotient by $M' = \bigoplus_{x \in X \setminus G \cdot x_i} H_x$, we may assume that $n = 1$, that is, we consider a wreath product of the form $\Gamma = H \wr_X G$ with $X = G \cdot x_1$.

Let Y and Z be finite generating sets for H and G, respectively. Since $X = G \cdot x_1$ it is clear that $Y \cup Z$ is a finite generating set for Γ (we see Y as a subset of the copy H_{x_1}). Then $Cay(\Gamma; Y \cup Z)_\chi$ must be connected, since $[\chi] \in \Sigma^1(\Gamma)$ by hypothesis.

First, we show that M can be generated by the left conjugates of elements of $Y^{\pm 1}$ by elements of G_χ. Indeed if $m \in M$, then there is a path in $Cay(\Gamma; Y \cup Z)_\chi$ connecting 1 to m, since $m \in M \subseteq ker(\chi) \subseteq \Gamma_\chi$. Such a path has as label a word with letters in $Y^{\pm 1} \cup Z^{\pm 1}$, so we can write:

$$m = w_1 v_1 w_2 v_2 \cdots w_k v_k,$$

where each w_j is a word in $Y^{\pm 1}$ and each v_j is a word in $Z^{\pm 1}$ (possibly trivial). We rewrite:

$$m = w_1 ({}^{v_1} w_2)({}^{v_1 v_2} w_3) \cdots ({}^{v_1 \cdots v_{k-1}} w_k)(v_1 \cdots v_k).$$

Now, $w_1 ({}^{v_1} w_2)({}^{v_1 v_2} w_3) \cdots ({}^{v_1 \cdots v_{k-1}} w_k) \in M$ and $v_1 \cdots v_k \in G$. But $m \in M$ and Γ is the semi-direct product $M \rtimes G$, so $v_1 \cdots v_k = 1_G$. Moreover, since $\chi|_M = 0$, it is clear that $\chi(v_1 \cdots v_j) \geq 0$ for all $1 \leq j \leq k$, so:

$$m = w_1 ({}^{v_1} w_2)({}^{v_1 v_2} w_3) \cdots ({}^{v_1 \cdots v_{k-1}} w_k) \in \langle {}^{G_\chi}(Y^{\pm 1}) \rangle,$$

as we wanted.

But then

$$M = \langle {}^{G_\chi}(Y) \rangle \subseteq \langle {}^{G_\chi}(H_{x_1}) \rangle = \bigoplus_{x \in G_\chi \cdot x_1} H_{x_1},$$

that is, $X = G_\chi \cdot x_1$. Finally, as $\chi|_G \neq 0$, there is some $g_1 \in G$ such that $\chi(g_1) < 0$. On the other hand, there must be some $g_0 \in G_\chi$ such that $g_0 \cdot x_1 = g_1 \cdot x_1$. It follows that $g_1^{-1} g_0 \in stab_G(x_1)$, with $\chi(g_1^{-1} g_0) = -\chi(g_1) + \chi(g_0) > 0$, hence $\chi|_{stab_G(x_1)} \neq 0$. □

We obtain part 1 of Theorem A1 by combining Propositions 2.14 and 2.16.

2.3 The Σ^1-invariant and Renz's criterion

We shall use the combinatorial criterion described by Renz [68] to consider the characters $\chi : H \wr_X G \to \mathbb{R}$ such that $\chi|_M \neq 0$. We recall here the theory.

Let Γ be any finitely generated group and let $\mathcal{X} \subseteq \Gamma$ be a finite generating set. For a non-zero character $\chi : \Gamma \to \mathbb{R}$ and for any word $w = x_1 \cdots x_n$, with $x_i \in \mathcal{X}^{\pm 1}$, we denote:

$$v_\chi(w) := min\{\chi(x_1 \cdots x_j) \mid 0 \leq j \leq n\}.$$

We say that $v_\chi(w)$ is the χ-*valuation* of w.

Theorem 2.17 ([68], Theorem 1)**.** *With the notations above, $[\chi] \in \Sigma^1(\Gamma)$ if and only if there exists $t \in \mathcal{X}^{\pm 1}$ with $\chi(t) > 0$ and such that for all $x \in \mathcal{X}^{\pm 1} \smallsetminus \{t, t^{-1}\}$ the conjugate $t^{-1}xt$ can be represented by a word w_x in $\mathcal{X}^{\pm 1}$ such that*

$$v_\chi(t^{-1}xt) < v_\chi(w_x).$$

The theorem is stated as above by Renz, but the proof actually shows that the statement holds for *any* $t \in \mathcal{X}^{\pm 1}$ with $\chi(t) > 0$

Example 2.18. *Suppose that $z \in \Gamma$ is a central element and let $\chi : \Gamma \to \mathbb{R}$ be a character such that $\chi(z) > 0$. Then $[\chi] \in \Sigma^1(\Gamma)$: it suffices to choose a generating set $\mathcal{X}$ containing z and apply the theorem with $t = z$ and $w_x = x$ for all $x \in \mathcal{X}^{\pm 1}$.*

The example above suggests that, given $\chi : \Gamma \to \mathbb{R}$, finding an element $t \in \Gamma$ that commutes with many elements of a generating set of Γ and such that $\chi(t) > 0$ might be a good start. In the case of a wreath product $\Gamma = H \wr_X G$, we may use for instance an element of $M = \oplus_{x \in X} H_x$, keeping in mind that any two copies of H commute in Γ.

Proposition 2.19. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and let $[\chi] \in S(\Gamma)$. Suppose that there is some $x_1 \in X$ such that $G \cdot x_1 \neq \{x_1\}$ and $\chi|_{H_{x_1}} \neq 0$. Then $[\chi] \in \Sigma^1(\Gamma)$.*

Proof. Let Y and Z be finite generating sets for H and G, respectively, and choose $x_1, \ldots, x_n \in X$ such that $X = \bigsqcup_{j=1}^{n} G \cdot x_j$ (the element x_1 is already chosen to satisfy the hypotheses). For each $1 \leq j \leq n$ let Y_j be a copy of Y inside H_{x_j}. It is clear that Γ is generated by $Y_1 \cup \ldots \cup Y_n \cup Z$.

Now, since $G \cdot x_1 \neq \{x_1\}$ we can choose $g_1 \in G$ such that $g_1 \cdot x_1 \neq x_1$. Furthermore, since $\chi|_{H_{x_1}} \neq 0$, we can choose a generator $h \in Y_1$ such that $\chi(h) \neq 0$. We may assume without loss of generality that $\chi(h) > 0$. Define $t := {}^{g_1}h \in H_{g_1 \cdot x_1}$. We take $\mathcal{X} = Y_1 \cup \ldots \cup Y_n \cup Z \cup \{t\}$ as a generating set for Γ and we show that the conditions of Theorem 2.17 are satisfied.

If $y \in (Y_1 \cup \ldots \cup Y_n)^{\pm 1}$, then t and y commute in Γ, hence $w_y := y$ is word that represents $t^{-1}yt$. Also, $v_\chi(w_y) = min\{0, \chi(y)\}$ and

$$v_\chi(t^{-1}yt) = min\{-\chi(t), \chi(y) - \chi(t)\} = v_\chi(w_y) - \chi(t) < v_\chi(w_y)$$

so the words w_y satisfy what is required in the theorem.

If $z \in Z^{\pm 1}$, there are two cases: $z \in stab_G(g_1 \cdot x_1)$ or $z \notin stab_G(g_1 \cdot x_1)$. In the first case z and t commute in Γ, so we may proceed as in the previous paragraph: we take the word $w_z := z$, which represents $t^{-1}zt$ and satisfies $v_\chi(t^{-1}zt) < v_\chi(w_z)$. If $z \notin stab_G(g_1 \cdot x_1)$ notice that ${}^z t$ and t^{-1} lie in different copies of H in Γ, therefore they commute, so:

$$t^{-1}zt = t^{-1}({}^z t)z = ({}^z t)t^{-1}z = ztz^{-1}t^{-1}z.$$

In this case define $w_z := ztz^{-1}t^{-1}z$. Observe that $v_\chi(w_z) = min\{0, \chi(z)\}$. If this minimum is 0, then $\chi(z) \geq 0$, and so $v_\chi(t^{-1}zt) = -\chi(t) < 0$. Otherwise $v_\chi(w_z) = \chi(z) < 0$ and so $v_\chi(t^{-1}zt) \leq \chi(t^{-1}z) = \chi(z) - \chi(t) < \chi(z)$. In both cases $v_\chi(t^{-1}zt) < v_\chi(w_z)$.

Thus $[\chi] \in \Sigma^1(\Gamma)$ by Theorem 2.17. □

In order to complete the proof of Theorem A1, we only need to consider the cases where the restriction of χ to the copies of H is non-zero only for copies associated to orbits that are composed by only one element. This is done using the direct product formula, as follows.

Theorem 2.20. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and let $M = \oplus_{x \in X} H_x \subseteq \Gamma$. Let $\chi : \Gamma \to \mathbb{R}$ be a character such that $\chi|_M \neq 0$. Then $[\chi] \in \Sigma^1(\Gamma)$ if and only if at least one the following conditions holds:*

1. *The set $T = \{x \in X \mid \chi|_{H_x} \neq 0\}$ has at least two elements;*
2. *$T = \{x_1\}$ and $\chi|_G \neq 0$;*
3. *$T = \{x_1\}$ and $[\chi|_{H_{x_1}}] \in \Sigma^1(H)$.*

Proof. By Proposition 2.19 it is enough to consider the case where $G \cdot x = \{x\}$ for all $x \in T$. Notice that in this case T must be finite, since each of its elements is an entire orbit of the action of G on X, and there are finitely many of those. Let $P = \prod_{x \in T} H_x$ and $X' = X \setminus T$. Then

$$\Gamma = H \wr_X G \simeq P \times (H \wr_{X'} G).$$

If T has at least two elements, then $[\chi|_P] \in \Sigma^1(P)$ and hence $[\chi] \in \Sigma^1(\Gamma)$, by two applications of Theorem 2.8. If $T = \{x_1\}$, the same theorem gives us exactly that $[\chi] \in \Sigma^1(\Gamma)$ if and only if one of conditions 2 or 3 holds, since $\chi|_G \neq 0$ if and only if $\chi|_{H \wr_{X'} G} \neq 0$. □

Example 2.21. *When $\Gamma = H \wr G$ is a regular wreath product, the stabilizers of the action are always trivial, thus $\Sigma^1(\Gamma) = \{[\chi] \in S(\Gamma) | \chi(M) \neq 0\}$. This was obtained before by Strebel [76, Prop. C1.18]. In particular, if $\Gamma = \mathbb{Z} \wr \mathbb{Z}$, then $S(\Gamma) = S^1$ and $\Sigma^1(\Gamma)$ is everything but two antipodal points.*

2.4 Graph-wreath products

We now digress a bit and obtain a generalization of the results of Section 2.2 to a wider class of groups. Besides being interesting in its own right, this will be useful in the analysis of the Σ^2-invariants of wreath products.

Recall that given a graph K and a family $\mathcal{H} = \{H_k\}_{k \in V(K)}$ of groups indexed by the vertex set $V(K)$ of K, the *graph-product* $\mathcal{H}^{\langle K \rangle}$ is the quotient of the free product $*_{k \in V(K)} H_k$ by the normal subgroup generated by $[H_{k_1}, H_{k_2}]$ whenever k_1 and k_2 are vertices of K that are connected by an edge.

Given two groups G and H and K a G-graph, the *graph-wreath product* $H \wr_K G$ is defined by Kropholler and Martino [50] as the semi-direct product $\mathcal{H}^{\langle K \rangle} \rtimes G$, where $H_k = H$ for all $k \in K$. The action of G is given by permutation of the copies of H according to the G-action on the vertex set of K. When K is the complete graph, $H \wr_K G$ is simply $H \wr_X G$, where X is the vertex set of K.

Kropholler and Martino showed that $H \wr_K G$ is finitely generated if and only if G and H are finitely generated and G acts with finitely many orbits of vertices on K, that is, $H \wr_K G$ is finitely generated under the same conditions as $H \wr_X G$ is, where X is the vertex set of K.

In what follows we fix $\Gamma = H \wr_K G$ and $M = \mathcal{H}^{\langle K \rangle} \subseteq \Gamma$. We assume that Γ is finitely generated and we decompose $X = V(K)$ in orbits as $X = G \cdot x_1 \sqcup \ldots \sqcup G \cdot x_n$. Moreover, we choose finite generating sets Z for G and Y_i for H_{x_i} for all $i = 1, \ldots, n$ and we denote $\mathcal{X} = (\cup_{i=1}^n Y_i) \cup Z$, which is seen as a generating set for Γ.

Theorem 2.22. *Let $\chi : H \wr_K G \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. Then $[\chi] \in \Sigma^1(H \wr_K G)$ if and only if $[\chi|_G] \in \Sigma^1(G)$ and $\chi|_{stab_G(x)} \neq 0$ for all $x \in X$.*

Proof. Let N_K be the kernel of the obvious homomorphism $M \twoheadrightarrow \oplus_{x \in X} H_x$. Note that $N_K \subseteq ker(\chi)$ and that $\bar{\Gamma} := \Gamma / N_K \simeq H \wr_X G$. It follows that χ induces a character $\bar{\chi} : \bar{\Gamma} \to \mathbb{R}$. For an element $\gamma \in \Gamma$, we denote by $\bar{\gamma}$ its image in $\bar{\Gamma}$.

If $[\chi] \in \Sigma^1(\Gamma)$, then $[\bar{\chi}] \in \Sigma^1(\bar{\Gamma})$ by Lemma 2.7. Thus $[\chi|_G] \in \Sigma^1(G)$ and $\chi|_{stab_G(x)} \neq 0$ for all $x \in X$ by Theorem A1.

Conversely, suppose that $[\chi|_G] \in \Sigma^1(G)$ and that $\chi|_{stab_G(x)} \neq 0$ for all $x \in X$. Then $[\bar{\chi}] \in \Sigma^1(\bar{\Gamma})$. We will show that this implies that $Cay(\Gamma; \mathcal{X})_\chi$ is connected.

We need to show that for all $\gamma \in \Gamma_\chi$, there is a path in $Cay(\Gamma; \mathcal{X})_\chi$ connecting 1 and γ. Given such a γ, notice that $\bar{\gamma} \in \bar{\Gamma}_{\bar{\chi}}$, so there must be a path from 1 to $\bar{\gamma}$ in $Cay(\bar{\Gamma}; \bar{\mathcal{X}})_{\bar{\chi}}$. Its obvious lift to $Cay(\Gamma; \mathcal{X})$ with 1 as initial vertex is a path in $Cay(\Gamma; \mathcal{X})_\chi$ that ends at an element of the form γn, with $n \in N_K$. If we can connect γ to γn inside $Cay(\Gamma; \mathcal{X})_\chi$ we are done. For that it suffices to find a path in $Cay(\Gamma; \mathcal{X})_\chi$ connecting 1 and n, and then act with γ on the left.

Since $N_K \subseteq M$, each $n \in N_K$ can be written as:

$$n = ({}^{g_1}h_1)({}^{g_2}h_2)\cdots({}^{g_k}h_k), \tag{2.1}$$

with $h_j \in \cup_{i=1}^n Y_i^{\pm 1}$ and $g_j \in G$ for all j. Even more, we may assume that each $\chi(g_j) \geq 0$. Indeed, since $\chi|_{stab_G(x)} \neq 0$ for all x, we can always pick $t_j \in G$ such that $\chi(t_j) > 0$ and ${}^{t_j}h_j = h_j$. Then we may change g_j for $g_j t_j^{k_j}$ in (2.1), where k_j is some integer such that $k_j\chi(t_j) \geq -\chi(g_j)$.

But if $\chi(g_j) \geq 0$, then $g_j \in G_{\chi|_G}$, and since $[\chi|_G] \in \Sigma^1(G)$, we can choose words w_j in $Z^{\pm 1}$ representing g_j and such that $v_\chi(w_j) \geq 0$. Finally, the word

$$w = (w_1 h_1 w_1^{-1})(w_2 h_2 w_2^{-1})\cdots(w_k h_k w_k^{-1})$$

is the label for a path connecting 1 and n in $Cay(\Gamma; \mathcal{X})_\chi$, by the choice of each w_j together with the fact that $\chi(h_j) = 0$ for all j by hypothesis. □

The above result will be needed only in a special case, namely when K is a graph without edges, so that $\Gamma \simeq (*_{x\in X} H_x) \rtimes G$.

2.5 The Σ^2-invariant

Renz's paper [68] also brings a criterion for the invariant Σ^2. In order to state it, we need to introduce the concept of a diagram over a group presentation, for which we follow [19]. Fix an orientation on $\mathbb{R}^2$. Define a *diagram* to be a subset $M \subseteq \mathbb{R}^2$ endowed with the structure of a finite combinatorial 2-complex. Thus to each 1-cell of M correspond two opposite directed edges. If $\langle \mathcal{X} | \mathcal{R} \rangle$ is a presentation for a group Γ, a *labeled diagram over* $\langle \mathcal{X} | \mathcal{R} \rangle$ is a diagram M endowed with an edge labeling satisfying:

1. The edges of M are labeled by elements of $\mathcal{X}^{\pm 1}$;
2. If an edge e has label x, then its opposite edge has label x^{-1};
3. The boundary of each face of M, read as a word in $\mathcal{X}^{\pm 1}$, beginning at any vertex and proceeding with either orientation, is either a cyclic permutation of some $r \in \mathcal{R}^{\pm 1}$, or a word of the form $tt^{-1}t^{-1}t$ for some $t \in \mathcal{X}^{\pm 1}$.

A labeled diagram M is said to be *simple* if it is connected and simply connected.

Remark 2.23. *This is a weakening of the definition of the usual van Kampen diagrams. In fact, a simple diagram M, with a vertex chosen as a base point, differs from a van Kampen diagram only by the fact that it can have what we call trivial faces, that is, those labeled by $tt^{-1}t^{-1}t$ for some $t \in \mathcal{X}^{\pm 1}$. This weakening has the effect of simplifying the drawing of some diagrams that we will consider in the sequence (see [68], Subsection 3.3).*

Suppose that we are given a simple diagram M with a base point u (a vertex in the boundary of M) and an element $\gamma \in \Gamma$. Then to each vertex u' of M corresponds a unique element of Γ, given by $\gamma\eta$, where η is the image in Γ of the label of any path connecting u and u' inside M. In particular, the given group element γ corresponds to the base point u. For any character $\chi : \Gamma \to \mathbb{R}$ we define the χ-*valuation of M with respect to u and* γ, denoted by $v_\chi(M)$, to be the minimum value of $\chi(g)$ when g runs over the elements of Γ corresponding to the vertices of M.

Now, suppose that Γ is finitely presented, with $\langle \mathcal{X} | \mathcal{R} \rangle$ a finite presentation, and assume that $[\chi] \in \Sigma^1(\Gamma)$. Then we can distinguish an element $t \in \mathcal{X}^{\pm 1}$ with $\chi(t) > 0$ with which we can apply Renz's criterion for Σ^1: for each $x \in \mathcal{X}^{\pm 1} \smallsetminus \{t, t^{-1}\}$ we can associate a word w_x in $\mathcal{X}^{\pm 1}$ that represents $t^{-1}xt$ and for which $v_\chi(t^{-1}xt) < v_\chi(w_x)$. Additionally, we put $w_t := t$ and $w_{t^{-1}} := t^{-1}$. If $r = x_1 \cdots x_n \in \mathcal{R}^{\pm 1}$, we define:

$$\hat{r} := w_{x_1} \cdots w_{x_n}.$$

Notice that $\hat{r}$ is a relator too. We are now ready to state the criterion for Σ^2.

Theorem 2.24 ([68], Theorem 3)**.** *Let Γ, $\mathcal{X}$ and t be as above. Suppose that the set $\mathcal{R}$ of defining relators contains some cyclic permutation of the words $t^{-1}xtw_x^{-1}$, for all $x \in \mathcal{X}^{\pm 1}$. Then $[\chi] \in \Sigma^2(\Gamma)$ if and only if for each $r \in \mathcal{R}^{\pm 1}$ there exist a simple diagram $M_{\hat{r}}$ and vertex u in its boundary, such that both the following conditions hold:*

1. *The boundary path of $M_{\hat{r}}$, read from u, has as label the word $\hat{r}$;*

2. *$v_\chi(r) < v_\chi(M_{\hat{r}})$, where the valuation of $M_{\hat{r}}$ is taken with respect to the base point u and the element $t \in \Gamma$.*

Intuitively, the criterion works as follows. If ρ is a closed path in $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ with base point 1, the we can always find a simple diagram D over $\langle \mathcal{X} | \mathcal{R} \rangle$ whose boundary is exactly ρ (this is the 1-connectedness of the Cayley complex). It may happen that $v_\chi(D) \geq 0$, in which case ρ is nullhomotopic in $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$. Otherwise, the theorem provides a way of changing internal cells of D by diagrams with greater valuation (we substitute cells e_r, associated to the relator r, with M_r, which is composed by $M_{\hat{r}}$ and some cells associated to relators $t^{-1}xtw_x^{-1}$).

Example 2.25. *As before, if $z \in \Gamma$ is central and $\chi(z) > 0$, then $[\chi] \in \Sigma^2(\Gamma)$. Indeed, for each $r \in \mathcal{R}$, the diagram with the single 2-cell associated to r is a choice for $M_{\hat{r}}$.*

Recall that a wreath product $H \wr_X G$ is finitely presented if and only if G and H are finitely presented, G acts diagonally on X^2 with finitely many orbits and the stabilizers of the G-action on X are finitely generated (Theorem 1.7).

We will apply Theorem 2.24 to show that if $\Gamma = H \wr_X G$ is finitely presented and if $\chi : \Gamma \to \mathbb{R}$ is a character such that $\chi|_{H_{x_1}} \neq 0$ for some $x_1 \in X$ with $|G \cdot x_1| = \infty$, then $[\chi] \in \Sigma^2(\Gamma)$.

We start by assuming that G acts transitively on X, with $X = G \cdot x_1$. We think of $\Gamma = H \wr_X G$ with a presentation obtained as in Section 1.2.

Let $\langle Y|R\rangle$ and $\langle Z|S\rangle$ be finite presentations for H and G, respectively. We assume that Z contains a generating set E for the stabilizer subgroup $stab_G(x_1)$ and a set J of representatives for the non-trivial double cosets of $(stab_G(x_1), stab_G(x_1))$ in G. So Γ is generated by the set $Y \cup Z$, subject to the following defining relators:

(1) r, for all $r \in R$ (defining relators for H);

(2) s, for all $s \in S$ (defining relators for G);

(3) $[{}^g y_1, y_2]$, for $g \in J$, $y_1, y_2 \in Y$;

(4) $[e, y]$, for $e \in E$ and $y \in Y$.

Let us adapt a bit this presentation. We are under the hypothesis that $\chi|_{H_{x_1}} \neq 0$ and $|G \cdot x_1| = \infty$. We may assume without loss of generality that $\chi(h) > 0$ for some $h \in Y$. Choose $g_i \in Z$, for $1 \leq i \leq 5$, such that $\{x_1\} \cup \{g_i \cdot x_1 \mid 1 \leq i \leq 5\}$ is a set with exactly six elements (of course we may assume that Z contains elements g_i with this property). Define

$$t_i := {}^{g_i} h,$$

for $i = 1, \ldots, 5$. Then Γ is generated by $Y \cup Z \cup \{t_i \mid 1 \leq i \leq 5\}$, subject to the following defining relators:

(1) r, for all $r \in R$ (defining relators for H);

(2) s, for all $s \in S$ (defining relators for G);

(3) $[{}^g y_1, {}^{g'} y_2]$, for all $y_1, y_2 \in Y \cup \{t_i \mid 1 \leq i \leq 5\}$ and $g, g' \in Z \cup \{1\}$ whenever the commutator $[{}^g y_1, {}^{g'} y_2]$ is indeed a relator in Γ;

(4) $[e, y]$, for $e \in E$ and $y \in Y$ and
$[z, t_1]$, for $z \in Z \cap stab_G(g_1 \cdot x_1)$;

(5) $g_i h g_i^{-1} t_i^{-1}$, for $1 \leq i \leq 5$.

Remark 2.26. *We could write the conditions of item* (3) *in a more precise way, but it would require writing many cases. If $y_1 \in Y$ and $y_2 = t_1$, for example, then $[{}^g y_1, {}^{g'} y_2]$ is a defining relator if $g \cdot x_1 \neq (g' g_1) \cdot x_1$.*

Note that we have added a few relators of the types (3) and (4), but clearly they are consequences of the others. Furthermore, the set of relators is clearly still finite.

Set $t = t_1$. We will continue using the notation of Proposition 2.19. Thus for $y \in Y^{\pm 1}$ we have chosen $w_y = y$. If $z \in Z^{\pm 1}$, then $w_z = z$ if $z \in stab_G(g_1 \cdot x_1)$ and $w_z = ztz^{-1}t^{-1}z$ otherwise. Moreover, since t_i and t commute in Γ for all $1 \leq i \leq 5$, we can define $w_{t_i} := t_i$ and $w_{t_i^{-1}} := t_i^{-1}$.

Let us check that the chosen presentation satisfies the conditions of Theorem 2.24. First, the set of defining relators contains the relators $t^{-1} x t w_x^{-1}$. Indeed if $y \in Y^{\pm 1} \cup \{t_i \mid 1 < i \leq 5\}^{\pm 1}$, then $t^{-1} y t w_y^{-1}$ is a relator of type (3), since $w_y = y$. If $z \in Z^{\pm 1} \cap stab_G(g_1 \cdot x_1)$, then $w_z = z$ and $t^{-1} z t z^{-1}$ is a relator of type (4). Finally, if $z \in Z^{\pm 1}$ but $z \notin stab_G(g_1 \cdot x_1)$, then $w_z = ztz^{-1}t^{-1}z$ and

$$t^{-1} z t w_z^{-1} = t^{-1} z t z^{-1} t z t^{-1} z^{-1} = t^{-1} ({}^z t) t ({}^z t)^{-1},$$

which is a cyclic permutation of $[{}^z t, t]$, a relator of type (3).

According to Theorem 2.24, now we need to apply the transformation $r \mapsto \hat{r}$ to each defining relator and then find a simple diagram $M_{\hat{r}}$ satisfying the stated conditions. The following subsections are devoted to the verification of the existence of these diagrams. Observe that we do not need to consider the inverses of defining relators, since any simple diagram for $\hat{r}$ is a simple diagram for the inverse of $\hat{r}$ if we read its boundary backwards.

2.5.1 Relations of type (1)

Note that the relators of type (1) involve only generators in $Y^{\pm 1}$. But $w_y = y$ for all $y \in Y^{\pm 1}$, so $\hat{r} = r$ whenever r is a relator of type (1). Thus the one-faced diagram M that represents the relator r, with base point corresponding to t, is already a choice for $M_{\hat{r}}$, since its χ-valuation is increased by $\chi(t) > 0$.

In Figure 1 we represent the diagram $M_{\hat{r}}$, for $r = \hat{r} = y_1 y_2 y_3 y_4$, as the internal square of the figure. The external boundary represents the path beginning at the base point $1 \in \Gamma$ and with label the original relator r. The edges labeled by t indicate that $\hat{r}$ is obtained from r by conjugation by t.

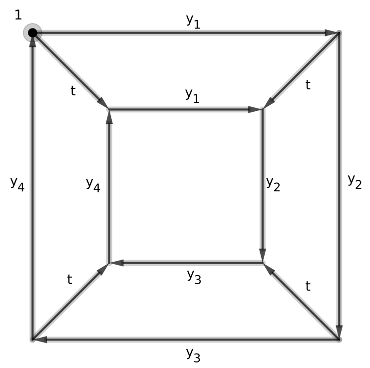


Figure 1 – Diagram for a relator of type (1), $r = \hat{r} = y_1y_2y_3y_4$

2.5.2 Relations of type (2)

Since $\chi(h) \neq 0$, the order of h in Γ is infinite. Consider the subgroup

$$\Gamma_0 := \langle h, G\rangle \leqslant \Gamma.$$

Notice that $\Gamma_0 \simeq \mathbb{Z} \wr_X G$. Let χ_0 be the restriction of χ to Γ_0. The group Γ_0 is an extension of an abelian group $A = \oplus_{x\in X}\mathbb{Z}$ by G, so it follows from Theorem 2.12 that $[\chi_0] \in \Sigma^2(\Gamma)$, since $\chi_0|_A \neq 0$. Now choose a presentation for Γ_0 that is compatible with the chosen presentation for Γ: write the same presentation with $Y = \{h\}$ and discard the relators of type (1). Naturally, this presentation satisfies the hypothesis of Theorem 2.24.

Let $r = z_1 \cdots z_n$ be a relator of type (2). We can see r as a relator in Γ_0. By Theorem 2.24 there is a simple diagram $M_{\hat{r}}$, with respect to the presentation of Γ_0, whose base point corresponds to t and such that $\partial M_{\hat{r}} = \hat{r}$ and $v_\chi(r) < v_\chi(M_{\hat{r}})$. But all the relators in the chosen presentation of Γ_0 are also relators in the original presentation of Γ, after identifying the generating sets. Then $M_{\hat{r}}$, if seen as a diagram over the presentation of Γ, is the diagram we wanted.

2.5.3 Relations of the types (4) or (5)

All cases are similar: we can obtain simple diagrams whose only vertices are those of the boundary, that is, those that are defined by the word $\hat{r}$. In this case, the diagram automatically satisfies the hypothesis about its χ-valuation, exactly as in the case of the relators of type (1). See the diagram for the relator $g_1hg_1^{-1}t^{-1}$ in Figure 2. As before, the external boundary represents $r = g_1hg_1^{-1}t^{-1}$, and the diagram $M_{\hat{r}}$ itself is the internal diagram composed by the five squares. Again, the edges labeled by t and with origin at some point in the external path indicate conjugation by t and represent the growth of v_χ from r to $\hat{r}$.

Figure 2 is an illustration of the case when $g = g_1 \notin stab_G(g_1 \cdot x_1)$, when the word w_g is more complicated. If the letter representing an element of G lies in $stab_G(g_1 \cdot x_1)$,

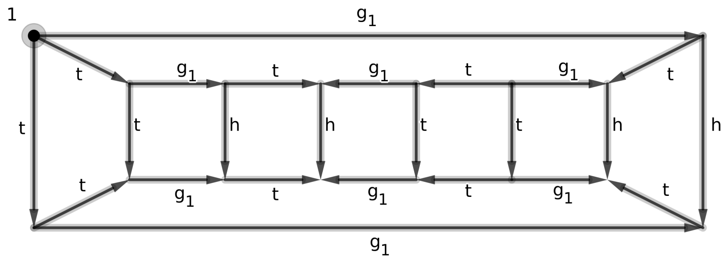


Figure 2 – Diagram for relators of type (5)

the argument is simpler: all the letters involved in the relator commute with t, so $r = \hat{r}$ and the argument follows as in the case of the relators of type (1).

2.5.4 Relations of type (3)

Let $y \in Y \cup \{t_i \mid 1 \le i \le 5\}$ and $g \in Z$. Let $\eta_{g,y}$ be the word obtained from gyg^{-1} by applying the transformation that takes each letter α to w_α:

$$\eta_{g,y} = ({}^g t)t^{-1}({}^g y)t({}^g t)^{-1}, \tag{2.2}$$

if $g \notin stab_G(g_1 \cdot x_1)$, or

$$\eta_{g,y} = {}^g y, \tag{2.3}$$

if $g \in stab_G(g_1 \cdot x_1)$. If $g = 1$, put $\eta_{1,y} := y$. In all cases we see that $\eta_{g,y}$ is a product of subwords representing elements of at most 3 copies of H in Γ. Indeed, ${}^g t$ and $({}^g t)^{-1}$ are elements of $H_{gg_1 \cdot x_1}$, while t and t^{-1} are elements of $H_{g_1 \cdot x_1}$ and, finally, ${}^g y$ is an element of $H_{g \cdot x_1}$ or $H_{gg_i \cdot x_1}$ for some $1 \le i \le 5$, depending on y.

Consider $r = [{}^g y_1, {}^{g'} y_2]$, a relator of type (3). The word $\hat{r} = [\eta_{g,y_1}, \eta_{g',y_2}]$ is a relator in Γ, so we can always find a simple diagram M_1 with some base point corresponding to t and such that $\partial M_1 = \hat{r}$. If $v_\chi(M_1) > v_\chi(r)$ we are done. Otherwise there is a vertex p of M_1 such that $\chi(p) \le v_\chi(r)$. Notice that this vertex can not lie on the boundary of M_1, since $v_\chi(\hat{r}) > v_\chi(t^{-1}rt)$.

Now, the commutator between the words η_{g,y_1} and η_{g',y_2} is a product of elements of the form ${}^z y$, with $z \in Z \cup \{1\}$ and $y \in Y^{\pm 1} \cup \{t_1, \dots, t_5\}^{\pm 1}$. By the remarks above, these elements lie in at most five different copies of H, one of them being indexed by $g_1 \cdot x_1$ when five copies do pop up. It follows that for some $u \in \{h, t_2, t_3, t_4, t_5\}$, the words $[{}^z y, u]$ are defining relators for all the subwords ${}^z y$ appearing in $\hat{r} = [\eta_{g,y_1}, \eta_{g',y_2}]$ (we consider the subwords ${}^z y$ that appear when η_{g,y_1} and η_{g',y_2} are written exactly as in (2.2) or (2.3)). Observe that $\chi(u) = \chi(h) > 0$ in all cases. So we can build a diagram M_2 by surrounding M_1 with faces representing the commutators $[{}^z y, u]$, for all these subwords ${}^z y$.

Clearly the boundary of M_2 is also labeled by $\hat{r}$. If we set as base point the vertex on the new boundary corresponding to the base point of M_1 (that is, the one joined to it by an edge with label u), then the χ-valuation of the interior points (including p) is increased by $\chi(u) > 0$, so that either $v_\chi(M_2) = v_\chi(M_1) + \chi(t)$, or $v_\chi(M_2)$ is attained at the boundary. In the latter case, M_2 satisfies the conditions of the theorem. Otherwise we repeat the process now starting with M_2 in place of M_1. After finitely many repetitions, we eventually obtain a simple diagram M_n whose boundary is labeled by $\hat{r}$ and $v_\chi(M_n)$ is attained at the boundary, thus it satisfies the conditions of the theorem. This will occur for the smallest integer n such that $v_\chi(M_1) + (n-1)\chi(t) \geq v_\chi(\hat{r})$.

We record what we have proved in the following proposition.

Proposition 2.27. *Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $M = \oplus_{x\in X} H_x \subseteq \Gamma$. Suppose that G acts transitively on the infinite set X. If $\chi : \Gamma \to \mathbb{R}$ is a character with $\chi|_M \neq 0$, then $[\chi] \in \Sigma^2(\Gamma)$.*

The arguments above essentially contain what we need when $G \cdot x$ is infinite for some $x \in X$ such that $\chi|_{H_x} \neq 0$ (but the G-action on X is not necessarily transitive), so we will only indicate in the proof of the following theorem how to deal with this case.

Recall that we denote by T the set of elements $x \in X$ such that $\chi|_{H_x} \neq 0$. Notice that if $T = \{x_1\}$, then Γ is a direct product

$$\Gamma \simeq H_{x_1} \times (H \wr_{X'} G),$$

where $X' = X \setminus \{x_1\}$. Then the direct product formulas and the results on the Σ^1-invariants of wreath products already contain all the information we need. The remaining cases are all part of the following theorem, which includes Theorem A2.

Theorem 2.28. *Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $M = \oplus_{x\in X} H_x \subseteq \Gamma$. Suppose that the set*

$$T = \{x \in X \mid \chi|_{H_x} \neq 0\}.$$

has at least two elements. Then $[\chi] \in \Sigma^2(\Gamma)$ if and only if at least one of the following conditions holds:

(1) $[\chi|_{H_x}] \in \Sigma^1(H)$ for some $x \in T$;

(2) $\chi|_G \neq 0$;

(3) T has at least three elements.

Proof. Suppose first that T is a finite set and consider the subgroup $B = \cap_{x\in T} stab_G(x) \leqslant G$. It is of finite index in G, so $\Gamma_1 = H \wr_X B$ is of finite index in Γ. Notice that

$$\Gamma_1 \simeq (\prod_{x\in T} H_x) \times (H \wr_{X'} B).$$

Denote $P = \prod_{x \in T} H_x$ and $Q = H \wr_{X'} B$. The fact that T has at least two elements implies that $[\chi|_P] \in \Sigma^1(P)$, by Theorem 2.8. By applying the theorem again, now to the product $\Gamma_1 = P \times Q$, we get that $[\chi|_{\Gamma_1}] \in \Sigma^2(\Gamma_1)$ if and only if $[\chi|_P] \in \Sigma^2(P)$ or $\chi|_Q \neq 0$. The former happens if and only if at least one of conditions (1) or (3) is satisfied (once again, by the direct product formula), while the latter clearly happens if and only if $\chi|_B \neq 0$, which in turn is equivalent with $\chi|_G \neq 0$, since B is a subgroup of finite index. Finally, since the index of Γ_1 in Γ is finite, we are done by Theorem 2.10.

We are left with the case where T is infinite and we want to show that $[\chi] \in \Sigma^2(\Gamma)$. Since G acts on X with finitely many orbits, there must be some $x_1 \in T$ such that $|G \cdot x_1| = \infty$. We adapt the proof of Proposition 2.27 putting the orbit of x_1 in a distinguished position.

Choose $x_2, \ldots, x_n \in X$ such that $X = \bigsqcup_{j=1}^{n} G \cdot x_j$. For each j choose a finite generating set E_j for the stabilizer subgroup $stab_G(x_j)$. For each pair par (i, j), with $1 \leq i, j \leq n$, choose a finite set $J_{i,j}$ of representatives of the non-trivial double cosets of $(stab_G(x_i), stab_G(x_j))$ in G. Finally, choose finite presentations $\langle Y|R \rangle$ and $\langle Z|S \rangle$ for H and G respectively. We may assume that Z contains E_j and $J_{i,j}$ for all $1 \leq i, j \leq n$.

A finite presentation for Γ, adapted from the presentation given by Cornulier [26], can be given as follows. For each $1 \leq i \leq n$ we associate a copy $\langle Y_i|R_i \rangle$ of the presentation for H and, as before, we define $t_i := {}^{g_i}h$ for some $g_i \in Z$ and $h \in Y_1$ with $\chi(h) > 0$ and $|\{x_1\} \cup \{g_i \cdot x_1 | 1 \leq i \leq 5\}| = 6$. We think of Γ as generated by $(\bigcup_{i=1}^{n} Y_i) \cup Z \cup \{t_i \mid 1 \leq i \leq 5\}$ and subject to the defining relators given by:

(1) r, for all $r \in \bigcup_{i=1}^{n} R_i$ (defining relators for the copies of H);

(2) s, for all $s \in S$ (defining relators for G);

(3) $[{}^{g}y_1, {}^{g'}y_2]$, for $y_1, y_2 \in (\bigcup_{i=1}^{n} Y_i) \cup \{t_i \mid 1 \leq i \leq 5\}$ and $g, g' \in Z \cup \{1\}$ whenever $[{}^{g}y_1, {}^{g'}y_2]$ is indeed a relator in Γ;

(4) $[e_i, y_i]$, for all $e_i \in E_i$, $y_i \in Y_i$ and $1 \leq i \leq n$ and
$[z, t_1]$, for all $z \in Z \cap stab_G(g_1 \cdot x_1)$;

(5) $g_i h g_i^{-1} t_i^{-1}$, for $1 \leq i \leq 5$.

Set $t = t_1$. We use again the notation of Proposition 2.19. So we use the same words w_z if $z \in Z^{\pm 1}$, and $w_y = y$ for all other generators y. It is clear that the set of defining relators above still satisfies the hypothesis of Theorem 2.24. The construction of the diagrams associated to each defining relator can be done exactly as in the case where the action is transitive, as we will argue below. The key fact is that the generators coming from copies of H associated to all other orbits of G (other than $G \cdot x_1$) commute with $t = t_1$.

First, notice that the construction of the diagrams associated to the relators of types (1) or (4) in the case of a transitive action depends only on the fact that $[t, y]$ is a defining relator for all $y \in Y = Y_1$. But t_1 commutes also with all elements of $Y_2 \cup \ldots \cup Y_n$, so the construction can be carried out in the same way. For the case of relators of type (3), it was only necessary that for any generators $g, g' \in Z \cup \{1\}$ and $y, y' \in Y_1$, we could find some $u \in \{h, t_2, \ldots, t_5\}$ that commutes with all the following elements: t, ${}^{g}t$, ${}^{g'}t$, ${}^{g}y$ and ${}^{g'}y'$. If we allow y to be an element of $Y_2 \cup \ldots \cup Y_n$, then any u that commutes with t, ${}^{g}t$, ${}^{g'}t$ and ${}^{g'}y'$ will do it, since ${}^{g}y$ commutes any choice of u. Thus the five options for u, coming from different copies of H, are enough to let us repeat the argument. Similar considerations cover the cases where either only y', or both y and y' are elements of $Y_2 \cup \ldots \cup Y_n$.

This is all we needed to check, since relators of types (2) and (5) do not involve any of the new generators. □

2.6 Some observations about Σ^2

Let Γ be a finitely presented group and let $[\chi] \in S(\Gamma)$. Let $\langle \mathcal{X} | \mathcal{R} \rangle$ be a finite presentation for Γ. Denote by $C = Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)$ the associated Cayley complex and by C_χ the full subcomplex of C spanned by Γ_χ. The canonical action of Γ on C restricts to an action by the monoid Γ_χ on C_χ.

Remark 2.29. *If a monoid K acts on some set X we still say that the sets $K \cdot x$ are orbits. By "K has finitely many orbits on X" we mean that there are elements $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{j=1}^{n} K \cdot x_j$.*

The following lemma can be found in Renz's thesis [67].

Lemma 2.30. *C_χ has finitely many Γ_χ-orbits of k-cells for $k \leq 2$.*

Proof. Denote by D and D_χ the sets of k-cells of C and C_χ, respectively (for a fixed $k \leq 2$). We know that Γ acts on D with finitely many orbits. Choose representatives $d_1, \ldots, d_n$ for these orbits so that $d_j \in D_\chi$ but $\gamma \cdot d_j \notin D_\chi$ for all j and for all $\gamma \in \Gamma$ with $\chi(\gamma) < 0$. For this it suffices to take any representatives $\tilde{d}_1, \ldots, \tilde{d}_n$ and then put $d_j := \gamma_j^{-1} \cdot \tilde{d}_j$, where $\gamma_j \in \Gamma$ is the vertex of $\tilde{d}_j$ with lowest χ-value. Thus if $d \in D_\chi$, then $d = \gamma \cdot d_j$ for some j and, by choice of d_j, we have that $\chi(\gamma) \geq 0$. So $D_\chi = \bigcup_{j=1}^{n} \Gamma_\chi \cdot d_j$. □

Denote by $F(\mathcal{X},\chi)$ the submonoid of $F(\mathcal{X})$ consisting of the classes of reduced words w with $v_\chi(w) \geq 0$. Note that $F(\mathcal{X},\chi)$ is indeed closed under the product, since $w_1, w_2 \in F(\mathcal{X},\chi)$ implies $v_\chi(w_1 w_2) \geq 0$, and this property is preserved by elementary reductions (that is, canceling out terms of the form xx^{-1} or $x^{-1}x$). Let $R(\chi)$ be the subgroup of $F(\mathcal{X})$ consisting of the classes of reduced words w that represent relators (that is $w \in \langle \mathcal{R} \rangle^{F(\mathcal{X})}$) and such that $v_\chi(w) \geq 0$. Observe that $R(\chi) \subseteq F(\mathcal{X},\chi)$ and notice that $R(\chi)$ is indeed a subgroup, since $v_\chi(w) \geq 0$ implies $v_\chi(w^{-1}) \geq 0$ whenever w is a relator. Finally, observe that $R(\chi)$ admits an action by the monoid $F(\mathcal{X},\chi)$ via left conjugation.

Now, let r be a reduced word in $\mathcal{X}^{\pm 1}$ representing a relator in Γ, that is, $r \in \langle \mathcal{R} \rangle^{F(\mathcal{X})}$. Suppose that M is a van Kampen diagram over $\langle \mathcal{X} | \mathcal{R} \rangle$ whose boundary, read in some orientation from some base point p, is exactly r. Then it holds in $F(\mathcal{X})$ that

$$r = {}^{w_1}r_1 \cdots {}^{w_n}r_n, \tag{2.4}$$

where each r_i is a word read on the boundary of some face of M and w_i is the label for a path in M connecting p to a base point of the face associated to r_i. Both the facts that such a diagram exists and that r can be written as above are consequences of van Kampen's lemma (see Proposition 4.1.2 and Theorem 4.2.2 in [19], for instance).

Lemma 2.31. *If $\chi : \Gamma \to \mathbb{R}$ is a character such that $C_\chi = Cay(G, \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ is 1-connected, then $R(\chi)$ is finitely generated over $F(\mathcal{X},\chi)$.*

Remark 2.32. *By "$R(\chi)$ is finitely generated over $F(\mathcal{X},\chi)$" we mean that every element of $R(\chi)$ can be written as a product of elements of the form ${}^{w}s$, where $w \in F(\mathcal{X},\chi)$ and $s \in S$ for some finite set $S \subseteq R(\chi)$.*

Proof. Let $r \in R(\chi)$ and consider the path ρ in C beginning at 1 and with label r. Notice that this path runs inside C_χ, since $v_\chi(r) \geq 0$. Also, ρ is clearly a loop and it must be nullhomotopic in C_χ, since C_χ is 1-connected. A homotopy from ρ to the trivial path can then be realized by a van Kampen diagram M with $v_\chi(M) \geq 0$ (the valuation is taken with respect to 1, seen both as base point in C and group element). This is made precise by Theorem 2 in [68].

Write r as in (2.4). Thus r is a product of relators corresponding to the faces of M conjugated on the left by elements of $F(\mathcal{X},\chi)$. Since $v_\chi(M) \geq 0$, such faces are faces of C_χ, so by Lemma 2.30 and using that every element of Γ_χ can be written as a word in $F(\mathcal{X},\chi)$, each ${}^{w_j}r_j$ can be rewritten as ${}^{u_j}s_j$ where $u_j \in F(\mathcal{X},\chi)$ and each s_j is a word read on the boundary of a face in a finite set S of representatives of Γ_χ-orbits of faces of C_χ. It follows that S is a finite generating set for $R(\chi)$ modulo the action of $F(\mathcal{X},\chi)$. ☐

2.7 Σ^2 for characters with $\chi|_M = 0$

Recall that for a group G, a $K(G,1)$-*complex* (or *Eilenberg-MacLane space for* G) is a connected CW-complex Y such that $\pi_1(Y) = G$ and $\pi_i(Y) = 1$ for all $i \geq 2$. If $G = \langle \mathcal{X} | \mathcal{R} \rangle$, then we can build a $K(G,1)$-complex Y as follows. To start with, Y will have exactly one 0-cell v. For each $x \in \mathcal{X}$, we associate a 1-cell e_x both whose endpoints are v. For each $r \in \mathcal{R}$, we glue a 2-cell c_r along the boundary that we can read for r as word in the letters $\mathcal{X}$. This already gives a connected CW-complex with fundamental group G. Then we keep adding cells of higher dimensions to ensure that the higher homotopy groups are trivial.

If G is finitely presentable, the procedure above gives a $K(G,1)$-complex with finitely many cells of dimension at most 2.

We get back to a finitely presented wreath product $\Gamma = H \wr_X G = M \rtimes G$. We consider now the non-zero characters $\chi : \Gamma \to \mathbb{R}$ such that $\chi|_M = 0$.

In order to find sufficient conditions for $[\chi] \in \Sigma^2(\Gamma)$, we consider a nice action of Γ on a complex. We will briefly describe the construction in the proof of Theorem A in [50], with the simplifications allowed by the fact that our situation is less general than what is considered in that paper.

We are assuming that $\Gamma = H \wr_X G$ is finitely presented, so H is also finitely presented. Choose a $K(H,1)$-complex Y, with base point $*$, having a single 0-cell and finitely many 1-cells and 2-cells. Let $Z = \oplus_{x \in X} Y_x$ be the *finitary product* of copies of Y indexed by X, that is, Z is the subset of the cartesian product $\prod_{x \in X} Y_x$ consisting on the families $(y_x)_{x \in X}$ such that y_x is not the base point $*$ only for finitely many indices $x \in X$. It follows from the results in [28] and [29] that Z is an Eilenberg-MacLane space for $M = \oplus_{x \in X} H_x$. Notice that Z has a natural cell structure. There is a single 0-cell, given by the family $(y_x)_{x \in X}$ with $y_x = *$ for all x. For $n \geq 1$, the n-cells can be identified with products $c_1 \times \cdots \times c_k$ of cells of Y, supported by some tuple $(x_1, \ldots, x_k) \in X^k$, such that $dim(c_1) + \ldots + dim(c_k) = n$.

There is an obvious action of G on Z. On the other hand, M acts freely on the universal cover E of Z. By putting together these two actions, we get an action of $\Gamma = M \rtimes G$ on E. Notice that, since we are assuming that Γ is finitely presented (in particular G acts on X^2 with finitely many orbits by Cornulier's results), the 2-skeleton of E has finitely many Γ-orbits of cells. Moreover, since the action of M is free, the stabilizer subgroups are all conjugate to subgroups of G, and can be described as follows:

1. The stabilizer subgroup of any 0-cell is a conjugate of G;

2. For $n \geq 1$, the stabilizer subgroup of each n-cell contains a conjugate of the stabilizer

$stab_G(\bar{x})$ of some $\bar{x} = (x_1, \ldots, x_n) \in X^n$ as a finite index subgroup.

We make stabilizers of n-cells correspond to stabilizers of n-tuples (rather than k-tuples, for $k \leq n$) by repeating some indices if necessary. The reason why we need to pass to a finite index subgroup is that cells of Z written as products of cells of Y may contain some repetition. For instance, a cell of Z that arises as a product $c \times c$, supported by (x_1, x_2), is also fixed by elements of G that interchange x_1 and x_2. This will also happen in the Γ-action on the universal cover E.

For groups admitting sufficiently nice actions on complexes, there is a criterion for the Σ-invariants.

Theorem 2.33 ([60], Theorem B)**.** *Let E' be a 2-dimensional 1-connected complex. Suppose that a group Γ acts on E' with finitely many orbits of cells. If $\chi : \Gamma \to \mathbb{R}$ is a character such that $[\chi|_{stab_\Gamma(c)}] \in \Sigma^{2-dim(c)}(stab_\Gamma(c))$ for all cells c in E', then $[\chi] \in \Sigma^2(\Gamma)$.*

We apply the theorem above with E' being the 2-skeleton of E. We obtain:

Proposition 2.34. *Suppose that $\Gamma = H \wr_X G$ is finitely presented and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. Suppose also that all properties below hold:*

(1) $[\chi|_G] \in \Sigma^2(G)$*;*

(2) $[\chi|_{stab_G(x)}] \in \Sigma^1(stab_G(x))$ *for all $x \in X$ and*

(3) $\chi|_{stab_G(x,y)} \neq 0$ *for all $(x, y) \in X^2$.*

Then $[\chi] \in \Sigma^2(\Gamma)$.

The fact that we can state the proposition above with reference only to the stabilizers contained in G follows from the invariance of the Σ^1-invariants under isomorphisms ([76, Prop. B1.5]). It is also clear that item (3) is equivalent with asking that the restriction of χ to the actual stabilizers is non zero.

By Theorem 2.11, if $[\chi] \in \Sigma^2(\Gamma)$ and $\chi|_M = 0$, then $[\chi|_G] \in \Sigma^2(G)$. We can also show that condition (3) of Proposition 2.34 is necessary.

Lemma 2.35. *If $\chi|_{stab_G(x,y)} = 0$, then the monoid G_χ can not have finitely many orbits on $G \cdot (x, y)$.*

Proof. Suppose that $G \cdot (x, y) = \bigcup_{j=1}^{n} G_\chi \cdot (x_j, y_j)$ and choose $g_1, \ldots, g_n \in G$ such that $(x_j, y_j) = g_j \cdot (x, y)$. Choose $g \in G$ such that

$$\chi(g) < min\{\chi(g_j) \mid 1 \leq j \leq n\}.$$

Since $g \cdot (x,y) \in \bigcup_{j=1}^{n} G_\chi \cdot (x_j, y_j)$, there must be some $g_0 \in G_\chi$ and $1 \leq j \leq n$ such that $g \cdot (x,y) = g_0 \cdot (x_j, y_j) = g_0 g_j (x,y)$. But then $g^{-1} g_0 g_j \in stab_G(x,y)$, with:

$$\chi(g^{-1}g_0 g_j) = \chi(g_0) + (\chi(g_j) - \chi(g)) > 0,$$

so $\chi|_{stab_G(x,y)} \neq 0$. □

Proposition 2.36. *Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character. Let $M = \oplus_{x \in X} H_x \subseteq \Gamma$ and suppose that $\chi|_M = 0$. If $\chi|_{stab_G(x,y)} = 0$ for some $(x,y) \in X^2$, then $[\chi] \notin \Sigma^2(\Gamma)$.*

Proof. We may assume that $[\chi] \in \Sigma^1(\Gamma)$, otherwise there is nothing to do. Thus $[\chi|_G] \in \Sigma^1(G)$ and $\chi|_{stab_G(x)} \neq 0$ for all $x \in X$ by Proposition 2.16.

Let $\Gamma_0 = (*_{x \in X} H_x) \rtimes G$ and let $\mathcal{X} \subseteq \Gamma_0$ be a finite generating set. Note that Γ is a quotient of Γ_0, so we can consider the following diagram:

$$\begin{array}{ccc} F(\mathcal{X}) & \xrightarrow{\pi_0} & \Gamma_0 \\ & \searrow^{\pi} & \downarrow \\ & & \Gamma \end{array}$$

The homomorphism π defines presentations for Γ with generating set $\mathcal{X}$. We first show that for finite presentations of type $\Gamma = \langle \mathcal{X} | \mathcal{R} \rangle$ (with $ker(\pi) = \langle \mathcal{R} \rangle^{F(\mathcal{X})}$) the complex $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ can not be 1-connected.

Fix $\langle \mathcal{X} | \mathcal{R} \rangle$ such a presentation. We use the notations $F(\mathcal{X}, \chi)$ and $R(\chi)$ defined in Section 2.6. We want to show that $R(\chi)$ is not finitely generated over $F(\mathcal{X}, \chi)$, from what follows that $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ is not 1-connected by Lemma 2.31.

If $\chi|_{stab_G(x,y)} = 0$, then by Lemma 2.35 we can build a strictly increasing sequence

$$I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_j \subsetneq \ldots$$

of G_χ-invariant subsets of X^2 such that $X^2 = \bigcup_j I_j$.

Let N be the normal subgroup of $*_{x \in X} H_x$ such that $M = (*_{x \in X} H_x)/N$. Note that N admits an action by $(*_{x \in X} H_x) \rtimes G$ (which defines the wreath product $H \wr_X G$). Let N_j be the normal subgroup of $*_{x \in x} H_x$ generated by the commutators $[H_x, H_y]$ with $(x,y) \in I_j$. Note that $N_1 \subsetneq N_2 \subsetneq \ldots$, that $N = \bigcup_j N_j$ and that each N_j is $(*_{x \in X} H_x) \rtimes G_\chi$-invariant.

Put

$$\Gamma_{j,\chi} = \frac{*_{x \in X} H_x}{N_j} \rtimes G_\chi.$$

This is a well defined monoid under the operation we use to define the semi-direct product. This defines a sequence $\{\Gamma_{j,\chi}\}_j$ of monoids that converges to Γ_χ.

Now, remember we have chosen $\mathcal{X}$ so that the projection $\pi_0 : F(\mathcal{X}) \twoheadrightarrow \Gamma_0$ is well defined. Passing to monoids, we obtain a homomorphism

$$p_0 : F(\mathcal{X}, \chi) \to (\Gamma_0)_{\chi_0},$$

where χ_0 is the obvious lift of χ to Γ_0. From p_0 we define

$$p_j : F(\mathcal{X}, \chi) \to \Gamma_{j,\chi}$$

for each $j \geq 1$. Let

$$R_j = p_j^{-1}(\{1\}).$$

Notice that $R_j \subsetneq R(\chi)$ for all j. Indeed, it is clear that χ_0 and χ restrict to the same homomorphisms on G and $stab_G(x)$, for all $x \in X$. Also, χ_0 restricts to zero on $*_{x\in X} H_x$ by construction. So it follows from Theorem 2.22 that $[\chi_0] \in \Sigma^1(\Gamma_0)$, since we are assuming that $[\chi] \in \Sigma^1(\Gamma)$. Thus p_0 is surjective and any $n \in N \smallsetminus N_j$ defines an element in $R(\chi) \smallsetminus R_j$. Observe further that each that R_j is actually a $F(\mathcal{X}, \chi)$-invariant subgroup of $R(\chi)$ and that $\bigcup_j R_j = R(\chi)$. The existence of the sequence $\{R_j\}_j$ implies that $R(\chi)$ can not be finitely generated over $F(\mathcal{X}, \chi)$.

For the general case, let $\langle \mathcal{X} | \mathcal{R} \rangle$ be any finite presentation for Γ and suppose by contradiction that $Cay(\Gamma; \langle \mathcal{X} | \mathcal{R} \rangle)_\chi$ is 1-connected. From $\langle \mathcal{X} | \mathcal{R} \rangle$ we build another finite presentation $\langle \mathcal{X}' | \mathcal{R}' \rangle$ for Γ with $\mathcal{X} \subseteq \mathcal{X}'$, $\mathcal{R} \subseteq \mathcal{R}'$ and satisfying the previous hypothesis (that is, $\mathcal{X}'$ is actually a generating set for Γ_0). For this, it suffices to add the necessary generators and include the relators that define them in Γ in terms of the previous generating set $\mathcal{X}$. It may be the case that $Cay(\Gamma; \langle \mathcal{X}' | \mathcal{R}' \rangle)_\chi$ is not 1-connected anymore, but by [68, Lemma 3], we can always enlarge $\mathcal{R}'$ to a (still finite) set $\mathcal{R}''$ so that $Cay(\Gamma; \langle \mathcal{X}' | \mathcal{R}'' \rangle)_\chi$ is indeed 1-connected. This is done by adding the relators of the form $t^{-1}xtw_x^{-1}$, as in Theorem 2.24. We arrive at a contradiction with the first part of the proof, since $\mathcal{X}'$ satisfies the previous hypothesis, that is, $\mathcal{X}'$ can be lifted to a generating set for Γ_0. □

The above proposition completes the proof of Theorem A3 as stated in the introduction, since its last assertion (when we assume that H has infinite abelianization) follows from Theorem 2.13.

2.8 Applications to twisted conjugacy

We now derive some consequences of the previous results to twisted conjugacy, more specifically to the study of Reidemeister numbers of automorphisms of wreath products. For this we start by considering the Koban invariant Ω^1.

Given a finitely generated group Γ, endow $Hom(\Gamma, \mathbb{R})$ with an inner product structure, so that it makes sense to talk about angles in $S(\Gamma)$. Denote by $N_{\pi/2}([\chi])$ the open neighborhood of angle $\pi/2$ and centered at $[\chi] \in S(\Gamma)$. Following Koban [40], we can define the invariant $\Omega^1(\Gamma)$ in terms of $\Sigma^1(\Gamma)$:

$$\Omega^1(\Gamma) = \{[\chi] \in S(\Gamma) \mid N_{\pi/2}([\chi]) \subseteq \Sigma^1(\Gamma)\}.$$

A proof of the fact that this does not depend on the inner product can be found in the above-mentioned paper, which contains the original definition of the invariant.

Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. With some restrictions on the action by G on X, we can obtain nice descriptions of $\Omega^1(\Gamma)$. Notice that, since the invariant does not depend on the choice of inner product, we can assume that characters $[\chi], [\eta] \in S(\Gamma)$ such that $\chi|_G = 0$ and $\eta|_M = 0$ are always orthogonal, and this will be done in the proposition below.

Proposition 2.37. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. Suppose that*

$$\Sigma^1(\Gamma) = \{[\chi] \in S(\Gamma) \mid \chi|_M \neq 0\},$$

where $M = \oplus_{x \in x} H_x \subseteq \Gamma$. Then

$$\Omega^1(\Gamma) = \{[\chi] \in S(\Gamma) \mid \chi|_G = 0\}.$$

Proof. Let $[\chi] \in S(\Gamma)$ with $\chi|_G = 0$. Clearly $\chi|_M \neq 0$, so $[\chi] \in \Sigma^1(\Gamma)$. Furthermore, if $[\eta] \in N_{\pi/2}([\chi])$, then $\eta|_M \neq 0$, otherwise χ and η would be orthogonal. So $N_{\pi/2}([\chi]) \subseteq \Sigma^1(\Gamma)$ whenever $\chi|_G = 0$. On the other hand, if there were some $[\chi] \in \Omega^1(\Gamma)$ with $\chi|_G \neq 0$, then by taking $\eta : \Gamma \to \mathbb{R}$ defined by $\eta|_M = 0$ and $\eta|_G = \chi|_G$, we would have that $[\eta] \in N_{\pi/2}([\chi])$, but $[\eta] \notin \Sigma^1(\Gamma)$. □

For any group V, we denote by V^{ab} its abelianization. By Theorem A1, if the G-action on X does not contain orbits composed by only one element, then many conditions imply the hypothesis on the description of $\Sigma^1(\Gamma)$, such as:

1. $(stab_G(x))^{ab}$ is finite for some $x \in X$, or
2. The set $\{[\chi] \in \Sigma^1(G) \mid \chi|_{stab_G(x)} \neq 0\}$ is empty for some $x \in X$.

This includes the cases where the G-action is free (in particular the regular wreath products $\Gamma = H \wr G$) and the case where $\Sigma^1(G) = \emptyset$.

Given a group isomorphism $\varphi : V \to V$, the *φ-twisted conjugacy action* of V on itself is defined by

$$v_1 \cdot v_2 = v_1 v_2 \varphi(v_1)^{-1}$$

for all $v_1, v_2 \in V$. The *Reidemeister number* $R(\varphi)$ of φ is defined as the number of orbits of this action. Finally, a group V is *of type* R_∞ if $R(\varphi) = \infty$ for all isomorphisms $\varphi : V \to V$.

A connection between the invariant Ω^1 and Reidemeister numbers was studied by Koban and Wong [42]. Recall that a character χ is *discrete* if its image is infinite cyclic.

Theorem 2.38. *[42, Thm. 4.3] Let G be a finitely generated group and suppose that $\Omega^1(G)$ contains only discrete characters.*

(1) If $\Omega^1(G)$ contains only one element, then G is of type R_∞;

(2) If $\Omega^1(G)$ has exactly two elements, then there is a subgroup $N \subseteq Aut(G)$, with $[Aut(G) : N] = 2$, such that $R(\varphi) = \infty$ for all $\varphi \in N$.

Corollary 2.39. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and suppose that the G-action on X is transitive. Suppose further that $\Sigma^1(\Gamma)$ is as described in Proposition 2.37 and that H^{ab} has torsion-free rank 1. Then there is a subgroup $N \subseteq Aut(\Gamma)$, with $[Aut(\Gamma) : N] = 2$, such that $R(\varphi) = \infty$ for all $\varphi \in N$.*

Proof. By the hypothesis on H^{ab} we have that

$$\Omega^1(\Gamma) = \{[\nu_1], [\nu_2]\},$$

where $\nu_j(G) = 0$, $\nu_1(h) = 1$ and $\nu_2(h) = -1$ for some lift $h \in H$ of a generator for the infinite cyclic factor of H^{ab}. It suffices then to apply part (2) of Theorem 2.38. □

The applications that we keep in mind are the finitely generated regular wreath products of the form $\mathbb{Z} \wr G$.

Gonçalves and Kochloukova [34] exhibited other connections between the Σ-theory and the property R_∞. Below we denote by $\Sigma^1(G)^c$ the complement of $\Sigma^1(G)$ in $S(G)$, that is, $\Sigma^1(G)^c = S(G) \smallsetminus \Sigma^1(G)$.

Theorem 2.40. *[34, Cor. 3.4] Let G be a finitely generated group and suppose that*

$$\Sigma^1(G)^c = \{[\chi_1], \ldots, [\chi_n]\},$$

where $n \geq 1$ and each χ_j is a discrete character. Then there is a subgroup of finite index $N \subseteq Aut(G)$ such that $R(\varphi) = \infty$ for all $\varphi \in N$.

Corollary 2.41. *Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. Once again, suppose that $\Sigma^1(\Gamma)$ is as described in Proposition 2.37. Suppose further that G^{ab} has torsion-free rank 1. Then there is a subgroup of finite index $N \subseteq Aut(\Gamma)$ such that $R(\varphi) = \infty$ for all $\varphi \in N$.*

Proof. Under the hypothesis above, we have

$$\Sigma^1(\Gamma)^c = \{[\chi_1], [\chi_2]\},$$

where $\chi_j|_M = 0$ and $\chi_1(g) = 1$ and $\chi_2(g) = -1$ for some $g \in G$ whose image in G^{ab} is a generator of the infinite cyclic factor. Then Theorem 2.40 applies. □

This time we can take as an example the regular wreath product $\Gamma = H \wr \mathbb{Z}$.

We note that Gonçalves and Wong [35] and Taback and Wong [78] had already obtained some results about the property R_∞ for regular wreath products of the form $H \wr \mathbb{Z}$, with H abelian or finite. Our results complement theirs in the sense that it considers other basis groups H and non-regular actions, but here we were limited to talk about Reidemeister numbers of automorphisms contained in subgroups of finite index in the automorphism group. In the above-mentioned papers, on the other hand, the authors were able to determine positively the property R_∞ for some choices of H.

3 Weak commutativity for Lie algebras

In this chapter we discuss the weak commutativity construction in the category of Lie algebras over a field K with $char(K) \neq 2$. We fix K once and for all. All Lie algebras in this chapter are Lie algebras over K.

We will discuss the definition of $\chi(\mathfrak{g})$, the series of ideals $R \subseteq W \subseteq L \subseteq \chi(\mathfrak{g})$, and we will establish the results announced in the introduction. The results of this chapter are the content of the articles [62, 63].

3.1 The weak commutativity construction

Let $\mathfrak{g}$ be any Lie algebra and let $\mathfrak{g}^\psi$ be an isomorphic copy. For any $x \in \mathfrak{g}$, we denote by x^ψ its image in $\mathfrak{g}^\psi$. We define $\chi(\mathfrak{g})$ by the presentation

$$\chi(\mathfrak{g}) = \langle \mathfrak{g}, \mathfrak{g}^\psi \mid [x, x^\psi] = 0 \ \forall x \in \mathfrak{g} \rangle.$$

This must be understood as the quotient of the free Lie sum of $\mathfrak{g}$ and $\mathfrak{g}^\psi$ by the ideal generated by the elements $[x, x^\psi]$, for all $x \in \mathfrak{g}$.

The only case in which we can immediately understand $\chi(\mathfrak{g})$ from the definition is $\mathfrak{g} = K$, when clearly $\chi(\mathfrak{g}) \simeq K \oplus K$. Non-trivial concrete examples will appear in Section 3.5, after we establish some structural theory of $\chi(-)$. We will try now to understand $\chi(\mathfrak{g})$ via some ideals and the associated quotients.

Let $L = L(\mathfrak{g})$ be the ideal of $\chi(\mathfrak{g})$ generated by the elements of the form $x - x^\psi$, for all $x \in \mathfrak{g}$. Equivalently, L is the kernel of the homomorphism $\alpha : \chi(\mathfrak{g}) \to \mathfrak{g}$ defined by $\alpha(x) = \alpha(x^\psi) = x$ for all $x \in \mathfrak{g}$. It is clear that this homomorphism is split, that is, $\chi(\mathfrak{g}) \simeq L \rtimes \mathfrak{g}$.

Lemma 3.1. *Let $\mathfrak{g}$ be any Lie algebra. Then:*

1. *$[x, y^\psi] = [x^\psi, y]$ as elements of $\chi(\mathfrak{g})$;*
2. *L is generated as a Lie subalgebra by the elements $x - x^\psi$, for all $x \in \mathfrak{g}$.*

Proof. By the relations that define $\chi(\mathfrak{g})$ we have that $[x + y, (x + y)^\psi] = 0$ for all $x, y \in \mathfrak{g}$, so

$$0 = [x + y, (x + y)^\psi] = [x, x^\psi] + [x, y^\psi] + [y, x^\psi] + [y, y^\psi] = [x, y^\psi] + [y, x^\psi].$$

Thus $[x, y^\psi] = [x^\psi, y]$ for all $x, y \in \mathfrak{g}$.

Denote by A the Lie subalgebra of $\chi(\mathfrak{g})$ generated by the elements $y - y^{\psi}$, for all $y \in \mathfrak{g}$. We want to show that $A = L$, and for that it is enough to show that $[x, y - y^{\psi}] \in A$ and $[x^{\psi}, y - y^{\psi}] \in A$ for all $x, y \in \mathfrak{g}$. Even more, as

$$[x, y - y^{\psi}] - [x^{\psi}, y - y^{\psi}] = [x - x^{\psi}, y - y^{\psi}] \in A,$$

it suffices to show that one of these brackets is an element of A.

Now

$$a_1 := [x - x^{\psi}, y - y^{\psi}] = [x, y] - 2[x, y^{\psi}] + [x^{\psi}, y^{\psi}] \in A$$

and

$$a_2 := [x, y] - [x, y]^{\psi} = [x, y] - [x^{\psi}, y^{\psi}] \in A.$$

But then

$$2[x, y - y^{\psi}] = a_1 + a_2 \in A,$$

which proves that $[x, y - y^{\psi}] \in A$ provided that $char(K) \neq 2$. □

Similarly, let $D = D(\mathfrak{g})$ be the ideal of $\chi(\mathfrak{g})$ generated by the elements of the form $[x, y^{\psi}]$ for all $x, y \in \mathfrak{g}$. It can be seen as the kernel of the homomorphism $\beta : \chi(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g}$ defined by $\beta(x) = (x, 0)$ and $\beta(x^{\psi}) = (0, x)$ for all $x \in \mathfrak{g}$.

Lemma 3.2. *For all* $\mathfrak{g}$*, we have* $[D, L] = 0$.

Proof. By Lemma 3.1, the ideal L is generated as a Lie algebra by the elements $x - x^{\psi}$, for $x \in \mathfrak{g}$. Thus it is enough to show that

$$[[y, z^{\psi}], x - x^{\psi}] = 0$$

for all $x, y, z \in \mathfrak{g}$. We will make repeated use of the fact that $[x, y^{\psi}] = [x^{\psi}, y]$ for all $x, y \in \mathfrak{g}$.

Consider the element $[[x, y], z^{\psi}] \in \chi(\mathfrak{g})$, for some $x, y, z \in \mathfrak{g}$. By the Jacobi identity we have

$$[[x, y], z^{\psi}] = [[x, z^{\psi}], y] + [x, [y, z^{\psi}]]. \tag{3.1}$$

On the other hand, as $[[x, y], z^{\psi}] = [[x, y]^{\psi}, z] = [[x^{\psi}, y^{\psi}], z]$, we have

$$[[x, y], z^{\psi}] = [[x^{\psi}, z], y^{\psi}] + [x^{\psi}, [y^{\psi}, z]]. \tag{3.2}$$

By subtracting (3.2) from (3.1) we obtain

$$0 = [[x, z^{\psi}], y - y^{\psi}] + [x - x^{\psi}, [y, z^{\psi}]],$$

and then

$$[[x, z^{\psi}], y - y^{\psi}] = [[y, z^{\psi}], x - x^{\psi}] \tag{3.3}$$

for all $x, y, z \in \mathfrak{g}$.

Now

$$\begin{aligned} [[x, z^\psi], y - y^\psi] &= [[x^\psi, z], y - y^\psi] \\ &= -[[z, x^\psi], y - y^\psi] \\ &= -[[y, x^\psi], z - z^\psi]. \end{aligned}$$

by (3.3). If we apply once again this reasoning we get

$$\begin{aligned} [[x, z^\psi], y - y^\psi] &= -[[y, x^\psi], z - z^\psi] \\ &= [[x, y^\psi], z - z^\psi] \\ &= [[z, y^\psi], x - x^\psi]. \end{aligned}$$

The equality above, together with (3.3), gives us that

$$[[y, z^\psi], x - x^\psi] = -[[y, z^\psi], x - x^\psi]$$

for all $x, y, z \in \mathfrak{g}$, which completes the proof, since $char(K) \neq 2$. □

The lemma above tell us that the intersection $L \cap D$ is an *abelian ideal* of $\chi(\mathfrak{g})$. Let

$$W = W(\mathfrak{g}) = L(\mathfrak{g}) \cap D(\mathfrak{g}).$$

We can easily cook a homomorphism defined on $\chi(\mathfrak{g})$ that has W as its kernel. Define

$$\rho : \chi(\mathfrak{g}) \to \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$$

with $\rho(x) = (x, x, 0)$ and $\rho(x^\psi) = (0, x, x)$ for all $x \in \mathfrak{g}$. Notice that

$$\rho(z) = (\beta_1(z), \alpha(z), \beta_2(z))$$

for all $z \in \chi(\mathfrak{g})$, where β_1 and β_2 are the two components of β. Then clearly $ker(\rho) = ker(\alpha) \cap ker(\beta)$, that is, $W = L \cap D$.

Remark 3.3. *The constructions above show that $\chi(\mathfrak{g})$ is an extension of an abelian Lie algebra by a certain subalgebra of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. This fact will be essential in the proof of almost anything in this chapter. Notice that this already makes use of the fact that $char(K) \neq 2$. See Section 3.10 for some comments about the characteristic 2 case.*

The first property that we can show that $\chi(-)$ preserves is solvability.

Proposition 3.4. *If $\mathfrak{g}$ is solvable of derived length n, then $\chi(\mathfrak{g})$ is solvable of derived length at most $n + 1$.*

Proof. In this case the subalgebra $Im(\rho) \subseteq \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ also has derived length n. The conclusion is clear, since $\chi(\mathfrak{g})/W \simeq Im(\rho)$ and W is abelian. □

Now we analyze the quotient $\chi(\mathfrak{g})/W$ or, equivalently, the image $Im(\rho)$. It admits a quite concrete description:

$$Im(\rho) = \{(x, y, z) \in \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \mid x - y + z \in \mathfrak{g}'.\} \tag{3.4}$$

Indeed, one inclusion is direct by the definition of ρ. On the other hand, for any $x, y, z \in \mathfrak{g}$ we have

$$(x - y + z, 0, 0) = (x, y, z) - (0, z, z) - (y - z, y - z, 0) = (x, y, z) - \rho(z^{\psi}) - \rho(y - z)$$

thus $(x, y, z) \in Im(\rho)$ if and only if $(x - y + z, 0, 0) \in Im(\rho)$. But for any $x, y \in \mathfrak{g}$, we have

$$([x, y], 0, 0) = [(x, x, 0), (0, y, y)] = [\rho(x), \rho(y^{\psi})] \in Im(\rho),$$

thus $(\mathfrak{g}', 0, 0) \subseteq Im(\rho)$. This shows the other inclusion.

Denote by p_1, p_2 and p_3 the projections of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ onto its first, second and third coordinate, respectively. For any $x \in \mathfrak{g}$ we have

$$x = p_1(\rho(x)) = p_2(\rho(x)) = p_3(\rho(x^{\psi})),$$

thus the image $Im(\rho)$ is a *subdirect sum* in $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. This puts us in the context of the work of Kochloukova and Martínez-Pérez [46].

Proposition 3.5. *If $\mathfrak{g}$ is finitely presentable (resp. of type FP_2), then $Im(\rho)$ is finitely presented (resp. of type FP_2) as well.*

Proof. For convenience, in this proof we denote by $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ the range of ρ. Denote by $p_{(i,j)} : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \twoheadrightarrow \mathfrak{g}_i \oplus \mathfrak{g}_j$ the projection for $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$. It is clear by the description (3.4) that $p_{(i,j)}(Im(\rho)) = \mathfrak{g}_i \oplus \mathfrak{g}_j$ for all i, j. It follows from Theorem 1.18 that $Im(\rho)$ will be finitely presented (resp. of type FP_2) as soon as $Im(\rho) \cap \mathfrak{g}_i \neq 0$ for $i = 1, 2, 3$ (where $\mathfrak{g}_i$ is seen as a subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$).

Notice that, again by (3.4), the image $Im(\rho)$ contains $\mathfrak{g}_i'$ for all i. Thus if $\mathfrak{g}'$ is non-trivial, Theorem 1.18 is applicable and we are done. Otherwise it is easily seen that $Im(\rho)$ is abelian. But it is also finitely generated, because $\chi(\mathfrak{g})$ is, so $Im(\rho)$ is again finitely presented (resp. of type FP_2) in this case. □

Notice that $\mathfrak{g}$ is a split quotient of $Im(\rho)$, so the converse to the proposition above also holds by Proposition 1.16.

One could hope to use this result to study the finite presentability of $\chi(\mathfrak{g})$ by means of the short exact sequence $W(\mathfrak{g}) \rightarrowtail \chi(\mathfrak{g}) \twoheadrightarrow Im(\rho)$. The problem is that, though W is abelian, we do not know if it is of finite dimension in general. We study this question in the next section.

3.2 A condition that forces W to be finite dimensional

We fix $\mathfrak{g}$ and we study $W = W(\mathfrak{g})$ by means of the extension $W \rightarrowtail L \twoheadrightarrow \rho(L)$. Since W is central in L, the associated 5-term exact sequence can be written as

$$H_2(L; K) \to H_2(\rho(L); K) \to W \to H_1(L; K) \to H_1(\rho(L); K) \to 0.$$

It follows that W is finite dimensional if both $H_2(\rho(L); K)$ and $H_1(L; K) \simeq L/L'$ are.

3.2.1 Bounding the dimension of $H_2(\rho(L); K)$

Notice that $\rho(L)$ is the subalgebra of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ generated by the elements of the form

$$\rho(x - x^{\psi}) = (x, x, 0) - (0, x, x) = (x, 0, -x)$$

for $x \in \mathfrak{g}$. The same argument we used in the previous section shows that we can identify it with the following subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$:

$$\rho(L) \simeq S := \{(x, y) \in \mathfrak{g} \oplus \mathfrak{g} \mid x + y \in \mathfrak{g}'\}. \tag{3.5}$$

It is clear that S is a subdirect sum in $\mathfrak{g} \oplus \mathfrak{g}$, but it is not in general of type FP_2 (which would imply that $H_2(S; K)$ is finite dimensional). For instance, this is not the case if $\mathfrak{g}$ is free non-abelian by [46, Thm. A]. We need to impose some restrictions.

Lemma 3.6. *If $\mathfrak{g}$ is of type FP_2 and $\mathfrak{g}'/\mathfrak{g}''$ is finite dimensional, then $H_2(S; K)$ is finite dimensional as well.*

Proof. Notice that $S/(\mathfrak{g}' \oplus \mathfrak{g}') \simeq \mathfrak{g}/\mathfrak{g}'$, and an isomorphism can be given by projection on the first coordinate. Consider the Lyndon-Hochschild-Serre spectral sequence associated to this quotient:

$$E^2_{p,q} = H_p(\mathfrak{g}/\mathfrak{g}'; H_q(\mathfrak{g}' \oplus \mathfrak{g}'; K)) \Rightarrow H_{p+q}(S; K).$$

If we want to show that $H_2(S; K)$ is finite dimensional, it is enough to show that $E^2_{p,q}$ is finite dimensional for all $p, q \geq 0$ with $p + q = 2$.

First consider $(p, q) = (2, 0)$. Clearly $H_0(\mathfrak{g}' \oplus \mathfrak{g}'; K) \simeq K$, so $E^2_{2,0} \simeq H_2(\mathfrak{g}/\mathfrak{g}'; K)$. But $\mathfrak{g}/\mathfrak{g}'$ is finite dimensional, hence of type FP_2, so $H_2(\mathfrak{g}/\mathfrak{g}'; K)$ is finite dimensional.

Let $(p, q) = (1, 1)$. First of all, $H_1(\mathfrak{g}' \oplus \mathfrak{g}'; K) \simeq \mathfrak{g}'/\mathfrak{g}'' \oplus \mathfrak{g}'/\mathfrak{g}''$. The action of $\mathfrak{g}/\mathfrak{g}'$ on $H_1(\mathfrak{g}' \oplus \mathfrak{g}'; K)$ is then converted in an action by $\mathfrak{g}/\mathfrak{g}'$ on $\mathfrak{g}'/\mathfrak{g}'' \oplus \mathfrak{g}'/\mathfrak{g}''$, which is induced by the adjoint action on the first coordinate and the same on the second coordinate, but with opposite sign.

Now, $\mathfrak{g}'/\mathfrak{g}''$ is clearly finitely generated as a $\mathfrak{g}/\mathfrak{g}'$-module, and since $\mathfrak{g}/\mathfrak{g}'$ is of finite dimension, $\mathcal{U}(\mathfrak{g}/\mathfrak{g}')$ is noetherian. This implies that $\mathfrak{g}'/\mathfrak{g}'' \oplus \mathfrak{g}'/\mathfrak{g}''$ is actually of type

FP_∞ as a $\mathcal{U}(\mathfrak{g}/\mathfrak{g}')$-module. Thus $H_1(\mathfrak{g}/\mathfrak{g}'; H_1(\mathfrak{g}' \oplus \mathfrak{g}'; K)) \simeq H_1(\mathfrak{g}/\mathfrak{g}'; \mathfrak{g}'/\mathfrak{g}'' \oplus \mathfrak{g}'/\mathfrak{g}'')$ is finite dimensional.

Finally, let $(p, q) = (0, 2)$. Now we want to show that $H_0(\mathfrak{g}/\mathfrak{g}'; H_2(\mathfrak{g}' \oplus \mathfrak{g}'; K))$ is finite dimensional. By the Künneth formula [80, Ex. 7.3.8] we have

$$H_2(\mathfrak{g}' \oplus \mathfrak{g}'; K) \simeq \oplus_{0 \leq i \leq 2}(H_i(\mathfrak{g}'; K) \otimes_K H_{2-i}(\mathfrak{g}'; K)).$$

Clearly it is enough to show that each of the components of the direct sum above is finitely generated as a $\mathfrak{g}/\mathfrak{g}'$-module.

One of the components is

$$H_1(\mathfrak{g}'; K) \otimes_K H_1(\mathfrak{g}'; K) \simeq \mathfrak{g}'/\mathfrak{g}'' \otimes_K \mathfrak{g}'/\mathfrak{g}'',$$

which is clearly finitely generated, since $\mathfrak{g}'/\mathfrak{g}''$ is of finite dimension.

The other two components are isomorphic and we have

$$H_2(\mathfrak{g}'; K) \otimes_K H_0(\mathfrak{g}'; K) \simeq H_2(\mathfrak{g}'; K) \otimes_K K \simeq H_2(\mathfrak{g}'; K).$$

Consider a projective resolution

$$\mathcal{P} : \ldots \to P_n \to P_{n-1} \to \ldots \to P_2 \to P_1 \to P_0 \to K \to 0$$

of K as a $\mathcal{U}(\mathfrak{g})$-module, with P_j finitely generated for $j \leq 2$. By applying the functor $K \otimes_{\mathcal{U}(\mathfrak{g}')} -$, we get a complex of $\mathcal{U}(\mathfrak{g}/\mathfrak{g}')$-modules, and the modules are still finitely generated up to degree 2. As $\mathcal{U}(\mathfrak{g}/\mathfrak{g}')$ is noetherian, the homologies of this complex are also finitely generated up to degree 2. But

$$H_i(K \otimes_{\mathcal{U}(\mathfrak{g}')} \mathcal{P}) \simeq H_i(\mathfrak{g}'; K).$$

Thus $H_2(\mathfrak{g}'; K)$ is finitely generated as a $\mathcal{U}(\mathfrak{g}/\mathfrak{g}')$-module, as we wanted. □

3.2.2 Bounding the dimension of L/L'

Lima and Oliveira proved in [53] that, in the group-theoretic case, the quotient L/L' is finitely generated as soon as the original group G is finitely generated. The idea was to realize L/L' as a certain quotient of the augmentation ideal $Aug(\mathbb{Z}G)$, where a finite generating set could be detected more easily. We could adapt their argument (this was done in [62, Section 4.2]), but here we follow a simpler route, which actually says more about the structure of L.

Lemma 3.7. *Let $\mathfrak{g}$ be any Lie algebra and let $x, y, z \in \mathfrak{g}$. Then:*

$$[x - x^\psi, [y - y^\psi, z - z^\psi]] = [x, [y, z]] - [x, [y, z]]^\psi.$$

Proof. Since $x - x^\psi \in L$ and $[y, z^\psi] = [y^\psi, z] \in D$, and these two ideals commute, we have

$$[x - x^\psi, [y - y^\psi, z - z^\psi]] = [x - x^\psi, [y, z] - [y^\psi, z^\psi]].$$

But $[x, [y^\psi, z^\psi]] = [x, [y, z]^\psi] = [x^\psi, [y, z]]$. By expanding the right-hand side of the equation above we obtain the result. □

Proposition 3.8. *If $\mathfrak{g}$ is a finitely generated Lie algebra, then L is finitely generated. In particular, L/L' is finite dimensional.*

Proof. By Proposition 3.1 L is generated as a Lie algebra by $x - x^\psi$, for all $x \in \mathfrak{g}$. If $x_1, \ldots, x_n$ is a generating set for $\mathfrak{g}$, then by linearity L is actually generated by the elements $u - u^\psi$, where $u = [x_{i_1}, \ldots, x_{i_m}]$ is a right-normed bracket involving these generators. But by Lemma 3.7 we only need brackets of length 1 and 2. Indeed, it follows from that lemma that

$$u - u^\psi = \begin{cases} [x_{i_1} - x_{i_1}^\psi, \ldots, x_{i_m} - x_{i_m}^\psi] & \text{if n is odd,} \\ [x_{i_1} - x_{i_1}^\psi, \ldots, x_{i_{m-2}} - x_{i_{m-2}}^\psi, [x_{i_{m-1}}, x_{i_m}] - [x_{i_{m-1}}, x_{i_m}]^\psi] & \text{otherwise.} \end{cases}$$

Thus $x_i - x_i^\psi$ and $[x_i, x_j] - [x_i, x_j]^\psi$, for $1 \le i < j \le n$, generate L. □

By putting together Lemma 3.6, Proposition 3.8 and the comments in the beginning of Section 3.2, we obtain the following result.

Theorem 3.9. *Suppose that $\mathfrak{g}$ is a Lie algebra of type FP_2 such that $\mathfrak{g}'/\mathfrak{g}''$ is finite dimensional. Then $W(\mathfrak{g})$ is finite dimensional.*

This is Theorem B1 in the introduction. The condition "$\mathfrak{g}'/\mathfrak{g}''$ is finite dimensional" is very restrictive, but we can deduce some nice consequences of this theorem. For instance, we have:

Corollary 3.10. *If $\mathfrak{g}$ is finite dimensional, then so is $\chi(\mathfrak{g})$.*

Proof. Clearly $\mathfrak{g}'/\mathfrak{g}''$ is also finite dimensional and $\mathfrak{g}$ is of type FP_2, so $W(\mathfrak{g})$ is finite dimensional. On the other hand $\chi(\mathfrak{g})/W(\mathfrak{g}) \simeq Im(\rho)$ is a subalgebra of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$, which is also clearly finite dimensional. Thus $\chi(\mathfrak{g})$ itself is of finite dimension. □

We do not have any bounds for the dimension of $\chi(\mathfrak{g})$ in terms of the dimension of $\mathfrak{g}$ in the general case.

Similarly to the group-theoretic case in [49], the weak commutativity construction for Lie algebras preserves the property FP_∞ *for solvable Lie algebras*. Indeed, by [36, Thm. 1], if $\mathfrak{g}$ is solvable of type FP_∞, then it is finite dimensional. In this case of course $\mathfrak{g}'/\mathfrak{g}''$ is also finite dimensional, and then so is W, by Theorem 3.9. Moreover, $\chi(\mathfrak{g})/W$ is clearly finite dimensional and solvable, being a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$.

Corollary 3.11. *If $\mathfrak{g}$ is solvable of type FP_∞, then $\chi(\mathfrak{g})$ has the same property.*

Remark 3.12. *The converse of Theorem 3.9 does not hold as it will become clear later. For instance, if $\mathfrak{g}$ is the free Lie algebra on two generators, then $\mathfrak{g}'/\mathfrak{g}''$ is infinite dimensional, but $W(\mathfrak{g})$ is actually trivial (see Theorem 3.19 and Proposition 3.24).*

3.3 Finiteness properties of $\chi(\mathfrak{g})$

In the last section we showed that $\chi(-)$ takes a finite dimensional Lie algebra into a another Lie algebra of finite dimension. It is also clear $\chi(\mathfrak{g})$ is finitely generated if $\mathfrak{g}$ is. Now we will consider other finiteness properties of Lie algebras, such as finite presentability and the homological types FP_m, that is, we will prove Theorem B4. We will make use of the results about W, L and D that we have already deduced.

3.3.1 Finite presentability

The first step towards a proof of the first part of Theorem B4 is to establish it for free Lie algebras.

Proposition 3.13. *If $\mathfrak{f}$ is a free Lie algebra of finite rank, then $\chi(\mathfrak{f})$ is finitely presented.*

Proof. Recall that $\chi(\mathfrak{f}) \simeq L \rtimes \mathfrak{f}$. Let $\{x_1, \ldots, x_m\}$ be a free basis for $\mathfrak{f}$. By Proposition 3.8 L is finitely generated, say $L = \langle \ell_1, \ldots, \ell_n \rangle$. Notice that the quotient $\chi(\mathfrak{f})/D \simeq \mathfrak{f} \oplus \mathfrak{f}$ is finitely presented, so D is finitely generated as an ideal. Put $D = \langle\langle d_1, \ldots, d_s \rangle\rangle$. Each d_i can be written as a sum of brackets involving the generators $\ell_1, \ldots, \ell_n, x_1, \ldots, x_m$; we denote by δ_i one of such sums.

Similarly, $\chi(\mathfrak{f})/W \simeq Im(\rho)$ is finitely presented by Proposition 3.5, thus we can write

$$Im(\rho) = \langle \ell_1, \ldots, \ell_n, x_1, \ldots, x_m \mid \tau_1, \ldots, \tau_k \rangle$$

for some τ_i's. If we denote by F the free lie algebra on $\{\ell_1, \ldots, \ell_n, x_1, \ldots, x_m\}$, then the obvious homomorphism $F \twoheadrightarrow Im(\rho)$ is injective on the subalgebra generated by $\{x_1, \ldots, x_m\}$, as $\mathfrak{f}$ is free on this set. It follows that each τ_i is an element of the ideal of F generated by $\{\ell_1, \ldots, \ell_n\}$.

Finally, since L is an ideal, we can choose words $\mu_{i,j}$ in $\ell_1, \ldots, \ell_n$ representing $[x_i, \ell_j]$, for each i, j.

Let Γ be the Lie algebra generated by the symbols ℓ_i, x_i, d_i, w_j, where i runs through the appropriate indices and $1 \leq j \leq k$, subject to the following defining relations:

1. $d_i = \delta_i$ for $1 \leq i \leq s$;

2. $w_i = \tau_i$ for $1 \leq i \leq k$;

3. $[x_i, \ell_j] = \mu_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$;

4. $[d_i, \ell_j] = 0$ for $1 \leq i \leq s$ and $1 \leq j \leq n$;

5. $[w_i, \ell_j] = 0$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Denote by L_0 the subalgebra of Γ generated by $\{\ell_i | 1 \leq i \leq n\}$, by D_0 the ideal generated by $\{d_i | 1 \leq i \leq s\}$ and by W_0 the ideal generated by $\{w_i | 1 \leq i \leq k\}$. Notice that L_0 is actually an ideal of Γ, by the relations of types 3, 4 and 5, together with the definition of the words $\mu_{i,j}$. The relations of type 4 imply that $[D_0, L_0] = 0$. From the relations of type 5 we conclude that W_0 commutes with L_0, while the relations of the types 2, 3 and 4 imply that W_0 is commutes with D_0, since each w_i represents an element of the ideal generated by $\ell_1, \dots, \ell_n$, that is, the subalgebra $L_0 \subseteq \Gamma$, which commutes with D_0. So W_0 is central in $L_0 + D_0$, and in particular it is an abelian ideal of Γ.

It is clear that there is a well-defined surjective homomorphism $\phi : \Gamma \to \chi(\mathfrak{f})$ that takes the generators of Γ to the corresponding elements in $\chi(\mathfrak{f})$. The choice of $\tau_1, \dots, \tau_k$ implies that ϕ induces an isomorphism $\Gamma/W_0 \simeq \chi(\mathfrak{f})/W$. Thus $ker(\phi) \subseteq W_0$. Also $\phi(L_0) = L$ and $\phi(D_0) = D$, and since $W_0 \subseteq L_0 + D_0$, we have

$$\frac{\Gamma}{L_0 + D_0} \simeq \frac{\chi(\mathfrak{f})}{L + D} \simeq \frac{\mathfrak{f}}{\mathfrak{f}'}.$$

Now W_0 is a module over the universal enveloping algebra of $\Gamma/(L_0 + D_0) \simeq \mathfrak{f}/\mathfrak{f}'$, and it is generated by $w_1, \dots, w_k$. The fact that $\mathcal{U}(\mathfrak{f}/\mathfrak{f}')$ is noetherian implies that $ker(\phi)$, being a submodule of W_0, is finitely generated too. But then $\chi(\mathfrak{f}) \simeq \Gamma/ker(\phi)$ is finitely presented. □

Corollary 3.14. *If $\mathfrak{g}$ is finitely presented, then so is $\chi(\mathfrak{g})$.*

Proof. Let $\mathfrak{g} = \mathfrak{f}/\langle\langle R \rangle\rangle$, where $\mathfrak{f}$ is a free Lie algebra of finite rank and R is a finite set. By Proposition 3.13 we know that $\chi(\mathfrak{f})$ is finitely presented. But then

$$\chi(\mathfrak{g}) \simeq \chi(\mathfrak{f})/\langle\langle r, r^{\psi};\ \text{for } r \in R \rangle\rangle$$

is also finitely presented. □

The converse to the corollary is clearly also true since $\mathfrak{g}$ is a split quotient of $\chi(\mathfrak{g})$.

3.3.2 Property FP_2

The version of this result for the property FP_2 can be obtained from the lemmas in Section 1.3.

Proposition 3.15. *If $\mathfrak{g}$ is of type FP_2, then so is $\chi(\mathfrak{g})$.*

Proof. By Proposition 1.15, there is a finitely presented Lie algebra $\mathfrak{h}$ and an ideal $\mathfrak{r} \subseteq \mathfrak{h}$ such that $\mathfrak{g} \simeq \mathfrak{h}/\mathfrak{r}$ and $\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}]$. Now, $\chi(\mathfrak{h})$ is finitely presented by Corollary 3.14, and clearly $\chi(\mathfrak{g}) \simeq \chi(\mathfrak{h})/I$, where I is the ideal generated by x and x^{ψ}, for all $x \in \mathfrak{r}$. This ideal is perfect, since $\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] \subseteq [I, I]$, and the same holds for $\mathfrak{r}^{\psi}$. Thus $\chi(\mathfrak{g})$ is of type FP_2 by Proposition 1.15. □

Again, the converse is true by the Proposition 1.16.

3.3.3 $\chi(-)$ does not preserve FP_3

Fix $\mathfrak{f}$ a non-abelian free Lie algebra of finite rank. Recall that subalgebras of a free Lie algebra are again free (see for instance [74]).

Lemma 3.16. *Let $S = \langle \{(x, -x) \mid x \in \mathfrak{f}\} \rangle \subseteq \mathfrak{f} \oplus \mathfrak{f}$. Then $H_2(S; K)$ is infinite dimensional.*

Proof. Let $\pi : \mathfrak{f} \oplus \mathfrak{f} \to \mathfrak{f}$ be the projection onto the second coordinate and let $N = ker(\pi) \cap S$. Notice that $N = \mathfrak{f}' \oplus 0$ (by (3.5), for instance). The sequence $N \rightarrowtail S \twoheadrightarrow \mathfrak{f}$ is exact, so there is a spectral sequence

$$E^2_{p,q} = H_p(\mathfrak{f}, H_q(N; K)) \Rightarrow H_{p+q}(S; K).$$

The fact that $\mathfrak{f}$ is free implies that $E^2_{p,q} = 0$ for all $p \geq 2$, so $E^2 = E^{\infty}$.

Suppose that $H_2(S; K)$ is actually finite dimensional. Then the subquotient $E^2_{1,1}$ is also finite dimensional, and

$$E^2_{1,1} = H_1(\mathfrak{f}; H_1(N; K)) \simeq H_1(\mathfrak{f}; N/N').$$

Now fix any $c \in \mathfrak{f}' \setminus \{0\}$. Notice that c acts trivially on N/N', so $dim_K(c \cdot N/N') = 0$, where c is identified with its image in $\mathcal{U}(\mathfrak{f})$. It follows from Lemma 1.17 that N/N' itself is finite dimensional. This is a contradiction, since $N \simeq \mathfrak{f}'$ is an infinitely generated free Lie algebra. □

Lemma 3.17. *If $\chi(\mathfrak{f})$ is of type FP_3, then $H_2(L; K)$ is of finite dimension.*

Proof. The short exact sequence $L \rightarrowtail \chi(\mathfrak{f}) \twoheadrightarrow \mathfrak{f}$ gives rise to a spectral sequence:

$$E^2_{p,q} = H_p(\mathfrak{f}; H_q(L; K)) \Rightarrow H_{p+q}(\chi(\mathfrak{f}); K).$$

Notice that, as in the proof of Lemma 3.16, the fact that $\mathfrak{f}$ is free implies that $E^2 = E^\infty$. Now $H_3(\chi(\mathfrak{f}); K)$ is finite dimensional, thus $E^2_{p,q} = E^\infty_{p,q}$ is finite dimensional as well whenever $p+q=3$. In particular, $E^2_{1,2} = H_1(\mathfrak{f}; H_2(L;K))$ is finite dimensional.

Now for any $x, y \in \mathfrak{f}$ the element $[x,y]$ acts trivially on L, since it is the image of the element $[x, y^\psi] \in D \subseteq \chi(\mathfrak{f})$, and $[D, L] = 0$. It follows that $[x,y]$ acts trivially on $H_2(L;K)$ as well. Finally since $\mathfrak{f}$ is non-abelian, $[x,y]$ can be taken to be non-trivial, so Lemma 1.17 applies again: $H_2(L;K)$ is finite dimensional. □

Proposition 3.18. *$\chi(\mathfrak{f})$ is not of type FP_3.*

Proof. Suppose on the contrary that $\chi(\mathfrak{f})$ is of type FP_3. Consider the spectral sequence associated to $W \rightarrowtail L \twoheadrightarrow S$:

$$E^2_{p,q} = H_p(S; H_q(W;K)) \Rightarrow H_{p+q}(L;K).$$

Since $S \subseteq \mathfrak{f} \oplus \mathfrak{f}$, it follows that $E^2_{p,q} = 0$ for all $p \geq 3$. Consider the term $E^2_{1,1}$. Recall that the bidegree of the differential map of the spectral sequence $\{E^r\}$ is $(-r, r-1)$. Thus the differential maps that involve $E^2_{1,1}$ are

$$d^2_{1,1} : E^2_{1,1} \to E^2_{-1,2}$$

and

$$d^2_{3,0} : E^2_{3,0} \to E^2_{1,1}.$$

Note that $E^2_{-1,2} = 0 = E^2_{3,0}$, so $E^2_{1,1} = E^3_{1,1}$. The fact that E^3 is non-trivial only on the columns $p = 0, 1, 2$, together with the knowledge of the bidegree implies that d^r is trivial for $r \geq 3$. Thus $E^3 = E^\infty$. It follows that $E^2_{1,1} = E^\infty_{1,1}$ is a subquotient of $H_2(L;K)$, which is finite dimensional by Lemma 3.17.

On the other hand

$$E^2_{1,1} = H_1(S; H_1(W;K)) = H_1(S;W).$$

Notice that S acts trivially on W, so $H_1(S;W) \simeq S/S' \otimes_K W$. Thus W must be finite dimensional, since S/S' is not trivial (S projects onto $\mathfrak{f}/\mathfrak{f}'$).

Finally, the 5-term exact sequence associated to $W \rightarrowtail L \twoheadrightarrow S$ can be written as:

$$H_2(L;K) \to H_2(S;K) \to W \to H_1(L;K) \to H_1(S;K) \to 0.$$

But $H_2(L;K)$ and W are both finite dimensional, so $H_2(S;K)$ is finite dimensional as well. This contradicts Lemma 3.16. □

3.4 Stem extensions and the Schur multiplier

Given any Lie algebra $\mathfrak{g}$, denote

$$R = R(\mathfrak{g}) := [\mathfrak{g}, L, \mathfrak{g}^\psi] \subseteq \chi(\mathfrak{g}).$$

This is the subalgebra of $\chi(\mathfrak{g})$ generated by the triple brackets $[x, [\ell, y^\psi]]$, for all $x, y \in \mathfrak{g}$ and $\ell \in L = L(\mathfrak{g})$. It follows from the facts that L is an ideal and $[L, D] = 0$ that R is actually an ideal of $\chi(\mathfrak{g})$. Notice also that $R \subseteq W = W(\mathfrak{g})$. Our goal here is to prove the following theorem.

Theorem 3.19. *For any Lie algebra $\mathfrak{g}$, we have $W(\mathfrak{g})/R(\mathfrak{g}) \simeq H_2(\mathfrak{g}; K)$.*

Recall that D is the ideal of $\chi(\mathfrak{g})$ generated by the elements $[y, z^\psi]$. In general it is not generated by these elements as a Lie subalgebra, but it will be modulo R. Indeed, notice that for $x, y, z \in \mathfrak{g}$ we have

$$[x, [y - y^\psi, z^\psi]] = [x, [y, z^\psi]] - [x, [y^\psi, z^\psi]] = [x, [y, z^\psi]] - [x, [y, z]^\psi]$$

Since $[x, [y - y^\psi, z^\psi]] \in R$, it follows that $[x, [y, z^\psi]]$ is congruent to $[x, [y, z]^\psi]$. The same holds for $[x^\psi, [y, z^\psi]]$. Thus D/R is actually generated as an algebra by the image of the brackets $[x, y^\psi]$, for $x, y \in \mathfrak{g}$.

Now we consider the quotient W/R. Since $W \subseteq D$, it follows from the comments above that the elements of W/R are of the form:

$$w + R = \sum_\alpha \lambda_\alpha [[x_{\alpha,1}, y^\psi_{\alpha,1}], \dots, [x_{\alpha,n_\alpha}, y^\psi_{\alpha,n_\alpha}]] + R, \tag{3.6}$$

with $\lambda_\alpha \in K$ and $x_{\alpha,j}, y_{\alpha,j} \in \mathfrak{g}$. Also, as $W \subseteq L$, it must be true as well that

$$\sum_\alpha \lambda_\alpha [[x_{\alpha,1}, y_{\alpha,1}], \dots, [x_{\alpha,n_\alpha}, y_{\alpha,n_\alpha}]] = 0, \tag{3.7}$$

and this describes completely the elements of W/R.

3.4.1 W/R is a quotient of $H_2(\mathfrak{g}; K)$

Following Ellis [32], we consider the non-abelian exterior product $\mathfrak{g} \wedge \mathfrak{g}$. It is defined as the Lie algebra generated by the symbols $x \wedge y$, with $x, y \in \mathfrak{g}$, subject to the following defining relations:

1. $(x_1 + x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$;

2. $x \wedge (y_1 + y_2) = x \wedge y_1 + x \wedge y_2$;

3. $\lambda(x \wedge y) = (\lambda x) \wedge y = x \wedge (\lambda y)$;

4. $x \wedge x = 0$;
5. $[x_1, x_2] \wedge y = [x_1, y] \wedge x_2 + x_1 \wedge [x_2, y]$;
6. $x \wedge [y_1, y_2] = [x, y_1] \wedge y_2 + y_1 \wedge [x, y_2]$;
7. $[x_1 \wedge y_1, x_2 \wedge y_2] = [x_1, y_1] \wedge [x_2, y_2]$;

for all $x, x_1, x_2, y, y_1, y_2 \in \mathfrak{g}$ and $\lambda \in K$.

Let $\phi : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ be the Lie algebra homomorphism defined by $\phi(x \wedge y) = [x, y]$. The main result in [32] is that $ker(\phi)$ is isomorphic to the Schur multiplier $H_2(\mathfrak{g}; K)$.

Notice that an element in $ker(\phi)$ is written as

$$\sum_\alpha \lambda_\alpha [x_{\alpha,1} \wedge y_{\alpha,1}, \dots, x_{\alpha,n_\alpha} \wedge y_{\alpha,n_\alpha}], \tag{3.8}$$

with $\lambda_\alpha \in K$ and $x_{\alpha,j}, y_{\alpha,j} \in \mathfrak{g}$ such that

$$\sum_\alpha \lambda_\alpha [[x_{\alpha,1}, y_{\alpha,1}], \dots, [x_{\alpha,n_\alpha}, y_{\alpha,n_\alpha}]] = 0 \tag{3.9}$$

in $\mathfrak{g}$. Consider the homomorphism $\theta : \mathfrak{g} \wedge \mathfrak{g} \to \chi(\mathfrak{g})/R$ defined by

$$\theta(x \wedge y) = [x, y^\psi] + R.$$

It is not hard to see that θ is well defined. Moreover, it follows from (3.6), (3.7), (3.8) and (3.9) that θ induces a surjective homomorphism

$$\theta_1 : ker(\phi) \to W/R,$$

thus W/R is a quotient of $H_2(\mathfrak{g}; K)$.

3.4.2 $H_2(\mathfrak{g}; K)$ is a quotient of W/R

Now we adapt the arguments in [75], Section 4.1. Suppose that

$$0 \to Z \to \mathfrak{h} \to \mathfrak{g} \to 0 \tag{3.10}$$

is a stem extension of Lie algebras, that is, (3.10) is an exact sequence, Z is a central ideal of $\mathfrak{h}$ and $Z \subseteq \mathfrak{h}'$. Consider

$$P = \langle \{(x, x, 0), (0, x, x) \mid x \in \mathfrak{h}\} \rangle \subseteq \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h}.$$

In other words, P is the image of $\rho_\mathfrak{h} : \chi(\mathfrak{h}) \to \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h}$. Recall that P can be described also as

$$P = \{(x, y, z) \in \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h} \mid x - y + z \in \mathfrak{h}'\}.$$

Define

$$B = \{(z, z + z', z') \mid z, z' \in Z\} \subseteq P. \tag{3.11}$$

Notice that B is a central subalgebra of P, since Z is central in $\mathfrak{h}$. It follows that P/B is a quotient of $\chi(\mathfrak{h})$, by means of the homomorphism $\nu : \chi(\mathfrak{h}) \to P/B$ such that $\nu(x) = (x, x, 0) + B$ and $\nu(x^\psi) = (0, x, x) + B$. Since $\nu(z) = \nu(z^\psi) = 0$ for all $z \in Z$, it follows that ν factors through a homomorphism $\lambda : \chi(\mathfrak{g}) \to P/B$, thus making the following diagram commutative:

$$\begin{array}{ccc} \chi(\mathfrak{h}) & \xrightarrow{\ \nu\ } & P/B \\ \downarrow & \nearrow_{\lambda} & \\ \chi(\mathfrak{g}) & & \end{array}$$

Lemma 3.20. *We have:*

1. $R(\mathfrak{g}) \subseteq ker(\lambda) \subseteq W(\mathfrak{g})$,
2. $\lambda(W(\mathfrak{g})) \simeq Z$.

Proof. The first inclusion in item 1 is clear, since $R(\mathfrak{g})$ is the image of $R(\mathfrak{h})$, and $R(\mathfrak{h}) \subseteq ker(\nu)$. The second inclusion is also clear, since $\rho(\chi(\mathfrak{g})) \simeq P/(Z \oplus Z \oplus Z)$, which clearly is a quotient of P/B, and ρ can be written as the composite

$$\chi(\mathfrak{g}) \to P/B \twoheadrightarrow \rho(\chi(\mathfrak{g}))$$

where the second map is the canonical projection.

As to item 2, notice that since $\lambda(W(\mathfrak{g})) \subseteq ker(P/B \twoheadrightarrow \rho(\chi(\mathfrak{g})))$, every element of $\lambda(W(\mathfrak{g}))$ is of the form $(z_1, z_2, z_3) + B$, with $z_i \in Z$. Such elements are clearly equivalent to elements of the form $(0, z, 0) + B$ in P/B for some $z \in Z$. Conversely, any such element must be in the image of λ, and actually it must be the image of some element of $W(\mathfrak{g})$, since it projects to 0 in $\rho(\chi(\mathfrak{g}))$.

Thus $\lambda(W(\mathfrak{g})) = \{(0, z, 0) + B \mid z \in Z\} \subseteq P/B$. The homomorphism $Z \to \lambda(W(\mathfrak{g}))$ that takes z to $(0, z, 0) + B$ is clearly well defined and surjective, and it also injective, by the description of B in (3.11). Thus $\lambda(W(\mathfrak{g})) \simeq Z$. □

By the lemma above we see that $W/R = W(\mathfrak{g})/R(\mathfrak{g})$ has Z as quotient for every Z that occurs as the kernel of some stem extension of $\mathfrak{g}$. Now we show that $H_2(\mathfrak{g}; K)$ is one of such kernels, from what follows that $H_2(\mathfrak{g}; K)$ is a quotient of W/R.

Lemma 3.21. *Any Lie algebra $\mathfrak{g}$ fits into a stem extension written as*

$$0 \to H_2(\mathfrak{g}; K) \to \mathfrak{h} \to \mathfrak{g} \to 0.$$

Proof. Write $\mathfrak{g} = \mathfrak{f}/\mathfrak{n}$, where $\mathfrak{f}$ is a free Lie algebra. Then by the Hopf formula

$$H_2(\mathfrak{g}; K) = \frac{[\mathfrak{f}, \mathfrak{f}] \cap \mathfrak{n}}{[\mathfrak{f}, \mathfrak{n}]}.$$

Thus we can see $H_2(\mathfrak{g}; K)$ as a subalgebra of the abelian Lie algebra $\mathfrak{n}/[\mathfrak{f}, \mathfrak{n}]$. It follows that $H_2(\mathfrak{g}; K)$ admits a complement, that is, there is some $\mathfrak{a} \subseteq \mathfrak{n}$, with $[\mathfrak{f}, \mathfrak{n}] \subseteq \mathfrak{a}$, such that

$$\frac{\mathfrak{n}}{[\mathfrak{f}, \mathfrak{n}]} \simeq H_2(\mathfrak{g}; K) \oplus \frac{\mathfrak{a}}{[\mathfrak{f}, \mathfrak{n}]}. \tag{3.12}$$

Notice that $[\mathfrak{f}, \mathfrak{a}] \subseteq [\mathfrak{f}, \mathfrak{n}] \subseteq \mathfrak{a}$, so $\mathfrak{a}$ is an ideal of $\mathfrak{f}$. Consider the exact sequence:

$$0 \to \mathfrak{n}/\mathfrak{a} \to \mathfrak{f}/\mathfrak{a} \to \mathfrak{f}/\mathfrak{n} \to 0. \tag{3.13}$$

The choice of $\mathfrak{a}$ implies that $\mathfrak{n}/\mathfrak{a} \simeq H_2(\mathfrak{g}; K)$. Since $[\mathfrak{f}, \mathfrak{n}] \subseteq \mathfrak{a}$, the extension is central. By the direct sum description (3.12), any element of $\mathfrak{n}$ is equivalent modulo $\mathfrak{a}$ to some $w \in [\mathfrak{f}, \mathfrak{f}]$, so $\mathfrak{n}/\mathfrak{a} \subseteq [\mathfrak{f}/\mathfrak{a}, \mathfrak{f}/\mathfrak{a}]$. Thus (3.13) is a stem extension. □

Let $\lambda_2 : W(\mathfrak{g})/R(\mathfrak{g}) \to H_2(\mathfrak{g}; K)$ be the homomorphism induced by the homomorphism arising in Lemma 3.20 when we take Z to be $H_2(\mathfrak{g}; K)$. By thinking of $H_2(\mathfrak{g}; K)$ given by the Hopf formula for a fixed presentation of $\mathfrak{g}$, as in Lemma 3.21, we can write explicit expressions for λ_2 and for the isomorphism $\alpha : H_2(\mathfrak{g}; K) \to ker(\phi)$ of [32]. It is not hard to see then that the composition

$$H_2(\mathfrak{g}; K) \to ker(\phi) \to W(\mathfrak{g})/R(\mathfrak{g}) \to H_2(\mathfrak{g}; K)$$

of the maps that we defined is the identity. So $W(\mathfrak{g})/R(\mathfrak{g}) \simeq H_2(\mathfrak{g}; K)$, as we wanted.

3.5 Examples

In this section we finally consider some examples. This is mostly done by restricting our attention to $W(\mathfrak{g})$ and $R(\mathfrak{g})$ under some particular hypotheses, so that we can understand $\chi(\mathfrak{g})$ in terms of $Im(\rho)$ by means of the short exact sequences

$$0 \to W(\mathfrak{g}) \to \chi(\mathfrak{g}) \to Im(\rho) \to 0$$

and

$$0 \to R(\mathfrak{g}) \to W(\mathfrak{g}) \to H_2(\mathfrak{g}; K) \to 0.$$

This section was delayed to this point of the chapter because we will make heavy use of Theorem 3.19.

3.5.1 Abelian Lie algebras

Let $\mathfrak{g}$ be a finite dimensional abelian Lie algebra and let $x_1, \ldots, x_n$ be a basis. Then $\chi(\mathfrak{g})$ is generated by the symbols $x_1, \ldots, x_n$ and $x_1^\psi, \ldots, x_n^\psi$ with defining relations given by:

1. $[x_i, x_j] = 0$ for all $i > j$;
2. $[x_i^\psi, x_j^\psi] = 0$ for all $i > j$;
3. $[x_i, x_i^\psi] = 0$ for all i;
4. $[x_i, x_j^\psi] = [x_i^\psi, x_j]$ for all $i > j$.

Notice that the elements $[x_i, x_j^\psi]$ are central. Indeed

$$[x_i, x_j^\psi] = -\frac{1}{2}[x_i - x_i^\psi, x_j - x_j^\psi],$$

and by the Jacobi identity, together with the fact that D and L commute, we get that $[[x_i - x_i^\psi, x_j - x_j^\psi], x_k] = 0$ for all k. Similarly, we have $[[x_i - x_i^\psi, x_j - x_j^\psi], x_k^\psi] = 0$ for all k.

Proposition 3.22. *If $\mathfrak{g}$ is an abelian Lie algebra of dimension n, then $\chi(\mathfrak{g})$ is a Lie algebra of dimension $2n + \binom{n}{2}$. More specifically, $W = D$ is a central ideal of dimension $\binom{n}{2}$, with $\chi(\mathfrak{g})/W \simeq K^{2n}$. Finally, $R = 0$.*

Proof. The remarks above the proposition imply that D is linearly generated by $[x_i, x_j^\psi]$, for $i > j$. Each of these elements is clearly in the kernel of ρ, that is, in W, so $D = W$. Now $D/R \simeq W/R \simeq H_2(\mathfrak{g}; K) \simeq \bigwedge^2(\mathfrak{g})$ (see Example 1.10), thus $dim(D) \geq \binom{n}{2}$. But then the elements $[x_i, x_j^\psi]$ with $1 \leq j < i \leq n$ must be linearly independent and $R = 0$. Finally, it is clear by (3.4) that $Im(\rho) \simeq \mathfrak{g} \oplus \mathfrak{g}$. □

3.5.2 Perfect Lie algebras

Let $\mathfrak{g}$ be *perfect*, that is, $\mathfrak{g} = \mathfrak{g}'$. Notice that in this case $Im(\rho) = \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. Moreover, W is a central ideal of $\chi(\mathfrak{g})$. In fact, for $x, y \in \mathfrak{g} \subseteq \chi(\mathfrak{g})$ and $w \in W$, we have

$$[[x, y], w] = [[x, w], y] + [x, [y, w]] = [[x, w], y^\psi] + [x, [y^\psi, w]] = [[x, y^\psi], w] = 0.$$

The first and the third equalities are instances of the Jacobi identity; the second and the fourth are consequences of $[L, D] = 0$. Thus W commutes with $\mathfrak{g}' = \mathfrak{g} \subseteq \chi(\mathfrak{g})$, and similarly with $\mathfrak{g}^\psi$. In this case $R(\mathfrak{g}) = [\mathfrak{g}, [L, \mathfrak{g}^\psi]] = 0$. Indeed:

$$R(\mathfrak{g}) = [\mathfrak{g}, [L, \mathfrak{g}^\psi]] = [\mathfrak{g}', [L, \mathfrak{g}^\psi]] \subseteq [\mathfrak{g}, [\mathfrak{g}, [L, \mathfrak{g}^\psi]]] = 0,$$

since $[\mathfrak{g}, [L, \mathfrak{g}^\psi]] \subseteq W$.

We conclude that $\chi(\mathfrak{g})$ is a central (in fact stem) extension of $H_2(\mathfrak{g}; K)$ by $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$. In particular, if $\mathfrak{g}$ is *superperfect*, that is, $\mathfrak{g}$ is perfect and $H_2(\mathfrak{g}; K) = 0$, then $\chi(\mathfrak{g}) \simeq \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$.

3.5.3 Lie algebras generated by two elements

We will show that $R(\mathfrak{f}) = 0$ if $\mathfrak{f}$ is free of rank 2.

Remark 3.23. *We will use repeatedly that for any $d \in D$ and $[x_1, \dots, x_i, \dots, x_n]$ an arbitrarily arranged bracket of elements $x_i \in \mathfrak{f} \cup \mathfrak{f}^\psi$, we have*

$$[[x_1, \dots, x_i, \dots, x_n], d] = [[x_1, \dots, x_i^\psi, \dots, x_n], d]$$

for any i, as a consequence of $[D, L] = 0$. If x_i is already an element of $\mathfrak{f}^\psi$, we interpret ψ as an automorphism of order 2, that is, $x_i = y^\psi \in \mathfrak{f}^\psi$ and $x_i^\psi = (y^\psi)^\psi = y \in \mathfrak{f}$.

Let $\{x, y\}$ be a free basis of $\mathfrak{f}$ and let M be the set of monomials in these generators. We want to show that $R(\mathfrak{f}) = [\mathfrak{f}, [L, \mathfrak{f}^\psi]] = 0$. Clearly it is enough to show that

$$\mathcal{R}(g, \ell, h) := [g, [\ell, h^\psi]] = 0 \tag{3.14}$$

for all $g, h \in \mathfrak{f}$ and $\ell \in L$. By linearity, it suffices to consider $g, h \in M$. We will show that actually it is enough to consider indecomposable monomials, that is, $g, h \in \{x, y\}$. For this, it suffices to show that if g or h can be written as a non-trivial bracket, then (3.14) follows from the identities with respect to each of the terms of the bracket.

First, if we have

$$[[g_1, g_2], [\ell, h^\psi]] = [g_1, [g_2, [\ell, h^\psi]]] - [g_2, [g_1, [\ell, h^\psi]]].$$

So if $\mathcal{R}(g_i, \ell, h) = 0$ for $i = 1, 2$, then $\mathcal{R}([g_1, g_2], \ell, h) = 0$ as well.

Similarly, suppose $h = [h_1, h_2]$. By the Jacobi identity we have:

$$[g, [\ell, [h_1, h_2]^\psi]] = [[\ell, h_1^\psi], [g, h_2^\psi]] + [[g, [\ell, h_1^\psi]], h_2^\psi] - [[g, [\ell, h_2^\psi]], h_1^\psi] - [[\ell, h_2^\psi], [g, h_1^\psi]].$$

The first and the fourth terms in the right-hand side of the equation above vanish because $[D, L] = 0$. The second and the third terms vanish if we assume that $\mathcal{R}(g, \ell, h_1) = \mathcal{R}(g, \ell, h_2) = 0$.

Now we want to do the same with respect to ℓ. We will show that is enough to consider the elements $\ell = m - m^\psi$, with $m \in \{x, y\}$.

Clearly it is enough to let ℓ run through a linear spanning set for L. We know that L is generated as an algebra by the elements $m - m^\psi$ with $m \in M$. Thus a spanning set for L can be obtained by considering the long brackets involving these elements.

Now recall that given $m, n, p \in M$, we have by Lemma 3.7:

$$[m - m^\psi, [n - n^\psi, p - p^\psi]] = [m, [n, p]] - [m, [n, p]]^\psi.$$

From this follows that L is linearly spanned by elements of the form $m - m^\psi$ and $[m - m^\psi, n - n^\psi]$ with $m, n \in M$. Now:

$$[g, [[m - m^\psi, n - n^\psi], h^\psi]] = [g, [[m - m^\psi, h^\psi], n - n^\psi]] + [g, [m - m^\psi, [n - n^\psi, h^\psi]]] = (*)$$

If $\mathcal{R}(n, m - m^\psi, h) = \mathcal{R}(m, n - n^\psi, h) = 0$, then:

$$(*) = -[g, [[m - m^\psi, h^\psi], n^\psi]] - [g, [m^\psi, [n - n^\psi, h^\psi]]]$$

Then by the Jacobi identity, together with $[D, L] = 0$, we have:

$$(*) = -[[g, [m - m^\psi, h^\psi]], n^\psi] - [m^\psi, [g, [n - n^\psi, h^\psi]]],$$

so $\mathcal{R}(g, [m - m^\psi, n - n^\psi], h) = 0$ if we also assume that $\mathcal{R}(g, m - m^\psi, h) = 0$ and $\mathcal{R}(g, n - n^\psi, h) = 0$.

We are down to: if $\mathcal{R}(g, m - m^\psi, h) = 0$ for all $g, h \in \{x, y\}$ and $m \in M$, then $R(\mathfrak{f}) = 0$.

Finally, it is enough to consider $m \in \{x, y\}$. In fact, if $m = [u, v]$, then

$$[g, [[u, v], h^\psi]] = [[g, [u, h^\psi]], v] + [[u, h^\psi], [g, v]] + [[g, u], [v, h^\psi]] + [u, [g, [v, h^\psi]]].$$

To see that $\mathcal{R}(g, [u, v] - [u, v]^\psi, h) = 0$, we need to show that we can change any instance of u (resp. v) for u^ψ (resp. v^ψ) in the right-hand side of the equation above. For the first term we use that $\mathcal{R}(g, u - u^\psi, h) = 0$ and then Remark 3.23 (to change v for v^ψ). The fourth term is analogous, but we use that $\mathcal{R}(g, v - v^\psi, h) = 0$. For the second and third terms we apply Remark 3.23 twice.

By the arguments above, for $R(\mathfrak{f}) = 0$, it is enough that $\mathcal{R}(g, m - m^\psi, h) = 0$ with $g, h, m \in \{x, y\}$. But this can verified directly, being consequence of the relations $[x, x^\psi] = 0$, $[y, y^\psi] = 0$ and $[D, L] = 0$. Thus:

Proposition 3.24. *If $\mathfrak{g}$ can be generated by two elements, then $R(\mathfrak{g}) = 0$.*

Proof. We have proved for $\mathfrak{f}$ free of rank two. In general, if $\mathfrak{g}$ is generated by two elements, then there is a surjective homomorphism $\varphi : \mathfrak{f} \to \mathfrak{g}$, which induces $\varphi_* : \chi(\mathfrak{f}) \to \chi(\mathfrak{g})$. It is clear that $\varphi_*(R(\mathfrak{f})) = R(\mathfrak{g})$, so the result follows. □

Remark 3.25. *Observe that the proof above does not work for a free Lie algebra of rank greater than 2, since we can not guarantee the base step: $[x^\psi, [y - y^\psi, z]]$ will not be trivial if x, y and z are three independent generators.*

Thus, in general, if $\mathfrak{g}$ is generated by two elements, then $\chi(\mathfrak{g})$ is an extension of $H_2(\mathfrak{g}; K)$ by $Im(\rho)$. In particular, if $\mathfrak{f}$ is the free Lie algebra on two generators, then $\chi(\mathfrak{f}) \simeq Im(\rho)$.

3.6 A presentation for $L(\mathfrak{g})$

At this point we have some non-trivial examples of concrete descriptions of $\chi(\mathfrak{g})$, but they all rely in showing somehow that $R(\mathfrak{g}) = 0$. In this section we develop a new approach that consists in writing down a presentation for $L(\mathfrak{f})$, where $\mathfrak{f}$ is a free Lie algebra. Besides being interesting on its own right, this allows us to understand very well the structure of $L(\mathfrak{g})$ for some nilpotent Lie algebras of small class. We will find in particular the first examples where $R(\mathfrak{g}) \neq 0$.

Recall that our convention is that unspecified brackets denote right-normed brackets, that is:

$$[x_1, x_2, \ldots, x_{n-1}, x_n] = [x_1, [x_2, \ldots, [x_{n-1}, x_n] \ldots]]$$

for any elements $x_1, \ldots, x_n$ of a Lie algebra.

Let $\mathfrak{f}$ be the free Lie algebra with free basis $x_1, \ldots, x_m$. Recall that by the proof of Proposition 3.8, $L = L(\mathfrak{f})$ is generated as a subalgebra by the elements $x_i - x_i^{\psi}$ and $[x_i, x_j] - [x_i, x_j]^{\psi}$, for $1 \leq i < j \leq m$.

We introduce the notation:

$$\tilde{a}_i := x_i - x_i^{\psi}$$

for $1 \leq i \leq m$ and

$$\tilde{b}_{i,j} := [x_i, x_j] - [x_i, x_j]^{\psi}$$

for $1 \leq i < j \leq m$. For convenience we set $\tilde{b}_{i,i} = 0$ and $\tilde{b}_{i,j} = -\tilde{b}_{j,i}$ for all $1 \leq j < i \leq m$.

In the next two lemmas we deduce some relations between these generators. They all come from suitable interpretations of the identity of Lemma 3.7. We will show later that they actually form a full set of relations for L with the generators that we chose.

Lemma 3.26. *For any i, j, k, l, we have:*

$$[\tilde{a}_i, \tilde{a}_j, \tilde{b}_{k,l}] = [\tilde{a}_i, \tilde{b}_{j,k}, \tilde{a}_l] + [\tilde{a}_i, \tilde{a}_k, \tilde{b}_{j,l}]$$

Proof. We use Lemma 3.7, the Jacobi identity and then Lemma 3.7 again:

$$\begin{aligned}
[\tilde{a}_i, \tilde{a}_j, \tilde{b}_{k,l}] &= [x_i - x_i^{\psi}, x_j - x_j^{\psi}, [x_k, x_l] - [x_k, x_l]^{\psi}] \\
&= [x_i, x_j, x_k, x_l] - [x_i, x_j, x_k, x_l]^{\psi} \\
&= [x_i, [x_j, x_k], x_l] - [x_i, [x_j, x_k], x_l]^{\psi} + [x_i, x_k, x_j, x_l] - [x_i, x_k, x_j, x_l]^{\psi} \\
&= [\tilde{a}_i, \tilde{b}_{j,k}, \tilde{a}_l] + [\tilde{a}_i, \tilde{a}_k, \tilde{b}_{j,l}],
\end{aligned}$$

as we wanted. □

Lemma 3.27. *Let $n > 1$ be an odd integer and let u_i be either some $\tilde{a}_j$ or some $\tilde{b}_{j,j'}$, for all i. Then:*

$$[u_1, \dots, \tilde{b}_{j,j'}, \dots, \tilde{b}_{k,k'}, \dots, u_n] = [u_1, \dots, [\tilde{a}_j, \tilde{a}_{j'}], \dots, [\tilde{a}_k, \tilde{a}_{k'}], \dots, u_n]$$

and

$$[u_1, \dots, \tilde{b}_{j,j'}, \dots, u_{n-2}, \tilde{a}_{n-1}, \tilde{a}_n] = [u_1, \dots, [\tilde{a}_j, \tilde{a}_{j'}], \dots, u_{n-2}, \tilde{b}_{n-1,n}].$$

Proof. Notice that Lemma 3.7 actually gives a way of writing any right-normed bracket of odd length involving elements of the form $v - v^{\psi}$ as a single element of this same form. It suffices then to apply this reasoning to each of the brackets in the statement of the lemma and verify that the two sides of each equation actually coincide. □

Define $\mathcal{L}$ as the Lie algebra generated by the symbols a_i and $b_{i,j}$, for $1 \leq i < j \leq m$, subject to the relators as in Lemmas 3.26 and 3.27 (without the tildes), that is,

$$r = [a_i, a_j, b_{k,l}] - [a_i, b_{j,k}, a_l] - [a_i, a_k, b_{j,l}] \tag{3.15}$$

for all i, j, k, l,

$$r = [u_1, \dots, b_{j,j'}, \dots, b_{k,k'}, \dots, u_n] - [u_1, \dots, [a_j, a_{j'}], \dots, [a_k, a_{k'}], \dots, u_n] \tag{3.16}$$

for all $u_i \in \{a_t, b_{s,t}\}_{s,t}$, all indices and $n \geq 3$ an odd integer and

$$r = [u_1, \dots, b_{j,j'}, \dots, u_{n-2}, a_{n-1}, a_n] - [u_1, \dots, [a_j, a_{j'}], \dots, u_{n-2}, b_{n-1,n}], \tag{3.17}$$

for all $u_i \in \{a_t, b_{s,t}\}_{s,t}$, all indices and $n \geq 3$ an odd integer. Again we are using the convention that $b_{i,j} = -b_{j,i}$ and $b_{i,i} = 0$, so that all expressions above make sense. We will show that $\mathcal{L} \simeq L(\mathfrak{g})$.

Lemma 3.28. *The following formulas define an action of $\mathfrak{f}$ on $\mathcal{L}$:*

$$x_s \cdot a_i = \frac{1}{2}([a_s, a_i] + b_{s,i})$$

and

$$x_s \cdot b_{i,j} = \frac{1}{2}([a_s, b_{i,j}] + [a_s, a_i, a_j]),$$

for all s, i, j.

Proof. Let F be the free Lie algebra on a_i, $b_{i,j}$, for $1 \leq i < j \leq m$. It is clear that the formulas above give well-defined derivations $D_s : F \to F$. To see that they induce derivations $D_s : \mathcal{L} \to \mathcal{L}$ we need to verify that the relations are respected.

First notice that

$$D_s([a_i, a_j, b_{k,l}]) = \frac{1}{2}([a_s, a_i, a_j, b_{k,l}] + [b_{s,i}, a_j, b_{k,l}] + [a_i, b_{s,j}, b_{k,l}] + [a_i, a_j, a_s, a_k, a_l]).$$

Modulo relators of type (3.16) we have

$$[b_{s,i}, a_j, b_{k,l}] = [[a_s, a_i], a_j, a_k, a_l]$$

and

$$[a_i, b_{s,j}, b_{k,l}] = [a_i, [a_s, a_j], a_k, a_l],$$

thus

$$D_s([a_i, a_j, b_{k,l}]) = \frac{1}{2}([a_s, a_i, a_j, b_{k,l}] + [a_s, a_i, a_j, a_k, a_l]).$$

It becomes clear from this expression that for

$$r = [a_i, a_j, b_{k,l}] - [a_i, b_{j,k}, a_l] - [a_i, a_k, b_{j,l}] \in F,$$

the image $D_s(r)$ is a consequence of the defining relators of $\mathcal{L}$.

Now let

$$r_{21} = [u_1, \ldots, b_{j,j'}, \ldots, b_{k,k'}, \ldots, u_n]$$

$$r_{22} = [u_1, \ldots, [a_j, a_{j'}], \ldots, [a_k, a_{k'}], \ldots, u_n],$$

with $n > 1$ odd. We need to verify that $D_s(r_{21} - r_{22})$ is a consequence of the defining relators of $\mathcal{L}$. We compute each $D_s(r_{2i})$ by applying the derivation property. Notice that

$$D_s(r_{21}) - \frac{1}{2}[a_s, r_{21}] = \frac{1}{2}([u_1, \ldots, [a_s, a_j, a_{j'}], \ldots, b_{k,k'}, \ldots, u_n] \\ + [u_1, \ldots, b_{j,j'}, \ldots, [a_s, a_k, a_{k'}], \ldots, u_n] + A),$$

where A is the sum of the terms

$$[u_1, \ldots, D_s(u_i) - [a_s, u_i], \ldots, b_{j,j'}, \ldots, b_{k,k'}, \ldots, u_n]$$

with $i \neq j, k$. Similarly we have:

$$D_s(r_{22}) - \frac{1}{2}[a_s, r_{22}] = \frac{1}{2}([u_1, \ldots, [b_{s,j}, a_{j'}] + [a_j, b_{s,j'}], \ldots, [a_k, a_k'], \ldots, u_n] \\ + [u_1, \ldots, [a_j, a_{j'}], \ldots, [b_{s,k}, a_{k'}] + [a_k, b_{s,k'}], \ldots, u_n] + B),$$

where B is the sum of the terms

$$[u_1, \ldots, D_s(u_i) - [a_s, u_i], \ldots [a_j, a_{j'}], \ldots, [a_k, a_{k'}], \ldots, u_n]$$

with $i \neq j, k$.

Now notice that each of the terms of $D_s(r_{22}) - \frac{1}{2}[a_s, r_{22}]$ can be transformed into the respective term of $D_s(r_{21}) - \frac{1}{2}[a_s, r_{21}]$ modulo a relator of type (3.16) or (3.17) (or that after an application of the Jacobi identity). We recall that n is an odd integer to begin with, so all brackets have length n or $n+2$ and the application of the defining relators is licit. Thus

$$D_s(r_{21} - r_{22}) \equiv \frac{1}{2}[a_s, r_{21} - r_{22}] \equiv 0,$$

where the congruence is taken modulo the defining relators of $\mathcal{L}$. The proof for a relator of type (3.17) is completely analogous, so we omit.

Finally, since each $D_s : \mathcal{L} \to \mathcal{L}$ is a well defined derivation and $\mathfrak{g}$ is free, the association $x_s \mapsto D_s$ defines a Lie algebra homomorphism $\mathfrak{g} \to Der(\mathcal{L})$, that is, an action of $\mathfrak{g}$ on $\mathcal{L}$. □

We will abandon the notation D_s for these derivations; we will simply write $x_s \cdot \ell$ to denote the action of D_s on some $\ell \in \mathcal{L}$. We may consider the semi-direct product $\mathcal{L} \rtimes \mathfrak{f}$ defined with respect to this action, so that $x_s \cdot \ell$ is identified with $[x_s, \ell]$ for all $\ell \in \mathcal{L}$.

In order to prove that $\mathcal{L} \simeq L(\mathfrak{f})$, we first deduce some formulas for the action of $\mathfrak{f}$ on $\mathcal{L}$. For instance, we have this nice formula for the action of long right-normed brackets involving the generators of $\mathfrak{f}$ on the generators of $\mathcal{L}$.

Lemma 3.29. *For all $n \geq 2$ and for any $u \in \{a_i, b_{i,j}\}$ we have:*

$$[x_{i_1}, \dots, x_{i_n}] \cdot u = -\frac{1}{2}([u, a_{i_1}, \dots, a_{i_n}] + [u, a_{i_1}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}])$$

Proof. This can be proved by induction on n. The formulas are easily verified for $n = 2$. For $n > 2$ we use the following fact: if α and β are derivations of a Lie algebra L, and $\beta(x) = [x, b]$ for some $b \in L$ and for all $x \in L$ (that is, β is inner), then $[\alpha, \beta](x) = [x, \alpha(b)]$ for all $x \in L$.

In order to use the induction hypothesis, we write

$$[x_{i_1}, \dots, x_{i_n}] = [x_{i_1}, [x_{i_2}, \dots, x_{i_n}]].$$

The derivation associated to $[x_{i_2}, \dots, x_{i_n}]$ is inner and the multiplying element is given by the statement. Thus

$$[x_{i_1}, \dots, x_{i_n}] \cdot u = -\frac{1}{2}([u, x_{i_1} \cdot ([a_{i_2}, \dots, a_{i_n}] + [a_{i_2}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}])])$$

for all u.

By computing via the derivation property, we actually obtain

$$x_{i_1} \cdot ([a_{i_2}, \dots, a_{i_n}] + [a_{i_2}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}]) = [a_{i_1}, \dots, a_{i_n}] + [a_{i_1}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}] + s$$

modulo the defining relators of $\mathcal{L}$, where s in an element such that $[u, s]$ is a defining relator of $\mathcal{L}$ for all $u \in \{a_i, b_{i,j}\}$ (in other words, s is central). This gives the desired result. □

Analogously to Lemma 3.7, we have a nice way of writing some long brackets of $\mathcal{L}$.

Lemma 3.30. *We have:*

$$[x_{i_1} - a_{i_1}, \dots, x_{i_n} - a_{i_n}] = \begin{cases} [x_{i_1}, \dots, x_{i_n}] - [a_{i_1}, \dots, a_{i_n}] & \text{if } n \text{ is odd,} \\ [x_{i_1}, \dots, x_{i_n}] - [a_{i_1}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}] & \text{otherwise.} \end{cases}$$

Proof. We prove it by induction of n. If $n = 1$ this is clear. If $n > 1$ is even, then by induction hypothesis we have

$$[x_{i_1} - a_{i_1}, \dots, x_{i_n} - a_{i_n}] = [x_{i_1} - a_{i_1}, [x_{i_2}, \dots, x_{i_n}] - [a_{i_2}, \dots, a_{i_n}]].$$

By Lemma 3.29 we have

$$-[a_{i_1}, [x_{i_2}, \dots, x_{i_n}]] = -\frac{1}{2}([a_{i_1}, \dots, a_{i_n}] + [a_{i_1}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}]). \tag{3.18}$$

Also:

$$-[x_{i_1}, [a_{i_2}, \dots, a_{i_n}]] = -\frac{1}{2}([a_{i_1}, \dots, a_{i_n}] + \sum_{j=2}^{n} [a_{i_2}, \dots, b_{i_1, i_j}, \dots, a_{i_n}]). \tag{3.19}$$

By relations 3.15 and 3.17 it follows that

$$\sum_{j=2}^{n} [a_{i_2}, \dots, b_{i_1, i_k j}, \dots, a_{i_n}] = [a_{i_1}, a_{i_2}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}].$$

Finally, by summing (3.18) and (3.19), we get

$$[x_{i_1} - a_{i_1}, \dots, x_{i_n} - a_{i_n}] = [x_{i_1}, \dots, x_{i_n}] - [a_{i_1}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}],$$

as we wanted.

If n is odd we have similarly

$$[x_{i_1} - a_{i_1}, \dots, x_{i_n} - a_{i_n}] = [x_{i_1} - a_{i_1}, [x_{i_2}, \dots, x_{i_n}] - [a_{i_2}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}]].$$

This time we have:

$$\begin{aligned} -[x_{i_1}, [a_{i_2}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}]] = -\frac{1}{2}(&[a_{i_1}, \dots, a_{i_{n-2}}, b_{i_{n-1}, i_n}] \\ &+ \sum_{j=1}^{n-2} [a_{i_2}, \dots, b_{i_1, i_j}, \dots a_{i_{n-2}}, b_{i_{n-1}, i_n}] \\ &+ [a_{i_2}, \dots, a_{i_{n-2}}, a_{i_1}, a_{i_{n-1}}, a_{i_n}]) \end{aligned} \tag{3.20}$$

Now, by relations (3.16) we get

$$\sum_{j=1}^{n-2} [a_{i_2}, \dots, b_{i_1, i_j}, \dots a_{i_{n-2}}, b_{i_{n-1}, i_n}] + [a_{i_2}, \dots, a_{i_{n-2}}, a_{i_1}, a_{i_{n-1}}, a_{i_n}] = [a_{i_1}, \dots, a_{i_n}].$$

Thus by (3.18) and (3.20) we get the result. The proof is complete. □

Let $\mathcal{M}$ be the set of right-normed brackets involving $x_1, \ldots, x_m$. For $u = [x_{i_1}, \ldots, x_{i_n}] \in \mathcal{M}$, with $n \geq 2$, denote

$$\mu(u) = \begin{cases} [a_{i_1}, \ldots, a_{i_n}] & \text{if n is odd,} \\ [a_{i_1}, \ldots, a_{i_{n-2}}, b_{i_{n-1}, i_n}] & \text{otherwise.} \end{cases}$$

Similarly, let:

$$\xi(u) = \begin{cases} [a_{i_1}, \ldots, a_{i_{n-2}}, b_{i_{n-1}, i_n}] & \text{if n is odd,} \\ [a_{i_1}, \ldots, a_{i_n}] & \text{otherwise.} \end{cases}$$

By Lemma 3.29, for $u = [x_{i_1}, \ldots, x_{i_n}]$ with $n \geq 2$, we have:

$$u \cdot \ell = -\frac{1}{2}[\ell, \mu(u) + \xi(u)]$$

for all $\ell \in \mathcal{L}$.

Remark 3.31. *Notice that if $\mathfrak{f}$ is free on $x_1, \ldots, x_m$, then a full set of relations for $\chi(\mathfrak{f})$, with generating set $x_1, \ldots x_m, x_1^\psi, \ldots, x_m^\psi$, is given by:*

$$[u, v^\psi] = [u^\psi, v]$$

for all right-normed brackets u, v involving the generators $x_1, \ldots, x_m$ (including the case $u = v$).

Theorem 3.32. *Let $\mathfrak{f}$ be free on the set $\{x_1, \ldots, x_m\}$. Then $\chi(\mathfrak{f}) \simeq \mathcal{L} \rtimes \mathfrak{f}$.*

Proof. Define

$$\sigma : \mathcal{L} \rtimes \mathfrak{f} \to \chi(\mathfrak{f})$$

by

$$\sigma(x_i) = x_i, \ \sigma(a_i) = x_i - x_i^\psi, \ \sigma(b_{i,j}) = [x_i, x_j] - [x_i, x_j]^\psi.$$

By choice, this is a well-defined surjective homomorphism of Lie algebras.

Similarly, define

$$\theta : \chi(\mathfrak{f}) \to \mathcal{L} \rtimes \mathfrak{f}$$

by

$$\theta(x_i) = x_i, \ \theta(x_i^\psi) = x_i - a_i.$$

We need show that θ is well-defined. We will verify that the relations as in Remark 3.31 are preserved.

Let $u, v \in \mathcal{M}$. If u and v have length 1, say $u = x_i$ and $v = x_j$, we can see directly by the definition of the action of $\mathfrak{f}$ on $\mathcal{L}$ that $[x_i, a_j] = -[a_i, x_j]$, which implies that $[\theta(u), \theta(v^\psi)] = [\theta(u^\psi), \theta(v)]$.

So suppose that v has length at least 2. By Lemma 3.30 we have

$$[\theta(u), \theta(v^{\psi})] = [u, v - \mu(v)] = [u, v] - [u, \mu(v)].$$

If $u = x_t$ has length 1, this reduces to $[x_t, v] - [x_t, \mu(v)]$ and we have on the other hand:

$$[\theta(u^{\psi}), \theta(v)] = [x_t - a_t, v] = [x_t, v] - [a_t, v].$$

Now write $v = [x_{i_1}, \ldots, x_{i_n}]$. If n is odd, we have:

$$[x_t, \mu(v)] = [x_t, [a_{i_1}, \ldots, a_{i_n}]] = \frac{1}{2}\sum_{j=1}^{n}[a_{i_1}, \ldots, [a_t, a_{i_j}] + b_{t,i_j}, \ldots, a_{i_n}].$$

For $1 \leq j \leq n-2$, as a consequence of relation (3.17) we have

$$[a_{i_1}, \ldots, b_{t,i_j}, \ldots, a_{i_n}] = [a_{i_1}, \ldots, [a_t, a_{i_j}], \ldots, a_{i_{n-2}}, b_{i_{n-1},i_n}].$$

Also, by relation (3.15) we have:

$$[a_{i_1}, \ldots, a_{i_{n-2}}, b_{t,i_{n-1}}, a_{i_n}] + [a_{i_1}, \ldots, a_{i_{n-2}}, a_{i_{n-1}}, b_{t,i_n}] = [a_{i_1}, \ldots, a_{i_{n-2}}, a_t, b_{i_{n-1},i_n}]$$

Thus:

$$[x_t, \mu(v)] = \frac{1}{2}([a_t, a_{i_1}, \ldots, a_{i_n}] + [a_t, a_{i_1}, \ldots, a_{i_{n-2}}.b_{i_{n-1},i_n}]) = \frac{1}{2}[a_t, \mu(v) + \xi(v)]$$

This coincides with the formula given on Lemma 3.29 for $[a_t, v]$, so we have $[\theta(x_i), \theta(v^{\psi})] = [\theta(x_i^{\psi}), \theta(v)]$. The same reasoning gives the result if n is even, but we use the relations (3.16).

Finally, suppose that both u and v have length at least two. Since $\mu(v) \in \mathcal{L}$, we have by the comment above the theorem:

$$[\theta(u), \theta(v)^{\psi}] = [u, v] - [u, \mu(v)] = [u, v] + \frac{1}{2}[\mu(v), \mu(u) + \xi(u)].$$

By looking similarly at $[\theta(u^{\psi}), \theta(v)]$, we see that in order to show that the relation $[u, v^{\psi}] = [u^{\psi}, v]$ is preserved by θ, we need only to verify that

$$[\mu(u), \xi(v)] = [\xi(u), \mu(v)]$$

for all u, v. But this is an instance of relation (3.17), after opening up the brackets as right-normed ones (note that $[\mu(u), \xi(v)]$ is always a bracket of odd length).

Thus θ is well defined and clearly $\sigma \circ \theta = id$ and $\theta \circ \sigma = id$, that is, θ is an isomorphism of Lie algebras. □

By restriction we get an isomorphism $L(\mathfrak{f}) \simeq \mathcal{L}$.

In general, if $\mathfrak{g} = \mathfrak{f}/N$, where $N \subseteq \mathfrak{f}$ is an ideal, then $\chi(\mathfrak{g})$ is the quotient of $\mathcal{L} \rtimes \mathfrak{f}$ by the ideal generated by $\theta(N \cup N^{\psi})$, where θ is the homomorphism defined in the proof of the theorem. Clearly $\theta(N)$ generates a copy of N inside $\mathfrak{f}$. On the other hand, if $r \in N$, then

$$\theta(r^{\psi}) = r - \mu(r),$$

where μ is extended by linearity (or $\mu : \mathfrak{f} \to \mathcal{L} \rtimes \mathfrak{f}$ is *defined* as $\mu = inc - \theta|_{\mathfrak{f}^{\psi}}$, where $inc : \mathfrak{f} \to \mathcal{L} \rtimes \mathfrak{f}$ is the obvious inclusion). It follows that $\chi(\mathfrak{g}) = \mathcal{L}/J \rtimes \mathfrak{g}$, where J is the ideal of $\chi(\mathfrak{f})$ generated by $\mu(r)$, for all $r \in N$. This gives a presentation for $L(\mathfrak{g})$ as well.

The presentation of $L(\mathfrak{f})$ that we obtained is of course infinite. For a non-abelian free Lie algebra $\mathfrak{f}$ in fact $L(\mathfrak{f})$ does not admit a finite presentation, as we can show using arguments similar to those in Section 3.3.3.

Proposition 3.33. *If $\mathfrak{f}$ is free non-abelian, then $L(\mathfrak{f})$ does not admit a finite presentation.*

Proof. It suffices to show that $H_2(L(\mathfrak{f}); K)$ is infinite dimensional. Suppose, on the contrary, that it is of finite dimension. Recall that $W/R \simeq H_2(\mathfrak{f}; K)$. In particular $W = R$ if $\mathfrak{f}$ is free. By analyzing R in terms of the generators of $\mathcal{L}$, it becomes clear that $R \subseteq [L, L]$. In particular, $L^{ab} \simeq \rho(L)^{ab}$. Now, the 5-term exact sequence associated to the LHS spectral sequence arising from $R \rightarrowtail L \twoheadrightarrow \rho(L)$ reduces to

$$H_2(L; K) \to H_2(\rho(L); K) \to R \to 0.$$

By Lemma 3.16, the homology $H_2(\rho(L); K)$ is infinite dimensional. Thus $W = R$ is infinite dimensional (we are under the hypothesis that $H_2(L; K)$ is finite dimensional).

Now, consider the spectral sequence itself

$$E^2_{p,q} = H_p(\rho(L); H_q(W; K)) \Rightarrow H_{p+q}(L; K).$$

Notice that $E^2_{1,1} = E^{\infty}_{1,1}$. Indeed, the differentials involved are $d_{1,1} : E^2_{1,1} \to E^2_{-1,2}$ and $d_{1,1} : E^2_{3,0} \to E^2_{1,1}$. Clearly $E^2_{-1,2} = 0$, but also $E^2_{3,0} = H_3(\rho(L); K) = 0$, since $\rho(L) \subseteq \mathfrak{g} \oplus \mathfrak{g}$. Then $E^2_{1,1} = E^3_{1,1} = E^{\infty}_{1,1}$. But:

$$E^2_{1,1} = H_1(\rho(L); H_1(W; K)) \simeq \rho(L)^{ab} \otimes_K W,$$

since $\rho(L)$ acts trivialy on W. Thus, if W is infinite dimensional, then so is $E^{\infty}_{1,1}$, and finally so is $H_2(L; K)$. This is a contradiction. □

Remark 3.34. *Notice that if $\mathfrak{f}$ is free of rank 2 the conclusion was more immediate: $W = R = 0$, so $L \simeq \rho(L)$, and we already knew that $H_2(\rho(L); K)$ was infinite dimensional.*

3.7 Nilpotent Lie algebras

For $\mathfrak{g}$ an abelian Lie algebra, $\chi(\mathfrak{g})$ is completely described in Proposition 3.22. We consider here nilpotent Lie algebras of class $c \geq 2$. We will show that if $\mathfrak{g}$ is nilpotent of class c, then $\chi(\mathfrak{g})$ is nilpotent of class at most $c + 2$.

Denote by $\mathfrak{n}_{m,c}$ the free nilpotent Lie algebra of rank m and class c. By the comments in the previous section, we can obtain $L(\mathfrak{n}_{m,c})$ by taking the quotient of $\mathcal{L} \simeq L(\mathfrak{g})$ (where $\mathfrak{g}$ is free of rank m) by the ideal generated by the elements $\mu(u)$, for brackets u of length at least $c+1$ involving the generators of $\mathfrak{g}$.

Consider the generators a_i, $b_{i,j}$ of $\mathcal{L}$. Define

$$d(a_i) := 1,\ d(b_{i,j}) := 2,$$

for all $i < j$. For a right-normed bracket $\ell = [\ell_1, \dots, \ell_n]$ involving some of these generators, we define the *degree* $d(\ell)$ of ℓ as

$$d(\ell) = \sum_{j=1}^{n} d(\ell_j).$$

Now we are ready to determine the (class of) nilpotency of $\chi(\mathfrak{g})$.

Theorem 3.35. *Suppose that $\mathfrak{h}$ is nilpotent of class c. Then $\chi(\mathfrak{h})$ is nilpotent and its nilpotency class is bounded by the smallest even integer greater than c.*

Proof. Clearly we can assume that $\mathfrak{h}$ is free nilpotent of class c, so that all the comments above the theorem make sense. Suppose that we want to show that $\chi(\mathfrak{h})$ is nilpotent of class n, where n may be $c+1$ or $c+2$ depending on the parity of c. Let $x_1, \dots, x_m$ be a set of generators for $\mathfrak{h}$. It is enough then to show that

$$w = [x_{i_{n+1}}^{\theta_{n+1}}, \dots, x_{i_2}^{\theta_2}, x_{i_1}^{\theta_1}] = 0$$

for all $1 \leq i_j \leq m$ and $\theta_j \in \{id, \psi\}$. Clearly if $\theta_i = id$ for all i, then $w = 0$. Similarly, $w = 0$ if $\theta_i = \psi$ for all i. We can assume without loss of generality that $\theta_1 = id$. Let $k = min\{j | \theta_j \neq id\}$. Since $[x_{i_k}^{\psi}, x_{i_{k-1}}, \dots, x_{i_1}] \in D$ and $[D, L] = 0$, we have

$$w = [x_{i_{n+1}}, \dots, x_{i_{k+1}}, x_{i_k}^{\psi}, x_{i_{k-1}}, \dots, x_{i_2}, x_{i_1}].$$

It follows by induction on k that w is a linear combination of terms of the form $[x_{j_{n+1}}, \dots, x_{j_2}, x_{j_1}^{\psi}]$, for some $1 \leq j_t \leq m$. Indeed, this is clear if $k = 2$. If $k > 2$, by the Jacobi identity we have:

$$\begin{aligned} w = {} & [x_{i_{n+1}}, \dots, x_{i_{k+1}}, x_{i_{k-1}}, x_{i_k}^{\psi}, x_{i_{k-2}}, \dots, x_{i_2}, x_{i_1}] \\ & + [x_{i_{n+1}}, \dots, x_{i_{k+1}}, [x_{i_k}^{\psi}, x_{i_{k-1}}], x_{i_{k-2}}, \dots, x_{i_2}, x_{i_1}]. \end{aligned} \tag{3.21}$$

By induction hypothesis we can rewrite the first term on the right-hand side of the equation above in the form we want. By antisymmetry the second term is

$$[x_{i_{n+1}}, \dots, x_{i_{k+1}}, [x_{i_{k-2}}, \dots, x_{i_2}, x_{i_1}], [x_{i_{k-1}}, x_{i_k}^{\psi}]],$$

which can be rewritten by the Jacobi identity as a linear combination of terms of the form

$$[x_{i_{n+1}}, \dots, x_{i_{k+1}}, x_{\sigma(i_{k-2})}, \dots, x_{\sigma(i_2)}, x_{\sigma(i_1)}, x_{i_{k-1}}, x_{i_k}^{\psi}]$$

for some permutations $\sigma \in S_{k-2}$. All of this means that $\chi(\mathfrak{h})$ is nilpotent of class n if

$$w = [x_{i_{n+1}}, \dots, x_{i_2}, x_{i_1}^{\psi}] = 0$$

for all $1 \leq i \leq m$.

Now we interpret this in terms of the isomorphism $\theta : \chi(\mathfrak{h}) \to \mathcal{L}/J \rtimes \mathfrak{h}$. We have:

$$\theta(w) = \theta([x_{i_{n+1}}, \dots, x_{i_2}, x_{i_1}^{\psi}]) = [x_{i_{n+1}}, \dots, x_{i_2}, x_{i_1} - a_{i_1}] = -[x_{i_{n+1}}, \dots, x_{i_2}, a_{i_1}],$$

since $[x_{i_{n+1}}, \dots, x_{i_2}, x_{i_1}] = 0$. By induction we see that $-[x_{i_{n+1}}, \dots, x_{i_2}, a_{i_1}]$ is a linear combination of brackets $\ell = [\ell_1, \dots, \ell_k]$, involving the generators of $\mathcal{L}$, with $d(\ell) = n+1$.

Finally, we consider the parity of c. If c is odd, we are trying to prove that $\chi(\mathfrak{g})$ is nilpotent of class $c+1$, that is, $n = c+1$. Given a bracket $\ell = [\ell_1, \dots, \ell_k]$ with $d(\ell) = c+2$, we can use the defining relations of $\mathcal{L}$ to rewrite it as a linear combination of elements of the forms

$$[a_{i_1}, \dots, a_{i_{c+2}}]$$

and

$$[a_{i_1}, \dots, a_{i_c}, b_{i_{c+1}, i_{c+2}}].$$

Notice that it essential the fact that $c+2$ is an odd integer, otherwise we would not be able to get rid of brackets of the form

$$[b_{i_1, i_2}, a_{i_3}, \dots, a_{i_c}, b_{i_{c+1}, i_{c+2}}].$$

Now, as observed before, $\mu(u)$ is trivial in $\mathcal{L}$ for any u a bracket involving the generators of $\mathfrak{g}$ with length at least $c+1$. In particular, for $u = [x_{i_1}, \dots, x_{i_{c+2}}]$ and $v = [x_{i_2}, \dots, x_{i_{c+2}}]$ we get

$$\mu(u) = [a_{i_1}, \dots, a_{i_{c+1}}, a_{i_{c+2}}]$$

and

$$\mu(v) = [a_{i_2}, \dots, a_{i_c}, b_{i_{c+1}, i_{c+2}}].$$

Clearly $\mu(u) = 0$ and $\mu(v) = 0$ for all u and v of those forms implies that $\ell = 0$. Thus $\chi(\mathfrak{g})$ is nilpotent of class at most $c+1$.

Similarly, suppose that c is even. Now we want to show that $\chi(\mathfrak{g})$ is nilpotent of class at most $n = c+2$. Once again $n+1 = c+3$ is an odd integer, so as before we only need to show that brackets of the forms

$$[a_{i_1}, \dots, a_{i_{c+3}}]$$

and

$$[a_{i_1}, \dots, a_{i_{c+1}}, b_{i_{c+2}, i_{c+3}}].$$

The same argument works: the fact that $\mu([x_{i_1}, \dots, x_{i_{c+1}}])$ and $\mu([x_{i_1}, \dots, x_{i_{c+2}}])$ must be trivial in $\mathcal{L}$ is enough to guarantee what we want. In this case the proof works to show that $\mathfrak{g}$ must be nilpotent of class at most $n = c + 2$, as we wanted. □

These bounds are sharp in the generality of the statement of the theorem, as we will see in the next section. We can, however, obtain a sharper result for 2-generated Lie algebras by a very simple argument.

Proposition 3.36. *If $\mathfrak{g}$ is 2-generated and nilpotent of class c, then $\chi(\mathfrak{g})$ is nilpotent of class $c + 1$.*

Proof. As in the proof of Theorem 3.35, it is enough to show that right-normed brackets of the form

$$w = [x_{i_1}, \dots, x_{i_{c+1}}, x^{\psi}_{i_{c+2}}]$$

are trivial. Now, by Proposition 3.24 we know that $R(\mathfrak{g}) = [\mathfrak{g}, L(\mathfrak{g}), \mathfrak{g}^{\psi}] = 0$ whenever $\mathfrak{g}$ is 2-generated. In particular,

$$[u, v, w^{\psi}] = [u, v^{\psi}, w^{\psi}]$$

for all $u, v, w \in \mathfrak{g}$. But then, by induction, we have:

$$w = [x_{i_1}, \dots, x_{i_c}, x^{\psi}_{i_{c+1}}, x^{\psi}_{i_{c+2}}] = \dots = [x_{i_1}, x^{\psi}_{i_2}, \dots, x^{\psi}_{i_{c+1}}, x^{\psi}_{i_{c+2}}] = 0,$$

since $[x^{\psi}_{i_2}, \dots, x^{\psi}_{i_{c+1}}, x^{\psi}_{i_{c+2}}]$ is trivial in $\mathfrak{g}^{\psi}$. □

3.8 More examples

For the classes of nilpotency $c \leq 3$, we can actually get from the proofs in the previous sections a concrete description of $\chi(\mathfrak{n}_{m,c})$.

3.8.1 Free nilpotent of class 2

Proposition 3.37. *If $\mathfrak{h} = \mathfrak{n}_{m,2}$, then $L(\mathfrak{h})$ is free nilpotent of rank $m + \binom{m}{2}$ and class 2. In particular, we have*

$$dim(\chi(\mathfrak{h})) = 2k + \binom{k}{2},$$

where $k = m + \binom{m}{2}$. Finally, if $m \geq 3$, then $\chi(\mathfrak{h})$ is nilpotent of class at exactly 4.

Proof. By the previous section, $\mu(u)$ is trivial if u has length at least 3. Thus:

$$\mu([x_{i_1}, x_{i_2}, x_{i_3}]) = [a_{i_1}, a_{i_2}, a_{i_3}] = 0 \tag{3.22}$$

and

$$\mu([x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]) = [a_{i_1}, a_{i_2}, b_{i_3,i_4}] = 0 \tag{3.23}$$

for all i_j. It is clear then that any $[a_i, a_j]$ is a central element in $\mathcal{L}$. Moreover, by relation (3.16) we have

$$[b_{i_1,i_2}, a_{i_3}, b_{i_4,i_5}] = [[a_{i_1}, a_{i_2}], a_{i_3}, a_{i_4}, a_{i_5}]$$

and

$$[b_{i_1,i_2}, b_{i_3,i_4}, b_{i_5,i_6}] = [[a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], a_{i_5}, a_{i_6}]$$

which also become trivial by (3.22). This is enough to conclude that both $[a_i, b_{j,k}]$ and $[b_{i,j}, b_{k,l}]$ are also central in $\mathcal{L}$. Finally, it is clear that the original relations of $\mathcal{L}$ and all $\mu(u)$, with u a bracket of length greater than 4, are actually consequences of (3.22) and (3.23). Thus $L(\mathfrak{h})$ is free nilpotent of class 2 with basis a_i and $b_{i,j}$, for all $1 \leq i < j \leq m$.

The formula for the dimension follows clearly from the fact that $\chi(\mathfrak{h}) \simeq L(\mathfrak{h}) \rtimes \mathfrak{h}$. Finally, if $m \geq 3$, then we can consider the element

$$[[x_1, a_2], [x_1, a_3]] = \frac{1}{4}[b_{1,2}, b_{1,3}] \neq 0,$$

which is clearly a non-trivial element of $\gamma_4(\chi(\mathfrak{h}))$. □

Putting the dimension of $\chi(\mathfrak{n}_{m,2})$ in terms of m, it is not hard to see that

$$dim(\chi(\mathfrak{n}_{m,2})) = \frac{1}{8}(m^4 + 2m^3 + 7m^2 + 6m). \tag{3.24}$$

In order to compute the formula for the dimension of $R(\mathfrak{n}_{m,2})$ it suffices to subtract from the dimension of $\chi(\mathfrak{n}_{m,2})$ the dimensions of $Im(\rho)$ and $H_2(\mathfrak{n}_{m,2}; K)$. The former can be computed by observing that

$$Im(\rho) = \{(x, y, z) \in (\mathfrak{n}_{m,2})^3 | \ \ x - y + z \in \mathfrak{n}'_{m,2}\},$$

so $dim(Im(\rho)) = 2dim(\mathfrak{n}_{m,2}) + dim(\mathfrak{n}'_{m,2})$. Thus

$$dim(Im(\rho) = 2(m + \binom{m}{2}) + \binom{m}{2} = \frac{1}{2}(3m^2 + m). \tag{3.25}$$

Regarding the other term, we have:

$$H_2(\mathfrak{n}_{m,2}; K) \simeq \gamma_3/\gamma_4$$

where the γ_i are the terms of the lower central series of F (the free Lie algebra on m generators). By Witt's dimension formula the dimension of γ_3/γ_4 is $\frac{1}{3}(m^3 - m)$. By (3.24) and (3.25) we have:

$$dim(R) = \frac{1}{24}(3m^4 - 2m^3 - 15m^2 + 14m)$$

as stated in the introduction.

3.8.2 Heisenberg algebras

Let $\mathfrak{h}_n$ be the Heisenberg Lie algebra of dimension $2n+1$, that is, $\mathfrak{h}_n$ is the nilpotent of class 2 Lie algebra generated by the symbols $x_1, \ldots, x_{2n}$, subject to the relations

$$[x_i, x_{n+i}] = [x_j, x_{n+j}]$$

for all $1 \leq i, j \leq n$ and

$$[x_i, x_j] = 0$$

for $1 \leq i < j \leq 2n$ such that $j \neq n+i$, as well as

$$[x_i, x_j, x_k] = 0$$

for any i, j, k. We can compute $L(\mathfrak{h}_n)$ from the case $L(\mathfrak{n}_{2n,2})$. In fact it suffices to include the relations

$$\mu([x_i, x_{n+i}] - [x_j, x_{n+j}]) = b_{i,n+i} - b_{j,n+j} = 0,$$

for $1 \leq i, j \leq n$, and

$$\mu([x_i, x_j]) = b_{i,j} = 0$$

if $1 \leq i < j \leq 2n$ such that $j \neq n+i$. Then clearly $L(\mathfrak{h}_n)$ is free nilpotent of class 2 on the generators $a_1, \ldots, a_{2n}, b_{1,n+1}$. As in the free nilpotent case, we can then easily compute the dimensions of $\chi(\mathfrak{h}_n)$ and its ideals.

3.8.3 Free nilpotent of class 3

Proposition 3.38. *If $\mathfrak{h} = \mathfrak{n}_{m,3}$, then $L(\mathfrak{h})$ is a central extension of K^t, where $t = m\binom{m}{2}$, by $\mathfrak{n}_{m,4} \oplus \mathfrak{n}_{\binom{m}{2},2}$.*

Proof. Once again we must have:

$$\mu([x_{i_1}, \ldots, x_{i_4}]) = [a_{i_1}, a_{i_2}, b_{i_3,i_4}] = 0 \tag{3.26}$$

and

$$\mu([x_{i_1}, \ldots, x_{i_5}]) = [a_{i_1}, \ldots, a_{i_5}] = 0. \tag{3.27}$$

Also, by the defining relation (3.16) we have

$$[b_{i_1,i_2}, a_{i_3}, b_{i_4,i_5}] = [[a_{i_1}, a_{i_2}], a_{i_3}, a_{i_4}, a_{i_5}]$$

Thus, imposing (3.26) and (3.27) as relators, we get that $[a_i, b_{l,k}]$ is central and that the Lie algebra generated by the a_i's is nilpotent of class 4. Furthermore, again by (3.16) we have

$$[b_{i_1,i_2}, b_{i_3,i_4}, b_{i_5,i_6}] = [[a_{i_1}, a_{i_2}], [a_{i_3}, a_{i_4}], a_{i_5}, a_{i_6}] = 0,$$

so the Lie algebra generated by the $b_{i,j}$'s is nilpotent of class 3. It is clear that the relations of $\mathcal{L}$ become trivial in the presence of the relations described in the assumption of the proposition, and also clearly no relations of smaller degree involving the a_i can exist. □

Remark 3.39. *It is immediate in this case that $\chi(\mathfrak{h})$ is nilpotent of class 4, since $L(\mathfrak{h})$ contains a copy of $\mathfrak{n}_{m,4}$.*

Corollary 3.40. *$dim R(\mathfrak{n}_{m,2}) = dim R(\mathfrak{n}_{m,3})$ for all m.*

Proof. We can proceed as in the previous subsection and compute the exact dimension of $R(\mathfrak{n}_{m,3})$ in terms of m. Here we use Witt's dimension formula to compute both $dim(\mathfrak{n}_{m,3})$ and $dim H_2(\mathfrak{n}_{m,3}; K) \simeq dim \gamma_4/\gamma_5$. □

For $c \geq 4$ the situation is more complicated and we cannot expect to describe $\chi(\mathfrak{h})$ as nicely as in the cases above. The reason for this is that the expression of type $\mu(u) = 0$ for u of length at least 5 will not trivialize the defining relator (3.15) of $\mathcal{L}$, since the elements $[a_i, b_{j,k}]$ will not be central in general.

3.9 The ideal R

Let $\mathfrak{g}$ be free with generators x_1, x_2, x_3. Consider the presentation $L(\mathfrak{g}) = \langle X|S\rangle$ described in Section 3.6, where $X = \{a_i, b_{i,j}\}_{i,j}$. So the relators $s \in S$ are those defined in (3.15), (3.16) and (3.17). We will assume that relators which are already trivial (in the free Lie algebra with free basis X) are not elements of S. For instance, for a relator of type (3.15):

$$s = [a_i, a_j, b_{k,l}] - [a_i, b_{j,k}, a_l] - [a_i, a_k, b_{j,l}],$$

we assume that j, k and l are distinct indices.

For an even integer $n \geq 2$, consider the element

$$f_n = [b_{1,2}, a_2, \ldots, a_2, a_3] - [[a_1, a_2], a_2, \ldots, a_2, b_{2,3}], \tag{3.28}$$

where a_2 appears n times in each bracket. By applying the homomorphism ρ we deduce that each f_n lies in $R(\mathfrak{g})$ (recall that $R(\mathfrak{g}) = W(\mathfrak{g}) = ker(\rho)$, since $\mathfrak{g}$ is free).

Proposition 3.41. *The element f_n is non-trivial in $L(\mathfrak{g})$ for every even integer $n \geq 2$.*

Proof. Define $d : X \to \mathbb{N}$ by $d(a_i) = 1$ and $d(b_{i,j}) = 2$. Then clearly the set S is homogeneous with respect to the degree function (in the sense of Section 1.3.1) extending d.

Let S_m be the set of elements in S with degree at most m. Clearly it suffices to show that f_n does not lie in the ideal generated by S_m with $m = d(f_n) = n + 3$, because it cannot be a consequence of relators of degree higher than itself.

Consider the weight lexicographic ordering on the associative words with letters in X, where

$$b_{1,2} > b_{1,3} > b_{2,3} > a_1 > a_2 > a_3$$

and the degrees are defined by the function d.

The set S_m is finite and homogeneous, so we are in the situation described in the end of Section 1.3.1. Let $\widehat{S_m}$ be the resulting reduced set with the property that any composition between two of its elements either lies in $\widehat{S_m}$ or has degree greater than f_n. This set inherits the following property from S_m: if $s \in \widehat{S_m}$, then no monomial involved in the expression of s can have only a single ocurrence of $b_{1,2}$ and some occurrences of a_2 as letters. Indeed, if $g, h \in \widehat{S_m}$, then any monomial involved in a composition of g and h contains all the letters (counting multiplicity) of some monomial involved in g or h, and similarly with reduction. The fact that S_m actually has such property to begin with can be verified directly by inspection of (3.15), (3.16) and (3.17).

The same reasoning implies that no monomial involved in the expression of any $s \in \widehat{S_m}$ can have only a_2 and a single occurrence of a_3 as letters (that is, if all letters of the monomial are a_2 and a_3, then a_3 must appear at least twice).

Notice that $\overline{f}_n = b_{1,2}a_2^n a_3$, since the term $[b_{1,2}, a_2, \ldots, a_2, a_3]$ is the unique regular bracketing of the regular associative word $b_{1,2}a_2^n a_3$ and the other term of f_n does not involve $b_{1,2}$, which is the highest letter in the lexicographic ordering. Suppose that, for some $s \in \widehat{S_m}$, the associative word $\overline{s}$ is a subword $\overline{f_n} = b_{1,2}a_2^n a_3$. Clearly $\overline{s}$ cannot be of type a_2^k. It cannot be neither $\overline{s} = b_{1,2}a_2^k$ nor $\overline{s} = a_2^k a_3$ as well, by the previous paragraph.

The last thing we must check is that there is no element $s \in \widehat{S_m}$ with $\overline{s} = \overline{f}_n = b_{1,2}a_2^n a_3$. If such an element existed, it would not be an element of S_m, because all elements $s_0 \in S_m$ result in a word $\overline{s}_0$ of odd length. Thus s should be the result of some composition or some reduction. In any case in we conclude that there is some $g \in \widehat{S_m}$ of lower degree and a monomial u that has non-zero coefficient in the expression of g and such that all letters of u are letters of $\overline{f}_n$, with at most the same number of occurrences.

First notice that u must involve a_3, otherwise we would a have a monomial of some element of $\widehat{S_m}$ involving only $b_{1,2}$ and a_2, which we already argued that cannot happen. Similarly, it must involve $b_{1,2}$, otherwise u would a be a monomial with letters a_2 and a single occurrence of a_3.

So u involves *all* letters of $\overline{f}_n$. The only possibilities for $\overline{u}$ to be regular are $\overline{u} = b_{1,2}a_2^i a_3 a_2^j$ for some i, j with $i + j < n$. The bracketing of such a word is of the form

$$u = [\ldots[[b_{1,2}, a_2, \ldots, a_2, a_3], a_2], \ldots, a_2]$$

where a_2 appears i times to the left of a_3, and j times to the right. It follows that the monomial corresponding to $\overline{f}_n$ in s is obtained in the composition or reduction process by taking brackets of u with x_2 on the right or on the left. In any case the resulting monomial is up to sign the regular bracketing of the word $b_{1,2}a_2^i a_3 a_2^k$ for some $k > j$, so $u = b_{1,2}a_2^n a_3$ can never be achieved. Thus $\overline{f}_n$ cannot actually be the associative carrier of a monomial involved in some composition or reduction of elements of $\widehat{S_m}$.

By Theorem 1.29 it follows that f_n does not lie in the ideal generated by S_m, and consequently $f_n \notin \langle\langle S \rangle\rangle$. Thus f_n, as an element of $L(\mathfrak{g})$, is non-trivial. □

Theorem 3.42. *If $\mathfrak{f}$ is free non-abelian of rank at least 3, then $R(\mathfrak{g})$ is infinite dimensional.*

Proof. For a free Lie algebra $\mathfrak{g}$ of rank 3, Proposition 3.41 says that none of the f_n defined in (3.28) are trivial. Furthermore, they are all of different degree with respect to the function d defined in the proof of the proposition, so they make up an infinite linearly independent set inside $R(\mathfrak{g})$. In general, if $\mathfrak{f}$ is free of rank more than 3, then there is an epimorphism $\phi : \mathfrak{f} \to \mathfrak{g}$ and the induced homomorphism $\phi_* : \chi(\mathfrak{f}) \to \chi(\mathfrak{g})$ satisfies $\phi_*(R(\mathfrak{f})) = R(\mathfrak{g})$, thus $R(\mathfrak{f})$ is of infinite dimension as well. □

Corollary 3.43. *If $\mathfrak{g}$ is free non-abelian of rank at least 3, then $\chi(\mathfrak{g})$ is of infinite cohomological dimension.*

Proof. This is clear, since $\chi(\mathfrak{g})$ contains an abelian subalgebra of infinite dimension. □

Remark 3.44. *It is clear by the proofs that Theorem 3.42 and its corollary hold if we assume only that the free Lie algebra of rank 3 is a quotient of $\mathfrak{g}$.*

3.10 Remarks about the characteristic 2 case

We will show that the conclusion of part 2 of Lemma 3.1, which is essential for the development of the results of this chapter, fails almost always in characteristic 2. That is, if $\chi(\mathfrak{g})$ is defined in the same way as it was done for $char(K) \neq 2$, then the version of Lemma 3.1 does not hold in general.

Proposition 3.45. *Suppose that $char(K) = 2$. Then the ideal L is generated by $\{x - x^\psi \mid x \in \mathfrak{g}\}$ as a subalgebra if and only if $\mathfrak{g}$ is abelian and $\chi(\mathfrak{g}) \simeq \mathfrak{g} \oplus \mathfrak{g}$.*

Proof. The "if" direction is clear. Denote by A the subalgebra of $\chi(\mathfrak{g})$ generated by the elements $x - x^\psi$ and suppose $L = A$.

First notice that $\rho(A)$ is generated by the elements $\rho(x - x^\psi) = (x, 0, -x)$ for $x \in \mathfrak{g}$. In characteristic 2 the set of these elements is closed by the bracket, as well as sum and multiplication by scalar:

$$[(x, 0, -x), (y, 0, -y)] = ([x, y], 0, [x, y]) = ([x, y], 0, -[x, y]).$$

It follows that $\rho(A) = \{(x, 0, x) \mid x \in \mathfrak{g}\}$. Now let $x, y \in \mathfrak{g}$. Then $[x, y - y^\psi] \in L = A$. But

$$\rho([x, y - y^\psi]) = [(x, x, 0), (y, 0, -y)] = ([x, y], 0, 0) \in \rho(A),$$

therefore $[x, y] = 0$, that is, $\mathfrak{g}$ is abelian.

Now define $\sigma : \chi(\mathfrak{g}) \to \mathfrak{g}$ by $\sigma(x) = x$ and $\sigma(x^\psi) = 0$. It is clear that $\sigma(A) = \mathfrak{g}$. Notice that $[x, y^\psi] = [x^\psi, y]$ still holds. Then:

$$[x - x^\psi, y - y^\psi] = [x, y] - [x, y^\psi] - [x^\psi, y] + [x, y]^\psi = 2[x, y^\psi] = 0.$$

The inverse for $\sigma|_A$ is then well defined. Now for any $x, y \in \mathfrak{g}$, we have $[x, y^\psi] = [x, y - y^\psi] \in L = A$. But $\sigma([x, y^\psi]) = 0$, so $[x, y^\psi] = 0$. Thus $[\mathfrak{g}, \mathfrak{g}^\psi] = 0$ in $\chi(\mathfrak{g})$, as we wanted. □

4 Weak commutativity for pro-p groups

In this chapter we consider a version of the weak commutativity construction in the category of pro-p groups for a fixed prime p. In the first section we deduce some results about subdirect products of pro-p groups that we shall need in the sequence. The results of the other sections are analogous to those of Chapter 3. The work in this chapter is joint with D. H. Kochloukova and has been submitted for publication [47].

4.1 Pro-p fiber and subdirect products

We deduce here some results of subdirect products of pro-p groups which are analogous to those of Kochloukova and Martínez-Pérez [46] that we used in Chapter 3 in the case of Lie algebras. We are guided here by the work of Kuckuck [51], who discussed the case of discrete groups.

In his work, Kuckuck used the notion of weak FP_m type for discrete groups: H is *weak* FP_m if for every subgroup H_0 of finite index in H the homologies $H_i(H_0;\mathbb{Z})$ are finitely generated for all $i \leq m$. In the case of pro-p groups the analogous notion turns out to be equivalent with the usual type FP_m by Lemmas 1.32 and 1.33. For this reason, the methods of Kuckuck lead to stronger results for pro-p groups.

Lemma 4.1. *Let $N \to \Gamma \to Q$ be a short exact sequence of pro-p groups, where N is of type FP_{n-1} and Q is of type FP_{n+1}. Then Γ is of type FP_n if and only if $H_0(Q;H_n(N;\mathbb{F}_p))$ is finite.*

Proof. This is proved with the aid of the associated LHS spectral sequence

$$E^2_{i,j} = H_i(Q;H_j(N;\mathbb{F}_p)) \Rightarrow H_{i+j}(\Gamma;\mathbb{F}_p).$$

Notice that $E^2_{0,n} = H_0(Q;H_n(N;\mathbb{F}_p))$. The hypotheses on N and Q, together with Lemma 1.32, give that $E^2_{i,j}$ and thus $E^k_{i,j}$ and $E^\infty_{i,j}$ are finite whenever $i \leq n+1$ and $j < n$, for all k. Indeed, for any $j < n$, the $\mathbb{Z}_p[[Q]]$-module $H_j(N;\mathbb{F}_p)$ is finite, thus it admits a filtration where each quotient W_s is a simple $\mathbb{Z}_p[[Q]]$-module. Any simple $\mathbb{Z}_p[[Q]]$-module is isomorphic to the trivial $\mathbb{Z}_p[[Q]]$-module $\mathbb{F}_p$. Finally, since Q is of type FP_{n+1} we see that $H_i(Q;\mathbb{F}_p)$ is finite for $i \leq n+1$, so $H_i(Q;H_j(N;\mathbb{F}_p))$ itself is finite.

By looking at all differentials involving $E^k_{0,n}$, we see that $E^\infty_{0,n}$ is finite if and only if $E^2_{0,n}$ is. Indeed, all the differentials either take or assume values in a finite p-group for all k, by the previous paragraph. It is clear then that $H_k(\Gamma;\mathbb{F}_p)$ is finite for all $k \leq m+1$ if and only if $E^\infty_{i,j}$ is finite whenever $i+j \leq n+1$. We conclude by applying Lemma 1.32 to Γ. ∎

Let $p_1 : G_1 \to Q$ and $p_2 : G_2 \to Q$ be surjective homomorphisms of pro-p groups. The *fiber product* of p_1 and p_2 is the pro-p group

$$G = \{(g_1, g_2) \in G_1 \times G_2 \mid \ p_1(g_1) = p_2(g_2)\}.$$

The following theorem is a pro-p version of what has been called *the* $(n-1)-n-(n+1)$ *Theorem*.

Theorem 4.2. *Let $p_1 : G_1 \to Q$ and $p_2 : G_2 \to Q$ be surjective homomorphisms of pro-p groups. Suppose that $ker(p_1)$ is of type FP_{n-1}, both G_1 and G_2 are of type FP_n and Q is of type FP_{n+1}. Then the fiber product of p_1 and p_2 is of type FP_n.*

Proof. Denote by G the fiber product of p_1 and p_2 and consider the short exact sequence

$$ker(p_1) \to G \to G_2$$

where $G \to G_2$ is the projection onto the second coordinate. The corresponding LHS spectral sequence is

$$E^2_{i,j} = H_i(G_2; H_j(ker(p_1); \mathbb{F}_p)) \Rightarrow H_{i+j}(G; \mathbb{F}_p).$$

By hypothesis $ker(p_1)$ is of type FP_{n-1} and G_2 is of type FP_n, so we can argue as in Lemma 4.1 to deduce that $E^2_{i,j}$ is finite for $i \leq n$ and $j \leq n-1$.

Notice also that

$$E^2_{0,n} = H_0(G_2; H_n(ker(p_1); \mathbb{F}_p)) = H_0(Q; H_n(ker(p_1); \mathbb{F}_p)),$$

since both Q and G_2 act on $H_n(ker(p_1); \mathbb{F}_p)$ via conjugation by elements G_1 (that is, both modules above are the module of coinvariants of the natural action of G_1 on $H_n(ker(p_1); \mathbb{F}_p)$). By applying Lemma 4.1 to the exact sequence $ker(p_1) \to G_1 \to Q$ we deduce that $E^2_{0,n}$ is also finite. By the convergence of the spectral sequence, we get that $H_k(G; \mathbb{F}_p)$ is finite for $k \leq n$, so G is of type FP_n by Lemma 1.32. □

We record the following particular case in order to use it in the sequence of the text.

Corollary 4.3. *Let H be a finitely presented pro-p group and let*

$$S = \overline{\langle (h, h^{-1}) \mid h \in H \rangle} = \{(h_1, h_2) \in H \times H \mid h_1 h_2 \in H'\} \subset H \times H.$$

Suppose that H' is finitely generated as a pro-p group. Then S is a finitely presented pro-p group.

Proof. Let $p_1 : H \to Q = H/H'$ be the canonical projection and $p_2 = \sigma \circ p_1$, where $\sigma : Q \to Q$ is the antipodal homomorphism sending g to g^{-1}.Then S is the fiber product of the maps p_1 and p_2. It follows from Theorem 4.2 that S is of type FP_2, which is the same as finite presentability. □

Let $G_1, \dots, G_n$ be pro-p groups for some $n \geq 2$. Denote by

$$\pi_i : G_1 \times \cdots \times G_n \to G_i$$

the canonical projections for $1 \leq i \leq n$. A *subdirect product* of $G_1, \dots G_n$ is a closed subgroup $P \leq G_1 \times \cdots \times G_n$ such that $\pi_i(P) = G_i$ for all $1 \leq i \leq n$.

Let $I = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, n\}$ with exactly k elements. Denote by π_I the projection

$$\pi_I : G_1 \times \cdots \times G_n \to G_{i_1} \times \cdots \times G_{i_k}.$$

We say that a subgroup $P \leq G_1 \times \cdots \times G_n$ is *virtually k-surjective* if $p_I(P)$ is of finite index in $G_{i_1} \times \cdots \times G_{i_k}$ for all possible I.

Lemma 4.4. *Suppose that $P \leq G_1 \times \cdots \times G_n$ is a virtually k-surjective subdirect product for some $k \geq 2$. Denote by N_i the intersection $P \cap G_i$, where G_i is identified with its image in the direct product $G_1 \times \cdots \times G_n$. Then G_i/N_i is virtually nilpotent for all i.*

Proof. First notice that N_i is indeed normal in G_i, since for any $x \in N_i$ and $g \in G_i$, we can find some $\gamma \in P$ such that $\pi_i(\gamma) = g$, and in this case $x^g = x^\gamma$.

By symmetry it is enough to prove the lemma for $i = 1$. Let $I_1, \dots, I_m$ be all the subsets of $\{1, \dots, n\}$ with exactly k elements and $1 \in I_j$. By hypothesis the following subgroup has finite index in G_1:

$$S_1 := \bigcap_{j=1}^{m} (\pi_{I_j}(P) \cap G_1) \leq G_1.$$

Notice that $N_1 \subseteq S_1$, since $1 \in I_j$ for all j.

Let $g_1, \dots, g_m \in S_1$. Choose $\gamma_1, \dots, \gamma_m \in P$ such that $\pi_1(\gamma_j) = g_j$ and $\pi_i(\gamma_j) = 1$ for all $i \in I_j \setminus \{1\}$. This is possible by the definition of S_1. Then

$$[g_1, [g_2, \cdots [g_{m-1}, g_m] \cdots]] = \pi_1([\gamma_1, [\gamma_2 \cdots [\gamma_{m-1}, \gamma_m] \cdots]]).$$

Let $c = [\gamma_1, [\gamma_2 \cdots [\gamma_{m-1}, \gamma_m] \cdots]]$. Notice that $\pi_i(c) = 1$ for all $i \neq 1$, since any such i lies in some I_j, thus $\pi_i(\gamma_j) = 1$. Thus $c = \pi_1(c) \in G_1 \cap P = N_1$. This proves that S_1/N_1 is nilpotent of class at most $m - 1$. □

Lemma 4.5. *Let $G_1, \dots, G_n$ be finitely generated pro-p groups for some $n \geq 1$. Let $P \leq G_1 \times \cdots \times G_n$ be a subgroup such that $\pi_i(P)$ has finite index in G_i and $\pi_i(P)/(P \cap G_i)$ is virtually nilpotent for all i. Then P is finitely generated.*

Proof. By substituting G_i with $\pi_i(P)$ for all i, we can assume that P is actually a subdirect product of $G_1 \times \cdots \times G_n$.

Let $T = \pi_{\{1,\dots,n-1\}}(P) \subseteq G_1 \times \cdots \times G_{n-1}$. Notice that now P is subdirect product of $T \times G_n$. In fact, it is the fiber product of the homomorphisms $p_1 : T \to Q$ and $p_2 : G_n \to Q$ for

$$Q := G_n/N_n \simeq P/(N' \times N_n) \simeq T/N',$$

where $N_n = P \cap G_n$ and $N' = P \cap T$.

We will prove the lemma by induction on n. For $n = 1$ there is nothing to do. For $n > 1$, the last paragraph gives a decomposition of P as a fiber product of $p_1 : T \to Q$ and $p_2 : G_n \to Q$, where T is a subdirect product of $G_1 \times \cdots \times G_{n-1}$. We can apply the induction hypothesis to T, since $G_i/(T \cap G_i)$, being a quotient of the virtually nilpotent group $G_i/(P \cap G_i)$, is virtually nilpotent. Thus T is finitely generated.

Now, $Q = G_n/(P \cap G_n)$ is finitely generated and virtually nilpotent, thus it is finitely presented and the kernel $N_n = P \cap G_n = ker(p_2)$ is finitely generated as a normal subgroup of G_n. It clear that N_n must be finitely generated also as a normal subgroup of P. Thus P, which fits into the short exact sequence $N_n \rightarrowtail P \twoheadrightarrow T$, is finitely generated. □

The result that we will really need is the following corollary. This is Corollary 5.5 in [51] in the case of discrete groups (keeping in mind that here we do not distinguish between "weak FP_m" and FP_m, as discussed in the beginning of this section).

Corollary 4.6. *Let $G_1, \dots, G_n$ be pro-p groups of type FP_k for some $n \geq 1$. Suppose that $P \leq G_1 \times \cdots \times G_n$ is a virtually k-surjective subgroup for some $k \geq 2$. Then P is of type FP_k.*

Proof. Again we can assume that P is a subdirect product of $G_1 \times \cdots \times G_n$. We will prove the result by induction on $k + n$. The base cases are covered by Lemma 4.5 (using also Lemma 4.4).

If $k \geq n$, then P is a subgroup of finite index in $G_1 \times \cdots \times G_n$ and there is nothing to do. Let $1 < k < n$. As in the proof of Lemma 4.5, we can write P as a fiber product of $p_1 : T \to Q$ and $p_2 : G_n \to Q$, where T is a subdirect product of $G_1 \times \cdots \times G_{n-1}$. In this case, T is again virtually k-surjective, but the number of factors in the direct product has decreased. By induction T is of type FP_k.

On the other hand, it is not hard to see that $N' = P \cap T$ is a subgroup of $G_1 \times \cdots \times G_{n-1}$ that is virtually $(k-1)$-surjective. To see this, it suffices to consider for any subset $I \subseteq \{1, \dots, n-1\}$ with exactly $k - 1$ elements the k-surjectivity of P with respect to $I \cup \{n\}$. Thus we can also assume by induction that N' is of type FP_{k-1}. Finally, $Q = G_n/N_n$ is finitely generated and virtually nilpotent (by Lemma 4.4), so it is of type FP_m for all m. Thus Theorem 4.2 implies that P is of type FP_k. □

4.2 Weak commutativity: finite presentability and completions

Let G be a pro-p group. As in the cases of discrete groups and Lie algebras, we define the weak commutativity construction $\mathfrak{X}_p(G)$ by the pro-p presentation

$$\mathfrak{X}_p(G) = \langle G, G^\psi \mid [g, g^\psi] = 1 \text{ for all } g \in G \rangle_p,$$

where G^ψ is an isomorphic copy of G via $g \mapsto g^\psi$.

We use the subscript p to distinguish the pro-p construction $\mathfrak{X}_p(-)$ from the discrete construction $\mathfrak{X}(-)$, originally defined by Sidki. Recall that for a discrete group H, we have by definition

$$\mathfrak{X}(H) = \langle H, H^\psi \mid [h, h^\psi] = 1 \text{ for all } h \in H \rangle,$$

where H^ψ is an isomorphic copy of H via $h \mapsto h^\psi$, and here $\langle - | - \rangle$ denotes a presentation by generators and relators in the category of discrete groups.

The distinction between the two constructions could be necessary when they can be considered at the same time for a single object, that is, for finite p-groups. We show in the following lemma that actually the constructions coincide in this case.

Lemma 4.7. *For a finite p-group P we have $\mathfrak{X}(P) \simeq \mathfrak{X}_p(P)$.*

Proof. The construction $\mathfrak{X}(G)$ for a discrete group G is characterized by the following property: for any discrete group Γ and two homomorphisms $\sigma, \tau : G \to \Gamma$ such that $[\sigma(g), \tau(g)] = 1$ for all $g \in G$, there exists a unique homomorphism $\varphi : \mathfrak{X}(G) \to \Gamma$ such that $\varphi|_G = \sigma$ and $\psi \circ (\varphi|_{G^\psi}) = \tau$. The pro-$p$ construction has the analogous property where "discrete" is substituted with "pro-p", and "homomorphism" with "continuous homomorphism". This, together with the fact that $\mathfrak{X}(P)$ is a finite p-group ([75, Thm. C]), allows us to construct the obvious maps between $\mathfrak{X}(P)$ and $\mathfrak{X}_p(P)$, in both directions, whose compositions are the identity maps and whose restriction on $P \cup P^\psi$ is the identity map. □

In principle a pro-p group is determined by all its finite quotients. The weak commutativity pro-p group $\mathfrak{X}_p(G)$ is actually determined by its finite quotients of type $\mathfrak{X}(Q)$, where Q is a finite quotient of G.

Lemma 4.8. *Let G be a pro-p group. Then $\mathfrak{X}_p(G) \simeq \varprojlim \mathfrak{X}(G/U)$, where the inverse limit runs over all finite quotients G/U of G.*

Proof. Let U be a closed normal subgroup of G such that G/U is a finite p-group. Then the epimorphism $G \to G/U$ induces an epimorphism $\mathfrak{X}_p(G) \to \mathfrak{X}_p(G/U)$ and we identify $\mathfrak{X}_p(G/U)$ with $\mathfrak{X}(G/U)$. This induces an epimorphism

$$\mathfrak{X}_p(G) \to \varprojlim \mathfrak{X}(G/U).$$

To show that this map is an isomorphism it suffices to show that every finite p-group V that is a quotient of $\mathfrak{X}_p(G)$ is a quotient of suitable $\mathfrak{X}(G/U)$. Let $\mu : \mathfrak{X}_p(G) \to V$ be the quotient map and $U_1 = ker(\mu) \cap G$ and $U_2 = ker(\mu)^\psi \cap G$. Set $U = U_1 \cap U_2$. Notice that that since V is a finite p-group, the groups G/U_1, G/U_2 and thus G/U are finite p-groups. By construction there is an epimorphism $\mathfrak{X}_p(G/U) \to V$. Finally, $\mathfrak{X}_p(G/U) \simeq \mathfrak{X}(G/U)$ by Lemma 4.7, which completes the proof. □

Recall that for a discrete group H, we denote by $\widehat{H}$ its pro-p completion.

Proposition 4.9. *For any discrete group H, we have*

$$\mathfrak{X}_p(\widehat{H}) \simeq \widehat{\mathfrak{X}(H)}.$$

Proof. Let $H = \langle X \mid R \rangle$ be a presentation of H as a discrete group. Notice that $\mathfrak{X}(H)$ is the discrete group generated by $X \cup X^\psi$, with $\{[h, h^\psi]|h \in H\} \cup R \cup R^\psi$ as a set defining relators. Clearly $\widehat{\mathfrak{X}(H)}$ is the group with this same presentation in the category of pro-p groups, that is:

$$\widehat{\mathfrak{X}(H)} = \langle X, X^\psi \mid R, R^\psi, [h, h^\psi] \text{ for } h \in i(H) \subset \widehat{H} \rangle_p,$$

where $i : H \to \widehat{H}$ is the canonical map. The group $\mathfrak{X}_p(\widehat{H})$, on the other hand, has by definition a presentation with the same set of generators, but with defining relators $[h, h^\psi]$ for all $h \in \widehat{H}$ (rather that only $h \in i(H)$). Thus there is a continuous epimorphism $\widehat{\mathfrak{X}(H)} \twoheadrightarrow \mathfrak{X}_p(\widehat{H})$.

To show that this is an isomorphism it suffices to show that any finite p-group that is a quotient of $\widehat{\mathfrak{X}(H)}$ is also a quotient of $\mathfrak{X}_p(\widehat{H})$. Let V be such a quotient. Notice that the composite $\mu = \pi \circ j$ of the canonical homomorphisms $\pi : \widehat{\mathfrak{X}(H)} \twoheadrightarrow V$ and $j : \mathfrak{X}(H) \to \widehat{\mathfrak{X}(H)}$ is also surjective, since the image of j is dense in $\widehat{\mathfrak{X}(H)}$.

Let $U_1 = ker(\mu) \cap H$, $U_2 = ker(\mu)^\psi \cap H$ and $U = U_1 \cap U_2$. Define $P = H/U$. Then clearly P is a finite p-group and $\mathfrak{X}(P) \simeq \mathfrak{X}_p(P)$ is quotient of $\mathfrak{X}_p(\widehat{H})$. But also V is a quotient of $\mathfrak{X}(P)$ by choice, which gives the result. □

Corollary 4.10. *If G is a finitely presented pro-p group, then so is $\mathfrak{X}_p(G)$.*

Proof. Let F be a finitely generated free discrete group and let $\widehat{F}$ be its pro-p completion. We can see $\widehat{F}$ as the free pro-p group with the same free basis as F. By Proposition 4.9 $\mathfrak{X}_p(\widehat{F})$ is the pro-p completion of $\mathfrak{X}(F)$, which is finitely presented as a discrete group by Theorem A in [24]. Thus $\mathfrak{X}_p(\widehat{F})$ also admits a finite presentation (as a pro-p group).

In general, if $G \simeq \widehat{F}/R$, where $R \leq \widehat{F}$ is the normal closure of the closed subgroup generated by a finite set $\{r_1, \ldots, r_n\} \subset R$, then $\mathfrak{X}_p(G)$ is the quotient of $\mathfrak{X}_p(\widehat{F})$ by the normal closure of the closed subgroup generated by $\{r_1, \ldots, r_n, r_1^\psi, \ldots, r_n^\psi\}$. Thus $\mathfrak{X}_p(G)$ is also finitely presented as a pro-p group. □

In the case of discrete groups, it was shown by Gupta, Rocco and Sidki [37] that if H is a nilpotent and finitely generated group, then $\mathfrak{X}(H)$ is also nilpotent. Their proof involves long commutator calculations. We give here a pro-p version of this result as a corollary of the fact that it holds for discrete groups.

Proposition 4.11. *Let G be a finitely generated nilpotent pro-p group. Then $\mathfrak{X}_p(G)$ is a nilpotent pro-p group.*

Proof. Let X be a finite generating set of G as a pro-p group and let H be the discrete subgroup of G generated by X. Clearly the closure $\overline{H}$ of H is G. It follows that there is an epimorphism of pro-p groups $\widehat{H} \to \overline{H} = G$, which in turn induces an epimorphism of pro-p groups $\mathfrak{X}_p(\widehat{H}) \to \mathfrak{X}_p(G)$. Since G is nilpotent, H is nilpotent and hence $\mathfrak{X}(H)$ is nilpotent too. Then $\mathfrak{X}_p(\widehat{H}) \simeq \widehat{\mathfrak{X}(H)}$ and its quotient $\mathfrak{X}_p(G)$ are nilpotent pro-p groups. □

4.3 Some structural results

Recall that in the discrete case $L(H) \leq \mathfrak{X}(H)$ denotes the subgroup generated by the elements $h^{-1}h^{\psi}$ for all $h \in H$. It can also be described as the kernel of the homomorphism $\alpha : \mathfrak{X}(H) \to H$ defined by $\alpha(h) = \alpha(h^{\psi}) = h$ for all $h \in H$.

For a pro-p group G, let $\alpha : \mathfrak{X}_p(G) \to G$ be the homomorphism of pro-p groups defined by $\alpha(g) = \alpha(g^{\psi}) = g$ for all $g \in G$. We define

$$L_p = L_p(G) := ker(\alpha) \subset \mathfrak{X}_p(G),$$

in analogy with the discrete case.

Lemma 4.12. *Let G be a pro-p group. Then $L_p(G)$ is the closed subgroup generated by the elements $g^{-1}g^{\psi}$, for all $g \in G$.*

Proof. Denote by $A(G)$ the closed subgroup of $\mathfrak{X}_p(G)$ generated by the elements $g^{-1}g^{\psi}$ for $g \in G$. Note that $A(Q) = L_p(Q)$ for a finite p-group Q. Clearly $A(G) \subseteq ker(\alpha)$. Let $\pi_0 : G \twoheadrightarrow Q$ be a continuous epimorphism onto a finite p-group Q. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{X}_p(G) & \xrightarrow{\pi} & \mathfrak{X}(Q) \\ \alpha \downarrow & & \downarrow \alpha_Q \\ G & \xrightarrow[\pi_0]{} & Q \end{array}$$

where π is induced by π_0. If follows that $\pi(L_p(G)) = ker(\alpha_Q) = L_p(Q) = A(Q) = \pi(A(G))$.

Thus $L_p(G)$ and $A(G)$ cannot be distinguished by the epimorphisms $\pi : \mathfrak{X}_p(G) \to \mathfrak{X}(Q)$ with Q a finite quotient of G. Since $\mathfrak{X}_p(G)$ is actually the inverse limit of the system of images of all these epimorphisms (Lemma 4.8), it follows that $L_p(G) = A(G)$. □

Proposition 4.13. *If G is a finitely generated pro-p group, then $L_p(G)$ is a finitely generated pro-p group.*

Proof. Let F be a finitely generated free discrete group and let $\widehat{F}$ be its pro-p completion, that is, the free pro-p group with the same free basis. We already know that $\mathfrak{X}_p(\widehat{F}) \simeq \widehat{\mathfrak{X}(F)}$. By [24, Prop. 2.3] we know that $L(F) \subseteq \mathfrak{X}(F)$ is a finitely generated discrete group. If $Y \subseteq L(F)$ is a finite generating set, then its image $i(Y)$ in $\mathfrak{X}_p(\widehat{F})$, where $i : \mathfrak{X}(F) \to \widehat{\mathfrak{X}(F)} \simeq \mathfrak{X}_p(\widehat{F})$ is the canonical map, is a generating set for $L_p(\widehat{F})$ as a pro-p group. Indeed, it suffices to verify that the image of $i(Y)$ in each quotient of $\widehat{\mathfrak{X}(F)}$ of the type $\mathfrak{X}(P)$, where P is a finite p-group, is a generating set for $L(P)$. This follows from the fact that $L(P)$ is the image of $L(F)$ by the homomorphism $\mathfrak{X}(F) \twoheadrightarrow \mathfrak{X}(P)$ induced by the projection $F \twoheadrightarrow P$.

Thus $L_p(\widehat{F})$ is a finitely generated pro-p group. In general, if G is a quotient of $\widehat{F}$, then $L_p(G)$ is a quotient of $L_p(\widehat{F})$, thus it is finitely generated as a pro-p group too. □

Similarly, we define

$$D_p = D_p(G) := ker(\beta) \subset \mathfrak{X}_p(G),$$

where $\beta : \mathfrak{X}_p(G) \to G \times G$ is the homomorphism of pro-p groups defined by $\beta(g) = (g, 1)$ and $\beta(g^\psi) = (1, g)$ for all $g \in G$. Arguing as in Lemma 4.12, we obtain:

Lemma 4.14. *Let G be a pro-p group. Then $D_p(G)$ is generated by $[g, h^\psi]$, for $g, h \in G$, as normal closed subgroup of $\mathfrak{X}_p(G)$.*

Proposition 4.15. *Let G be a pro-p group. Then $[L_p(G), D_p(G)] = 1$.*

Proof. As in [75, Lemma 4.1.6], for any $g_1, g_2, g_3 \in G$, the relation

$$[g_1^{-1} g_1^\psi, [g_2, g_3^\psi]] = 1$$

can be obtained as a consequence of the defining relations $[h, h^\psi] = 1$ for h runs through the set of products of two or three elements in $\{g_1, g_2, g_2\}$. Thus the (topological) generators of L_p commute with the (topological) generators of D_p as a normal subgroup of $\mathfrak{X}_p(G)$, so $[L_p, D_p] = 1$. □

Finally, let

$$\rho_p : \mathfrak{X}_p(G) \to G \times G \times G$$

be the homomorphism of pro-p groups defined by $\rho_p(g) = (g, g, 1)$ and $\rho_p(g^\psi) = (1, g, g)$.

We give another description of ρ_p. Denote by $\beta_1, \beta_2 : \mathfrak{X}_p(G) \to G$ be components of β, that is, β_1 and β_2 are the unique maps such that $\beta(z) = (\beta_1(z), \beta_2(z)) \in G \times G$ for all $z \in \mathfrak{X}_p(G)$. Then

$$\rho_p(z) = (\beta_1(z), \alpha(z), \beta_2(z)) \in G \times G \times G$$

for all $z \in \mathfrak{X}_p(G)$. In particular $ker(\rho_p) = D_p(G) \cap L_p(G)$.

We set

$$W_p = W_p(G) := ker(\rho_p) = D_p(G) \cap L_p(G).$$

This is a normal subgroup of $\mathfrak{X}_p(G)$ which is central in $D_p(G)L_p(G)$ by Proposition 4.15. In particular, $W_p(G)$ is abelian. Furthermore, analogously to the Lie algebra case, we can show that the image of ρ_p can be written as

$$Im(\rho_p) = \{(g_1, g_2, g_3) \in G \times G \times G \mid g_1 g_2^{-1} g_3 \in \overline{[G, G]}\}. \tag{4.1}$$

Proposition 4.16. *If G is solvable, then so is $\mathfrak{X}_p(G)$.*

Proof. This is clear, since W_p is abelian and $\mathfrak{X}_p(G)/W_p$ is a subgroup of $G \times G \times G$, therefore is solvable. □

It is clear by (4.1) that $Im(\rho_p)$ is a subdirect product of $G \times G \times G$, that is, it maps surjectively on each copy of G in the direct product.

Corollary 4.17. *If G is a finitely presented pro-p group, then $\mathfrak{X}_p(G)/W_p(G) \simeq Im(\rho_p)$ is a finitely presented pro-p group.*

Proof. Recall that a pro-p group is finitely presented if and only if it is FP_2. It is clear by (4.1) that

$$\pi_{1,2}(Im(\rho_p)) = \pi_{1,3}(Im(\rho_p)) = \pi_{2,3}(Im(\rho_p)) = G \times G.$$

Thus Corollary 4.6 applies for $k = 2$. □

Proposition 4.18. *If G is a finitely presented pro-p group such that $\overline{[G, G]}$ is a finitely generated pro-p group, then $W_p(G)$ is a finitely generated pro-p group.*

Proof. Consider the beginning of the 5-term exact sequence associated to the LHS spectral sequence arising from the central extension of pro-p groups $W_p \rightarrowtail L_p \twoheadrightarrow \rho_p(L_p)$:

$$H_2(\rho_p(L_p); \mathbb{Z}_p) \to W_p \to H_1(L_p; \mathbb{Z}_p) \to H_1(\rho_p(L_p); \mathbb{Z}_p) \to 0.$$

Clearly W_p is finitely generated as a pro-p group if $H_2(\rho_p(L_p); \mathbb{Z}_p)$ and $H_1(L_p; \mathbb{Z}_p) \simeq L_p/\overline{[L_p, L_p]}$ are finitely generated.

Note that by Proposition 4.13 L_p is a finitely generated pro-p group, thus so is $H_1(L_p; \mathbb{Z}_p)$. It remains to prove that $H_2(\rho_p(L_p); \mathbb{Z}_p)$ is a finitely generated pro-p group. Now

$$\rho_p(L_p) \simeq \overline{\langle \{(g^{-1}, 1, g) \mid g \in G\} \rangle} \simeq \{(g_1, g_2) \in G \times G \mid g_1 g_2 \in \overline{[G, G]}\}.$$

This is exactly the group S considered in Corollary 4.3. So $\rho_p(L_p)$ is finitely presented and thus $H_2(\rho_p(L_p); \mathbb{Z}_p)$ is finitely generated, as we wanted. □

4.4 The Schur multiplier

4.4.1 The non-abelian exterior square

In [65] Moravec defined the *non-abelian tensor square* $G \hat{\otimes} G$ of a pro-p group G as the pro-p group generated (topologically) by the symbols $g \hat{\otimes} h$, for $g, h \in G$, subject to the defining relations

$$(g_1 g) \hat{\otimes} h = (g_1^g \hat{\otimes} h^g)(g \hat{\otimes} h) \tag{4.2}$$

and

$$g \hat{\otimes} (h_1 h) = (g \hat{\otimes} h)(g^h \hat{\otimes} h_1^h) \tag{4.3}$$

for all $g, g_1, h, h_1 \in G$. Furthermore the *non-abelian exterior square* $G \hat{\wedge} G$ of G is

$$G \hat{\wedge} G = G \hat{\otimes} G / \Delta(G),$$

where $\Delta(G)$ is the normal (closed) subgroup generated by $g \hat{\otimes} g$, for all $g \in G$. We denote by $g \hat{\wedge} h$ the image of $g \hat{\otimes} h$ in $G \hat{\wedge} G$.

Let

$$\mu_G : G \hat{\wedge} G \to \overline{[G, G]}$$

be the homomorphism of pro-p groups defined by

$$\mu_G(g \hat{\wedge} h) = [g, h] \text{ for all } g, h \in G.$$

Proposition 4.19 ([64]; [65], Proposition 2.2)**.** *For any pro-p group G we have*

$$H_2(G; \mathbb{Z}_p) \simeq ker(\mu_G).$$

4.4.2 The subgroup $R_p(G)$

Consider the pro-p group

$$R_p = R_p(G) = \overline{[G, [L_p(G), G^\psi]]} \subseteq \mathfrak{X}_p(G).$$

As in the discrete case, it is immediate that $R_p(G) \subseteq W_p(G)$, thus it is an abelian subgroup of $\mathfrak{X}_p(G)$. Also, we can show that R_p is actually normal in $\mathfrak{X}_p(G)$, as a consequence of the fact that D_p and L_p commute.

Lemma 4.20. *Let G be a pro-p group and let $g, h, k \in G$. Then:*

1. $[g, h^\psi] = [g^\psi, h]$ *in* $\mathfrak{X}_p(G)$;
2. $[[g, h^\psi], k]R_p = [[g, h], k^\psi]R_p$ *in* $\mathfrak{X}_p(G)/R_p(G)$.

Proof. Item 1 is is proved in the discrete case ([75, Lemma 4.1.6]) using the commutator formulas

$$[ab, c] = [a, c]^b.[b, c] \text{ and } [a, bc] = [a, c].[a, b]^c. \tag{4.4}$$

The same proof applies here.

By applying item 1 we obtain

$$[[g, h], k^\psi] = [[g^\psi, h^\psi], k] = [[g(g^{-1}g^\psi), h^\psi], k] = [[g, h^\psi]^{g^{-1}g^\psi}.[g^{-1}g^\psi, h^\psi], k] =: \alpha.$$

Also, by Lemma 4.15, we have $[[g, h^\psi], g^{-1}g^\psi] = 1$ and $[[[g, h^\psi], k], [g^{-1}g^\psi, h^\psi]] = 1$, so

$$\alpha = [[g, h^\psi][g^{-1}g^\psi, h^\psi], k] = [[g, h^\psi], k]^{[g^{-1}g^\psi, h^\psi]}.[[g^{-1}g^\psi, h^\psi], k],$$

which proves that $\alpha \in [[g, h^\psi], k]R_p$. □

Proposition 4.21. *There is an epimorphism of pro-p groups $H_2(G; \mathbb{Z}_p) \twoheadrightarrow W_p/R_p$.*

Proof. Consider the homomorphism of pro-p groups

$$\phi : G\widehat{\wedge}G \to \mathfrak{X}_p(G)/R_p$$

defined by $\phi(g\hat{\wedge}h) = [g, h^\psi]R_p$ for $g, h \in G$. The fact that ϕ is well-defined follows from (4.2), (4.3) and (4.4). Recall that $\alpha : \mathfrak{X}_p(G) \to G$ is the homomorphism defined by $\alpha(x) = \alpha(x^\psi) = x$ and $L_p = ker(\alpha)$. Now, consider the commutative diagram:

$$\begin{array}{ccc} G\widehat{\wedge}G & \xrightarrow{\mu_G} & \overline{[G,G]} \\ \phi\big\downarrow & & \big\downarrow inc \\ \mathfrak{X}_p(G)/R_p & \xrightarrow[\overline{\alpha}]{} & G \end{array}$$

where $inc : \overline{[G, G]} \to G$ is the inclusion map and the homomorphism $\overline{\alpha}$ is induced by α. It follows that $\phi(ker(\mu_G)) \subseteq ker(\overline{\alpha}) = ker(\alpha)/R_p = L_p/R_p$. Since ϕ clearly takes values in D_p/R_p, we have $\phi(ker(\mu_G)) \subseteq (L_p \cap D_p)/R_p = W_p/R_p$. Thus ϕ induces

$$\bar{\phi} : H_2(G; \mathbb{Z}_p) \simeq ker(\mu_G) \to W_p/R_p. \tag{4.5}$$

To see that $\bar{\phi}$ is surjective, we only need to observe that $Im(\phi)$ generates D_p modulo R_p. Indeed, D_p is generated *as a normal closed subgroup* by the elements $[g, h^\psi]$, with $g, h \in G$, but modulo R_p we have

$$[g, h^\psi]^k \equiv [g, h^\psi][[g, h], k^\psi] \text{ for all } g, h, k \in G.$$

This follows from Lemma 4.20 and $[g, h^\psi]^k = [g, h^\psi][[g, h^\psi], k]$.

We conclude that the image of $\{[g, h^\psi]; g, h \in G\}$ generates (topologically) the quotient D_p/R_p. Thus for any $w \in W_p$, there is some $\xi \in G \widehat{\wedge} G$ such that $\phi(\xi) = wR_p \subseteq W_p/R_p \subseteq L_p/R_p = ker(\overline{\alpha})$ and, by the commutativity of the diagram, $\xi \in ker(\mu_G)$. □

Theorem 4.22. *For any pro-p group G we have $H_2(G; \mathbb{Z}_p) \simeq W_p(G)/R_p(G)$.*

Proof. In order to construct an inverse map $W_p/R_p \twoheadrightarrow H_2(G; \mathbb{Z}_p)$ we can proceed as in Section 3.4.2, where we studied the case of Lie algebras. An outline of the procedure is as follows. We start by building a stem extension of pro-p groups

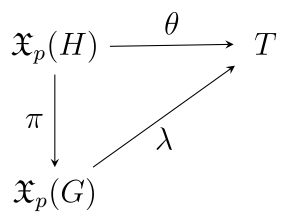
$$1 \to H_2(G; \mathbb{Z}_p) =: M \hookrightarrow H \to G \to 1. \tag{4.6}$$

This can be done by considering a free presentation $G = F/N$ and taking $H = F/A$, where A is a pro-p subgroup of N that contains $\overline{[F, N]}$ such that $A/\overline{[F, N]}$ is a complement of $H_2(G; \mathbb{Z}_p) \simeq (\overline{[F, F]} \cap N)/\overline{[F, N]}$ inside $N/\overline{[F, N]}$. This complement exists because $N/(N \cap \overline{[F, F]}) \subseteq F/\overline{[F, F]}$ is free abelian pro-p and $N/\overline{[F, N]}$ is a pro-p abelian group. We are using implicitly that subgroups of free abelian pro-p groups are again free abelian, which follows for instance from [69, Thm. 4.3.4].

Consider the map $\rho_p^H : \mathfrak{X}_p(H) \to H \times H \times H$. The superscript H is used to distinguish from G, the group in the statement of the theorem. By composing it with the projection onto the quotient $T = Im(\rho_p^H)/B$, where B is defined by

$$B = \{(x, xy, y) \mid x, y \in M\},$$

we obtain a map θ that factors through $\mathfrak{X}_p(G)$:



where π is induced by the epimorphism $\gamma : H \twoheadrightarrow G$ with kernel M. It is not hard then to verify $R_p(G) \subseteq ker(\lambda)$ and

$$\lambda(W_p(G)) = \{(1, m, 1)B \mid m \in M\} \simeq M.$$

Indeed

1) $\rho_p^H(R_p(H)) \subseteq \rho_p^H(W_p(H)) = 1$, hence $\theta(R_p(H)) = 1$. And since $\pi(R_p(H)) = R_p(G)$ we conclude that $\lambda(R_p(G)) = \theta(R_p(H)) = 1$.

2) Let $q : T \to Im(\rho_p^G)$ be the quotient map. Notice that $\rho_p^G = q \circ \lambda$. If $w \in W_p(G)$, then clearly $q \circ \lambda(w) = \rho_p^G(w) = 1$, thus $\lambda(w) \in ker(p)$. It follows that $\lambda(w) = (x, y, z)B$ for some $x, y, z \in M$, and thus $\lambda = (0, m, 0)B$ for some $m \in M$ by the definition of B. Finally, since λ is surjective, we have

$$\lambda(W_p(G)) = \{(1, m, 1)B \mid m \in M\} \simeq M,$$

as we wanted.

Thus λ induces a map

$$\bar{\lambda} : W_p(G)/R_p(G) \to M \simeq H_2(G; \mathbb{Z}_p).$$

If we realize both the homomorphism (4.5) and the stem extension (4.6) by means of the Hopf formula for a fixed free presentation $G = F/N$, it is not hard to see that we have actually constructed maps that are inverse to each other. □

Note that Theorem 4.22 is consistent with the discrete case: if G is a finite p-group, then the isomorphism $\mathfrak{X}_p(G) \to \mathfrak{X}(G)$ identifies $W_p(G)$ with $W(G)$ and $R_p(G)$ with $R(G)$, but also $H_2(G; \mathbb{Z}_p) \simeq H_2(G; \mathbb{Z})$ (the first homology is continuous, the second discrete).

4.5 Pro-p analytic groups

Recall that for a pro-p group G, we denote by $d(G)$ the cardinality of a minimal (topological) generating set. The *rank* of G is

$$rk(G) = sup\{d(H) \mid H \text{ is a closed subgroup of } G\},$$

and G is *p-adic analytic* if and only if $rk(G) < \infty$. From now on we write analytic for p-adic analytic.

Recall also that a *procyclic pro-p group* is an inverse limit of finite cyclic p-groups. A pro-p group is *poly-procyclic* if it can be build as an (iterated) extension of procyclic pro-p groups.

Proposition 4.23. *Let G be a pro-p group. Then $\mathfrak{X}_p(G)$ is analytic (resp. poly-procyclic) if and only if G is analytic (resp. poly-procyclic).*

Proof. The property of being analytic behaves well with respect to extensions, that is, if $N \rightarrowtail G \twoheadrightarrow Q$ is an exact sequence of pro-p groups, then G is analytic if and only if both

N and Q are ([54, Corollary 2.4]). Thus one implication of the proposition is immediate, since G is a quotient of $\mathfrak{X}_p(G)$.

Suppose that G is analytic. Then G is of type FP_∞ (see [39] or [77] for instance) and $\overline{[G,G]}$ is finitely generated as a pro-p group. Thus Proposition 4.18 applies and $W_p(G)$ is a finitely generated abelian pro-p group. In particular, $W_p(G)$ is also analytic. But $\mathfrak{X}_p(G)/W_p(G) \simeq Im(\rho_p)$ must be analytic as well, being a closed subgroup of $G \times G \times G$. Thus $\mathfrak{X}_p(G)$ is analytic.

The result for poly-(procyclic) pro-p groups follows from the observation that these groups are exactly the solvable pro-p groups of finite rank ([81, Proposition 8.2.2]). Thus we only need to combine the first part of the proof with Proposition 4.16. □

Corollary 4.24. *Suppose that G is solvable pro-p group of type FP_∞. Suppose further that G is torsion-free or metabelian. Then $\mathfrak{X}_p(G)$ is a solvable pro-p group of type FP_∞.*

Proof. Notice that $\mathfrak{X}_p(G)$ must be solvable, since G is solvable. Assume first that G is torsion-free. Then G is analytic by the main result of Corob Cook in [27]. By Proposition 4.23 we know that $\mathfrak{X}_p(G)$ is analytic, so in particular it is of type FP_∞.

Suppose now that G is metabelian. Then it fits into an exact sequence of pro-p groups

$$1 \to A \to G \to Q \to 1,$$

where A is abelian and Q has finite rank (and in our case is abelian). By a result of King ([39, Theorem 6.2]), we know that if G is of type FP_∞, then G is actually of finite rank, that is, analytic. Then we can apply again Proposition 4.23 to deduce that $\mathfrak{X}_p(G)$ is analytic and thus of type FP_∞. □

It is plausible that Corob Cook's result from [27] holds for pro-p groups that are not torsion-free, but this is still an open problem. If that is the case, then the condition that G is torsion-free in Corollary 4.24 is redundant.

4.6 $\mathfrak{X}_p$ does not preserve FP_3

We begin by proving an auxiliary result about $R_p(G)$, which may be of independent interest.

Proposition 4.25. *If G is a 2-generated pro-p group, then $R_p(G) = 1$.*

This can be obtained as a corollary of the analogous result for discrete groups, which is a consequence of Lemma 4.26. The discrete case of Proposition 4.25 was first proved with different methods by Bridson and Kochloukova [23]. We write $y^{-\psi}$ for $(y^\psi)^{-1}$.

Lemma 4.26. *Let G be a discrete group and let $X \subset G$ be a generating set. Suppose that X is symmetric (with respect to inversion). Then $R(G) = [G, [L, G^\psi]]$ is the normal subgroup of $\mathfrak{X}(G)$ generated by the set*

$$Z = \{[x_1, [yy^{-\psi}, x_2^\psi]] \mid x_1, x_2, y \in X\}.$$

Proof. The proof relies on commutator calculations that use the following commutator identities

$$[a, bc] = [a, c].[a, b]^c \text{ and } [ab, c] = [a, c]^b.[b, c].$$

Recall that $R(G)$ is generated as a subgroup by the elements of the form $r = [g, [\ell, h^\psi]]$, with $g, h \in G$ and $\ell \in L = L(G)$. If $g = g_1 g_2$, then r is a consequence of $[g_1, [\ell, h^\psi]]$ and $[g_2, [\ell, h^\psi]]$. The analogous claim holds "in the other variable", that is, for $h = h_1 h_2$. Indeed $[g, [\ell, (h_1 h_2)^\psi]]$ is a consequence of $[g, [\ell, h_2^\psi]]$ and

$$r_1 = [g, [\ell, h_1^\psi]^{h_2^\psi}] = [g[g, h_2^{-\psi}], [\ell, h_1^\psi]]^{h_2^\psi} = [g, [\ell, h_1^\psi]]^{[g, h_2^{-\psi}] h_2^\psi},$$

where the last equality follows from the fact that $[[g, h_2^{-\psi}], [\ell, h_1^\psi]] \in [D, L] = 1$, where $D = D(G)$.

If $\ell = \ell_1 \ell_2$, with $\ell_2 = yy^{-\psi}$, then by applying the commutator formulas we obtain that r is a consequence of $[g, [yy^{-\psi}, h^\psi]]$ and $r_2 = [g, [\ell_1, h^\psi]^{yy^{-\psi}}]$. But

$$r_2 = [g, ([\ell_1, h^\psi][[\ell_1, h^\psi], y])^{y^{-\psi}}],$$

thus r_2 is a consequence of $[y, [\ell_1, h^\psi]]$ and $r_3 = [g, [\ell_1, h^\psi]^{y^{-\psi}}]$. Again

$$r_3 = [g[g, y^\psi], [\ell_1, h^\psi]]^{y^{-\psi}} = [g, [\ell_1, h^\psi]]^{[g, y^\psi] y^{-\psi}},$$

where the last equality follows from the fact that $[[g, y^\psi], [\ell_1, h^\psi]] \in [D, L] = 1$. A similar argument works for $\ell = \ell_1 \ell_2$, with $\ell_2 = (yy^{-\psi})^{-1}$.

Finally, if $r = [g, [bb^{-\psi}, h^\psi]]$ for some $b = uv \in G$, then

$$r = [g, [(vv^{-\psi} u^{-\psi} u)^{u^{-1}}, h^\psi]] = [g', [vv^{-\psi} uu^{-\psi}, (h^\psi)^u]]^{u^{-1}}$$

for $g' = g^u$. But

$$[vv^{-\psi} uu^{-\psi}, (h^\psi)^u] = [vv^{-\psi} uu^{-\psi}, [u, h^{-\psi}] h^\psi] = [vv^{-\psi} uu^{-\psi}, h^\psi],$$

since $[vv^{-\psi} uu^{-\psi}, [u, h^{-\psi}]] \in [L, D] = 1$. Thus r is a consequence of $[g', [vv^{-\psi} uu^{-\psi}, h^\psi]]$, and we fall in the previous case, that is, r is a consequence of $[g', [vv^{-\psi}, h^\psi]]$ and $[g', [uu^{-\psi}, h^\psi]]$.

The arguments above imply that any $r = [g, [\ell, h^\psi]]$ is an element of the normal subgroup of $\mathfrak{X}(G)$ generated by Z. □

Lemma 4.27. *If G is a discrete group generated by $\{x, y\}$, then $R(G) = 1$.*

Proof. It is enough to show that all elements of Z as in Lemma 4.26 are trivial for $X = \{x, y, x^{-1}, y^{-1}\}$. Consider $r = [x, [yy^{-\psi}, x^{\psi}]]$. By the standard commutator identities we have

$$r = [x, [y^{-\psi}, x^{\psi}]][x, [y, x^{\psi}]^{y^{-\psi}}]^{[y^{-\psi}, x^{\psi}]}. \tag{4.7}$$

Using the identity $[x, y^{\psi}] = [x^{\psi}, y]$ we deduce that $[x, [y, x^{\psi}]^{y^{-\psi}}] = [x, [y^{\psi}, x]^{y^{-\psi}}]$. Furthermore since $x^{-1}x^{\psi} \in L$ we have $[y^{-\psi}, x^{\psi}] \in [y^{-\psi}, x]L$. Then since $[D, L] = 1$ and $[x, [y^{\psi}, x]^{y^{-\psi}}] \in D$, we can reduce the second term of the above product in (4.7) to $[x, [y^{\psi}, x]^{y^{-\psi}}]^{[y^{-\psi}, x]}$.

Applying the Hall-Witt identity, we can see that the first term of the above product in (4.7) reduces to $[x, [y^{-\psi}, x]]$. Indeed

$$1 = [[y^{-\psi}, x^{\psi}], x]^{x^{-\psi}}[[x^{-\psi}, x^{-1}], y^{-\psi}]^{x}[[x, y^{\psi}], x^{-\psi}]^{y^{-\psi}},$$

so

$$[x, [y^{-\psi}, x^{\psi}]] = ([[y^{-\psi}, x^{\psi}], x])^{-1} = [[x, y^{\psi}], x^{-\psi}]^{y^{-\psi}x^{\psi}}.$$

Now using twice that $[D, L] = 1$ we have

$$[x, [y^{-\psi}, x^{\psi}]] = [[x, y^{\psi}], x^{-\psi}]^{y^{-\psi}x^{\psi}} = [[x, y^{\psi}], x^{-1}]^{y^{-\psi}x^{\psi}} = [[x, y^{\psi}], x^{-1}]^{y^{-\psi}x}.$$

Similarly, the Hall-Witt identity gives

$$[x, [y^{-\psi}, x]] = ([[y^{-\psi}, x], x])^{-1} = [[x, y^{\psi}], x^{-1}]^{y^{-\psi}x},$$

so $[x, [y^{-\psi}, x^{\psi}]] = [x, [y^{-\psi}, x]]$ and by (4.7) we get

$$r = [x, [y^{-\psi}, x]].[x, [y^{\psi}, x]^{y^{-\psi}}]^{[y^{-\psi}, x]} \tag{4.8}$$

Using the commutator formulas and (4.8) we obtain

$$1 = [x, [y^{\psi}y^{-\psi}, x]] = [x, [y^{\psi}, x]^{y^{-\psi}}.[y^{-\psi}, x]] = [x, [y^{-\psi}, x]].[x, [y^{\psi}, x]^{y^{-\psi}}]^{[y^{-\psi}, x]} = r$$

All the other generators of $R(G)$, given by Lemma 4.26, are either immediately trivial or can be shown to be trivial exactly as above. So $R(G) = 1$. □

Let $\widehat{F}$ be the free pro-p group on two generators. Then we can write $\mathfrak{X}_p(\widehat{F})$ as an inverse limit of the quotients $\mathfrak{X}(P)$, where each P is a 2-generated p-group. If follows by Lemma 4.27 that $R(P) = 1$. Clearly $R(P)$ is the image of $R_p(\widehat{F})$ by the homomorphism $\mathfrak{X}_p(\widehat{F}) \twoheadrightarrow \mathfrak{X}(P)$ induced by the epimorphism $\widehat{F} \twoheadrightarrow P$. Then it follows that $R_p(\widehat{F})$ must be trivial as well. In general, if G is a 2-generated pro-p group, then $R_p(G)$ is a quotient of $R_p(\widehat{F})$, so $R_p(G) = 1$. This completes the proof of Proposition 4.25.

Proposition 4.28. *If G is a non-abelian free pro-p group, then $\mathfrak{X}_p(G)$ is not of type FP_3.*

Proof. Let $\widehat{F}$ be the free pro-p group of rank 2. Then $R_p(\widehat{F}) = 1$, which combined with Theorem 4.22 and $H_2(\widehat{F};\mathbb{Z}_p) = 0$ implies that $W_p(\widehat{F}) = 1$ and hence $\mathfrak{X}_p(\widehat{F}) \simeq Im(\rho_p^{\widehat{F}})$. Note that $Im(\rho_p^{\widehat{F}})$ is a subdirect product of $\widehat{F} \times \widehat{F} \times \widehat{F}$ which is clearly not of type FP_3 (by Theorem A in [48], for instance), but it is of type FP_2 by Corollary 4.6. Then $H_i(\mathfrak{X}_p(\widehat{F});\mathbb{F}_p)$ is finite for $i \leq 2$ and is infinite for $i = 3$.

More generally, if G is any non-abelian free pro-p group, then the epimorphism $\pi : \mathfrak{X}_p(G) \to \mathfrak{X}_p(\widehat{F})$ induced by any epimorphism $\gamma : G \twoheadrightarrow \widehat{F}$ splits, that is, there is a homomorphism $\sigma : \mathfrak{X}_p(\widehat{F}) \to \mathfrak{X}_p(G)$ such that $\pi \circ \sigma = id$ and σ is induced by a splitting of γ. The same holds for the induced homomorphisms on the homologies. In particular $\pi_* : H_3(\mathfrak{X}_p(G);\mathbb{F}_p) \to H_3(\mathfrak{X}_p(\widehat{F});\mathbb{F}_p)$ is surjective, thus $H_3(\mathfrak{X}_p(G);\mathbb{F}_p)$ is not finite if $H_3(\mathfrak{X}_p(\widehat{F});\mathbb{F}_p)$ is not. Thus $\mathfrak{X}_p(G)$ cannot be of type FP_3 either. $\square$

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