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**Author:** Janusz Matkowski

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# ITERATIONS OF MEAN-TYPE MAPPINGS AND INVARIANT MEANS

JANUSZ MATKOWSKI

*To the memory of Professor György Targonski*

**Abstract.** It is shown that, under some general conditions, the sequence of iterates of every mean-type mapping on a finite dimensional cube converges to a unique invariant mean-type mapping. Some properties of the invariant means and their applications are presented.

## Introduction

The sequence of iterates of a selfmap of a metric space often appears in fixed point theory and, in general, the assumed conditions imply its convergence to a constant map the value of which is a fixed point. In this context the questions whether there are nontrivial selfmaps with non-constant limits of the sequences of iterates, and what are the properties of their limits, seem to be interesting.

To give an answer, in section 2, we consider a class of *mean-type* self-mappings  $M$  of a finite dimensional cube  $I^p$ , where  $I \subseteq \mathbb{R}$  is an interval and  $p \geq 2$  a fixed integer, showing that (under some general assumptions) the sequence of iterates  $(M^n)_{n=1}^{\infty}$  converges to a unique non-constant mapping  $K$  which is an invariant mean-type with respect to  $M$  (shortly  $M$ -invariant). Since the coordinate functions of  $M$  are means, every point of the diagonal of  $I^p$  is a fixed point of  $M$ . In section 3 we apply these results to determine the limits of the sequence of iterates for some special

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classes of mean-type mappings. In section 4 we present some examples of nonexpansive mean-type mappings and we show that the mean-type mapping  $M = (A, G)$  (for which the sequence of iterates converges) is neither nonexpansive nor expansive.

The subject considered here is related to the papers by J. Borwein [2], and P. Flor, F. Halter-Koch [4] where a problem concerning some recurrence sequences, posed by J. Aczél [1], was considered.

## 1. Means and auxiliary results

Let  $I \subset \mathbb{R}$  be an interval, and  $p \in \mathbb{N}$ ,  $p \geq 2$  fixed. A function  $M : I^p \rightarrow \mathbb{R}$  is said to be a *mean* on  $I^p$  if for all  $x = (x_1, \dots, x_p) \in I^p$ ,

$$\min(x_1, \dots, x_p) \leq M(x_1, \dots, x_p) \leq \max(x_1, \dots, x_p);$$

in particular,  $M : I^p \rightarrow I$ , and, for all  $x \in I$ ,

$$M(x, \dots, x) = x.$$

A mean  $M$  on  $I^p$  is called *strict* if whenever  $x = (x_1, \dots, x_p) \in I^p$  such that  $x_i \neq x_j$  for some  $i, j \in 1, \dots, p$ , then

$$\min(x_1, \dots, x_p) < M(x_1, \dots, x_p) < \max(x_1, \dots, x_p);$$

in particular we have the following

REMARK 1. Let  $M : I^p \rightarrow I$  be a strict mean and let  $(x_1, \dots, x_p) \in I^p$ . If

$$M(x_1, \dots, x_p) = \min(x_1, \dots, x_p) \quad \text{or} \quad M(x_1, \dots, x_p) = \max(x_1, \dots, x_p)$$

then  $x_1 = \dots = x_p$ .

LEMMA 1. Let  $p \in \mathbb{N}$ ,  $p \geq 2$ , be fixed. Suppose that  $M_i : I^p \rightarrow \mathbb{R}$ ,  $i = 1, \dots, p$ , are continuous means on  $I^p$  such that at most one of them is not strict. Let the functions  $M_{i,n} : I^p \rightarrow I$ ,  $i = 1, \dots, p$ ,  $n \in \mathbb{N}$ , be defined by

$$(1) \quad M_{i,1} := M_i, \quad i = 1, \dots, p,$$

$$(2) \quad M_{i,n+1}(x_1, \dots, x_p) := M_i(M_{1,n}(x_1, \dots, x_p), \dots, M_{p,n}(x_1, \dots, x_p)).$$

Then

1<sup>0</sup> for every  $n \in \mathbb{N}$  and for each  $i = 1, \dots, p$ , the function  $M_{i,n}$  is a continuous mean on  $I^p$ ;

2<sup>0</sup> there is a continuous mean  $K : I^p \rightarrow I$  such that for each  $i = 1, \dots, p$ ,

$$\lim_{n \rightarrow \infty} M_{i,n}(x_1, \dots, x_p) = K(x_1, \dots, x_p), \quad x_1, \dots, x_p \in I;$$

3<sup>0</sup> if  $M_1, \dots, M_p$  are strict means, then so is  $K$ .

PROOF. Part 1<sup>0</sup> is obvious. To prove 2<sup>0</sup> assume that, for instance,  $M_p$  is strict, and define  $\alpha_n, \beta_n : I^p \rightarrow I$ ,  $n \in \mathbb{N}$ , by

$$\alpha_n := \min(M_{1,n}, \dots, M_{p,n}), \quad \beta_n := \max(M_{1,n}, \dots, M_{p,n}).$$

The functions  $\alpha_n, \beta_n$  are continuous means. Since  $M_1, \dots, M_p$  are means we have

$$\alpha_n \leq M_{i,n+1} \leq \beta_n, \quad i = 1, \dots, p; \quad n \in \mathbb{N},$$

and, consequently,

$$\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n, \quad n \in \mathbb{N}.$$

Now we show the following

CLAIM. For every  $x_1, \dots, x_p \in I$ , either

(a) there is some  $k \in \mathbb{R}$  such that

$$\alpha_n(x_1, \dots, x_p) = \beta_n(x_1, \dots, x_p), \quad n \in \mathbb{N}, \quad n \geq k;$$

or

(b) for all  $n \in \mathbb{R}$ ,

$$\alpha_n(x_1, \dots, x_p) < \alpha_{n+1}(x_1, \dots, x_p) \text{ or } \beta_{n+1}(x_1, \dots, x_p) < \beta_n(x_1, \dots, x_p)$$

This claim is obvious if  $x_1 = \dots = x_p$ . Take arbitrary  $x_1, \dots, x_p \in I$  such that  $x_i \neq x_j$  for some  $i, j \in 1, \dots, p$ . Suppose, for an indirect argument, that the statement (b) does not hold, i.e. that there is a  $k \in \mathbb{N}$  such that

$$\alpha_k(x_1, \dots, x_p) = \alpha_{k+1}(x_1, \dots, x_p) < \beta_{k+1}(x_1, \dots, x_p) = \beta_k(x_1, \dots, x_p).$$

By the definition of  $\alpha_k$  and  $\beta_k$  we hence get

$$\begin{aligned} \min(M_{1,k}, \dots, M_{p,k}) &= \min(M_{1,k+1}, \dots, M_{p,k+1}) \\ &< \max(M_{1,k+1}, \dots, M_{p,k+1}) = \max(M_{1,k}, \dots, M_{p,k}), \end{aligned}$$

and, consequently, there are  $i, j, r, s \in \{1, \dots, p\}$ ,  $i \neq r$ ,  $j \neq s$ , such that

$$\begin{aligned} M_{i,k} &= \min (M_{1,k}, \dots, M_{p,k}) = \min (M_{1,k+1}, \dots, M_{p,k+1}) = M_{j,k+1} \\ &< M_{r,k} = \max (M_{1,k}, \dots, M_{p,k}) = \max (M_{1,k+1}, \dots, M_{p,k+1}) = M_{s,k+1}, \end{aligned}$$

(where the values of the occurring functions are taken at the chosen point  $(x_1, \dots, x_p)$ ). Hence, since

$$M_{j,k+1}(x_1, \dots, x_p) = M_j(M_{1,k}(x_1, \dots, x_p), \dots, M_{p,k}(x_1, \dots, x_p)),$$

$$M_{s,k+1}(x_1, \dots, x_p) := M_s(M_{1,k}(x_1, \dots, x_p), \dots, M_{p,k}(x_1, \dots, x_p)),$$

and at least one of the means  $M_j$  and  $M_s$  is strict, applying Remark 1, we infer that

$$M_{1,k}(x_1, \dots, x_p) = \dots = M_{p,k}(x_1, \dots, x_p).$$

Hence, by the definition of  $M_{i,n+1}$ ,  $i = 1, \dots, p$ , and the fact that the restriction of every mean on  $I^p$  to the diagonal of  $I^p$  is the identity function on  $I$ , we obtain

$$M_{i,n}(x_1, \dots, x_p) = M_{j,k}(x_1, \dots, x_p), \quad n \geq k, \quad i, j \in \{1, \dots, p\}.$$

Now the definitions of  $\alpha_n$  and  $\beta_n$  give

$$\alpha_n(x_1, \dots, x_p) = \beta_n(x_1, \dots, x_p), \quad n \in \mathbb{N}, \quad n \geq k,$$

showing that relation (a) is true. This completes the proof of our claim.

Since the sequences  $(\alpha_n)$  and  $(\beta_n)$  are monotonic and bounded, there exist  $\alpha, \beta : I^p \rightarrow I$  defined by

$$\alpha := \lim_{n \rightarrow \infty} \alpha_n, \quad \beta := \lim_{n \rightarrow \infty} \beta_n.$$

We shall show that  $\alpha = \beta$ . For an indirect argument suppose that there exist  $x_1, \dots, x_p \in I$  such that

$$\alpha(x_1, \dots, x_p) < \beta(x_1, \dots, x_p).$$

We can assume, without any loss of generality, that, for each  $j \in \{2, \dots, p\}$ ,  $M_j$  is a strict mean. Then for every  $j \in \{2, \dots, p\}$  we have

$$\alpha(x_1, \dots, x_p) < M_j(\gamma_1(x_1, \dots, x_p), \dots, \gamma_p(x_1, \dots, x_p)) < \beta(x_1, \dots, x_p),$$

where

$$\gamma_i(x_1, \dots, x_p) = \alpha(x_1, \dots, x_p) \quad \text{or} \quad \gamma_i(x_1, \dots, x_p) = \beta(x_1, \dots, x_p)$$

and  $\gamma_r(x_1, \dots, x_p) \neq \gamma_s(x_1, \dots, x_p)$  for some  $r, s \in \{1, \dots, p\}$ . Take arbitrary positive  $\delta > 0$ . Then there is  $n(\delta)$  such that for all  $n \geq n(\delta)$ ,

$$\alpha(x_1, \dots, x_p) - \delta < M_{i,n}(x_1, \dots, x_p) < \beta(x_1, \dots, x_p) + \delta, \quad i = 1, \dots, p,$$

Hence, choosing  $\delta$  small enough, by the continuity of  $M_j$ , we infer that

$$\alpha(x_1, \dots, x_p) < M_{j,n+1}(x_1, \dots, x_p) < \beta(x_1, \dots, x_p),$$

$$j = 2, \dots, p, \quad n \geq n(\delta).$$

It follows that for every  $n > n(\delta)$  either

$$\alpha(x_1, \dots, x_p) < \alpha_n(x_1, \dots, x_p) < \beta(x_1, \dots, x_p)$$

or

$$\alpha(x_1, \dots, x_p) < \beta_n(x_1, \dots, x_p) < \beta(x_1, \dots, x_p),$$

which contradicts the definition of  $\alpha$  and  $\beta$ . Thus we have shown that

$$\alpha = \beta \quad \text{in } I^p.$$

Since  $\alpha_n, \beta_n$  are continuous,  $(\alpha_n)$  is increasing and  $(\beta_n)$  is decreasing, the function  $\alpha$  is lower semicontinuous, and  $\beta$  is upper semicontinuous on  $I^p$ . It follows that the function  $K : I^p \rightarrow I$  defined by

$$K(x_1, \dots, x_p) := \alpha(x_1, \dots, x_p), \quad x_1, \dots, x_p \in I,$$

is continuous on  $I^p$ . It is obvious that  $K$  is a mean on  $I^p$ . □

LEMMA 2. *Let  $p \in \mathbb{N}$ ,  $p \geq 2$ , be fixed. Suppose that  $M_i : I^p \rightarrow \mathbb{R}$ ,  $i = 1, \dots, p$ , are continuous means on  $I^p$  such that for some  $j \in \{1, \dots, p\}$ ,  $M_j$  is strict and either*

$$(3) \quad M_i \leq M_j, \quad i = 1, \dots, p,$$

or

$$(4) \quad M_j \leq M_i, \quad i = 1, \dots, p.$$

Then the functions  $M_{i,n} : I^p \rightarrow I$ ,  $i = 1, \dots, p$ ,  $n \in \mathbb{N}$ , defined by (1)–(2) in Lemma 1 satisfy the conclusions 1<sup>0</sup>–3<sup>0</sup> of Lemma 1.

PROOF. Assume that condition (3) is satisfied. Without any loss of generality we can assume that  $j = p$ , i.e. that

$$M_i \leq M_p, \quad i = 1, \dots, p.$$

Part 1<sup>0</sup> is obvious. To prove 2<sup>0</sup> define  $\alpha_n$ ,  $\beta_n$ ,  $\alpha$  and  $\beta$ , in the same way as in the proof of Lemma 1. Of course we have

$$\beta_n = M_{p,n}, \quad \beta_{n+1} \leq \beta_n, \quad (n \in \mathbb{N}), \quad \beta = \lim_{n \rightarrow \infty} M_{p,n}.$$

$$\alpha_n = \min (M_{1,n}, \dots, M_{p-1,n}), \quad \alpha_n \leq \alpha_{n+1}, \quad (n \in \mathbb{N}),$$

$$\alpha = \lim_{n \rightarrow \infty} \alpha_n, \quad \alpha \leq \beta.$$

Suppose that there is a point  $(x_1, \dots, x_p) \in I^p$  such that

$$\alpha(x_1, \dots, x_p) < \beta(x_1, \dots, x_p).$$

Since  $M_p$  is a strict mean we hence get

$$\alpha(x_1, \dots, x_p) < M_p(\alpha(x_1, \dots, x_p), \dots, \alpha(x_1, \dots, x_p), \beta(x_1, \dots, x_p))$$

$$< \beta(x_1, \dots, x_p).$$

Now the continuity of  $M_p$  implies that, for sufficiently large  $n$ ,

$$\alpha(x_1, \dots, x_p) < M_{p,n}(x_1, \dots, x_p) < \beta(x_1, \dots, x_p).$$

This contradiction proves that  $\alpha = \beta$ . The remaining argument is similar to that of Lemma 1.

Since in the case when condition (4) is satisfied the reasoning is analogous, the proof is completed.  $\square$

## 2. The main results

Let  $I \subseteq \mathbb{R}$  be an interval and let  $p \in \mathbb{N}$ ,  $p \geq 2$ , be fixed. A function  $M: I^p \rightarrow \mathbb{R}^p$ ,  $M = (M_1, \dots, M_p)$ , is called a *mean-type mapping* if each coordinate function  $M_i$ ,  $i = 1, \dots, p$ , is a mean on  $I^p$ ; in particular,  $M: I^p \rightarrow I^p$ . A mean type mapping  $M = (M_1, \dots, M_p)$  is *strict* if each of its coordinate functions  $M_i$  is a strict mean.

REMARK 2. Note that the restriction of an arbitrary mean-type mapping  $M: I^p \rightarrow I^p$  to the diagonal of the cube  $I^p$  coincides with the identity function i.e., for every  $x \in I$ ,

$$M(x, \dots, x) = (x, \dots, x).$$

It follows that for any function  $\mathbf{K}: I^p \rightarrow I^p$ ,  $\mathbf{K} = (K_1, \dots, K_p)$ , with equal coordinates, i.e. such that  $K_1 = \dots = K_p = K$ , we have

$$\mathbf{M} \circ \mathbf{K} = \mathbf{K}.$$

The first result on the convergence of the sequences of iterates of the mean-type mappings reads as follows.

**THEOREM 1.** *Let an interval  $I \subseteq \mathbb{R}$  and  $p \in \mathbb{N}$ ,  $p \geq 2$ , be fixed. If  $\mathbf{M}: I^p \rightarrow \mathbb{R}^p$ ,  $\mathbf{M} = (M_1, \dots, M_p)$ , is a continuous mean-type mapping such that at most one of the coordinate means  $M_i$  is not strict, then:*

- 1<sup>0</sup> *for every  $n \in \mathbb{N}$ , the  $n$ -th iterate of  $\mathbf{M}$  is a mean-type mapping;*
- 2<sup>0</sup> *there is a continuous mean  $K: I^p \rightarrow I$  such that the sequence of iterates  $(\mathbf{M}^n)_{n=1}^\infty$  converges (pointwise) to a continuous mean-type mapping  $\mathbf{K}: I^p \rightarrow I^p$ ,  $\mathbf{K} = (K_1, \dots, K_p)$ , such that*

$$K_1 = \dots = K_p = K;$$

- 3<sup>0</sup>  *$\mathbf{K}$  is an  $\mathbf{M}$ -invariant mean-type mapping i.e.,*

$$\mathbf{K} \circ \mathbf{M} = \mathbf{K},$$

or, equivalently, the mean  $K$  is  $\mathbf{M}$ -invariant i.e., for all  $x_1, \dots, x_p \in I$ ,

$$K(M_1(x_1, \dots, x_p), \dots, M_p(x_1, \dots, x_p)) = K(x_1, \dots, x_p);$$

- 4<sup>0</sup> *a continuous  $\mathbf{M}$ -invariant mean-type mapping is unique;*
- 5<sup>0</sup> *if  $\mathbf{M}$  is a strict mean-type mapping then so is  $\mathbf{K}$ ;*
- 6<sup>0</sup> *if  $I = (0, \infty)$  and  $\mathbf{M}$  is positively homogeneous, then  $\mathbf{K}$  is positively homogeneous.*

**PROOF.** Define  $M_{i,n}: I^p \rightarrow I$ ,  $i = 1, \dots, p$ ,  $n \in \mathbb{N}$ , by formulas (1)-(2). By induction it is easy to verify that

$$\mathbf{M}^n = (M_{1,n}, \dots, M_{p,n}), \quad n \in \mathbb{N}.$$

Now, applying Lemma 1.1<sup>0</sup> - 2<sup>0</sup>, we get the conclusions 1<sup>0</sup> and 2<sup>0</sup>. Thus, for all  $(x_1, \dots, x_p) \in I^p$ , we have

$$\mathbf{K}(x_1, \dots, x_p) = \lim_{n \rightarrow \infty} \mathbf{M}^n(x_1, \dots, x_p).$$

Hence, making use of (2) and the continuity of  $\mathbf{K}$ , we get

$$\mathbf{K} = \lim_{n \rightarrow \infty} \mathbf{M}^{n+1} = \mathbf{M}(\lim_{n \rightarrow \infty} \mathbf{M}^n) = \mathbf{M} \circ \mathbf{K}.$$



Since  $\mathbf{K} = (K_1, \dots, K_p)$  where  $K_1 = \dots = K_p = K$ , this relation is equivalent to

$$K(M_1(x_1, \dots, x_p), \dots, M_p(x_1, \dots, x_p)) = K(x_1, \dots, x_p),$$

for all  $(x_1, \dots, x_p) \in I^p$ , and the proof of  $3^0$  is completed.

To prove  $4^0$  take an arbitrary continuous mean-type mapping  $L: I^p \rightarrow I^p$  that is  $M$ -invariant. Thus we have  $L = L \circ M$ , and, by an obvious induction,

$$L = L \circ M^n, \quad n \in \mathbb{N}.$$

Hence, letting  $n \rightarrow \infty$ , making use of  $2^0$  and the continuity of  $L$  gives

$$L = \lim_{n \rightarrow \infty} L \circ M^n = L \circ \left( \lim_{n \rightarrow \infty} M^n \right) = L \circ K.$$

Since  $\mathbf{K} = (K, \dots, K)$ , in view of Remark 2, we hence get  $L = K$  which proves the desired uniqueness of the  $M$ -invariant mean.

Part  $5^0$  is an immediate consequence of Lemma 1.3<sup>0</sup>. Since part  $6^0$  is obvious, the proof is completed.

**REMARK 3.** The assumption of Theorem 1 that at most one of the means  $M_1, \dots, M_p$  is not strict is essential. To show this consider the following

**EXAMPLE 1.** Take  $p = 3$  and define  $L, M, N: \mathbb{R}^3 \rightarrow \mathbb{R}$  by

$$L(x, y, z) := \min(x, y, z), \quad M(x, y, z) := \frac{x + y + z}{3}, \quad N(x, y, z) := \max(x, y, z).$$

Then  $\mu := (L + N)/2$  is a mean and for all  $x, y, z \in \mathbb{R}$ ,

$$L(x, y, z) = L(L(x, y, z), \mu(x, y, z), N(x, y, z))$$

$$\mu(x, y, z) = M(L(x, y, z), \mu(x, y, z), N(x, y, z))$$

$$N(x, y, z) = N(L(x, y, z), \mu(x, y, z), N(x, y, z)).$$

Thus, setting  $\mathbf{M} := (L, M, N)$  and  $\mathbf{K} := (L, \mu, N)$ , we have,  $\mathbf{K} = \mathbf{M} \circ \mathbf{K}$ , i.e.  $\mathbf{K}$  is an  $\mathbf{M}$ -invariant mean-type mapping. However the coordinate means of  $\mathbf{K}$  are not equal.

In Theorem 1 we assume that only one of the means  $M_1, \dots, M_p$  is not strict. The next result shows that, under some additional conditions, this assumption can be essentially relaxed.

**THEOREM 2.** *Let  $p \in \mathbb{N}$ ,  $p \geq 2$ , be fixed. Suppose that  $M: I^p \rightarrow I^p$ ,  $M = (M_1, \dots, M_p)$ , is a continuous mean-type mapping. Let  $(M^n)_{n=0}^\infty$  be the sequence of iterations of  $M$ . If there is an  $j \in 1, \dots, p$  such that  $M_j$  is strict and either*

$$(5) \quad M_i \leq M_j, \quad i = 1, \dots, p,$$

or

$$(6) \quad M_j \leq M_i, \quad i = 1, \dots, p,$$

then

1<sup>0</sup> for every  $n \in \mathbb{N}$ , the iterate  $M^n$  is a mean type mapping on  $I^p$ ;

2<sup>0</sup> the sequence  $(M^n)_{n=1}^\infty$  converges (pointwise) to a mean type mapping  $K: I^p \rightarrow I^p$ ,  $K = (K_1, \dots, K_p)$ , such that

$$K_1 = \dots = K_p;$$

3<sup>0</sup>  $K$  is  $M$ -invariant i.e.,

$$K \circ M = K;$$

4<sup>0</sup> a continuous  $M$ -invariant mean-type mapping is unique,

5<sup>0</sup> if  $M$  is a strict mean-type mapping then so is  $K$ ;

6<sup>0</sup> if  $I = (0, \infty)$  and  $M$  is positively homogeneous, then  $K$  is positively homogeneous.

**PROOF.** It is enough to apply Lemma 2 and argue along the same line as in the case of Theorem 1.

**REMARK 4.** Example 1 shows that the existence of a strict coordinate mean of a mean-type mapping  $M$  such that either condition (5) or (6) is satisfied is an essential assumption of Theorem 2.

### 3. Invariant means and applications of main results

According to Theorem 1 and Theorem 2, the problem to determine the limit of the sequence of iterates of a mean-type mapping  $M$  reduces to finding an  $M$ -invariant mean-type mapping (or an  $M$ -invariant mean). To show that this fact can be helpful in determining the limit of the sequence  $(M^n)$  we begin this section by presenting the following

EXAMPLE 2. Take  $I = (0, \infty)$  and  $p = 2$ . Let  $M: I^2 \rightarrow I^2$  be defined by  $M = (A, H)$ , where  $A$  and  $H$  are respectively the arithmetic and harmonic means:

$$A(x, y) = \frac{x + y}{2}, \quad H(x, y) = \frac{2xy}{x + y}, \quad x, y \in I.$$

By Theorem 1 there exists a unique mean-type mapping  $K: I^2 \rightarrow I^2$  which is invariant with respect to  $M$ . Let  $G$  be the geometric mean,  $G(x, y) = (xy)^{1/2}$ ,  $(x, y \in I)$ . Since (cf. P. Kahlig, J. Matkowski [5])

$$G(A(x, y), H(x, y)) = \left( \frac{x + y}{2} \frac{2xy}{x + y} \right)^{1/2} = G(x, y), \quad x, y > 0,$$

$G$  is an  $M$ -invariant mean and, by the uniqueness of the invariant mean, we have  $K = (G, G)$ . Moreover,

$$\lim_{n \rightarrow \infty} M^n(x, y) = \lim_{n \rightarrow \infty} \left( \frac{x + y}{2}, \frac{2xy}{x + y} \right)^n = (\sqrt{xy}, \sqrt{xy}), \quad x, y > 0.$$

This example can be easily deduced from more general facts presented below as Propositions 1-3 in which we consider some special classes of means.

Given  $r \in \mathbb{R}, r \neq 0$ , the function  $M^{[r]}: (0, \infty)^2 \rightarrow (0, \infty)$ ,

$$M^{[r]}(x, y) := \left( \frac{x^r + y^r}{2} \right)^{1/r}, \quad x, y > 0,$$

is called the *power mean*.

Now we prove

PROPOSITION 1. Let  $r \in \mathbb{R}, r \neq 0$ , be fixed. Then

$$G(M^{[r]}(x, y), M^{[-r]}(x, y)) = G(x, y), \quad x, y > 0,$$

i.e., for all  $r \in \mathbb{R}$ , the geometric mean  $G$  is invariant with respect to the mean-type mapping  $M = (M^{[r]}, M^{[-r]})$ . Moreover,

$$\lim_{n \rightarrow \infty} M^n = (G, G).$$

PROOF. By simple calculation, we verify the invariance. The remaining part of the proposition follows from Theorem 1.  $\square$

For a fixed  $r \in \mathbb{R}$  define  $D^{[r]} : (0, \infty)^2 \rightarrow (0, \infty)$  by

$$D^{[r]}(x, y) := \begin{cases} \frac{x-y}{\log x - \log y}, & r = 0 \\ \frac{r}{r+1} \frac{x^{r+1} - y^{r+1}}{x^r - y^r}, & -1 \neq r \neq 0, \quad (x, y > 0). \\ xy \frac{\log x - \log y}{x-y}, & r = -1 \end{cases}$$

$D^{[r]}$  is called the difference quotient mean.

PROPOSITION 2. For all  $r \in \mathbb{R}$ ,

$$G(D^{[r]}(x, y), D^{[-r-1]}(x, y)) = G(x, y), \quad x, y > 0,$$

i.e., the geometric mean  $G$  is invariant with respect to the mean-type mapping  $M = (D^{[r]}, D^{[-r-1]})$ . Moreover,

$$\lim_{n \rightarrow \infty} M^n = (G, G).$$

(We omit an easy proof of Proposition 2, as well as Proposition 3, below).

For a fixed  $r \in \mathbb{R}$  the function  $G^{[r]} : (0, \infty)^2 \rightarrow (0, \infty)$  given by

$$G^{[r]}(x, y) := \frac{x^{r+1/2} + y^{r+1/2}}{x^{r-1/2} + y^{r-1/2}}, \quad x, y > 0,$$

is the Gini mean (Bullen–Mitrinović–Vasić [3], p. 189). Note that  $G^{[0]} = G$ .

PROPOSITION 3. For all  $r \in \mathbb{R}$ ,

$$G(G^{[r]}(x, y), G^{[-r]}(x, y)) = G(x, y), \quad x, y > 0,$$

i.e., the geometric mean  $G$  is invariant with respect to the mean-type mapping  $M = (G^{[r]}, G^{[-r]})$ . Moreover,

$$\lim_{n \rightarrow \infty} M^n = (G, G).$$

In connection with Propositions 1-3 let us note a general

REMARK 5. Let  $I \subset (0, \infty)$  be an interval. If  $M : I^2 \rightarrow I$  is a mean then  $N : I^2 \rightarrow \mathbb{R}$ , defined by

$$N(x, y) := \frac{xy}{M(x, y)}, \quad x, y \in I,$$

is a mean. Moreover, the geometric mean  $G$  is invariant with respect to the mean-type  $M := (M, N)$ , and  $\lim_{n \rightarrow \infty} M^n = (G, G)$ .

The next result (which is easy to verify) gives a broad class of mean-type mappings  $M: I^2 \rightarrow I^2$  for which the  $M$ -invariant means are quasi-arithmetic.

**PROPOSITION 4.** *Let  $\phi: I \rightarrow \mathbb{R}$  be continuous and strictly monotonic. Suppose that  $M: I^2 \rightarrow I$ , is a mean. Then the function  $N: I^2 \rightarrow I$  defined by*

$$N(x, y) := \phi^{-1}(\phi(x) + \phi(y) - \phi(M(x, y)))$$

*is a mean. Moreover, the quasi-arithmetic mean  $K: I^2 \rightarrow I$ , defined by*

$$K(x, y) := \phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right),$$

*is  $M$ -invariant for a mean-type mapping  $M = (M, N)$ .*

Example 2 and Propositions 1-4 were concerned with the case  $p = 2$ . If  $p \geq 3$  the situation is a little more complicated. However, the following counterpart of Proposition 4 is easily verified.

**PROPOSITION 5.** *Let  $p \geq 3$ ,  $p \in \mathbb{N}$ , and a continuous strictly increasing function  $\phi: I \rightarrow \mathbb{R}$  be fixed. Suppose that  $M_i: I^p \rightarrow I$ ,  $i = 1, \dots, p-1$ , are symmetric means which are increasing with respect to each variable. Then the function  $M_p: I^p \rightarrow I$  defined by*

$$M_p(x_1, \dots, x_p) := \phi^{-1}\left(\sum_{i=1}^p \phi(x_i) - \sum_{i=1}^{p-1} \phi(M_i(x_1, \dots, x_p))\right)$$

*is a mean if, and only if, the following two conditions are satisfied:*

(a) *for all  $x_2, \dots, x_p \in I$ ,*

$$x_2 < \dots < x_p \Rightarrow \sum_{i=1}^{p-1} \phi(M_i(x_2, x_2, x_3, \dots, x_p)) \leq \sum_{i=2}^p \phi(x_i);$$

(b) *for all  $x_1, \dots, x_{p-1} \in I$ ,*

$$x_1 < \dots < x_{p-1} \Rightarrow \sum_{i=1}^{p-1} \phi(x_i) \leq \sum_{i=1}^{p-1} \phi(M_i(x_1, \dots, x_{p-1}, x_{p-1})).$$

*Moreover, the quasi-arithmetic mean  $K: I^p \rightarrow I$ , defined by*

$$K(x_1, \dots, x_p) := \phi^{-1}\left(\frac{1}{p} \sum_{i=1}^p \phi(x_i)\right), \quad x_1, \dots, x_p \in I,$$

*is  $M$ -invariant for the mean-type mapping  $M = (M_1, \dots, M_p)$ .*

EXAMPLE 3. Taking  $p = 3, I = (0, \infty), \phi(x) = x^2 (x > 0), M_1 = A, M_2 = R$ , where  $A$  is the arithmetic mean and  $R$  is the square-root mean, i.e.

$$A(x, y, z) := \frac{x + y + z}{3}, \quad R(x, y, z) := \left( \frac{x^2 + y^2 + z^2}{3} \right)^{1/2},$$

it is easy to verify that the conditions (a)-(b) of Proposition 5 are fulfilled. Therefore  $M_3 = N$ ,

$$N(x, y, z) := \frac{1}{3} [3(x^2 + y^2 + z^2) + (x - y)^2 + (y - z)^2 + (z - x)^2]^{1/2},$$

is a mean and the mean-type mapping  $M: (0, \infty)^3 \rightarrow (0, \infty)^3, M = (A, R, N)$ , is  $K$ -invariant with  $K = R$ , i.e.

$$R(A(x, y, z), R(x, y, z), N(x, y, z)) = R(x, y, z), \quad x, y, z > 0.$$

Moreover, in view of Theorem 1 (or Theorem 2),

$$\lim_{n \rightarrow \infty} M^n = (R, R, R).$$

#### 4. Mean-type mappings and nonexpansivity. Examples

According to Remark 2, every mean-type mapping restricted to the diagonal is the identity map. The identity of  $I^p$  is an example of a mean-type mapping which, being an isometry, is of course nonexpansive. The following example is less trivial:

EXAMPLE 4. The map  $M: I^p \rightarrow I^p$ , defined by

$$M(x_1, x_2, \dots, x_p) := (x_1, x_1, x_2, \dots, x_{p-1}), \quad x_1, \dots, x_p \in I,$$

is, of course, a nonexpansive (with respect to the Euclidean norm) mean-type mapping, and we have

$$\lim_{n \rightarrow \infty} M^n(x_1, \dots, x_p) = M^{p-1}(x_1, \dots, x_p) = (x_1, x_1, \dots, x_1).$$

The next example shows that there are mean-type mappings which are neither nonexpansive nor expansive.

EXAMPLE 5. Take  $p = 2$  and  $I = (0, \infty)$ . Then the mean-type mapping  $M: (0, \infty)^2 \rightarrow (0, \infty)^2, M = (A, G)$ , where  $A$  and  $G$  are, respectively, the

arithmetic and geometric mean, is neither nonexpansive nor expansive in the sense of the Euclidean norm. In fact, for  $x, y \in (0, \infty)^2$  such that

$$x = (a, a + h), \quad y = (b, b + h), \quad a, b, h > 0, \quad a \neq b,$$

we have

$$\mathbf{M}(x) = \left( a + \frac{h}{2}, \sqrt{a(a+h)} \right), \quad \mathbf{M}(y) = \left( b + \frac{h}{2}, \sqrt{b(b+h)} \right),$$

$$\|x - y\|^2 = 2(a - b)^2,$$

$$\|\mathbf{M}(x) - \mathbf{M}(y)\|^2 = 2(a - b)^2 + 2ab + ah + bh - 2\sqrt{ab(a+h)(b+h)},$$

and, since

$$2\sqrt{ab(a+h)(b+h)} < 2ab + ah + bh, \quad a, b, h > 0,$$

(which can be easily verified by taking the second power of both sides) we infer that

$$\|\mathbf{M}(x) - \mathbf{M}(y)\| > \|x - y\|.$$

On the other hand, taking

$$x, y \in (0, \infty)^2, \quad x = (a, b), \quad y = (ta, tb), \quad a, b, t > 0, \quad t \neq 0,$$

we have

$$\mathbf{M}(x) = \left( \frac{a+b}{2}, \sqrt{ab} \right), \quad \mathbf{M}(y) = \left( t\frac{a+b}{2}, t\sqrt{ab} \right),$$

$$\|x - y\|^2 = (t-1)^2(a^2 + b^2), \quad \|\mathbf{M}(x) - \mathbf{M}(y)\|^2 = (t-1)^2\left[\left(\frac{a+b}{2}\right)^2 + ab\right],$$

and, clearly,

$$\|\mathbf{M}(x) - \mathbf{M}(y)\| < \|x - y\|.$$

Actually we have shown that  $\mathbf{M}$  is neither nonexpansive nor expansive in each of the sets  $\{x = (a, b) : a, b > 0, a < b\}$  and  $\{x = (a, b) : a, b > 0, a > b\}$ . Note that  $\mathbf{M}(a, b) = \mathbf{M}(b, a)$ .

### 5. Remark on iterative functional equations

In the theory of iterative functional equations (cf. M. Kuczma [6]) a very important role is played by the following

**FACT.** Let  $I \subseteq \mathbb{R}$  be an interval and  $a \in \mathbb{R}$  a point belonging to the closure of  $I$ . If  $f : I \rightarrow \mathbb{R}$  a continuous function such that

$$(7) \quad 0 < \frac{f(x) - a}{x - a} < 1, \quad x \in I \setminus \{a\},$$

then  $f : I \rightarrow I$ , and for every  $x \in I$ ,

$$\lim_{n \rightarrow \infty} f^n(x) = a.$$

Note that condition (7) can be written in the equivalent form

$$\min(x, a) < f(x) < \max(x, a), \quad x \in I \setminus \{a\}.$$

This observation leads immediately to the following finite-dimensional counterpart of the above fact (which is easily verified):

**REMARK 6.** Let  $p \in \mathbb{N}$  be fixed. Suppose that  $I \subseteq \mathbb{R}$  is an interval and  $a \in \mathbb{R}$  a point belonging to the closure of  $I$ . If  $f : I^p \rightarrow \mathbb{R}^p$ ,  $f = (f_1, \dots, f_p)$  is a continuous map such that

$$\min(x_1, \dots, x_p, a) < f_i(x_1, \dots, x_p) < \max(x_1, \dots, x_p, a), \\ x_i \neq a, \quad i = 1, \dots, p,$$

then  $f : I^p \rightarrow I^p$ , and for every  $x \in I^p$ ,

$$\lim_{n \rightarrow \infty} f^n(x) = (a, \dots, a).$$

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INSTITUTE OF MATHEMATICS

SILESIA UNIVERSITY

BANKOWA 14

PL-40-007 KATOWICE

POLAND

e-mail: [matkow@omega.im.wsp.zgora.pl](mailto:matkow@omega.im.wsp.zgora.pl)