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Title: On a problem of Wu Wei Chao

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ON A PROBLEM OF WU WEI CHAO

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Abstract. Answering a question posed by Wu Wei Chao [2], we determine all solutions $f : \mathbb{R} \rightarrow \mathbb{R}$ of the equation

$$f(x^2 + y + f(y)) = f(x)^2 + 2y, \quad x, y \in \mathbb{R}.$$

In volume 108 (No 10, December 2001) of the American Mathematical Monthly, Wu Wei Chao [2] posed the following problem: find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the functional equation

$$(1) \quad f(x^2 + y + f(y)) = f(x)^2 + 2y, \quad x, y \in \mathbb{R}.$$

We will show that the only solution of (1) is an identity function i.e. $f(x) = x$, $x \in \mathbb{R}$. Our proof is based upon two lemmas.

LEMMA 1. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution of the functional equation (1), then it is odd.*

PROOF. Firstly we shall show that

$$(2) \quad c := f(0) = 0.$$

By (1) we have

$$(3) \quad f(c) = c^2;$$

$$(4) \quad f(x^2 + c) = f(x)^2, \quad x \in \mathbb{R};$$

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and

$$(5) \quad f(c + c^2) = c^2 + 2c.$$

By virtue of (3) and (4), $f(c + c^2) = c^4$, which together with (5) implies

$$(6) \quad c(c^3 - c - 2) = 0.$$

Putting $x = 0$ in (1), we get

$$(7) \quad f(y + f(y)) = c^2 + 2y, \quad y \in \mathbb{R},$$

and whence $f(\mathbb{R}) = \mathbb{R}$.

Let y_0 be chosen such that $f(y_0) = 0$. It follows from (7) that $0 = c^2 + 2y_0$ or, equivalently,

$$(8) \quad y_0 = -\frac{c^2}{2}.$$

Putting y_0 instead of x in (1), we obtain

$$f\left(\frac{c^4}{4} + y + f(y)\right) = 2y, \quad y \in \mathbb{R}.$$

Therefore $f\left(\frac{c^4}{4} + c\right) = 0$, and using (8) we get $c(c^3 + 2c + 4) = 0$. According to (6), we infer that $c = 0$, which finishes the proof of condition (2).

Putting $y = 0$ in (1) and on account of (2), we get

$$(9) \quad f(x^2) = f(x)^2, \quad x \in \mathbb{R}.$$

This implies that

$$(10) \quad f(u) \geq 0, \quad u \in [0, \infty).$$

According to (9), we have the following alternative

$$f(x) = f(-x) \quad \text{or} \quad f(-x) = -f(x), \quad x \in \mathbb{R}.$$

Assume that for some $z > 0$ we have $f(z) = f(-z)$. It follows from (1) and (9) that

$$(11) \quad f(x^2 - z + f(z)) = f(x^2) - 2z, \quad x \in \mathbb{R}.$$

Choose x such that $x^2 = z$. By virtue of (10)

$$0 \leq f(f(z)) = f(z) - 2z.$$

Taking into account (11) with $x = 0$ and using (2), we obtain

$$f(-z + f(z)) = -2z,$$

and, therefore (cf. (10)) $f(z) \leq z$. Thus

$$0 \leq f(z) - 2z \leq z - 2z = -z,$$

which means that $z \leq 0$. This contradicts our assumption that $z > 0$ and proves that $f(-x) = -f(x)$ for each $x \in \mathbb{R}$. This ends the proof of Lemma 1.

LEMMA 2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function fulfilling the equation (1). Then function $F : \mathbb{R} \rightarrow \mathbb{R}$ given by the formula $F(x) := f(x) + x$, $x \in \mathbb{R}$, satisfies the following conditions:*

$$(12) \quad F \text{ is odd};$$

$$(13) \quad F(x^2 + F(y)) = F(x^2) + F(y) + 2y, \quad x, y \in \mathbb{R};$$

$$(14) \quad F((0, \infty)) = (0, \infty) \quad \text{and} \quad F((-\infty, 0)) = (-\infty, 0).$$

PROOF. Condition (12) is a consequence of the definition of F and Lemma 1. Rewrite (1) in the form

$$f(x^2 + y + f(y)) + x^2 + y + f(y) = f(x)^2 + 2y + x^2 + y + f(y), \quad x, y \in \mathbb{R}.$$

By definition of F and on account of (9), we obtain (13). Now, let us put $y = -x^2$ in (13). Then, by (12) we get

$$(15) \quad F(u - F(u)) = -2u, \quad u \in \mathbb{R},$$

and whence $F(\mathbb{R}) = \mathbb{R}$. It follows from (10) and the definition of F that $\text{sgn}(u)(u - F(u)) \leq 0$ for each $u \neq 0$. Thus, (14) follows directly from (15). The proof of Lemma 2 is completed.

THEOREM. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional equation (1) then $f(x) = x$ for every $x \in \mathbb{R}$.*

PROOF. Let F be function defined in Lemma 2. It follows from (13) and (12) that

$$F(x^2 - F(y)) = F(x^2) - F(y) - 2y, \quad x, y \in \mathbb{R}.$$

Summing up this equality and (13), we get

$$F(x^2 + F(y)) + F(x^2 - F(y)) = 2F(x^2), \quad x, y \in \mathbb{R}.$$

Since $F((0, \infty)) = (0, \infty)$ (comp. (14)), the above equality means that F satisfies Jensen's functional equation

$$F\left(\frac{u+v}{2}\right) = \frac{F(u) + F(v)}{2}, \quad u, v \in (0, \infty),$$

and therefore F , being bounded below on $(0, \infty)$, has to be of the following form

$$F(u) = ku + b, \quad u \in (0, \infty),$$

where $k, b \geq 0$ are constants (cf. [1], pages 315, 316, for example). Since F is odd and $F(0) = 0$, b has to be equal to zero and $F(u) = ku$ for every $u \in \mathbb{R}$. Moreover, k has to be equal to 2, because F is a solution of (13). Now, our assertion follows immediately from the definition of F .

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