Thesis for the Degree of Doctor of Philosophy

Symplectic methods for isospectral flows and 2D ideal hydrodynamics

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Abstract

The numerical solution of non-canonical Hamiltonian systems is an active and still growing field of research. At the present time, the biggest challenges concern the realization of structure preserving algorithms for differential equations on infinite dimensional manifolds. Several classical PDEs can indeed be set in this framework, and in particular the 2D hydrodynamical Euler equations. In this thesis, I have developed a new class of numerical schemes for Hamiltonian and non-Hamiltonian isospectral flows, in order to solve the 2D hydrodynamical Euler equations. The use of a conservative scheme has revealed new insights in the 2D ideal hydrodynamics, showing clear connections between geometric mechanics, statistical mechanics and integrability theory. The results are presented in four papers.

In the first paper, we derive a general framework for the isospectral flows, providing a new class of numerical methods of arbitrary order, based on the Lie–Poisson reduction of Hamiltonian systems. Avoiding the use of any constraint, we obtain geometric integrators for a large class of Hamiltonian and non-Hamiltonian isospectral flows. One of the advantages of these methods is that, together with the isospectrality, they exhibit near conservation of the Hamiltonian and, indeed, they are Lie–Poisson integrators.

In the second paper, using the results of paper I and III, we present a numerical method based on the geometric quantization of the Poisson algebra of the smooth functions on a sphere, which gives an approximate solution of the Euler equations with a number of discrete first integrals which is consistent with the level of discretization. The conservative properties of these schemes have allowed a more precise analysis of the statistical state of a fluid on a sphere. On the one hand, we show the link of the statistical state with some conserved quantities, on the other hand, we suggest a mechanism of formation of coherent structures related to the integrability theory of point-vortices.

In the third paper, I present and analyse a minimal variable isospectral Lie–Poisson integrator for quadratic matrix Lie algebras. This result comes from a more careful analysis of the isospectral midpoint method derived in paper I. I also present a detailed description of quadratic Lie algebras, showing under which conditions the related Lie–Poisson systems are also isospectral flows.

In the fourth paper, we give a survey on the integrability theory of the point-vortex dynamics. In particular, we show that all the results found in literature

can be derived in the framework of symplectic reduction theory. Furthermore, our work aims to connect the 2D Euler equations with the point-vortex dynamics, as suggested in paper II.

Keywords: Geometric integration, Symplectic methods, Structure preserving algorithms, Lie–Possion systems, Hamiltonian systems, Isospectral flows, Euler equations, Fluid dynamics, Integrability theory.

List of appended papers

Paper I K. Modin and **M. Viviani**, Lie–Poisson methods for isospectral flows, *Found. Comput. Math.*, 2019. DOI:10.1007/s10208-019-09428-w

Paper II K. Modin and **M. Viviani**, A Casimir preserving scheme for long-time simulation of spherical ideal hydrodynamics, *Journal of Fluid Mechanics*, 2019. DOI:10.1017/jfm.2019.944

Paper III **M. Viviani**, A minimal-variable method for isospectral flows, *BIT journal*, 2019, DOI: 10.1007/s10543-019-00792-1

Paper IV K. Modin, **M. Viviani**, Integrability of point-vortex dynamics via symplectic reduction: a survey, *arXiv*, 2020 - submitted to Arnold Mathematical Journal, Springer

My contribution to the appended papers:

Paper I: I have developed the numerical methods and most of the theoretical framework, drafted the manuscript and, after consultation, produced the final manuscript, except for the first two introductory sections. I have implemented the code, under the supervision of professor K. Modin.

Paper II: I have developed and implemented the numerical methods presented and drafted the manuscript. Under the supervision of professor K. Modin, I have conducted several numerical experiments on a remote server at Chalmers and contributed substantially to the theoretical framework presented in the paper.

Paper III: Independently developed and written.

Paper IV: Under the supervision of professor K. Modin, I have proved most of the results. I have produced the first draft of the manuscript.

Acknowledgements

Taking a PhD program of five years might be seen as a long journey into a specific topic, with the hope of understanding and unveiling a little bit more of what is already known. Luckily, I have not been alone during this journey. I acknowledge everyone who has been part of it, even though I do not have the occasion of mentioning him/her here.

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Milo Viviani
Göteborg, May 2020

Contents

Introduction

History

The following thesis summarises five years of doctoral studies, spent swinging between the geometric numerical integration of non-canonical Hamiltonian systems and the ideal 2D hydrodynamics. In this section, the (academic) history of my PhD is presented.

The starting point of the present research dates back to my master thesis, defended in September 2015. In that work I was interested in the numerical solution of the hydrodynamical Euler equations on a rotating sphere with continuous and singular (point vortices) vorticity fields. The aim of the master thesis was to get a numerical method which retained the main geometric properties of the continuous equations in the discrete case. The hydrodynamical Euler equations are indeed a classical example of an infinite dimensional Lie–Poisson system. This means that the equations encode a lot of symmetries and therefore conservation laws. What had motivated my research was that there was not yet an established and efficient way to integrate the Euler equations respecting those symmetries.

Eventually, the master thesis did not give a satisfactory result and the research had to be continued during my PhD studies, started in October 2015 under the supervision of prof. Klas Modin. During the first one and half years the results obtained were quite satisfactory but still not really innovative. The main reason was that our simulations of the Euler equations required very large matrices and the algorithm developed still had too many implicit equations to be solved in order to be really applicable.

Finally, we came to a turning point. In our approach, it was clear that to retain the symmetries of our problem, we needed to put some constraints on the equations. However, what if the constraints could be instead intrinsically encoded into the numerical method? This was not in general a feasible approach but surprisingly it turned out that, in this case, aiming for simplicity was rewarding. Working with this idea in mind it was possible to generate several numerical methods in a much simpler and efficient way. Moreover, a lot of

Lie–Poisson systems could then be solved with the same approach and in fact, for any quadratic semisimple Lie algebra, it was easy to derive a Lie–Poisson integrator of any order. An encouraging fact is also that the methods developed looked like to be the natural ones, requiring only the information coming from the underlying Lie algebra. In conclusion, we have derived a powerful tool to integrate the Euler equations for the 2D spherical ideal hydrodynamics, respecting their Lie–Poisson structure. The initial numerical experiments have been promising and in June 2018 I presented my Licentiate exam.

Finally, after the summer, the first paper was submitted to the Journal of Foundations of Computational Mathematics (JFoCM). In the autumn, a more careful study of the 2D spherical ideal hydrodynamics revealed new unexpected insights into the Euler equations. Thanks to the conservative schemes derived, we could in fact determine and characterize new statistical states of the fluid. This result was so puzzling and unexpected that it would have occupied the rest of my PhD work. In February 2019, our second paper was submitted to the Journal of Fluid Mechanics. After that, it did not take long time until the reviews from the first paper sent to JFoCM arrived. The comments were enthusiastic for the new results presented in the article. A comment of one the reviewers motivated me to pursue further analysis of the Lie–Poisson integrators. This led me to publish a new paper with a detailed analysis of a new "isospectral" midpoint rule, which was submitted to BIT Numerical Mathematics.

Once the article in JFoCM was accepted, I have signed up for attending the biannual conference of computational mathematics SciCADE. The conference featured a prize, called "New Talent Award", given to the best paper of a young researcher to be presented at the conference in July. While I was in Cuba on vacation with my family for my mother's 60th birthday, totally unexpectedly, I got an email via the stuttered Cuban internet that I had received the award! I then had to prepare a plenary talk of my work, which in less than two years went from looking like a dead end, to receive international recognition and appreciation.

In September, I started to attend a three months program on the Mathematics of Climate and Environment at the Institut Henri Poincaré in Paris. While there, I could proceed in my studies on the 2D ideal hydrodynamics and discuss my ideas with many significant researchers in the field. In the Autumn, all the papers submitted had been accepted for publication, and in March 2020 a further survey article on the integrability of the point-vortex dynamics was finished.

This positive recognition and the opening up of new possible projects and ideas motivated me to proceed in the academic career. I applied for several postdoc positions in Italy, finally being admitted for the two year Junior Visiting Position at the Scuola Normale Superiore in Pisa. The conclusion of my PhD seems to delineate a not foreseeable path along these five years, which I can undoubtedly say have been a milestone for my professional and personal life.

Motivation

The problems here presented are a classical and widely studied topic in numerical analysis. However, it may (or may not) be surprising that several questions are still unsolved. It should be clear while reading the thesis that the work here presented aims to connect different threads, and to give fresh ideas and critiques on classical questions. In particular, the thesis is divided into two main branches: the numerical integration of Hamiltonian isospectral flows and the 2D hydrodynamical Euler equations. The first one will be focused on the possibility of having intrinsic arbitrarily high order methods for Hamiltonian isospectral flows. The positive answer obtained will lead to a direct application to the second one and will provide a strong evidence on the advantages of having a conservative scheme for analysing and understanding the ideal fluid dynamics.

Chapter 1

Lie–Poisson systems

Since its foundation, mathematical physics has been built upon the language and the concepts coming from geometry. However, between the the XVIII and the XIX century, Leonard Euler, Giuseppe Lodovico Lagrangia and William Hamilton tried instead to develop an analytical formulation of the fundamental laws of nature. Surprisingly, it did not take long to realize that the equations they had derived were actually hiding even more geometry than before. Sophus Lie, Emmy Noether and later Vladimir Arnold showed that differential geometry is indeed the natural language of physics.

In this section, one of the most intriguing and ubiquitous structure arising in differential geometry and mathematical physics, named after the French mathematician Siméon Denis Poisson, is introduced and analysed.

1.1 Poisson structures and Hamiltonian systems

Definition 1 (Poisson bracket)**.** Let M be a smooth manifold and $C^\infty(M)$ the real vector space of smooth real valued functions defined on M. The Poisson bracket is a bilinear operation $\{\cdot,\cdot\}$ on $C^\infty(M)$, satisfying the following conditions:

- $\{F,G\} = -\{G,F\}$ skew symmetry;
- $\{F, G\cdot H\} = \{F,G\}\cdot H + \{F,H\}\cdot G$ Leibniz rule;
- $\{\{F,G\},H\} + \{\{H,F\},G\} + \{\{G,H\},F\} = 0$ Jacobi identity.

A manifold M equipped with a Poisson bracket is said to be a *Poisson manifold*. The Poisson bracket can be represented by a form $\mathrm{P} \in \bigwedge^2 TM$ by[1]:

$$\{F,G\}(x) = \mathrm{P}_x(dF(x), dG(x)), \tag{1.1}$$

[1] Note that we will denote by $\bigwedge^2 TM$ the space of the sections from M to the alternating

for any $x \in M$.

Definition 2 (Symplectic form)**.** Let M be a smooth manifold. $\omega \in \bigwedge^2 M$ is said to be a symplectic form if it is closed and non degenerate.

A manifold M equipped with a symplectic form ω is said to be a *symplectic manifold*, and it is denoted as (M, ω).

Remark 1. We observe that M always admits a trivial Poisson bracket, i.e., the zero one, but not always a symplectic form. In fact M has to be of even dimension and orientable (e.g., $\mathbb{R}^{2n}$, $n > 0$). Moreover, if M is compact, then the second group of De Rahm cohomology of M must be non zero (e.g., $\mathbb{S}^2$ and $\mathbb{T}^{2n}$ are symplectic but neither $\mathbb{RP}^2$ nor $\mathbb{S}^{2n}$ for $n > 1$ are). Furthermore, a symplectic form induces a canonical Poisson bracket as we will see below.

Definition 3 (Hamiltonian vector field)**.** Let (M, ω) be a symplectic manifold. A vector field $X \in TM$ is said to be Hamiltonian if there exists a function $H \in C^\infty(M)$ such that:

$$\iota_X \omega = dH, \tag{1.2}$$

where $\iota_X \omega$ is the contraction of ω by X, i.e., $\iota_X : \bigwedge^2 M \to \bigwedge^1 M$ such that, for all $p \in M$ and $v \in T_p M$, $\iota_X \omega_p(v) = \omega_p(X_p, v)$.

Remark 2. We observe that (1.2) are nothing else than the *Hamilton's equations*. In fact, for the sake of simplicity, assume $M = \mathbb{R}^{2n}$. Then a symplectic form can be represented in the *canonical coordinates* $q_1, ..., q_n, p_1, ..., p_n$ by the constant skew matrix $J \in \mathbb{M}(2n, \mathbb{R})$ with coefficients as follows: $J_{ij} = 0$ if $i, j \le n$ or $i, j > n$, $J_{ij} = \delta_{ij+n}$ if $i > n, j \le n$ and $J_{ij} = -\delta_{i+nj}$ if $i \le n, j > n$. The vector field $X(q, p) = (\dot{q}, \dot{p})^T$, where (q, p) are the flow lines of X from some initial values. Hence, (1.2) becomes

$$(\dot{q}, \dot{p}) \cdot J = dH_{q,p}, \tag{1.3}$$

which are the Hamilton equations after inversion of J.

Furthermore, we observe that ω induces a diffeomorphism $\hat{\omega} : TM \to T^*M$ defined as $\hat{\omega}(v) = \omega_p(v, \cdot)$ for every $v \in T_p M$. Hence, given $F \in C^\infty(M)$, we define the *Hamiltonian vector field associate to* F as $X_F = \hat{\omega}^{-1}(dF)$. Finally, given a symplectic manifold (M, ω), we define, for every $F, G \in C^\infty(M)$ the following Poisson bracket:

$$\{F, G\} = \omega(X_F, X_G). \tag{1.4}$$

To compute (1.4) in local coordinates we need the following fundamental theorem.

2-tensor on the tangent bundle of M while by $\bigwedge^2 T^*M =: \bigwedge^2 M$ the space of the sections from M to the the alternating 2-tensor on the cotangent bundle of M, which are the usual 2-forms.

Theorem 1 (Darboux). *Let (M, ω) be a symplectic manifold of dimension $2n$. Then for every $x \in M$, there exists a local chart $(V, \varphi = (q_1, ..., q_n, p_1, ..., p_n))$ centred in x, such that:*

$$\omega_{|V} = \textstyle\sum_{i=1}^n dq_i \wedge dp_i,$$

i.e., ω is represented by the matrix J defined above.

Such coordinates are called *canonical* or *Darboux coordinates.* Now we can write (1.4) in coordinates. Let (V, φ) be a chart given by the Darboux theorem, then, in this chart, $X_F = \sum_{i=1}^n \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}$. A similar expression holds for X_G. Then, a straightforward computation leads to write (1.4) as:

$$\{F, G\} = \sum_{i=1}^n \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}, \tag{1.5}$$

where the relations $dq_i(\partial_{q_i}) = 1, dq_i(\partial_{p_i}) = 0, dp_i(\partial_{q_i}) = 0, dp_i(\partial_{p_i}) = 1$, for $i = 1, ...n$, have been used. The canonical coordinates satisfy:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \text{ and } \{q_i, p_j\} = -\{p_j, q_i\} = \delta_{ij}, \text{ for } i, j = 1, ..., n.$$

An obvious consequence is that, for every $F \in C^\infty(M)$, and any Hamiltonian vector field X_H we have:

$$X_H(F) = \{F, H\}.$$

Hence, for p_i, q_i, $i = 1, ..., n$ integral curves (or trajectories) of X_H we have that:

$$\dot{q_i} = X_H(q_i) = \{q_i, H\} \text{ and } \dot{p_i} = X_H(p_i) = \{p_i, H\} \text{ for } i = 1, ..., n,$$

which is another formulation of the Hamilton equations (1.3).

A triple (M, ω, H), where (M, ω) is a symplectic manifold and H is a smooth function on M, is called *Hamiltonian system*, with Hamiltonian H. A Hamiltonian system represents the typical setting when studying a closed physical system. In particular, its conservation laws can be understood in terms of symmetries of the Hamiltonian vector field. Let us recall here the precise definition of symmetry and conservation law, and the fundamental Noether theorem.

Definition 4. Let (M, ω, H) be a Hamiltonian system. A function $f \in C^\infty(M)$ constant on any integral curve of X_H is said to be a *first integral* of the system. A vector field $X \in TM$ is said to be an *infinitesimal symmetry* if both ω and H are invariant under the flow of X.

In terms of the Poisson bracket, F is a first integral of a Hamiltonian system if F commutes with H, which means $\{F, H\} = 0$. Hence, H is clearly a first integral, since the bracket is skew-symmetric. A vector field X is an infinitesimal symmetry of a Hamiltonian system if $X[H] = 0$ and $L_X\omega = 0$, where L_X is the Lie derivative.

Theorem 2 (Noether theorem). *Let (M, ω, H) be a Hamiltonian system.*

- *if f is a first integral, then X_f is an infinitesimal symmetry;*
- *conversely, if $H^1(M) = 0$ (where $H^1(M)$ is the first group of De Rham cohomology of M), then any infinitesimal symmetry is a Hamiltonian vector field of a first integral, uniquely defined, except for an additive constant for any connected component of M.*

1.2 Lie–Poisson systems

In this section, we show that on vector spaces that are the dual of a Lie algebra it is possible to define a canonical Poisson structure, which will be called *Lie–Poisson bracket*. We recall that a Lie algebra is a vector space $\mathfrak{g}$ with a skew-symmetric bilinear form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which satisfies the Jacobi identity.

Consider a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, not necessarily of finite dimension, and let $\mathfrak{g}^*$ be its dual. Then on $C^\infty(\mathfrak{g}^*)$ we have the following (canonical) Poisson bracket[2]:

$$\{F, G\}_\pm(v) = \pm\langle v, [dF(v), dG(v)]\rangle \tag{1.6}$$

where $v \in \mathfrak{g}^*$ and we have canonically identified $\mathfrak{g} \cong \mathfrak{g}^{**}$. The Lie–Poisson bracket is a very important example of a generally neither trivial nor symplectic Poisson bracket. In this case, the Poisson form $\mathrm{P} \in \bigwedge^2 T\mathfrak{g}^*$ is linear and can be expressed by:

$$\mathrm{P}_{ij}(v) = \pm C_{ij}^k v_k \tag{1.7}$$

where C_{ij}^k are the structure constants of $\mathfrak{g}$, and we have used Einstein's notation for repeated indices.

Let H be a smooth function on $\mathfrak{g}^*$. Then, the system:

$$\dot{F} = \{F, H\}_\pm \tag{1.8}$$

which has to be satisfied for any $F \in C^\infty(\mathfrak{g}^*)$, will be called *Lie–Poisson system* with Hamiltonian function H. Remarkably, accordingly to the rank of the form P, we have a certain number of first integrals of the motion, that are the same for any Hamiltonian. These functions that commute with any other one, i.e., $\{C, \cdot\} = 0$ are called *Casimir functions*. As we will discuss further in the thesis, the conservation of the Casimir functions and the Hamiltonian by a numerical scheme applied to (1.8) is crucial in order to guarantee good predictions for long times.

[2]The $\pm$ sign depends on the fact that the Poisson bracket here defined can also be obtained via the reduction of the canonical ones on the left (-) or right (+) invariant functions on T^*G (see section 1.2.2 below).

1.2.1 Co-adjoint representation

We now want to express (1.8) in terms of the co-adjoint representation of a Lie algebra. We have first to recall some definitions. Let G be a Lie group, and consider the map

$$\begin{aligned} C : G \times G &\longrightarrow G \\ (g,h) &\mapsto C_g(h) := ghg^{-1}. \end{aligned}$$

Then, for each $g \in G$, we have the internal automorphism C_g. If we take the differential of this map in the identity e we get the *adjoint representation*, Ad of G in $\mathrm{End}(\mathfrak{g})$, that is defined by:

$$Ad_g(X) = \frac{d}{dt}_{|t=0}(g \exp^{tX} g^{-1}),$$

for every $g \in G, X \in \mathfrak{g}$. Finally, differentiating $Ad : G \to \mathrm{End}(\mathfrak{g})$ and identifying $\mathrm{End}(\mathfrak{g})$ with its tangent, we obtain the map:

$$\begin{aligned} ad : &\mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) \\ &X \mapsto ad_X = [X, \cdot]. \end{aligned}$$

Let us now define the dual of the adjoint representation, i.e., the representation of the group G and the Lie algebra $\mathfrak{g}$ on the endomorphism of the dual of the Lie algebra $\mathfrak{g}$. We define the *co-adjoint representation* $Ad^* : G \to \mathrm{End}(\mathfrak{g}^*)$ by:

$$\langle Ad^*(g)(\phi), X\rangle = \langle \phi, Ad(g^{-1})X\rangle$$

for every $g \in G, X \in \mathfrak{g}, \phi \in \mathfrak{g}^*$. Proceeding as before, one can find the infinitesimal version $ad^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, given by $ad^*_X = -(ad_X)^*$, i.e.,

$$\langle ad^*_X(\phi), Y\rangle = \langle \phi, -ad_X(Y)\rangle$$

for every $X, Y \in \mathfrak{g}, \phi \in \mathfrak{g}^*$.

Let $\mathcal{O}$ be an orbit of the co-adjoint action $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. It holds the remarkable fact that the co-adjoint orbits have a canonical symplectic structure, called *Kirillov-Kostant-Souriau form*. Let $p \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$, then the two form:

$$\omega_p(\mathrm{ad}^*_X(p), \mathrm{ad}^*_Y(p)) = \langle p, [X,Y]\rangle,$$

is a symplectic form on $\mathcal{O}$, where we have used the canonical identification of $\mathfrak{g}^{**}$ with $\mathfrak{g}$, from which we have obtained that $\mathrm{T}^*\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}$. We conclude noticing that the co-adjoint orbits are immersed submanifold[3], which are contained in the

[3] If the action of the group G is also proper, e.g., G compact, then the co-adjoint orbits are embedded submanifold.

level sets of the Casimir functions. However, in general, the Casimir functions do not characterize the co-adjoint orbits[4].

Let us go back to the Lie–Poisson system (1.8). We notice that the bracket can be expressed in terms of the co-adjoint representation of $\mathfrak{g}$:

$$\pm \langle v, [dF(v), dH(v)]\rangle = \mp\langle v, \mathrm{ad}_{dH(v)}(dF(v))\rangle = \pm\langle \mathrm{ad}^*_{dH(v)}(v), dF(v)\rangle. \quad (1.9)$$

Hence, the trajectories of a Lie–Poisson system evolve precisely on the co-adjoint orbits given by the Ad^* action. In fact, let us consider $F = F(v(t))$, where $v(t)$ is a curve in $\mathfrak{g}^*$, and $v(0) = v_0$. Applying the chain rule we get:

$$dF(\dot{v}) = \pm\langle \mathrm{ad}^*_{dH(v)}(v), dF(v)\rangle$$

for any $F \in C^\infty(\mathfrak{g}^*)$. Hence it is true that:

$$\dot{v} = \pm\mathrm{ad}^*_{dH(v)}(v). \quad (1.10)$$

Integrating this system we get:

$$v(t) = \mathrm{Ad}^*_{\exp(\pm\int_0^t dH(v(s))ds)}(v_0).$$

1.2.2 Momentum maps

In this section we will briefly recall the concept and the main properties of the *momentum map* of a Lie group action on a Poisson manifold. For further details we refer to [18] and [19].

Let G be a Lie group acting to the left on a Poisson manifold P, such that for any $g \in G$ the action Φ_g is a Poisson map, i.e., $\{\cdot,\cdot\} \circ \Phi_g = \{\cdot \circ \Phi_g, \cdot \circ \Phi_g\}$. Let the *infinitesimal action* of G be the map $\rho : \mathfrak{g} \times P \to TP$ defined by:

$$\rho_\xi(p) = \frac{d}{dt}_{|t=0} \exp(t\xi)p, \quad (1.11)$$

for any $\xi \in \mathfrak{g}, p \in P$. Hence, ρ_ξ is a vector field on P. Furthermore, let us assume ρ_ξ to be Hamiltonian, i.e., there exists a Lie algebra homomorphism $J : \mathfrak{g} \to C^\infty(P)$ such that $\rho_\xi = \{\cdot, J_\xi\}$, for any $\xi \in \mathfrak{g}$. Then we define the momentum map $\mu : P \to \mathfrak{g}^*$ by:

$$\langle \mu(p), \xi\rangle = J_\xi(p). \quad (1.12)$$

We remark that, if the Poisson bracket is induced by a symplectic form ω, then the momentum map can be defined by the formula:

$$d\langle \mu(p), \xi\rangle = \iota_{\rho_\xi(p)}\omega_p. \quad (1.13)$$

[4] [19], pag. 479.

For a right action one can use the same formalism. The main difference is that the map $J : \mathfrak{g} \to C^\infty(P)$ has to be a Lie algebra anti-morphism for a right action (cfr. [18]). Let us now denote μ_L (respectively μ_R) the momentum map coming from the left (respectively right) G-action on P. Let also $\mathfrak{g}^*_-$ (respectively $\mathfrak{g}^*_+$) be the dual of the Lie algebra $\mathfrak{g}$ endowed with the $-$ (respectively $+$) Lie–Poisson bracket. The main property of the momentum maps is stated in the following proposition:

Proposition 1 (Prop 2.1, [18])**.** *Let $\mu_L : P \to \mathfrak{g}^*_+$ (respectively $\mu_R : P \to \mathfrak{g}^*_-$) be the momentum map defined above. Then μ_L (respectively μ_R) is a Poisson map.*

Proof. Let us consider the left case. By definition of the Lie–Poisson bracket:

$$\begin{aligned}\{F, G\}_+(\mu(p)) &= \langle \mu(p), [dF(\mu(p)), dG(\mu(p))]\rangle \\ &= J_{[dF(\mu(p)),dG(\mu(p))]}(p).\end{aligned}$$

Now, since $J : \mathfrak{g} \to C^\infty(P)$ is a Lie algebra homomorphism, we have that:

$$J_{[dF(\mu(p)),dG(\mu(p))]}(p) = \{J_{dF(\mu(p))}, J_{dG(\mu(p))}\}(p).$$

Finally, by the definition of the Lie–Poisson bracket, it is enough to prove that:

$$d(J_{dF(\mu(p))}) = d(F \circ \mu)(p).$$

Indeed, we have:

$$\begin{aligned}\langle d(F \circ \mu)(p), v_p\rangle &= \langle dF(\mu(p)) \circ d\mu(x), v_p\rangle \\ &= \langle d\langle \mu(p), dF(\mu(p))\rangle, v_p\rangle \\ &= \langle d(J_{dF(\mu(p))}), v_p\rangle.\end{aligned}$$

for any $v_p \in T_pP$. □

1.2.3 Lie–Poisson reduction

Now that we have introduced the momentum maps, we can show how Lie–Poisson systems are related to the canonical Hamilton equations.

Let $(P, \{\cdot, \cdot\}, H)$ be a Poisson Hamiltonian system and let (M, ω, H_ψ) be a Hamiltonian system, where $H_\psi = H \circ \psi$ and $\psi : M \to P$ is a Poisson map. Consider G a Lie group with a symplectic left (resp. right) Hamiltonian action on M and assume that H_ψ is left (respectively right) G-invariant and G is transitive on the fibres of ψ. Suppose that there exists a left momentum map $\mu : M \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the associated Lie algebra of G. Then, by Proposition 1, we know that μ is a Poisson map between the canonical Poisson bracket on M

and the Lie–Poisson bracket on $\mathfrak{g}_+^*$. Since H_ψ is $G-$invariant, the momentum map μ is a conserved quantity of the dynamical system [19, Thm. 11.4.1]. It is shown in [18] that, assuming there are no singularities in the quotient with respect to the group action, given a co-adjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$, the map $\Psi_{\mathcal{O}} = \psi_{|\mu^{-1}\mathcal{O}}$ induces a symplectic embedding $\widehat{\Psi}_{\mathcal{O}} : \mu^{-1}\mathcal{O}/G \to P$ to a symplectic leaf of P[5] (see the diagram here below). Hence, in this setting, a canonical Hamiltonian system with symmetries on M has a reduced dynamics on the symplectic leaves of P.

$$\begin{array}{ccccc} \mathcal{O} & \xleftarrow[\mu]{} & \mu^{-1}\mathcal{O} & \xrightarrow{\pi} & \mu^{-1}\mathcal{O}/G \\ \Big\downarrow \iota & & \Big\downarrow \iota & \searrow^{\Psi_{\mathcal{O}}} & \Big\downarrow \hat{\Psi}_{\mathcal{O}} \\ \mathfrak{g}_+^* & \xleftarrow[\mu]{} & M & \xrightarrow{\psi} & P \end{array}$$

In particular, when $M = T^*G$ and $P = \mathfrak{g}_-^*$ (resp. $P = \mathfrak{g}_+^*$), we can take $\psi = \mu_R$ (resp. μ_L) and $\mu = \mu_L$ (resp. μ_R). In this case the reduction is called *Lie–Poisson reduction*. Then, the canonical Hamilton equations in T^*G w.r.t to the Hamiltonian $\widetilde{H}$ become the equations (1.8) on $\mathfrak{g}_-^*$ (resp. $\mathfrak{g}_+^*$) with respect to to the Hamiltonian H on $\mathfrak{g}^*$, defined by $H \circ \mu_L = \widetilde{H}$ (resp. $H \circ \mu_R = \widetilde{H}$). In the diagram here below we summarize these concepts.

$$\begin{array}{ccccc} \mathcal{O} & \xleftarrow[\mu_L]{} & G \times \mathcal{O} & \xrightarrow{\mu_R} & \mathcal{O} \\ \Big\downarrow \iota & & \Big\downarrow \iota & \searrow^{\mu_{R|\mathcal{O}}} & \Big\downarrow \hat{\mu}_{R|\mathcal{O}} \equiv \iota \\ \mathfrak{g}_+^* & \xleftarrow[\mu_L]{} & T^*G & \xrightarrow{\mu_R} & \mathfrak{g}_-^* \end{array}$$

The Lie–Poisson reduction theory shows why Lie–Poisson systems arise naturally in physics. In particular, in [18] it is shown that the Euler equations on a mainfold M are also a Lie–Poisson system on the dual space of the Lie algebra of divergence free vector fields, coming from the Lie–Poisson reduction of the canonical equations on $T^*Diff_{\text{vol}}(M)$, where $Diff_{\text{vol}}(M)$ is the group of volume preserving diffeomorphisms of M to itself. In chapter 3, we will discuss in detail the Lie–Poisson structure of the 2D Euler equations.

1.2.4 Lie–Poisson systems on $\mathfrak{gl}(n, \mathbb{C})^*$

In this section, in view of the applications, we recall some facts about Lie–Poisson systems on the dual of the general matrix Lie algebra $\mathfrak{gl}(n, \mathbb{C})^*$. In particular, we explicitly describe the identification between $\mathfrak{gl}(n, \mathbb{C})^*$ and $\mathfrak{gl}(n, \mathbb{C})$ and how this affects the representation of the equations of a Lie–Poisson system.

[5] It is a general fact that any Poisson manifold admits a foliation of symplectic submanifolds, called *symplectic leaves*. A trajectory of X_H starting in a particular leaf necessarily lies entirely in the same leaf. For $\mathfrak{g}_\pm^*$ the symplectic leaves coincide with the respective co-adjoint orbits.

ad **vs** ad*

Considering the adjoint representation of $\mathfrak{gl}(n,\mathbb{C})$ on itself:

$$\mathrm{ad}_A(B) = [A,B] = AB - BA,$$

for any $A, B \in \mathfrak{gl}(n,\mathbb{C})$. Let us now look at the co-adjoint representation of $\mathfrak{gl}(n,\mathbb{C})$ on $\mathfrak{gl}(n,\mathbb{C})^*$. Consider the two different identifications of $\mathfrak{gl}(n,\mathbb{C})^*$ with $\mathfrak{gl}(n,\mathbb{C})$:

$$\begin{aligned}\langle A,B\rangle_1 &= \mathrm{Tr}(AB)\\ \langle A,B\rangle_2 &= \mathrm{Tr}(A^\dagger B),\end{aligned}$$

for $A, B \in \mathfrak{gl}(n,\mathbb{C})$. The second one comes from the Frobenius inner product on $\mathfrak{gl}(n,\mathbb{C})$ (in terms of the canonical basis, the first one says that the dual element of a given one is its complex adjoint whereas the second one says that it is itself)[6]. Recalling that $\mathrm{ad}^*_A = -(\mathrm{ad}_A)^*$, the respective co-adjoint representations are:

$$\begin{aligned}\mathrm{ad}^{*_1}_A B &= -[B,A] = \mathrm{ad}_A B\\ \mathrm{ad}^{*_2}_A B &= [B,A^\dagger] = -\mathrm{ad}_{A^\dagger} B.\end{aligned}$$

Lie–Poisson equations and their representations

In this paragraph, we consider Lie–Poisson systems with quadratic non-degenerate Hamiltonian and we show that the dynamics is independent from the identification of $\mathfrak{gl}(n,\mathbb{C})^*$ with $\mathfrak{gl}(n,\mathbb{C})$. To define the Lie–Poisson equations with quadratic non-degenerate Hamiltonian, we need a symmetric positive-definite linear map $\mathcal{A} : \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})^*$. An explicit form of this map depends on the way we identify the algebra with its dual. Let us denote with $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, the respective form, with respect to ad^{*_2} and respectively, ad^{*_1}. We then have that the following inner products are identically defined:

$$\begin{aligned}\langle A,B\rangle_{\mathcal{A}} &:= \langle \mathcal{A}A, B\rangle_2 = \mathrm{Tr}(((\mathcal{A}A)^\dagger B)\\ \langle A,B\rangle_{\widetilde{\mathcal{A}}} &:= \langle \widetilde{\mathcal{A}}A, B\rangle_1 = \mathrm{Tr}(\widetilde{\mathcal{A}}AB).\end{aligned}$$

for $A, B \in \mathfrak{gl}(n,\mathbb{C})$. Therefore, we have to have that $\widetilde{\mathcal{A}} = {}^\dagger \circ \mathcal{A}$. Then, for $\Psi \in \mathfrak{gl}(n,\mathbb{C})$, the Lagrangian function can be defined as:

$$\mathcal{L}(\Psi) = \frac{1}{2}\langle \Psi,\Psi\rangle_{\mathcal{A}} = \frac{1}{2}\langle \Psi,\Psi\rangle_{\widetilde{\mathcal{A}}}.$$

[6]Here with $\dagger$ we understand the complex adjoint.

The respective momentum variables in $\mathfrak{gl}(n,\mathbb{C})^*$ are:

$$\Omega_{\mathcal{A}} = \frac{\partial \mathcal{L}(\Psi)}{\partial \Psi} = \mathcal{A}\Psi$$
$$\Omega_{\widetilde{\mathcal{A}}} = \left(\frac{\partial \mathcal{L}(\Psi)}{\partial \Psi}\right)^{\dagger} = \widetilde{\mathcal{A}}\Psi$$

and we observe that $(\Omega_{\mathcal{A}})^{\dagger} = \Omega_{\widetilde{\mathcal{A}}}$. From these calculations, we get the Hamiltonian functions:

$$H_{\mathcal{A}}(\Omega_{\mathcal{A}}) = \frac{1}{2}\langle \Omega_{\mathcal{A}}, \mathcal{A}^{-1}\Omega_{\mathcal{A}}\rangle_2$$
$$H_{\widetilde{\mathcal{A}}}(\Omega_{\widetilde{\mathcal{A}}}) = \frac{1}{2}\langle \Omega_{\widetilde{\mathcal{A}}}, \widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}}\rangle_1.$$

So we have the identities:

$$\frac{\partial H_{\mathcal{A}}(\Omega_{\mathcal{A}})}{\partial \Omega_{\mathcal{A}}} = \mathcal{A}^{-1}\Omega_{\mathcal{A}} = \Psi$$
$$\frac{\partial H_{\widetilde{\mathcal{A}}}(\Omega_{\widetilde{\mathcal{A}}})}{\partial \Omega_{\widetilde{\mathcal{A}}}} = \widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}} = \Psi.$$

Finally, we get the equation of motion ([19], Chapt. 13):

$$\langle \dot{\Psi}, Y\rangle_{\mathcal{A}} = -\langle \Psi, \operatorname{ad}_{\Psi} Y\rangle_{\mathcal{A}} = \langle \mathcal{A}^{-1} \operatorname{ad}^{*_2}_{\Psi} \mathcal{A}\Psi, Y\rangle_{\mathcal{A}},$$
$$\langle \dot{\Psi}, Y\rangle_{\widetilde{\mathcal{A}}} = \langle \Psi, \operatorname{ad}_{\Psi} Y\rangle_{\widetilde{\mathcal{A}}} = -\langle \widetilde{\mathcal{A}}^{-1} \operatorname{ad}^{*_1}_{\Psi} \widetilde{\mathcal{A}}\Psi, Y\rangle_{\widetilde{\mathcal{A}}},$$

for any $Y \in \mathfrak{gl}(n,\mathbb{C})$. These can also be written in the strong form as:

$$\dot{\Psi} = \mathcal{A}^{-1} \operatorname{ad}^{*_2}_{\Psi} \mathcal{A}\Psi = \mathcal{A}^{-1}[\mathcal{A}\Psi, \Psi^{\dagger}],$$
$$\dot{\Psi} = \widetilde{\mathcal{A}}^{-1} \operatorname{ad}^{*_1}_{\Psi} \widetilde{\mathcal{A}}\Psi = -\widetilde{\mathcal{A}}^{-1}[\widetilde{\mathcal{A}}\Psi, \Psi],$$

or, considering the dual version for $\Omega_{\mathcal{A}}, \Omega_{\widetilde{\mathcal{A}}}$:

$$\dot{\Omega}_{\mathcal{A}} = \operatorname{ad}^{*_2}_{\mathcal{A}^{-1}\Omega_{\mathcal{A}}} \Omega_{\mathcal{A}} = [\Omega_{\mathcal{A}}, (\mathcal{A}^{-1}\Omega_{\mathcal{A}})^{\dagger}],$$
$$\dot{\Omega}_{\widetilde{\mathcal{A}}} = \operatorname{ad}^{*_1}_{\widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}}} \Omega_{\widetilde{\mathcal{A}}} = -[\Omega_{\widetilde{\mathcal{A}}}, \widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}}].$$

Remark 3. If we transpose the second equation, we get:

$$\dot{\Omega}^{\dagger}_{\widetilde{\mathcal{A}}} = [\Omega^{\dagger}_{\widetilde{\mathcal{A}}}, (\widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}})^{\dagger}],$$

and, using the fact that $(\Omega_{\mathcal{A}})^{\dagger} = \Omega_{\widetilde{\mathcal{A}}}$, and $\widetilde{\mathcal{A}}^{-1}\Omega_{\widetilde{\mathcal{A}}} = \mathcal{A}^{-1}\Omega_{\mathcal{A}}$, we see that the Lie–Poisson equations are independent from the choice of the pairing.

Lie–Poisson maps on $\mathfrak{gl}(n,\mathbb{C})^*$

Consider the identification of $\mathfrak{gl}(n,\mathbb{C})$ with its dual, via the Frobenius pairing. After this identification, the Lie–Poisson structure on $\mathfrak{gl}(n,\mathbb{C})^*$ is completely determined by the structure constants of $\mathfrak{gl}(n,\mathbb{C})$. Therefore any Lie algebra morphism of $\mathfrak{gl}(n,\mathbb{C})$ will be a *Lie Poisson map* on $\mathfrak{gl}(n,\mathbb{C})^*$ and viceversa. We now want to check how it looks with respect to ad^*. Let consider $a:\mathfrak{gl}(n,\mathbb{C})\to\mathfrak{gl}(n,\mathbb{C})$ invertible Lie algebra morphism, $A,B\in\mathfrak{gl}(n,\mathbb{C})$ and $\phi\in\mathfrak{gl}(n,\mathbb{C})^*\equiv\mathfrak{gl}(n,\mathbb{C})$ (via the Frobenius identification). Then we get:

$$\begin{aligned}\mathrm{Tr}((a\,\mathrm{ad}^*_A(\phi))^\dagger B) &= -\mathrm{Tr}((a[A^\dagger,\phi])^\dagger B)\\ &= -\mathrm{Tr}(\phi^\dagger[A,a^\dagger B])\\ &= -\mathrm{Tr}((a\phi)^\dagger[a^{-T}A,B])\\ &= -\mathrm{Tr}(([A^\dagger a^{-1},a\phi])^\dagger B)\\ &= \mathrm{Tr}((\mathrm{ad}^*_{a^{-T}A}(a\phi))^\dagger B).\end{aligned}$$

So we have the formula:

$$a\,\mathrm{ad}^*_A(\phi)=\mathrm{ad}^*_{a^{-T}A}(a\phi). \tag{1.14}$$

Consider A to be equal to $dH(\phi)$, for a smooth function H defined on $\mathfrak{gl}(n,\mathbb{C})^*$, i.e., we have a Lie–Poisson system. Then the action on an invertible linear map on the (Lie–Poisson) Hamiltonian vector field is:

$$a\cdot X_H := Da\circ X_H\circ a^{-1},$$

where $X_H(\phi)=\mathrm{ad}^*_{dH(\phi)}(\phi)$. Then, using the formula (1.14), we get:

$$\begin{aligned}a\cdot X_H(\phi) &= a\,\mathrm{ad}^*_{dH(a^{-1}\phi)}(a^{-1}\phi)\\ &= \mathrm{ad}^*_{a^{-T}dH(a^{-1}\phi)}(\phi)\\ &= \mathrm{ad}^*_{d(H\circ a^{-1})(\phi)}(\phi)\\ &= X_{H\circ a^{-1}}(\phi),\end{aligned}$$

which is again a Lie–Poisson system.

Chapter 2

Isospectral flows and their numerical solution

2.1 Isospectral flows and their properties

The isospectral flows are a central class of dynamical systems with symmetries. They arise in fact in different contexts: Lie–Poisson reduction, matrix factorization, Lax pairs of integrable systems, and so on [8],[11],[24]. As the name suggests, isospectral flows represent a dynamical system on linear operators such that the spectrum of the operator is fixed during the whole evolution. If the operators are diagonalizable, then, at each time, the operator is similar to the initial one. Let the flow

$$\begin{aligned} \Phi : [0,\infty) \times \mathcal{L}(V) &\to \mathcal{L}(V) \\ (t,W) &\mapsto \Phi_t(W) \end{aligned}$$

be an isospectral flow on $\mathcal{L}(V)$, where V is a finite dimensional vector space of dimension n. Let W_0 be the initial value. Then, for any $t \geq 0$, assume that exists $U(t)$ such that:

$$W(t) = \Phi_t(W_0) = U(t)^{-1} W_0 U(t). \tag{2.1}$$

By differentiation of (2.1), one find that W is the solution of:

$$\begin{aligned} &\dot{W} = [B(W), W] \\ &W(0) = W_0, \end{aligned} \tag{2.2}$$

where $B(W) = -U^{-1}\dot{U}$ and the bracket is the usual matrix commutator $[A, B] = AB - BA$. Other than the eigenvalues of the operator, one can choose a different set of generators for the first integrals of (2.2). This is provided by the moments of W. In fact:

$$\frac{d}{dt}\operatorname{Tr}(W^k) = \operatorname{Tr}(W^{k-1}[B(W), W]) = \operatorname{Tr}(B(W)[W^{k-1}, W]) = 0,$$

for $k = 1, 2, ...$ Since W is represented by a $n \times n$ matrix, its first n moments are independent, and for $k > n$ they are related by the Cayley–Hamilton theorem (in fact $\operatorname{Tr}(W^k) = \sum_{i=1}^{n} \lambda_i^k$, for λ_i the eigenvalues of W). When $B(W)$ takes the form of (the transpose of) a gradient of a function, the equation (2.2) will be said *Hamiltonian-Isospectral flow*.[1] The word Hamiltonian is because the function H such that $B = -\nabla H^\dagger$ is a conserved quantity of (2.2). In fact:

$$\frac{d}{dt}H(W) = -\operatorname{Tr}(\nabla H(W)^\dagger[\nabla H(W)^\dagger, W]) = -\operatorname{Tr}(W[\nabla H(W)^\dagger, \nabla H(W)^\dagger]) = 0.$$

A further reason to use the word Hamiltonian is that $\mathcal{L}(V)$, endowed with the bracket $[\cdot,\cdot]$, can be seen as the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ and the equations (2.2) as the reduced form of a canonical Hamiltonian system, as shown in section 1.2.3. Indeed, if we identify the dual of $\mathfrak{gl}(n, \mathbb{C})$ with itself, via the Frobenius inner product $\langle A, B\rangle = \operatorname{Tr}(A^\dagger B)$, the equations (2.2) above form a Lie–Poisson system with respect to the co-adjoint representation of $\mathfrak{gl}(n, \mathbb{C})$ on $\mathfrak{gl}(n, \mathbb{C})^*$ given by:

$$\operatorname{ad}^*_A B = [B, A^\dagger] = -\operatorname{ad}_{A^\dagger} B$$

for $A \in \mathfrak{gl}(n, \mathbb{C}), B \in \mathfrak{gl}(n, \mathbb{C})^* \cong \mathfrak{gl}(n, \mathbb{C})$.

2.1.1 Restriction to a subspace of $\mathfrak{gl}(n, \mathbb{C})$

It is interesting, both for theoretical and practical purposes, to analyse the case when W evolves on a subspace S of $\mathfrak{gl}(n, \mathbb{C})$. It is clear that if $\{W(t)\}_{t\in\mathbb{R}} \subset S$ then $B(W)$ has to be in $\mathfrak{n}(S)$, i.e., the normalizer of S in $\mathfrak{gl}(n, \mathbb{C})$, which is the largest subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ such that $[\mathfrak{n}(S), S] \subseteq S$. This framework is used in Paper I to encompass at the same time the classical isospectral flows, e.g., $W \in Sym(n), B(W) \in \mathfrak{o}(n)$, and the Lie–Poisson systems on reductive Lie-algebras.

2.2 Numerical approximation of isospectral flows

As we have shown above, the main feature of the isospectral flows is to have a set of first integrals that can be expressed as polynomials of different orders. A direct application of a Runge–Kutta method to (2.2) would not preserve these invariants. It has actually been proved that no Runge–Kutta method can be

[1]The transpose on the gradient depends on the identification of a Lie algebra with its dual for Lie–Poisson systems, as explained in section 1.2.4.

isospectral for all flows (2.2) when $n \geq 3$, [8, Cor. 6.1]. A popular method to overcome this issue is the so called Runge-Kutta-Munthe-Kaas scheme [11],[24]. The idea is to solve:

$$\dot{U} = UB(U^{-1}W_0U), \tag{2.3}$$

for U and then find W using (2.1). Since U is in a Lie group G, the Munthe-Kaas method consists in lifting (2.3) to its Lie algebra $\mathfrak{g}$ via some map from $\mathfrak{g} \to G$ (e.g., $\exp, \mathrm{Cay}$). Then, on $\mathfrak{g}$, any classical Runge-Kutta method can be applied. This method allows to preserve the isospectrality of the flow but in general not its Lie–Poisson structure and therefore, for example, we cannot expect (near) conservation of the Hamiltonian H. Another disadvantage is that the lifting can be expensive to compute. However, a clear pro of the RK-MK methods is that they provide explicit isospectral methods. A related technique is given by the symplectic Lie group methods on T^*G as developed by Bogfjellmo and Martinsen [5]. These methods rely on an invertible mapping between the Lie algebra and (an identity neighbourhood of) the Lie group, such as the exponential map (works in general) or the Cayley map (works for quadratic Lie groups). Another approach for solving (2.2) is given by the so called RATTLE method [11]. RATTLE is a general method for Hamiltonian systems with constraints. To use RATTLE for (2.1), one has to pull-back the equations from $\mathfrak{g}^*$ to T^*G and then solve the constrained Hamiloninan system with G as a constraint manifold in the vector space of all matrices. It indeed provides a Lie–Poisson integrator for (3.1) but with the burden of solving implicit equations to constrain the system on the right manifold. Some attempts of removing the constraints can be found for example in [21]. Our approach, presented in Paper I, has (independently) followed exactly that thread. Indeed, starting from some simple cases, it turns out that, with some manipulations of the canonical symplectic Runge-Kutta methods, in many cases the removal of the constraints is possible. This has led to a large class of isospectral methods directly defined on the Lie algebra. We refer to chapter 5 for further details.

Chapter 3

2D Euler equations

3.1 Spherical ideal hydrodynamics

Consider a homogeneous, incompressible, inviscid, two-dimensional fluid which is constrained to move on a spherical surface, embedded in the standard Euclidean $\mathbb{R}^3$, which is rotating with constant angular speed, with respect to a fixed normal axis. The equations of motion of such a fluid are given by the well known Euler equations of hydrodynamics:

$$\begin{aligned} &\dot{v} + v \cdot \nabla v = -\nabla p - 2\widetilde{\Omega} \times v \\ &\nabla \cdot v = 0 \end{aligned} \tag{3.1}$$

where v is the velocity vector field of the fluid, p is its internal pressure and $\widetilde{\Omega} = (\Omega \cdot n)n$ is the projection of the angular rotation of the sphere Ω to the normal n at a point of the sphere. The last term in the first equation of (3.1), $F_c = -2\widetilde{\Omega} \times v$ is called *Coriolis force*. The underlying geometry of equations (3.1) turns out to play a central role in understanding the behaviour of the fluid [2], [3], [18] and in the investigation of numerical methods to solve it [1], [20], [25], [26]. In particular the Euler equations (3.1) can be equivalently expressed in terms of the one form $v^\flat$ as a Lie–Poisson system on the dual of the infinite-dimensional Lie-algebra of divergence-free vector fields. The respective Poisson tensor is degenerate so that there is an infinite number of independent first integrals (Casimir functions) [3]. On the other hand, the simple connectedness of the spheres allows an equivalent formulation of (3.1) in terms of the vorticity $\omega = (\nabla \times v) \cdot n$. We notice that by the Stokes' theorem it must be that $\int \omega = 0$. Then the Euler equations (3.1) can be written as:

$$\begin{aligned} &\dot{\omega} = \{\psi, \omega\} \\ &\Delta\psi = \omega - f, \end{aligned} \tag{3.2}$$

where $f = 2\Omega \cdot n$ and ψ is the unique solution to the Poisson equation in $C^\infty(\mathbb{S}^2)$, such that $\int \psi = 0$. In this form, the Euler equations are a Lie–Poisson system defined on the smooth functions on the sphere, which integrate to 0. The Hamiltonian is given by

$$H(\omega) = \frac{1}{2}\int(\omega - f)\psi.$$

The (infinitely many) Casimir functions are given, for any smooth function $g : \mathbb{R} \to \mathbb{R}$, by $F(\omega) = \int g(\omega)$. In fact, it is easy to check:

$$\frac{d}{dt}\int g(\omega) = -\int g'(\omega)v \cdot \nabla\omega = -\int v \cdot \nabla g(\omega) = \int(\nabla \cdot v)g(\omega) = 0,$$

where we have used the following identity:

$$\{\psi, \cdot\}_p = (X_\psi)_p(\cdot) = p \cdot (\nabla\psi \times \nabla\cdot) = (p \times \nabla\psi) \cdot \nabla\cdot = -v_p \cdot \nabla \cdot .$$

for all $p \in \mathbb{S}^2$. The presence of all these first integrals is a major challenge in giving a suitable discretization of (3.2). In chapter 4, we will present a possible approach to tackle this issue.

3.1.1 Equivalence of rotating and non-rotating Euler equations

In this section, we show that for the spherical ideal hydrodynamics one can always reduce to study the non-rotating equations. Indeed, the $SO(3)$ equivariance of equations (3.2) allows a simple change of coordinates which takes away the Coriolis term. Consider the rotating Euler equations (3.2), for which the Poisson bracket has been replaced by the ad^* operator (see section 3.3):

$$\begin{aligned} \partial_t\omega &= \mathrm{ad}^*_{\Delta^{-1}(\omega-f)}\omega \\ \omega(0) &= \omega_0, \end{aligned} \tag{3.3}$$

for $\omega \in C^1([0,\infty), C^\infty(\mathbb{S}^2))$ and $f = 2\Omega\cos\vartheta$, $\Omega > 0$ and (φ, ϑ) are the azimuthal and the colatitude angle respectively. These can be written as:

$$\begin{aligned} \partial_t\omega &= \mathrm{ad}^*_{\Delta^{-1}(\omega-f)}\omega \\ &= \mathrm{ad}^*_{\Delta^{-1}\omega}\omega + \mathrm{ad}^*_{\frac{1}{2}f}\omega \end{aligned}$$

The first term at the RHS of the latter equation is equivariant w.r.t. the coadjoint action of $SO(3)$ and $\exp(-\frac{1}{2}ft) \in SO(3)$, for any $t \geq 0$.[1] Hence, applying

[1] Here with $\exp(-\frac{1}{2}ft)$ we indicate the diffeomorphism generated by Hamiltonian vector field $X_{-\frac{1}{2}f}$, which corresponds to a rotation with respect to the z axis of an angle Ωt.

both sides by $\mathrm{Ad}^*_{\exp(-\frac{1}{2}ft)}(\cdot)$, we obtain:

$$\begin{aligned} \mathrm{Ad}^*_{\exp(-\frac{1}{2}ft)}\partial_t\omega &= \mathrm{Ad}^*_{\exp(-\frac{1}{2}ft)}(\mathrm{ad}^*_{\Delta^{-1}\omega}\omega + \mathrm{ad}^*_{\frac{1}{2}f}\omega) \\ &= \mathrm{ad}^*_{\Delta^{-1}\widetilde{\omega}}\widetilde{\omega} + \mathrm{ad}^*_{\frac{1}{2}f}\widetilde{\omega}, \end{aligned}$$

where $\widetilde{\omega} = \mathrm{Ad}^*_{\exp(-\frac{1}{2}ft)}\omega$. From the identity:

$$\partial_t \mathrm{Ad}^*_{\exp(-\frac{1}{2}ft)}\omega = \mathrm{ad}^*_{-\frac{1}{2}f}\widetilde{\omega} + \mathrm{Ad}^*_{\exp(-\frac{1}{2}ft)}\partial_t\omega,$$

we conclude that $\widetilde{\omega}$ satisfies the following non-rotating Euler equations:

$$\begin{aligned} \partial_t\widetilde{\omega} &= \mathrm{ad}^*_{\Delta^{-1}\widetilde{\omega}}\widetilde{\omega} \\ \widetilde{\omega}(0) &= \omega(0). \end{aligned} \tag{3.4}$$

Hence, we have proved the following proposition:

Proposition 2. *Let $\omega \in C^1([0,\infty), C^\infty(\mathbb{S}^2))$ be a solution of* (3.3)*, for $f = 2\Omega\cos\vartheta$, $\Omega > 0$. Then $\widetilde{\omega} := Ad^*_{\exp(-\frac{1}{2}ft)}\omega = \omega(\varphi, \vartheta - \Omega t)$ is a solution of* (3.4).

3.2 Double periodic domain ideal hydrodynamics

In this section, we will consider another well studied model of 2D ideal hydrodynamics, which is the double periodic domain. Despite its less clear physical interpretation, the double periodic domain, also known as flat torus $\mathbb{T}^2$, has the advantage of allowing the simplest possible Fourier analysis and studying the fluid dynamics on a compact flat domain without boundary.

The Euler equations on $\mathbb{T}^2 := [-1,1)^2$ are given by:

$$\begin{aligned} &\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p \\ &\nabla\cdot\mathbf{v} = 0, \end{aligned} \tag{3.5}$$

where $\mathbf{v} : \mathbb{T}^2 \to \mathbb{R}^2$ is the velocity vector field and $p : \mathbb{T}^2 \to \mathbb{R}$ is the internal pressure field. The momentum vector is defined as:

$$\mathbf{L}(\mathbf{v}) := \int_{\mathbb{T}^2} \mathbf{v}\, dS.$$

It is known that $\mathbf{L}$ is a first integral of (3.5). We want write (3.5) in terms of vorticity, defined as

$$\omega := \nabla\times\mathbf{v} = \partial_x v_y - \partial_y v_x.$$

In terms of vorticity the Euler equations take the form:

$$\begin{aligned} &\partial_t\omega + \mathbf{v}\cdot\nabla\omega = 0 \\ &\nabla\cdot\mathbf{v} = 0, \end{aligned} \tag{3.6}$$

and the momentum vector:

$$\mathbf{M}(\omega) := \begin{pmatrix} \int_{\mathbb{T}^2} & \omega y dS + 2\int_{-1}^{1} v_x(x,1)dx \\ -\int_{\mathbb{T}^2} & \omega x dS + 2\int_{-1}^{1} v_y(1,y)dy \end{pmatrix}. \tag{3.7}$$

We notice that on $\mathbb{T}^2$ there is a two dimensional gauge in associating a velocity to a vorticity field, i.e. $\eta := \alpha\mathbf{e}_x + \beta\mathbf{e}_y$, where $\mathbf{e}_x, \mathbf{e}_y$ are the constant vector fields with respect to the x and y direction. The divergence free condition on $\mathbf{v}$ ensures that there exists a *stream function* $\psi : \mathbb{T}^2 \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ such that:

$$\mathbf{v} = \nabla^{\perp}\psi + \alpha\mathbf{e}_x + \beta\mathbf{e}_y.$$

Then it follows that $\Delta\psi = \omega$. By the Kelvin circulation theorem, it is clear that at any time $\int \omega = 0$. Without loss of generality, we can assume $\int \psi = 0$. From which it follows that it is well defined $\psi = \Delta^{-1}\omega$. Hence the momentum (3.7) can be written as:

$$\begin{aligned} \mathbf{M}(\omega) &= \begin{pmatrix} \int_{\mathbb{T}^2} & \omega y dS + 2\int_{-1}^{1} \partial_y \Delta^{-1}\omega(x,1)dx + \alpha \\ -\int_{\mathbb{T}^2} & \omega x dS - 2\int_{-1}^{1} \partial_x \Delta^{-1}\omega(1,y)dy + \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \end{aligned} \tag{3.8}$$

In conclusion, given a vorticity ω and a velocity $\mathbf{v}$ such that $\nabla \times \mathbf{v} = \omega$, using the conservation of momenta $\mathbf{L}$ and $\mathbf{M}$ we can require $\mathbf{L}(\mathbf{v}) = \mathbf{M}(\omega)$. Under this assumption equations (3.5) and (3.6) are equivalent.

Analogously to the sphere, equations (3.6) have infinitely many Casimir functions given by $F(\omega) = \int g(\omega)$, for any smooth function $g : \mathbb{R} \to \mathbb{R}$. Moreover, whenever $\alpha, \beta = 0$, equations (3.6) are Hamiltonian, with respect to $H(\omega) = \int_{\mathbb{T}^2} \psi\omega$.

3.3 Geometric structure of the Euler equations

The geometric picture of fluid dynamics dates back to Arnold [2]. The velocity vector field of a 2D incompressible fluid moving on a compact orientable Riemannian surface (S, g) may indeed be seen as a trajectory in the Lie algebra of divergence free vector fields, denoted by $\mathrm{sdiff}(S)$. The Euler equations (3.1) can be seen in this picture as a Lie–Poisson system on the dual space of $\mathrm{sdiff}(S)$. Given a volume form α, the pairing of 1-forms and vector fields

$$\langle \beta, X \rangle = \int_S \beta(X)\alpha,$$

where $\beta \in \bigwedge^1 S$ and X is a vector field on S. Then one gets that, for $X \in \mathrm{sdiff}(S)$, the pairing is invariant with respect to any exact translation of β:

$$\langle \beta + df, X \rangle = \langle \beta, X \rangle,$$

for any $f \in C^\infty(S)$.[2] Therefore we have that $\mathrm{sdiff}^*(S) = \bigwedge^1 S / d \bigwedge^0 S$.[3]

To avoid topological complications, let us continue to work on $S = \mathbb{S}^2$, which is simply connected. In [3] it is shown that the Euler equations (3.1) are equivalent to a Lie–Poisson system on $\bigwedge^1 \mathbb{S}^2 / d \bigwedge^0 \mathbb{S}^2 = \mathrm{sdiff}^*(\mathbb{S}^2)$ (which is isomorphic to the kernel of the 1-form divergence operator δ), with respect to the Hamiltonian function:

$$H([\eta]) = \frac{1}{2} \langle \eta - c^\flat, \eta^\sharp - c \rangle,$$

which represents the kinetic energy in the non inertial frame. Here $\eta = (v + c)^\flat$, $[\eta]$ is its respective class in $\bigwedge^1 \mathbb{S}^2 / d \bigwedge^0 \mathbb{S}^2$ and c is the velocity due to the rotation of the sphere. The Lie–Poisson system can be written as:

$$\dot{F}([\eta]) = \langle \mathrm{ad}^*_{dH}([\eta]), dF \rangle,$$

for any $F : \mathrm{sdiff}^*(\mathbb{S}^2) \to \mathbb{R}$, where $\mathrm{ad}^* : \mathrm{sdiff}(\mathbb{S}^2) \to \mathrm{End}(\mathrm{sdiff}^*(\mathbb{S}^2))$ is the co-adjoint representation of $\mathrm{sdiff}(\mathbb{S}^2)$, or equivalently (see I.6-7, [3]):

$$\dot{[\eta]} = -L_{dH}([\eta]).$$

where L is the Lie derivative. In our case, we have $dH = \eta^\sharp - c = v$. Hence:

$$\dot{[\eta]} = -L_v([\eta]). \tag{3.9}$$

Note that Lie–Poisson system above defined is respect to the dual pairing in $\mathrm{sdiff}(\mathbb{S}^2)$, being $dF \in (\mathrm{sdiff}^*(\mathbb{S}^2))^* \cong \mathrm{sdiff}(\mathbb{S}^2)$. To get rid of the equivalence class, we take the exterior derivative of (3.9) and, using the fact that $[L_v, d] = 0$, we get the Euler equations in the vorticity form:

$$\dot{\beta} = -L_v \beta, \tag{3.10}$$

where $\beta = d[\eta] \in \bigwedge^2 \mathbb{S}^2$ represents the vorticity of v. We write the vorticity in terms of the volume form α such that $\beta = \omega\alpha$, where the $\omega \in C^\infty(\mathbb{S}^2)$ and has zero mean. Then we get

$$L_v \beta = L_v(\omega\alpha) = (L_v \omega)\alpha + \omega L_v \alpha = (L_v \omega)\alpha,$$

being v volume preserving. By taking the Hodge star of (3.10), via the identification $\bigwedge^2 \mathbb{S}^2 \cong \bigwedge^0 \mathbb{S}^2 = C^\infty(\mathbb{S}^2)$, we can understand (3.10) in $C^\infty_0(\mathbb{S}^2)$, i.e., the space of smooth functions which integrate to 0. Hence, we get a map $*d : \mathrm{sdiff}^*(\mathbb{S}^2) \to C^\infty_0(\mathbb{S}^2)$ between two Lie–Poisson systems. If we call ad^*_1 the Lie–Poisson structure in $\mathrm{sdiff}^*(\mathbb{S}^2)$ and ad^*_2 the Lie–Poisson structure in $C^\infty_0(\mathbb{S}^2)$, then we have:

[2] This is easily checked as $\int df(X)\alpha = \int (\iota_X) df \alpha = \int (L_X f)\alpha = \int f(L_X \alpha) = 0$, where $f \in C^\infty(S)$ and we have used the fact that X is volume preserving.

[3] See Chp. 1 for this notation.

Theorem 3. *The map* $\pi \equiv *d : \mathrm{sdiff}^*(\mathbb{S}^2) \to C_0^\infty(\mathbb{S}^2)$ *is a Lie–Poisson isomorphism.*

Proof. Let $v \in \mathrm{sdiff}(\mathbb{S}^2)$ and $[\eta] \in \mathrm{sdiff}^*(\mathbb{S}^2)$. Let us call $\omega = d[\eta]$ and, as above, again $*\omega = \omega$.

$$\pi \circ \mathrm{ad}^*_{1v}[\eta] = - * dL_v[\eta] = - * L_v d[\eta] = -L_v\omega = L_{X_\psi}\omega = \{\psi, \omega\} = \mathrm{ad}^*_{2\psi}\omega,$$

where ψ is the only function in $C_0^\infty(\mathbb{S}^2)$ such that $X_\psi = -v$ (it exists being v divergence free and being $\mathbb{S}^2$ simply connected.). □

3.4 Point-vortex dynamics

In this section we present a class of highly singular solutions to the 2D Euler equations, called *point-vortices*. Since their discovery by Helmholtz in 1858 [12], extensively studies on these objects have been conducted. On the one hand, a well-defined limit of an infinite number of point-vortices is known to approximate the continuous fluid (see [17, Chp. 5.3]) and, on the other hand, for a small number of point-vortices one gets a surprisingly rich dynamical system in terms of geometric mechanics (see [22]).

In paper II and IV, we have focused our attention in connecting the statistical state of a 2D fluid with the integrability theory of point-vortices. This connection shows that, after the initial turbulent regime of a 2D fluid, the core of the fluid dynamics can be very well represented in terms of few point-vortices. Moreover, the number of occurring vortices highly depends on the topology of the ambient space of the fluid.

Let (M, g) be an orientable 2D Riemannian manifold, and let $G : M \times M \to \mathbb{R}$ the Green's function of the Laplace operator:

$$\Delta G(x, \cdot) = \delta_x(\cdot),$$

for any $x \in M$. Then, for any $N > 1$ and $\Gamma_i \in \mathbb{R} \setminus \{0\}$, for $i = 1, \ldots, N$, the point-vortex equations are defined as:

$$\dot{x}_i = \sum_{i \neq j} \Gamma_j \nabla^\perp_{x_i} G(x_i, x_j), \tag{3.11}$$

where ∇_{x_i} is the skew gradient with respect to x_i, i.e., the operation which first takes the gradient with respect to x_i and then rotates this vector by $\pi/2$ in positive orientation. Equations (3.11) correspond to the 2D Euler equations on M for an initial vorticity $\omega = \sum_{i=1}^N \Gamma_i \delta_{x_i}$. For example on the sphere equations (3.11) are given by:

$$\dot{x}_i = \frac{1}{2\pi} \sum_{i \neq j} \Gamma_j \frac{x_i \times x_j}{1 - x_i \cdot x_j}. \tag{3.12}$$

It is important to notice that the Hamiltonian structure of the 2D Euler equations descends towards an Hamiltonian structure for (3.11). Given on M a compatible symplectic structure Ω_M with the metric g, equations (3.12) are a Hamiltonian system on $(M^N, \Omega := \oplus_{i=1}^N \Gamma_i \Omega_{M^i})$, with Hamiltonian function:

$$H = \sum_{i \neq j} \Gamma_i \Gamma_j G(x_i, x_j).$$

Therefore, in order to understand the qualitative behaviour of the dynamical system (3.11), it is crucial to look at the symmetries of the Hamiltonian H, which preserve the symplectic form Ω. These symmetries depend on the topology and the Riemannian structure of the manifold M and are equivalent for the 2D Euler equations and the point-vortex dynamics. In paper II and IV, this connection is thoroughly analysed and discussed.

Chapter 4

Quantization of 2D Euler equations

In this chapter, we present a discrete model for the 2D Euler equations, known in the literature as *consistent truncation* of the Euler equations [25, 26]. The main feature of this approach is to obtain a spatial discretization of the Euler equations (3.2) such that the finite model obtained is again a Lie–Poisson system which converges in some sense to the exact one. In paper II, it is shown, for the 2D Euler equations on a sphere, how to get a space-time discretization such that the numerical approximation is still a Lie–Poisson system (in the backward analysis sense). Moreover, it is proved, showing new insights in the long-time evolution of the fluid, the advantages in using a conservative numerical scheme.

In this chapter, we recall the main results in literature on the quantization of 2D Euler equations, and we show how this approach can be extremely powerful in understanding the original continuous model, though with some limitations depending on the fluid domain. In particular, we will treat the two cases of the sphere and the torus. For the first one, we will also present a new theory for the formation of quasi-zonal flows. For the torus, we will show that the quantized model of the Euler equations is qualitatively not such a good approximation. Indeed, despite retaining the Lie–Poisson structure, the discrete model loses the fundamental translational symmetries, causing Gibbs phenomena in the numerical simulations.

4.1 L_α-convergence

In this section, we recall the spatial discretization introduced in [25, 26]. The space we want to approximate is the one of smooth functions with zero mean on a compact orientable surface S. This space will be denoted with $C_0^\infty(S,\mathbb{C})$. On this space, it is naturally defined a Lie algebra structure, coming from a symplectic form Ω defined on S:

$$\{f,g\} := \Omega(X_f, X_g). \tag{4.1}$$

for any $f,g \in C_0^\infty(S,\mathbb{C})$. Hence, we would like to have a discretization of $C_0^\infty(S,\mathbb{C})$ on which it could be defined a suitable Lie algebra structure. This can be precisely defined in terms of L_α-convergence.

Let us consider a Lie-algebra $(\mathfrak{g},[\cdot,\cdot])$ and a family of labelled Lie algebras $(\mathfrak{g}_\alpha,[\cdot,\cdot]_\alpha)_{\alpha\in I}$, where $\alpha \in I = \mathbb{N}$ or $\mathbb{R}$. Furthermore, assume then that to any element of this family it is associated a distance d_α and a surjective projection map $p_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$. Then we will say that $(\mathfrak{g},[\cdot,\cdot])$ is an L_α-limit of $(\mathfrak{g}_\alpha,[\cdot,\cdot]_\alpha)_{\alpha\in I}$ if:

- if $x,y \in \mathfrak{g}$ and $d_\alpha(p_\alpha(x),p_\alpha(y)) \to 0$, for $\alpha \to \infty$, then $x = y$,
- for all $x,y \in \mathfrak{g}$ we have $d_\alpha(p_\alpha([x,y]),[p_\alpha(x),p_\alpha(y)]_\alpha) \to 0$, for $\alpha \to \infty$.

The above definition is given in [6] and it is a quite weak requirement to get a limit for a sequence of Lie algebras. Indeed the same sequence may converge in the L_α sense to different Lie algebras [7]. The information that relates the approximating sequence and the target algebra is encoded in the projections p_α, that are not canonical. However, since we have already the target $(\mathfrak{g},[\cdot,\cdot]) = (C_0^\infty(S,\mathbb{C}),\{\cdot,\cdot\})$, we are not interest in the uniqueness of the limit of an approximating sequence.[1] In [6], it is shown that $C_0^\infty(S,\mathbb{C})$ can always be approximated by the matrix Lie algebra $(\mathfrak{sl}(N,\mathbb{C}),[\cdot,\cdot]_N)_{N\in\mathbb{N}}$, where $[\cdot,\cdot]_N$ is a suitably rescaled commutator of matrices. In order to get an explicit definition of the projections p_α, it must be chosen a basis for $C_0^\infty(S,\mathbb{C})$ and $\mathfrak{sl}(N,\mathbb{C})$, for any $N > 0$.

Specific calculations for the sphere and the flat 2-torus have been carried out in [14] and [10, 23]. In particular, on the sphere, a L^2-basis for $C_0^\infty(\mathbb{S}^2,\mathbb{C})$ is given by the complex spherical harmonics, which will be denoted in the standard notation and azimuthal-inclination coordinates (ϕ,ϑ) as:

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\vartheta)e^{im\phi},$$

where P_l^m are the associate Legendre polynomials, for $l \geq 1$ and $m = -l,\ldots,l$. On $\mathfrak{sl}(N,\mathbb{C})$, define $[\cdot,\cdot]_N = N^{3/2}[\cdot,\cdot]$ and the distances d_α given by a suitable

[1] The ad and the ad* representations on $C_0^\infty(S,\mathbb{C})$ differ only by a sign. Therefore, we identify the Lie-Poisson system on $C_0^\infty(S,\mathbb{C})$ with its dual representation on the Lie algebra $(C_0^\infty(S,\mathbb{C}),\{\cdot,\cdot\})$.

matrix norm. Then, the projections are defined by associating to any spherical harmonic a respective matrix, for any $N \in \mathbb{N}$, i.e., $p_N : Y_{lm} \mapsto T^N_{lm}$, where

$$(T^N_{lm})_{m_1 m_2} = (-1)^{N/2-m_1}\sqrt{2l+1}\begin{pmatrix} N/2 & l & N/2 \\ -m_1 & m & m_2 \end{pmatrix},$$

where the round bracket is the Wigner 3j-symbol.

On the flat 2-torus, a L^2-basis for $C^\infty_0(\mathbb{T}^2, \mathbb{C})$ is given by the standard Fourier basis:

$$f_{\mathbf{n}}(x,y) = e^{2\pi i n_1 x} e^{2\pi i n_2 y},$$

for $x, y \in [-1, 1)$ and $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$. On $\mathfrak{sl}(N, \mathbb{C})$, define $[\cdot,\cdot]_N = N[\cdot,\cdot]$ and the distances d_α given by a suitable matrix norm. Then, the projections are defined by associating to any basis element a respective matrix, for any $N \in \mathbb{N}$, i.e., $p_N : f_{\mathbf{n}} \mapsto T^N_{\mathbf{n}}$, where

$$T^N_{\mathbf{n}} = \omega^{n_1 n_2/2} g^{n_1} h^{n_2},$$

and

$$\omega = e^{\frac{2\pi i}{N}} Id, \qquad g = \mathrm{diag}(1, \omega, \dots, \omega^{N-1}), \qquad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

4.2 The quantized 2D Euler equations

We can now derive the spatial discretization of the Euler equations via the L_α-approximation. Consider the Euler equations

$$\begin{aligned} \dot\omega &= \{\psi, \omega\} \\ \Delta\psi &= \omega, \end{aligned}$$

where $\omega \in C^\infty_0(S, \mathbb{R})$. Then, for any $N \in \mathbb{N}$, we get the analogous matrix equations:

$$\dot W = [\Delta_N^{-1} W, W]_N, \tag{4.2}$$

where $W \in \mathfrak{su}(N)$ and Δ_N^{-1} is the inverse of a suitable discrete Laplacian. The definition of Δ_N is not unique, but it is often taken such that it keeps the spectral properties of the Laplace operator. In particular, on the sphere, it can be taken such that:

$$\Delta_N^{-1} T^N_{lm} = -1/(l(l+1)) T^N_{lm},$$

for any $l = 1, ..., N$, $m = -l, ..., l$, and on the 2-torus, such that:

$$\Delta_N^{-1} T^N_{\mathbf{n}} = -1/(n_1^2 + n_2^2) T^N_{\mathbf{n}},$$

for any $\mathbf{n} \in \mathbb{Z}^2$. We remark that, for a real valued vorticity, W is actually in $\mathfrak{su}(N)$, which means that $\overline{W}_{lm} = (-1)^m W_{l-m}$. The discrete Hamiltonian takes the following form:

$$H(W) = \frac{1}{2}\operatorname{Tr}(\Delta_N^{-1} W W^\dagger).$$

The discrete system has the following independent $N-1$ first integrals[2]

$$F_k(W) = \operatorname{Tr}(W^k)$$

for $k = 2, ..., N$, which, up to a normalization constant depending on N, converge to the powers of the continuous vorticity. Moreover, it has been proven in [6, Thm. 4.1] that there exists a constant $C \geq 0$, independent of N, such that

$$\|W\| \leq \|\omega\|_\infty \leq \|W\| + \frac{C}{N}$$

where $\|W\|$ is the matrix (operator) norm of $W \in \mathfrak{su}(N)$ and ω is the vorticity function corresponding to W. Since $\|W\|$ is the largest eigenvalue (in magnitude) of W, and since all the eigenvalues are conserved by the quantized flow (the isospectral property), we get that $\|\omega\|_\infty$ is nearly conserved in the quantized system.

4.3 The case of the sphere

4.3.1 The role of the Coriolis force

The quantized 2D Euler equations (4.2), in the presence of the Coriolis force, take the form:

$$\begin{aligned} \partial_t W &= [\Delta^{-1}(W - F), W]_N \\ W(0) &= W_0, \end{aligned} \tag{4.3}$$

for $C^1([0,\infty), \mathfrak{su}(N))$ and $F = 2\Omega i T_{10}$, $\Omega > 0$ is the Coriolis vorticity.

As we have seen in section 3.1.1, the Euler equations in the presence of the Coriolis force can be written as the non-rotating equations in a suitable frame. Here we show that the same property holds for the quantized model. Let us write equations (4.3) as:

$$\begin{aligned} \partial_t W &= [\Delta^{-1}(W - F), W]_N \\ &= [\Delta^{-1} W, W]_N + [\tfrac{1}{2} F, W]_N \end{aligned}$$

The first term of the RHS of the latter equation is equivariant w.r.t. the coadjoint action of $SO(3)$ and $\exp(-\frac{1}{2}Ft) \in SO(3)$, for any $t \geq 0$. In particular,

[2]One should notice that by definition $\operatorname{Tr}(W) = 0$ for all $W \in \mathfrak{sl}(N, \mathbb{C})$ and for $k > N$, $\operatorname{Tr}(W^k)$ are dependent to those with $k \leq N$, by the Cayley–Hamilton theorem.

multiplying both sides by $\exp(\frac{1}{2}Ft)(\cdot)\exp(-\frac{1}{2}Ft)$, we obtain:

$$\begin{aligned}\exp(-\tfrac{1}{2}Ft)\partial_t W \exp(\tfrac{1}{2}Ft) &= \exp(-\tfrac{1}{2}Ft)[\Delta^{-1}W, W]_N \exp(\tfrac{1}{2}Ft)+ \\ &+ \exp(-\tfrac{1}{2}Ft)[\tfrac{1}{2}F, W]_N \exp(\tfrac{1}{2}Ft) \\ &= [\Delta^{-1}\widetilde{W}, \widetilde{W}]_N + [\tfrac{1}{2}F, \widetilde{W}]_N,\end{aligned}$$

where $\widetilde{W} = \exp(-\frac{1}{2}Ft)W\exp(\frac{1}{2}Ft)$. From the identity:

$$\partial_t\widetilde{W} = -\frac{1}{2}F\widetilde{W} + \exp(-\frac{1}{2}Ft)\partial_t W \exp(\frac{1}{2}Ft) + \widetilde{W}\frac{1}{2}F,$$

we conclude that $\widetilde{W}$ satisfies the following quantized non-rotating Euler equations:

$$\begin{aligned}\partial_t\widetilde{W} &= [\Delta^{-1}\widetilde{W}, \widetilde{W}]_N \\ \widetilde{W}(0) &= W(0).\end{aligned} \tag{4.4}$$

Hence, we have proved the following proposition:

Proposition 3. *Let $W \in C^1([0,\infty), \mathfrak{su}(N))$ be a solution of (4.3), for $F = 2\Omega i T_{10}$, $\Omega > 0$. Then $\widetilde{W} := \exp(-\frac{1}{2}Ft)W\exp(\frac{1}{2}Ft)$ is a solution of (4.4).*

Therefore, all the phenomena happening in the non-rotating Euler equations can be found in the rotating ones and viceversa. In particular, in the next section we will show that the quasi-zonal band structure appearing in some fast rotating planets, like Jupiter, can be obtained by some specific perturbation of the Rossby-Haurwitz waves.

4.3.2 Sliced matrix subalgebras and quasi-zonal flows

In this section, we show how quasi-zonal flows can be obtained by some specific perturbation of the Rossby-Haurwitz waves. Here with quasi-zonal flow we understand a zonal flow with possibly other vortices trapped into the zonal bands. The interest in this derivation is twofold. On the one hand, it is shown the mechanism that cause the drift of a Rossby-Haurwitz wave into a quasi-zonal flow, which in virtue of proposition 3 cannot depend on the rotation of the sphere.[3] On the other hand, we show that quasi-zonal flows correspond to a reduced dynamics into some Lie subalgebra of $\mathfrak{su}(N)$ of sliced (or banded) matrices. Since these subalgebras are reductive, we can perform a complete classification of them, which gives a deep insight in the kind of quasi-zonal flow obtained. In particular, the quasi-zonal flow will be characterized by a certain amount of bands and a vortices trapped into the bands.

[3]However, the rotation of the sphere could play a role in the stability of the quasi-zonal flows. This question is still under investigation.

As noticed in the appendix of paper II, Rossby-Haurwitz (RH) waves are exact solutions to the Euler equations. They are defined in terms of spherical harmonics as:

$$\omega(\phi,\vartheta,t) = Cf + \sum_{m=-l}^{l} \omega^{lm} Y_{lm}(\phi + 2\Omega\alpha_l t, \vartheta) \tag{4.5}$$

where $\alpha_l = \frac{1}{2}\left(\frac{2C}{l(l+1)} - C + 1\right)$, $\omega^{lm} \in \mathbb{C}$, $C \in \mathbb{R}$ and $l = 1, 2, \ldots$. That (4.5) are exact solutions to (3.2) depends only on the algebraic properties of the Poisson bracket of the spherical harmonics. Indeed, it is not hard to check that we get an analogous class of exact solutions to (4.3) in terms of T^N_{lm}, as defined in section 4.1:

$$W(t) = C \cdot F + \exp(-\alpha_l N^{3/2} F \cdot t) \sum_{m=-l}^{l} W^{lm} \mathrm{i} T^N_{lm} \exp(\alpha_l N^{3/2} F \cdot t) \tag{4.6}$$

where $\alpha_l = \frac{1}{2}\left(\frac{2C}{l(l+1)} - C + 1\right)$, $W^{lm} \in \mathbb{C}$, $C \in \mathbb{R}$ and $l = 1, 2, \ldots, N$ and exp is the usual matrix exponential. We call these solutions *quantized RH waves.*

Since we only consider real vorticity, we have the further symmetry of the coefficients: $W^{l-m} = (-1)^m \overline{W}^{lm}$. This symmetry comes from the fact that $(iT^N_{lm})^\dagger = (-1)^m iT^N_{l-m}$. Hence, W is in $\mathfrak{su}(N)$ and we can consider the respective basis elements in $\mathfrak{su}(N)$ defined as:

$$R^N_{lm} = \begin{cases} \frac{1}{\sqrt{2}}(T^N_{lm} - (-1)^m T^N_{l-m}) & \text{for } m > 0 \\ iT^N_{l0} & \text{for } m = 0 \\ \frac{i}{\sqrt{2}}(T^N_{l-m} + (-1)^m T^N_{lm}) & \text{for } m < 0 \end{cases}$$

for any $l = 1, \ldots N, m = -l, \ldots, l$. The matrices R^N_{lm} have a banded structure, having non-zero entries only in the $\pm m$ diagonals.

Let us define the spaces $\mathfrak{D}(N,k)$. Let $\mathfrak{D}(N,k)$ be the subalgebra of $\mathfrak{u}(N)$, for some $N > 0$ and $0 < k \leq N$, defined as follows. Let $\mathfrak{D}(N,k)$ be the set of all the matrices in $\mathfrak{u}(N)$ (seen as the $N \times N$ skew-Hermitian matrices) that have non-zero entries only in the $0^{th}, \pm k^{th}, \pm 2k^{th}, \ldots$ diagonals. Then $\mathfrak{D}(N,k)$ is a reductive Lie algebra, being closed under complex conjugate transpose.

Lemma 1. *The bracket closure of $R^N_{lm}, R^N_{l'm'}$ is included in the $\mathfrak{D}(N,k)$, for $k = gcd(m,m')$.*

Proof. The bracket closure of some collection of square matrices $\mathcal{A} = \{A_i\}_{i\in I}$ is the smallest Lie algebra containing the repeated bracketing of elements in $\mathcal{A}$.

Let $\mathcal{A} = \{A, B\}$, for $A = R^N_{lm}, B = R^N_{l'm'}$, and define $C = [A, B]$. Then, for any $i, j = 1, .., N$:

$$C_{i,j} = \sum_{k=1}^{N} A_{i,k}\delta^k_{i\pm m}\delta^k_{j\pm m'}B_{k,j} - B_{i,k}\delta^k_{i\pm m'}\delta^k_{j\pm m}A_{k,j}.$$

Hence, $C_{i,j} \neq 0$ only if $i = j \pm m' \mp m$ or $i = j \pm m \mp m'$. This implies that the bracket of two banded matrices is still banded. Iterating the bracketing, we notice that the smallest bands gap which is possible to reach is:

$$k = \min_{a,b\in\mathbb{Z}} \{|am + bm'|, \text{ s.t. } |am + bm'| > 0\},$$

which is equal to the greatest common divisor of m, m'. $\square$

As shown below, the spaces $\mathfrak{D}(N, k)$ are invariant under the Euler equations. Therefore, a perturbation of a *quantized RH waves* in some specific direction R^N_{lm} will evolve in some space $\mathfrak{D}(N, k)$. On the one hand, since the diagonal matrices correspond to zonal-flows, it follows from the definition of the $\mathfrak{D}(N, k)$ that the larger the ratio k/N is, the more the vorticity will be zonal. On the other hand, the number k determines the blobs trapped into these zonal bands. Indeed, the lowest non-zero off-diagonal components correspond to the spherical harmonics Y_{lk}, for $l = k, \ldots, N$ which have k latitudinal blobs.

As an example, in figure 4.1, we report the evolution of an unstable Rossby-Haurwitz wave presented in paper II. The quantized RH wave is defined by:

$$C = 1, \quad W^{10} = 12.9487, \quad W^{54} = W^{5(-4)} = 7.7300, \tag{4.7}$$

for $N = 501$. Here, the perturbation is in the direction of some linear combination of $\{R^N_{l4}\}_{l\geq 4}$, due to a numerical rounding error which caused the loss of linear proportionality, up to the Coriolis vorticity term, between the stream function and the vorticity. Hence, the RH wave evolves in $\mathfrak{D}(501, 4)$, which creates a quasi-zonal flow with four blobs trapped in the zonal bands.

The following theorem gives the explicit reductive structure of the $\mathfrak{D}(N, k)$.

Theorem 4.

$$\mathfrak{D}(N, k) \cong \mathfrak{u}\,(d)^{k-r} \oplus \mathfrak{u}\,(d+1)^r \cong \mathfrak{su}\,(d)^{k-r} \oplus \mathfrak{su}\,(d+1)^r \oplus \mathfrak{u}(1)^k.$$

where $d = \lfloor \frac{N}{k} \rfloor$ *and* $r = N - kd$, *where* $\lfloor\cdot\rfloor$ *denotes the integer part. In particular,* $dim(\mathfrak{D}(N, k)) = kd^2 + 2rd + r = d(N + r) + r$.

Proof. **Step 1.** The Lie algebra $\mathfrak{D}(N, k)$ can be decomposed into k subalgebras. Let $A, B \in \mathfrak{D}(N, k)$ and consider $C = [A, B] \in \mathfrak{D}(N, k)$. Then, for any

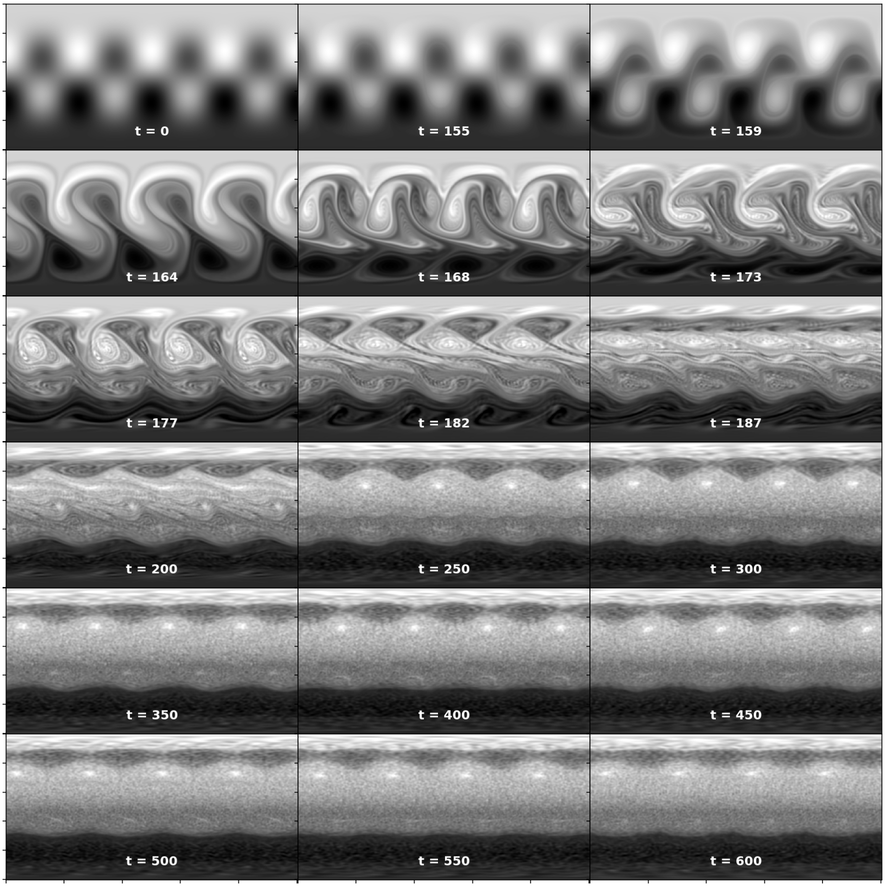

Figure 4.1: Unsteady quantized RH wave, for the initial conditions as in (4.6) with parameters (4.7). Due to numerical rounding errors, the wave eventually breaks up, goes through an intermediate transition, and then reaches a quasi-periodic asymptotic with sliding zonal vortex belts.

$a = 1, .., k$:

$$C_{a+ik,a+jk} = \sum_{h=0}^{\lfloor \frac{N-a}{k} \rfloor} A_{a+ik,a+hk} B_{a+hk,a+jk} - B_{a+ik,a+hk} A_{a+hk,a+jk},$$

for all $i, j = 0, ..., \lfloor \frac{N-a}{k} \rfloor$. Hence, defining

$$\mathfrak{g}_a = \{M \in \mathfrak{D}(N,k) \text{ s.t. } M_{i,j} \neq 0 \text{ iff } i \equiv a \bmod k, j \equiv a \bmod k\},$$

we see that for any $a = 1, ..., k$ the $\mathfrak{g}_a$ are closed under the matrix commutator and that $[\mathfrak{g}_a, \mathfrak{g}_b] = 0$, for $a \neq b$. In fact, $AB = 0$, for any $A \in \mathfrak{g}_a, B \in \mathfrak{g}_b$, for $a \neq b$. Finally, since any element in $\mathfrak{D}(N, k)$ is a linear combination of elements in the $\mathfrak{g}_a$, for some $a = 1, ..., k$, we conclude that:

$$\mathfrak{D}(N, k) = \oplus_{a=1}^{k} \mathfrak{g}_a.$$

$$\mathfrak{D}(7,3) = \left\{ A = \begin{bmatrix} * & 0 & 0 & * & 0 & 0 & * \\ 0 & x & 0 & 0 & x & 0 & 0 \\ 0 & 0 & = & 0 & 0 & = & 0 \\ * & 0 & 0 & * & 0 & 0 & * \\ 0 & x & 0 & 0 & x & 0 & 0 \\ 0 & 0 & = & 0 & 0 & = & 0 \\ * & 0 & 0 & * & 0 & 0 & * \end{bmatrix}, A \in \mathfrak{u}(7) \right\} \cong \mathfrak{u}\,(2)^2 \oplus \mathfrak{u}\,(3)$$

Figure 4.2: Example of a decomposition of $\mathfrak{D}(7, 3)$ as explained in Step 1, where the symbols $\{*, x, =\}$ denote the only possible non-zero entries.

Step 2. It is clear that $\mathfrak{g}_a$ are closed under complex conjugate transpose, and no further restriction is present. Therefore, $\mathfrak{g}_a \cong \mathfrak{u}(n_a)$, for some $0 < n_a \leq N$.
Step 3. It is straightforward to check that $n_a = \lfloor \frac{N-a}{k} \rfloor + 1$, for any $a = 1, ..., k$. Therefore, defining $r = N - k\lfloor \frac{N}{k} \rfloor$, we have that for $a = 1, ..., r$, $n_a = \lfloor \frac{N}{k} \rfloor + 1$ and for $a = r + 1, ..., k$, $n_a = \lfloor \frac{N}{k} \rfloor$. □

Let $\mathcal{B} = \{X_1, X_2, X_3\}$ be a basis of $\mathfrak{su}(2)$ and let B a non-degenerate, ad-inviariant, bilinear form B on $\mathfrak{su}(2)$ (e.g. B can be assumed to be the Killing form). Then let $\mathcal{B}^\vee = \{X^1, X^2, X^3\}$ be the dual basis of $\mathcal{B}$ with respect to B. Then, the Casimir element C of $\mathfrak{su}(2)$ is defined as an element of the universal enveloping algebra $\mathfrak{U}(\mathfrak{su}(2))$ of $\mathfrak{su}(2)$ and can be written as

$$C = \sum_{i=1}^{3} X_i X^i.$$

Notice that this definition of C does not depend neither on the choice of $\mathcal{B}$ nor B. Consider the irreducible representation ϱ of $\mathfrak{su}(2)$ on $\mathbb{C}^n$ and the adjoint representation of $\mathfrak{su}(N)$ on itself. Define the discrete Laplacian as:

$$\Delta_N := \mathrm{ad}_{\varrho(C)} := \sum_{i=1}^{3} \mathrm{ad}_{\varrho(X_i)} \circ \mathrm{ad}_{\varrho(X_i)} : \mathfrak{su}(N) \to \mathfrak{su}(N).$$

Finally, in this notation, the discrete Laplacian Δ_N acts on $\mathfrak{su}(N)$ as:

$$\Delta_N(Y) = \mathrm{ad}_{\varrho(C)}(Y) = [\varrho(X_1), [\varrho(X_1), Y]] + [\varrho(X_2), [\varrho(X_2), Y]] + [\varrho(X_3), [\varrho(X_3), Y]],$$

for any $Y \in \mathfrak{su}(N)$.

Theorem 5. *The discrete Laplacian* Δ_N *is a vector space endomorphism of* $\mathfrak{D}(N,k)$.

Proof. It is clear that:

$$\mathfrak{D}(N,k) = \text{Span}\left\{ R^N_{l\pm\alpha k}, \text{ for } l = 1,\dots,N, \alpha = 0,\dots,\left\lfloor \frac{l}{k} \right\rfloor \right\} \oplus i\mathbb{R}Id.$$

Hence, since $\{R^N_{l\pm\alpha k}, Id\}$ are eigenvectors of Δ_N, $\mathfrak{D}(N,k)$ is invariant with respect to the action of Δ_N. □

Corollary 1. *The discrete Laplacian* Δ_N *is a vector space automorphism of* $\mathfrak{D}(N,k) \cap \mathfrak{su}(N)$. *Hence, the quantized Euler equations* (4.2) *can be restricted to* $\mathfrak{D}(N,k) \cap \mathfrak{su}(N)$.

4.4 The case of the torus

In this section, we analyse in detail the algebraic structure of the 2D Euler equations on a flat-torus and their quantized version. In particular, we will show that the translational invariance of the continuous equations is not preserved under the quantization discretization. This fact shows that the quantization procedure is much more natural on the sphere, where the $SO(3)$ invariance of the Euler equations is preserved in the discrete model.

In section 4.4.3, we show that the quantized Euler equations on the torus have only a finite symmetry group. This may be a crucial feature of the discretization. Indeed, numerical evidences, still under investigation and not presented in this thesis, suggest that schemes that preserve energy and enstrophy, like the Arakawa's finite difference scheme [1], and which are translational invariant perform much better than the scheme proposed in [20, 25].

4.4.1 Euler equations in Fourier space

Let us consider the Euler equations (3.5). The velocity vector field can always be uniquely decomposed as:

$$\mathbf{v} = \nabla^\perp \psi + \alpha \mathbf{e}_x + \beta \mathbf{e}_y,$$

for some function $\psi : \mathbb{T}^2 \to \mathbb{R}$ called *stream function*, and some $\alpha, \beta \in \mathbb{R}$. Using the stream function, equations (3.6) can be written as:

$$\partial_t \omega + \nabla^\perp \psi \cdot \nabla \omega + \alpha \partial_x \omega + \beta \partial_y \omega = 0, \tag{4.8}$$

or, in coordinates:

$$\partial_t \omega = \partial_x \psi \partial_y \omega - \partial_y \psi \partial_x \omega - \alpha \partial_x \omega - \beta \partial_y \omega. \tag{4.9}$$

If we write the vorticity in terms of Fourier modes, i.e. $\omega = \frac{1}{2} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \omega_{n_1,n_2} e^{2\pi i n_1 x} e^{2\pi i n_2 y}$, equations (4.9) become:

$$\partial_t \omega_{\mathbf{n}} = \sum_{\mathbf{k} \in \mathbb{Z}^2} \frac{\mathbf{n} \times \mathbf{k}}{|\mathbf{k}|^2} \omega_{\mathbf{n}-\mathbf{k}} \omega_{\mathbf{k}} - \alpha n_1 \omega_{\mathbf{n}} - \beta n_2 \omega_{\mathbf{n}}, \tag{4.10}$$

for each $\mathbf{n} \in \mathbb{Z}^2$, where $\mathbf{n} = (n_1, n_2)$ and $\mathbf{n} \times \mathbf{k} = n_1 k_2 - n_2 k_1$. Notice that equations (4.10) are well defined, being $\int_{\mathbb{T}^2} \omega dS = \omega_{(0,0)} = 0$. Being $\psi_{n_1,n_2} = \frac{\omega_{n_1,n_2}}{-(n_1^2 + n_2^2)}$, the momentum (3.8) can be expressed in Fourier modes as:

$$\begin{aligned} \mathbf{M}(\omega) &= \frac{1}{2\pi i} \begin{pmatrix} \sum_{n_2=-\infty}^{\infty} \frac{\omega_{0,n_2}}{n_2} + \sum_{n_2=-\infty}^{\infty} n_2 \frac{\omega_{0,n_2}}{-n_2^2} \\ -\sum_{n_1=-\infty}^{\infty} \frac{\omega_{n_1,0}}{n_1} + \sum_{n_1=-\infty}^{\infty} n_1 \frac{\omega_{n_1,0}}{-n_1^2} \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \end{aligned} \tag{4.11}$$

Hence, the conservation of the momentum does not imply any further condition on the Fourier coefficients of ω.

4.4.2 The symmetry group of the Hamiltionian of the Euler equations on $\mathbb{T}^2$

The Hamiltionian of the Euler equations (4.9), for $\alpha = \beta = 0$, is defined as:

$$H(\omega) = \int_{\mathbb{T}^2} \psi \omega dS.$$

The Hamiltionan H has symmetry group isomorphic to $\mathbb{R}^2$, acting on the vorticity as $(a,b).\omega(x,y) = \omega(x+a, y+b)$, for any $(x,y) \in \mathbb{T}^2$ and $(a,b) \in \mathbb{R}^2$. The infinitesimal symmetry of this action corresponds to the two dimensional Abelian Lie algebra generated by $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. This two vector fields do not correspond to a momentum map. In fact, the "formal" generators would be $\{-\int \omega y, \int \omega x\}$, but since $f(x,y) = x, g(x,y) = y$ do not have a strong derivative on $\mathbb{T}^2$, the formal generators are ill-posed.

4.4.3 The sine-bracket case

Let us now consider the *sine-bracket* truncation:

$$\partial_t \omega_{\mathbf{n}} = \sum_{k_1,k_2=-N}^{N} \sin\left(\frac{\mathbf{n} \times \mathbf{k}}{N}\right) \frac{N}{|\mathbf{k}|^2} \omega_{\mathbf{n}-\mathbf{k}} \omega_{\mathbf{k}}, \tag{4.12}$$

for each $n_1, n_2 \in [-N, N]$, for $N > 0$ odd integer. We want to show that equations (4.12) do not have $\mathbb{R}^2$ symmetry group, but rather a finite subgroup of $SU(N)$.

In order to find the symmetry group of the Hamiltonian in the sine-bracket model, we should look at symmetries of the Hamiltonian which come from a Lie–Poisson action on $\mathfrak{su}(N)^*$. Any Lie–Poisson action on $\mathfrak{su}(N)^*$ must be an automorphism of $\mathfrak{su}(N)$. This group is known to be $\mathrm{Aut}(\mathfrak{su}(N)) \cong SU(N) \rtimes \mathbb{Z}_2$, for $N > 2$. Since the complex conjugation do not preserve the spectrum (it sends any eigenvalue $\lambda = i\alpha$ to $-\lambda$), the only Lie–Poisson maps on $\mathfrak{su}(N)^*$ are the conjugation via an element in $SU(N)$.

We want to prove that the discrete Laplacian[4] Δ_N is equivariant only by the conjugate action of G, where $G = \{\omega^i g^j h^k, \text{ for } i, j, k = 1, \ldots, N\}$ is a finite group of cardinality N^3, where:

$$\omega = e^{\frac{2\pi i}{N}} Id, \qquad g = \mathrm{diag}(1, \omega, \ldots, \omega^{N-1}), \qquad h = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Notice that $\omega^N = g^N = h^N = Id$ and $gh = \omega hg$, therefore G is a well defined group. Moreover, $\omega\omega^* = gg^* = hh^* = Id$ and $\det(\omega) = \det(g) = \det(h) = 1$. Hence $G < SU(N)$.

We want to show that the discrete Laplacian does not admit a continuous symmetry group $K < SU(N)$. More explicitly, we want to prove that if $\Delta_N(U^*WU) = U^*\Delta_N(W)U$, for every W in $\mathfrak{su}(N)$, then $U \in G$. Without loss of generality, we can restrict to the case when W is any basis element $T_{\mathbf{n}}$, for $\mathbf{n} = (n_1, n_2)$. In this case, we have that:

$$\Delta_N(U^*T_{\mathbf{n}}U) = U^*\Delta_N(T_{\mathbf{n}})U = -(n_1^2 + n_2^2)(U^*T_{\mathbf{n}}U).$$

This means that $U^*T_{\mathbf{n}}U$ is an eigenvector of Δ_N, with eigenvalue $-(n_1^2 + n_2^2)$. Therefore, it must be that:

$$U^*T_{\mathbf{n}}U = \sum_{\mathbf{k} \text{ s.t. } |\mathbf{k}|=|\mathbf{n}|} c_{\mathbf{k}} T_{\mathbf{k}}. \tag{4.13}$$

Notice that this procedure on the sphere gives that the group of symmetry of the Hamiltonian is $SO(3)$. In fact, if the eigenspaces are invariant under rotations (which have the physical meaning of Rossby-Haurwitz waves). In conclusion, it only remains to prove that to satisfy (4.13), for each $\mathbf{n} \in [-N, N] \times [-N, N]$, then $U \in G$.

[4]We recall that the discrete Laplacian is defined as: $\Delta_N T_{\mathbf{n}} = -(n_1^2 + n_2^2)T_{\mathbf{n}}$, for every $T_{\mathbf{n}}$ basis-element.

Theorem 6. *Let $T_{\mathbf{n}}$ denote a basis element for the sine-bracket model, for $\mathbf{n} \in [-N, N]$. Then, $U \in SU(N)$ satisfies* (4.13), *for every $\mathbf{n} \in [-N, N] \times [-N, N]$, if and only if $U \in G$.*

Proof. Suppose $U \in SU(N)$ satisfies (4.13), for every $\mathbf{n} \in [-N, N] \times [-N, N]$.

Step 1. The basis element $T_{\mathbf{n}}$ are defined as:

$$T_{\mathbf{n}} = \omega^{n_1 n_2/2} g^{n_1} h^{n_2}.$$

Hence, it is not hard to check that the $T_{\mathbf{n}} \in SU(N)$ and have the same spectrum:

$$\sigma(T_{\mathbf{n}}) = \{1, \omega, \ldots, \omega^{N-1}\},$$

for every $\mathbf{n}$.[5] Consider (4.13) for $|\mathbf{n}| = 1$ and take $T_{\mathbf{n}} = T_{(1,0)}$. Then (4.13) becomes:

$$U^* T_{(1,0)} U = c_{(1,0)} T_{(1,0)} + c_{(-1,0)} T_{(-1,0)} + c_{(0,1)} T_{(0,1)} + c_{(0,-1)} T_{(0,-1)}. \quad (4.14)$$

Hence, since the LHS of equation (4.14) is unitary, it must be $Id = MM^*$, where M is the RHS of (4.14). Using the fact that $T_{(1,0)} = g$ and $T_{(0,1)} = h$, we obtain a set of algebraic equations in the $c_{\mathbf{k}}$:

$$\begin{aligned}
&|c_{(1,0)}|^2 + |c_{(-1,0)}|^2 + |c_{(0,1)}|^2 + |c_{(0,-1)}|^2 = 1\\
&c_{(1,0)}\overline{c}_{(-1,0)} = 0\\
&c_{(0,1)}\overline{c}_{(0,-1)} = 0\\
&c_{(1,0)}\overline{c}_{(0,1)} = 0\\
&c_{(1,0)}\overline{c}_{(0,-1)} = 0\\
&c_{(-1,0)}\overline{c}_{(0,1)} = 0\\
&c_{(-1,0)}\overline{c}_{(0,-1)} = 0.
\end{aligned} \quad (4.15)$$

Equations (4.15) are satisfied only when one and only one of the $|c_{\mathbf{k}}| = 1$ and the others are equal to zero. Then, since $\sigma(U^* T_{(1,0)} U) = \sigma(M) = \{1, \omega, \ldots, \omega^{N-1}\}$, it can be one and only one $c_{\mathbf{k}} = \omega^j$, for some $j = 1, \ldots, N$, and the others are equal to zero.

Step 2. It is not hard to check, with explicit calculations, that if $U^* T_{(1,0)} U = \omega^j T_{(0,\pm 1)}$ or $U^* T_{(1,0)} U = \omega^j T_{(-1,0)}$, for some $j = 1, \ldots, N$, then $U^* T_{(0,\pm 1)} U$ or $U^* T_{(-1,0)} U$ do not belong to the eigenspace in which $|\mathbf{n}| = 1$. Hence, any U found in Step 1 can be a symmetry of the discrete Laplacian only if U stabilizes $T_{(1,0)}$, up to a multiplication by a $N-$th root of the unity.

Step 3. Reasoning as Step 1-2, for $T_{(0,1)}$, U can be a symmetry of the discrete Laplacian only if U stabilizes $T_{(0,1)}$, up to a multiplication by a $N-$th root of the unity. In conclusion, U must satisfy:

$$U^* g U = \omega^j g \qquad U^* h U = \omega^k h, \quad (4.16)$$

[5]Notice that the $\mathfrak{sl}(n, \mathbb{C})$ basis elements of the sine-bracket model are also in $SU(n)$.

for some $j, k = 1, \dots, N$. Notice that $\mathrm{stab}_{SU(N)}(g) \cap \mathrm{stab}_{SU(N)}(h) = \{Id\}$, being $SU(N)$ centerless. Therefore, it follows via a direct computation from (4.16) that U must be in G.

Vice-versa, suppose $U \in G$. Then it is easy to check that it satisfies (4.16) for any basis element $T_{\mathbf{n}}$, being g, h algebraic generators for $\mathfrak{su}(N)$. □

Remark 4. Notice that the conjugate action of h on g and, respectively, the conjugate action of g on h, correspond for $N \to \infty$ to the 1-dimensional translation group with respect to the x axis and, respectively, the 1-dimensional translation group with respect to the y axis. This finite symmetries of the sine-bracket model may indicate that the sine-bracket model better represents the Euler equations on a "polyhedral" torus rather than a smooth one.

Corollary 2. *Let Δ_N denote the discrete Laplacian on $\mathfrak{su}(N)$. If $U \in SU(N)$ satisfies $\Delta_N(U^*WU) = U^*\Delta_N(W)U$, for every W in $\mathfrak{su}(N)$, then $U \in G$. In particular, the sine-bracket model does not have $\mathbb{R}^2$ symmetry group and hence they are not translational invariant.*

Chapter 5

Summary of the results in the papers

Paper I: Lie–Poisson methods for isospectral flows

In paper I, we treat isospectral flows and Lie–Poisson systems together, obtaining a new general theory of numerical geometric integration for these type of ODEs. Numerical methods for isospectral flows is a classical theme within the field of geometric numerical integration, which is why the simplicity of the methods developed in the paper is surprising. Let us recall the results in this paper.

A general isospectral flow is defined as:

$$\begin{aligned} &\dot{W} = [B(W), W] \\ &W(0) = W_0, \end{aligned} \tag{5.1}$$

for $W \in S \subset \mathfrak{gl}(n,\mathbb{C})$ linear subspace. In order for (5.1) to be well defined, $B(W)$ has to belong to $\mathfrak{n}(S)$, i.e., the $\mathfrak{gl}(n,\mathbb{C})$-normalizer of S. Consider the following Hamiltonian isospectral flow for $W \in \mathfrak{g}$, Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ and H smooth function on $\mathfrak{g}$, such that $\nabla H(W)^\dagger \in \mathfrak{n}(\mathfrak{g})$:

$$\begin{aligned} &\dot{W} = [\nabla H(W)^\dagger, W] \\ &W(0) = W_0. \end{aligned} \tag{5.2}$$

Then we have the following fact:

Proposition 4. *If $\mathfrak{g}$ is a semisimple Lie algebra then $\nabla H(W)^\dagger \in \mathfrak{g}$ and, via the Frobenius inner product identification,* (5.2) *is a Lie–Poisson system on $\mathfrak{g}^*$.*

In both cases, we require the following assumption to hold (the Hamiltonian case is for $B = \nabla H^\dagger$):

Assumption 1. Let S_ε be a $\varepsilon-$neighbourhood of S in $\mathfrak{gl}(n,\mathbb{C})$. We assume that $B(\cdot)$ can be extended to S_ε such that $B(W) \in \mathfrak{n}(S)$ for all $W \in S_\varepsilon$.

Then, the main result of Paper 1 is:

Theorem 7. *Consider an isospectral flow of the form* (5.1) *evolving on a linear subspace* $S \subset \mathfrak{gl}(n,\mathbb{C})$. *Let* $\Phi_h\colon T^*GL(n,\mathbb{C}) \to T^*GL(n,\mathbb{C})$ *be a symplectic B-series (or P-series) method for the corresponding system:*

$$\begin{aligned}\dot{Q} &= QB(Q^\dagger P)^\dagger \\ \dot{P} &= -PB(Q^\dagger P),\end{aligned} \tag{5.3}$$

obtained by extension from S *in accordance with Assumption 1.*

1. *If* Φ_h *is equivariant with respect to the cotangent lifted action of* $\mathfrak{gl}(n,\mathbb{C})$, *i.e.,*
$$G \cdot \Phi_h(Q,P) = \Phi_h(G \cdot (Q,P)).$$
then it descends to an isospectral integrator ϕ_h *on* $\mathfrak{gl}(n,\mathbb{C})$,
2. *If, in addition,* Φ_h *preserves the foliation*
$$\mathcal{F}_G = \{Q \mid GQ^\dagger \in N(S)\}, \qquad G \in GL(n,\mathbb{C})$$
then ϕ_h *restricts to an integrator on* S.

The second constructive result is that any symplectic Runge-Kutta method gives a isospectral (Lie–Poisson) integrator for $\mathfrak{gl}(n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C})$ and any of their quadratic reductive subalgebras. The general s-stage method is given by the following scheme. Given a Butcher tableau:

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

of a s-stage symplectic Runge-Kutta method with time step h, we get the following isospectral (Lie–Poisson) integrator:

$$X_i = -h(W_n + \textstyle\sum_{j=1}^s a_{ij}X_j)B(\widetilde{W}_i), \qquad \text{for } i = 1, ..., s.$$

$$Y_i = hB(\widetilde{W}_i)(W_n + \textstyle\sum_{j=1}^s a_{ij}Y_j), \qquad \text{for } i = 1, ..., s.$$

$$K_{ij} = hB(\widetilde{W}_i)(\textstyle\sum_{j'=1}^s (a_{ij'}X_{j'} + a_{jj'}K_{ij'})), \qquad \text{for } i,j = 1, ..., s.$$

$$\widetilde{W}_i = W_n + \textstyle\sum_{j=1}^s a_{ij}(X_j + Y_j + K_{ij}), \qquad \text{for } i = 1, ..., s.$$

$$W_{n+1} = W_n + h\textstyle\sum_{i=1}^s b_i[B(\widetilde{W}_i), \widetilde{W}_i],$$

where the unknowns are X_i, Y_i, K_{ij} for $i, j = 1, ..., s$ and the last two lines are explicit. In the paper, it is shown how this scheme can be simplified in several cases. In the article, several applications of the method to the rigid body equations, the point vortex equations, the Heisenberg spin chain equations, the Euler equations, the Toda lattice and the Toeplitz inverse eigenvalue problem are presented. In the figure below we show the results for one of our methods applied to the generalized rigid body equations.

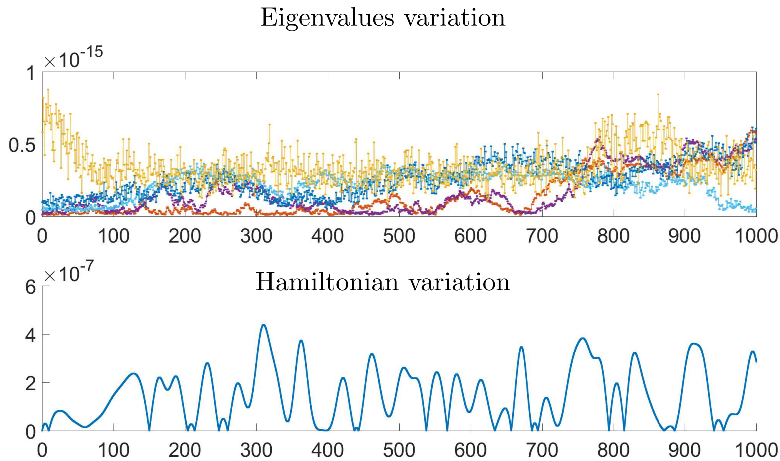

Figure 5.1: Generalized 45-dimensional rigid body in $\mathfrak{so}(10)$. Eigenvalues (which occur in pair) and Hamiltonian variation; $h = 10^{-1}$; inertia tensor $\mathbb{I} = diag(1 : 10)$; initial value $(W_0)_{ij} = 1/10$ if $i < j$, $(W_0)_{ij} = -1/10$ if $i > j$, $(W_0)_{ij} = 0$ if $i = j$.

Paper II: A Casimir preserving scheme for long-time simulation of spherical ideal hydrodynamics

In paper II, we introduce a new class of numerical schemes for the 2D Euler equations on a sphere. The integrators developed have the advantage of preserving the symmetries of the original equations and of being computationally efficient, making them a powerful tool for analysing the long-term behaviour of the fluid. Indeed, taking advantage of this feature, we have provided convincing evidences of persistent unsteadiness of the fluid. Moreover, we have found that the kind of statistical state on the sphere strongly depends on the value of the conserved quantities of the equations. In particular, the most crucial ones seem to be the linear momentum and the enstrophy. However, our results suggest that these are not enough to characterize the statistical state of the system revealing the relevance of the higher order Casimirs in determining the statistical state of the fluid. Finally, we have proposed a connection of long-time dynamics of the fluid with integrability theory of point-vortices, which has been developed in paper IV.

The 2D Euler equations on a sphere, written in the vorticity form are:

$$\begin{aligned} \dot{\omega} &= \{\psi, \omega\} \\ \Delta\psi &= \omega, \end{aligned} \tag{5.4}$$

for a smooth vorticity $\omega \in C^\infty(\mathbb{S}^2)$. As shown in chapter 3, these equations are a Lie–Poisson system on the dual of the Lie algebra of divergence-free vector fields. In chapter 4, we have introduced the quantized model of the Euler equations, which gives a Lie–Poisson system on $\mathfrak{su}(N)^*$:

$$\dot{W} = [\Delta_N^{-1}W, W]_N, \tag{5.5}$$

where Δ_N^{-1} is the discrete Laplacian operator. Combining the results of Paper I and Paper III, we provided efficient Lie–Poisson integrators for (5.5). The numerical methods developed have the advantage to preserve, up to machine precision, the discrete Casimir functions and the linear momentum, and nearly conserving the Hamiltonian. The simplest scheme for the quantized Euler equations (5.5) that we propose is the 2nd order isospectral midpoint rule, as described in paper III. For a time step $h > 0$, this is:

$$\begin{aligned} W_n &= (I - \tfrac{h}{2}\Delta_N^{-1}\widetilde{W})\widetilde{W}(I + \tfrac{h}{2}\Delta_N^{-1}\widetilde{W}) \\ W_{n+1} &= (I + \tfrac{h}{2}\Delta_N^{-1}\widetilde{W})\widetilde{W}(I - \tfrac{h}{2}\Delta_N^{-1}\widetilde{W}), \end{aligned} \tag{5.6}$$

where the only implicit unknown is $\widetilde{W}$. A crucial aspect in the scheme (5.6) is the efficient evaluation of Δ_N^{-1}, which as explained in Paper II can be done in $\mathcal{O}(N^2)$ operations.

As mentioned above, our study on the long-term behaviour of the Euler equations on a sphere provides strong hints against the statistical mechanics theory *MRS* (see [13]), which predicts a final steady state. Moreover, the conservative scheme (5.6) shows that the effect of unsteadiness observed in [9] cannot be caused by artificial numerical dissipation.

Here below, we show the different kind of statistical states we have found (see figure 5.2). The most common configurations present three or four large unsteady vortices, that may reduce to two in case of very large linear momentum. We predict that, in the first approximation, they can be determined looking at the ratio between the linear momentum and the enstrophy (see figure 5.3).

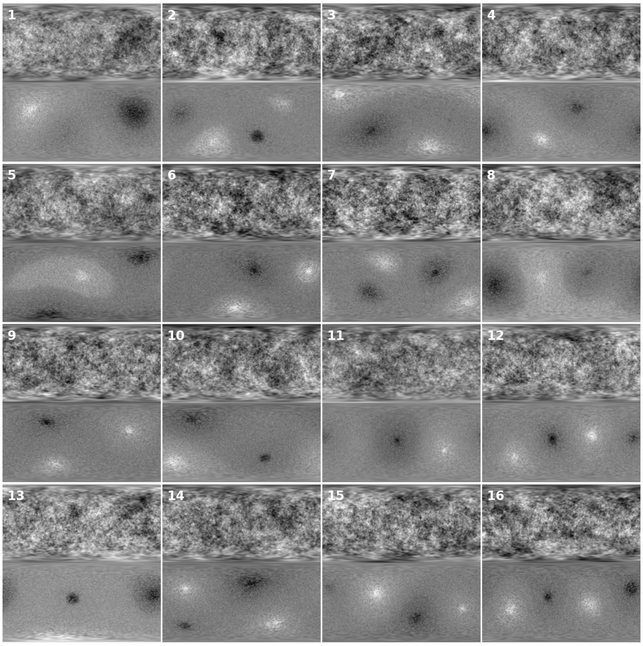

Figure 5.2: Pairs of initial (upper) and final (lower) vorticities for the 16 generic simulations with $L^2(\mathbb{S}^2)$ random initial data. The numbers labelling the simulations correspond to those in Figure 5.3.

However, is clear from figure 5.3 that these quantity are not enough to determine exactly the long-term behaviour of the fluid, suggesting that other higher order Casimir play a crucial role in it.

Finally, we propose the following principle to determine the final state of the fluid: the turbulent mixing of vorticity continues accumulating it at low frequencies (inverse energy cascade), until an integrable configuration of large vortex blobs is reached. This integrable configuration corresponds to the highest integrable configuration of point-vortices, which for zero-linear momentum is given by four vortices and for non-zero linear momentum is given by three vortices.

Paper III: A minimal-variable symplectic method for isospectral flows

In paper III, we have carried out a detailed analysis of the isospectral midpoint method, defined in paper I. We have shown that this scheme can be simpli-

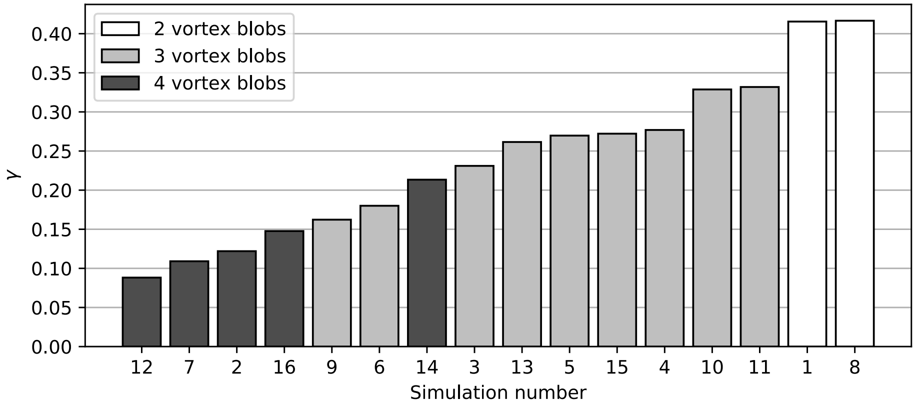

Figure 5.3: Values of $\gamma = \|\mathbf{L}\|/(R\sqrt{\mathcal{C}_2})$ for the simulations of Figure 5.2. The grey-scale correspond to the number of vortex blobs observed in the final state: 2, 3, or 4. Notice that the value of γ largely determines the number of vortex blobs in the final state: 4 when $\gamma \lesssim 0.15$, 3 when $0.15 \lesssim \gamma \lesssim 0.4$, and 2 when $\gamma \gtrsim 0.4$.

fied to get what we called a *minimal variable* integrator, which means that the number of the implicit unknowns to be determined have the same dimension as the manifold where the dynamical system evolves. This is a quite remarkable feature, since only a few Lie–Poisson integrators are know to have this property. In particular, we have explicated our scheme for $\mathfrak{so}(3)$ and $\mathfrak{sl}(2)$, obtaining symplectic integrators for the sphere and the hyperbolic plane. We have also tested our scheme for the genearlized rigid body, the Brockett flow, the Heisenberg spin chain and the point-vortex equations on the hyperbolic plane.

Furthermore, the calculations in this paper have shown an unexpected connection between the methods defined in paper I and the Cayley map. Further investigations on this connection are still under study. To this aspect, since the Cayley map is a parametrization of quadratic Lie groups, we have determined under which conditions Lie–Poisson systems on quadratic Lie algebras are isospectral flows. This result extends the range of applicability of the schemes presented in paper I.

Here we present the main result of the paper. Consider the isospectral flow:

$$\begin{aligned} &\dot{W} = [B(W), W] \\ &W(0) = W_0. \end{aligned} \tag{5.7}$$

Definition 5. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Furthermore, let

$V \subseteq \mathfrak{g}$ be a linear subspace. Then the two sets

$$N(V) = \{g \in G \mid g^{-1}Vg \subseteq V\}$$
$$\mathfrak{n}(V) = \{\xi \in \mathfrak{g} \mid [\xi, V] \subseteq V\}$$

are respectively called the G-normalizer and the $\mathfrak{g}$-normalizer of V. Notice that $N(V)$ is a subgroup of G and $\mathfrak{n}(V)$ is a Lie subalgebra of $\mathfrak{g}$.

A related concept to normalizer is the centralizer Lie algebra.

Definition 6. Let $\mathfrak{g}$ be a Lie algebra and let $V \subseteq \mathfrak{g}$ be a linear subspace. Then the set

$$\mathfrak{c}(V) = \{\xi \in \mathfrak{g} \mid [\xi, V] = 0\}$$

is called the $\mathfrak{g}$-centralizer of V. Notice that $\mathfrak{c}(V)$ is a Lie subalgebra of $\mathfrak{g}$.

We now recall the definition of a $J-$quadratic Lie algebra.

Definition 7. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n, \mathbb{C})$ is called $J-$quadratic Lie algebra if there exists an invertible matrix J, such that

$$W \in \mathfrak{g} \iff W^\dagger J + JW = 0. \tag{5.8}$$

The main result of paper III is summarized here below.

Theorem 8. *Let $W_k \in D \subset V$, for a domain D in the linear subspace $V \subset \mathfrak{gl}(n, \mathbb{C})$. Assume that the normalizer splits as $\mathfrak{n}(V) = \mathfrak{g}_0 \oplus \mathfrak{c}(V)$, for some Lie algebra $\mathfrak{g}_0$, which satisfies*

$$N \in \mathfrak{g}_0 \iff N^\dagger P + PN = 0. \tag{5.9}$$

for some constant matrix P. Furthermore, let $B : D \subset \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{n}(V)$ be continuously differentiable. Then, for some $h > 0$, there exists $\widetilde{W} \in V$ such that the numerical scheme $W_k \to W_{k+1}$, implicitly defined by:

$$\boxed{\begin{aligned} W_k &= (Id - \tfrac{h}{2}B(\widetilde{W}))\widetilde{W}(Id + \tfrac{h}{2}B(\widetilde{W})) \\ W_{k+1} &= (Id + \tfrac{h}{2}B(\widetilde{W}))\widetilde{W}(Id - \tfrac{h}{2}B(\widetilde{W})), \end{aligned}} \tag{5.10}$$

is a second order isospectral integrator for (5.7), for any $k \geq 0$.[1]*. Moreover, when (5.7) is a Lie–Poisson system on $\mathfrak{gl}(n, \mathbb{C})^*$ or on the dual of some $J-$quadratic Lie algebra $\mathfrak{g}$ such that $J^2 \in \mathfrak{c}(\mathfrak{g})$ (or even on $\mathfrak{g} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is an Abelian Lie algebra), then (5.10) is a Lie-Poisson integrator for (5.7) which preserves the coadjoint orbits in $\mathfrak{g}^*$.*

[1] Here Id denotes the $n \times n$ identity matrix.

Paper IV: Integrability of point-vortex dynamics via symplectic reduction: a survey

In paper IV, we have provided a survey on the integrability of the point-vortex dynamics via symplectic reduction. We have shown that the scattered results of the integrability of the point vortex dynamics present in literature can all be covered and understood in terms of symplectic reduction. This approach wants to emphasize the role of the symmetries of the point-vortex equations and to show that the fact that a configuration of point-vortices is integrable can be entirely expressed in terms of geometric concepts. In particular, since the symplectic reduction theory is based on the momentum map of a Lie group action, all the integrability results boil down to a study of the properties of the momentum map. More specifically, these are related to topological obstruction to existence of a momentum map, to its equivariance to the Lie group action, and to the freeness and the properness of this action.

Our geometric approach gives more transparent proofs for the integrability results. Moreover, in view of connecting the point-vortex dynamics to real fluids, it is clear that the geometry of the equations play a central role. In particular, as stated in paper II, we have found a direct connection between the long-term behaviour of the 2D Euler equations on a sphere and the integrable states of point-vortices. However, the same connection does not seem possible on a flat torus. In fact, the point-vortex dynamics on a flat torus has a dual interpretation as a Hamiltonian system on the plane. The different topology of the plane and the torus makes a big difference in terms of momentum map, which exists in the first case but not in the second one. However, the 2D Euler equations on the torus do not have a dual interpretation on the plane, which implies that the phase space of the point-vortex equations and the fluid equations cannot be directly compared as for the sphere.

Here below, we summarize the integrability results given in the article. We use the following acronyms:

- EQ = Equilibrium;
- ND = Equations not defined;
- RE = Relative equilibrium;
- INT = Integrable;
- NON-INT = Non-integrable;
- ? = Not-known.

In the columns are specified the number of vortices N, while in the row the conditions fulfilled by the point-vortices. $\Gamma = \sum_{i=1}^{N} \Gamma_i$ is the total circulation, whereas $\mathbf{L} = \sum_{i=1}^{N} \Gamma_i x_i$ is the linear momentum.

Euclidean plane $\mathbb{R}^2$				
Cond \ N	2	3	4	≥ 5
$\Gamma = 0, \mathbf{L} = 0$	ND	RE	INT	?
$\Gamma = 0, \mathbf{L} \neq 0$	RE	INT	?	?
$\Gamma \neq 0$	RE	INT	NON-INT[a]	ARN-DIF[b]
Flat-torus $\mathbb{R}^2/\mathbb{Z}^2$				
Cond \ N	2	3	4	≥ 5
$\Gamma = 0$	RE	INT	?	?
$\Gamma \neq 0$	INT	NON-INT[c]	?	?
Sphere $\mathbb{S}^2$				
Cond \ N	2	3	4	≥ 5
$\mathbf{L} = 0$	EQ	RE	INT	?
$\mathbf{L} \neq 0$	RE	INT	NON-INT[d]	?
Hyperbolic plane $\mathbb{H}^2$				
Cond \ N	2	3	4	≥ 5
$\mathbf{L} = 0$	ND	RE	INT	?
$\mathbf{L} \neq 0$	RE	INT	?	?

Table 5.1: Integrability results for point-vortex dynamics on different surfaces.

[a] [16]
[b] [16]
[c] Conjecture, [15]
[d] Conjecture, [4]

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