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A THEOREM ON SPACES OF FINITE SUBSETS

Abstract. We give conditions under which iterated hyperspaces of finite subsets, with Ochan's topology, are homeomorphic.

Introduction. In [2] and [3] Ochan introduced a new topology on the space of subsets of a given space X. His topology is generated by sets $[x, V] = \{y \subset X : x \subset y \subset V\}$, where x is a closed subset of X and V is an open subset of X. Then Pixley and Roy [4] proved that non-void finite subsets of reals, with the Ochan's topology creates an important example of a Moore space. Later some other authors investigated the Pixley-Roy hyperspaces and generalizations of the Pixley and Roy's construction (see for instance Douven [1], Przymusiński [6] or Plewik [5]).

The main theorem. Let $\mathcal{F}[X]$ be the set of non-void finite subsets of a T_1-space X. Equip $\mathcal{F}[X]$ by topology induced from the Ochan's topology. Let $\langle x, V\rangle = [x, V] \cap \mathcal{F}[X]$. Observe that sets $\langle x, V\rangle$ are closed-open and that they form a base.

LEMMA. *Let X be a T_1-space and let λ be a regular cardinal. If for each point $x \in X$ there exists a decreasing and well ordered family $U(x) = \{x(\alpha) : \alpha < \lambda\}$ of open neighbourhoods such that $\bigcap U(x) = \{x\}$, then for every n there exists a collection $\mathcal{D}_n$ of open subsets of $\mathcal{F}[\mathcal{F}[X]]$ such that:*

(1) *every collection $\mathcal{D}_n$ covers the subspace $\{y \in \mathcal{F}[\mathcal{F}[X]] : |y| = n\}$,*

(2) *every collection $\mathcal{D}_n$ is discrete in the subspace $\{y \in \mathcal{F}[\mathcal{F}[X]] : |y| \geq n\}$,*

(3) $|B \cap \{y \in \mathcal{F}[\mathcal{F}[X]] : |y| = n\}| = 1$ *for each* $B \in \mathcal{D}_n$.

Proof. If $y = \{y_1, \ldots, y_n\}$, then let $y(\alpha) = \langle y, y_1(\alpha) \cup \ldots \cup y_n(\alpha)\rangle$, $y_k = \{y_k^1, \ldots, y_k^r\}$ $r = r(k)$, and $y_k(\alpha) = \langle y_k, y_k^1(\alpha) \cup \ldots \cup y_k^r(\alpha)\rangle$.

Let $\alpha = \alpha(y)$ be the least ordinal such that if $t \in y_i$ and $t \notin y_k$, then $t \notin y_k^1(\alpha) \cup \ldots \cup y_k^r(\alpha)$, i.e. $\{t\} \cup y_k \notin y_k(\alpha)$.

Let $\mathcal{D}_n = \{y(\alpha) : |y| = n$ and $\alpha = \alpha(y)\}$. So, it is easy to verify, that collections $\mathcal{D}_n$ satisfied conditions (1), (2), (3).

Received November 14, 1982.

AMS (MOS) subject classification (1980). Primary 54B20.

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Any space $\mathscr{F}[Z]$ can be partitioned into closed-open sets as follows. Let A_* be the set of isolated points of $\mathscr{F}[Z]$ and let $A_0 = \{x \in \mathscr{F}[Z]$: there is an open subset $V^x \subset Z$ such that $|\langle x, V^x\rangle| \leqslant \aleph_0\} \setminus A_*$.

If sets A_β are defined for $\beta < \alpha$, then let $A_\alpha = \{x \in \mathscr{F}[Z]$: there is an open subset $V^x \subset Z$ such that $|\langle x, V^x\rangle| \leqslant \aleph_\alpha\} \setminus \bigcup \{A_\beta : \beta < \alpha\} \cup A_*$.

THEOREM. *Let λ be a regular cardinal and let X be a T_1-space with no or infinite many of isolated points such that for each point $x \in X$ there exists a decreasing and well ordered base $\{x(\alpha) : \alpha < \lambda\}$ of open neighbourhoods, then $\mathscr{F}[\mathscr{F}[X]]$ is homeomorphic with $\mathscr{F}[\mathscr{F}[\mathscr{F}[X]]]$.*

Proof. Denote by A_α and $\mathscr{A}_\alpha$ elements of the above defined partition for spaces $\mathscr{F}[X]$ and $\mathscr{F}[\mathscr{F}[X]]$, respectively, instead of a space Z. Observe that $|A_\alpha| = |\mathscr{A}_\alpha|$ for all $\alpha \geqslant 0$ and $|A_*| = |\mathscr{A}_*|$.

Let $\alpha \geqslant 0$ and let $y(\beta)$ be defined as in the proof of Lemma and let $\mathscr{D}_n$ denotes families which satisfy conditions (1), (2), (3). We define partitions $R_\beta = \{\langle x, V(x, \beta)\rangle : x \in B_\beta\}$ of A_α consisting of closed-open sets for $\beta < \lambda$ such that:

(i) R_β is a refinement of R_γ iff $\gamma \leqslant \beta$,
(ii) $B_\beta \subset B_\gamma$ iff $\beta \leqslant \gamma$,
(iii) $|R_1| = |A_\alpha|$,
(iv) $\{V : V \in R_\beta$ and $\beta < \lambda\}$ is a base for A_α,
(v) $|\bigcap \{\langle x, V(x, \beta)\rangle : \beta < \gamma\} \cap B_\gamma| = \aleph_\alpha$ for each $x \in \bigcup \{B_\beta : \beta < \gamma\}$.

We can do this as follows: Let $R_1^1 = \{\langle x, V(x, 1)\rangle \subset A_\alpha : |x| = 1\}$ refines $\mathscr{D}_1$ and $\{y(1) : |y| = 1\}$. If collections R_1^k are defined for $k < n$, then let $R_1^n = \{\langle x, V(x, 1\rangle \subset A_\alpha \setminus \bigcup \{\bigcup R_1^k : k < n\} : |x| = n\}$ refines $\mathscr{D}_n$ and $\{y(1) : |y| = n\}$. Let $R_1 = \bigcup \{R_1^n : n = 1, 2, \ldots\}$ and $B_1 = \{x : \langle x, V(x, 1)\rangle \in R_1\}$.

Assume that there are defined partitions R_β for $\beta < \gamma$. Let $P_\gamma = \{\langle x, \bigcap \{V(x, \beta) : \beta < \gamma\}\rangle : x \in \bigcup \{B_\beta : \beta < \gamma\}$. Let $R_\gamma^1 = \{\langle x, V(x, \gamma)\rangle \subset A_\alpha : |x| = 1\}$ refines P_γ and $\{y(\gamma) : |y| = 1\}$. If collections R_γ^k are defined for $k < n$, then let $R_\gamma^n = \{\langle x, V(x, \gamma)\rangle \subset \subset A_\alpha \setminus \bigcup \{\bigcup R^k : k < n\} : |x| = n\}$ refines P_γ and $\mathscr{D}_n$ and $\{y(\gamma) : |y| = n\}$ in a such way that $|\bigcap \{V_\gamma(x, \beta) : \beta < \gamma\} \setminus V(x, \gamma)| = \aleph_\alpha$ for each $x \in \bigcup \{B_\beta : \beta < \gamma\}$. Let $R_\gamma = \bigcup \{R_\gamma^n : n = 1, 2, \ldots\}$ and $B_\gamma = \{x : \langle x, V(x, \gamma)\rangle \in R_\gamma\}$. Analogously we define sets $\mathscr{B}_\beta$ and partitions $\mathscr{R}_\beta = \{\langle x, V(x, \beta)\rangle : x \in \mathscr{B}_\beta\}$ of $\mathscr{A}_\alpha$ for $\beta < \lambda$.

Let us define a one-to-one function $f \colon A_\alpha \to \mathscr{A}_\alpha$ step by step on sets B_β. Let f be a one-to-one function from B_1 onto $\mathscr{B}_1$. Further, by induction, let f be a one-to-one function from $B_\gamma \setminus \bigcup \{B_\beta : \beta < \gamma\}$ onto $\mathscr{B}_\gamma \setminus \bigcup \{\mathscr{B}_\beta : \beta < \gamma\}$ such that if $y \in \langle z, \bigcap \{V(z, \beta) : \beta < \gamma\}\rangle$, then $f(y) \in \langle f(z), \bigcap \{V(f(z), \beta) : \beta < \gamma\}$ (there is a finite many of such points z only).

Observe that $f(A_\alpha) = \mathscr{A}_\alpha$ and $f(\langle x, V(x, \beta)\rangle) = \langle f(x), V(f(x), \beta)\rangle$ for every $\beta < \lambda$ and each $x \in A_\alpha$. Therefore the required homeomorphism is defined for α was taken arbitrarily.

The assumption of Theorem do not imply that $\mathscr{F}[X]$ is homeomorphic with $\mathscr{F}[\mathscr{F}[X]]$. For example, let X be the unit interval I, then $\mathscr{F}[I]$ satisfied the countable chain condition, see [3], but $\mathscr{F}[\mathscr{F}[I]]$ contains a family $\{\langle\{\{t\}\}, \langle\{t\}, \mathscr{F}[\mathscr{F}[I]]\rangle : t \in I\}$ of open pairwise disjoint sets of cardinality $2^{\aleph_0}$.

Let us note, that the proof of our main theorem is a generalization of methods from [5].

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