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On the theory of Hausdorff measures in metric spaces.

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Abstract.

In this work the main objective is to extend the theory of Hausdorff measures in general metric spaces. Throughout the thesis Hausdorff measures are defined using premeasures. A condition on premeasures of 'finite order' is introduced which enables the use of a Vitali type covering theorem. Weighted Hausdorff measures are shown to be an important tool when working with Hausdorff measures defined by a premeasure of finite order.

The main result of this thesis is the existence of subsets of finite positive Hausdorff measure for compact metric spaces when the Hausdorff measure has been generated by a premeasure of finite order. This result then extends to analytic subsets of complete separable metric spaces by standard techniques in the case when the increasing sets lemma holds. The proof of this result uses techniques from functional analysis. In this respect the proof presented is quite different from those of the previous literature.

A discussion on Hausdorff–Besicovitch dimension is also to be found. In particular the problem of whether

$$\dim(X) + \dim(Y) \leq \dim(X \times Y)$$

is solved in complete generality. Generalised dimensions involving partitions of Hausdorff functions are also discussed for product spaces. These results follow from a study of the weighted Hausdorff measure on product spaces.

An investigation is made of the sufficiency of some conditions for the increasing sets lemma to hold. Some counterexamples are given to show insufficiency of some of these conditions. The problem of finding a counterexample to the increasing sets lemma for Hausdorff measures generated by Hausdorff functions is also examined. It is also proved that for compact metric spaces we may also approximate the weighted Hausdorff measure by finite Borel measures that are 'dominated' by the premeasure generating the weighted Hausdorff measure.

I am greatly indebted to Professor David Preiss for proposing this investigation and for much wise advice. In particular he suggested the study of the weighted Hausdorff measures. I should like to express my gratitude to Professor C.A. Rogers for his help and constructive criticism on the manuscript. I am also grateful to the Science and Engineering Research Council for the funding of my studies.

J.D.H.

Contents

1 Introduction.

1.1 General discussion.

When dealing with Hausdorff measures it is often useful to be assured of the existence of a set of finite positive measure. For example, the modern day proof of the Frostman Lemma uses this existence. The Frostman Lemma relates capacities to Hausdorff–Besicovitch dimension. Frostman's original proof used properties of the net measure given by the dyadic cubes of Euclidean space. The techniques of comparable net measures were used by Besicovitch to prove the existence of subsets of finite positive Hausdorff measure. These techniques were then formalised by C.A. Rogers in [26]. In fact the Frostman Lemma follows from Theorem 6.3 of this thesis without the use of subsets of finite positive Hausdorff measure, see Corollary 6.8 of this thesis.

In [7] R.O. Davies and C.A. Rogers give an example of a compact metric space and a continuous Hausdorff function h such that the space has infinite Hausdorff h-measure but no subsets of finite positive Hausdorff h-measure. In this example the function h decreases rapidly to zero, in the sense that

$$\lim_{t \searrow 0} \frac{h(3t)}{h(t)} = \infty.$$

We say that a Hausdorff function h is of finite order if there exists a constant η such that

$$\limsup_{t \searrow 0} \frac{h(3t)}{h(t)} \leq \eta.$$

The question then naturally arises of whether we are assured of the existence of subsets of finite positive Hausdorff h-measure, when h is of finite order. In particular, are we assured of the existence of subsets of finite positive s-dimensional Hausdorff measure? We show in Corollary 11.5 that if X is an analytic subset of a complete separable metric space of infinite Hausdorff

h-measure, where h is a continuous Hausdorff function of finite order, then there exists a (compact) subset of X of finite positive measure (as large as we wish). Thus if X (an analytic subset of a complete separable metric space) is of infinite s-dimensional Hausdorff measure then we are assured of the existence of a subset of finite positive measure. This result (Corollary 11.5) follows from Theorem 11.2 by the increasing sets lemma, as proved by R.O. Davies in [5]. Theorem 11.2 deals with the case when X is compact and the Hausdorff measure is generated by a 'premeasure of finite order' and may be considered as the main result of this work.

The methods which we have introduced to prove the existence of subsets of finite positive measure use standard techniques from functional analysis. In this respect they are rather different from the methods used to prove previous results of this nature. The use of the measures that we have termed 'the weighted Hausdorff measures' has enabled the functional analytic techniques to be applied. In general, the weighted Hausdorff measure and the Hausdorff measure (defined by the same premeasure or Hausdorff function) may differ, see Note 6.5. H. Federer gives sufficient conditions under which the two measures coincide in 2.10.24 of [10], see Theorem 9.7 of this thesis. As far as we are aware the first systematic study of weighted Hausdorff measures is by J.D. Kelly in [17]. Kelly uses the term 'method III measures' for the weighted Hausdorff measures.

The existence of a subset of finite positive Hausdorff measure has also been used in the proof of the proposition that

$$\dim(X) + \dim(Y) \leq \dim(X \times Y)$$

where $\dim(X)$ is the Hausdorff–Besicovitch dimension of X. For some time this has been unanswered in full generality. We are now able to answer in the affirmative the above problem for arbitrary metric spaces. There must however, be some restriction on how the space $X \times Y$ is metrised, see Note 10.24. The proof we give does not require the existence of subsets of finite positive Hausdorff measure.

The condition on premeasures of finite order in the proof of the existence of subsets of finite positive Hausdorff measure ensures that we have a 'differentiation basis' for Borel measures 'dominated' by the premeasure. The differentiation theorems given in Section 9 rely on a Vitali type covering theorem namely Theorem 7.3. Theorem 7.3 generalises the standard Vitali covering theorem by using the concept of overflow as may be found in [13]. The results of Section 9 are in the spirit of [28], Theorems 2.10.17 and 2.10.18 of [10] and of [8].

A central result in the theory of Hausdorff measures is that of the increasing sets lemma. However this result is only known to hold under fairly restrictive conditions. We have proved a very much weakened version for Hausdorff measures generated by premeasures of 'strong finite order'. This has enabled me to strengthen some results. However this weakened form of the increasing sets lemma does not imply the approximation in Hausdorff measure of analytic subsets of a complete separable metric space by compact sets. Other conditions for the increasing sets lemma are examined in the final section of this work. Some counterexamples have been given to show insufficiency. As far as we are aware the only counterexamples to the increasing sets lemma other than Example 12.1 (of this thesis) have failed only for a given size δ. Example 12.1 however fails the increasing sets lemma for all sizes.

1.2 Literature review.

The existence of subsets of finite positive measure has been shown in the previous literature for the following cases. A.S. Besicovitch proved, in [1], that a closed subset E (of the real line) of infinite Λ^s-measure has subsets of any finite measure. He was then able to extend the result to the case when E is $F_{\sigma\delta\sigma}$. It was also noted in the paper that the method used in the proof of these results is easily extended to n-dimensional Euclidean space. In a subsequent paper [3] R.O. Davies was able to further extend the above result to the case when E is an analytic subset of the line. The reader is also referred to Corollary 28 of [9] for these results. Theorems 54–57 of [26] deal with the cases when the Hausdorff measure is a net measure and when the Hausdorff measure is defined by a Hausdorff function h and may be approximated by a net measure $\Lambda^{\tau}(\,\cdot\,;\mathcal{N})$ in the sense that there exists a finite positive constant M such that for all positive δ and subsets E,

$$\Lambda^{\tau}_{M\delta}(E;\mathcal{N}) \leq M\Lambda^{h}_{\delta}(E) \quad \text{and} \quad \Lambda^{h}_{M\delta}(E) \leq M\Lambda^{\tau}_{\delta}(E;\mathcal{N}).$$

In particular, these results apply to analytic subsets of complete separable ultrametric spaces in the case when the Hausdorff measure is generated by a continuous Hausdorff function. D.G. Larman was able to apply the results concerning net measures to the case when E is a finite dimensional (in the sense of Larman) compact metric space and the Hausdorff measure is defined by a Hausdorff function, see Theorem 2 of [22]. This was then easily extended (Theorem 3 of [22]) to the case when E is an analytic subset of a finite dimensional compact metric space.

The reader is referred to page 131 of [26] for a comprehensive list of references on the problem of the dimension of product sets and related problems. Of special mention is the paper [23] of J.M. Marstrand. In a personal communication J.M. Marstrand tells me that for premeasures of finite-order the method he used in [23] generalizes to subsets of compact metric spaces. It is also noted that D.G. Larman answers, in [22], the corresponding problem with generalized Hausdorff–Besicovitch dimension when the factor spaces are compact and finite dimensional (in the sense of Larman). This result then immediately gives the required result for Hausdorff–Besicovitch dimension, when the factor spaces are compact and finite dimensional, by the same arguments as in the proof of Corollary 10.23 of this thesis. An excellent account on dimension, capacities and Hausdorff measures may be found in [9].

Further results on capacities and related topics may be found in [14] and [20].

J. Hawkes considers, in [12], s-dimensional Hausdorff measures in $\mathbb{R}^+ \times \mathbb{R}$ with appropriate metrics which reflect the scaling properties of stable subordinators of index α (where $0 < \alpha < 1$). This is a generalisation of the work of R.P. Kaufman [16] on the stable subordinator given by Brownian motion. Under these metrics $\mathbb{R}^+ \times \mathbb{R}$ is neither ultrametric nor finite dimensional (in the sense of Larman). The method of net measures employed by Rogers, to prove the existence of subsets of finite positive measure in [26], does not immediately translate to this situation. However since the Hausdorff functions $h(t) = t^s$ are of finite order the Hausdorff measures generated by these metrics certainly fall into the framework presented here.

Many authors have contributed to the knowledge on the increasing sets lemma and the approximation in Hausdorff measure by compact sets. Of particular mention are the following. A.S. Besicovitch proved a weak form in [1] for Euclidean space and s-dimensional Hausdorff measure. R.O. Davies proved the stronger form for this case in [3]. R.O. Davies was then able to extend these result firstly to 'measures of Hausdorff type' in [4] and then in [5] to arbitrary metric spaces in the cases listed in Theorem 8.3 of this thesis. 'Measures of Hausdorff type' in the terminology of this thesis are Hausdorff measures generated by premeasures that are regular with respect to a family of sets that is finite in the large. In [5] R.O. Davies also makes a searching investigation of the circumstances in which the increasing sets lemma holds for Hausdorff measures generated by Hausdorff functions. C.A. Rogers proved corresponding results for net measures in [26]. Also of note is the paper [25] by M. Sion and D. Sjerve. The results on the increasing sets lemma given by Sion and Sjerve follow as a consequence of the paper [4]. However Sion and Sjerve proved much more than this and show that under their conditions analytic subsets of non-σ-finite Hausdorff measure contain a compact subset of non-σ-finite Hausdorff measure. Finally J.D. Kelly gave corresponding results to those of [4] for the weighted Hausdorff measure in [19].

2 Preliminary Definitions.

2.1 Real notation.

The set of reals, positive reals and non-negative reals are denoted by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ respectively. The algebraic operations and partial ordering of real valued functions are defined pointwise.

2.2 The arena of thought.

Throughout the following, X will denote a metric space with metric ρ. Any conditions on the space X will always be stated in full. We denote the closed ball centre x and radius δ by $B(x, \delta)$. For $E \subseteq X$ we denote by $B(E, \delta)$ the set

$$\{x \in X | \exists y \in E \colon \rho(x, y) \leq \delta\} = \bigcup_{x \in X} B(x, \delta).$$

We use $\hat{\rho}$ to denote the Hausdorff pseudometric induced by ρ, which is defined for $E \subseteq X$ and $F \subseteq X$ by

$$\hat{\rho}(E, F) = \inf\{\delta \in \mathbb{R}^+ | \mathrm{E} \subseteq \mathrm{B}(\mathrm{F}, \delta) \text{ and } \mathrm{F} \subseteq \mathrm{B}(\mathrm{E}, \delta)\},$$

with the convention $\inf \emptyset = \infty$.

Remark. As is well known $\hat{\rho}$ restricted to the family of non-empty closed subsets of X is a metric. Also if X is compact then $\hat{\rho}$ induces a compact topology.

For $E \subseteq X$ we denote the diameter of E by $\operatorname{diam} E$ defined as

$$\sup\{\rho(x, y) | x, y \in E\},$$

with the convention $\sup \emptyset = 0$. Where it is necessary to distinguish between metrics we write $\operatorname{diam}_\rho E$ to denote the diameter with respect to ρ.

For $E \subseteq X$ and $F \subseteq X$ we denote the distance between E and F by $\operatorname{dist}(E, F)$ defined as

$$\inf\{\rho(x, y) | x \in E \text{ and } y \in F\},$$

with the convention $\inf \emptyset = \infty$.

We say that X has finite structural dimension if and only if for all positive κ, there exists N such that every subset of X of sufficiently small diameter δ can be covered by N sets of diameter not greater than $\kappa\delta$. This notion of finite structural dimension is similar to finite dimensional in the sense of Larman. Moreover if X is compact and finite dimensional in the sense of Larman then X has finite structural dimension, see [21] for further details.

The space X is said to be ultrametric if and only if

$$\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$$

for all x, y and z in X.

2.3 Particular families of sets

We say that a subset B of X is a Borel set if and only if B is a member of the smallest σ-algebra containing the family of all open sets.

We define $\mathbf{I}$ to be the set of all sequences $\mathbf{i} = (i_n)_{n\geq 1}$ of positive integers. Given an element $\mathbf{i}$ of $\mathbf{I}$ we use $\mathbf{i}|n$ to denote the n-tuple $(i_1, \ldots, i_n)$. For a family $\mathcal{E}$ of subsets of X, we define the Souslin-$\mathcal{E}$ sets to be those sets of the form

$$E = \bigcup_{\mathbf{i}\in\mathbf{I}} \bigcap_{n=1}^{\infty} E_{\mathbf{i}|n}$$

where $E_{\mathbf{i}|n} \in \mathcal{E}$ for all $n \geq 1$ and $\mathbf{i}$ in $\mathbf{I}$.

We say that a subset A of X is analytic if and only if A is a Souslin-$\mathcal{F}$ set where $\mathcal{F}$ is the family of all closed subsets of X.

Note 2.1 It is well known that every Borel set is analytic. $\dashv$

2.4 Some properties of families of sets.

It will be useful to consider particular families of sets. A family $\mathcal{E}$ of subsets of X is said to be a covering of a subset S of X if and only if

$$S \subseteq \bigcup \mathcal{E}.$$

We say that $\mathcal{E}$ is a fine covering of S if and only if for each $\delta > 0$ the set

$$\{E \in \mathcal{E} | \operatorname{diam} E \leq \delta\}$$

is a covering of S.

A family $\mathcal{E}$ of subsets of X is termed θ-adequate (for X) where θ is a positive constant if and only if the following conditions are satisfied:

1. $\emptyset$ and X are members of $\mathcal{E}$,
2. for every subset S of X of positive diameter and E in $\mathcal{E}$ with $S \subseteq E$, there exists F in $\mathcal{E}$ such that
$$S \subseteq F \subseteq E \quad \text{and} \quad \operatorname{diam} F \leq \theta \operatorname{diam} S,$$
3. $\mathcal{E}$ is a fine covering of X.

We simply say that $\mathcal{E}$ is adequate (for X) if for some positive constant θ we have $\mathcal{E}$ is θ-adequate (for X).

A family $\mathcal{E}$ of subsets of X is called finite in the large if and only if for all $\varepsilon > 0$ the set

$$\{E \in \mathcal{E} | \operatorname{diam} E > \varepsilon\}$$

has finitely many members.

Remark. J.D. Kelly uses the term 'of countable type' for 'finite in the large'. A natural generalisation of this property could be 'locally finite in the large' or 'point finite in the large' meaning that for all $\varepsilon > 0$ the set

$$\{E \in \mathcal{E} | \operatorname{diam} E > \varepsilon\}$$

is locally finite or point finite. However we have found no 'practical' use for these generalisations.

2.5 Set enlargement.

Given a family $\mathcal{E}$ of subsets of X, positive τ and a subset S of X we define $\widehat{S}^{\tau}_{\mathcal{E}}$ the τ-enlargement of S with respect to $\mathcal{E}$ by

$$\widehat{S}^{\tau}_{\mathcal{E}} = \bigcup\{E \in \mathcal{E} | S \cap E \neq \emptyset \text{ and } \operatorname{diam} E \leq \tau \operatorname{diam} S\}.$$

In the case when $\mathcal{E}$ is the family of all subsets of X we simply write $\widehat{S}^{\tau}$ for $\widehat{S}^{\tau}_{\mathcal{E}}$ and $\widehat{S}$ for $\widehat{S}^{1}$.

Remark. It would be equivalent to define $\widehat{S}^{\tau}$ by $\widehat{S}^{\tau} = B(S, \tau \operatorname{diam} S)$.

3 Premeasures.

3.1 Definition.

A premeasure on X is a function ξ mapping the subsets of X to the non-negative reals satisfying

1. $\xi(\emptyset) = 0$,

2. $U \subseteq V \Longrightarrow \xi(U) \leq \xi(V)$ for all $U, V \subseteq X$.

We say that the premeasure ξ is of finite τ-order if and only if for some constant η we have

1. $\xi(\widehat{S}^\tau) \leq \eta\xi(S)$ for all $S \subseteq X$,

2. $\inf\{\xi(B(x,\delta))|\delta > 0\} \leq \eta\xi(\{x\})$ for all $x \in X$.

We simply say that ξ is of finite order when ξ is of finite 1-order. We also say that ξ is of strong finite order when ξ is of finite τ-order for some $\tau > 1$.

Remark. It is noted that if we restrict our attention to premeasures ξ which satisfy, $\xi(\{x\}) = 0$ for every x in X then the condition

$$\inf\{\xi(B(x,\delta))|\delta > 0\} \leq \eta\xi(\{x\})$$

is equivalent to

$$\lim_{\delta \searrow 0} \xi(B(x,\delta)) = 0.$$

3.2 Hausdorff functions.

A function $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Hausdorff function if and only if the following conditions are satisfied

1. $h(t) > 0$ for all $t > 0$,

2. $h(t) \geq h(s)$ for all $t \geq s$,

3. h is continuous from the right for all $t \geq 0$.

For such a function and a positive constant θ, we define a premeasure ξ on X by

$$\begin{aligned} \xi(S) &= \min\{h(\operatorname{diam} S), h(\theta)\} \qquad \text{for } S \neq \emptyset \\ \xi(\emptyset) &= 0. \end{aligned}$$

It will be convenient to describe ξ as the premeasure, on X, defined by (the Hausdorff function) h and cut off size θ. We simply say that ξ is a premeasure defined by h when for some positive θ, ξ is the premeasure defined by h and cut off size θ. We note that if h satisfies the condition that for some constant η,

$$\limsup_{t \searrow 0} \frac{h(3t)}{h(t)} \leq \eta$$

then for sufficiently small θ the premeasure defined by h and cut off size θ is of finite-order. In this case we say that h is of finite-order. Finally we denote by $\mathbf{H}$ the family of all Hausdorff functions.

Note 3.1 In the above definition of ξ, a premeasure defined by some Hausdorff function h, we have introduced the value θ solely to ensure that the resulting premeasure assigns (finite) values to all sets. It is clear that the resulting Hausdorff ξ-measure and weighted Hausdorff ξ-measure are independent of the choice of θ. Thus in the case when h is of finite order we may assume that θ has been chosen sufficiently small to ensure that ξ is of finite order. ⊣

Remark. In the above premeasures have been defined to map subsets to the non-negative reals. This definition could be extended to mappings from subsets to the non-negative extended reals thus allowing the value ∞ to be assigned to subsets. In this case it would be more natural to define the premeasure, ξ say, defined by some Hausdorff function h by

$$\begin{aligned}\xi(S) &= h(\operatorname{diam} S) \qquad \text{for } S \neq \emptyset \\ \xi(\emptyset) &= 0\end{aligned}$$

with the convention that

$$h(\infty) = \lim_{t \to \infty} h(t),$$

although this change in ξ would not change the resulting measures Λ_δ^ξ and λ_δ^ξ. In much of what follows it would be necessary to impose the condition that premeasures assign a finite value to every set of finite diameter. Since this condition is dependent on how the space, on which the premeasure is defined, is metrised we prefer to restrict the definition of premeasures to maps into the non-negative reals. Moreover, if X is metrised by ρ then we may define a new metric ρ' by

$$\rho'(x, y) = \min\{\theta, \rho(x, y)\}$$

where θ is some (finite) positive constant. Furthermore ρ and ρ' define the same topology and the same Hausdorff and weighted Hausdorff measures on X, and all subsets are ρ'-bounded.

3.3 Regularity.

For ξ a premeasure and $\mathcal{E}$ a family of subsets of X, we say that ξ is $\mathcal{E}$-regular if and only if for all subsets S of X we have

$$\xi(S) = \inf\{\xi(E) | E \in \mathcal{E} \text{ and } S \subseteq E\}.$$

In the case when $\mathcal{E}$ is the family of all open subsets of X we say open regular. Similarly when $\mathcal{E}$ is the family of all Borel subsets of X we say Borel regular.

Note 3.2 If ξ is a premeasure on X defined by some Hausdorff function then ξ is open regular, since Hausdorff functions were defined to be continuous from the right. $\dashv$

3.4 Subspaces.

For ξ a premeasure on X and S a subset of X we define $\xi|S$ to be the restriction of ξ to the family of subsets of S. Also for $\mathcal{E}$ a family of subsets of X and S as above we define $\mathcal{E}|S$ to be the family of subsets of S

$$\{E \cap S | E \in \mathcal{E}\}.$$

Note 3.3 For ξ and S as above, we note that $\xi|S$ is a premeasure on S. For $\mathcal{E}$ a family of subsets of X, if ξ is $\mathcal{E}$-regular then $\xi|S$ is $(\mathcal{E}|S)$-regular. $\dashv$

4 Hausdorff Measures.

4.1 Measures.

A measure on X is a function μ mapping the subsets of X to the (non-negative) extended reals satisfying

1. $\mu(\emptyset) = 0$
2. $U \subseteq V \implies \mu(U) \leq \mu(V)$ for all $U, V \subseteq X$
3. $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$ for all sequences $(E_i)_{i \geq 1}$ of subsets of X.

We say that μ is a finite measure (on X) if and only if $\mu(X) < \infty$.

Remark. Our definition of a measure is what most authors term an outer measure.

Given a measure μ on X, we say that a set S is μ-measurable if and only if

$$\mu(E) = \mu(E \cap S) + \mu(E \setminus S) \qquad (\forall E \subseteq X).$$

Note 4.1 The family $\mathcal{M}$ of all μ-measurable sets is a σ-algebra on which μ is additive and every Souslin-$\mathcal{M}$ set is a member of $\mathcal{M}$. $\dashv$

4.2 Some properties of measures.

For μ a measure on X and $\mathcal{E}$ a family of subsets of X, we say that μ is $\mathcal{E}$-regular if and only if for all subsets S of X we have

$$\mu(S) = \inf\{\mu(E) | E \in \mathcal{E} \text{ and } S \subseteq E\}.$$

In the case when $\mathcal{E}$ is the family of all Borel subsets of X we say Borel regular. We say that a measure μ is a regular measure if it is regular with respect to the family of all μ-measurable sets. A Borel measure is a measure that is Borel regular and such that all Borel sets are measurable.

Throughout what follows it will be convenient to refer to the following inner regularity condition on a measure μ of X,

(∗) *For every Borel set B of X*

$$\mu(B) = \sup\{\mu(K) | K \subseteq B \text{ and } K \text{ compact}\}.$$

We simply say that μ satisfies the Radon condition(∗).

Note 4.2 For μ a regular measure on X and an increasing sequence $(E_i)_{i\geq 1}$ of sets (that is $E_i \subseteq E_{i+1}$ for $i \geq 1$) we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{i\geq 1} \mu(E_i). \qquad \dashv$$

Note 4.3 For μ a Borel measure on X that is $\mathcal{G}_\delta$-regular (where $\mathcal{G}_\delta$ is the family of all countable non-empty intersections of open sets) and $E \subseteq X$ a μ-measurable set of finite μ-measure we have

$$\mu(E) = \sup\{\mu(F) | F \subseteq E \text{ and } F \text{ closed}\}.$$

Also, whenever μ is a finite Borel measure on X we have that μ is open regular, and hence for all μ-measurable subsets E of X

$$\mu(E) = \sup\{\mu(F) | F \subseteq E \text{ and } F \text{ closed}\}.$$

Furthermore if X is an analytic subset of a complete separable metric space then μ a finite Borel measure satisfies the Radon condition(∗). $\dashv$

For μ a measure on X and S a subset of X we define $\mu|S$ to be the restriction of μ to the family of subsets of S.

Note 4.4 For μ and S as above, we note that $\mu|S$ is a measure on S. For $\mathcal{E}$ a family of subsets of X, if μ is $\mathcal{E}$-regular then $\mu|S$ is $(\mathcal{E}|S)$-regular. Also if E is a μ-measurable subset of X then $E \cap S$ is $(\mu|S)$-measurable. $\dashv$

For μ a measure of X and a subset S of X we define $\mu \llcorner S$ by

$$(\mu \llcorner S)(E) = \mu(E \cap S)$$

for all subsets E of X.

Note 4.5 For μ and S as above, we note that $\mu \llcorner S$ is a measure on X (whereas $\mu|S$ is a measure on S) and if E is μ-measurable then E is $(\mu \llcorner S)$-measurable. Now if μ is finite and $\mathcal{E}$-regular and S is μ-measurable then $\mu \llcorner S$ is $\mathcal{E}$-regular. Hence if μ is a finite Borel measure and S is μ-measurable then $\mu \llcorner S$ is a Borel measure. $\dashv$

4.3 Hausdorff measures.

We say that a sequence $(E_i)_{i\geq 1}$ of subsets of X is a δ-cover of a set S if $S \subseteq \bigcup_{i=1}^{\infty} E_i$ and $\operatorname{diam} E_i \leq \delta$ for $i \geq 1$. We use $\Omega_\delta(S)$ to denote the family of all such (countable) δ-covers of S. The measures Λ_δ^ξ are defined for $\delta > 0$ by

$$\Lambda_\delta^\xi(S) = \inf \left\{ \sum_{i=1}^{\infty} \xi(E_i) \,\middle|\, (E_i)_{i\geq 1} \in \Omega_\delta(S) \right\},$$

with the convention that $\inf \emptyset = \infty$. The Hausdorff ξ-measure Λ^ξ is then defined as

$$\Lambda^\xi(S) = \sup_{\delta>0} \Lambda_\delta^\xi(S).$$

When ξ is a premeasure defined by some Hausdorff function h, we simply write Λ^h for Λ^ξ. As is well known, see for example [26], Λ^ξ is a metric outer measure, in particular all Borel sets are measurable.

4.4 Weighted Hausdorff measures.

In a similar fashion, we say a sequence $(c_i, E_i)_{i\geq 1}$ of pairs, with c_i a non-negative real number and E_i a subset of X, is a weighted cover of a set S

if

$$\mathcal{X}_S \leq \sum_{i=1}^{\infty} c_i \mathcal{X}_{E_i},$$

that is for all points x of S we have $\sum\{c_i | x \in E_i\} \geq 1$. Furthermore we say that $(c_i, E_i)_{i\geq 1}$ is a weighted δ-cover of a set S if it is a weighted cover of S and $\operatorname{diam} E_i \leq \delta$ for $i \geq 1$. We use $\Upsilon_\delta(S)$ to denote the family of all weighted δ-covers of S. The measures λ^ξ_δ are defined for $\delta > 0$ by

$$\lambda^\xi_\delta(S) = \inf \left\{ \sum_{i=1}^{\infty} c_i \xi(E_i) \,\middle|\, (c_i, E_i)_{i\geq 1} \in \Upsilon_\delta(S) \right\},$$

with the convention that $\inf \emptyset = \infty$. The measure λ^ξ is then defined as

$$\lambda^\xi(S) = \sup_{\delta>0} \lambda^\xi_\delta(S)$$

which may be described as the Hausdorff ξ-measure defined by using weighted covers. It is convenient to call this the weighted Hausdorff ξ-measure. When ξ is a premeasure defined by some Hausdorff function h, we simply write λ^h for λ^ξ. It can also be shown, see for example [18], that λ^ξ is a metric outer measure.

4.5 Some properties of Hausdorff and weighted Hausdorff measures.

Note 4.6 For S a subset of X and ξ a premeasure, we have

$$\Lambda^\xi_\delta(S) \nearrow \Lambda^\xi(S) \quad \text{and} \quad \lambda^\xi_\delta(S) \nearrow \lambda^\xi(S) \qquad \text{as } \delta \searrow 0,$$

$$\lambda^\xi_\delta(S) \leq \Lambda^\xi_\delta(S) \quad \text{and} \quad \lambda^\xi(S) \leq \Lambda^\xi(S). \qquad \dashv$$

Note 4.7 For $\mathcal{E}$ a family of subsets of X that is adequate for X and ξ a premeasure on X that is $\mathcal{E}$-regular then the measure Λ^ξ is $\mathcal{E}_{\sigma\delta}$-regular and the measure λ^ξ is $\mathcal{F}_{\sigma\delta}$-regular where $\mathcal{F}$ is the closure of $\mathcal{E}$ under finite intersections, see [26] and [18] respectively. Here we use the notation $\mathcal{E}_\sigma$ to represent the family of all countable unions of sets E_i, in $\mathcal{E}$, and similarly $\mathcal{E}_\delta$ for countable unions replaced by (non-empty) countable intersections. Thus if ξ is Borel regular then Λ^ξ and λ^ξ are Borel measures and if ξ is open regular then Λ^ξ and λ^ξ are $\mathcal{G}_\delta$-regular where $\mathcal{G}$ is the family of all open subsets of X. $\dashv$

Note 4.8 For $\mathcal{E}$ a family of subsets of X, ξ a premeasure on X and a subset S of X we have

$$\Lambda^{\xi|S} = \Lambda^\xi|S \quad \text{and} \quad \lambda^{\xi|S} = \lambda^\xi|S.$$

Also if $\mathcal{E}$ is adequate for X then $\mathcal{E}|S$ is adequate for S. $\dashv$

Remark. In the previous literature premeasures are often defined on some family of sets and then covers taken from this family of sets. To distinguish between these two concepts we refer, for this remark, to premeasure defined on all subsets of X as outer premeasure and those defined on some particular family of subsets as restricted premeasures. Given an restricted premeasure, ζ say, defined on some family $\mathcal{E}$ of subsets of X that is adequate on X we may define an outer premeasure ξ by

$$\xi(S) = \inf\{\zeta(E) | S \subseteq E \in \mathcal{E}\},$$

for all subsets S of X. Then using either ξ or ζ results in the same measures for the Hausdorff construction and also for weighted Hausdorff construction. Also ξ is $\mathcal{E}$-regular. However without the condition that $\mathcal{E}$ is adequate things can go badly wrong. For example, if $(E_i)_{i\geq 1}$, is a cover of a set S with $\operatorname{diam} E_i \leq \delta$ for $i \geq 1$ there may be no cover of the set S from $\mathcal{E}$ with a bound on their diameters. Furthermore, it seems to me that the condition that $\mathcal{E}$ be adequate for X, or something similar, is required by some parts of the previous work. To illustrate this further, in the proof of 2.10.18 (1) of [10] we are told that

$$\phi_\delta(B) \leq \sum_{S \in H} \zeta(S) + \sum_{S \in G \setminus H} \zeta(\widehat{S})$$

where ζ is defined on a family of closed sets F and

$$\widehat{S} = \bigcup \{T \in F | T \cap S \neq \emptyset \text{ and } \operatorname{diam} T \leq 2 \operatorname{diam} S\}.$$

It is only tacitly implied in the hypothesis of the theorem that $\widehat{S}$ is in F by the condition that $\zeta(\widehat{S}) < \eta\zeta(S) < \infty$ whenever S is in F, as the domain of ζ was defined to be F. Moreover in the hypothesis of 2.10.17 (3) it is asserted that $S \cap T$ is in F whenever S and T are in F. The notion of adequacy is a weaker condition than that of requiring $\widehat{S}$ and $S \cap T$ to be in F whenever S and T are in F.

5 New Premeasures from Old.

5.1 Finite order to finite in the large.

Proposition 5.1 *For X a compact metric space and a premeasure ξ on X which is of finite-order, there exists a family of open sets, $\mathcal{G}$ say, such that $\mathcal{G}$ is 3-adequate (for X) and finite in the large and there exists a premeasure ζ on X, which is of finite-order and $\mathcal{G}$-regular such that*

$$\Lambda_\delta^\xi \le \Lambda_\delta^\zeta \le \eta \Lambda_\delta^\xi$$

$$\lambda_\delta^\xi \le \lambda_\delta^\zeta \le \eta \lambda_\delta^\xi$$

where η is some positive constant. ⊣

Proof. We prove this by construction. Inductively for every integer $i \ge 1$,

$$\{G \subseteq X | G \text{ open, } \operatorname{diam} G \le \frac{1}{3^i} \operatorname{diam} X\}$$

covers X, and hence we may define $\mathcal{U}_i$ to be a finite subcover of X, since X is compact. For E a subset of X of positive diameter we define $\tilde{E}^{\mathcal{U}}$ by

$$\tilde{E}^{\mathcal{U}} = \bigcup \{U \in \mathcal{U}_i | U \cap E \neq \emptyset\}$$

where i is the unique positive integer such that

$$\frac{1}{3^i} \operatorname{diam} X < \operatorname{diam} E \le \frac{1}{3^{i-1}} \operatorname{diam} X.$$

We define $\mathcal{H}$ to be the set

$$\{\tilde{E}^{\mathcal{U}} | E \subseteq X \text{ and } \operatorname{diam} E > 0\} \cup \{\{x\} | x \in X \text{ and } \{x\} \text{ is open}\} \cup \{\emptyset\},$$

$\mathcal{G}$ to be the completion of $\mathcal{H}$ under all finite intersections, that is the set

$$\left\{ \bigcap \mathcal{V} \middle| \mathcal{V} \subseteq \mathcal{H} \text{ and } \mathcal{V} \text{ is non-empty and finite} \right\}$$

and ζ by

$$\zeta(E) = \inf\{\xi(G) | G \in \mathcal{G} \text{ and } E \subseteq G\}$$

for all subsets E of X.

By the finite order of ξ there exists η such that for all subsets E of X of positive diameter $\xi(\widehat{E}) \leq \eta\xi(E)$. Thus there exists G in $\mathcal{G}$ such that

$$E \subseteq G \subseteq \widehat{E} \quad \text{and} \quad \zeta(E) \leq \xi(G) \leq \eta\xi(E).$$

The required results now follow easily. □

5.2 Premeasures on metric spaces of finite order.

Lemma 5.2 *For X a metric space and ξ a premeasure on X, ζ defined by*

$$\zeta(U) = \inf\left\{ \sum_{i=1}^{\infty} \xi(E_i) \middle| U \subseteq \bigcup_{i=1}^{\infty} E_i, E_i \subseteq X \textit{ for } i \geq 1 \right\}$$

is a premeasure on X and

$$\Lambda^\zeta = \Lambda^\xi \quad \text{and} \quad \lambda^\zeta = \lambda^\xi. \qquad \dashv$$

Proof. It is immediate that ζ is a premeasure on X. Now for $(E_i)_{i\geq1}$ a δ-cover of some set U and $(F_{i,j})_{j\geq1}$ a cover of E_i (for each $i \geq 1$), it is clear that $(F_{i,j} \cap E_i)_{i,j\geq1}$ is a δ-cover of U. The required result follows. □

Proposition 5.3 *Let X be a metric space, h a Hausdorff function and $\tau > 0$. Then, if X has finite structural dimension then there exists a premeasure ζ on X that is of finite τ-order and*

$$\Lambda^\zeta = \Lambda^h \quad \text{and} \quad \lambda^\zeta = \lambda^h. \qquad \dashv$$

Proof. By the definition of finite structural dimension, there exist positive constants ε and K such that for $\delta \leq \varepsilon$ every set of diameter not greater than

$(1+2\tau)\delta$ can be covered by K sets of diameter not greater than δ. We let ξ be the premeasure on X defined by h and cut off size ε. We define ϑ by

$$\vartheta(S) = \inf\left\{ \sum_{i=1}^{\infty} \xi(E_i) \middle| S \subseteq \bigcup_{i=1}^{\infty} E_i, E_i \subseteq X \text{ for } i \geq 1 \right\}.$$

We are assured by the previous lemma that ϑ is a premeasure and

$$\Lambda^{\vartheta} = \Lambda^{\xi} \quad \text{and} \quad \lambda^{\vartheta} = \lambda^{\xi}.$$

Now for every subset S of X we have $\vartheta(S) \leq \xi(S)$ and if the $\operatorname{diam} S \leq \varepsilon$ then

$$\vartheta(\widehat{S}^{\tau}) \leq K \sup\{\vartheta(E) | E \subseteq X \text{ and } \operatorname{diam} E \leq \operatorname{diam} S\} \leq K\xi(S).$$

We define ζ by

$$\zeta(S) = \sup\{\vartheta(E) | E \subseteq X \text{ and } \operatorname{diam} E \leq \operatorname{diam} S\}$$

for all non-empty subsets S of X and $\zeta(\emptyset) = 0$. Then ζ is a premeasure. For S a subset of X we see that

$$\begin{aligned} \zeta(\widehat{S}^{\tau}) &\leq K\zeta(S) \qquad && \text{if } \operatorname{diam} U \leq \varepsilon \\ \zeta(S) &\leq h(\varepsilon) && \text{if } \operatorname{diam} U > \varepsilon \end{aligned}$$

For our convenience we set

$$c = \sup\{\zeta(S) | S \subseteq X \text{ and } \operatorname{diam} S \leq \varepsilon\}.$$

There are two cases, either $c = 0$ or $c > 0$. If $c = 0$ then we must redefine ζ to assign the value 0 to every set. The result then follows by noting that for S a subset of X, either S admits a δ-cover for every $\delta \leq \varepsilon$ or S does not: if S admits a δ-cover for every $\delta \leq \varepsilon$ then $\Lambda^{\xi}(S) = \Lambda^{\zeta}(S) = 0$, otherwise there is no δ-cover of S for some $\delta \leq \varepsilon$ and thus $\Lambda^{\xi}(S) = \Lambda^{\zeta}(S) = \infty$. If $c > 0$ then it suffices to observe that $\vartheta \leq \zeta \leq \xi$ and

$$\zeta(\widehat{S}^{\tau}) \leq \max(K, \tfrac{1}{c}h(\varepsilon))\zeta(S) \qquad (\forall S \subseteq X).$$

□

Proposition 5.4 *For any ultrametric space and ξ a premeasure defined by some Hausdorff function, ξ is of finite-order.* $\dashv$

Proof. This follows immediately from the fact that in an ultrametric space for all subsets U we have $\operatorname{diam}\widehat{U} = \operatorname{diam} U$. $\square$

6 Approximation by Finite Borel Measures.

6.1 Notation.

It is convenient to denote the set of all finite (non-negative) Borel measures on X by $\mathrm{M}^+(X)$. For a premeasure ξ and a family of sets $\mathcal{E}$ we also employ the notation

$$\mathrm{M}^\xi_\delta(X;\mathcal{E}) = \{\mu \in \mathrm{M}^+(X) | \forall E \in \mathcal{E}, \operatorname{diam} E \leq \delta \Longrightarrow \mu(E) \leq \xi(E)\}$$

and

$$\mathrm{M}^\xi(X;\mathcal{E}) = \bigcup_{\delta>0} \mathrm{M}^\xi_\delta(X;\mathcal{E}).$$

When ξ is a premeasure defined by some Hausdorff function h, we simply write $\mathrm{M}^h(X;\mathcal{E})$ for $\mathrm{M}^\xi(X;\mathcal{E})$. When $\mathcal{E}$ is the family of all subsets of X, we simply write $\mathrm{M}^\xi_\delta(X)$ and $\mathrm{M}^\xi(X)$. The set $\mathrm{M}^\xi(X)$ may be thought of as the set of finite Borel measures on X that are 'dominated' by ξ.

Throughout the following we denote by $C(X)$ the Banach space of continuous bounded functions from X to the reals with supremum norm $\|\cdot\|$. In the case when X is compact we identify $\mathrm{M}^+(X)$ with the positive linear functionals in $C^*(X)$, the dual space of $C(X)$, as given by Riesz's Representation Theorem. That is, for μ in $\mathrm{M}^+(X)$, we may identify μ with the linear functional on $C(X)$ defined by

$$f \mapsto \int f \,\mathrm{d}\mu.$$

We refer to this identification as the identification given by Riesz's Representation Theorem.

6.2 Approximation Theorem.

Lemma 6.1 *Let X be a metric space, ξ a premeasure on X, and $\delta > 0$. Then for all μ in $\mathrm{M}^\xi_\delta(X)$ we have $\mu \leq \lambda^\xi_\delta$.* ⊣

Proof. For X, ξ and μ as above, E a subset of X and $(c_i, E_i)_{i\geq 1}$ in $\Upsilon_\delta(E)$ we choose U_i, for $i \geq 1$, so that U_i is μ-measurable,

$$E_i \subseteq U_i \quad \text{and} \quad \mu(U_i) = \mu(E_i).$$

Hence we calculate

$$\mu(E) \leq \int \sum_{i=1}^{\infty} c_i \mathcal{X}_{U_i} \, \mathrm{d}\mu = \sum_{i=1}^{\infty} c_i \mu(E_i) \leq \sum_{i=1}^{\infty} c_i \xi(E_i).$$

The required result follows. □

Lemma 6.2 *Let X be a metric space, $\mathcal{E}$ be a family of subsets of X that is θ-adequate, for some θ, and ξ a premeasure on X that is $\mathcal{E}$-regular. Then for all $\delta > 0$ we have*

$$\mathbf{M}^{\xi}_{\theta\delta}(X) \subseteq \mathbf{M}^{\xi}_{\theta\delta}(X;\mathcal{E}) \subseteq \mathbf{M}^{\xi}_{\delta}(X),$$

and hence $\mathbf{M}^{\xi}(X;\mathcal{E}) = \mathbf{M}^{\xi}(X)$. ⊣

Proof. For X, θ, $\mathcal{E}$, ξ and δ as above,

$$\{E \in \mathcal{E} | \operatorname{diam} E \leq \theta\delta\} \subseteq \{E \subseteq X | \operatorname{diam} E \leq \theta\delta\}$$

and hence

$$\mathbf{M}^{\xi}_{\theta\delta}(X) \subseteq \mathbf{M}^{\xi}_{\theta\delta}(X;\mathcal{E}).$$

Now for S a subset of X with diameter not exceeding δ,

$$\xi(S) = \inf\{\xi(E) | E \in \mathcal{E}, \operatorname{diam} E \leq \theta\delta \text{ and } S \subseteq E\}$$

by the adequacy of $\mathcal{E}$ and the $\mathcal{E}$-regularity of ξ. Hence for μ in $\mathbf{M}^{\xi}_{\theta\delta}(X;\mathcal{E})$ we have

$$\begin{aligned} \mu(S) &\leq \inf\{\mu(E) | E \in \mathcal{E}, \operatorname{diam} E \leq \theta\delta \text{ and } S \subseteq E\} \\ &\leq \inf\{\xi(E) | E \in \mathcal{E}, \operatorname{diam} E \leq \theta\delta \text{ and } S \subseteq E\} \\ &\leq \xi(S) \end{aligned}$$

and the required result follows. □

Theorem 6.3 *For X a compact metric space, $\mathcal{G}$ a family of open sets, which is θ-adequate on X, and a premeasure ξ (on X) which is $\mathcal{G}$-regular we have*

$$\lambda^{\xi}_{\theta\delta}(X) \leq \sup\{\mu(X) | \mu \in \mathrm{M}^{\xi}_{\theta\delta}(X;\mathcal{G})\} \leq \lambda^{\xi}_{\delta}(X)$$

and this supremum is attained. $\dashv$

Proof. We fix $\delta > 0$ and mimic the definition of $\lambda^{\xi}_{\theta\delta}$ to give a function $P: C(X) \to \mathrm{I\!R}^{+}_{0}$. For f in $C(X)$, we say that a sequence of pairs $(c_i, G_i)_{i\geq 1}$, with c_i a non-negative real number and G_i a set in $\mathcal{G}$, is a weighted sequence from $\mathcal{G}$ that majorises f if

$$f \leq \sum_{i=1}^{\infty} c_i \mathcal{X}_{G_i},$$

that is $f(x) \leq \sum\{c_i | x \in G_i\}$ for all x in X. We use $\Upsilon_{\theta\delta}(f;\mathcal{G})$ to denote the family of all weighted sequences from $\mathcal{G}$ that majorise f, with

$$\operatorname{diam} G_i \leq \theta\delta \qquad \text{for } i \geq 1.$$

We define the function P on $C(X)$ by

$$P(f) = \inf\left\{ \sum_{i=1}^{\infty} c_i \xi(G_i) \,\middle|\, (c_i, G_i) \in \Upsilon_{\theta\delta}(f;\mathcal{G}) \right\},$$

for all f in $C(X)$. It is easily seen that P is a positive semilinear functional on $C(X)$.

Now $\lambda^{\xi}_{\theta\delta}(X) \leq P(\mathcal{X}_X)$ since if $(c_i, E_i)_{i\geq 1}$ majorises $\mathcal{X}_X$ then $(c_i, E_i)_{i\geq 1}$ is a weighted cover of X. Also for μ in $\mathrm{M}^{\xi}_{\theta\delta}(X;\mathcal{G})$, $\mu(X) \leq P(\mathcal{X}_X)$ since if $(c_i, G_i)_{i\geq 1}$ is in $\Upsilon_{\theta\delta}(\mathcal{X}_X;\mathcal{G})$ we have

$$\mu(X) = \int \mathcal{X}_X \,\mathrm{d}\mu \leq \sum_{i=1}^{\infty} c_i \int \mathcal{X}_{G_i} \,\mathrm{d}\mu \leq \sum_{i=1}^{\infty} c_i \xi(G_i).$$

This follows from the fact that for $i \geq 1$, G_i is open and thus μ-measurable. Hence by the previous two lemmas if we can prove that there exists μ in $\mathbf{M}^{\xi}_{\theta\delta}(X;\mathcal{G})$ such that $P(\mathcal{X}_X) = \mu(X)$ then the required result follows.

By the Hahn-Banach Theorem there exists a positive linear functional, m^* say, on $C(X)$ such that

$$m^*(\mathcal{X}_X) = P(\mathcal{X}_X)$$

$$m^*(f) \leq P(f) \qquad (\forall f \in C(X).$$

Hence by Riesz's Representation Theorem there exists a positive Borel regular measure on X, μ say, such that for all f in $C(X)$ we have

$$m^*(f) = \int f \,\mathrm{d}\mu.$$

Hence $\mu(X) = P(\mathcal{X}_X)$ and all that remains to be proved is that μ is in $\mathbf{M}^{\xi}_{\theta\delta}(X;\mathcal{G})$.

For G in $\mathcal{G}$ with $\operatorname{diam} G \leq \theta\delta$, we choose a sequence $(f_i)_{i\geq 1}$ of positive functions in $C(X)$ such that

$$\begin{array}{lll} f_i(x) = 0 & & \forall x \in X \setminus G \\ f_i(x) \nearrow 1 & \text{as } i \to \infty & \forall x \in G \end{array}$$

that is $f_i \nearrow \mathcal{X}_G$. It follows that this is possible from the fact that X is a metric space and G is open. Therefore by the Dominated Convergence Theorem

$$\mu(G) = \lim_{i\to\infty} \int f_i \,\mathrm{d}\mu \leq \lim_{i\to\infty} P(f_i) \leq \xi(G).$$

Thus indeed, we have μ in $\mathbf{M}^{\xi}_{\theta\delta}(X;\mathcal{G})$. □

Corollary 6.4 *For X, $\mathcal{G}$ and ξ satisfying the conditions of the above theorem we have*

$$\lambda^{\xi}(X) = \sup\{\mu(X) | \mu \in \mathbf{M}^{\xi}(X)\}. \qquad \dashv$$

Note 6.5 In [7] an example is given of a compact metric space X and a continuous Hausdorff function h where X has infinite Λ^h measure and no subsets of finite positive Λ^h measure. It is proved that there are no finite positive measures on X that are absolutely continuous with respect to Λ^h. Hence by the previous corollary X has zero λ^h measure. This is also proved by direct analysis in [18]. $\dashv$

Remark. Given a premeasure ξ on a metric space X we may define for $\delta > 0$ the measure Ψ_δ^ξ by

$$\Psi_\delta^\xi(S) = \sup\{\mu(S) | \mu \in \mathbf{M}_\delta^\xi(X)\}$$

and then

$$\Psi^\xi(S) = \sup_{\delta>0} \Psi_\delta^\xi(S) = \sup\{\mu(S) | \mu \in \mathbf{M}^\xi(X)\}.$$

It is easily proved that

1. Ψ^ξ is a metric measure,
2. Ψ^ξ satisfies the Radon condition($*$),
3. when ever $(E_i)_{i\geq 1}$ is an increasing sequence of sets with union E we have $\Psi^\xi(E) = \sup_{i\geq 1} \Psi^\xi(E_i)$.

Furthermore we see by the previous corollary that for a compact subset K of X we have $\Psi^\xi(K) = \lambda^\xi(K)$ provided ξ is open regular.

6.3 Capacities.

For X a compact metric space with metric ρ, $\mu \in \mathbf{M}^+(X)$ and h a Hausdorff function we define the h-potential $\psi_\mu^h(x)$ of μ at the point x by

$$\psi_\mu^h(x) = \int_X \frac{\mathrm{d}\mu(y)}{h(\rho(x,y))}\ .$$

The h-capacity $C^h(X)$ of X is defined by

$$C^h(X) = \sup\Big\{\mu(X) \;\Big|\; \mu \in \mathbf{M}^+(X) \text{ and } \psi_\mu^h(x) \le 1 \text{ for all } x \in X\Big\}.$$

Note 6.6 We note that if there exists $\mu \in \mathbf{M}^+(X)$ such that $\mu(X) > 0$ and $\psi_\mu^h(x)$ is bounded for $x \in X$ then by setting

$$c = \sup\{\psi_\mu^h(x) \mid x \in X\}$$

and $\nu = \frac{1}{c}\mu$ we have $C^h(X) \ge \frac{1}{c}\mu(X) > 0$. $\dashv$

Proposition 6.7 *Let h, g be Hausdorff functions such that for some $\varepsilon > 0$ we have*

$$\int_0^\varepsilon \frac{1}{g(t)}\, \mathrm{d}h(2t) < \infty.$$

Suppose ξ is the premeasure on X defined by h and cut off size 2ε and suppose $\mu \in \mathbf{M}_{2\delta}^\xi(X)$ where $\delta \le \varepsilon$. Then $\psi_\mu^g(x)$ is bounded for $x \in X$. $\dashv$

Proof. This is proved in [15]. We give the proof here for completeness. We fix $x \in X$ and define $m(r)$ by $m(r) = \mu(B(x,r))$. By hypothesis $\mu \in \mathbf{M}_{2\delta}^\xi(X)$ and hence $m(r) \le h(2r)$ for $0 \le r \le \delta$. Thus

$$\begin{aligned} \psi_\mu^g(x) &= \int_X \frac{\mathrm{d}\mu(y)}{g(\rho(x,y))} \\ &= \int_0^\infty \frac{1}{g(r)}\, \mathrm{d}m(r) \\ &= \int_0^\delta \frac{1}{g(r)}\, \mathrm{d}m(r) + \int_\delta^\infty \frac{1}{g(r)}\, \mathrm{d}m(r) \\ &\le \int_0^\delta \frac{1}{g(r)}\, \mathrm{d}h(2r) + \frac{\mu(X)}{g(\delta)} \end{aligned}$$

and hence $\psi_\mu^g(x)$ is bounded for $x \in X$, as required. $\square$

Remark. In the case when h and g are of the form $h(r) = r^t$ and $g(r) = r^s$ then the condition that

$$\int_0^{\varepsilon} \frac{1}{g(t)}\, \mathrm{d}h(2t) < \infty$$

merely states that $s < t$ or $s = t = 0$.

Corollary 6.8 *Let X be a compact metric space and h, g be Hausdorff function such that $\lambda^h(X) > 0$ and for some $\varepsilon > 0$ we have*

$$\int_0^{\varepsilon} \frac{1}{g(t)}\, \mathrm{d}h(2t) < \infty.$$

Then $C^g(X) > 0$. $\dashv$

Proof. We let ξ to be the premeasure on X defined by h and cut off size 2ε. By the hypothesis of the corollary we choose $\delta \leq \varepsilon$ such that $\lambda^{\xi}_{2\delta}(X) > 0$. By Theorem 6.3 we choose μ in $\mathrm{M}^{\xi}_{2\delta}(X)$ such that $\mu(X) > 0$. By Proposition 6.7 we have $\psi^g_\mu(x)$ is bounded for $x \in X$, and the result follows by Note 6.6. □

Remark. In [15], S. Kametani shows that for X a Borel subset of $\mathbb{R}^n$ and for h, g Hausdorff functions such that for some $\varepsilon > 0$

$$\int_0^{\varepsilon} \frac{1}{g(t)}\, \mathrm{d}h(2t) < \infty,$$

if $\Lambda^h(X) > 0$ then $C^g(X) > 0$. We also remark that in Section 9.3 we give certain conditions which guarantee $\Lambda^h(X) = \lambda^h(X)$. In this respect Corollary 6.8 may be viewed as a generalisation of Kametani's result.

Remark. There are a number of different definitions of capacity in the previous literature. It is sometimes more convenient to consider the quantities $C^h_*(X)$ defined below, see for example [20] pages 131–133 and [24] pages 109–110. For $\mu \in \mathrm{M}^+(X)$ we define the h-energy $I^h(\mu)$ of μ by

$$I^h(\mu) = \int_X \psi^h_\mu(x)\, \mathrm{d}\mu(x) = \int_X \int_X \frac{\mathrm{d}\mu(x)\, \mathrm{d}\mu(y)}{h(\rho(x, y))},$$

and then $C_*^h(X)$ is defined by

$$C_*^h(X) = \sup\left\{ \frac{1}{I^h(\mu)} \middle| \mu \in \mathrm{M}^+(X) \text{ and } \mu(X) = 1 \right\}.$$

However, if $\mu \in \mathrm{M}^+(X)$ is such that $\mu(X) > 0$ and $\psi_\mu^h(x) \leq 1$ for all $x \in X$ then

$$I^h\left(\frac{\mu}{\mu(X)}\right) = \frac{1}{(\mu(X))^2} \int_X \int_X \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{h(\rho(x,y))} \leq \frac{1}{\mu(X)}.$$

Thus in all cases $C^h(X) \leq C_*^h(X)$. Finally we remark that the definition we give for $C^h(X)$ coincides with the definition of capacity given in [11] and is perhaps a more natural definition for general metric spaces (see Chapter VI § 2 of [20]).

7 Covering Theorems.

7.1 Vitali coverings.

Given a family $\mathcal{E}$ of sets we say that $\mathcal{F}$ is a τ-Vitali subfamily of $\mathcal{E}$ if and only if

1. $\mathcal{F} \subseteq \mathcal{E}$,
2. for all $E \in \mathcal{E}$ there exists $F \in \mathcal{F}$ such that

$$F \cap E \neq \emptyset \quad \text{and} \quad \operatorname{diam} E \leq \tau \operatorname{diam} F.$$

We say that $\mathcal{F}$ is a Vitali subfamily when $\mathcal{F}$ is a 1-Vitali subfamily. We also say that $\mathcal{F}$ is a strong Vitali subfamily when $\mathcal{F}$ is a τ-Vitali subfamily for some $\tau > 1$.

Theorem 7.1 *For $\mathcal{E}$ a family of subsets of X and $\tau > 1$ there exists a disjoint τ-Vitali subfamily of $\mathcal{E}$. Furthermore if $\mathcal{E}$ is finite in the large then there exists a disjoint Vitali subfamily of $\mathcal{E}$.* $\dashv$

Proof. This is proved in 2.8.4 of [10]. We consider the class Ω of all disjoint subfamilies $\mathcal{H}$ with the property: whenever $E \in \mathcal{E}$

$$\begin{array}{ll} \text{either} & E \cap H = \emptyset \quad \text{for all } H \in \mathcal{H}, \\ \text{or} & E \cap H \neq \emptyset \text{ and } \operatorname{diam} E \leq \tau \operatorname{diam} H \quad \text{for some } H \in \mathcal{H}. \end{array}$$

Applying Hausdorff's maximal principle we choose $\mathcal{F} \in \Omega$ so that $\mathcal{F}$ is not a proper subset of any member of Ω.

We define $\mathcal{K}$ by

$$\mathcal{K} = \{E \in \mathcal{E} | \forall F \in \mathcal{F}, E \cap F = \emptyset\}.$$

If $\mathcal{K} \neq \emptyset$ we could select $E \in \mathcal{K}$ so that

$$\tau \operatorname{diam} E \geq \sup\{\operatorname{diam} K | K \in \mathcal{K}\}$$

and verify that $\mathcal{F} \cup \{E\}$ is in Ω, contrary to the maximality of $\mathcal{F}$.

Furthermore if $\mathcal{E}$ is finite in the large then the above arguments hold with τ set at 1. The required results follow. □

Corollary 7.2 *For $\mathcal{F}$ a family of closed subsets of X which is a fine covering of a subset S of X and $\tau > 1$ there exists a disjoint τ-Vitali subfamily $\mathcal{G}$ of $\mathcal{F}$ such that for all finite subfamilies $\mathcal{H}$ of $\mathcal{G}$,*

$$S \subseteq \bigcup \mathcal{H} \cup \bigcup \{\widehat{G}^{\tau} | G \in \mathcal{G} \setminus \mathcal{H}\}.$$

Furthermore if $\mathcal{F}$ is finite in the large then $\mathcal{G}$ can be assumed to be a disjoint Vitali subfamily of $\mathcal{F}$. ⊣

Proof. This is proved in 2.8.6 of [10]. For $\mathcal{F}$ as above we may apply the previous theorem to obtain a maximal family $\mathcal{G}$ of Ω where Ω is defined as above. For $\mathcal{H}$ a finite subfamily of $\mathcal{G}$ we have $\bigcup \mathcal{H}$ is closed. Hence each point $x \in S \setminus \bigcup \mathcal{F}$ belongs to some $F \in \mathcal{F}$ with $F \cap \bigcup \mathcal{H} = \emptyset$. Thus F meets some $G \in \mathcal{G} \setminus \mathcal{H}$ with $\operatorname{diam} F \leq \tau \operatorname{diam} G$ and hence $F \subseteq \widehat{G}^{\tau}$.

Furthermore if $\mathcal{F}$ is finite in the large then the above arguments hold with τ set at 1. The required results follow. □

7.2 Overflow.

For a family $\mathcal{E}$ of subsets of X we define the overflow function $\phi_{\mathcal{E}}$ of $\mathcal{E}$ by

$$\phi_{\mathcal{E}}(x) = \sum_{E \in \mathcal{E}} \mathcal{X}_E(x) - \mathcal{X}_{\bigcup \mathcal{E}}(x).$$

We define the μ-overflow of $\mathcal{E}$ to be

$$\overline{\int} \phi_{\mathcal{E}} \, \mathrm{d}\mu$$

where $\overline{\int}$ denotes the upper integral.

Theorem 7.3 *Let μ be a finite measure on X and $\mathcal{E}$ be a family of sets each of positive measure such that $\mathcal{E}$ is a fine covering of a subset S of X and for all $E \in \mathcal{E}$ we have*

$$\mu(E) = \sup\{\mu(F) | F \subseteq E \text{ and } F \textit{ closed}\}.$$

Then for all $\tau > 1$ there exists a countable τ-Vitali subfamily $\mathcal{G}$ of $\mathcal{E}$ with μ-overflow as small as we wish such that for all finite subfamilies $\mathcal{H}$ of $\mathcal{G}$,

$$S \subseteq \bigcup \mathcal{H} \cup \bigcup\{\widehat{G}^{\tau} | G \in \mathcal{G} \setminus \mathcal{H}\}.$$

Furthermore if $\mathcal{E}$ is finite in the large then $\mathcal{G}$ can be assumed to be a Vitali subfamily of $\mathcal{E}$ with μ-overflow as small as we wish. $\dashv$

Proof. For $\varepsilon > 0$ by hypothesis we may choose a family $\mathcal{F}$ of closed subsets of X indexed by $\mathcal{E}$ such that

$$F_E \subseteq E \quad \text{and} \quad (1+\varepsilon)\mu(F_E) > \mu(E) \qquad (\forall E \in \mathcal{E}).$$

We consider the class Ω of all disjoint subfamilies $\mathcal{H}$ of $\mathcal{E}$ with the property: whenever $E \in \mathcal{E}$

either $\quad E \cap F_H = \emptyset$ for all $H \in \mathcal{H}$,
or $\quad E \cap F_H \neq \emptyset$ and $\operatorname{diam} E \leq \tau \operatorname{diam} H$ for some $H \in \mathcal{H}$.

Applying Hausdorff's maximal principle we choose $\mathcal{G} \in \Omega$ so that $\mathcal{G}$ is not a proper subset of any member of Ω.

We define $\mathcal{K}$ by

$$\mathcal{K} = \{E \in \mathcal{E} | \forall G \in \mathcal{G}, E \cap F_G = \emptyset\}.$$

If $\mathcal{K} \neq \emptyset$ we could select $E \in \mathcal{K}$ so that

$$\tau \operatorname{diam} E \geq \sup\{\operatorname{diam} K | K \in \mathcal{K}\}$$

and verify that $\mathcal{G} \cup \{E\}$ is in Ω, contrary to the maximality of $\mathcal{G}$.

For $\mathcal{H}$ a finite subfamily of $\mathcal{G}$ we have $\bigcup_{H \in \mathcal{H}} F_H$ is closed. Hence each point $x \in S \setminus \bigcup \mathcal{H}$ belongs to some $F \in \mathcal{F}$ with $F \cap \bigcup_{H \in \mathcal{H}} F_H = \emptyset$. Thus F meets F_G for some $G \in \mathcal{G}$ with $\operatorname{diam} F \leq \tau \operatorname{diam} G$ and hence $F \subseteq \widehat{G}^{\tau}$.

It immediately follows from the definitions that

1. $\{F_G | G \in \mathcal{G}\}$ is a disjoint family of closed subsets of X

2. $\mu(F_G) > 0$ for all $G \in \mathcal{G}$

and hence $\mathcal{G}$ is countable since μ is finite. Also we see that the μ-overflow of $\mathcal{G}$ is not greater than

$$\sum_{G \in \mathcal{G}} \mu(G) - \mu\left(\bigcup \mathcal{G}\right) \leq \sum_{G \in \mathcal{G}} \mu(G) - \sum_{G \in \mathcal{G}} \mu(F_G) \leq \varepsilon \sum_{G \in \mathcal{G}} \mu(F_G) \leq \varepsilon \mu(X)$$

and hence can be made as small as we wish by a suitable choice of ε.

Furthermore if $\mathcal{E}$ is finite in the large then the above arguments hold with τ set at 1. The required results follow. □

Corollary 7.4 *For μ a finite Borel measure on X and $\mathcal{E}$ a family of μ-measurable sets each of positive measure which is a fine covering of a subset S of X and $\tau > 1$ there exists a countable τ-Vitali subfamily $\mathcal{G}$ of $\mathcal{E}$ with μ-overflow as small as we wish such that for all finite subfamilies $\mathcal{H}$ of $\mathcal{G}$,*

$$S \subseteq \bigcup \mathcal{H} \cup \bigcup \{\widehat{G}^{\tau} | G \in \mathcal{G} \setminus \mathcal{H}\}.$$

Furthermore if $\mathcal{E}$ is finite in the large then $\mathcal{G}$ can be assumed to be a Vitali subfamily of $\mathcal{E}$ with μ-overflow as small as we wish. ⊣

Proof. This follows immediately from the previous theorem by Note 4.3. □

Corollary 7.5 *For μ a finite regular measure on X and $\mathcal{E}$ a family of open sets each of positive measure which is a fine covering of a subset S of X and $\tau > 1$ there exists a countable τ-Vitali subfamily $\mathcal{G}$ of $\mathcal{E}$ with μ-overflow as small as we wish such that for all finite subfamilies $\mathcal{H}$ of $\mathcal{G}$,*

$$S \subseteq \bigcup \mathcal{H} \cup \bigcup \{\widehat{G}^{\tau} | G \in \mathcal{G} \setminus \mathcal{H}\}.$$

Furthermore if $\mathcal{E}$ is finite in the large then $\mathcal{G}$ can be assumed to be a Vitali subfamily of $\mathcal{E}$ with μ-overflow as small as we wish. ⊣

Proof. This follows immediately from the previous theorem by Note 4.2. □

8 Increasing Sequences of Sets.

8.1 Definitions.

A sequence $(E_i)_{i\geq 1}$ of subsets of X is said to be increasing if $E_i \subseteq E_{i+1}$ for $i \geq 1$. For ξ a premeasure on X, we say that the increasing sets lemma holds for Λ^ξ if for all $0 < \varepsilon < \delta$ and for all increasing sequences $(E_i)_{i\geq 1}$ of subsets of X we have

$$\Lambda^\xi_\delta\left(\bigcup_{i\geq 1} E_i\right) \leq \sup_{i\geq 1} \Lambda^\xi_\varepsilon(E_i).$$

Similarly, we say the increasing sets lemma holds for λ^ξ if

$$\lambda^\xi_\delta\left(\bigcup_{i\geq 1} E_i\right) \leq \sup_{i\geq 1} \lambda^\xi_\varepsilon(E_i).$$

If there exist (finite) constants $\theta > 1$ and $\eta > 1$ such that for all increasing sequences $(E_i)_{i\geq 1}$ of subsets of X and for all $\delta > 0$ we have

$$\Lambda^\xi_{\theta\delta}\left(\bigcup_{i\geq 1} E_i\right) \leq \eta \sup_{i\geq 1} \Lambda^\xi_\delta(E_i)$$

then we say the weak increasing sets lemma holds for Λ^ξ.

8.2 Previous results.

Proposition 8.1 *For X a metric space, ζ a premeasure on X and $\mathcal{E}$ an adequate family of subsets of X such that ζ is $\mathcal{E}$ regular and $\mathcal{E}$ is finite in the large, the increasing sets lemma holds for Λ^ζ and λ^ζ.* ⊣

Proof. See Theorem 1 of [19] and the final *Remark* of [19] for details. A proof for Hausdorff measures may also be found in [4]. □

Corollary 8.2 *Let X be a compact metric space, and let ξ be a premeasure on X which is continuous with respect to the Hausdorff distance and satisfies the condition that*

$$\xi(E) = 0 \implies \operatorname{diam} E = 0 \qquad (\forall E \subseteq X).$$

Then the increasing sets lemma holds for Λ^ξ and λ^ξ. ⊣

Proof. This follows from the previous proposition by the results of [6]. This was originally proved by direct analysis in [25]. □

Theorem 8.3 *Let X be a metric space and h be a continuous Hausdorff function such that one of the following is satisfied*

1. *h is of finite-order;*

2. *X has finite structural dimension;*

3. *X is ultrametric.*

Then the increasing sets lemma holds for Λ^h. ⊣

Proof. This is shown in [5]. □

Theorem 8.4 *Let X be an analytic subset of a complete separable metric space, and let ξ be a premeasure on X which is open regular. Suppose the increasing sets lemma holds for Λ^ξ. Then Λ^ξ satisfies the Radon condition(*). Similarly if the increasing sets lemma holds for λ^ξ then λ^ξ satisfies the Radon condition(*).* ⊣

Proof. This is shown in [26] for Λ^ξ and in [19] for λ^ξ. □

Example 8.5 *There exists a metric space and a continuous Hausdorff function h of finite order such that the Radon condition(*) fails for Λ^h and λ^h.* ⊣

Proof. This is well known in the mathematical 'folklore'. We let X be any uncountable set metrised by the discrete metric and h be the function defined by $h(t) = t$. It is clear that the only compact sets are those sets that contain finitely many points and that

$$\Lambda^h(E) = \lambda^h(E) = \begin{cases} 0 & \text{if } E \text{ is countable;} \\ \infty & \text{otherwise,} \end{cases}$$

for all subsets E of X. $\square$

8.3 Strong finite order.

Theorem 8.6 *For X a metric space and ξ a premeasure of strong finite order, Λ^ξ satisfies the weak increasing sets lemma.* $\dashv$

Proof. Given $\delta > 0$ and an increasing sequence of sets $(E_i)_{i\geq 1}$ we denote $\bigcup_{i=1}^{\infty} E_i$ by E. We wish to show that for some $\theta > 1$ and some $\kappa > 1$

$$\Lambda^\xi_{\theta\delta}(E) \leq \kappa \sup_{i\geq 1} \Lambda^\xi_\delta(E_i)$$

where θ and κ are independent of $(E_i)_{i\geq 1}$ and δ. We may suppose that $\Lambda^\xi_\delta(E_i)$ is bounded. Now if $\Lambda^\xi_\delta(E_i) = 0$ for all $i \geq 1$ then

$$\Lambda^\xi_\delta(E) \leq \sum_{i=1}^{\infty} \Lambda^\xi_\delta(E_i) = 0.$$

Thus we may suppose that $\ell \in \mathbb{R}^+$ is such that

$$\ell = \sup_{i\geq 1} \Lambda^\xi_\delta(E_i).$$

Given $\varepsilon > 0$ we choose δ-covers $(F_{i,j})_{j\geq 1}$ of E_i for each $i \geq 1$ such that

$$\sum_{j=1}^{\infty} \xi(F_{i,j}) \leq \Lambda^\xi_\delta(E_i) + \frac{\varepsilon}{2^i}.$$

Since ξ is of strong finite order there exist $\tau > 1$ and $\eta > 0$ such that $\xi(\widehat{S}^\tau) \leq \eta\xi(S)$ for all subsets S of X. For convenience we use $\mathcal{F}$ to denote the family of sets $\{F_{i,j} | i, j \geq 1\}$. We let $\mathcal{H} = \{\widehat{F}^\tau_{\mathcal{F}} | F \in \mathcal{F}\}$ which is clearly indexed by $\mathcal{F}$. We apply Theorem 7.1 to obtain a maximal disjoint τ-Vitali subfamily $\mathcal{K}$ of $\mathcal{H}$. Hence we may choose a subfamily $\mathcal{G}$ of $\mathcal{F}$ uniquely indexing $\mathcal{K}$; that is for each $K \in \mathcal{K}$ there exists a unique $G \in \mathcal{G}$ such that $K = \widehat{G}^\tau_{\mathcal{F}}$. Clearly

$$E \subseteq \bigcup_{K \in \mathcal{K}} \widehat{K}^\tau \qquad \text{and} \qquad \operatorname{diam} \widehat{K}^\tau \leq (1+2\tau)^2\delta \quad (\forall K \in \mathcal{K}).$$

Hence the result follows with $\theta = (1+2\tau)^2$ and $\kappa = \eta^2$ if we can show that

$$\sum_{G \in \mathcal{G}} \xi(G) \leq \ell + \varepsilon.$$

For $G \in \mathcal{G}$ we define i_G to be the least integer $i \geq 1$ such that $G = F_{i,j}$ for some $j \geq 1$. Given a finite subfamily $\mathcal{E}$ of $\mathcal{G}$ we let n be any integer such that $n > i_G$ for each $G \in \mathcal{E}$. By the disjointness of $\mathcal{K}$ above we have

$$\sum_{G \in \mathcal{E}} \sum_{\substack{F_{n,j} \cap G \neq \emptyset \\ j \geq 1}} \xi(F_{n,j}) \leq \ell + \frac{\varepsilon}{2^n}\,.$$

Now for $k \geq 1$ we let $\mathcal{E}_k = \{G \in \mathcal{E} | i_G = k\}$. Since for all $G \in \mathcal{E}_k$

$$E_k \cap G \subseteq \bigcup \{F_{n,j} | j \geq 1 \text{ and } F_{n,j} \cap G \neq \emptyset\}$$

and $\sum_{j=1}^\infty \xi(F_{k,j}) \leq \Lambda^\xi_\delta(E_k) + \varepsilon/2^k$ we have

$$\sum_{G \in \mathcal{E}_k} \xi(G) \leq \sum_{G \in \mathcal{E}_k} \sum_{\substack{F_{n,j} \cap G \neq \emptyset \\ j \geq 1}} \xi(F_{n,j}) + \frac{\varepsilon}{2^k}\,.$$

Hence

$$\sum_{G \in \mathcal{E}} \xi(G) = \sum_{k=1}^{n-1} \sum_{G \in \mathcal{E}_k} \xi(G) \leq \ell + \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k} + \frac{\varepsilon}{2^n} < \ell + \varepsilon.$$

This completes the proof. $\square$

9 Differentiation.

9.1 Definition.

For a family of sets $\mathcal{E}$, a premeasure ξ and a positive set function μ, we define for each point x in X

$$\mathrm{D}^{\xi}_{\delta}\mu(x;\mathcal{E}) = \sup\left\{ \frac{\mu(E)}{\xi(E)} \middle| x \in E \in \mathcal{E} \text{ and } \operatorname{diam} E \leq \delta \right\}$$

with the convention that

$$\frac{a}{0} = \begin{cases} \infty & \text{if} \quad a > 0; \\ 0 & \text{if} \quad a = 0. \end{cases}$$

The case $a < 0$ is not considered since μ is positive. The differential of μ by ξ with respect to (the basis) $\mathcal{E}$ is then defined by

$$\mathrm{D}^{\xi}\mu(x;\mathcal{E}) = \lim_{\delta \searrow 0} \mathrm{D}^{\xi}_{\delta}\mu(x;\mathcal{E}).$$

In what follows μ will always be a measure. When $\mathcal{E}$ is the family of all subsets of X, we simply write $\mathrm{D}^{\xi}\mu(x)$ for $\mathrm{D}^{\xi}\mu(x;\mathcal{E})$.

9.2 Differentiation of measures by premeasures.

Theorem 9.1 *Let $\mathcal{E}$ be an adequate family of subsets of X, let ξ be a premeasure which is $\mathcal{E}$-regular, and μ be a regular measure. Suppose S is a subset of X and $\mathrm{D}^{\xi}\mu(x;\mathcal{E}) < t$ whenever $x \in S$. Then $\mu(S) \leq t\lambda^{\xi}(S)$.* $\dashv$

Proof. For $\delta > 0$ we define $A(\delta)$ by

$$A(\delta) = \{x \in S | \mathrm{D}^{\xi}_{\delta}\mu(x;\mathcal{E}) < t\}.$$

The hypothesis of the theorem and Note 4.2 imply that

$$S \subseteq \bigcup_{\delta > 0} A(\delta) \quad \text{and} \quad \mu(S) \leq \sup_{\delta > 0} \mu(A(\delta)).$$

Now by the adequacy of $\mathcal{E}$, there exists θ such that for $\delta > 0$ and a weighted δ-cover $(c_i, E_i)_{i\geq 1}$ of $A(\theta\delta)$ we can choose a sequence of sets $(F_i)_{i\geq 1}$ in $\mathcal{E}$ such that $E_i \subseteq F_i$ and $\operatorname{diam} F_i \leq \theta \operatorname{diam} E_i$ for all $i \geq 1$ and $\sum_{i\geq 1} c_i\xi(F_i)$ is as close as we wish to $\sum_{i\geq 1} c_i\xi(E_i)$. Hence by the regularity of μ we have $\mu(A(\theta\delta)) \leq t\lambda_\delta^\xi(A(\theta\delta))$. Thus $\mu(A(\theta\delta)) \leq t\lambda^\xi(S)$. The required result follows. □

Corollary 9.2 *For an adequate family $\mathcal{E}$ of Borel subsets of X, a premeasure ξ which is regular with respect to $\mathcal{E}$, a subset S of X, and $\mu = \Lambda^\xi$ or $\mu = \lambda^\xi$ if $\mu(S) < \infty$ then*

$$\mathrm{D}^\xi(\mu \llcorner S)(x;\mathcal{E}) \geq 1 \qquad \text{for } \mu\text{-almost all } x \text{ in } S. \qquad \dashv$$

Proof. For $t \in (0,1)$ we define $A(t)$ by

$$A(t) = \{x \in S | \mathrm{D}^\xi(\mu \llcorner S)(x;\mathcal{E}) < t\}.$$

By Note 4.7 and the hypothesis of the corollary μ is a Borel measure. Hence if $\mu(S) < \infty$ we may choose a Borel set B such that $S \subseteq B$ and $\mu(S) = \mu(B)$. Thus by Notes 4.5 and 4.2 $\mu \llcorner B$ is a regular measure and hence we may apply the previous theorem to $\mu \llcorner B$ to deduce $\mu(A(t)) \leq t\lambda^\xi(A(t))$. The required result follows. □

Remark. In the above if X is separable then the condition that $\mu(S) < \infty$ may be removed.

Theorem 9.3 *Let $S \subseteq X$, let $\mathcal{E}$ be a family of subsets of X which is a fine covering of S, let $\tau > 1$, let ξ be a premeasure of finite τ-order, and let μ be a finite measure on X such that for all $E \in \mathcal{E}$*

$$\mu(E) = \sup\{\mu(F) | F \subseteq E \text{ and } F \text{ closed}\}.$$

Suppose $\mathrm{D}^\xi\mu(x;\mathcal{E}) > t$ whenever $x \in S$. Then for all open sets U containing S we have $\mu(U) \geq t\Lambda^\xi(S)$. Furthermore if $\mathcal{E}$ is finite in the large then τ may be set at one in the above. $\dashv$

Proof. For t as above, and positive δ it is immediate that

$$\mathcal{F} = \{E \in \mathcal{E} | \mu(E) > t\xi(E),\ E \subseteq U \text{ and } \operatorname{diam} E \leq \delta\}$$

is a fine cover of S. Hence by Theorem 7.3 for all positive ε there exists a countable τ-Vitali subfamily $\mathcal{G}$ of $\mathcal{F}$ with μ-overflow less than ε such that for all finite subfamilies $\mathcal{H}$ of $\mathcal{G}$,

$$S \subseteq \bigcup \mathcal{H} \cup \bigcup \{\widehat{G}^\tau | G \in \mathcal{G} \setminus \mathcal{H}\}.$$

Hence we may choose a finite subfamily $\mathcal{H}$ of $\mathcal{G}$ such that $t\sum_{G\in\mathcal{G}\setminus\mathcal{H}} \xi(G) \leq \varepsilon$. By the finite order of ξ there exists $\eta > 0$ such that

$$\xi(\widehat{E}^\tau) \leq \eta\xi(E) \qquad (\forall E \subseteq X).$$

Thus we may calculate

$$\begin{aligned}
t\Lambda_\delta^\xi(S) &\leq t\sum_{H\in\mathcal{H}} \xi(H) + t\sum_{G\in\mathcal{G}\setminus\mathcal{H}} \xi(\widehat{G}^\tau) \\
&\leq t\sum_{G\in\mathcal{G}} \xi(G) + (\eta - 1)\varepsilon \\
&\leq \sum_{G\in\mathcal{G}} \mu(G) + (\eta - 1)\varepsilon \\
&\leq \mu\left(\bigcup \mathcal{G}\right) + \eta\varepsilon \\
&\leq \mu(U) + \eta\varepsilon.
\end{aligned}$$

Hence on letting first ε and then δ tend to zero we see that $\mu(U) \geq t\Lambda^\xi(S)$.

Furthermore if $\mathcal{E}$ is finite in the large then the above arguments hold with τ set at 1. The required results follow. □

Corollary 9.4 *For a family $\mathcal{E}$ of subsets of X which is a fine covering of X, $\tau > 1$, a premeasure ξ of finite τ-order, a finite Borel regular measure μ and a μ-measurable subset S of X if one of the following holds*

1. *E is μ-measurable for all $E \in \mathcal{E}$*

2. *ξ is Borel regular*

then $\mathrm{D}^{\xi}(\mu \llcorner S)(x;\mathcal{E}) = 0$ *for Λ^{ξ}-almost all $x \in X \setminus S$. Furthermore if $\mathcal{E}$ is finite in the large then τ can be set at* 1. $\dashv$

Proof. For $t > 0$ we define $A(t)$ by

$$A(t) = \{x \in X \setminus S | \mathrm{D}^{\xi}(\mu \llcorner S)(x;\mathcal{E}) > t\}.$$

We let $\mathcal{A}(t)$ be the family of sets

$$\{E \in \mathcal{E} | E \cap A(t) \neq \emptyset \text{ and } \mu(E \cap S) > t\xi(E)\}.$$

We can assume that each set $E \in \mathcal{A}(t)$ is μ-measurable since if ξ is Borel regular we can replace each set E by a Borel set B such that $E \subseteq B$, $\operatorname{diam} E = \operatorname{diam} B$ and $\xi(E) = \xi(B)$. By the hypothesis of the corollary and Note 4.3 if $\Lambda^{\xi}(A(t)) > 0$ then there exists a closed set F of X such that $\mu(S \setminus F) < t\Lambda^{\xi}(A(t))$ but then the previous theorem with μ, U replaced by $\mu \llcorner S$, $X \setminus F$ would imply the opposite inequality $\mu(S \setminus F) \geq t\Lambda^{\xi}(A(t))$. The required results follow by noting that if $\mathcal{E}$ is finite in the large then the above arguments hold with τ set at 1. $\square$

Corollary 9.5 *Let $S \subseteq X$, let $\mathcal{E}$ be a family of Borel subsets of X which is a fine covering of S, let $\tau > 1$, and let ξ be a Borel regular premeasure of finite τ-order. Suppose $\mu = \Lambda^{\xi}$ or $\mu = \lambda^{\xi}$, and suppose $\mu(S) < \infty$. Then*

$$0 \leq \mathrm{D}^{\xi}(\mu \llcorner S)(x;\mathcal{E}) \leq 1 \qquad \textit{for } \Lambda^{\xi}\textit{-almost all } x \in S.$$

Hence if S is also μ-measurable, $\mathcal{E}$ is also adequate on X and ξ is also $\mathcal{E}$-regular then we have

$$\mathrm{D}^{\xi}(\mu \llcorner S)(x;\mathcal{E}) = \begin{cases} 1 & \textit{for } \mu\textit{-almost all } x \in S; \\ 0 & \textit{for } \Lambda^{\xi}\textit{-almost all } x \notin S. \end{cases}$$

Furthermore if $\mathcal{E}$ is finite in the large then τ can be set at 1. $\dashv$

Proof. By Note 4.7 and the hypothesis of the corollary μ is a Borel measure. Hence if $\mu(S) < \infty$ then we may choose a Borel set B such that $S \subseteq B$ and $\mu(S) = \mu(B)$. Thus for $E \in \mathcal{E}$ and $\varepsilon > 0$ we can choose a closed set $F \subseteq E$ such that $(\mu \llcorner B)(E \setminus F) \leq \varepsilon$. Hence $(\mu \llcorner S)(E \setminus F) \leq \varepsilon$ and thus by the μ-measurability of F we have

$$\begin{aligned}(\mu \llcorner S)(E) = \mu(E \cap S) &= \mu(F \cap S) + \mu(E \cap S \setminus F) \\ &\leq (\mu \llcorner S)(F) + \varepsilon.\end{aligned}$$

For $t > 1$ we define $A(t)$ by

$$A(t) = \{x \in S | \mathrm{D}^{\xi}(\mu \llcorner S)(x; \mathcal{E}) > 1\}$$

Thus by the previous theorem for all open sets U containing $A(t)$ we have

$$(\mu \llcorner S)(U) \geq t\Lambda^{\xi}(A(t)).$$

Hence if $\Lambda^{\xi}(A(t)) > 0$ we could find an open set U containing $A(t)$ such that $(\mu \llcorner B)(U) < t\Lambda^{\xi}(A(t))$, since $(\mu \llcorner B)(A(t)) \leq \Lambda^{\xi}(A(t))$. Thus we would gain a contradiction.

Hence by Corollary 9.2 if $\mathcal{E}$ is also adequate on X and ξ is also $\mathcal{E}$-regular then

$$\mathrm{D}^{\xi}(\mu \llcorner S)(x; \mathcal{E}) = 1 \qquad \text{for } \mu\text{-almost all } x \in S.$$

Also if S is μ-measurable and $\mathcal{E}$ is adequate on X and hence a fine covering of X then by Corollary 9.4

$$\mathrm{D}^{\xi}(\mu \llcorner S)(x; \mathcal{E}) = 0 \qquad \text{for } \Lambda^{\xi}\text{-almost all } x \notin S.$$

The required results follow by noting that if $\mathcal{E}$ is finite in the large then the above arguments hold with τ set at 1. □

9.3 Equality of the Hausdorff and the weighted Hausdorff measures.

Lemma 9.6 *For X a metric space and ξ a Borel regular premeasure on X of finite-order if for all subsets S of X we have*

$$\lambda^\xi(S) = 0 \Longleftrightarrow \Lambda^\xi(S) = 0$$

then $\Lambda^\xi = \lambda^\xi$. $\dashv$

Proof. For E a subset of X if $\lambda^\xi(E) = \infty$ then $\Lambda^\xi(E) = \infty$. If $\lambda^\xi(E) = 0$ then by hypothesis $\Lambda^\xi(E) = 0$. Hence we may assume that E has finite positive λ^ξ-measure. If there exists a Borel set B containing E such that

$$\lambda^\xi(E) = \lambda^\xi(B) = \Lambda^\xi(B)$$

then it is immediate that $\lambda^\xi(E) = \Lambda^\xi(E)$. Hence we may assume that E is Borel.

For $\tau > 0$ we let $(\delta_j)_{j\geq 1}$ be a decreasing sequence of positive numbers tending to zero. For each $j \geq 1$ we choose a weighted δ_j-cover $(c_{i,j}, E_{i,j})_{i\geq 1}$ of E with $E_{i,j}$ a Borel set such that

$$\sum_{i=1}^{\infty} c_{i,j}\xi(E_{i,j}) \leq \lambda^\xi_{\delta_j}(E) + \delta_j.$$

For the sake of notation we let $\mathcal{E}_j$ be the set $\{E_{i,j} | i \geq 1\}$ and $\mathcal{F}_j$ be the set

$$\{E_{i,j} \in \mathcal{E}_j | (1+\tau)\lambda^\xi(E_{i,j} \cap E) \leq \xi(E_{i,j})\}.$$

We calculate

$$\begin{aligned}
\lambda^\xi_{\delta_j}(E) &\leq \sum_{E_{i,j}\in\mathcal{F}_j} c_{i,j}\lambda^\xi_{\delta_j}(E_{i,j} \cap E) + \sum_{E_{i,j}\in\mathcal{E}_j\setminus\mathcal{F}_j} c_{i,j}\xi(E_{i,j}) \\
&\leq \sum_{E_{i,j}\in\mathcal{F}_j} c_{i,j}\lambda^\xi(E_{i,j} \cap E) + \sum_{E_{i,j}\in\mathcal{E}_j\setminus\mathcal{F}_j} c_{i,j}\xi(E_{i,j})
\end{aligned}$$

$$\leq \sum_{E_{i,j}\in\mathcal{E}_j} c_{i,j}\xi(E_{i,j}) \;-\; \tau \sum_{E_{i,j}\in\mathcal{F}_j} c_{i,j}\lambda^{\xi}(E_{i,j}\cap E)$$

$$\leq \lambda^{\xi}_{\delta_j}(E) \;+\; \delta_j \;-\; \tau \sum_{E_{i,j}\in\mathcal{F}_j} c_{i,j}\lambda^{\xi}(E_{i,j}\cap E)$$

Hence

$$\tau \sum_{E_{i,j}\in\mathcal{F}_j} c_{i,j}\lambda^{\xi}(E_{i,j}\cap E) \leq \delta_j.$$

We let F_j be the set

$$\{x \in E | \forall E_{i,j} \in \mathcal{E}_j, x \in E_{i,j} \Longrightarrow (1+\tau)\lambda^{\xi}(E_{i,j}\cap E) \leq \xi(E_{i,j})\}.$$

By the above we have $\tau\lambda^{\xi}(F_j) \leq \delta_j$. Hence $F = \bigcup_{n\geq 1}\bigcap_{j\geq n} F_j$ is a Borel set and has zero λ^{ξ}-measure.

In a similar fashion we choose N_j so that

$$\sum_{i=N_j}^{\infty} c_{i,j}\xi(E_{i,j}) \leq \delta_j$$

and let G_j be the set $E \setminus \bigcup_{i=1}^{N_j} E_{i,j}$. It is immediate that $G = \bigcup_{n\geq 1}\bigcap_{j\geq n} G_j$ is a Borel set and has zero λ^{ξ}-measure.

We let H be the set $E \setminus (F \cup G)$. We let $\mathcal{E}$ be the family of sets

$$\{E_{i,j} | E_{i,j} \notin \mathcal{F}_j, 1 \leq i \leq N_j \text{ and } j \geq 1\}.$$

Since E is a Borel set and has finite λ^{ξ}-measure and λ^{ξ} is a Borel measure it is clear that the measure $\mu = \lambda^{\xi} \llcorner E$ is a finite Borel measure. Also by the construction we have

$$\mathrm{D}^{\xi}\mu(x,\mathcal{E}) > \frac{1}{1+\tau} \qquad (\forall x \in H)$$

where $\mathcal{E}$ is finite in the large. Hence by Theorem 7.3 we have for all open sets U containing H

$$\mu(U) = \lambda^{\xi}(H) \geq \frac{1}{1+\tau}\Lambda^{\xi}(H)$$

and thus on letting τ tend to zero we have $\lambda^{\xi}(H) = \Lambda^{\xi}(H)$. The result follows since by hypothesis F and G have zero Λ^{ξ}-measure. $\square$

Theorem 9.7 *Let X be a metric space and ξ be a premeasure on X of finite-order. Suppose the weak increasing sets lemma holds for Λ^{ξ}. Then for some positive constant η we have $\Lambda^{\xi} \leq \eta\lambda^{\xi}$, and hence in the case when ξ is also Borel regular we have $\Lambda^{\xi} = \lambda^{\xi}$.* $\dashv$

Proof. First suppose that for E a subset of X and $\delta > 0$ there exists a (finite) weighted δ-cover $(c_i, E_i)_{i=1}^{n}$ of E such that

$$\sum_{i=1}^{n} c_i \xi(E_i) < \ell$$

for some $\ell > 0$. Then we may approximate the c_i by rationals with a common denominator $a_i/N > c_i$ so that

$$\sum_{i=1}^{n} \frac{a_i}{N} \xi(E_i) \leq \ell$$

For the sake of notation we let $\mathcal{F}$ be the family of sets $\{E_i | 1 \leq i \leq n\}$. We consider a multiplicity function m_0 mapping $\mathcal{F}$ and to the non-negative integers defined by

$$m_0(F) = \sum_{E_i = F} a_i,$$

for all $F \in \mathcal{F}$. We define for $1 \leq j \leq N$ subfamilies $\mathcal{G}_j$ of $\mathcal{F}$ and functions m_j mapping $\mathcal{F}$ to the non-negative integers by

$$\mathcal{G}_j \text{ a maximal disjoint Vitali subfamily of } \{F \in \mathcal{F} | m_{j-1}(F) \geq 1\}$$

$$m_j(F) = \begin{cases} m_{j-1}(F) - 1 & \text{whenever } F \in \mathcal{G}_j \\ m_{j-1}(F) & \text{whenever } F \notin \mathcal{G}_j \end{cases}$$

The existence of $\mathcal{G}_j$ is implied by Theorem 7.1. Since $(a_i/N, E_i)_{i=1}^n$ is a weighted cover of E we have

$$E \subseteq \bigcup_{G \in \mathcal{G}_j} \widehat{G} \qquad (1 \leq j \leq N).$$

Thus by the finite order of ξ we have

$$\begin{aligned} N\Lambda^{\xi}_{3\delta}(E) &\leq \sum_{j=1}^{N} \sum_{G \in \mathcal{G}_j} \xi(\widehat{G}) \\ &\leq \eta \sum_{j=1}^{N} \sum_{F \in \mathcal{F}} (m_{j-1} - m_j)\xi(F) \\ &\leq \eta \sum_{i=1}^{n} a_i \xi(E_i) \\ &\leq N\eta\ell. \end{aligned}$$

where $\eta > 0$ is dependent only on ξ.

For $\delta > 0$, $\tau > 1$ and $(c_i, E_i)_{i \geq 1}$ a weighted δ-cover of a subset E of X we define H_n by

$$H_n = \left\{ x \in E \,\middle|\, \sum_{i=1}^{n} \tau c_i \mathcal{X}_{E_i} \geq 1 \right\}.$$

Thus if the weak increasing sets lemma holds for Λ^{ξ} then for some positive constants $\theta > 1$ and $\kappa > 1$ independent of E, $(c_i, E_i)_{i \geq 1}$ and δ

$$\Lambda^{\xi}_{3\theta\delta}(E) \leq \kappa \sup_{n \geq 1} \Lambda^{\xi}_{3\delta}(H_n) \leq \tau\eta\kappa\lambda^{\xi}_{\delta}(E).$$

Hence the required results follow on letting first τ tend to 1 and then δ tend to zero and the previous lemma. □

Corollary 9.8 *For X a metric space and ξ a premeasure on X of strong finite order there exists a positive constant η such that $\Lambda^{\xi} \leq \eta\lambda^{\xi}$ and hence in the case when ξ is also Borel regular we have $\Lambda^{\xi} = \lambda^{\xi}$.* ⊣

Proof. This follows immediately from the previous theorem and Theorem 8.6 and in the case when ξ is Borel regular Lemma 9.6. □

Corollary 9.9 *For X a compact metric space and ξ a premeasure on X of finite order there exists a positive constant η such that $\Lambda^{\xi} \leq \eta\lambda^{\xi}$ and hence in the case when ξ is also Borel regular we have $\Lambda^{\xi} = \lambda^{\xi}$.* ⊣

Proof. By Proposition 5.1 there exists a family of open sets $\mathcal{G}$ such that $\mathcal{G}$ is adequate and finite in the large and there exists a premeasure ζ on X, which is of finite-order and $\mathcal{G}$-regular such that for all $\delta > 0$

$$\Lambda_{\delta}^{\xi} \leq \Lambda_{\delta}^{\zeta} \leq \eta\Lambda_{\delta}^{\xi}$$

$$\lambda_{\delta}^{\xi} \leq \lambda_{\delta}^{\zeta} \leq \eta\lambda_{\delta}^{\xi}$$

where η is some positive constant only dependent on ξ. By Proposition 8.1 the increasing sets lemma holds for Λ^{ζ}. Thus by the previous theorem we have $\Lambda^{\zeta} = \lambda^{\zeta}$ and hence $\Lambda^{\xi} \leq \eta\lambda^{\xi}$. In the case when ξ is Borel regular we can apply Lemma 9.6. The required results follow. □

Corollary 9.10 *Let X be a metric space and ξ be a Borel regular premeasure on X of finite-order. Suppose Λ^{ξ} satisfies the Radon condition(*). Then $\Lambda^{\xi} = \lambda^{\xi}$.* ⊣

Proof. For a subset E of X we may choose (by the Borel regularity of ξ and Note 4.7) a Borel set B containing E and of the same λ^{ξ}-measure. Hence if Λ^{ξ} satisfies the Radon condition(*) then by the preceding corollary we see that

$$\begin{aligned}\Lambda^{\xi}(E) \leq \Lambda^{\xi}(B) &= \sup\{\Lambda^{\xi}(K) | K \subseteq B \text{ and } K \text{ compact}\} \\ &\leq \sup\{\eta\lambda^{\xi}(K) | K \subseteq B \text{ and } K \text{ compact}\} \\ &\leq \eta\lambda^{\xi}(B) = \eta\lambda^{\xi}(E)\end{aligned}$$

where $\eta > 0$ is dependent only on ξ. The required result follows. □

10 Dimension and Product Spaces.

10.1 Dimension.

We now make some definitions in order to discuss dimension. We recall that **H** denotes the set of all Hausdorff functions. We now define the Hausdorff–Besicovitch dimension of a metric space (X, ρ) to be the supremum of all non-negative s for which $\Lambda^{h(s)}(X) > 0$, where $h(s)$ is defined on all non-negative t by $h(s)(t) = t^s$. We denote the Hausdorff–Besicovitch dimension of X (with respect to the metric ρ) by $\dim(X, \rho)$.

Note 10.1 By Theorem 8.3, for any metric space (X, ρ), the increasing sets lemma holds for $\Lambda^{h(s)}$. Thus by Theorem 9.7 the Hausdorff–Besicovitch dimension is equal to the supremum of all non-negative s for which $\lambda^{h(s)}(X) > 0$. ⊣

We may partition **H**, for a metric space (X, ρ), into two sets $B^0(X, \rho)$ and $B^+(X, \rho)$, by

$$B^0(X,\rho) = \{h \in \mathbf{H} | \Lambda^h(X) = 0\},$$

$$B^+(X,\rho) = \{h \in \mathbf{H} | \Lambda^h(X) > 0\}.$$

We call the partition $\{B^0(X,\rho), B^+(X,\rho)\}$ the generalised Hausdorff–Besicovitch dimension (of (X, ρ)).

In a similar fashion we may partition **H**, for the metric space (X, ρ), into two sets $W^0(X, \rho)$ and $W^+(X, \rho)$, by

$$W^0(X,\rho) = \{h \in \mathbf{H} | \lambda^h(X) = 0\},$$

$$W^+(X,\rho) = \{h \in \mathbf{H} | \lambda^h(X) > 0\}.$$

Similarly we call the partition $\{W^0(X,\rho), W^+(X,\rho)\}$ the generalised weighted Hausdorff–Besicovitch dimension (of (X, ρ)).

Note 10.2 It is noted that the two partitions

$$\{B^0(X,\rho), B^+(X,\rho)\} \quad \text{and} \quad \{W^0(X,\rho), W^+(X,\rho)\}$$

may differ, as is shown by the example in [7] (see Note 6.5). ⊣

Note 10.3 For X a metric space metrised by ρ and τ, if $\rho \leq \tau$ then

$$\dim(X,\rho) \leq \dim(X,\tau)$$

and

$$W^+(X,\rho) \subseteq W^+(X,\tau)$$

and if there exist positive constants η_1 and η_2 such that

$$\eta_1\rho \leq \tau \leq \eta_2\rho$$

then

$$\dim(X,\rho) = \dim(X,\tau).$$ ⊣

Given a metric space X, we denote by $\mathbf{0}$ the set consisting solely of the trivial measure assigning the value 0 to every subset of X.

Proposition 10.4 *For X a compact metric space with metric ρ we have the following characterisation of dimension*

$$\dim(X,\rho) = \sup\{s \geq 0 | \mathbf{M}^{h(s)}(X) \neq \mathbf{0}\}$$

and similarly

$$W^0(X,\rho) = \{h \in \mathbf{H} | \mathbf{M}^h(X) = \mathbf{0}\},$$

$$W^+(X,\rho) = \{h \in \mathbf{H} | \mathbf{M}^h(X) \neq \mathbf{0}\}.$$ ⊣

Proof. This is immediate from the definitions of dimension, Note 3.2 and Theorem 6.3. □

Note 10.5 The condition that $\mathrm{M}^h(X) \neq \mathbf{0}$ above merely states that for some $\delta > 0$, there exists a non-trivial Borel measure μ such that for every subset E of X with $\operatorname{diam} E \leq \delta$ we have $\mu(E) \leq h(\operatorname{diam} E)$. ⊣

Proposition 10.6 *For any metric space X and premeasure ξ on X which is regular with respect to the family of all open sets, $\mathrm{M}^\xi(X) \neq \mathbf{0}$ if and only if there exists a finite non-trivial Borel measure μ such that for some constant k the set*

$$\left\{x \in X \,\middle|\, \mathrm{D}^\xi\mu(x) \leq k\right\}$$

has non-zero μ-measure. ⊣

Proof. Now suppose $\mathrm{M}^\xi(X) \neq \mathbf{0}$ and hence by definition there exists positive δ and a finite non-trivial Borel measure μ such that for every subset E of X with $\operatorname{diam} E \leq \delta$, $\mu(E) \leq \xi(E)$. Thus indeed for every x in X we have $\mathrm{D}^\xi\mu(x) \leq 1$.

Conversely suppose μ is a finite non-trivial Borel measure with the set

$$\left\{x \in X \,\middle|\, \mathrm{D}^\xi\mu(x) \leq k\right\}$$

of non-zero μ-measure, for some fixed k. We define the functions f_n, mapping X into the non-negative extended reals, by

$$f_n(x) = \sup\left\{\frac{\mu(E)}{\xi(E)} \,\middle|\, x \in E \text{ and } \operatorname{diam} E < \frac{1}{n}\right\}$$

for all positive integers n. It is clear that

$$\left\{x \in X \,\middle|\, \mathrm{D}^\xi\mu(x) \leq k\right\} \subseteq \bigcup_{n=1}^{\infty} f_n^{-1}([0, 2k])$$

and hence we may choose N so large that $f_N^{-1}([0, 2k])$ is of non-zero μ-measure. Thus for all subsets E of X with diameter not greater than $\frac{1}{N+1}$,

$$(\mu \llcorner f_n^{-1}([0, 2k]))(E) \leq 2k\xi(E)$$

since if E has non-empty intersection with $f_n^{-1}([0, 2k])$ then

$$\mu(E) \leq 2k\xi(E)$$

and otherwise E has zero $(\mu \llcorner f_n^{-1}([0, 2k]))$-measure. Using the regularity of ξ one sees that $f_n^{-1}([0, 2k])$ is closed and hence μ-measurable. Thus, by Note 4.5, $\mu \llcorner f_n^{-1}([0, 2k])$ is a finite non-trivial Borel measure. The required result follows on noting that $\frac{1}{2k}\mu \llcorner f_n^{-1}([0, 2k])$ is in $\mathbf{M}_\delta^\xi(X)$ for $\delta \leq \frac{1}{1+N}$. □

10.2 Definitions for Product Spaces.

For two metric spaces X and Y, with metrics ρ and σ respectively, there are many ways to metrise the space $X \times Y$. We may define the metric, $\rho \times \sigma$ say, on $X \times Y$ by

$$(\rho \times \sigma)((x, y), (w, z)) = \max\{\rho(x, w), \sigma(y, z)\}$$

for any pairs (x, y) and (w, z) in $X \times Y$. A metric v on $X \times Y$ is termed a strict product metric if and only if for some constant η,

$$\rho \times \sigma \leq v \leq \eta(\rho \times \sigma),$$

and is termed a weak product metric if and only if for some constant η,

$$\eta(\rho \times \sigma) \leq v.$$

We reserve the term product metric for v when

$$\eta_1(\rho \times \sigma) \leq v \leq \eta_2(\rho \times \sigma)$$

with η_1, η_2 positive constants. The idea of the above terminology is that 'strict' refers to η_1 set at 1 and that 'weak' refers to the validity of the left hand inequality only. Throughout the remainder of this section we denote by p_X and p_Y the projections of $X \times Y$ onto X and Y respectively.

Note 10.7 For X, Y metric spaces, with metrics ρ, σ respectively and $X \times Y$ with metric υ, if $\eta_1(\rho \times \sigma) \leq \upsilon$ for some constant η_1 then we have for all subsets G of $X \times Y$,

$$\eta_1 \max(\operatorname{diam}_\rho p_X(G), \operatorname{diam}_\sigma p_Y(G)) \leq \operatorname{diam}_\upsilon G.$$

Also for $E \subseteq X$ and $F \subseteq Y$ if $\upsilon \leq \eta_2(\rho \times \sigma)$ for some constant η_2 then we have

$$\operatorname{diam}_\upsilon (E \times F) \leq \eta_2 \max(\operatorname{diam}_\rho E, \operatorname{diam}_\sigma F).$$

It follows that if υ is a product metric then the topology generated by υ is precisely the topology generated by $\rho \times \sigma$; that is

$$\mathcal{T}(X \times Y, \upsilon) = \mathcal{T}(X \times Y, \rho \times \sigma)$$

where $\mathcal{T}(X, \rho)$ denotes the topology on X generated by the metric ρ. ⊣

For $\mathcal{E}$, $\mathcal{F}$ any families of subsets of X, Y respectively we define $\mathcal{E} \otimes \mathcal{F}$ to be the family of sets

$$\{E \times F | E \in \mathcal{E} \text{ and } F \in \mathcal{F}\}.$$

Note 10.8 Given a product metric on $X \times Y$, if $\mathcal{E}$ is adequate on X and $\mathcal{F}$ is adequate on Y then $\mathcal{E} \otimes \mathcal{F}$ is adequate on $X \times Y$. ⊣

For premeasures ξ and ζ on X and Y we may define the product premeasure $\xi \bowtie \zeta$ on $X \times Y$ by

$$(\xi \bowtie \zeta)(G) = \xi(p_X(G))\zeta(p_Y(G)),$$

for all subsets G of $X \times Y$.

Note 10.9 We reserve the symbol $\times$ for the product measure since measures may also be considered as premeasures and in this case we have in general the product premeasure different to the product measure. ⊣

Note 10.10 For all subsets G of $X \times Y$ we have

$$(\xi \bowtie \zeta)(G) = \inf\{\xi(E)\zeta(F) | E \subseteq X, F \subseteq Y \text{ and } S \subseteq E \times F\}.$$

It immediately follows that if ξ is $\mathcal{E}$-regular and ζ is $\mathcal{F}$-regular then $\xi \bowtie \zeta$ is $\mathcal{E} \otimes \mathcal{F}$-regular. $\dashv$

Throughout the rest of this section we employ the convention of

$$0.\infty = \infty.0 = 0.$$

For measures μ and ν on X and Y we may define the measure $\mu \times \nu$ on $X \times Y$ by

$$(\mu \times \nu)(G) = \inf\left\{ \sum_{i=1}^{\infty} \mu(E_i)\nu(F_i) \middle| G \subseteq \bigcup_{i=1}^{\infty} E_i \times F_i \right\}.$$

Note 10.11 It is clear that, for E and F subsets of X and Y respectively,

$$(\mu \times \nu)(E \times F) \leq \mu(E)\nu(F),$$

and $\mu \times \nu$ is the largest measure satisfying this inequality. As is well known, if E is μ-measurable and F is ν-measurable then $E \times F$ is $(\mu \times \nu)$-measurable and if μ is $\mathcal{E}$-regular and ν is $\mathcal{F}$-regular then $\mu \times \nu$ is $(\mathcal{E} \otimes \mathcal{F})_\sigma$-regular. Furthermore if μ satisfies the condition that for every set S and weighted cover $(c_i, E_i)_{i \geq 1}$ of S we have

$$\mu(S) \leq \sum_{i=1}^{\infty} c_i \mu(E_i),$$

then for S, T subsets of X, Y respectively

$$(\mu \times \nu)(S \times T) = \mu(S)\nu(T),$$

and similarly for ν. Now if μ is a regular measure then μ satisfies the condition that

$$\mu(S) \leq \sum_{i=1}^{\infty} c_i \mu(E_i),$$

for every set S and weighted cover $(c_i, E_i)_{i\geq 1}$ of S. Thus if $X \times Y$ is metrised by a product metric and μ and ν are Borel then so is $\mu \times \nu$. $\dashv$

10.3 Hausdorff and weighted Hausdorff measures on Product spaces.

Proposition 10.12 *For X, Y metric spaces and premeasures ξ on X and ζ on Y, if $X \times Y$ is metrised by a product metric then*

$$\mathbf{M}^{\xi}(X) \otimes \mathbf{M}^{\zeta}(Y) \subseteq \mathbf{M}^{\xi \bowtie \zeta}(X \times Y),$$

where $\mathbf{M}^{\xi}(X) \otimes \mathbf{M}^{\zeta}(Y)$ is the set

$$\{\mu \times \nu | \mu \in \mathbf{M}^{\xi}(X) \text{ and } \nu \in \mathbf{M}^{\zeta}(Y)\}. \qquad \dashv$$

Proof. Suppose ρ and σ are the metrics on X and Y respectively and υ is a product metric on $X \times Y$. For μ in $\mathbf{M}^{\xi}(X)$ and ν in $\mathbf{M}^{\zeta}(Y)$ we have μ in $\mathbf{M}^{\xi}_{\delta}(X)$ and ν in $\mathbf{M}^{\zeta}_{\varepsilon}(Y)$ for some positive δ and ε. Thus by Note 10.7 there exists a positive constant η such that for all subsets G of $X \times Y$ with $\operatorname{diam}_{\upsilon} G \leq \eta \min(\delta, \varepsilon)$ we see that

$$\operatorname{diam}_{\rho} p_X(G), \operatorname{diam}_{\sigma} p_Y(G) \leq \frac{1}{\eta} \operatorname{diam}_{\upsilon} G \leq \delta, \varepsilon.$$

Hence we may calculate

$$\begin{aligned} (\mu \times \nu)(G) &\leq (\mu \times \nu)(p_X(G) \times p_Y(G)) \\ &\leq \mu(p_X(G))\nu(p_Y(G)) \\ &\leq \xi(p_X(G))\zeta(p_Y(G)) = (\xi \bowtie \zeta)(G). \end{aligned}$$

The required result follows by Note 10.11. $\square$

Lemma 10.13 *For X, Y metric spaces and Hausdorff functions h, g, if $X \times Y$ is metrised by a strict product metric then*

$$\lambda^{h \bowtie g} \leq \lambda^{hg},$$

where $h \bowtie g$ is the product premeasure of a premeasure defined by h on X and a premeasure defined by g on Y. ⊣

Proof. This result is given by Lemma 5 of [18]. It follows immediately from the fact that for all subsets G of $X \times Y$,

$$h(\operatorname{diam}_\rho p_X(G))g(\operatorname{diam}_\sigma p_Y(G)) \leq (hg)(\operatorname{diam}_v G). \qquad \square$$

Lemma 10.14 *For X, Y metric spaces and premeasures ξ on X and ζ on Y, if $X \times Y$ is metrised by a strict product metric then*

$$(\lambda^\xi \times \lambda^\zeta)(S \times T) = \sup_{\delta > 0} (\lambda^\xi_\delta \times \lambda^\zeta_\delta)(S \times T)$$

for all subsets S of X and T of Y. ⊣

Proof. It is clear that for every set S and weighted cover $(c_i, E_i)_{i \geq 1}$ of S we have

$$\lambda^\xi_\delta(S) \leq \sum_{i=1}^\infty c_i \lambda^\xi_\delta(E_i),$$

for all positive δ and hence

$$\lambda^\xi(S) \leq \sum_{i=1}^\infty c_i \lambda^\xi(E_i).$$

Thus by Note 10.11 we have

$$\sup_{\delta > 0}(\lambda^\xi_\delta \times \lambda^\zeta_\delta)(S \times T) = \sup_{\delta > 0} \lambda^\xi_\delta(S) \lambda^\zeta_\delta(T) = \lambda^\xi(S)\lambda^\zeta(T) = (\lambda^\xi \times \lambda^\zeta)(S \times T).$$

This gives the required result. $\square$

Proposition 10.15 *For X, Y metric spaces and premeasures ξ on X and ζ on Y, if $X \times Y$ is metrised by a strict product metric then for all subsets S of X and T of Y,*

$$(\lambda^{\xi} \times \lambda^{\zeta})(S \times T) \leq \lambda^{\xi \bowtie \zeta}(S \times T),$$

and equality holds provided that the following conditions are satisfied:

1. *if $\lambda^{\xi}(S) = 0$ then for every positive δ there exists a (countable) δ-cover of T,*
2. *if $\lambda^{\zeta}(T) = 0$ then for every positive δ there exists a (countable) δ-cover of S.* ⊣

Proof. This is essentially proved in Theorem 2 of [18] but we give the proof for completeness. We suppose ρ and σ are the metrics on X and Y respectively and υ is a strict product metric on $X \times Y$ with $\upsilon \leq \eta(\rho \times \sigma)$ for some η. Given subsets S of X and T of Y and positive δ we have for all weighted δ-cover $(c_i, G_i)_{i \geq 1}$ of $S \times T$,

$$\sum_{i=1}^{\infty} c_i(\xi \bowtie \zeta)(G_i) = \sum_{i=1}^{\infty} c_i \xi(p_X(G_i))\zeta(p_Y(G_i)).$$

Now for each y in F,

$$\{(c_i, p_X(G_i)) | y \in p_Y(G_i) \text{ and } i \geq 1\}$$

forms a weighted cover of S and hence

$$\sum \{c_i \xi(p_X(G_i)) | y \in p_Y(G_i) \text{ and } i \geq 1\} \geq \lambda^{\xi}_{\delta}(S).$$

Thus

$$\sum_{i=1}^{\infty} c_i(\xi \bowtie \zeta)(G_i) \geq \lambda^{\xi}_{\delta}(S)\lambda^{\zeta}_{\delta}(T) \geq (\lambda^{\xi}_{\delta} \times \lambda^{\zeta}_{\delta})(S \times T)$$

and hence on letting δ tend to 0 we see, by the previous lemma, that

$$(\lambda^{\xi} \times \lambda^{\zeta})(S \times T) \leq \lambda^{\xi \bowtie \zeta}(S \times T).$$

If for some positive δ there does not exist a (countable) δ-cover of S then there does not exist a weighted δ-cover of $S \times T$ and hence $\lambda^{\xi \bowtie \zeta}(S \times T) = \infty$. Now by the above hypothesis $\lambda^{\xi}(S) = \infty$ and hence, by Note 10.11, if $\lambda^{\zeta}(T) \neq 0$ then

$$(\lambda^{\xi} \times \lambda^{\zeta})(S \times T) = \infty.$$

Similarly if for some positive δ there does not exist a (countable) δ-cover of T and $\lambda^{\xi}(S) \neq 0$ then

$$(\lambda^{\xi} \times \lambda^{\zeta})(S \times T) = \lambda^{\xi \bowtie \zeta}(S \times T).$$

Now suppose $\lambda^{\xi}(S) = 0$, so that if $(F_j)_{j \geq 1}$ is a δ-cover of T then for every positive ε we may choose for every $j \geq 1$ a weighted δ-cover $(a_{i,j}, E_{i,j})_{i \geq 1}$ of S such that

$$\zeta(F_j) \sum_{i=1}^{\infty} a_{i,j} \xi(E_{i,j}) < \frac{\varepsilon}{2^j}.$$

Thus by Note 10.7, $(a_i, E_{i,j} \times F_j)_{i,j \geq 1}$ forms a $\eta\delta$-weighted cover of $S \times T$ with

$$\sum_{i,j \geq 1} a_{i,j} (\xi \bowtie \zeta)(E_{i,j} \times F_j) < \varepsilon.$$

Hence by Note 10.11, on letting first ε and then δ tend to 0 we have

$$\lambda^{\xi \bowtie \zeta}(S \times T) = 0 = (\lambda^{\xi} \times \lambda^{\zeta})(S \times T).$$

Similarly if $\lambda^{\zeta}(T) = 0$ and for every positive δ there exists a (countable) δ-cover of S then

$$\lambda^{\xi \bowtie \zeta}(S \times T) = 0 = (\lambda^{\xi} \times \lambda^{\zeta})(S \times T).$$

Hence we may assume that $\lambda^{\xi}(S)$ and $\lambda^{\zeta}(T)$ are both non-zero and for every positive δ there exist (countable) δ-covers of both S and T. Thus for positive δ such that $\lambda^{\xi}_{\delta}(S)$ and $\lambda^{\zeta}_{\delta}(T)$ are both non-zero we have the following. For any weighted δ-covers $(a_i, E_i)_{i\geq 1}$ of S and $(b_j, F_j)_{j\geq 1}$ of T we see that $(a_i b_j, E_i \times F_j)_{i,j\geq 1}$ forms a weighted cover of $S \times T$ and

$$\sum_{i,j\geq 1} a_i b_j (\xi \bowtie \zeta)(E_i \times F_j) = \left(\sum_{i\geq 1} a_i \xi(E_i)\right)\left(\sum_{j\geq 1} b_j \zeta(F_j)\right).$$

Thus by Notes 10.7 and 10.11

$$\lambda^{\xi\bowtie\zeta}_{\eta\delta}(S \times T) \leq (\lambda^{\xi}_{\delta} \times \lambda^{\zeta}_{\delta})(S \times T).$$

The required result follows on taking the limit as δ tends to 0 and the previous lemma. □

Note 10.16 With some slight changes to the above proof it is clear that for all $\delta > 0$ and all $U \subseteq X \times Y$ we have

$$(\lambda^{\xi}_{\delta} \times \lambda^{\zeta}_{\delta})(U) \leq \lambda^{\xi\bowtie\zeta}_{\delta}(U),$$

and if the following conditions are satisfied

1. whenever $\lambda^{\xi}_{\delta}(p_X(U)) = 0$ there exists a (countable) δ-cover of $p_Y(U)$

2. whenever $\lambda^{\zeta}_{\delta}(p_Y(U)) = 0$ there exists a (countable) δ-cover of $p_X(U)$

then

$$\lambda^{\xi\bowtie\zeta}_{\eta\delta}(U) \leq (\lambda^{\xi}_{\delta} \times \lambda^{\zeta}_{\delta})(U). \qquad \dashv$$

Corollary 10.17 *For X, Y separable metric spaces and premeasures ξ on X and ζ on Y, if $X \times Y$ is metrised by a strict product metric then*

$$\lambda^{\xi\bowtie\zeta} \leq \lambda^{\xi} \times \lambda^{\zeta}. \qquad \dashv$$

Proof. We denote the family of all rectangles of $X \times Y$ by $\mathcal{R}$, that is

$$\mathcal{R} = \{E \times F | E \subseteq X \text{ and } F \subseteq Y\}.$$

By Notes 4.7, 10.10 and 10.11 we have $\lambda^{\xi \bowtie \zeta}$ is $\mathcal{R}_{\sigma\delta}$-regular and $\lambda^{\xi} \times \lambda^{\zeta}$ is $\mathcal{R}_{\sigma}$-regular. The result follows by the previous proposition. □

Corollary 10.18 *For X, Y arbitrary metric spaces and Hausdorff functions h, g, if $X \times Y$ is metrised by a strict product metric then for all subsets S of X and T of Y we have*

$$(\lambda^h \times \lambda^g)(S \times T) \leq \lambda^{hg}(S \times T),$$

and hence if $\lambda^h(X)$ and $\lambda^g(Y)$ are non-zero then $\lambda^{hg}(X \times Y)$ is non-zero. ⊣

Proof. This is immediate from the above proposition and Lemma 10.13. This result is given in Theorem 9 of [18]. □

Note 10.19 In the above, if E and F are compact then this also follows by Notes 3.2, 4.7 and 4.8, Theorem 6.3, Proposition 10.12 and Lemma 10.13. ⊣

Example 10.20 *There exists a compact metric space X and a Hausdorff function h such that*

1. *there exists a subset U of $X \times X$ such that*

$$(\lambda^h \times \lambda^h)(U) > \lambda^{h \bowtie h}(U) = \lambda^{hh}(U),$$

2. *there exists subsets S and T of X such that*

$$(\lambda^h \times \lambda^h)(S \times T) = \lambda^{h \bowtie h}(S \times T) < \lambda^{hh}(S \times T).$$ ⊣

Proof. We let X be $[0, 1]$ and h be the function defined by $h(t) = \sqrt{t}$. We let U be the set $\{(x, x) | x \in X\}$, S be the set $\{0\}$ and T be the set X. It is easily checked that *1* and *2* above hold. □

10.4 Dimension of Product spaces.

Theorem 10.21 *For X, Y arbitrary metric spaces with metrics ρ, σ (respectively) and $X \times Y$ with strict product metric v the following set inequality holds*

$$W^+(X,\rho) \odot W^+(Y,\sigma) \subseteq W^+(X \times Y, v),$$

where $W^+(X,\rho) \odot W^+(Y,\sigma)$ is the set

$$\{fg | f \in W^+(X,\rho) \text{ and } g \in W^+(Y,\sigma)\}. \qquad \dashv$$

Proof. Let X, Y be as above and f in $W^+(X,\rho) \odot W^+(Y,\sigma)$. Thus f is equal to hg, for some h in $W^+(X,\rho)$ and g in $W^+(Y,\sigma)$. Hence by definition $\lambda^h(X), \lambda^g(Y)$ are non-zero. Thus, by Corollary 10.18, $\lambda^f(X \times Y)$ is non-zero and the result follows. $\square$

Note 10.22 It is noted that the above theorem applies only to the partition given as the generalised weighted Hausdorff–Besicovitch dimension and not that given as the generalised Hausdorff–Besicovitch dimension. However, under certain conditions on a metric space X we have $\Lambda^h(X) = \lambda^h(X)$ for all h in $\mathbf{H}$. By the work of J.D. Kelly in [18] we have for X, Y arbitrary metric spaces with metrics ρ, σ (respectively) and $X \times Y$ with strict product metric v,

$$W^+(X,\rho) \odot B^+(Y,\sigma) \subseteq B^+(X \times Y, v).$$

Now by Corollary 9.8, and Proposition 5.3 if X has finite structural dimension then for all h in $\mathbf{H}$ we have $\Lambda^h(X) = \lambda^h(X)$. Also by Corollary 9.9, and Proposition 5.4, if Z is a compact ultrametric space then for all h in $\mathbf{H}$ we have $\Lambda^h = \lambda^h$ (on Z). Hence if X has finite structural dimension or X is a subset of Z with the induced metric ρ where Z is as above and Y is any metric space with metric σ we have

$$B^+(X,\rho) \odot B^+(Y,\sigma) \subseteq B^+(X \times Y, v)$$

given that υ is a strict product metric. The condition that Z be compact is required since the functions in $\mathbf{H}$ may not be continuous. $\dashv$

Corollary 10.23 *For X, Y arbitrary metric spaces with metrics ρ, σ (respectively) and $X \times Y$ with weak product metric υ the following inequality holds*

$$\dim(X, \rho) + \dim(Y, \sigma) \leq \dim(X \times Y, \upsilon). \qquad \dashv$$

Proof. We denote by $h(s)$ the function defined on all non-negative t by $h(s)(t) = t^s$. By the finite order of $h(s)$ and Note 10.3, we may assume without loss of generality that υ is a strict product metric. By Note 10.1

$$\dim(X, \rho) = \sup\{s \in \mathbb{R}_0^+ | \mathrm{h(s)} \in \mathrm{W}^+(\mathrm{X}, \rho)\}.$$

The corollary now immediately follows from the previous theorem. $\square$

Note 10.24 There must be some restriction on how the space $X \times Y$ is metrised since we can remetrise X and Y to give as large a dimension as we like for X and Y respectively. Thus without some condition on how $X \times Y$ is metrised the above corollary would clearly be false. $\dashv$

Example 10.25 *There exist compact metric spaces X and Y such that*

$$\dim(X, \rho) + \dim(Y, \sigma) < \dim(X \times Y, \upsilon). \qquad \dashv$$

Proof. Such an example is given in [2]. $\square$

Conjecture 10.26 *There exist compact metric spaces X and Y such that*

$$B^+(X, \rho) \odot B^+(Y, \sigma) \not\subseteq B^+(X \times Y, \upsilon)$$

Remark. We believe this may be true for the case when $X = Y$ is as in the example given in [7] or something similar.

11 On the Existence of Sets of Finite Positive Hausdorff Measure.

Lemma 11.1 *For X a compact metric space, $\mathcal{G}$ a family of open sets of X that is adequate for X, a premeasure ξ on X that is $\mathcal{G}$-regular and positive δ, the set $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$ under the identification given by Riesz's Representation Theorem is convex and compact in the weak*-topology.* $\dashv$

Proof. The convexity of $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$ follows trivially from the definitions. Under the identification given by Riesz's Representation Theorem we have for μ in $\mathrm{M}^{+}(X)$

$$\|\mu\| = \mu(X)$$

and hence, by Theorem 6.3, $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$ is bounded. Thus it only remains to be proved that $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$ is weak*-closed, since by the Banach-Alaoglu Theorem, see [27], any weak*-closed and bounded set of the dual space to a normed vector space is compact in the weak*-topology.

Let $(\mu_i)_{i\geq 1}$, be a sequence in $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$ which is convergent to m^* in the weak*-topology. For all f in $C(X)$, such that $f \geq 0$ we have by convergence $m^*(f) \geq 0$. Hence m^* may be represented by a measure, μ say. Now for G in $\mathcal{G}$, such that $\operatorname{diam} G \leq \delta$ we have

$$\mu(G) \leq \liminf_{i\to\infty} \mu_i(G) \leq \xi(G).$$

Thus μ is in $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$. Hence $\mathrm{M}^{\xi}_{\delta}(X;\mathcal{G})$ is weak*-closed as required, thus completing the proof. $\square$

Theorem 11.2 *For X a compact metric space, ξ a premeasure on X of finite-order, and all real ℓ with $\Lambda^{\xi}(X) > \ell$, there exists A a (compact) subset of X such that*

$$\ell < \lambda^{\xi}(A) \leq \Lambda^{\xi}(A) < \infty.$$

Furthermore if ξ is Borel regular then

$$\ell < \lambda^{\xi}(A) = \Lambda^{\xi}(A) < \infty. \qquad \dashv$$

Proof. Let X, ξ and ℓ be as above. If $\ell < 0$ then $A = \emptyset$ gives the result. Also if $\Lambda^{\xi}(X) < \infty$ then $A = X$ gives the required result. Hence we may suppose that $\ell \geq 0$ and $\Lambda^{\xi}(X) = \infty$.

By Proposition 5.1 there exists $\mathcal{G}$, a family of open subsets of X and ζ a premeasure of X such that $\mathcal{G}$ is adequate and finite in the large and ζ is of finite-order, $\mathcal{G}$-regular and there exists a positive constant η such that for all positive ε,

$$\lambda_{\varepsilon}^{\xi} \leq \lambda_{\varepsilon}^{\zeta} \leq \eta \lambda_{\varepsilon}^{\xi}.$$

We define Q to be the set $\{x \in X \,|\zeta(\{x\}) > 0\}$. Thus if $\Lambda^{\zeta}(Q)$ is infinite we have $\sum \{\zeta(\{x\})|\, x \in Q\}$ is infinite. Hence there exists a finite (and thus compact) subset A of Q such that $\sum \{\zeta(\{x\})|\, x \in A\} > \ell$, and the result follows. Otherwise $\sum \{\zeta(\{x\})|\, x \in Q\}$ is finite. Thus Q is countable and hence Λ^{ζ}-measurable. Therefore $X \setminus Q$ is Borel and $\Lambda^{\zeta}(X \setminus Q)$ is infinite. Thus if we can choose a (compact) subset A of $X \setminus Q$ such that

$$\eta\ell < \Lambda^{\zeta}(A) < \infty$$

then the required result follows by Corollary 9.9. Now by Proposition 8.1 and Theorem 8.4 we may choose compact K a subset of $X \setminus Q$ such that $\eta\ell < \Lambda^{\zeta}(K)$. Hence by Note 3.3 we may suppose that Q is the empty set since we may replace X, ξ, ζ and $\mathcal{G}$ by K, $\xi|K$, $\zeta|K$ and $\mathcal{G}|K$ respectively. That is, without loss of generality, we may assume that $\zeta(\{x\})$ is zero, for all x in X. It follows by the adequacy of $\mathcal{G}$ and the $\mathcal{G}$-regularity of ζ that for all μ in $\mathbf{M}^{\zeta}$ and x in X, $\mu(\{x\})$ is zero.

By Corollary 9.9 we may choose positive δ so small that $\lambda_{\delta}^{\zeta}(X) > \eta\ell$ where η is as above. We let h be

$$\sup\{\mu(X)|\mu \in \mathbf{M}_{\delta}^{\zeta}(X;\mathcal{G})\}$$

and define H^* to be the hyperplane

$$\{m^* \in C^*(X) | m^*(\mathcal{X}_X) = h\},$$

which is clearly weak*-closed. Throughout the following we take the identification given by Riesz's Representation Theorem. By Theorem 6.3, $h \geq \lambda_\delta^\zeta(X)$ and hence by the preceding lemma $\mathbf{M}_\delta^\zeta(X; \mathcal{G}) \cap H^*$ is non-empty, convex and compact in the weak*-topology. Thus by the Krein-Milman Theorem, see [27], we may choose μ in $\mathbf{M}_\delta^\zeta(X; \mathcal{G}) \cap H^*$ to be an extreme point. That is if

$$\mu = \kappa m_1^* + (1 - \kappa) m_2^*,$$

for some κ in $(0, 1)$ and distinct m_1^*, m_2^* in $C^*(X)$, then either m_1^* or m_2^* is not in $\mathbf{M}_\delta^\zeta(X; \mathcal{G}) \cap H^*$. We also define

$$B_\varepsilon^\tau = \bigcup \{G \in \mathcal{G} | \operatorname{diam} G \leq \varepsilon \text{ and } \tau\mu(G) > \zeta(G)\}.$$

We claim that for ε in $(0, \delta)$ and τ in $(1, 2)$, B_ε^τ is μ-almost all of X. To prove this suppose $\mu(X \setminus B_\varepsilon^\tau) > 0$, and let $G_1, \ldots, G_n$ be an enumeration of $\{G \in \mathcal{G} | \operatorname{diam} G > \varepsilon\}$. We let $N_1 = X \setminus B_\varepsilon^\tau$ and then for $i = 1, \ldots, n$

$$N_{i+1} = \begin{cases} N_i \setminus G_i & \text{if } \mu(N_i \setminus G_i) \geq \mu(N_i \cap G_i) \\ N_i \cap G_i & \text{if } \mu(N_i \setminus G_i) < \mu(N_i \cap G_i) \end{cases}$$

Now B_ε^τ is open and hence μ-measurable. Therefore $N = N_{n+1}$ is such that $\mu(N) > 0$, N is Borel and for all G in $\mathcal{G}$, with $\operatorname{diam} G > \varepsilon$,

$$N \subseteq G \text{ or } N \cap G = \emptyset.$$

Now by hypothesis for all x in X, $\zeta(\{x\}) = 0$ and hence $\mu(\{x\}) = 0$, see the above. So there exist M_1 and M_2 subsets of N with

$$M_1 \cap M_2 = \emptyset, \quad M_1 \cup M_2 = N \quad \text{and} \quad \mu(M_1) = \mu(M_2) = \frac{1}{2}\mu(N) > 0$$

and M_1, M_2 Borel.

We now define the following finite (positive) Borel measures by

$$\mu_1(E) = \tau\mu(E \cap M_1) + (2-\tau)\mu(E \cap M_2) + \mu(E \setminus N)$$

$$\mu_2(E) = (2-\tau)\mu(E \cap M_1) + \tau\mu(E \cap M_2) + \mu(E \setminus N)$$

for all subsets E of X. Thus for any such E, we have

$$\mu(E) = \frac{\mu_1(E) + \mu_2(E)}{2}$$

and

$$N \subseteq E \text{ or } N \cap E = \emptyset \quad \Longrightarrow \quad \mu_1(E) = \mu_2(E) = \mu(E).$$

In particular

$$\mu_1(X) = \mu_2(X) = \mu(X) = h.$$

Hence by the extremality of μ, there exists G in $\mathcal{G}$ such that $\operatorname{diam} G \leq \delta$ and one of the following holds,

$$\mu_1(G) \;>\; \zeta(G) \tag{1}$$

$$\mu_2(G) \;>\; \zeta(G) \tag{2}$$

Suppose that (1) holds then $\operatorname{diam} G < \varepsilon$ since otherwise

$$\mu_1(G) = \mu_1(G_i) = \mu(G)$$

for some i $(1 \leq i \leq n)$. Also

$$\tau\mu(G) \geq \tau\mu(G \cap M_1) + (2-\tau)\mu(G \cap M_2) + \mu(G \setminus N) = \mu_1(G) > \zeta(G),$$

since $\tau > 1 > 2 - \tau > 0$. Therefore G is a subset of B_ε^τ, which is a contradiction since $\mu(G \cap M_1) > 0$. Similarly we gain a contradiction if (2) holds. Thus indeed the claim is true.

It follows immediately from the definitions that

$$B = \bigcap_{1<\tau<2} \bigcap_{0<\varepsilon<\delta} B_\varepsilon^\tau$$

is equal to a countable intersection of sets B_ε^τ. Hence B is μ-almost all of X. Also $\mathrm{D}^\zeta\mu(x;\mathcal{G}) = 1$ for all x in B.

Thus by Theorem 9.3 and Note 4.3 we have

$$\mu(X) \geq \Lambda^\zeta(B) \geq \lambda^\zeta(B) \geq \mu(B) = \mu(X) = h.$$

But $h \geq \lambda_\delta^\zeta(X) > \eta\ell$. Thus by Proposition 8.1 and Theorem 8.4 we may choose A a compact subset of B such that $\eta\ell < \Lambda^\zeta(A)$. The required result follows. □

Corollary 11.3 *For X a metric space and ξ a premeasure on X of finite-order, if Λ^ξ satisfies the Radon condition(*) then for all real ℓ with $\Lambda^\xi(X) > \ell$ there exists A a (compact) subset of X such that*

$$\ell < \lambda^\xi(A) \leq \Lambda^\xi(A) < \infty.$$

Furthermore if ξ is Borel regular then

$$\ell < \lambda^\xi(A) = \Lambda^\xi(A) < \infty. \qquad \dashv$$

Proof. This follows immediately from Note 4.8 and the preceding theorem. □

Corollary 11.4 *For X an analytic subset of a complete separable metric space and ξ a premeasure on X of finite-order that is regular with respect to the family of all open sets, if the increasing sets lemma holds for Λ^ξ then for all real ℓ with $\Lambda^\xi(X) > \ell$ there exists A a (compact) subset of X such that*

$$\ell < \lambda^\xi(A) = \Lambda^\xi(A) < \infty. \qquad \dashv$$

Proof. This follows immediately from Theorem 8.4 and the preceding corollary. □

Corollary 11.5 *For X an analytic subset of a complete separable metric space, h a continuous Hausdorff function, if one of the following is satisfied,*

1. *h is of finite-order*

2. *X has finite structural dimension*

3. *X is ultrametric*

then for all real ℓ with $\Lambda^h(X) > \ell$, there exists A a (compact) subset of X such that

$$\ell < \lambda^h(A) = \Lambda^h(A) < \infty. \qquad \dashv$$

Proof. This follows immediately from Theorem 8.3, Proposition 5.3, Proposition 5.4 and the preceding corollary. □

Remark. The Examples 8.5 and 12.1 together with the example given in [7] show that the conditions given in this thesis for the existence of subsets of finite positive Hausdorff measure are in some sense best possible. However there is no counterpart to the example given in [7] for the weighted Hausdorff measure. Thus the existence of subsets of finite positive weighted Hausdorff measure may be assured under some more general conditions. However we give the following as a conjecture.

Conjecture 11.6 *There exists a compact metric space X and a continuous Hausdorff function h such that $\lambda^h(X) = \infty$ and there are no subsets of X of finite positive λ^h-measure.*

12 More on the Increasing Sets Lemma.

12.1 A counterexample.

Example 12.1 *There exists a complete separable metric space X and a premeasure ξ on X such that:*

1. $\Lambda^\xi(X) = \lambda^\xi(X) = \infty$,

2. $\Lambda^\xi(K) = \lambda^\xi(K) = 0$ *for all compact subsets K of X,*

3. *ξ is open regular and continuous with respect to the Hausdorff metric.*

Hence the increasing sets lemma fails for Λ^ξ and λ^ξ and there are no subsets of finite positive Λ^ξ-measure or of finite positive λ^ξ-measure. ⊣

Proof. We prove this by construction. We let X be the Banach space of summable real sequences (this is usually denoted by ℓ^1). For each member $\mathbf{x} = (x_i)_{i\geq 1}$ of X the norm $\|\mathbf{x}\|$ is defined by

$$\|\mathbf{x}\| = \sum_{i=1}^{\infty} |x_i|.$$

We let ρ denote the metric induced by this norm. We define $\mathcal{Y}$ to be the family of all non-empty finite subsets of X. We define ξ by

$$\xi(E) = \inf(\{\delta \in \mathbb{R} | \exists \mathrm{Y} \in \mathcal{Y} \colon \mathrm{E} \subseteq \mathrm{B}(\mathrm{Y}, \delta)\} \cup \{1\}).$$

It is easily checked that ξ is a premeasure and open regular. Continuity of ξ with respect to the Hausdorff metric follows trivially from the fact that if $\hat{\rho}(E, F) \leq \delta$ then $E \subseteq B(F, \delta)$ and $F \subseteq B(E, \delta)$ where $\hat{\rho}$ denotes the Hausdorff metric induced by ρ. This proves (3).

To prove (2) we let $U(x, \delta)$ denote the open ball centre x and radius δ. For $\delta > 0$, $x \in X$ and K a compact subset of X we have $\xi(K \cap U(x, \delta)) = 0$

since for all $\varepsilon > 0$ there exists a finite subcover of $K \cap B(x, \delta)$ from the family of sets $\{U(y, \varepsilon) | y \in X\}$. The result follows by taking finite covers of K from the family of sets $\{U(y, \delta) | y \in X\}$ and deducing that $\Lambda^{\xi}_{2\delta}(K) = 0$.

To prove (1) we show that $\lambda^{\xi}(B(\mathbf{0}, 1)) \geq \frac{1}{2}$ where $\mathbf{0}$ denotes the point $\mathbf{x} = (x_i = 0)_{i \geq 1}$. The result follows from this by noting that there are countably many disjoint closed balls of radius 1 contained in X. Let us suppose that $\lambda^{\xi}_1(B(\mathbf{0}, 1)) < \frac{1}{2}$. We let D be the set of all sequences $\mathbf{x} = (x_i)_{i \geq 1}$ with finitely many non-zero terms: that is there exists $N(x)$ such that for $i \geq N(x)$ we have $x_i = 0$. Now suppose that $(c_j, E_j)_{j \geq 1}$ is a weighted 1-cover of $B(\mathbf{0}, 1)$ such that $\sum_{j=1}^{\infty} c_j \xi(E_j) < \frac{1}{2}$. Since D is dense in X we can choose a sequence of subsets $(D_j)_{j \geq 1}$ of D and a sequence of positive numbers $(\delta_j)_{j \geq 1}$ such that

$$E_j \subseteq B(D_j, \delta_j) \qquad (\forall j \geq 1)$$

$$\sum_{j=1}^{\infty} c_j \delta_j < \frac{1}{2}.$$

Hence we can choose a sequence of positive numbers $(\varepsilon_j)_{j \geq 1}$ such that

$$\varepsilon_j > \delta_j \qquad (\forall j \geq 1)$$

$$\sum_{j=1}^{\infty} c_j \varepsilon_j \leq \frac{1}{2}.$$

Now we can choose an increasing sequence of positive integers $(N_j)_{j \geq 1}$ such that for all $i \geq N_j$ we have $d_i = 0$ whenever $(d_i)_{i \geq 1} \in D_j$ and $j \geq 1$. We let $\mathbf{z} = (z_i)_{i \geq 1}$ be the point defined by

$$z_i = \begin{cases} 2c_j\varepsilon_j & \text{when } i = N_j \\ 0 & \text{otherwise.} \end{cases}$$

By definition we have $\mathbf{z} \in B(\mathbf{0}, 1)$. Also if $\mathbf{z} \in E_i$ then we must have

$$\sum_{j=i}^{\infty} 2c_j\varepsilon_j \leq \delta_i < \varepsilon_i.$$

We let $\mathcal{I}$ be the set

$$\left\{ i \geq 1 \,\middle|\, \sum_{j=i}^{\infty} 2c_j\varepsilon_j < \varepsilon_i \right\}.$$

We may calculate, provided $\mathcal{I} \neq \emptyset$

$$\sum_{i\in\mathcal{I}} c_i\varepsilon_i > \sum_{i\in\mathcal{I}} \sum_{j=i}^{\infty} 2c_ic_j\varepsilon_j \geq \sum_{i\in\mathcal{I}} c_i^2\varepsilon_i + \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{I}} c_ic_j\varepsilon_j > \left(\sum_{i\in\mathcal{I}} c_i\right) \left(\sum_{j\in\mathcal{I}} c_j\varepsilon_j\right) > 0.$$

Thus in all cases $\sum_{i\in\mathcal{I}} c_i < 1$. Hence we see that

$$\sum_{j=1}^{\infty} c_j \mathcal{X}_{E_j}(z) < 1$$

which is a contradiction since $(c_j, E_j)_{j\geq 1}$ was chosen to be a weighted cover of $B(\mathbf{0}, 1)$. This completes the proof. □

12.2 The increasing sets condition for premeasures.

We say that a premeasure satisfies the increasing sets condition if whenever $(E_i)_{i\geq 1}$ is an increasing sequence of subsets of X, that is $E_i \subseteq E_{i+1}$ for $i \geq 1$,

$$\xi\left(\bigcup_{i\geq 1} E_i\right) = \sup_{i\geq 1} \xi(E_i).$$

In the previous example we see that ξ does not satisfy the increasing sets condition.

Remark. The counterexamples to the increasing sets lemma given in [3] and in [25] both fail the above condition on their premeasures. In [3] the premeasure fails the above condition simply because it is not defined for all sets.

Note 12.2 We note that a premeasure ξ on X satisfies the increasing sets condition if and only if for all sequences $(E_i)_{i\geq 1}$ of subsets of X (not necessarily increasing) we have

$$\xi\left(\liminf_{i\to\infty} E_i\right) = \xi\left(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} E_i\right) \leq \liminf_{i\to\infty} \xi(E_i). \qquad \dashv$$

In view of the above definition we define the operator ϕ on the set functions of a space X by

$$\phi(\xi)(E) = \inf\left\{\sup_{i\geq 1}\xi(E_i)\,\middle|\, E\subseteq\bigcup_{i=1}^{\infty} E_i \text{ and } \forall i\geq 1, E_i\subseteq E_{i+1}\right\}$$

Hence a premeasure satisfies the increasing sets condition if and only if we have $\phi(\xi)=\xi$. It is also clear that the increasing sets lemma holds for Λ^ξ if and only if for all ε,δ with $0<\varepsilon<\delta$ we have $\Lambda^\xi_\delta \leq \phi(\Lambda^\xi_\varepsilon)$ and similarly for λ^ξ.

Proposition 12.3 *For X a metric space, ξ a premeasure on X and $\delta>0$ we have*

$$\phi(\Lambda^\xi_\delta) \leq \Lambda^{\phi(\xi)}_\delta \quad\text{and}\quad \phi(\lambda^\xi_\delta)\leq \lambda^{\phi(\xi)}_\delta. \qquad \dashv$$

Proof. For a subset S of X, $\varepsilon>0$ and a weighted δ-cover (c_i,E_i) of S we may choose for each $i\geq 1$ an increasing sequence of sets $(F_{i,j})_{j\geq 1}$ such that

$$\sup_{j\geq 1}\xi(F_{i,j}) \leq \phi(\xi)(E_i) + \frac{\varepsilon}{2^i}\ .$$

Now for $j\geq 1$ we define S_j by

$$S_j = \left\{x\in S\,\middle|\,\sum_{i=1}^{\infty}(1+\varepsilon)c_i\mathcal{X}_{F_{i,j}}(x)\geq 1\right\}.$$

It is clear that $S = \bigcup_{j=1}^{\infty} S_j$. Also

$$\sum_{i=1}^{\infty}(1+\varepsilon)c_i\xi(F_{i,j}) \leq (1+\varepsilon)\sum_{i=1}^{\infty} c_i\phi(\xi)(E_i) + (1+\varepsilon)\varepsilon.$$

Hence the result follows for λ^{ξ} on first letting ε tend to 0 and then taking the infimum over all weighted δ-covers $(c_i, E_i)_{i\geq 1}$ of S. The result follows immediately for Λ^{ξ} by taking the infimum over all 'weighted' δ-covers $(c_i, E_i)_{i\geq 1}$ with $c_i = 1$ for all $i \geq 1$. □

We now introduce another condition on premeasures which may be of use in the study of the increasing sets lemma. We say that a premeasure ξ on X is lower set continuous if and only if ξ satisfies the increasing sets condition and for every sequence of sets $(E_i)_{i\geq 1}$ (not necessarily increasing) there exists a subsequence $(i_k)_{k\geq 1}$, that is $i_{k+1} > i_k$ for all $k \geq 1$, such that

$$\xi\left(\limsup_{i\to\infty} E_{k_i}\right) = \xi\left(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} E_{k_i}\right) = \xi\left(\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} E_{k_i}\right) = \xi\left(\liminf_{i\to\infty} E_{k_i}\right).$$

Example 12.4 *There exists a space X and a premeasure ξ on X such that ξ is lower set continuous and the increasing sets lemma does not hold for Λ^{ξ}. Furthermore X can be metrised so that ξ is open regular and continuous with respect to the Hausdorff metric.* ⊣

Proof. We let $\mathbf{J}$ be the family of all sequences of integers $(j_i)_{i\geq 1}$ satisfying $0 \leq j_i \leq i$ for all $i \geq 1$. We let X be the set of $(j_i)_{i\geq 1} \in \mathbf{J}$ which satisfies for some $n \geq 1$

$$j_i \begin{cases} = 0 & \text{for } i < n \\ \geq 1 & \text{for } i \geq n \end{cases}$$

We define $E_{i,n}$ for $1 \leq n \leq i$ by

$$E_{i,n} = \{(j_k)_{k\geq 1} \in X | j_i = n\}.$$

Letting $Y_i = \bigcup_{n=1}^{i} E_{i,n}$ for $i \geq 1$, we see that $(Y_i)_{i\geq 1}$ is an increasing sequence of sets with union X. For convenience we define $\mathbf{J}^+$ to be the family of all sequence of positive integers $(j_i)_{i\geq 1} \in \mathbf{J}$; that is $1 \leq j_i \leq i$ for all $i \geq 1$. For $\mathbf{j} = (j_i)_{i\geq 1} \in \mathbf{J}^+$ we define

$$F_{\mathbf{j}} = \limsup_{i\to\infty} E_{i,j_i} = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_{i,j_i}.$$

First we define $\xi(E_{i,j}) = \frac{1}{i}$ for $1 \leq j \leq i$ and $\xi(X) = 2$. For every countable subfamily $\mathbf{K}$ of $\mathbf{J}^+$ we define

$$\xi\left(\bigcup_{\mathbf{k}\in\mathbf{K}} F_{\mathbf{k}}\right) = 0.$$

We let $\mathcal{E}$ be the family of sets

$$\{E_{i,j} | i, j \geq 1\} \cup \left\{ \bigcup_{\mathbf{k}\in\mathbf{K}} F_{\mathbf{k}} \,\middle|\, \mathbf{K} \subseteq \mathbf{J}^+ \text{ and } \mathbf{K} \text{ countable} \right\}.$$

Finally we define ξ for all $F \subseteq X$ by

$$\xi(F) = \inf\left\{\xi(E) \,\middle|\, F \subseteq E \in \mathcal{E} \cup \{X\}\right\}.$$

It is easily checked that ξ satisfies the conditions of the example. Furthermore if X is metrised by the discrete metric then ξ is continuous with respect to the Hausdorff metric and open regular.

By the above definition it is immediate that $\Lambda_\infty^\xi(Y_i) \leq 1$ for $i \geq 1$. It is also clear that $\Lambda_\infty^\xi(X) \leq 2$. The result follows on showing $\Lambda_\infty^\xi(X) \geq 2$. Suppose that $(G_i)_{i\geq 1}$ is a cover of X with $\sum_{i=1}^{\infty} \xi(G_i) < 2$. Thus $\xi(G_i) < 2$ for each $i \geq 1$ and hence we can choose a sequence of sets $(H_i)_{i\geq 1}$ with $H_i \in \mathcal{E}$ for $i \geq 1$ such that $G_i \subseteq H_i$ for $i \geq 1$ and $\sum_{i=1}^{\infty} \xi(H_i) < 2$. We partition the sets H_i by setting

$$\mathcal{H}^+ = \{H_i | \xi(H_i) > 0 \text{ and } i \geq 1\}$$

$$\mathcal{H}^0 = \{H_i | \xi(H_i) = 0 \text{ and } i \geq 1\}.$$

Now for each $H \in \mathcal{H}^+$ we have $H = E_{i,j}$ for some $i, j \geq 1$. Thus for $i \geq 1$ we define N_i to be the set $\{j \geq 1 | E_{i,j} \notin \mathcal{H}^+\}$. Hence

$$\sum_{i=1}^{\infty} \sum_{j \notin N_i} \xi(E_{i,j}) = \sum_{H \in \mathcal{H}^+} \xi(H) < 2.$$

Thus the number of elements in N_i tends to infinity as i tends to infinity. Hence given $(j_i)_{i \geq 1} \in \mathbf{J}^+$ such that $j_i \in N_i$ for all but finitely many $i \geq 1$ we have

$$\liminf_{i \to \infty} E_{i,j_i} = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_{i,j_i} \not\subseteq \bigcup \mathcal{H}^+.$$

The result follows if we can show that there exists $(j_i)_{i \geq 1} \in \mathbf{J}^+$ such that $j_i \in N_i$ for all but finitely many $i \geq 1$ and

$$\bigcup \mathcal{H}^0 \cap \liminf_{i \to \infty} E_{i,j_i} = \bigcup \mathcal{H}^0 \cap \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_{i,j_i} = \emptyset$$

since this would imply a contradiction as $(E_i)_{i \geq 1}$ was chosen to be a cover of X.

Now for each $H \in \mathcal{H}^0$ we have H equal to a countable union of sets $F_{\mathbf{k}}$ where $\mathbf{k} \in \mathbf{J}^+$. Since a countable union of countable sets is countable there exists $\mathbf{K} \subseteq \mathbf{J}^+$ such that $\mathbf{K}$ is countable and

$$\bigcup \mathcal{H}^0 \subseteq \bigcup_{\mathbf{k} \in \mathbf{K}} F_{\mathbf{k}}.$$

We let $(\mathbf{k}_j = (k_{i,j})_{i \geq 1})_{j \geq 1}$ be an enumeration of $\mathbf{K}$ and choose an increasing sequence of integers $(m_n)_{n \geq 1}$ such that for all $i \geq m_n$ the number of elements in N_i is greater than n. Using countable choice we can select a sequence $(j_i)_{i \geq 1} \in \mathbf{J}^+$ such that

$$i \geq m_n \quad \Longrightarrow \quad j_i \neq k_{i,n} \text{ and } j_i \in N_i \qquad (\forall i \geq 1 \; \forall n \geq 1).$$

It is clear that

$$\bigcup \mathcal{H}^0 \cap \liminf_{i\to\infty} E_{i,j_i} = \bigcup \mathcal{H}^0 \cap \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} E_{i,j_i} = \emptyset$$

This completes the proof. $\square$

Note 12.5 The above construction for X may be considered to be universal in the following sense. If Z is any metric space and ζ is a premeasure on Z such that $\phi(\Lambda_\delta^\zeta(Z)) \neq \Lambda_\delta^\zeta(Z)$ then there exists a premeasure ξ on X and a metric ρ on X such that $\phi(\Lambda_\delta^\xi(X)) \neq \Lambda_\delta^\xi(X)$. To show this we may choose a sequence of increasing sets $(W_i)_{i\geq 1}$ with union Z and $\sup_{i\geq 1}\Lambda_\delta^\zeta(W_i) < \Lambda_\delta^\zeta(Z)$. For each $i \geq 1$ we let $(E_{i,j})_{j\geq 1}$ be a δ-cover of W_i such that

$$\sup_{i\geq 1}\sum_{i=1}^{\infty}\zeta(E_{i,j}) < \Lambda_\delta^\zeta(Z).$$

It is clear that we may assume that for each $i \geq 1$ the covers $(E_{i,j})_{j\geq 1}$ are disjoint and there exists n_i such that $(E_{i,j})_{j=1}^{n_i}$ covers W_i. We define for $i \geq 1$ and $0 \leq j \leq i$

$$F_{i,j} = \begin{cases} E_{k,j} & \text{if } 1 \leq j \leq n_k \leq i < n_{k+1}; \\ Z \setminus W_i & \text{if } j = 0 \text{ and } n_k \leq i < n_{k+1}; \\ \emptyset & \text{otherwise.} \end{cases}$$

We define a mapping $f\colon Z \to X$ by

$$f(z) = (j_i)_{i\geq 1} \quad \Longleftrightarrow \quad z \in \bigcap_{i=1}^{\infty} F_{i,j_i}.$$

We denote the metric on Z by σ and define the metric ρ on X by

$$\rho(x,y) = \begin{cases} 2\delta & \text{if } \operatorname{diam}_\sigma(f^{-1}(x) \cup f^{-1}(y)) > 2\delta \\ \delta & \text{if } f^{-1}(x) = \emptyset \text{ or } f^{-1}(y) = \emptyset \\ \operatorname{diam}_\sigma(f^{-1}(x) \cup f^{-1}(y)) & \text{otherwise} \end{cases}$$

for $x \neq y$ and $\rho(x,x) = 0$, for all $x \in X$ and $y \in X$. For $S \subseteq X$ we define $\xi(S) = \zeta(f^{-1}(S))$.

We also note that the above example may be realised in the real line. We let Z be the set $[0, e)$ where e denotes the base of the natural logarithm. We use $n!$ to denote n factorial; that is $n! = n(n-1)\ldots 1$ with the convention that $0! = 1$. We also use $e(n)$ to denote the nth partial sum of the expansion of e, that is $\sum_{i=0}^{n} \frac{1}{i!}$. For $i \geq 1$ and $1 \leq j \leq i$ we define the sets $F_{i,j}$ by

$$F_{i,j} = \bigcup_{k=1}^{(i-1)!e(i-1)} \left[\frac{(k-1)i + j - 1}{i!}, \frac{(k-1)i + j}{i!}\right).$$

It is easily checked that for $i \geq 1$ we have

$$\bigcup_{j=1}^{i} F_{i,j} = [0, e(i)).$$

First we define $\zeta(F_{i,j}) = \frac{1}{i}$ for $1 \leq j \leq i$ and $\zeta(Z) = 2$. Then extend ζ to a premeasure on Z as set out in the previous example. We let f be the function constructed exactly as above and note that f is 1–1 (an injection). Defining H to be the set of all $(j_i)_{i \geq 1} \in X$ such that $j_i = i$ for all but finitely many $i \geq 1$ we see that the range of f is precisely $X \setminus H$. $\dashv$

Remark. If we are looking for conditions which ensure that the increasing sets lemma holds for Λ^{ξ} (where ξ is a premeasure) then heuristic arguments suggest that any such conditions should imply at least one of the following:

1. ξ does not increase too much under a suitable notion of 'set enlargement'.

2. Given any sequence of covers $((E_{i,j})_{j \geq 1})_{i \geq 1}$ then for 'many' sequences of positive integers $(j_i)_{i \geq 1}$ we have

$$\bigcap_{i=1}^{\infty} E_{i,j_i} = \emptyset.$$

12.3 Approximation by compact sets.

For a premeasure ξ on a metric space X we define inductively the measures Φ_δ^n by $\Phi_\delta^0 = \Lambda_\delta^\xi$ and $\Phi_\delta^{n+1} = \phi(\Phi_\delta^n)$ for $n \geq 0$, then the measures Φ^n are defined to be $\sup_{\delta>0} \Phi_\delta^n$. Finally we define Φ_δ^∞ to be the largest measure less than Φ_δ^n for all $n \geq 1$ and define Φ^∞ to be $\sup_{\delta>0} \Phi_\delta^\infty$.

Conjecture 12.6 *For X a complete separable metric space and ξ an open regular premeasure on X we have for every analytic subset A of X*

$$\Phi_\delta^\infty(A) = \sup\{\Lambda^\xi(K) | K \subseteq A \text{ and } K \textit{ compact}\}.$$

Remark. We feel that the above conjecture may be of use in proving the following.

Conjecture 12.7 *For X a complete separable metric space and ξ an open regular premeasure on X of strong finite order the Radon condition(*) holds for Λ^ξ and hence also for λ^ξ.*

12.4 The problem of the increasing sets lemma and Hausdorff functions.

Conjecture 12.8 *For X a metric space and h a continuous Hausdorff function the increasing sets lemma holds for Λ^h.*

This conjecture is known to hold true in the cases when X is compact, X has finite structural dimension, X is an ultrametric space or h is of finite order. It would be of much interest if a counterexample could be found for which the space is a complete separable metric space. With this in mind it would be natural to look for a counterexample in a Banach space. The following proposition eliminates this case.

Proposition 12.9 *For X an infinite dimensional Banach space and h a Hausdorff function there exists a compact subset K of X such that $\Lambda^h(K) > 0$.* ⊣

Proof. Let X be as above with norm $\|\cdot\|$. By the infinite dimensionality of X there exists a sequence of points $(x_i)_{i\geq 1}$ satisfying

1. $\|x_i\| = 1$ for all $i \geq 1$,
2. For $n \geq 1$ we have $\|x_{n+1} - y\| \geq 1$ for all $y \in Y_n$ where Y_n is the subspace of X spanned by $\{x_1, \ldots, x_n\}$.

For h a Hausdorff function we may suppose that $\lim_{t\searrow 0} h(t) = 0$ since otherwise Λ^ξ is a multiple of the counting measure. For $x \in \mathbb{R}$ we denote by $\lceil x \rceil$ the least integer n such that $x \leq n$. We define inductively a function ϕ on the positive integers by

$$\phi(1) = \left\lceil \frac{h(\frac{1}{4})}{h(\frac{1}{4^2})} \right\rceil,$$

$$\phi(n+1) = \left\lceil \frac{h(\frac{1}{4})}{\phi(1)\cdots\phi(n)h(\frac{1}{4^{n+2}})} \right\rceil.$$

It is immediate that

$$\frac{h(\frac{1}{4})}{\phi(1)\cdots\phi(n)} \leq h(\tfrac{1}{4^{n+1}})$$

for all $n \geq 1$. We define a code space $\mathcal{I}$ to be the family of all sequences of positive integers $\mathbf{i} = (i_j)_{j\geq 1}$ such that $i_j \leq \phi(j)$ for $j \geq 1$. Given an element $(i_j)_{j\geq 1} \in \mathbf{I}$ we use $\mathbf{i}|n$ to denote the n-tuple $(i_1, \ldots, i_n)$ and define $B_{\mathbf{i}|n}$ by

$$B_{\mathbf{i}|n} = B\left(\sum_{j=1}^{n} \frac{3}{4^j} x_{i_j}, \frac{1}{4^n}\right).$$

It is immediate that for $\mathbf{i}$ and $\mathbf{j}$ in $\mathbf{I}$ and $n \geq 1$ we have

$$\mathbf{i}|n \neq \mathbf{j}|n \quad \Longrightarrow \quad \operatorname{dist}(B_{\mathbf{i}|n}, B_{\mathbf{j}|n}) \geq \frac{1}{4^n}$$

and

$$B_{\mathbf{i}|n+1} \subseteq B_{\mathbf{i}|n}.$$

We define a sequence $(F_n)_{n\geq 1}$ of subsets of X by

$$F_n = \bigcup_{\mathbf{i}\in\mathbf{I}} B_{\mathbf{i}|n}.$$

We define K by $K = \bigcap_{n=1}^{\infty} F_n$ which is clearly compact.

We show that $\Lambda^{\xi}(K) \geq h(\frac{1}{4})$. Suppose that $(E_j)_{j\geq 1}$ is a covering of K. Since h is continuous from the right there exists a sequence $(G_j)_{j\geq 1}$ such that $E_j \subseteq G_j$ for all $j \geq 1$ and $\sum_{j=1}^{\infty} h(\operatorname{diam} G_j)$ is as close as we wish to $\sum_{j=1}^{\infty} h(\operatorname{diam} E_j)$. Now by the compactness of K we can choose n so large that for each $\mathbf{i} \in \mathbf{I}$ we have

$$B_{\mathbf{i}|n} \cap K \subseteq G_j$$

for some $j \geq 1$. Since $\lim_{t\searrow 0} h(t) = 0$ we may also ensure that $\phi(n+1) > 1$. For each $j \geq 1$ we define g_j to be the number of distinct $B_{\mathbf{i}|n} \cap K$ contained in G_j. Now

$$\begin{array}{llll}
\text{if} & g_j = 1 & \text{then} & \operatorname{diam} G_j \geq \frac{1}{4^{n+1}} \\
\text{if} & 1 < g(j) \leq \phi(n) & \text{then} & \operatorname{diam} G_j \geq \frac{1}{4^n} \\
\text{if} & \phi(m+1)\cdots\phi(n) < g(j) \leq \phi(m)\cdots\phi(n) & \text{then} & \operatorname{diam} G_j \geq \frac{1}{4^m} \\
 & & & \text{for } 1 \leq m < n
\end{array}$$

Also

$$\begin{array}{llll}
\text{if} & g_j = 1 & \text{then} & h(\frac{1}{4^{n+1}}) \geq \frac{g_j}{\phi(1)\cdots\phi(n)} h(\frac{1}{4}) \\
\text{if} & 1 < g(j) \leq \phi(n) & \text{then} & h(\frac{1}{4^n}) \geq \frac{g_j}{\phi(1)\cdots\phi(n)} h(\frac{1}{4}) \\
\text{if} & \phi(m+1)\cdots\phi(n) < g(j) \leq \phi(m)\cdots\phi(n) & \text{then} & h(\frac{1}{4^m}) \geq \frac{g_j}{\phi(1)\cdots\phi(n)} h(\frac{1}{4}) \\
 & & & \text{for } 1 \leq m < n
\end{array}$$

Hence

$$\sum_{j=1}^{\infty} h(\operatorname{diam} G_j) \geq \sum_{j=1}^{\infty} h(\tfrac{1}{4}) \frac{g_j}{\phi(1) \cdots \phi(n)} \geq h(\tfrac{1}{4}).$$

The required result follows. □

Remark. To find a counterexample to Conjecture 12.8 for which the space is a complete separable metric space it is sufficient to consider only subsets of the space of all sequences of natural numbers with the product topology given by the discrete topology on the natural numbers. However the way this space is metrised is of course highly important (that is if the conjecture is false).

Remark. The following proposition may be of use in searching for a counterexample to Conjecture 12.8 for which the space is a complete separable metric space.

Proposition 12.10 *For X a complete separable metric space and h a continuous Hausdorff function satisfying*

1. *for every compact subset K of X we have $\Lambda^h(K) = 0$*
2. *for every Hausdorff function g such that*

$$\limsup_{t \searrow 0} \frac{h(t)}{g(t)} = 0$$

there exist $K \subseteq X$ such that K is compact and $\Lambda^g(K) > 0$

the increasing sets lemma fails for Λ^h. ⊣

Proof. This follows from Theorem 8.4 and the fact that if $\Lambda^h(X) = 0$ then there exist a continuous Hausdorff function g such that

$$\limsup_{t \searrow 0} \frac{h(t)}{g(t)} = 0$$

and $\Lambda^g(X) = 0$; see [26]. □

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