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Results in Mathematics

Functions Preserving the Biadditivity

Radosław Łukasik and Paweł Wójcik

Abstract. In this paper we consider the generalization of the orthogonality equation. Let S be a semigroup, and let H, X be abelian groups. For two given biadditive functions $A\colon S^2 \to X$, $B\colon H^2 \to X$ and for two unknown mappings $f, g\colon S \to H$ the functional equation

$$B(f(x), g(y)) = A(x, y)$$

will be solved under quite natural assumptions. This extends the well-known characterization of the linear isometry.

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1. Introduction

Let H, K be unitary spaces. It is easy to check that, if $f\colon H \to K$ satisfies $\langle f(x)|f(y)\rangle = \langle x|y\rangle$, then f is an linear isometry. The above equation was generalized in normed spaces X, Y by considering a norm derivative $\rho'_+(x,y) := \|x\| \cdot \lim\limits_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t}$ instead of inner product, i.e.

$$\rho'_+(f(x), f(y)) = \rho'_+(x, y), \quad x, y \in X, \tag{1}$$

with an unknown function $f\colon X \to Y$. Note that if the norm comes from an inner product $\langle\cdot,\cdot\rangle$, we obtain $\rho'_+(x,y) = \langle x|y\rangle$. Another generalization of the orthogonality equation in Hilbert spaces H, K is to look for the solutions of

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x, y \in H, \tag{2}$$

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where $f, g\colon H \to K$ are unknown functions. Solutions of (1) and (2) can be found in the authors' previous papers [3], [4], [6]. Another generalization of (2) we can find in the paper [5] where the author studies the equation

$$\langle f(x)|g(y^*)\rangle = \langle x|y^*\rangle, \quad x \in E, y^* \in F^*,$$

where $f\colon E \to F$, $g\colon E^* \to F^*$, E, F are Banach spaces, E^*, F^* are spaces dual to E and F respectively, and $\langle a|\varphi\rangle{:=}\varphi(a)$.

In this paper we will give a natural generalization of such functional equations in the case of abelian groups. In this case we will consider biadditive mappings instead of inner products.

2. Preliminaries

We start by recalling here some notions and results from the theory of groups and semigroups (see [2, Appendix A]).

Definition 1. A group is *torsion* if every element has the finite order.

A group is *torsion-free* if every element except the identity has the infinite order.

Definition 2. A semigroup $(H, +)$ is said to be *divisible* if

$$\forall_{x\in H}\, \forall_{n\in\mathbb{N}}\, \exists_{y\in H}\, x = ny.$$

Let p be a prime number. The Prüfer p-group is the unique p-group in which every element has p different p-th roots. Alternatively we can write $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$, where $\mathbb{Z}[1/p] = \{\frac{m}{p^n} : m \in \mathbb{Z},\ n \in \mathbb{N}_0\}$, $\mathbb{N}_0{:=}\mathbb{N} \cup \{0\}$. It is known fact that Prüfer p-groups are divisible and torsion.

Definition 3. Let A_i, $i \in I$, be groups. The *direct sum* $\bigoplus\limits_{i\in I} A_i$ is the set of tuples $(a_i)_{i\in I} \in \prod\limits_{i\in I} A_i$ such that $a_i \neq 0$ for finitely many $i \in I$.

Remark 1. There exist an abelian divisible group G and divisible subgroups D, K of G such that $D \cap K$ is not divisible.

Lemma 1. *Let G be an abelian group, D, K be divisible subgroups of G. Then $D + K$ is divisible.*

Proof. Let $x \in D$, $y \in K$, $n \in \mathbb{N}$. Then there exist $u \in D$ and $v \in K$ such that $x = nu$ and $y = nv$. Hence $x + y = n(u + v)$. □

Theorem 1. *Let G be an abelian divisible group, D_1, D_2 be divisible subgroups of G and $D_1 \cap D_2$ be divisible. Then there exist divisible groups K_0, K_1, K_2, K_3 such that $G = \bigoplus\limits_{i=0}^{3} K_i$, $D_2 = K_0 \oplus K_1$, $D_1 = K_0 \oplus K_2$.*

Proof. Let $K_0 = D_1 \cap D_2$. Then there exist divisible groups K_1, K_2 such that $D_2 = K_0 \oplus K_1$, $D_1 = K_0 \oplus K_2$. We show that $K_1 \cap K_2 = \{0\}$. Let $x \in K_1 \cap K_2$, then $x \in D_1 \cap D_2 = K_0$. Hence $x = 0$. Finally, there exists a divisible group K_3 such that $G = \left(\bigoplus_{i=0}^{2} K_i\right) \oplus K_3$. □

After these preparations we may now pass to multi-additive functions. By $\mathrm{Perm}(n)$ we denote the set of all bijections of the set $\{1, \dots, n\}$.

Definition 4. Let S be a semigroup, H be a group, $n \in \mathbb{N}$. The function $A\colon S^n \to H$ is called *n-additive* if

$$\begin{aligned} &A(x_1, \dots, x_{i-1}, x_i + y, x_{i+1}, \dots, x_n) \\ &\quad = A(x_1, \dots, x_n) + A(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n), \end{aligned}$$

for all $y, x_1, \dots, x_n \in S$ and $i \in \{1, \dots, n\}$.

Moreover, A is called *symmetric* if

$$A(x_1, \dots, x_n) = A_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $x_1, \dots, x_n \in S$ and $\sigma \in \mathrm{Perm}(n)$.

Lemma 2. *Let H, X be groups, H be divisible. Let further $B\colon H^2 \to X$ be a biadditive function. Then for every element $x \in H$ of the finite order we have*

$$B(x, y) = B(y, x) = 0, \ \ y \in H.$$

Remark 2. The previous lemma can be easily extended to the n-additive functions for $n \geq 2$.

We use following two lemmas to show the existence of some biadditive map from $\mathbb{Q}^2$ to $\mathbb{Z}(2^\infty)$.

Lemma 3. *Let $k \in \mathbb{N}$, $l \in 2\mathbb{N} - 1$. Then there exists exactly one number $\varphi(2^k, l) \in \{1, 3, \dots 2^k - 1\}$ such that $l\varphi(2^k, l) \equiv 1 (\mathrm{mod} 2^k)$.*

Proof. Let $l(2i - 1) \equiv r_i (\mathrm{mod} 2^k)$, $1 \leq r_i < 2^k$ for $i \in \{1, 2, \dots 2^{k-1}\}$. We observe that $r_i \in 2\mathbb{N} - 1$ and $r_i \neq r_j$ for $i \neq j$. Indeed, if $r_i = r_j$, then $l(2i - 2j) \equiv 0 (\bmod 2^k)$ which means that $i = j$. Hence there exists exactly one j such that $l(2j - 1) \equiv 1 (\mathrm{mod} 2^k)$. □

Lemma 4. *Let $k, m \in \mathbb{N}$, $l, n \in 2\mathbb{N} - 1$. Then*

$$\begin{aligned} n\varphi(2^k, ln) &\equiv \varphi(2^k, l) (\mathrm{mod} 2^k), \\ \varphi(2^{k+m}, l) &\equiv \varphi(2^k, l) (\mathrm{mod} 2^k). \end{aligned}$$

Proof. We have

$$l\Big(n\varphi(2^k, ln) - \varphi(2^k, l)\Big) = ln\varphi(2^k, ln) - l\varphi(2^k, l) \equiv 0 (\mathrm{mod} 2^k),$$

$$l\varphi(2^{k+m},l)=1+c2^{k+m}=1+(c2^m)2^k\equiv 1(\mathrm{mod}2^k)\equiv l\varphi(2^k,l)(\mathrm{mod}2^k),$$

for some $c\in\mathbb{N}_0$ so

$$l\Big(\varphi(2^{k+m},l)-\varphi(2^k,l)\Big)\equiv 0(\mathrm{mod}2^k),$$

which means that

$$\varphi(2^{k+m},l)-\varphi(2^k,l)\equiv 0(\mathrm{mod}2^k).$$

Theorem 2. *There exists a biadditive and symmetric function $C\colon\mathbb{Q}^2\to\mathbb{Z}(2^\infty)$ such that $C(1,1)=\frac{1}{2}+\mathbb{Z}$.*

Proof. A greatest common divisor in this proof will be denoted by GCD. Let $m,k\in\mathbb{Z}$, $n,l\in\mathbb{N}$, $\mathrm{GCD}(m,n)=\mathrm{GCD}(k,l)=1$. Let further $s_n,s_l\in\mathbb{N}_0$ be such that $2^{s_n}|n$, $2^{s_n+1}\not|l$, $2^{s_l}|l$, $2^{s_l+1}\not|l$. We define C by the formula

$$C\left(\frac{m}{n},\frac{k}{l}\right):=mk\frac{\varphi\left(2^{s_n+s_l+1},\frac{nl}{2^{s_n+s_l}}\right)}{2^{s_n+s_l+1}}+\mathbb{Z}.$$

It is easy to see that C is symmetric, so we only show that C is additive in the first variable. Let $p\in\mathbb{Z}$, $q\in\mathbb{N}$, $\mathrm{GCD}(p,q)=1$, $d=\mathrm{GCD}(mq+np,nq)$. Let further $s_q,s_d\in\mathbb{N}_0$ be such that $2^{s_q}|q$, $2^{s_q+1}\not|q$ and $2^{s_d}|d$, $2^{s_d+1}\not|d$. Using Lemma 4 we get

$$\begin{aligned}
C\left(\frac{m}{n}+\frac{p}{q},\frac{k}{l}\right)&=C\left(\frac{mq+np}{nq},\frac{k}{l}\right)=C\left(\frac{\frac{mq+np}{d}}{\frac{nq}{d}},\frac{k}{l}\right)\\
&=\left(\frac{mq+np}{d}\cdot k\right)\frac{\varphi\left(2^{s_n+s_q-s_d+s_l+1},\frac{nql}{d2^{s_n+s_q-s_d+s_l}}\right)}{2^{s_n+s_q-s_d+s_l+1}}+\mathbb{Z}\\
&=\left(\frac{mq+np}{d}\cdot k\frac{d}{2^{s_d}}\right)\frac{\varphi\left(2^{s_n+s_q-s_d+s_l+1},\frac{nql}{d2^{s_n+s_q-s_d+s_l}}\cdot\frac{d}{2^{s_d}}\right)}{2^{s_n+s_q-s_d+s_l+1}}+\mathbb{Z}\\
&=\left(\frac{mq+np}{d}\cdot k\frac{d}{2^{s_d}}\right)\frac{\varphi\left(2^{s_n+s_q-s_d+s_l+1},\frac{nql}{2^{s_n+s_q+s_l}}\right)}{2^{s_n+s_q-s_d+s_l+1}}+\mathbb{Z}\\
&=\left(\frac{mq+np}{d}\cdot k\frac{d}{2^{s_d}}\right)\frac{\varphi\left(2^{s_n+s_q+s_l+1},\frac{nql}{2^{s_n+s_q+s_l}}\right)}{2^{s_n+s_q-s_d+s_l+1}}+\mathbb{Z}\\
&=(mq+np)k\frac{\varphi\left(2^{s_n+s_q+s_l+1},\frac{nql}{2^{s_n+s_q+s_l}}\right)}{2^{s_n+s_q+s_l+1}}+\mathbb{Z}\\
&=(mqk)\frac{\varphi\left(2^{s_n+s_q+s_l+1},\frac{nql}{2^{s_n+s_q+s_l}}\right)}{2^{s_n+s_q+s_l+1}}+(npk)\frac{\varphi\left(2^{s_n+s_q+s_l+1},\frac{nql}{2^{s_q+s_n+s_l}}\right)}{2^{s_n+s_q+s_l+1}}+\mathbb{Z}\\
&=(mk2^{s_q}\frac{q}{2^{s_q}})\frac{\varphi\left(2^{s_n+s_q+s_l+1},\frac{nl}{2^{s_n+s_l}}\cdot\frac{q}{2^{s_q}}\right)}{2^{s_n+s_q+s_l+1}}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(pk2^{s_n}\frac{n}{2^{s_n}}\right)\frac{\varphi\left(2^{s_n+s_q+s_l+1}, \frac{ql}{2^{s_q+s_l}}\cdot\frac{n}{2^{s_n}}\right)}{2^{s_n+s_q+s_l+1}} + \mathbb{Z}\\
&= (mk)\frac{\varphi\left(2^{s_n+s_q+s_l+1}, \frac{nl}{2^{s_n+s_l}}\right)}{2^{s_n+s_l+1}} + (pk)\frac{\varphi\left(2^{s_n+s_q+s_l+1}, \frac{ql}{2^{s_q+s_l}}\right)}{2^{s_q+s_l+1}} + \mathbb{Z}\\
&= (mk)\frac{\varphi\left(2^{s_n+s_l+1}, \frac{nl}{2^{s_n+s_l}}\right)}{2^{s_n+s_l+1}} + \mathbb{Z} + (pk)\frac{\varphi\left(2^{s_q+s_l+1}, \frac{ql}{2^{s_q+s_l}}\right)}{2^{s_q+s_l+1}} + \mathbb{Z}\\
&= C\left(\frac{m}{n}, \frac{k}{l}\right) + C\left(\frac{p}{q}, \frac{k}{l}\right).
\end{aligned}
$$

The proof is complete. □

Now we introduce some theory of the adjoint operator on groups.

Definition 5. Let S, H, X be groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions. Let further $T\colon S \to H$ and

$$D(T^*) = \{v \in H : \exists_{y\in S}\forall_{x\in S}\, B(T(x), v) = A(x, y)\}.$$

A function $T^*\colon D(T^*) \to S$ is called a (B, A)-*adjoint operator* (to T) if and only if

$$B(T(x), v) = A(x, T^*(v)),\ x \in S,\ v \in D(T^*).$$

Lemma 5. *Let S, H, X be groups, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions. Let further $T\colon S \to H$ and $T^*\colon D(T^*) \to S$ be a (B, A)-adjoint operator to T,*

$$S_{AR} := \{y \in S : \forall_{x\in S}\, A(x, y) = 0\}, \tag{3}$$

$$S_{ALT^*} := \{x \in S : \forall_{y\in im\ T^*}\, A(x, y) = 0\}, \tag{4}$$

$$H_{BTR} := 3\{v \in H : \forall_{u\in im\ T}\, B(u, v) = 0\}, \tag{5}$$

$$H_{BLD^*} := \{u \in H : \forall_{v\in D(T^*)}\, B(u, v) = 0\}. \tag{6}$$

Then

1. *$D(T^*)$ is a group, S_{AR}, S_{ALT^*} are normal subgroups of S, H_{BTR}, H_{BLD^*} are normal subgroups of H. Moreover in the case when X is torsion-free, if H is divisible, then H_{BTR}, H_{BLD^*} are divisible, if S is divisible, then S_{AR}, S_{ALT^*} are divisible, if S, H are divisible, then $D(T^*)$ is divisible;*
2. *$\forall_{x,y\in S}\, T(x+y) - T(y) - T(x) \in H_{BLD^*}$;*
3. *$\forall_{x,y\in S}\, x - y \in S_{ALT^*} \Leftrightarrow T(x) - T(y) \in H_{BLD^*}$;*
4. *$\forall_{u,v\in D(T^*)}\, T^*(u+v) - T^*(v) - T^*(u) \in S_{AR}$;*
5. *$\forall_{u,v\in D(T^*)}\, u - v \in H_{BTR} \Leftrightarrow T^*(u) - T^*(v) \in S_{AR}$;*
6. *$H_{BTR} \subset D(T^*)$;*
7. *Assume that H is abelian and divisible. Let K be a subgroup of H such that $H = K \oplus H_{BTR}$, $\varkappa\colon S \to S/S_{AR}$ be a canonical homomorphism. Then $D(T^*) \cap K$ is a group and $\widetilde{T}^* := \varkappa \circ T^*\colon D(T^*) \cap K \to im\ T^*/S_{AR}$ is an isomorphism.*

Proof. 1. Since kernel of any homomorphism is a normal subgroup, then S_{AR}, S_{ALT^*} are normal subgroups of S, H_{BTR}, H_{BLD^*} are normal subgroups of H.
Moreover, if S is divisible and X is torsion-free, then for $x \in S_{ALT^*}$ and $n \in \mathbb{N}$ there exists $z \in S$ such that $nz = x$. We have

$$nA(z, T^*(u)) = A(nz, T^*(u)) = A(x, T^*(u)) = 0, \ u \in D(T^*).$$

Since X is torsion-free, then $z \in S_{ALT^*}$.

2. Let $x, y \in S$, $v \in D(T^*)$. Then

$$\begin{aligned} &B(T(x+y) - T(y) - T(x), v) \\ &\quad = B(T(x+y), v) - B(T(y), v) - B(T(x), v) \\ &\quad = A(x+y, T^*(v)) - A(y, T^*(v)) - A(x, T^*(v)) \\ &\quad = A(x+y-y-x, T^*(v)) = A(0, T^*(v)) = 0, \end{aligned}$$

which shows that $T(x+y) - T(y) - T(x) \in H_{BLD^*}$.

3. Let $x, y \in S$, $v \in D(T^*)$. Then

$$\begin{aligned} B(T(x) - T(y), v) &= B(T(x), v) - B(T(y), v) \\ &= A(x, T^*(v)) - A(y, T^*(v)) = A(x-y, T^*(v)), \end{aligned}$$

which shows that $x - y \in S_{ALT^*} \Leftrightarrow T(x) - T(y) \in H_{BLD^*}$.

4. Let $u, v \in D(T^*)$, $x \in S$.

$$\begin{aligned} &A(x, T^*(u+v) - T^*(v) - T^*(u)) \\ &\quad = A(x, T^*(u+v)) - A(x, T^*(v)) - A(x, T^*(u)) \\ &\quad = B(T(x), u+v) - B(T(x), v) - B(T(x), u) \\ &\quad = B(T(x), u+v-v-u) = B(T(x), 0) = 0, \end{aligned}$$

which shows that $T^*(u+v) - T^*(v) - T^*(u) \in S_{AR}$.

5. Let $u, v \in D(T^*)$, $x \in S$. Then

$$\begin{aligned} B(T(x), u-v) &= B(T(x), u) - B(T(x), v) = A(x, T^*(u)) - A(x, T^*(v)) \\ &= A(x, T^*(u) - T^*(v)), \end{aligned}$$

which shows that $u - v \in H_{BTR} \Leftrightarrow T^*(u) - T^*(v) \in S_{AR}$.

6. Let $u \in H_{BTR}$ and $y \in S_{AR}$. Then

$$B(T(x), u) = 0 = A(x, y), \ x \in S,$$

which shows that $u \in D(T^*)$.

7. Let $u, v \in D(T^*)$. Then using property 4 we obtain

$$\begin{aligned} &(T^*(u) + S_{AR}) + (T^*(v) + S_{AR}) = T^*(u+v) + S_{AR}, \\ &(T^*(u) + S_{AR}) + (T^*(-u) + S_{AR}) = T^*(0) + S_{AR} = S_{AR}, \end{aligned}$$

so im T^*/S_{AR} is a group. Using property 4 we obtain that $\widetilde{T}^*$ is a homomorphism, from 5 we get that $\widetilde{T}^*$ is injective. Let $y = T^*(u)$ for some

$u \in D(T^*)$. Since $H = K \oplus H_{BTR}$, then $u = u_1 + u_2$, where $u_1 \in K$, $u_2 \in H_{BTR}$. From 6 we have $u_1 = u - u_2 \in D(T^*)$. Using property 5 we get $T^*(u) - T^*(u_1) \in S_{AR}$, so

$$\widetilde{T}^*(u_1) = \varkappa(T^*(u_1)) = \varkappa(T^*(u)) = \varkappa(y),$$

which shows that $\widetilde{T}^*$ is surjective.

□

Using property 7 from Lemma 5 we can accept the following

Definition 6. Let S, H, X be groups, H be abelian and divisible, $A\colon S^2 \to X$, $B\colon H^2 \to X$ be biadditive functions. Let further $T\colon S \to H$ and $T^*\colon D(T^*) \to S$ be a (B, A)-adjoint operator to T, im $T^*/S_{AR} = S/S_{AR}$, K be a subgroup of H such that $H = K \oplus H_{BTR}$. We define the function $(T^*)^{-1}\colon S \to D(T^*) \cap K$ by the formula

$$(T^*)^{-1}(x) = (\widetilde{T}^*)^{-1}(\varkappa(x)),\ x \in S. \tag{7}$$

Remark 3. The function $(T^*)^{-1}$ from the above definition is additive and im $(T^*)^{-1} = D(T^*) \cap K$.

3. Main results

Assume that $(S, +)$ is a semigroup, $(H, +)$ is a divisible abelian group, $(X, +)$ is a torsion-free group, $A\colon S^2 \to X$, $B\colon H^2 \to X$ are biadditive functions.

Theorem 3. *Let $f, g\colon S \to H$. Then (f, g) satisfies*

$$B(f(x), g(y)) = A(x, y),\ x, y \in S, \tag{8}$$

if and only if there exist divisible groups H_0, H_1, H_2, H_3, additive functions $f_a\colon S \to H_2 \oplus H_3$, $g_a\colon S \to H_1 \oplus H_3$ and functions $f_r\colon S \to H_0 \oplus H_1$, $g_r\colon S \to H_0 \oplus H_2$ such that

$$H = \bigoplus_{i=0}^{3} H_i \text{ and } H_1, H_2, H_3 \text{ are torsion-free}, \tag{9}$$

$$f = f_a + f_r,\ g = g_a + g_r, \tag{10}$$

$$(H_0 \oplus H_1) \times (H_0 \oplus H_2) \subset B^{-1}(\{0\}), \tag{11}$$

$$im\ f_a \times (H_0 \oplus H_2) \subset B^{-1}(\{0\}), \tag{12}$$

$$(H_0 \oplus H_1) \times im\ g_a \subset B^{-1}(\{0\}), \tag{13}$$

$$B(f_a(x), g_a(y)) = A(x, y),\ x, y \in S. \tag{14}$$

Moreover, we can assume that $H_0 \oplus H_2 = \{v \in H : \forall_{u \in im\ f}\, B(u, v) = 0\}$.

Proof. ($\Rightarrow$) Let

$$\begin{aligned} D_1 &:= \{v \in H :\ \forall_{u \in \mathrm{im}\ f}\ B(u,v) = 0\},\\ D_2 &:= \{u \in H :\ \forall_{v \in \mathrm{im}\ g + D_1}\ B(u,v) = 0\}. \end{aligned}$$

It is easy to see that above sets are groups. We show that $D_1, D_2, D_1 \cap D_2$ are divisible.

Let $v \in D_1$ and $n \in \mathbb{N}$. Then there exists $w \in H$ such that $v = nw$. For every $u \in \mathrm{im}\ f$ we have

$$nB(u,w) = B(u,nw) = B(u,v) = 0,$$

and since X is torsion-free, then $w \in D_1$.

Let $u \in D_2$ and $n \in \mathbb{N}$. Then there exists $w \in H$ such that $u = nw$. For every $v \in \mathrm{im}\ g + D_1$ we have

$$nB(w,v) = B(nw,v) = B(u,v) = 0,$$

and since X is torsion-free, then $w \in D_2$.

Let $x \in D_1 \cap D_2$ and $n \in \mathbb{N}$. Then there exists $z \in H$ such that $x = nz$. Let $u \in \mathrm{im}\ f$ and $v \in \mathrm{im}\ g + D_1$. We have

$$\begin{aligned} nB(u,z) &= B(u,nz) = B(u,x) = 0,\\ nB(z,v) &= B(nz,v) = B(x,v) = 0, \end{aligned}$$

and since X is torsion-free, then $z \in D_1 \cap D_2$.

In view of Theorem 1 there exist divisible groups H_0, H_1, H_2, H_3 such that $D_2 = H_0 \oplus H_1$, $D_1 = H_0 \oplus H_2$ and $H = \bigoplus_{i=0}^{3} H_i$. In view of Lemma 2 every element of H of the finite order belongs to $D_1 \cap D_2 = H_0$, so H_1, H_2, H_3 are torsion-free. Let $f = f_0 + f_1 + f_2 + f_3$, $g = g_0 + g_1 + g_2 + g_3$, where $f_i, g_i \colon S \to H_i$ for $i \in \{0,1,2,3\}$. Let further $f_a{:=}f_2 + f_3$, $g_a{:=}g_1 + g_3$. Hence $f_r{:=}(f - f_a)\colon S \to H_0 \oplus H_1$ and $g_r{:=}(g - g_a)\colon S \to H_0 \oplus H_2$.

We observe also that

$$\begin{aligned} &(H_0 \oplus H_1) \times (H_0 \oplus H_2) = D_2 \times D_1 \subset B^{-1}(\{0\}),\\ &\mathrm{im}\ f_a \times (H_0 \oplus H_2) \subset (\mathrm{im}\ f + D_2) \times D_1 \subset B^{-1}(\{0\}),\\ &(H_0 \oplus H_1) \times \mathrm{im}\ g_a \subset D_2 \times (\mathrm{im}\ g + D_1) \subset B^{-1}(\{0\}). \end{aligned}$$

Now we show that f_a and g_a are additive. Let $x, y \in S$, $v \in D_1$. Then

$$\begin{aligned} B(f_a(x+y) - f_a(y) - f_a(x), g(z) + v) &= B(f(x+y) - f(y) - f(x), g(z))\\ &= B(f(x+y), g(z)) - B(f(y), g(z)) - B(f(x), g(z))\\ &= A(x+y,z) - A(y,z) - A(x,z) = 0, \quad z \in S, \end{aligned}$$

which means that $f_a(x+y)-f_a(y)-f_a(x)\in D_2$, so $f_a(x+y)=f_a(x)+f_a(y)$. Similarly for g_a we have

$$\begin{aligned}&B(f(z),g_a(x+y)-g_a(y)-g_a(x))=B(f(z),g(x+y)-g(y)-g(x))\\&\quad=B(f(z),g(x+y))-B(f(z),g(y))-B(f(z),g(x))\\&\quad=A(z,x+y)-A(z,y)-A(z,x)=0,\quad z\in S,\end{aligned}$$

which means that $g_a(x+y)-g_a(x)-g_a(y)\in D_1$, so $g_a(x+y)=g_a(x)+g_a(y)$.

Moreover, using (11)–(13) we have

$$\begin{aligned}B(f_a(x),g_a(y))&=B(f_a(x),g_a(y))+B(f_r(x),g_a(y))\\&\quad+B(f_a(x),g_r(y))+B(f_r(x),g_r(y))\\&=B(f_a(x)+f_r(x),g_a(y)+g_r(y))\\&=B(f(x),g(y))=A(x,y),\quad x,y\in S.\end{aligned}$$

($\Leftarrow$) Assume that there exist divisible groups H_0,H_1,H_2,H_3, additive functions $f_a\colon S\to H_2\oplus H_3$, $g_a\colon S\to H_1\oplus H_3$ and functions $f_r\colon S\to H_0\oplus H_1$, $g_r\colon S\to H_0\oplus H_2$ such that conditions (9)–(14) holds. Then

$$\begin{aligned}B(f(x),g(y))&=B(f_a(x)+f_r(x),g_a(y)+g_r(y))\\&=B(f_a(x),g_a(y))+B(f_a(x),g_r(y))\\&\quad+B(f_r(x),g_a(y))+B(f_r(x),g_r(y))\\&=B(f_a(x),g_a(y))=A(x,y),\quad x,y\in S.\end{aligned}$$

□

The following example shows that we cannot drop the assumption that X is torsion-free in the previous theorem.

Example 1. Let $S=\mathbb{Z}^2$, $H=\mathbb{Q}^2$, $X=\mathbb{Q}\times\mathbb{Z}(2^\infty)$, $f,g\colon S\to H$ be functions given by formulas

$$\begin{aligned}&f(n,m)=\begin{cases}(n,1) & n\in\mathbb{Z},\ m\in 2\mathbb{Z}+1\\(n,2^{|m|+1}) & n\in\mathbb{Z},\ m\in 2\mathbb{Z}\end{cases},\\&g(n,m)=(n,m),\ n,m\in\mathbb{Z}.\end{aligned}$$

Let further $B\colon H^2\to X$, $A\colon S^2\to X$ be functions given by formulas

$$\begin{aligned}B\Big((n,m),(p,q)\Big)&=(np,C(m,q)),\ n,m,p,q\in\mathbb{Q},\\A(x,y)&=B(f(x),g(y)),\ x,y\in S,\end{aligned}$$

where $C\colon\mathbb{Q}^2\to\mathbb{Z}(2^\infty)$ is a biadditive and symmetric function such that $C(1,1)=\frac{1}{2}+\mathbb{Z}$ (see Theorem 2).

It is easy to see that g is additive, B is biadditive and symmetric.

Since for all $x, y \in S$ we have $f(x+y) - f(x) - f(y) \in \{0\} \times 2\mathbb{Z}$, then for every $z = (z_1, z_2) \in S$ there is an $n \in \mathbb{Z}$ such that

$$\begin{aligned} &A(x+y,z) - A(x,z) - A(y,z) \\ &\quad = B(f(x+y),z) - B(f(x),z) - B(f(y),z) \\ &\quad = B(f(x+y) - f(x) - f(y), z) = (0 \cdot z_1, C(2n, z_2)) \\ &\quad = (0, 2nz_2C(1,1)) = \left(0, 2nz_2\frac{1}{2} + \mathbb{Z}\right) = (0, \mathbb{Z}). \end{aligned}$$

Hence A is biadditive and (f, g) solves (8).

Suppose that there exist divisible groups H_0, H_1, H_2, H_3, additive functions $f_a : S \to H_2 \oplus H_3$, $g_a : S \to H_1 \oplus H_3$ and functions $f_r : S \to H_0 \oplus H_1$, $g_r : S \to H_0 \oplus H_2$ such that conditions (9)–(13) holds. Since

$$\mathbb{Z}^2 = \text{im } g \subset \text{im } g_a + (H_0 \oplus H_2),$$

then from (11), (13) we obtain

$$(H_0 \oplus H_1) \times \mathbb{Z}^2 \subset B^{-1}(\{(0, \mathbb{Z})\}).$$

Let $(p, q) \in H_0 \oplus H_1$. Then there exists $k \in \mathbb{N}$ such that $(kp, kq) \in \mathbb{Z}^2$. Hence, since $(kp, kq) \in H_0 \oplus H_1$, we get

$$(0, \mathbb{Z}) = B\Big((kp, kq), (1, 1)\Big) = \left(kp, \frac{kq}{2} + \mathbb{Z}\right),$$

so $p = 0$ and $kq \in 2\mathbb{Z}$. On the other hand, if $q \neq 0$ and $(0, kq) \in H_0 \oplus H_1$, then, by Lemma 1, $(0, 1) \in H_0 \oplus H_1$. Consequently,

$$(0, \mathbb{Z}) = B\Big((0, 1), (0, 1)\Big) = \left(0, \frac{1}{2} + \mathbb{Z}\right), \tag{15}$$

a contradiction. Thus $H_0 = H_1 = \{0\}$ and $f_a = f$, but f is not additive, which give us a contradiction.

In the theorem below we investigate the preservation of the biadditivity by only one function, namely we solve the following generalization of the orthogonality equation.

Theorem 4. *Let $f : S \to H$. Then f satisfies*

$$B(f(x), f(y)) = A(x, y), \ \ x, y \in S, \tag{16}$$

if and only if there exist divisible groups H_0, H_1, an additive function $F_a : S \to H_1$, and a function $F_r : S \to H_0$ such that

$$H = H_0 \oplus H_1 \ \textit{and} \ H_1 \ \textit{is torsion-free}, \tag{17}$$

$$f = F_a + F_r, \tag{18}$$

$$H_0 \times (H_0 \oplus im\ F_a) \subset B^{-1}(\{0\}), \tag{19}$$

$$(H_0 \oplus im\ F_a) \times H_0 \subset B^{-1}(\{0\}), \tag{20}$$

$$B(F_a(x), F_a(y)) = A(x, y),\ x, y \in S. \tag{21}$$

Moreover, we can assume that $H_0 \subset \{v \in H :\ \forall_{u \in im\ f}\, B(u, v) = 0\}$.

Proof. ($\Rightarrow$) In view of Theorem 3 there exist divisible groups K_0, K_1, K_2, K_3, additive functions $f_a\colon S \to K_2 \oplus K_3$, $\widetilde{f}_a\colon S \to K_1 \oplus K_3$ and functions $f_r\colon S \to K_0 \oplus K_1$, $\widetilde{f}_r\colon S \to K_0 \oplus K_2$ such that

$$\begin{aligned}
&H = \bigoplus_{i=0}^{3} K_i \text{ and } K_1, K_2, K_3 \text{ are torsion-free},\\
&f = f_a + f_r = \widetilde{f}_a + \widetilde{f}_r,\\
&(K_0 \oplus K_1) \times (K_0 \oplus K_2) \subset B^{-1}(\{0\}),\\
&\text{im } f_a \times (K_0 \oplus K_2) \subset B^{-1}(\{0\}),\\
&(K_0 \oplus K_1) \times \text{im } \widetilde{f}_a \subset B^{-1}(\{0\}),\\
&B(f_a(x), \widetilde{f}_a(y)) = A(x, y),\ x, y \in S.
\end{aligned}$$

Let $f = f_0 + f_1 + f_2 + f_3$, where $f_i\colon S \to K_i$ for $i \in \{0, 1, 2, 3\}$. Then $f_a = f_2 + f_3$ and $\widetilde{f}_a = f_1 + f_3$. Hence f_1, f_2, f_3 are additive. Let $H_0 = K_0$, $H_1 = \bigoplus_{i=1}^{3} K_i$, $F_a = f_1 + f_2 + f_3$, $F_r = f_0$. Then $F_a\colon S \to H_1$ is additive. We have also

$$\begin{aligned}
&(H_0 \oplus K_1) \times H_0 \subset (K_0 \oplus K_1) \times (K_0 \oplus K_2) \subset B^{-1}(\{0\}),\\
&\text{im } f_a \times H_0 \subset \text{im } f_a \times (K_0 \oplus K_2) \subset B^{-1}(\{0\}),\\
&H_0 \times (H_0 \oplus K_2) \subset (K_0 \oplus K_1) \times (K_0 \oplus K_2) \subset B^{-1}(\{0\}),\\
&H_0 \times \text{im } \widetilde{f}_a \subset (K_0 \oplus K_1) \times \text{im } \widetilde{f}_a \subset B^{-1}(\{0\}),
\end{aligned}$$

and since B is biadditive we obtain that

$$\begin{aligned}
&(H_0 \oplus \text{im } F_a) \times H_0 \subset (\text{im } f_a \oplus H_0 \oplus K_1) \times H_0 \subset B^{-1}(\{0\}),\\
&H_0 \times (H_0 \oplus \text{im } F_a) \subset H_0 \times (\text{im } \widetilde{f}_a \oplus H_0 \oplus K_2) \subset B^{-1}(\{0\}).
\end{aligned}$$

Consequently

$$\begin{aligned}
B(F_a(x), F_a(y)) &= B(F_a(x), F_a(y)) + B(F_a(x), F_r(y)) + B(F_r(x), F_a(y))\\
&\quad + B(F_r(x), F_r(y))\\
&= B(F_a(x) + F_r(x), F_a(y) + F_r(y)) = A(x, y), \quad x, y \in S.
\end{aligned}$$

($\Leftarrow$) Assume that there exist divisible groups H_0, H_1, an additive function $F_a\colon S \to H_1$, and a function $F_r\colon S \to H_0$ such that conditions (17)–(21) holds. Then

$$\begin{aligned} B(f(x), f(y)) &= B(F_a(x) + F_r(x), F_a(y) + F_r(y)) \\ &= B(F_a(x), F_a(y)) + B(F_a(x), F_r(y)) + B(F_r(x), F_a(y)) \\ &\quad + B(F_r(x), F_r(y)) \\ &= B(F_a(x), F_a(y)) = A(x, y), \quad x, y \in S. \end{aligned}$$

□

It is a natural question whether given a function f there exists a function g such that (f, g) satisfies equation (8). The theorem below give us an answer for this question.

Theorem 5. *Assume that S is a group, $f, g\colon S \to H$. Then (f, g) satisfies equation (8) if and only if there exist divisible groups H_0, H_1, H_2, H_3, an additive function $T\colon S \to H_2 \oplus H_3$, functions $f_r\colon S \to H_0 \oplus H_1$, $g_r\colon S \to H_0 \oplus H_2$ such that*

$$H = \bigoplus_{i=0}^{3} H_i \text{ and } H_1, H_2, H_3 \text{ are torsion-free}, \tag{22}$$

$$\text{im } T^*/S_{AR} = S/S_{AR}, \tag{23}$$

$$f = T + f_r,\ g = (T^*)^{-1} + g_r, \tag{24}$$

$$(H_0 \oplus H_1) \times (H_0 \oplus H_2) \subset B^{-1}(\{0\}), \tag{25}$$

$$\text{im } T \times (H_0 \oplus H_2) \subset B^{-1}(\{0\}), \tag{26}$$

$$(H_0 \oplus H_1) \times (D(T^*) \cap K) \subset B^{-1}(\{0\}), \tag{27}$$

where $T^*\colon D(T^*) \to S$ is a (B, A)-adjoint operator to T, S_{AR} is given by (3), $(T^*)^{-1}$ is defined by the formula (7) and K is a subgroup of H such that $H_{BTR} \oplus K = H$, where H_{BTR} is given by (5).

Proof. ($\Rightarrow$) Assume that (f, g) satisfies equation (8). Then in view of Theorem 3 there exist divisible groups H_0, H_1, H_2, H_3, additive functions $f_a\colon S \to H_2 \oplus H_3$, $g_a\colon S \to H_1 \oplus H_3$ and functions $f_r\colon S \to H_0 \oplus H_1$, $g_r\colon S \to H_0 \oplus H_2$ which satisfy conditions (9)–(14). Let $T = f_a$. In view of (14) im $g_a \subset D(T^*)$. Let $y \in S$. We have

$$A(x, y) = B(T(x), g_a(y)) = A(x, T^*(g_a(y))),\ x \in S,$$

so $y - T^*(g_a(y)) \in S_{AR}$ and $\varkappa(y) = \widetilde{T}^*(g_a(y))$. Hence $S/S_{AR} = \text{im } T^*/S_{AR}$ and

$$(T^*)^{-1}(y) = (\widetilde{T}^*)^{-1}(\varkappa(y)) = (\widetilde{T}^*)^{-1}(\widetilde{T}^*(g_a(y))) = g_a(y).$$

In view of Remark 3 and (13) we get

$$\begin{aligned} (H_0 \oplus H_1) \times (D(T^*) \cap K) &= (H_0 \oplus H_1) \times \text{im } (T^*)^{-1} \\ &= (H_0 \oplus H_1) \times \text{im } g_a \subset B^{-1}(\{0\}). \end{aligned}$$

Conditions (25), (26) are exactly the same as (11) and (12).

($\Leftarrow$) Assume that there exist divisible groups H_0, H_1, H_2, H_3, an additive function $T\colon S \to H_2 \oplus H_3$, functions $f_r\colon S \to H_0 \oplus H_1$, $g_r\colon S \to H_0 \oplus H_2$ which satisfy conditions (22)–(27).

For $y \in S$ we have

$$\varkappa(T^*((T^*)^{-1}(y))) = \widetilde{T}^*((\widetilde{T}^*)^{-1}(\varkappa(y))) = \varkappa(y),$$

which means that $y - T^*((T^*)^{-1}(y)) \in S_{AR}$. From Remark 3 we get

$$(H_0 \oplus H_1) \times \mathrm{im}\,(T^*)^{-1} = (H_0 \oplus H_1) \times (D(T^*) \cap K) \subset B^{-1}(\{0\}).$$

We have

$$\begin{aligned}
A(x,y) &= A(x, y - T^*((T^*)^{-1}(y))) + A(x, T^*((T^*)^{-1}(y))) \\
&= 0 + B(T(x), (T^*)^{-1}(y)) = B(T(x), (T^*)^{-1}(y)) \\
&\quad + B(T(x), g_r(y)) + B(f_r(x), (T^*)^{-1}(y)) + B(f_r(x), g_r(y)) \\
&= B(T(x) + f_r(x), (T^*)^{-1}(y) + g_r(y)) = B(f(x), g(y)), \quad x, y \in S.
\end{aligned}$$

□

The following result shows us for which f defined on a group (16) holds.

Theorem 6. *Assume that S is a group, $f\colon S \to H$. Then f satisfies (16) if and only if there exist divisible groups H_0, H_1, an additive function $T\colon S \to H_1$, and a function $F_r\colon S \to H_0$ such that*

$$H = H_0 \oplus H_1 \text{ and } H_1 \text{ is torsion-free}, \tag{28}$$
$$im\ T \subset D(T^*),\ \forall_{y\in S}\,(T^* \circ T)(y) - y \in S_{AR}, \tag{29}$$
$$f = T + F_r, \tag{30}$$
$$H_0 \times (H_0 \oplus im\ T) \subset B^{-1}(\{0\}), \tag{31}$$
$$(H_0 \oplus im\ T) \times H_0 \subset B^{-1}(\{0\}), \tag{32}$$

where $T^*\colon D(T^*) \to S$ is a (B, A)-adjoint operator to T, S_{AR} is given by (3).

Proof. ($\Rightarrow$) Assume that f satisfies (16). In view of Theorem 4 there exist divisible groups H_0, H_1, an additive function $F_a\colon S \to H_1$, a function $F_r\colon S \to H_0$ which satisfy conditions (17)–(21). Let $T = F_a$. We notice that conditions (28), (30)–(32) hold. From (21) we obtain that im $T \subset D(T^*)$ and for $y \in S$ we have

$$\begin{aligned}
A(x, T^*(T(y)) - y) &= A(x, T^*(T(y))) - A(x, y) \\
&= B(T(x), T(y)) - B(T(x), T(y)) = 0,\ x \in S,
\end{aligned}$$

which means that $T^*(T(y)) - y \in S_{AR}$.

($\Leftarrow$) Assume that there exist divisible groups H_0, H_1, an additive function $T\colon S \to H_1$, and a function $F_r\colon S \to H_0$ which satisfy conditions (28)–(32). We have

$$\begin{aligned} B(f(x), f(y)) &= B(T(x) + F_r(x), T(y) + F_r(y)) \\ &= B(T(x), T(y)) + B(T(x), F_r(y)) + B(F_r(x), T(y) + F_r(y)) \\ &= B(T(x), T(y)) = A(x, T^*(T(y))) \\ &= A(x, T^*(T(y)) - y) + A(x, y) = A(x, y), \quad x, y \in S. \end{aligned}$$

□

4. Applications

In this part of the article we would like to present some applications of the main results from Section 3 in particular for normed spaces. It is helpful to recall (see [1, Theorem 2.1.1 and Remark 2.1.1]) that the following properties for a real normed space $(X, \|\cdot\|)$ are true:

$$\text{If } X \text{ is real and smooth, then } \rho'_+(x, \cdot) \text{ is linear for all } x \in X. \tag{33}$$

$$\text{If } X \text{ is real and smooth, then } \rho'_+(\cdot, y) \text{ is homogeneous for all } y \in X. \tag{34}$$

$$|\rho'_+(x, y)| \leq \|x\| \cdot \|y\| \quad \text{and} \quad \rho'_+(x, x) = \|x\|^2. \tag{35}$$

Theorem 7. *Let X, Y be real and smooth normed spaces, X be reflexive. Let $f\colon X \to Y$ be a mapping satisfying:*

$$\rho'_+(f(x), f(y)) = \rho'_+(x, y), \quad x, y \in X. \tag{36}$$

Suppose that $V \subset im\ f$ is a closed subspace of Y such that $\operatorname{co}\dim V = 1$ *and* $\operatorname{cl}\operatorname{span} f^{-1}(V) \neq X$. *Then f is a linear isometry.*

Before we start the proof, some comments are needed. In the paper [6] this result was proved under the surjectivity assumption (and that X and Y are Banach and Y is separable). However our assumption (that $\operatorname{cl}\operatorname{span} f^{-1}(V) \neq X$) is weaker than the surjectivity. As regards the smoothness, this assumption seems to be reasonable. Indeed (see [6]), there are both smooth and strictly convex normed spaces Z_1, Z_2 and nonlinear mappings $T\colon Z_1 \to Z_2$ satisfying (36).

Proof. Let $W{:=}\operatorname{cl}\operatorname{span} f^{-1}(V)$. By the reflexivity, there is $x \in X$ such that $\|x\| = 1$ and $\|x\| = \operatorname{dist}(x, W)$. We define two bilinear mappings $A_x\colon X^2 \to \mathbb{R}$, $B_{f(x)}\colon Y^2 \to \mathbb{R}$ by the formulas $A_x(u, w){:=}\rho'_+(x, u) \cdot \rho'_+(x, w)$, $B_{f(x)}(z, v){:=}\rho'_+(f(x), z) \cdot \rho'_+(f(x), v)$. It follows from (36) that

$$A_x(u, w) = B_{f(x)}(f(u), f(w)), \quad u, w \in X. \tag{37}$$

Put $D_1 := \{z \in Y : \forall_{u \in X}\, B_{f(x)}(z, f(u)) = 0\}$. From this we get

$$\begin{aligned} D_1 &= \{z \in Y : \forall_{u \in X}\, \rho'_+(f(x), z) \cdot \rho'_+(f(x), f(u)) = 0\} \\ &= \{z \in Y : \forall_{u \in X}\, \rho'_+(f(x), z) \cdot \rho'_+(x, u) = 0\} \\ &= \{z \in Y : \rho'_+(f(x), z) = 0\}. \end{aligned}$$

Thus D_1 is a closed linear subspace. In particular, D_1 is a divisible abelian group. We have also

$$B_{f(x)}(u, v) = B_{f(x)}(v, u) = 0,\ u \in D_1, v \in Y. \tag{38}$$

Moreover $Y = \operatorname{span}\{f(x)\} \oplus D_1$ and so $f = f_a + f_r$, where $f_a \colon X \to \operatorname{span}\{f(x)\}$, $f_r \colon X \to D_1$. From Theorem 4 there exist divisible groups H_0, H_1, an additive function $F_a \colon X \to H_1$, and a function $F_r \colon X \to H_0$ such that

$$\begin{aligned} &Y = H_0 \oplus H_1 \text{ and } H_0 \subset \{v \in Y : \forall_{u \in \operatorname{im} f}\, B_{f(x)}(u, v) = 0\} = D_1, \\ &f = F_a + F_r. \end{aligned}$$

We observe that for $y, z \in X$ we have

$$\begin{aligned} f_a(y+z) - f_a(y) - f_a(z) &= f(y+z) - f(y) - f(z) \\ &\quad - f_r(y+z) + f_r(y) + f_r(z) = F_a(y+z) - F_a(y) - F_a(z) \\ &\quad + F_r(y+z) - F_r(y) - F_r(z) - f_r(y+z) + f_r(y) + f_r(z) \\ &= F_r(y+z) - F_r(y) - F_r(z) - f_r(y+z) + f_r(y) + f_r(z) \in H_0 + D_1 \subset D_1, \end{aligned}$$

which means that f_a is additive.

Since $f_a(w) \in \operatorname{span}\{f(x)\}$ for $w \in X$, there exists a function $\varphi \colon X \to \mathbb{R}$ such that $f_a = \varphi f(x)$. Therefore, by the property of the set D_1 and by (34) we have $\rho'_+(f_a(w), f_r(y)) = 0$ for $w, y \in X$. So, it and (33) and (35) yield

$$\begin{aligned} \|f_a(y)\|^2 &= \rho'_+(f_a(y), f_a(y)) + 0 = \rho'_+(f_a(y), f_a(y)) + \rho'_+(f_a(y), f_r(y)) \\ &= \rho'_+(f_a(y), f_a(y) + f_r(y)) \le \|f_a(y)\| \cdot \|f_a(y) + f_r(y)\| \\ &= \|f_a(y)\| \cdot \|f(y)\|. \end{aligned}$$

Since $\|f(y)\| = \|y\|$, it follows from the above inequalities that $\|f_a(y)\| \le \|y\|$ for all $y \in X$, which implies that f_a is continuous and linear. Consequently $f_a(w) = \varphi(w) \cdot f(x)$ for every $w \in X$ with some $\varphi \in X^*$. Next, for all u, w in X we have

$$\begin{aligned} \rho'_+(u, w) &= \rho'_+(f(u), f(w)) = \rho'_+(f(u), f_a(w) + f_r(w)) \\ &= \rho'_+(f(u), \varphi(w) \cdot f(x) + f_r(w)) \\ &= \varphi(w) \cdot \rho'_+(f(u), f(x)) + \rho'_+(f(u), f_r(w)) \\ &= \varphi(w) \cdot \rho'_+(u, x) + \rho'_+(f(u), f_r(w)). \end{aligned}$$

For given $u \in X$ we define a $\gamma_u \in X^*$ by the formula

$$\gamma_u(w) := \rho'_+(u, w) - \varphi(w) \rho'_+(u, x), \quad w \in X.$$

It follows from the above equalities that $\gamma_u(w) = \rho'_+(f(u), f_r(w))$. Therefore for fixed $w, z \in X$ we get

$$\begin{aligned}
&\rho'_+(f(u), f_r(\alpha w + \beta z) - \alpha f_r(w) - \beta f_r(z)) \\
&\quad = \rho'_+(f(u), f_r(\alpha w + \beta z)) - \alpha\rho'_+(f(u), f_r(w)) - \beta\rho'_+(f(u), f_r(z)) \\
&\quad = \gamma_u(\alpha w + \beta z) - \alpha\gamma_u(w) - \beta\gamma_u(z) \\
&\quad = \gamma_u(\alpha w + \beta z - \alpha w - \beta z) = 0.
\end{aligned}$$

To summarize, we proved

$$\forall_{u\in X}\ \rho'_+(f(u), f_r(\alpha w + \beta z) - \alpha f_r(w) - \beta f_r(z)) = 0. \tag{39}$$

Since $\|x\| = \mathrm{dist}(x, W)$, we have the inequality $\|x\| \le \|x + w\|$ for all $w \in W$. In particular, for all $t > 0$ we obtain $0 \le \|x\| \cdot \frac{\|x+tw\|-\|x\|}{t}$. Letting $t \to 0^+$, we get $0 \le \rho'_+(x, w)$. Putting $-w$ in place of w (and applying again (33)) we get $0 \ge \rho'_+(x, w)$. So, we proved that $\rho'_+(x, c) = 0$ for all $c \in W$.

Clearly $f^{-1}(V) \subset W$. In particular, for all c in $f^{-1}(V)$ we have $0 = \rho'_+(x, c) = \rho'_+(f(x), f(c))$. Thus $V \subset D_1$. Since $\operatorname{co\,dim} V = 1 = \operatorname{co\,dim} D_1$, we obtain $V = D_1$. Since $f_r(\alpha w + \beta z) - \alpha f_r(w) - \beta f_r(z) \in D_1 = V \subset \operatorname{im} f$, there is a $b_0 \in X$ such that $f(b_0) = f_r(\alpha w + \beta z) - \alpha f_r(w) - \beta f_r(z)$. Hence, applying (39), we get

$$\begin{aligned}
\|f_r(\alpha w + \beta z) - \alpha f_r(w) - \beta f_r(z)\|^2 &= \|f(b_0)\|^2 = \rho'_+(f(b_0), f(b_0)) \\
&= \rho'_+(f(b_0), f_r(\alpha w + \beta z) - \alpha f_r(w) - \beta f_r(z)) = 0.
\end{aligned}$$

It holds for all $w, z \in X$ and $\alpha, \beta \in \mathbb{R}$, which means that f_r is linear. Since f_a, f_r are linear then f also is linear mapping. The equality $\|f(w)\| = \|w\|$ for all w in X implies that f is an isometry. □

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