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A NOTE ON THE CORE TOPOLOGY

Abstract. In the paper examples are given of some plane sets peculiar with respect to the core topology. Some simple topological properties of the cartesian product of sets lying in linear spaces endowed with the core topology are also proved.

Introduction. Let X be a real linear space and let $A \subset X$ be a set. A point $a \in A$ is said to be *algebraically interior* to A iff for every $b \in X$ there exists a positive number $\varepsilon = \varepsilon(a, b)$ such that

$$a + \lambda b \in A \quad \text{for} \quad \lambda \in (-\varepsilon, \varepsilon). \tag{1}$$

(In other words, in every direction A contains an open segment centered at a). The set of all points which are algebraically interior to A is denoted $\operatorname{core} A$:

$$\operatorname{core} A = \{a \in A \mid a \text{ is algebraically interior to } A\},$$

and a set $A \subset X$ is called *algebraically open* whenever $A = \operatorname{core} A$, that is, all points of A are algebraically interior to A. The family

$$\mathscr{T}(X) = \{A \subset X \mid A = \operatorname{core} A\}$$

of all algebraically open subsets of X is a topology in X and is called the *core topology* (cf. [1]—[3], [8]).

In the present note we investigate the iteration of the operation core:

$$\operatorname{core}^1 A = \operatorname{core} A, \quad \operatorname{core}^{n+1} A = \operatorname{core} \operatorname{core}^n A \quad \text{for} \quad n \in \mathbf{N}, \tag{2}$$

and we exhibit some examples of sets with a peculiar behaviour under iteration (2). In the last section we prove a few simple properties of cartesian products of sets in spaces endowed with the core topology.

1. It would be reasonable to conjecture that actually $\operatorname{core} A = \operatorname{int} A$, the interior being meant in the sense of the core topology. This, however, is not the case. We always have (cf. [1], [3])

$$\operatorname{int} A \subset \operatorname{core} A, \tag{3}$$

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but the equality need not hold in (3). In general, the operation core is not necessarily idempotent. K. Nikodem gave an example (cf. [1], [3] and also Section 2 below) of a planar set A such that (see (2))

$$\operatorname{core}^2 A \neq \operatorname{core} A. \tag{4}$$

In the present section we are going to give another example of a planar set A with property (4).

We take $X = \mathbf{R}^2$. In order to distinguish points in the plane from open intervals, in the sequel the point $a \in \mathbf{R}^2$ with the coordinates ξ and η will be denoted $a = \langle \xi, \eta \rangle$.

Let

$$A = \mathbf{R}^2 \setminus \mathbf{Q}^2 \text{ and } B = \mathbf{R}^2 \setminus [(\mathbf{Q} \times \mathbf{R}) \cup (\mathbf{R} \times \mathbf{Q})]$$

be the set of points with at least one coordinate irrational and the set of points with both coordinates irrational, respectively. Clearly

$$\operatorname{core} B = \varnothing, \tag{5}$$

no segment can consist only of points with both coordinates irrational. We are going to show that

$$\operatorname{core} A \neq \varnothing. \tag{6}$$

Choose $\xi, \eta \in \mathbf{R}$ such that the numbers $1, \xi, \eta$ are linearly independent over $\mathbf{Q}$ and consider the point $a = \langle \xi, \eta \rangle \in B \subset A$. The vertical line $x = \xi$ and the horizontal line $y = \eta$ passing through a are contained in A, thus it is enough to investigate only the straight lines with the equation

$$y - \eta = \alpha(x - \xi), \tag{7}$$

where $\alpha \in \mathbf{R}$, $\alpha \neq 0$. Let α be rational and suppose that a point $\langle x, y \rangle \in \mathbf{Q}^2$ fulfils (7). Then

$$(y - \alpha x) + \alpha \xi - \eta = 0,$$

which is incompatible with the linear independence of $1, \xi, \eta$. Consequently the whole straight line (7) is contained in A. Now consider the case of irrational α and suppose that points $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in \mathbf{Q}^2$ fulfil (7). Put $x_0 = x_1 - x_2$, $y_0 = y_1 - y_2$. The numbers x_0, y_0 are rational and by (7) $y_0 = \alpha x_0$, which implies $x_0 = y_0 = 0$ and consequently $x_1 = x_2$, $y_1 = y_2$. Thus in this case straight line (7) can intersect the set $\mathbf{Q}^2$ at at most one point, say $\langle \bar{x}, \bar{y} \rangle$, and the segment

$$\{\langle x, y \rangle \in \mathbf{R}^2 \mid (7) \text{ holds and } |x - \xi| < |\bar{x} - \xi|\}$$

is contained in A.

The above considerations show that for every $b \in \mathbf{R}^2$ relation (1) is fulfilled, which means that $a \in \operatorname{core} A$. This proves (6).

If one of the numbers ξ, η is rational, then the corresponding line $x = \xi$ or $y = \eta$ has a dense set of points in common with $\mathbf{Q}^2$ and no segment of it can be contained in A so that $\langle\xi, \eta\rangle \notin \operatorname{core} A$. Consequently $\operatorname{core} A \subset B$, whence by (2) and (5)

$$\operatorname{core}^2 A = \operatorname{core}\ \operatorname{core} A \subset \operatorname{core} B = \emptyset,$$

i.e., $\operatorname{core}^2 A = \emptyset$. Together with (6) this yields (4).

2. The natural question arises as to whether for every positive integer n a set $A \subset X$ can be found such that

$$\operatorname{core}^{n+1} A \neq \operatorname{core}^n A. \tag{8}$$

Observe that the sequence $\{\operatorname{core}^n A\}_{n\in\mathbf{N}}$ is decreasing and if for a positive integer m (m may even be nonnegative if we put $\operatorname{core}^0 A := A$) we have

$$\operatorname{core}^{m+1} A = \operatorname{core}^m A, \tag{9}$$

then this sequence is stationary:

$$\operatorname{core}^n A = \operatorname{core}^m A \quad \text{for} \quad n \geqslant m$$

and, moreover,

$$\operatorname{int} A = \operatorname{core}^m A \tag{10}$$

(the interior being taken in the sense of the core topology). In fact, since $\operatorname{int} A \in \mathscr{T}(X)$, we have $\operatorname{core}\operatorname{int} A = \operatorname{int} A$, whence in view of (3)

$$\operatorname{int} A \subset \operatorname{core}^m A. \tag{11}$$

(If $m = 0$ relation (11) is trivial). On the other hand, (9) means that $\operatorname{core}^m A \in \mathscr{T}(X)$, and clearly $\operatorname{core}^m A \subset A$, whence

$$\operatorname{core}^m A \subset \operatorname{int} A. \tag{12}$$

Relations (11) and (12) yield (10).

Now again we take $X = \mathbf{R}^2$ and let $A_1 \subset \mathbf{R}^2$ be Nikodem's set fulfilling (4):

$$A_1 = \{\langle\xi, \eta\rangle \in \mathbf{R}^2 \mid (\xi-1)^2 + \eta^2 < 1\} \cup \{\langle\xi, \eta\rangle \in \mathbf{R}^2 \mid (\xi+1)^2 + \eta^2 < 1\} \cup I,$$

where

$$I = \{\langle\xi, \eta\rangle \in \mathbf{R}^2 \mid \xi = 0,\ -1 < \eta < 1\}.$$

Let $\{\alpha_n\}_{n\in\mathbf{N}}$ be a decreasing sequence of numbers from the interval $(0, 1)$ converging to zero:

$$0 < \alpha_{n+1} < \alpha_n < 1 \quad \text{for} \quad n \in \mathbf{N}, \quad \lim_{n\to\infty} \alpha_n = 0, \tag{13}$$

let $\{\beta_n\}_{n\in\mathbf{N}}$ be an arbitrary sequence of positive numbers and put

$$\gamma_n = \frac{1}{2}(\alpha_n + \alpha_{n+1}), \quad \delta_n = \frac{1}{2}(\alpha_n - \alpha_{n+1}), \quad n \in \mathbf{N}.$$

On the segment I of the set A_1 we build a “ladder”

$$S_1 = \bigcup_{n=1}^{\infty}[\{\langle\xi,\ \eta\rangle\in\mathbf{R}^2 \mid -\beta_n<\xi<\beta_n,\ \eta = \alpha_n\}$$
$$\cup\{\langle\xi,\ \eta\rangle\in\mathbf{R}^2 \mid -\beta_n<\xi<\beta_n,\ \eta = -\alpha_n\}$$
$$\cup\{\langle\xi,\ \eta\rangle\in\mathbf{R}^2 \mid \xi^2+(\eta-\gamma_n)^2<\delta_n^2\}$$
$$\cup\{\langle\xi,\ \eta\rangle\in\mathbf{R}^2 \mid \xi^2+(\eta+\gamma_n)^2<\delta_n^2\}].$$

(If we chose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ more thoroughly, we could make S_1 to fulfil $S_1\cap(A_1\backslash I) = \emptyset$, but this condition is not essential for our construction.

We write

$$A_2 = A_1\cup S_1.$$

The points of the "rungs"

$$\{\langle\xi,\ \eta\rangle\in\mathbf{R}^2 \mid -\beta_n<\xi<\beta_n,\ \eta = \pm\alpha_n\}, \qquad n\in\mathbf{N},$$

(except for these contained in A_1) clearly are not algebraically interior to A_2 and consequently are not in coreA_2. Therefore the latter contains no horizontal segments centered at the points $\langle 0,\ \pm\alpha_n\rangle$, $n\in\mathbf{N}$, so that these points are not in core^2A_2. In view of (13) core^2A_2 contains no vertical segment centered at the origin. Thus $\langle 0,\ 0\rangle\notin\text{core}^3A_2$ and the set A_2 fulfils (8) with $n = 2$.

We can proceed further in the same manner. On the "rungs" of S_1 we build (horizontal) "ladders" of the second generation according to the same pattern. We denote by S_2 the union of all the "ladders" of the second generation and we write $A_3 = A_2\cup S_2$. Arguing similarly as above we check that the set A_3 fulfils (8) with $n = 3$.

Having defined in this way the set A_n for an $n\in\mathbf{N}$ we build "ladders" of the n-th generation on all the "rungs" of the "ladders" of the $(n-1)$-st generation and we denote by S_n the union of all the "ladders" of the n-th generation. It is readily seen that the set

$$A_{n+1} = A_n\cup S_n \tag{14}$$

fulfils (8) with n replaced by $n+1$.

In this way, using formula (14), we define by induction an increasing sequence $\{A_n\}_{n\in\mathbf{N}}$ of sets $A_n\subset\mathbf{R}^2$ fulfilling the condition

$$\text{core}^{n+1}A_n \neq \text{core}^nA_n, \qquad n\in\mathbf{N}.$$

It is also easy to see that every set A_n fulfils (9), and hence also (10) with $m = n+1$.

We can carry on this construction *ad infinitum* arriving thus at the set

$$\tilde{A} = \bigcup_{n=1}^{\infty}A_n. \tag{15}$$

But, contrary to what one could expect, set (15) does not fulfil (8) for all $n\in\mathbf{N}$. In fact, we have $\text{core}\,\tilde{A} = \tilde{A}$ and thus the set $\tilde{A}$ is algebraically open providing another example of an algebraically open subset of the plane with a rather peculiar structure from the point of view of the natural topology of $\mathbf{R}^2$.

However, by a slight modification of definition (15) we can obtain a set $A\subset\mathbf{R}^2$ fulfilling relation (8) for all $n\in\mathbf{N}$. We assume $\alpha_0 = 1$ and define the set A by the formula

(16) $$A = \bigcup_{n=1}^{\infty}(A_{n+1} \cap [\mathbf{R} \times (-\alpha_{n-1}, \alpha_{n-1})]).$$

In other words, to each point $\langle 0, \pm\alpha_n \rangle$ there is affixed the system of "ladders" up to the n-th generation. Consequently

(17) $$\langle 0, \pm\alpha_n \rangle \in \operatorname{core}^n A \backslash \operatorname{core}^{n+1} A, \quad n \in \mathbf{N},$$

and

(18) $$\langle 0, 0 \rangle \in \operatorname{core}^n A, \quad n \in \mathbf{N}.$$

Relation (17) shows that (8) holds for all $n \in \mathbf{N}$.

We consider also the set

$$E = \bigcap_{n=1}^{\infty} \operatorname{core}^n A,$$

where A is given by (16), and encounter yet another surprise. We might have expected that $E = \operatorname{int} A$ (this is true, in particular, when A is replaced by any one of the sets A_m), which, however, is not the case. By virtue of (18) we have $\langle 0, 0 \rangle \in E$, whereas relation (17) implies that $\langle 0, \pm\alpha_n \rangle \notin E$, $n \in \mathbf{N}$. In view of (13) this means that E cannot contain any vertical segment centered at the origin. Consequently $\langle 0, 0 \rangle \notin \operatorname{core} E$, which shows that $E \neq \operatorname{core} E$ and thus the set E is not algebraically open.

3. Let X and Y be real linear spaces and consider the product $X \times Y$ as a real linear space with algebraic operations defined in the usual manner (coordinatewise). We consider X, Y and $X \times Y$ as topological spaces endowed with the core topology $\mathscr{T}(X)$, $\mathscr{T}(Y)$ and $\mathscr{T}(X \times Y)$, respectively.

It was shown in [6] that $\mathscr{T}(X \times Y)$ is not the product (Tychonov) topology determined by $\mathscr{T}(X)$ and $\mathscr{T}(Y)$. Thus simple theorems connecting the topological properties of sets $A \subset X$ and $B \subset Y$ with those of $A \times B$ *a priori* need not be valid in the present situation. Therefore it may come as a surprise that nevertheless a number of such results remain true also for the core topologies as may be seen from the theorem below.

In the sequel points $a \in X \times Y$ are represented as $a = \langle a_x, a_y \rangle$ with $a_x \in X$ and $a_y \in Y$. The functions (projections) $\pi_x : X \times Y \to X$ and $\pi_y : X \times Y \to Y$ are defined by $\pi_x(a) = a_x$ and $\pi_y(a) = a_y$. For every set $E \subset X \times Y$ and points $a_x \in X$, $a_y \in Y$ the sets (sections) $E_x[a_x] \subset Y$ and $E^y[a_y] \subset X$ are defined by

(19) $$E_x[a_x] := \{\tilde{a}_y \in Y \mid \langle a_x, \tilde{a}_y \rangle \in E\}, \quad E^y[a_y] := \{\tilde{a}_x \in X \mid \langle \tilde{a}_x, a_y \rangle \in E\}.$$

THEOREM 1. (i) *If $A \in \mathscr{T}(X)$ and $B \in \mathscr{T}(Y)$, then $A \times B \in \mathscr{T}(X \times Y)$. (The product of algebraically open sets is algebraically open).*

(ii) *If $X \backslash A \in \mathscr{T}(X)$ and $Y \backslash B \in \mathscr{T}(Y)$, then $(X \times Y) \backslash (A \times B) \in \mathscr{T}(X \times Y)$. (The product of algebraically closed sets is algebraically closed).*

(iii) *If $E \in \mathscr{T}(X \times Y)$, then $\pi_x(E) \in \mathscr{T}(X)$ and $\pi_y(E) \in \mathscr{T}(Y)$. (The projections of an algebraically open set are algebraically open).*

(iv) *If $E \in \mathscr{T}(X \times Y)$, then $E_x[a_x] \in \mathscr{T}(Y)$ and $E^y[a_y] \in \mathscr{T}(X)$ for arbitrary points $a_x \in X$ and $a_y \in Y$. (Sections of an algebraically open set are algebraically open).*

(v) $\operatorname{int}(A\times B) = (\operatorname{int}A)\times(\operatorname{int}B)$ *for arbitrary sets* $A\subset X$ *and* $B\subset Y$ (int *denotes the interior in the sense of the respective core topology*).

(vi) $\operatorname{cl}(A\times B) = (\operatorname{cl}A)\times(\operatorname{cl}B)$ *for arbitrary sets* $A\subset X$ *and* $B\subset Y$ (cl *denotes the closure in the sense of respective core topology*).

Proof. (i) Take arbitrary points $a = \langle a_x, a_y\rangle\in A\times B$ and $b = \langle b_x, b_y\rangle\in X\times Y$. Thus $a_x\in A$, $a_y\in B$, $b_x\in X$, $b_y\in Y$. Since the sets A and B are algebraically open there exist positive numbers ε_1 and ε_2 such that (cf. (1))

$$a_x+\lambda b_x\in A \quad \text{for} \quad \lambda\in(-\varepsilon_1, \varepsilon_1)$$

and

$$a_y+\lambda b_y\in B \quad \text{for} \quad \lambda\in(-\varepsilon_2, \varepsilon_2).$$

Hence

$$a+\lambda b = \langle a_x, a_y\rangle+\lambda\langle b_x, b_y\rangle = \langle a_x+\lambda b_x, a_y+\lambda b_y\rangle\in A\times B \quad \text{for} \quad \lambda\in(-\varepsilon, \varepsilon),$$

where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)>0$. Due to the unrestricted choice of $b\in X\times Y$ this means that the point a is algebraically interior to $A\times B$. Because of the arbitrariness of $a\in A\times B$ we get hence $A\times B = \operatorname{core}(A\times B)$, that is, $A\times B\in \mathscr{T}(X\times Y)$.

(ii) results from (i) in view of the relation

$$(X\times Y)\backslash(A\times B) = [(X\backslash A)\times(Y\backslash B)]\cup[X\times(Y\backslash B)]\cup[(X\backslash A)\times Y].$$

(iii) Write $A = \pi_x(E)$ and take arbitrary points $a_x\in A$, $b_x\in X$ and $b_y\in Y$. By the definition of A there exists a point $a_y\in Y$ such that $\langle a_x, a_y\rangle\in E$. Further, since E is algebraically open, there exists an $\varepsilon>0$ such that

$$\langle a_x+\lambda b_x, a_y+\lambda b_y\rangle = \langle a_x, a_y\rangle+\lambda\langle b_x, b_y\rangle\in E \quad \text{for} \quad \lambda\in(-\varepsilon, \varepsilon).$$

Hence

$$a_x+\lambda b_x\in\pi_x(E) = A \quad \text{for} \quad \lambda\in(-\varepsilon, \varepsilon),$$

which implies, in view of the arbitrariness of $a_x\in A$ and $b_x\in X$, that the set $A = \pi_x(E)$ is algebraically open. The proof for the set $\pi_y(E)$ is similar.

(iv) Take an $\tilde{a}_y\in E_x[a_x]$ and a $b_y\in Y$. We have $\langle a_x, \tilde{a}_y\rangle\in E$ and $\langle 0_x, b_y\rangle\in X\times Y$, where 0_x denotes the zero in X. Since E is algebraically open, there exists an $\varepsilon>0$ such that

$$\langle a_x, \tilde{a}_y+\lambda b_y\rangle = \langle a_x, \tilde{a}_y\rangle+\lambda\langle 0_x, b_y\rangle\in E \quad \text{for} \quad \lambda\in(-\varepsilon, \varepsilon).$$

In other words

$$\tilde{a}_y+\lambda b_y\in E_x[a_x] \quad \text{for} \quad \lambda\in(-\varepsilon, \varepsilon).$$

Due to the arbitrariness first of $b_y\in Y$ and then of $\tilde{a}_y\in E_x[a_x]$ this means that the set $E_x[a_x]$ is algebraically open. The proof for the sets E^y is similar.

(v) We have $(\operatorname{int}A)\times(\operatorname{int}B)\subset A\times B$, and since by virtue of (i) the set $(\operatorname{int} A)\times(\operatorname{int}B)$ is algebraically open this implies that actually

$$(20) \qquad (\mathrm{int}A)\times(\mathrm{int}B)\subset \mathrm{int}(A\times B).$$

On the other hand, the inclusion $\mathrm{int}(A\times B)\subset A\times B$ implies the relations

$$\pi_x(\mathrm{int}(A\times B))\subset\pi_x(A\times B) = A,\ \pi_y(\mathrm{int}(A\times B))\subset\pi_y(A\times B) = B,$$

which in turn imply, by virtue of (iii),

$$(21) \qquad \pi_x(\mathrm{int}(A\times B))\subset\mathrm{int}A, \quad \pi_y(\mathrm{int}(A\times B))\subset\mathrm{int}B.$$

Since

$$\mathrm{int}(A\times B)\subset\pi_x(\mathrm{int}(A\times B))\times\pi_y(\mathrm{int}(A\times B)),$$

relation (21) yields

$$(22) \qquad \mathrm{int}(A\times B)\subset(\mathrm{int}A)\times(\mathrm{int}B).$$

Assertion (v) is an immediate consequence of (20) and (21).

(vi) Obviously $A\times B\subset(\mathrm{cl}A)\times(\mathrm{cl}B)$, whence by virtue of (ii) we obtain

$$(23) \qquad \mathrm{cl}(A\times B)\subset(\mathrm{cl}A)\times(\mathrm{cl}B).$$

In order to prove the converse inclusion take an arbitrary point $a = \langle a_x, a_y\rangle\in(\mathrm{cl}A)\times(\mathrm{cl}B)$ so that

$$(24) \qquad a_x\in\mathrm{cl}A, \quad a_y\in\mathrm{cl}B$$

and let $E\subset X\times Y$ be an arbitrary algebraically open neighbourhood of a:

$$(25) \qquad a = \langle a_x, a_y\rangle\in E\in\mathscr{T}(X\times Y).$$

By virtue of (iv) the set $E^y[a_y]$ is an algebraically open neighbourhood of a_x (cf. (19)) so that by (24) $A\cap E^y[a_y] \neq \varnothing$. Consequently there exists an $\tilde{a}_x\in A\cap E^y[a_y]$. In particular, in view of (19), we have $\langle\tilde{a}_x, a_y\rangle\in E$. Thus, again by (iv) and (24), we have $B\cap E_x[\tilde{a}_x] \neq \varnothing$ and consequently there exists an $\tilde{a}_y\in B\cap E_x[\tilde{a}_x]$. We have according to (19) $\langle\tilde{a}_x, \tilde{a}_y\rangle\in E$ and clearly $\langle\tilde{a}_x, \tilde{a}_y\rangle\in A\times B$. This means that

$$(26) \qquad (A\times B)\cap E \neq \varnothing,$$

and (26) holds for every set $E\subset X\times Y$ fulfilling (25). This implies that $a\in\mathrm{cl}(A\times B)$, whence it follows, due to the arbitrariness of $a\in(\mathrm{cl}A)\times(\mathrm{cl}B)$, that

$$(\mathrm{cl}\,A)\times(\mathrm{cl}\,B)\subset\mathrm{cl}(A\times B),$$

which together with (23) yields (vi).

This completes the proof of the theorem.

ACKNOWLEDGEMENT. I owe the above proof of property (vi) to Dr. A. Kucia. I use this opportunity to thank her for her consent to include this proof into the present paper.

THEOREM 2. *Assume that* $\dim Y = 1$ *and let* $E\subset X\times Y$ *be a set of the first category in the topological space* $(X\times Y, \mathscr{T}(X\times Y))$. *Then the set*

$$(27) \qquad \{a_x\in X \mid E_x[a_x] \text{ is of the second category in } (Y, \mathscr{T}(Y))\}$$

is of the first category in the topological space $(X, \mathscr{T}(X))$.

The proof of the above theorem does not differ from the proof of the analogous result in the case, where X and Y are arbitrary topological spaces and $X \times Y$ is endowed with the product (Tychonov) topology (cf., e.g., [5; pp. 29—30] or [7; p. 222]). The assumption that $\dim Y = 1$ replaces the assumption that the topological space $(Y, \mathscr{T}(Y))$ has a countable neighbourhood base appearing in the theorem referred to, because as A. Kucia has observed the topological space $(Y, \mathscr{T}(Y))$ has a countable neighbourhood base if and only if $\dim Y = 1$.

We derive yet from Theorem 2 a known theorem ([4] contains a more general result). Recall that a topological space is called a *Baire space* whenever the Baire category theorem (to the effect that every set of the first category is a frontier set) is true in this space.

THEOREM 3. *For every positive integer n the topological space* $(\mathbf{R}^n, \mathscr{T}(\mathbf{R}^n))$ *is a Baire space.*

Proof. As in [4] the proof runs by induction on n. For $n = 1$ the theorem is true by virtue of the Baire category theorem, since $\mathscr{T}(\mathbf{R})$ coincides with the natural topology of the real line (cf. [1], [2]). Now suppose that for an $n \in \mathbf{N}$ the topological space $(\mathbf{R}^n, \mathscr{T}(\mathbf{R}^n))$ is a Baire space, but $(\mathbf{R}^{n+1}, \mathscr{T}(\mathbf{R}^{n+1}))$ is not. We represent $\mathbf{R}^{n+1}$ as $X \times Y$, where $X = \mathbf{R}^n$ and $Y = \mathbf{R}$. Our supposition about $\mathbf{R}^{n+1}$ implies that there exists a non-empty algebraically open set $E \subset X \times Y$ of the first category in the topological space $(X \times Y, \mathscr{T}(X \times Y))$. By (iv) of Theorem 1 for every $a_x \in \pi_x(E)$ the set $E_x[a_x] \subset Y$ is non-empty and (algebraically) open and hence, by virtue of the Baire theorem, is of the second category in the topological space $(Y, \mathscr{T}(Y))$. Thus $\pi_x(E)$ is contained in set (27), whence it follows by Theorem 2 that the set $\pi_x(E)$ is of the first category in the topological space $(X, \mathscr{T}(X))$. Evidently it is also non-empty and by Theorem 1(iii) it is algebraically open. But this is incompatible with the condition that $(X, \mathscr{T}(X)) = (\mathbf{R}^n, \mathscr{T}(\mathbf{R}^n))$ is a Baire space. Consequently our supposition must have been false and together with $(\mathbf{R}^n, \mathscr{T}(\mathbf{R}^n))$ also $(\mathbf{R}^{n+1}, \mathscr{T}(\mathbf{R}^{n+1}))$ is a Baire space, $n \in \mathbf{N}$. Induction ends the proof.

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