Tangent Measure Distributions and the Geometry of Measures

Ph.D.-Thesis submitted by

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Abstract

In this thesis we investigate the geometry of measures in Euclidean spaces by means of their average densities, average tangent measures and tangent measure distributions. These notions were recently introduced into geometric measure theory by Bedford and Fisher, Bandt, Graf and others, as tools for the study of non-rectifiable measures.

Our main result yields a connection between tangent measure distributions of measures on the line and Palm distributions:

> Let μ be a measure on the line with positive and finite α-densities almost everywhere. Then at almost all points all tangent measure distributions are Palm distributions. Therefore the tangent measure distributions define α-self similar random measures in the axiomatic sense of U.Zähle.

This result enables us to give a complete description of the one-sided average α-densities of the measure in terms of its lower and upper circular average α-densities. It also enables us to give an example of a measure with positive and finite α-densities which has unique average tangent measures but non-unique tangent measure distributions almost everywhere.

If μ is a measure on n-dimensional Euclidean space with positive and finite α-densities almost everywhere we show that at almost all points the unique tangent measure distribution, if it exists, is a Palm distribution.

We illustrate the limitations of tangent measure distributions by means of an example of a non-zero measure that has no non-trivial tangent measure distributions almost everywhere. Such measures can be studied by means of normalized tangent measure distributions and we prove an existence and a shift-invariance result for these distributions.

Contents

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List of Symbols

This is a list of some symbols and basic notation used in this thesis.

Sets

We denote by

$\mathbb{N}$	the natural numbers $\{1,2,3,\ldots\}$,
$\mathbb{N}_0$	the nonnegative integers $\{0,1,2,3,\ldots\}$.

If A is a set then we denote by

$\#A$	cardinality of A,
1_A	the indicator function of A.

Metric Spaces

We shall consider the following metric spaces:

$\mathbb{R}^n$	n-dimensional Euclidean space,
$\mathcal{M}(\mathbb{R}^n)$	the space of (Radon–)measures on $\mathbb{R}^n$ with the metric d introduced in lemma 1.2.5,
$\mathcal{P}$	the space of probability distributions on $\mathcal{M}(\mathbb{R}^n)$ with the metric D introduced in lemma 1.2.8.

In a metric space (M,d) we denote by $\mathcal{B}(M)$ the Borel-σ-algebra on M.

For $A,B \subseteq M$, $x \in M$ and $r \geq 0$ we denote

$$d(x,B) = \inf\{d(x,y) : y \in B\}, \qquad d(A,B) = \inf\{d(x,B) : x \in A\},$$

$$U(A,r) = \{y \in M : d(y,A) < r\}, \qquad B(A,r) = \{y \in M : d(y,A) \leq r\},$$

$$U(x,r) = \{y \in M : d(y,x) < r\}, \qquad B(x,r) = \{y \in M : d(y,x) \leq r\},$$

$$\mathrm{cl}\,A = B(A,0), \text{ the closure of } A.$$

Euclidean Spaces

For $A \subseteq \mathbb{R}^n$, $t \geq 0$ and $u \in \mathbb{R}^n$ we denote by

$\lvert u\rvert$	the Euclidean norm of u,
$\lvert A\rvert$	the Euclidean diameter of A.

$$u + A = \{u + a : a \in A\},$$
$$tA = \{ta : a \in A\},$$
$A^c = \{y \in \mathbb{R}^n : y \notin A\}$, the complement of A in $\mathbb{R}^n$,

$\partial A = (\mathrm{cl}\, A) \setminus (\mathrm{cl}\, A^c)$, the boundary of A in $\mathbb{R}^n$.

On the real line $\mathbb{R}$ we use the following notation:

$[a, b], [a, b], [a, b), (a, b]$ are the closed, open and half-open intervals with endpoints a, b.

If $x \in \mathbb{R}$, $A, B \subseteq \mathbb{R}$ we write

$x < A$ if $x < a$ for all $a \in A$,

$A < B$ if $a < b$ for all $a \in A$ and $b \in B$,

$A < x$ if $a < x$ for all $a \in A$.

For all intervals $I \subseteq \mathbb{R}$, $\kappa \geq 0$ we denote by

$$I^-(\kappa) = \{x \in \mathbb{R} : \text{ there is } y \in I \text{ such that } 0 \leq y - x \leq \kappa \cdot |I|\} \setminus I,$$
$$I^+(\kappa) = \{x \in \mathbb{R} : \text{ there is } y \in I \text{ such that } 0 \leq x - y \leq \kappa \cdot |I|\} \setminus I,$$
$$I^0(\kappa) = I^-(\kappa) \cup I \cup I^+(\kappa).$$

Functions

We shall look at the following sets of functions on a metric space (M, d):

$C_b(M)$ the set of real valued continuous bounded functions on M,

$C_c(M)$ the set of real valued continuous functions with compact support on M .

If $f : (M, d) \to \mathbb{R}$ we denote by

$$\mathrm{supp}\, f = \mathrm{cl}\,\{x \in M : f(x) \neq 0\},$$
$$\|f\|_{\mathrm{sup}} = \sup_{x \in M} |f(x)|,$$
$$\mathrm{Lip}(f) = \sup_{\substack{x,y \in M \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Measures

The following measures appear repeatedly in this thesis:

ϕ the zero-measure,

$\mathcal{L}^n$ Lebesgue measure on $\mathbb{R}^n$,

$\mathcal{H}^\alpha|_E$ α-Hausdorff measure restricted to $E \subseteq \mathbb{R}^n$,

δ_x the Dirac measure with mass concentrated at x.

φ_ε the measure on the open interval $(0,1)$ defined by

$\varphi_\varepsilon(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_A(t)\, \frac{dt}{t}$,

ψ_ε the measure on $\mathbb{R}$ defined by

$\psi_\varepsilon(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_A(t) + 1_A(-t)\, \frac{dt}{t}$,

ψ_ε^x the measure on $\mathbb{R}$ defined by

$\psi_\varepsilon^x(A) = \psi_\varepsilon(A - x)$.

We write dx in place of $d\mathcal{L}^1(x)$.

For a measure μ on $\mathbb{R}^n$ we denote

supp μ the support of μ, i.e. the set of all $x \in \mathbb{R}^n$ such that $\mu(B(x,r)) > 0$ for all $r > 0$,

$\mu|_E$ the restriction of μ to the μ-measurable set E,

$\mu_{x,t}$ the measure defined by $\mu_{x,t}(A) = \mu(x + tA)$ for $x \in \mathbb{R}$ and $t \geq 0$.

For measures μ, ν on $\mathbb{R}^n$ we write

$\nu \ll \mu$ if ν is absolutely continuous with respect to μ.

Random Variables

For a random variable X we denote by

EX the expectation of X,

$\sigma^2(X)$ the variance of X.

Chapter 1

Fractal and Rectifiable Measures

In this chapter we introduce sets and measures in Euclidean spaces as our objects of study. We discuss the notions of density and tangent measure and recall some of the classical and newer results from geometric measure theory, which show how densities and tangent measures reflect the dichotomy in the behaviour of rectifiable or regular sets and measures on the one side, and non-rectifiable or fractal sets and measure on the other side.

In section 1.1 we give a brief introduction to the subject of this thesis. We define rectifiable and fractal measures and show that α-sets can be studied in this framework by means of Hausdorff measure. In section 1.2 we provide some measure theoretical tools and we endow the set of measures with a suitable topological and metric structure. In section 1.3 we introduce densities and tangent measures, discuss some of their properties and recall how they can be used to characterize rectifiable measures.

1.1 An Introduction

The aim of geometric measure theory is the study of geometric properties of sets and measures in Euclidean spaces by measure theoretical means. A measure in the sense of this thesis is an outer measure μ on $\mathbb{R}^n$ that fulfills

- all Borel sets are μ-measurable;
- μ is Borel-regular, i.e. every $A \subseteq \mathbb{R}^n$ is contained in a Borel set $B \subseteq \mathbb{R}^n$ such that $\mu(A) = \mu(B)$;
- μ is locally finite, i.e. every $x \in \mathbb{R}^n$ is contained in an open neighbourhood of finite measure or, equivalently, every compact set has finite measure.

Such a measure is sometimes called *Radon measure.* We denote the set of all such measures on $\mathbb{R}^n$ by $\mathcal{M}(\mathbb{R}^n)$. Observe that every locally finite measure on the Borel σ-algebra of $\mathbb{R}^n$ can be extended in a unique way to a measure in our sense (see e.g. [Rog70, theorems 6 and 13]). This will be used implicitly throughout the thesis.

The connection between the study of sets and measures is given by the fundamental notion of Hausdorff measure. For many sets $E \subseteq \mathbb{R}^n$ Hausdorff measure defines a kind of "equidistribution" on the set and we can study E by means of the Hausdorff measure on E. Recall the definition of Hausdorff measure (for more details see [Rog70], [Fal85, chapter 1] or [Mat95, chapter 4]).

Definition

Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$ and for $U \subseteq \mathbb{R}^n$ let

$$|U| = \sup_{x,y\in U} |x-y| \,,$$

the Euclidean diameter of the set U. Let $\alpha \geq 0$ and $E \subseteq \mathbb{R}^n$. Then for $t > 0$ put

$$\mathcal{H}_t^\alpha(E) = \inf\left\{ \sum_{i\in\mathbb{N}} |U_i|^\alpha \ : \ \text{the family } (U_i)_{i\in\mathbb{N}} \text{ covers } E \text{ and } |U_i| \leq t \text{ for all } i \in \mathbb{N} \right\}$$

and define the *α-Hausdorff measure* of E as

$$\mathcal{H}^\alpha(E) = \sup_{t>0} \mathcal{H}_t^\alpha(E) \,.$$

For a given $E \subseteq \mathbb{R}^n$ there is exactly one value $0 \leq \alpha \leq n$ such that

$$\alpha = \inf\{\alpha \ : \ \mathcal{H}^\alpha(E) < \infty\} = \sup\{\alpha \ : \ \mathcal{H}^\alpha(E) > 0\} \,.$$

α is called the *Hausdorff dimension* of the set E.

If E is $\mathcal{H}^\alpha$-measurable and $0 < \mathcal{H}^\alpha(E) < \infty$, then E is called an *α-set.*

Remarks:

1. Some authors (e.g. [Fal85]) chose a different normalization for Hausdorff measures.

2. For an α-set E the measure $\mu = \mathcal{H}^\alpha|_E$, the restriction of α-Hausdorff measure to E, is in $\mathcal{M}(\mathrm{I\!R}^n)$ and therefore the investigation of α-sets is included in the investigation of the measures $\mu \in \mathcal{M}(\mathrm{I\!R}^n)$ (see e.g. [Rog70, theorem 27]).

Starting with the work of Besicovitch in the twenties and thirties of this century (see [Bes28], [Bes38], [Bes39]) the geometric measure theorists have found several deep results that reveal a dichotomy between two classes of measures: Rectifiable or regular measures on the one side and non-rectifiable or fractal measures on the other side (see for example [Fed47], [Mar61], [Mat75] and [Pre87]).

Definition

Let d be a nonnegative integer. A measure $\mu \in \mathcal{M}(\mathrm{I\!R}^n)$ is called *d-rectifiable* or *d-regular* if there is a countable family $(K_i)_{i\in\mathrm{N}}$ of d-submanifolds of class 1 such that

$$\mu \ll \mathcal{H}^d|_{\bigcup K_i}.$$

A measure $\mu \in \mathcal{M}(\mathrm{I\!R}^n)$ is called *purely non-rectifiable* or *fractal* if μ does not have a d-rectifiable part for any nonnegative integer d, precisely: Whenever

$$\mu = \mu_1 + \mu_2$$

with μ_1, $\mu_2 \in \mathcal{M}(\mathrm{I\!R}^n)$, such that μ_1 is d-rectifiable for some nonnegative integer d, we have $\mu_1 = 0$.

Note that there are many different definitions of "fractal measures" in the literature. We have (probably) picked the most general one. In this thesis we are mostly interested in the geometry of fractal measures, rather than rectifiable measures. A small number of examples of fractal sets and measures are described in various chapters of this thesis but many more can be found, for example, in the textbooks [Fal85], [Fal90].

As a tool for the investigation of the local geometry of fractal measures C. Bandt ([Ban92]) introduced the concept of tangent measure distributions. Tangent measure distributions can be defined as follows:

> For $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we define the family of measures $(\mu_{x,t})_{t>0}$, the *enlargements of μ about x*, by
>
> $$\mu_{x,t}(A) = \mu(x + tA)\,.$$
>
> For an appropriately chosen $0 \leq \alpha \leq n$ we define probability distributions P_ε^x on $\mathcal{M}(\mathbb{R}^n)$ by
>
> $$P_\varepsilon^x(M) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_M\Big(\frac{\mu_{x,t}}{t^\alpha}\Big)\,\frac{dt}{t} \quad \text{for Borel sets } M \subseteq \mathcal{M}(\mathbb{R}^n).$$
>
> The *tangent measure distributions of μ at x* are the limit points in the weak topology of P_ε^x as $\varepsilon \downarrow 0$.

In this thesis we study the properties of tangent measure distributions of fractal measures. In particular we ask which features they have in common with the concept of tangent spaces, which is only available for rectifiable measures. We also study the closely related notions of average densities and average tangent measures.

Before we start with the investigation of tangent measure distributions we try to retrace some developments that led to their definition.

1.2 Some Tools from Measure Theory

Covering and differentiation of measures are useful tools in some proofs of this thesis and we provide the necessary statements now. We start with Besicovitch's covering theorem:

Lemma 1.2.1 Besicovitch's Covering Theorem
There is a constant $N \in \mathbb{N}$ depending only on n with the following property: If $A \subseteq \mathbb{R}^n$ is bounded and $\mathcal{B}$ is a family of closed balls such that each point of A is the centre of some ball of $\mathcal{B}$ then there are subfamilies $\mathcal{B}_1, \ldots, \mathcal{B}_N \subseteq \mathcal{B}$ such that

$$A \subseteq \bigcup_{i=1}^{N} \bigcup_{B \in \mathcal{B}_i} B$$

and each of these families is disjoint.

Proof The proof can be found, for example, in [Mat95, theorem 2.7]. ■

Lemma 1.2.2 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $f : \mathbb{R}^n \longrightarrow [0, \infty]$ be μ-measurable such that*

$$\int_K f \, d\mu < \infty \text{ for every compact } K \subseteq \mathbb{R}^n.$$

Then, for μ-almost every x, we have

$$\lim_{r \downarrow 0} \frac{\int_{B(x,r)} |f(y) - f(x)| \, d\mu(y)}{\mu(B(x,r))} = 0 \, .$$

In particular, if $\nu \in \mathcal{M}(\mathbb{R}^n)$ and $\nu \ll \mu$ with Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$ then, for μ-almost every x, we have

$$\lim_{r \downarrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = f(x) \, .$$

Proof The proof can be found, for example, in [Fed68, 2.9.9]. ■

Lemma 1.2.3 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$. A point $x \in E$ is called a μ-density point of E if*

$$\lim_{t \downarrow 0} \frac{\mu(B(x,t) \cap E)}{\mu(B(x,t))} = 1 \, .$$

μ-almost every point of E is a μ-density point.

Proof The proof can be found, for example, in [Fed68, 2.9.11]. ■

We will now look at the measures in $\mathcal{M}(\mathbb{R}^n)$ as a whole and fix a suitable topological and metric structure on $\mathcal{M}(\mathbb{R}^n)$.

Definition

The *vague topology* on $\mathcal{M}(\mathbb{R}^n)$ is the smallest topology such that, for every continuous function f with compact support, the functional $f_\star$ defined by

$$\begin{array}{rccc} f_\star : & \mathcal{M}(\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ & \mu & \mapsto & \mu(f) := \int f \, d\mu \,, \end{array}$$

is continuous.

The following lemma gives equivalent conditions for the convergence of a sequence of measures $(\mu_k) \subseteq \mathcal{M}(\mathbb{R}^n)$.

Lemma 1.2.4 *Let $\mu_k, \mu \in \mathcal{M}(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(1) *$\mu_k \longrightarrow \mu$ in the vague topology.*

(2) $\liminf \mu_k(G) \geq \mu(G)$ *for all open sets $G \subseteq \mathbb{R}^n$ and* $\limsup \mu_k(A) \leq \mu(A)$ *for all closed sets $A \subseteq \mathbb{R}^n$.*

(3) *For all $R > 0$ we have*

$$\sup \left\{ \left| \int f \, d\mu_k - \int f \, d\mu \right| : \operatorname{supp} f \subseteq B(0,R),\, f \geq 0,\, \operatorname{Lip}(f) \leq 1 \right\} \longrightarrow 0 \,.$$

Proof A proof of the equivalence of (1) and (2) can be found, for example, in [Kal83, 15.7.2], and a proof of the equivalence of (1) and (3) is given in [Mat95, lemma 14.13]. ∎

The following lemma defines a metric on $\mathcal{M}(\mathbb{R}^n)$, which induces the vague topology.

Lemma 1.2.5

1. *Let $M \subseteq \mathcal{M}(\mathbb{R}^n)$ be closed. Then M is compact if and only if*

$$\sup_{\mu \in M} \mu(B(0,R)) < \infty \quad \textit{for all } R > 0.$$

In particular, every sequence $(\mu_k) \subseteq \mathcal{M}(\mathbb{R}^n)$ *with*

$$\sup_{k\in\mathbb{N}} \mu_k(B(0,R)) < \infty \text{ for all } R > 0$$

contains a convergent subsequence.

2. *There is a sequence* $(f_k) \subseteq C_c(\mathbb{R}^n)$ *of nonnegative Lipschitz functions such that the metric* d *on* $\mathcal{M}(\mathbb{R}^n)$ *defined by*

$$d(\mu,\nu) = \sum_{k=1}^{\infty}(1/2)^k \cdot \min\{1, |\nu(f_k) - \mu(f_k)|\}$$

induces the vague topology.

3. $\mathcal{M}(\mathbb{R}^n)$ *with the metric* d *is a complete separable metric space.*

Proof The proof of (1) can be found in [Mat95, theorem 1.23] and separability of $\mathcal{M}(\mathbb{R}^n)$ is proved, for example, in [Mat95, lemma 14.14].

A sequence as in (2) can be constructed as follows: For every $R \in \mathbb{N}$ take the positive part of the polynomial functions with rational coefficients and multiply them by $x \mapsto \max\{0, 1 - d(x, B(0,R))\}$. In this way we get Lipschitz functions on $\mathbb{R}^n$ with support in $B(0, R+1)$. Denote the countable family of these functions by $(f_i)_{i\in\mathbb{N}}$.

Let $\varepsilon > 0$, $R \in \mathbb{N}$ and $f \in C_c(\mathbb{R}^n)$ with supp $f \subseteq B(0,R)$. By Weierstrass' theorem the polynomials are dense in $C(B(0,R+1))$ and thus there are f_i, f_j such that supp f_i, supp $f_j \subseteq B(0,R+1)$ and $\|f - (f_i - f_j)\|_{\sup} < \varepsilon$.

Now suppose $\nu_k(f_m) \to \nu(f_m)$ for all $m \in \mathbb{N}$. Then $C := \sup_{k\in\mathbb{N}} \nu_k(B(0,R+1)) < \infty$ and $|\nu_k(f) - (\nu_k(f_i) - \nu_k(f_j))| \leq C \cdot \varepsilon$. Therefore also $\nu_k(f) \to \nu(f)$. Hence if $\nu_k(f_m)$ converges to $\nu(f_m)$ for all $m \in \mathbb{N}$ then $\nu_k \to \nu$ and this implies (2).

To prove the completeness take a Cauchy sequence $(\mu_k)_{k\in\mathbb{N}} \subseteq \mathcal{M}(\mathbb{R}^n)$. By definition of the metric in (2) the sequences $(\mu_k(f_m))_{k\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$ for every $m \in \mathbb{N}$. In particular, $\sup_{k\in\mathbb{N}} \mu_k(B(0,R)) < \infty$ for all $R > 0$. By (1) we can find a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that a subsequence of (μ_k) converges to μ. Because any Cauchy sequence in $\mathbb{R}$ converges, we get $\mu_k(f_m) \to \mu(f_m)$ for all $m \in \mathbb{N}$ and hence μ is the limit of the sequence (μ_k) in the vague topology. ■

The following simple lemma will be of use, when we discuss examples in chapter 2.

Lemma 1.2.6 *Let $\alpha \geq 0$. For every $C > 0$ and $\varepsilon > 0$ there are numbers $1 > \delta > 0$ and $R > 1$ such that any two measures $\nu, \mu \in \mathcal{M}(\mathbb{R}^n)$ fulfill $d(\nu,\mu) < \varepsilon$ whenever*

- $\mu(B(0,2R)) \leq C \cdot R^\alpha$.
- *There is a countable, disjoint family $(U_i)_{i\in I}$ of Borel sets, which covers $B(0,R)$ such that $|U_i| < \delta$ and $\nu(U_i) = \mu(U_i)$ for all $i \in I$.*

Proof By lemma 1.2.5 there are nonnegative Lipschitz functions $f_1, \ldots, f_k \in C_c(\mathbb{R}^n)$ such that $d(\nu,\mu) < \varepsilon$ is implied by $|\nu(f_j) - \mu(f_j)| \leq \varepsilon/2$ for all $1 \leq j \leq k$.
There is $R > 1$ such that

$$\bigcup_{j=1}^{k} \text{supp } f_j \subseteq B(0,R)$$

and $1 > \delta > 0$ such that

$$\delta \leq \varepsilon \cdot (2C \cdot \text{Lip}(f_j) \cdot R^\alpha)^{-1} \quad \text{for all } 1 \leq j \leq k.$$

Then we have for all ν, μ fulfilling the conditions of the lemma and all $1 \leq j \leq k$

$$\begin{aligned}
|\nu(f_j) - \mu(f_j)| &\leq \sum_{i\in I} \left| \int_{U_i} f_j \, d\mu - \int_{U_i} f_j \, d\nu \right| \\
&\leq \sum_{i\in I} (\sup\{f_j(x) : x \in U_i\} - \inf\{f_j(x) : x \in U_i\}) \cdot \mu(U_i) \\
&\leq \sum_{i\in I} \text{Lip}(f_j) \cdot \delta \cdot \mu(U_i) \\
&\leq \text{Lip}(f_j) \cdot \delta \cdot \mu(B(0,2R)) \\
&\leq \varepsilon/2
\end{aligned}$$

and this finishes the proof of the lemma. ■

By definition of the vague topology the mapping $\nu \mapsto \nu(f)$ is continuous for all $f \in C_c(\mathbb{R}^n)$. The following lemma is a useful generalization of this fact.

Lemma 1.2.7 *Suppose the map*

$$\begin{aligned}
G: \quad \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\
(\nu, y) &\mapsto G(\nu, y)
\end{aligned}$$

is continuous and the set $A = \{y : \text{ there is } \nu \in \mathcal{M}(\mathbb{R}^n) \text{ such that } G(\nu, y) \neq 0\ \}$ *is bounded. Then the map*

$$\begin{array}{rcll} H: & \mathcal{M}(\mathbb{R}^n) & \longrightarrow & \mathbb{R}, \\ & \nu & \mapsto & \int G(\nu, y)\, d\nu(y) \end{array}$$

is continuous.

Proof Fix $\nu \in \mathcal{M}(\mathbb{R}^n)$ and $\varepsilon > 0$. Denote $f(y) = G(\nu, y)$. Since $f \in C_c(\mathbb{R}^n)$ there is $\delta_1 > 0$ such that $d(\nu, \mu) < \delta_1$ implies

$$|\nu(f) - \mu(f)| < \varepsilon/2 \qquad \text{and} \qquad \mu(\operatorname{cl} A) < \nu(\operatorname{cl} A) + \varepsilon\,.$$

Furthermore, using the compactness of $\operatorname{cl} A$, we find $\delta_2 > 0$ such that $d(\nu, \mu) < \delta_2$ implies

$$|G(\nu, y) - G(\mu, y)| < \frac{\varepsilon}{2 \cdot (\nu(\operatorname{cl} A) + \varepsilon)}$$

for all $y \in A$. Then for all $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that $d(\nu, \mu) < \min\{\delta_1, \delta_2\}$ we have

$$\begin{aligned} & \left|\int G(\nu, y)\, d\nu(y) - \int G(\mu, y)\, d\mu(y)\right| \\ \leq\ & \left|\int f\, d\nu - \int f\, d\mu\right| + \left|\int G(\nu, y)\, d\mu(y) - \int G(\mu, y)\, d\mu(y)\right| \\ <\ & \varepsilon/2 + \frac{\varepsilon}{2 \cdot (\nu(\operatorname{cl} A) + \varepsilon)} \cdot \mu(\operatorname{cl} A) \\ <\ & \varepsilon \end{aligned}$$

as required to show continuity. ■

Finally let us introduce the space of probability distributions on $\mathcal{M}(\mathbb{R}^n)$.

Definition

A *probability distribution* on $\mathcal{M}(\mathbb{R}^n)$ is a measure P on the Borel-σ-algebra of $\mathcal{M}(\mathbb{R}^n)$ such that $P(\mathcal{M}(\mathbb{R}^n)) = 1$. Let $\mathcal{P}$ be the set of all probability distributions on $\mathcal{M}(\mathbb{R}^n)$. The *weak topology* on $\mathcal{P}$ is the smallest topology such that for every continuous bounded function $F : \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ the functional $F_\star$, defined by

$$\begin{array}{rcll} F_\star: & \mathcal{P} & \longrightarrow & \mathbb{R} \\ & P & \mapsto & P(F) := \int F\, dP, \end{array}$$

is continuous.

Lemma 1.2.8

1. *For $P_k, P \in \mathcal{P}$ the following statements are equivalent:*

(a) $P_k \longrightarrow P$,

(b) $\liminf P_k(G) \geq P(G)$ *for all open sets* $G \subseteq \mathcal{M}(\mathbb{R}^n)$,

(c) $\limsup P_k(A) \leq P(A)$ *for all closed sets* $A \subseteq \mathcal{M}(\mathbb{R}^n)$.

2. *There is a family of functions* $(F_i)_{i\in\mathbb{N}} \subseteq C_b(\mathcal{M}(\mathbb{R}^n))$ *such that* $P_k \longrightarrow P$ *if and only if* $P_k(F_i) \longrightarrow P(F_i)$ *for all* $i \in \mathbb{N}$. *The metric* D *on* $\mathcal{P}$, *defined by*

$$D(P,Q) = \sum_{i=1}^{\infty}(1/2)^i \cdot \min\{1, |P(F_i) - Q(F_i)|\},$$

induces the weak topology. $\mathcal{P}$ *with the metric* D *is a separable metric space.*

3. (Prohorov's theorem)
Let $S \subseteq \mathcal{P}$. *Then* S *is relatively compact if and only if for every* $\varepsilon > 0$ *there is a compact set* $K \subseteq \mathcal{M}(\mathbb{R}^n)$ *such that* $P(K) \geq 1 - \varepsilon$ *for all* $P \in S$.

Proof By lemma 1.2.5 the space $\mathcal{M}(\mathbb{R}^n)$ is a separable metric space. Therefore we can apply [Par72, theorem II.6.1] to prove (1), [Par72, theorem II.6.2] to prove (2), and [Par72, theorem II.6.7] to prove (3). ■

1.3 Densities and Tangent Measures

We recall now two essential notions from geometric measure theory: Densities and tangent measures. Heuristically speaking, the densities measure the "concentration of μ about x".

Definition
Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0 \leq \alpha \leq n$.
For $x \in \mathbb{R}^n$ define the *lower* and *upper α-density of μ at x* as

$$\underline{d}^{\alpha}(\mu, x) = \liminf_{t\downarrow 0} \frac{\mu(B(x,t))}{t^{\alpha}} \quad \text{and} \quad \overline{d}^{\alpha}(\mu, x) = \limsup_{t\downarrow 0} \frac{\mu(B(x,t))}{t^{\alpha}}.$$

If

$$\underline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu, x) < \infty$$

we say that the *α-density of μ at x exists* and call the joint value $d^\alpha(\mu, x)$ the *α-density of μ at x*. If we have

$$0 < \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) < \infty$$

we say that μ has *positive and finite α-densities at x.*

Remarks

Some authors (e.g. Mattila in [Mat95]) chose a different normalization for the densities.

The following proposition relates the densities of absolutely continuous measures.

Proposition 1.3.1 *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$. If $\mu \ll \nu$ with Radon-Nikodym derivative $f = \frac{d\mu}{d\nu}$ then, for ν-almost every x, we have*

$$\overline{d}^\alpha(\mu, x) = f(x) \cdot \overline{d}^\alpha(\nu, x) \text{ and } \underline{d}^\alpha(\mu, x) = f(x) \cdot \underline{d}^\alpha(\nu, x).$$

In particular, if $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$ is a μ-measurable set then, for μ-almost every x,

$$\overline{d}^\alpha(\mu|_E, x) = 1_E(x) \cdot \overline{d}^\alpha(\mu, x) \text{ and } \underline{d}^\alpha(\mu|_E, x) = 1_E(x) \cdot \underline{d}^\alpha(\mu, x).$$

Proof The statement follows from

$$\frac{\mu(B(x,r))}{r^\alpha} = \frac{\mu(B(x,r))}{\nu(B(x,r))} \cdot \frac{\nu(B(x,r))}{r^\alpha}$$

by taking limits and using lemma 1.2.2. ∎

The following deep regularity theorem of D. Preiss shows that the densities contain all the information about the rectifiability of a measure.

Theorem 1.3.2 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. The following conditions are equivalent:*

(1) *The α-density of μ at x exists and is positive for μ-almost all x.*

(2) *μ is α-rectifiable.*

Proof See [Pre87, theorem 5.6] or, for a sketch of the proof, [Mat95, chapter 17]. ■

Theorem 1.3.2 comprises the following result of J.M. Marstrand ([Mar64]): If the α-density of μ at x exists and is positive for μ-almost every x then α must be an integer (see [Mat95, chapter 14] for a very readable proof of Marstrand's theorem). Theorem 1.3.2 is the climax in a long sequence of results in this direction, for example [Bes28], [Bes38], [Mar61] and [Mat75].

Whereas the existence of positive α-densities almost everywhere has turned out to be a very restrictive condition, positivity and finiteness of the α-densities almost everywhere is a much milder condition which holds for many fractal measures. Finiteness of upper densities almost everywhere is even fulfilled for all α-Hausdorff measures on α-sets.

Proposition 1.3.3 *Let $0 \le \alpha \le 1$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then the following statements are equivalent:*

(1) *$0 < \overline{d}^\alpha(\mu, x) < \infty$ for μ-almost every x.*

(2) *There is a disjoint family $(E_i)_{i\in\mathbb{N}}$ of α-sets such that $\mu \ll \mathcal{H}^\alpha|_{\bigcup E_i}$.*

Proof (1) $\Rightarrow$ (2) Without losing generality we can assume that μ is finite. For every integer i define the set

$$E_i = \{x \in \operatorname{supp} \mu \,:\, 2^{i-1} \le \overline{d}^\alpha(\mu, x) < 2^i\}\,.$$

The E_i are disjoint and cover supp μ. We look at the family consisting of those sets E_i which fulfill $\mu(E_i) > 0$. If $N \subseteq E_i$ has $\mathcal{H}^\alpha(N) = 0$ then, by [Mat95, theorem 6.9], we have $\mu(N) \le 2^i \cdot \mathcal{H}^\alpha(N) = 0$, and thus $\mu \ll \mathcal{H}^\alpha|_{\bigcup E_i}$.
Also by [Mat95, theorem 6.9] we have

$$\mathcal{H}^\alpha(E_i) \ge 2^{-i} \cdot \mu(E_i) > 0$$

and

$$\mathcal{H}^\alpha(E_i) \le 2^\alpha \cdot 2^{-(i-1)} \cdot \mu(E_i) < \infty\,.$$

Thus the E_i are α-sets, as required.

(2) $\Rightarrow$ (1) Denote $\nu = \mathcal{H}^\alpha|_{E_i}$ and let $f = \frac{d\mu}{d\nu}$ be the Radon-Nikodym derivative. By [Mat95, theorem 6.2] we know that $\overline{d}^\alpha(\nu, x) \le 2^\alpha$ for ν-almost every $x \in E_i$, and we can use lemma 1.3.1 to see

$$\overline{d}^\alpha(\mu, x) \le f(x) \cdot 2^\alpha < \infty \ \mu\text{-almost everywhere.}$$

By [Mat95, theorem 6.9] the sets $N_i = \{x \in E_i : \overline{d}^\alpha(\mu, x) = 0\}$ have $\mu(N_i) \le \lambda\mathcal{H}^\alpha(E_i)$ for all $\lambda > 0$. Hence $\mu(N_i) = 0$ and this proves the theorem. ∎

In order to take a closer look at the local geometry of a measure μ we define the notion of a tangent measure. Tangent measures are an extension of ideas used in [Mar61] and [Mat75]. They were introduced in the present form by D. Preiss in [Pre87] and used very effectively in his proof of the regularity theorem. Many applications and details on tangent measures can be found in [Mat95].

Roughly speaking, the set $\text{Tan}(\mu, x)$ of tangent measures of μ at a point x is defined as the set of limits of enlargements of μ about x. On the one hand this definition makes sense for all $\mu \in \mathcal{M}(\mathbb{R}^n)$ including fractal measures and on the other hand it turns out to be consistent with the notion of an approximate tangent space at μ-almost every point of a rectifiable measure (see [Fed68, 3.2.16] and theorem 1.3.12 below).

Definition

For every $\mu \in \mathcal{M}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $t > 0$, the *enlargement of μ about x of scale t* is the measure $\mu_{x,t} \in \mathcal{M}(\mathbb{R}^n)$ defined by

$$\mu_{x,t}(A) = \mu(x + tA) \quad \text{for all Borel sets } A \subseteq \mathbb{R}^n.$$

Lemma 1.3.4 *The map*

$$\begin{array}{rccc} M: & \mathcal{M}(\mathbb{R}^n)\times\mathbb{R}^n\times(0,\infty) & \longrightarrow & \mathcal{M}(\mathbb{R}^n) \\ & (\mu,x,t) & \mapsto & \mu_{x,t} \end{array}$$

is continuous.

Proof Fix $(\mu,x,t)\in\mathcal{M}(\mathbb{R}^n)\times\mathbb{R}^n\times(0,\infty)$ and $\varepsilon>0$ and let $f\in\mathcal{C}_c(\mathbb{R}^n)$ be nonnegative with supp $f\subseteq B(0,R)$ and $\mathrm{Lip}(f)\le 1$. Then

$$|\mu_{x,t}(f)-\nu_{y,s}(f)|\le|\mu_{x,t}(f)-\mu_{y,s}(f)|+|\mu_{y,s}(f)-\nu_{y,s}(f)|\,.$$

Let ν be such that we have

$$\sup\left\{\left|\int g\,d\nu-\int g\,d\mu\right| : \text{supp } g\subseteq B(0,|x|+1+2tR),\, g\ge 0,\, \mathrm{Lip}(g)\le 1\right\}<\varepsilon/(4t)\,.$$

Let s be such that $|1-s/t|,|1-t/s|<\varepsilon/(16tR\cdot\mu(B(x,2tR)))$ and $2t>s>t/2$. Let y be such that $|x-y|<(t\varepsilon)/(16\cdot\mu(B(x,2tR)))$ and $|x-y|<1$.

With these conditions fulfilled, $z\mapsto(t/2)\cdot f((z-y)/s)$ is continuous with support in $B(y,sR)\subseteq B(0,|x|+1+2tR)$ and Lipschitz constant at most 1. Therefore

$$\begin{aligned} |\mu_{y,s}(f)-\nu_{y,s}(f)| &\le (2/t)\left|\int(t/2)\cdot f\left(\frac{z-y}{s}\right)\,d\mu(z)-\int(t/2)\cdot f\left(\frac{z-y}{s}\right)\,d\nu(z)\right| \\ &\le (2/t)\cdot(\varepsilon/4t)=\varepsilon/2. \end{aligned}$$

Because $\mathrm{Lip}(f)\le 1$ we have

$$\begin{aligned} &|\mu_{x,t}(f)-\mu_{y,s}(f)| \\ &\le \int_{B(x,tR)}\left|\frac{z-x}{t}-\frac{z-y}{s}\right|d\mu(z)+\int_{B(y,sR)}\left|\frac{z-x}{t}-\frac{z-y}{s}\right|d\mu(z) \end{aligned}$$

and

$$\begin{aligned} &\int_{B(x,tR)}\left|\frac{z-x}{t}-\frac{z-y}{s}\right|d\mu(z) \\ &\le \int_{B(x,tR)}\left|\frac{z-x}{t}-\frac{z-x}{s}\right|d\mu(z)+\int_{B(x,tR)}\left|\frac{z-x}{s}-\frac{z-y}{s}\right|d\mu(z) \\ &\le \mu(B(x,tR))\Big[tR\cdot\left|1-\frac{t}{s}\right|+\frac{1}{s}\cdot|y-x|\Big] \\ &\le \frac{\varepsilon}{4} \end{aligned}$$

and similarly

$$\int_{B(y,sR)} \left| \frac{z-x}{t} - \frac{z-y}{s} \right| d\mu(z) \leq \frac{\varepsilon}{4}.$$

This implies continuity of M by lemma 1.2.4. ■

Definition

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. The set of all *tangent measures of* μ *at* x is defined as

$$\operatorname{Tan}(\mu, x) = \{\nu \in \mathcal{M}(\mathbb{R}^n) \,:\, \nu = \lim c_n \cdot \mu_{x,t_n} \text{ for some } t_n \downarrow 0 \text{ and } c_n > 0\}.$$

Note that for all $\lambda \geq 0$ and $\nu \in \operatorname{Tan}(\mu, x)$ we have $\lambda\nu \in \operatorname{Tan}(\mu, x)$, in other words $\operatorname{Tan}(\mu, x) \subseteq \mathcal{M}(\mathbb{R}^n)$ is a cone. We say that μ has a *unique tangent measure at* x if there is $\nu \in \operatorname{Tan}(\mu, x)$ such that

$$\operatorname{Tan}(\mu, x) = \{\lambda\nu \,:\, \lambda \geq 0\}.$$

Remarks:

1. Some authors explicitly exclude the zero-measure from $\operatorname{Tan}(\mu, x)$, which in our case would be inconvenient.

2. If $\nu \in \operatorname{Tan}(\mu, x)$ and $\lambda > 0$ then $\nu_{0,\lambda} \in \operatorname{Tan}(\mu, x)$. For $\nu = \lim c_n \cdot \mu_{x,t_n}$ this follows from $\nu_{0,\lambda} = \lim(c_n \cdot \mu_{x,t_n})_{0,\lambda} = \lim c_n \cdot \mu_{x,\lambda t_n}$.

Proposition 1.3.5 *For all* $x \in \mathbb{R}^n$ *the set* $\operatorname{Tan}(\mu, x) \subseteq \mathcal{M}(\mathbb{R}^n)$ *is closed.*

Proof Recall that the topology on $\mathcal{M}(\mathbb{R}^n)$ is generated by the metric d defined in lemma 1.2.5. If $(\nu_k) \subseteq \operatorname{Tan}(\mu, x)$ and $\nu_k \to \nu$ then there are sequences $(t_i^k)_{i\in\mathbb{N}}$ such that $0 < t_i^k < (1/i)$ and $d(\mu_{x,t_i^k}/(t_i^k)^\alpha, \nu_k) < (1/i)$. Picking the diagonal sequence yields

$$d(\mu_{x,t_k^k}/(t_k^k)^\alpha, \nu) \leq d(\mu_{x,t_k^k}/(t_k^k)^\alpha, \nu_k) + d(\nu_k, \nu) \longrightarrow 0,$$

and thus $\nu \in \operatorname{Tan}(\mu, x)$. Hence $\operatorname{Tan}(\mu, x)$ is closed. ■

T.C. O'Neil in [O'N95] has constructed an example of a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that $\mathrm{Tan}(\mu, x) = \mathcal{M}(\mathbb{R}^n)$ μ-almost everywhere. This example indicates that we have to impose additional restrictions on μ in order to get compactness properties of $\mathrm{Tan}(\mu, x)$.

Definition

A measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ fulfills a *doubling-condition at x* if there is a $C > 0$ such that

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < C\,.$$

We will see below that this condition has useful implications on $\mathrm{Tan}(\mu, x)$.

Definition

A function $\Delta : \mathbb{R}^n \longrightarrow [0, \infty)$ is a *normalizing function* if $\Delta \in \mathcal{C}_c(\mathbb{R}^n)$ and $\Delta(0) > 0$. For any normalizing function Δ define

$$\begin{aligned} M^\Delta :\ \mathcal{M}(\mathbb{R}^n) &\longrightarrow \mathcal{M}(\mathbb{R}^n)\,, \\ \nu &\mapsto \begin{cases} \frac{\nu}{\nu(\Delta)} & \text{if } \nu(\Delta) > 0\,, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Observe that M^Δ is continuous on the open set $\{\nu \in \mathcal{M}(\mathbb{R}^n) : \nu(\Delta) > 0\}$ and define the set of all *Δ-normalized tangent measures of μ at x* as

$$\mathrm{Tan}^\Delta(\mu, x) = \{\nu \in \mathcal{M}(\mathbb{R}^n) : \nu = \lim_{n \to \infty} M^\Delta(\mu_{x,t_n}) \text{ for some } t_n \downarrow 0\ \}\,.$$

For measures that fulfill a doubling-condition almost everywhere we can study $\mathrm{Tan}^\Delta(\mu, x)$ instead of $\mathrm{Tan}(\mu, x)$ without losing information.

Proposition 1.3.6 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ fulfills a doubling-condition at x. Then*

1. *The sets $cl\{M^\Delta(\mu_{x,t}) : t \in (0, 1)\}$ and $\mathrm{Tan}^\Delta(\mu, x) \subseteq \mathcal{M}(\mathbb{R}^n)$ are compact.*
2. *$\mathrm{Tan}^\Delta(\mu, x)$ is not empty.*
3. *For every non-zero $\nu \in \mathrm{Tan}(\mu, x)$ we have $0 \in \mathrm{supp}\ \nu$.*
4. *$\mathrm{Tan}^\Delta(\mu, x) = M^\Delta(\mathrm{Tan}(\mu, x) \setminus \{\phi\})$ and $\mathrm{Tan}(\mu, x) = \{\lambda\nu : \nu \in \mathrm{Tan}^\Delta(\mu, x), \lambda \geq 0\}$.*

5. The sets $\operatorname{Tan}(\mu,x)$ *and* $\operatorname{Tan}^{\Delta}(\mu,x)$ *are connected.*

Proof (1) By definition of a normalizing function there are $c>0$ and $\varepsilon>0$ such that $\Delta(x)>c$ for all $x\in B(0,\varepsilon)$. Let $D=\limsup_{t\downarrow 0}\mu(B(0,2t))/\mu(B(0,t))$.
Let $R>0$, say with $R<2^k\varepsilon$. Then

$$\limsup_{t\downarrow 0} M^{\Delta}(\mu_{x,t})(B(0,2R)) \le \limsup_{t\downarrow 0}\frac{\mu(B(x,2^{k+1}t\varepsilon))}{c\mu(B(x,t\varepsilon))}\le\frac{D^{k+1}}{c},$$

Hence there is $t_0>0$ such that $M^{\Delta}(\mu_{x,t})(B(0,2R))\le 2D^{k+1}/c$ for all $0<t\le t_0$. This implies, in particular, that $\nu(B(0,R))\le 2D^{k+1}/c$. By lemma 1.3.4 we also have

$$\sup_{t\in[t_0,1]} M^{\Delta}(\mu_{x,t})(B(0,R))<\infty\,.$$

Thus (1) follows with lemma 1.2.5.
(2) By (1) every sequence in $\{M^{\Delta}(\mu_{x,t})\,:\,t\in(0,1)\}$ has a convergent subsequence with limit in $\mathcal{M}(\mathbb{R}^n)$ and this implies (2).
(3) For every $\delta>0$ and every non-zero tangent measure $\nu=\lim c_n\cdot\mu_{x,t_n}$, say with $\nu(B(0,R))>0$ and $R<2^k\delta$, we have

$$\begin{aligned}\nu(B(0,\delta)) &\ge \liminf c_n\cdot\mu(B(x,\delta t_n))\\ &\ge \liminf c_n\mu(B(x,2Rt_n))\cdot\liminf\frac{\mu(B(x,\delta t_n))}{\mu(B(x,2^{k+1}\delta t_n))}\\ &\ge \nu(B(0,R))\cdot(1/D)^{k+1}>0\,,\end{aligned}$$

and therefore $0\in\operatorname{supp}\nu$. This proves (3).
(4) If $\nu\in\operatorname{Tan}^{\Delta}(\mu,x)$, then $\nu=M^{\Delta}(\nu)\in M^{\Delta}(\operatorname{Tan}(\mu,x)\setminus\{\phi\})$. If $\nu=\lim c_n\cdot\mu_{x,t_n}\in\operatorname{Tan}(\mu,x)$ is a non-zero measure then we have, using the continuity of M^{Δ} and (3),

$$M^{\Delta}(\nu)=\lim M^{\Delta}(c_n\mu_{x,t_n})=\lim M^{\Delta}(\mu_{x,t_n})\,,$$

and thus $M^{\Delta}(\nu)\in\operatorname{Tan}^{\Delta}(\mu,x)$. This proves the first equation in (4). If $\nu\in\operatorname{Tan}(\mu,x)$ and ν is not the zero-measure then

$$\nu=\nu(\Delta)\cdot M^{\Delta}(\nu)\,,$$

since $\nu(\Delta)>0$ by (3). Then $\lambda\nu\in\operatorname{Tan}^{\Delta}(\mu,x)$ for $\lambda=\nu(\Delta)^{-1}$. If $\nu\in\operatorname{Tan}^{\Delta}(\mu,x)$, then $\nu\in\operatorname{Tan}(\mu,x)$ and also $\lambda\nu\in\operatorname{Tan}(\mu,x)$ for all $\lambda\ge 0$. This finishes the proof of (4).

(5) Suppose $\mathrm{Tan}^{\Delta}(\mu, x)$ is not connected. Then there is $\varepsilon > 0$ and there are compact sets $D_1, D_2 \subseteq \mathrm{Tan}^{\Delta}(\mu, x)$ such that $D_1 \cup D_2 = \mathrm{Tan}^{\Delta}(\mu, x)$ and $d(D_1, D_2) > \varepsilon$. Let

$$E_1 = \{t \in (0,1) : M^{\Delta}(\mu_{x,t}) \in B(D_1, \varepsilon/2)\} \quad \text{and}$$

$$E_2 = \{t \in (0,1) : M^{\Delta}(\mu_{x,t}) \in B(D_2, \varepsilon/2)\}.$$

There is $\delta > 0$ such that $(0, \delta) \subseteq E_1 \cup E_2$ and the union is disjoint. Since $t \mapsto M^{\Delta}(\mu_{x,t})$ is continuous, the sets E_1, E_2 are closed in $(0,1)$ and this contradicts the connectedness of $(0, \delta)$. Hence $\mathrm{Tan}^{\Delta}(\mu, x)$ must be connected and from the second equality in (4) we see that $\mathrm{Tan}(\mu, x)$ is also connected. ■

Proposition 1.3.7 *If a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ has positive and finite α-densities at x then μ fulfills a doubling condition at x.*

Proof Observe that we have

$$\begin{aligned}
&\limsup_{r \downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} \\
\leq\ & 2^{\alpha} \cdot \limsup_{r \downarrow 0} \frac{\mu(B(x,2r))}{(2r)^{\alpha}} \cdot \limsup_{r \downarrow 0} \frac{r^{\alpha}}{\mu(B(x,r))} \\
\leq\ & 2^{\alpha} \cdot \overline{d}^{\alpha}(\mu, x) / \underline{d}^{\alpha}(\mu, x) < \infty .
\end{aligned}$$

■

There is another subset of $\mathrm{Tan}(\mu, x)$ which is particularly suitable for the investigation of measures with bounded densities.

Definition

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, $0 \leq \alpha \leq n$ and $x \in \mathbb{R}^n$. Then

$$\mathrm{Tan}_{\mathcal{S}}^{\alpha}(\mu, x) = \{\nu \in \mathcal{M}(\mathbb{R}^n) : \nu = \lim_{n \to \infty} \frac{\mu_{x,t_n}}{t_n^{\alpha}} \text{ for some } t_n \downarrow 0 \}$$

is the set of all *α-standardized tangent measures of μ at x*. If μ has positive and finite α-densities at x, then the sets $\mathrm{Tan}_{\mathcal{S}}^{\beta}(\mu, x)$, $\beta \neq \alpha$ contain no non-zero measures and we write $\mathrm{Tan}_{\mathcal{S}}(\mu, x)$ instead of $\mathrm{Tan}_{\mathcal{S}}^{\alpha}(\mu, x)$.

Let us collect a couple of properties of standardized tangent measures. We start with an easy scaling invariance property. For a fixed $\alpha \geq 0$ define the *rescaling group* $(T_\lambda)_{\lambda>0}$ by

$$\begin{array}{rccc} T_\lambda : & \mathcal{M}(\mathbb{R}^n) & \longrightarrow & \mathcal{M}(\mathbb{R}^n) \\ & \nu & \mapsto & \frac{\nu_{0,\lambda}}{\lambda^\alpha} \end{array}$$

and note that T_λ is continuous (by lemma 1.3.4) and $T_\lambda T_\kappa = T_{\lambda\kappa}$.

Proposition 1.3.8 *For every $\lambda > 0$ and $\nu \in \mathrm{Tan}_S^\alpha(\mu, x)$ we have $T_\lambda \nu \in \mathrm{Tan}_S^\alpha(\mu, x)$.*

Proof If $\nu = \lim \mu_{x,t_n}/t_n^\alpha$, then by lemma 1.3.4

$$T_\lambda \nu = \lim T_\lambda \left(\mu_{x,t_n}/t_n^\alpha\right) = \lim \mu_{x,\lambda t_n}/(\lambda t_n)^\alpha$$

and thus $T_\lambda \nu \in \mathrm{Tan}_S^\alpha(\mu, x)$. ■

Proposition 1.3.9 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ has finite upper α-density at x. Then*

1. *The sets $cl\{\mu_{x,t}/t^\alpha : t \in (0,1)\}$ and $\mathrm{Tan}_S^\alpha(\mu, x) \subseteq \mathcal{M}(\mathbb{R}^n)$ are compact.*
2. *The set $\mathrm{Tan}_S^\alpha(\mu, x) \subseteq \mathcal{M}(\mathbb{R}^n)$ is connected.*

If μ has also positive lower α-density at x then

3. *For every non-zero tangent measure $\nu \in \mathrm{Tan}(\mu, x)$ there is a $\lambda > 0$ such that $\lambda\nu \in \mathrm{Tan}_S(\mu, x)$.*
4. *$\mathrm{Tan}_S(\mu, x)$ is not empty and every $\nu \in \mathrm{Tan}_S(\mu, x)$ fulfills $0 \in \mathrm{supp}\ \nu$.*
5. *We have*

$$\underline{d}^\alpha(\mu, x) = \inf\{\nu(B(0,1)) : \nu \in \mathrm{Tan}_S(\mu, x)\}$$

and

$$\overline{d}^\alpha(\mu, x) = \sup\{\nu(B(0,1)) : \nu \in \mathrm{Tan}_S(\mu, x)\}.$$

Proof Denote $C = \overline{d}^\alpha(\mu, x)$.

(1) Let $R > 0$. Then $\limsup_{t\downarrow 0} \frac{\mu_{x,t}(B(0,2R))}{t^\alpha} \le C \cdot (2R)^\alpha$. Hence there is $t_0 > 0$ such that $\frac{\mu_{x,t}(B(0,2R))}{t^\alpha} \le 3^\alpha C \cdot R^\alpha$ for all $0 < t \le t_0$. In particular, this implies that $\nu(B(0,R)) \le 3^\alpha C \cdot R^\alpha$. By lemma 1.3.4 we also have $\sup_{t\in[t_0,1]} \frac{\mu_{x,t}(B(0,R))}{t^\alpha} < \infty$. Thus (1) follows with lemma 1.2.5.

(2) As in proposition 1.3.6(5) the compactness of $\mathrm{Tan}^\alpha(\mu, x)$ and continuity of the mapping $t \mapsto \frac{\mu_{x,t}}{t^\alpha}$ imply the connectedness of $\mathrm{Tan}_S^\alpha(\mu, x)$.

Suppose now that

$$c \le \underline{d}^\alpha(\mu, x) \le \overline{d}^\alpha(\mu, x) \le C\,.$$

(3) Let $\nu = \lim \mu_{x,t_n} \in \mathrm{Tan}(\mu, x)$ be a non-zero measure, say with $\nu(B(0, R/2)) > 0$. Then

$$\begin{aligned}\limsup c_n \cdot t_n^\alpha &\le \limsup c_n \cdot \mu(B(x, Rt_n)) \cdot \limsup \frac{t_n^\alpha}{\mu(B(x, Rt_n))}\\ &\le \nu(B(0,R)) \cdot 1/(cR^\alpha)\,,\end{aligned}$$

and

$$\begin{aligned}\liminf c_n \cdot t_n^\alpha &\ge \liminf c_n \cdot \mu(B(x, Rt_n)) \cdot \liminf \frac{t_n^\alpha}{\mu(B(x, Rt_n))}\\ &\ge \nu(B(0,R/2)) \cdot 1/(CR^\alpha)\,,\end{aligned}$$

and thus we can pick a subsequence $(c_{k_n} t_{k_n}^\alpha)$ such that $\lim c_{k_n} t_{k_n}^\alpha = \lambda > 0$. Then

$$\nu = \lim c_{k_n} \mu_{x,t_{k_n}} = \lambda \cdot \lim \frac{\mu_{x,t_{k_n}}}{t_{k_n}^\alpha}$$

and $\lim \frac{\mu_{x,t_{k_n}}}{t_{k_n}^\alpha} \in \mathrm{Tan}_S(\mu, x)$. This proves (3).

(4) Since by proposition 1.3.6 the set $\mathrm{Tan}(\mu, x)$ contains non-zero measures, we also have $\mathrm{Tan}_S^\alpha(\mu, x) \ne \emptyset$. For every $\nu \in \mathrm{Tan}_S^\alpha(\mu, x)$ and every $\delta > 0$ we have

$$\nu(B(0,\delta)) \ge \liminf_{r\downarrow 0} \frac{\mu(B(x, \delta r))}{r^\alpha} \ge \delta^\alpha \cdot c > 0\ ,$$

and thus $0 \in \mathrm{supp}\ \nu$ which completes the proof of (4).

(5) We have for every $\nu = \lim \mu_{x,t_n}/t_n^\alpha \in \mathrm{Tan}_S(\mu, x)$ and for every $\eta > 1$ that

$$\nu(B(0,1)) \le \liminf \mu(B(x, \eta t_n))/t_n^\alpha \le \overline{d}^\alpha(\mu, x) \cdot \eta^\alpha\,,$$

and with $\eta \to 1$ we get

$$\sup\{\nu(B(0,1))\ :\ \nu \in \mathrm{Tan}_S(\mu, x)\} \le \overline{d}^\alpha(\mu, x)\,.$$

If, on the other hand,

$$\overline{d}^{\alpha}(\mu, x) = \lim_{k\to\infty} \frac{\mu(B(x, t_k))}{t_k^{\alpha}},$$

we use (1) to pick a subsequence (r_k) of (t_k) such that

$$\nu = \lim \frac{\mu_{x,r_k}}{r_k^{\alpha}} \in \mathrm{Tan}_S(\mu, x)$$

exists. Then

$$\nu(B(0,1)) \geq \limsup \frac{\mu(B(x, r_k))}{r_k^{\alpha}} = \overline{d}^{\alpha}(\mu, x)$$

and this proves the first equality in (5).

The second equality is proved in the same way. ∎

Tangent measures and standardized tangent measures share a very useful shift-invariance property, which was discovered by D. Preiss (see [Pre87, 2.12]). Define the *shift-operator*

$$\begin{array}{rccc} T: & \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n & \longrightarrow & \mathcal{M}(\mathbb{R}^n) \\ & (\nu, u) & \mapsto & T^u\nu := \nu_{u,1}\,. \end{array}$$

Observe that T is continuous by lemma 1.3.4.

Theorem 1.3.10

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. The following statements hold for μ-almost every x:

1. *If $\nu \in \mathrm{Tan}(\mu, x)$ and $u \in \mathrm{supp}\ \nu$, then also $T^u\nu \in \mathrm{Tan}(\mu, x)$.*
2. *If $\nu \in \mathrm{Tan}_S^{\alpha}(\mu, x)$ and $u \in \mathrm{supp}\ \nu$, then also $T^u\nu \in \mathrm{Tan}_S^{\alpha}(\mu, x)$.*

Proof For the case of $\mathrm{Tan}(\mu, x)$ the proof can be found in [Pre87, 2.12]. This proof can be adapted as follows to prove the statement in the case of $\mathrm{Tan}_S^{\alpha}(\mu, x)$:

For $p, q \in \mathbb{N}$ let $E_{p,q}$ be the set of all $x \in \mathbb{R}^n$ for which there are $\nu(x) \in \mathrm{Tan}_S^{\alpha}(\mu, x)$ and $u(x) \in \mathrm{supp}\ \nu(x)$ such that

$$d(T^{u(x)}\nu(x), \frac{\mu_{x,r}}{r^{\alpha}}) > 1/p$$

for every $0 < r < (1/q)$.

Suppose the statement is wrong. Then there are $p, q \in \mathbb{N}$ such that $\mu(E_{p,q}) > 0$. We use the separability of $\mathcal{M}(\mathbb{R}^n)$ (see lemma 1.2.5(3)) to find a set $E \subseteq E_{p,q}$ such that $\mu(E) > 0$

and $d(T^{u(x)}\nu(x), T^{u(y)}\nu(y)) < 1/(2p)$ for all $x, y \in E$. By lemma 1.2.3 we can find an $x \in E$ which is a μ-density point of E. Let $r_k \downarrow 0$ be such that $\nu(x) = \lim_{k\to\infty} \frac{\mu_{x,r_k}}{r_k^\alpha}$ and pick $x_k \in E$ such that

$$|x_k - (x + r_k u(x))| < \inf_{y\in E} |y - (x + r_k u(x))| + r_k/k\,.$$

We prove that

$$\lim_{k\to\infty} \inf_{y\in E} |y - (x + r_k u(x))|/r_k = 0\,. \tag{1.1}$$

Assume that (1.1) does not hold. Then there is $0 < \delta < |u(x)|$ such that $\inf_{y\in E} |y - (x + r_k u(x))| > \delta r_k$ for infinitely many values of k. We have

$$\Big(E \cap B(x, 2r_k|u(x)|)\Big) \cup B(x + r_k u(x), \delta r_k) \subseteq B(x, 2r_k|u(x)|)\,,$$

the union on the left hand side being disjoint. Since x is a μ-density point of E we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{\mu(E \cap B(x, 2r_k|u(x)|))}{\mu(B(x, 2r_k|u(x)|))} \\ &\le 1 - \liminf_{k\to\infty} \frac{\mu(B(x + r_k u(x), \delta r_k))}{\mu(B(x, 2r_k|u(x)|))} \\ &\le 1 - \frac{\nu(x)(U(u(x), \delta))}{\nu(x)(B(0, 2|u(x)|))} \\ &< 1\,. \end{aligned}$$

This is a contradiction and thus (1.1) must hold.

By (1.1) we now have $\lim_{k\to\infty} \frac{x_k - x}{r_k} = u(x)$ and thus, using lemma 1.3.4,

$$\lim_{k\to\infty} \frac{\mu_{x_k,r_k}}{r_k^\alpha} = \lim_{k\to\infty} T^{\frac{x_k - x}{r_k}} \frac{\mu_{x,r_k}}{r_k^\alpha} = T^{u(x)}\nu(x)\,.$$

Therefore there is k such that $r_k < (1/q)$ and $d(T^{u(x)}\nu(x), \frac{\mu_{x_k,r_k}}{r_k^\alpha}) < 1/(2p)$. Consequently,

$$\begin{aligned} 1/p &< d(T^{u(x_k)}\mu(x_k), \frac{\mu_{x_k,r_k}}{r_k^\alpha}) \\ &\le d(T^{u(x_k)}\mu(x_k), T^{u(x)}\nu(x)) + d(T^{u(x)}\nu(x), \frac{\mu_{x_k,r_k}}{r_k^\alpha}) \\ &< (1/p)\,. \end{aligned}$$

This is a contradiction, and thus the statement holds. ■

Corollary 1.3.11 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ has positive and finite α-densities μ-almost everywhere. Then for μ-almost every x the following statement holds: For all $\nu \in \mathrm{Tan}_S(\mu, x)$, and all $u \in \mathrm{supp}\ \nu$ and $r > 0$, we have*

$$\underline{d}^\alpha(\mu, x) \cdot r^\alpha \leq \nu(B(u, r)) \leq \overline{d}^\alpha(\mu, x) \cdot r^\alpha$$

for all $r \geq 0$.

Proof For every $\nu \in \mathrm{Tan}_S(\mu, x)$ we have

$$\underline{d}^\alpha(\mu, x) \leq \nu(B(0, 1)) \leq \overline{d}^\alpha(\mu, x)$$

by proposition 1.3.9(5). Moreover, if x is such that the statement of theorem 1.3.10 holds and $\nu \in \mathrm{Tan}_S(\mu, x)$, we also have

$$\frac{\nu_{u,r}}{r^\alpha} \in \mathrm{Tan}_S(\mu, x)$$

for all $u \in \mathrm{supp}\ \nu$ and $r > 0$ (by theorem 1.3.10 and proposition 1.3.8). This implies the statement of the corollary. ∎

The following theorem, which is a part of Preiss' regularity theorem, shows that uniqueness of tangent measure distribution almost everywhere is a property which holds only for rectifiable measures.

Theorem 1.3.12 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ has positive and finite α-densities μ-almost everywhere. Then the following statements are equivalent:*

(1) *μ has a unique tangent measure at μ-almost all points.*

(2) *$\mathrm{Tan}_S(\mu, x)$ is a singleton for μ-almost all points.*

(3) *μ is α-rectifiable.*

If this holds then for μ-almost every x we have $\mathrm{Tan}_S(\mu, x) = \{\nu\}$ and

$$\nu = (1/2)^\alpha \cdot d^\alpha(\mu, x) \cdot \mathcal{H}^\alpha|_T\,,$$

where T is the approximate tangent space at x.

Proof The proof can be found in [Pre87] by combining theorems 4.11 and 5.6 of this paper in the following way:
Suppose (1) holds, i.e. for μ-almost every x there is a $\nu \in \mathcal{M}(\mathbb{R}^n)$ such that

$$\mathrm{Tan}(\mu, x) = \{\lambda\nu \,:\, \lambda \geq 0\}\,.$$

Then $\nu(\partial B(0,t)) = 0$ for all $t > 0$, otherwise proposition 1.3.5 would give a contradiction to local finiteness of ν. Therefore, for every $t > 0$, the limit

$$\lim_{r\downarrow 0} \frac{\mu(B(x,tr))}{\mu(B(x,r))} = \frac{\nu(B(0,t))}{\nu(B(0,1)}$$

exists and by [Pre87, theorem 4.11] this implies

$$\nu = c \cdot \mathcal{H}^m|_T$$

for some linear m-space $T \subseteq \mathcal{M}(\mathbb{R}^n)$ and $c > 0$ and $0 \leq m \leq n$. By [Pre87, theorem 5.6] this implies (3).
If (3) holds then, by [Fed68, 3.2.19], for μ-almost every x there is an approximate tangent space $T \subseteq \mathbb{R}^n$ at x and together with [Pre87, theorem 5.6] this implies $\mathrm{Tan}_{\mathcal{S}}(\mu, x) = \{\nu\}$ for

$$\nu = (1/2)^\alpha \cdot d^\alpha(\mu, x) \cdot \mathcal{H}^\alpha|_T\,.$$

In particular (2) holds. Finally, (2)⇒(1) is clear from proposition 1.3.9(3). ■

Theorem 1.3.12 shows that a measure μ is fractal if and only if $\mu_{x,t}/t^\alpha$ diverges as $t \downarrow 0$ for μ-almost every x. A similar idea has been used by K. Wicks (see [Wic91]) in his definition of "visual fractality".

Chapter 2

Average Densities and Tangent Measure Distributions

We have seen in the previous chapter how densities and tangent measures can be used to characterize rectifiabilty of measures. These properties of densities and tangent measures also indicate their limitations as a means to describe the local geometry of fractal measures: Densities do not enable us to measure the concentration of a fractal measure about its points since, by Preiss' regularity theorem, densities cannot exist and be positive on a set of positive measure. Also at almost all points there is no unique tangent measure and hence tangent measures do not define a natural generalization of the concept of tangent spaces. Therefore more refined tools seem to be necessary for the investigation of the local structure of fractal measures.

One class of tools for these studies is based on the idea of varying the classical notions by replacing ordinary limits by limits of averages:
T. Bedford and A.M. Fisher introduce in [BF92] the order-two or average density of a fractal measure. Several authors, like N. Patzschke and U. Zähle in [PZ90], T. Bedford and A.M. Fisher in [BF92], C. Bandt in [Ban92] and S. Graf in [Gra93], implicitly or explicitly suggested the investigation of random tangent measures based on the same averaging principle. These random tangent measures or, equivalently, probability distributions on

the set of tangent measures, are called tangent measure distributions in this thesis. Tangent measure distributions at a point x in many cases provide a good picture of the local geometry of the measure about x. The average tangent measures, which are the barycentres of the tangent measure distributions, provide another interesting local characteristic. The study of the properties of tangent measure distributions of general measures is the main issue of this thesis.

In section 2.1 we introduce the notion of average density. In section 2.2 we define the standardized tangent measure distributions and average tangent measures and give some of their basic properties. We illustrate the concept by means of an example. In section 2.3 we investigate the relationship between the existence of average densities and the uniqueness of tangent measure distributions and average tangent measures, and in the course of this investigation we study two further interesting examples.

2.1 Average Densities

By Preiss' regularity theorem (1.3.2) the α-densities of a fractal measure cannot exist and be positive except on a set of measure zero. Therefore one would like to find another concept, which assigns to every point x a number which gives an impression of the "density" of the fractal measure about the point and exists for a large class of fractal measures.

T. Bedford and A.M. Fisher introduce such a concept in [BF92]: The *average* or *order-two density*. They show that for suitable Hausdorff-measures on hyperbolic Cantor-sets and zero-sets of Brownian motion the average density exists almost everywhere (see also [Bed91]). By now different authors have shown that for various classes of fractal measures of self-similar type the average density exists, see for example [Fal92], [PZ92], [Spr94] and [FX93]. In some cases the value has been calculated explicitly, see for example [PZ93]. Average densities give a local characteristic for fractal measures closely connected to the heuristic notion of lacunarity (see [Man83, pp.315-318] and [BF92, p.96]). The average densities also contain information on the regularity of the measure (see [FS94] and [Spr94]).

The idea behind average densities is, roughly speaking, the following: Since the density function $f(t) = \mu(B(x,t))/t^\alpha$ oscillates as $t \downarrow 0$ one studies the limit of a family of suitably weighted averages of $f(t)$ as the centre of weight goes to 0.

There are several theoretical approaches to such "averaged limits" which are closely connected and this is discussed in [Fis87] and in particular in [Fis90], where more details can be found. We give here a short account of n-th order averaging operators, using the terminology of A.M. Fisher, in order to justify the averaging procedure T. Bedford and A.M. Fisher use to define the average density.

Definition

Let $\mathcal{P}_1 = \{\psi \in L^1(\mathbb{R}) : \psi \geq 0 \text{ and } \int \psi(t)\,dt = 1\}$. An *averaging operator of order 0* is a map

$$\begin{array}{rccc} A_\psi^0 : & L^\infty(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}), \\ & f & \mapsto & \psi * f \end{array}$$

for some $\psi \in \mathcal{P}_1$. Let

$$\begin{array}{rccc} E : & L^\infty(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}), \\ & f & \mapsto & f \circ \exp\,, \end{array}$$

and

$$\begin{array}{rccc} E^{-1} : & L^\infty(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}) \\ & f & \mapsto & \begin{cases} f \circ \log(t) & \text{if } t > 0\,, \\ 0 & \text{if } t \leq 0. \end{cases} \end{array}$$

Define an *averaging operator of order n* to be a map

$$\begin{array}{rccc} A_\psi^n : & L^\infty(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}), \\ & f & \mapsto & E^{-1} \circ A_\psi^{n-1} \circ E(f)\,. \end{array}$$

An *averaging method of order n* defines the averaged limit of $f \in L^\infty(\mathbb{R})$ as

$$\lim_{t \to \infty} [A_\psi^n f](t)\,.$$

This defines a hierarchy of averaging methods as the following lemma shows:

Lemma 2.1.1 *Let $\varphi \in \mathcal{P}_1$ be defined as $\varphi(x) = e^{-x}\mathbf{1}_{[0,\infty)}(x)$. For any bounded function $f : \mathbb{R} \to \mathbb{R}$ we have that $\lim_{t\to\infty}[A^n_\varphi f](t) = a$ implies $\lim_{t\to\infty}[A^k_\psi f](t) = a$ for all $k \geq n$ and $\psi \in \mathcal{P}_1$.*

Proof The proof is an application of Wiener's Tauberian theorem and can be found in [Fis90, lemmas 4.3 and 4.4]. ∎

Applying this procedure to the problem of defining a "density" means applying an averaging method of suitable order to the function

$$g(t) = \frac{\mu(B(x,(1/t)))}{(1/t)^\alpha}. \tag{2.1}$$

By lemma 2.1.1 one can concentrate on the averaging operators $A^n = A^n_\varphi$. What is the right n to define the average density? Let us have a look at the explicit formulas for the operators A^n: We have

$$[A^1 f](T) = (1/T)\int_0^T f(t)\,dt,$$

the Cesàro-average, and

$$[A^2 f](T) = (\log T)^{-1}\int_1^T f(t)\,\frac{dt}{t},$$

the logarithmic average, and generally

$$[A^n f](T) = (a_n(T))^{-1}\int_{b_n}^T f(t)K_n(t)\,dt,$$

where $b_n = \exp^{(n)}(-\infty)$, $a_n(T) = \log^{(n-1)}(T)$ and $K_n(x) = \frac{d}{dx}(a_n(x))$.

As Fisher points out, one way to understand these formulas is the following: A^1 is an average with respect to Haar-measure on $(\mathbb{R},+)$ restricted to the interval $[0,T]$, A^2 is an average with respect to Haar-measure on $(\mathbb{R}^+,\cdot)$ restricted to the interval $[1,T]$, A^3 is analogously defined for the next higher exponential conjugate of the group $(\mathbb{R},+)$, that is the set $(1,\infty)$ with the operation $(a,b)\mapsto a^{\log b}$, and so forth.

The ordinary ternary Cantor set has obvious self-similarites at scales $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$, and it seems natural to take an average that assigns equal weight to each of these scaling steps, i.e. an average with respect to Haar-measure on $(\mathbb{R}^+,\cdot)$. This is a heuristic argument in favour of an averaging procedure of order two. The heuristic idea is confirmed by the results in the papers mentioned before for the case of self-similar measures, and in [FS94] and chapters 3 and 5 of this thesis for the case of general fractal measures.

Definition

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0 \le \alpha \le n$. For $x \in \mathbb{R}^n$ define the *lower* and *upper average α-density* as

$$\underline{D}^\alpha(\mu,x) = \liminf_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu(B(x,t))}{t^\alpha}\frac{dt}{t},$$

and

$$\overline{D}^\alpha(\mu,x) = \limsup_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu(B(x,t))}{t^\alpha}\frac{dt}{t}.$$

If $\underline{D}^\alpha(\mu,x) = \overline{D}^\alpha(\mu,x) < \infty$ we say that the average α-density of μ at x exists and call the common value $D^\alpha(\mu,x)$ the *average α-density* of μ at x.

Moreover, if $\mu \in \mathcal{M}(\mathbb{R})$ we define the *one-sided lower average α-densities* as

$$\underline{D}^\alpha_-(\mu,x) = \liminf_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu([x-t,x])}{t^\alpha}\frac{dt}{t},$$

and

$$\underline{D}^\alpha_+(\mu,x) = \liminf_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t},$$

the *left-sided* and *right-sided lower average α-densities.* Analogously define the *left-sided* and *right-sided upper average α-densities* $\overline{D}^\alpha_-(\mu,x)$ and $\overline{D}^\alpha_+(\mu,x)$ and, if they exist, the *left-sided* and *right-sided average α-densities* $D^\alpha_-(\mu,x)$ and $D^\alpha_+(\mu,x)$.

Lemma 2.1.2 shows that the average densities indeed result from the application of the order-two averaging method to the function g and gives some equivalent expressions for the average densities.

Lemma 2.1.2 *For every $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have*

$$\begin{aligned} D^\alpha(\mu,x) &= \lim_{t\uparrow\infty}[A^2(g)](t) \\ &= \lim_{t\uparrow\infty}(\log t)^{-1}\int_1^t \frac{\mu(B(x,1/\tau))}{(1/\tau)^\alpha}\frac{d\tau}{\tau} \\ &= \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \frac{\mu(B(x,e^{-r}))}{e^{-r\alpha}}\,dr. \end{aligned}$$

for the function g defined in (2.1). Analogous formulas hold for the lower and upper average densities.

Proof By a substitution of variables, with $t = (1/\varepsilon)$ we get

$$(-\log\varepsilon)^{-1}\int_\varepsilon^1 \frac{\mu(B(x,s))}{s^\alpha}\frac{ds}{s} = (\log t)^{-1}\int_1^t \frac{\mu(B(x,1/\tau))}{(1/\tau)^\alpha}\frac{d\tau}{\tau},$$

and by another substitution we see for $T = \log t$

$$(\log t)^{-1}\int_1^t \frac{\mu(B(x,1/\tau))}{(1/\tau)^\alpha}\frac{d\tau}{\tau} = \frac{1}{T}\int_0^T \frac{\mu(B(x,e^{-r}))}{e^{-r\alpha}}\,dr,$$

and these two equalities imply the statement. ■

Lemma 2.1.3 *Let $f:(0,\infty)\longrightarrow \mathbb{R}$ be measurable such that*

$$\int_K f(x)\,dx < \infty \quad \textit{for all compact } K \subseteq (0,\infty).$$

Let $\varepsilon_n \downarrow 0$. Then the following implications hold: (1) ⇒ (2), (2) ⇔ (3) and, if f is bounded, (2) ⇒ (1) .

(1) $\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 |f(t)|\frac{dt}{t} = 0$

(2) *For every $\varepsilon > 0$ the set $A_\varepsilon = \{t\in(0,1) : |f(t)| > \varepsilon\}$ fulfills*

$$\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_{A_\varepsilon}(t)\frac{dt}{t} = 0.$$

(3) *There is a set $Z \subseteq (0,1)$ such that* $\lim_{\substack{t\downarrow 0\\ t\notin Z}} f(t) = 0$ *and*

$$\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_Z(t)\frac{dt}{t} = 0.$$

Proof The proof is the same as the proof of [Fis90, lemma 4.9].

(1)⇒(2) This implication follows easily from

$$(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_{A_\varepsilon}(t)\frac{dt}{t} \le (1/\varepsilon)\cdot(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 |f(t)|\frac{dt}{t}.$$

(3)⇒(2) If (3) holds then there is $t_0 > 0$ such that $|f(t)| \le \varepsilon$ for all $t \notin Z$, $0 < t < t_0$. Thus A_ε is contained in the set $B = [t_0, 1) \cup Z$ and

$$\begin{aligned}&\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_{A_\varepsilon}(t)\,\frac{dt}{t}\\ \le\ &\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{t_0}^1 \frac{dt}{t} + \lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_Z(t)\,\frac{dt}{t} = 0\,.\end{aligned}$$

(2)⇒(3) Suppose that (2) holds. By definition of the sets A_ε we have

$$A_1 \subseteq A_{\frac{1}{2}} \subseteq A_{\frac{1}{3}} \subseteq A_{\frac{1}{4}} \subseteq \ldots.$$

By (2) there are integers $0 = n_0 < n_1 < \ldots$ such that, for $n \ge n_k$,

$$(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_{A_{1/(k+1)}}(t)\,\frac{dt}{t} \le \frac{1}{k+1}\,.$$

Let

$$A = \bigcup_{k=0}^{\infty}\left(A_{1/(k+1)} \cap [\varepsilon_{n_{k+1}}, \varepsilon_{n_k})\right)\,.$$

If $t \notin A$ and $t < \varepsilon_{n_k}$ then $t \notin A_{1/(k+1)}$ and thus $|f(t)| \le 1/(k+1)$. Therefore we have

$$\lim_{\substack{t\downarrow 0\\ t\notin A}} |f(t)| = 0\,.$$

For all $n_{k+1} \ge n \ge n_k$ we have

$$A \cap [\varepsilon_n, 1) = (A \cap [\varepsilon_{n_k}, 1)) \cup (A \cap [\varepsilon_n, \varepsilon_{n_k})) \subseteq \left(A_{1/k} \cap [\varepsilon_{n_k}, 1)\right) \cup \left(A_{1/(k+1)} \cap [\varepsilon_n, \varepsilon_{n_k})\right)\,.$$

Hence,

$$\begin{aligned}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_A(t)\,\frac{dt}{t} &\le (|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_{A_{1/k}}(t)\,\frac{dt}{t} + (|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 1_{A_{1/(k+1)}}(t)\,\frac{dt}{t}\\ &\le (1/k) + (1/(k+1))\,,\end{aligned}$$

which yields (3).

(2)⇒(1) Suppose $\|f\|_{\sup} \le C$. Then, for every $\varepsilon > 0$, we have

$$\begin{aligned}&\limsup_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 |f(t)|\,\frac{dt}{t}\\ \le\ &\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 C\cdot 1_{A_\varepsilon}(t)\,\frac{dt}{t} + \varepsilon = \varepsilon\,.\end{aligned}$$

Hence, $\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 |f(t)|\,\frac{dt}{t} = 0$ as required. ■

For more details on average densities we refer to [BF92]. We now pass on to a more general concept.

2.2 Tangent Measure Distributions

In this section we extend the idea of applying an averaging method of order two from the concept of densities to the more general concept of tangent measures.
A straightforward approach leads to the following definition:

Definition

Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$, $0 \leq \alpha \leq n$ and $x \in \mathbb{R}^n$.
For any $\varepsilon \in (0,1)$ we define a measure $\bar{\nu}_\varepsilon^x \in \mathcal{M}(\mathbb{R}^n)$ by

$$\bar{\nu}_\varepsilon^x(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu_{x,t}(A)}{t^\alpha} \frac{dt}{t}$$

for all Borel sets $A \subseteq \mathbb{R}^n$. We call the limit points of $(\bar{\nu}_\varepsilon^x)$ as $\varepsilon \downarrow 0$ the *α-standardized average tangent measures* or *average tangent measures* of μ at x. If

$$\lim_{\varepsilon \downarrow 0} \bar{\nu}_\varepsilon^x = \bar{\nu}^x$$

exists it is called the *unique average tangent measure of μ at x.*

We will see in lemma 2.3.2(a) that average tangent measures of measures in Euclidean spaces of dimension $n \geq 2$ actually contain more information than average densities. But in general they are still too crude to convey a good picture of the local geometry of the measure.

In order to get a better picture, C. Bandt suggested in [Ban92] the application of the order-two averaging principle in order to get a "random tangent measure" at almost every point. This idea was applied by S. Graf in [Gra93] to standardized tangent measures. The idea of studying random tangents for self-similar sets is also present in papers of N. Patzschke, U. Zähle and M. Zähle (see e.g. [PZ90] or [PZ94]).

The approach of S. Graf is as follows: Define a family $(\varphi_\varepsilon)_{\varepsilon>0}$ of distributions on the scales $(0,1)$ such that, as $\varepsilon \downarrow 0$, the centre of weight of these distributions tends to 0. As in the definition of average densities a suitable family (φ_ε) is given by

$$\varphi_\varepsilon(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_A(t) \frac{dt}{t} \text{ for Borel sets } A \subseteq (0,1).$$

We look at the image distributions of φ_ε under the mapping

$$t \mapsto \frac{\mu_{x,t}}{t^\alpha}.$$

As $\varepsilon \downarrow 0$ we get limit distributions on the set $\mathrm{Tan}_{\mathcal{S}}(\mu, x)$, which, heuristically speaking, "remember" what happened during the procedure of blowing up and thus define a kind of "tangent measure with memory", the tangent measure distribution or random tangent. Distributions, which are defined in such a way, can provide a better means of understanding the local structure of a fractal measure than ordinary tangent measures in two principal ways:

(1) The tangent measure distributions give additional information on the "number of scales" for which $\mu_{x,t}/t^\alpha$ is close to a given $\nu \in \mathrm{Tan}^\alpha_{\mathcal{S}}(\mu, x)$ and therefore contain more information on the process $t \mapsto \frac{\mu_{x,t}}{t^\alpha}$ than the set $\mathrm{Tan}^\alpha_{\mathcal{S}}(\mu, x)$ alone.

(2) In many cases the set $\mathrm{Tan}^\alpha_{\mathcal{S}}(\mu, x)$ is too large to give a good picture of the local structure of μ about x. Many elements in $\mathrm{Tan}^\alpha_{\mathcal{S}}(\mu, x)$ stem from a small number of scales and represent merely marginal effects that appear during the process of blowing up. These elements might not appear in the support of the tangent measure distribution.

Both effects can be seen in the examples studied in [Gra93], [AP94] and [O'N94] and in the examples of this thesis. In order to define the notion of tangent measure distribution recall the definition and properties of the weak topology on the set $\mathcal{P}$ of probability distributions on $\mathcal{M}(\mathbb{R}^n)$ (see lemma 1.2.8).

Definition

Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$, $0 \le \alpha \le n$ and $x \in \mathbb{R}^n$. For any $\varepsilon > 0$ we define a probability distribution P^x_ε on $\mathcal{M}(\mathbb{R}^n)$ by

$$P^x_\varepsilon(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_A\left(\frac{\mu_{x,t}}{t^\alpha}\right) \frac{dt}{t}$$

for all Borel sets $A \subseteq \mathcal{M}(\mathbb{R}^n)$. We denote the set of all weak limit points of $(P^x_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \downarrow 0$ by $\mathcal{P}^\alpha(\mu, x)$. The elements of $\mathcal{P}^\alpha(\mu, x)$ are called the *α-standardized tangent measure* *distributions* or *tangent measure distributions* of μ at x. If the limit

$$\lim_{\varepsilon\downarrow 0} P^x_\varepsilon = P$$

exists in the weak topology, we call P the *unique tangent measure distribution* of μ at x. P is the *trivial distribution* if P is the Dirac distribution with mass concentrated at the zero measure.

Remarks:

1. If μ is an α-rectifiable measure then, by theorem 1.3.12 and the proposition 2.2.1 below, at μ-almost every point x there is a unique tangent measure distribution P_x of μ at x. P_x is the Dirac distribution with mass concentrated at

$$\nu = (1/2)^\alpha \cdot d^\alpha(\mu, x) \cdot \mathcal{H}^\alpha|_T$$

where T is the approximate tangent space of μ at x.

2. An important class of measures that have unique tangent measure distributions at almost all points are Hausdorff measures on self-similar sets fulfilling the open set condition (see [Ban92], [Gra93] or [AP94]). D. Krieg in [Kri95] has used methods similar to those in [BF92] to show that Hausdorff measures on hyperbolic Cantor sets also have unique tangent measure distributions almost everywhere.

We now formulate some general properties of the sets $\mathcal{P}^\alpha(\mu, x)$.

Proposition 2.2.1 *The set $\mathcal{P}^\alpha(\mu, x)$ is a weakly closed subset of $\mathcal{P}$. For every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ we have* supp $P \subseteq \mathrm{Tan}^\alpha_S(\mu, x)$.

Proof Recall that the weak topology on $\mathcal{P}$ is generated by the metric D defined in lemma 1.2.8. If $(P_k) \subseteq \mathcal{P}^\alpha(\mu, x)$ and $P_k \to P$ then there are sequences $(t_i^k)_{i\in\mathbb{N}}$ such that $0 < t_i^k < 1/i$ and $D(P^x_{t_i^k}, P_k) < (1/i)$. Picking the diagonal sequence yields

$$D\Big(P^x_{t_k^k}, P\Big) \le D\Big(P^x_{t_k^k}, P_k\Big) + D\Big(P_k, P\Big) \longrightarrow 0\,,$$

and thus $P \in \mathcal{P}^\alpha(\mu, x)$. Hence $\mathcal{P}^\alpha(\mu, x)$ is closed.

For every $\delta \in (0,1)$ we have

$$\begin{aligned} P(\text{cl}\,\{\mu_{x,t}/t^\alpha \,:\, t \in (0,\delta)\}) &\geq \limsup_{\varepsilon \downarrow 0} P_\varepsilon^x(\text{cl}\,\{\mu_{x,t}/t^\alpha \,:\, t \in (0,\delta)\}) \\ &\geq \limsup_{\varepsilon \downarrow 0} \frac{\log\delta - \log\varepsilon}{|\log\varepsilon|} = 1, \end{aligned}$$

and thus P is concentrated on the closed set

$$\bigcap_{\delta>0} \text{cl}\,\{\mu_{x,t}/t^\alpha \,:\, t \in (0,\delta)\} = \text{Tan}_S^\alpha(\mu, x)$$

as stated. ∎

The use of Haar measure on $(\mathbb{R}^+, \cdot)$ in the averaging procedure yields a scaling invariance property for the tangent measure distributions. Recall the definition of the rescaling group $(T_\lambda)_{\lambda>0}$ from section 1.3.

Definition

A random measure or its distribution $P \in \mathcal{P}$ is called *α-scale invariant* if, for every $\lambda > 0$,

$$P = P \circ T_\lambda^{-1},$$

where $T_\lambda \nu = \frac{\nu_{0,\lambda}}{\lambda^\alpha}$.

Proposition 2.2.2 *For every $x \in \mathbb{R}^n$ every $P \in \mathcal{P}^\alpha(\mu, x)$ is α-scale invariant.*

Proof Suppose $P = \lim P_{r_k}^x$ for $r_k \downarrow 0$. For every $F \in C_b(\mathcal{M}(\mathbb{R}^n))$ and $\lambda > 0$ we calculate

$$\begin{aligned} \int F\,dP &= \lim_{k\to\infty}(-\log r_k)^{-1} \int_{r_k}^1 F\left(\frac{\mu_{x,t}}{t^\alpha}\right)\frac{dt}{t} \\ &= \lim_{k\to\infty}(-\log r_k)^{-1} \int_{r_k/\lambda}^{1/\lambda} F\left(\frac{\mu_{x,\lambda t}}{(\lambda t)^\alpha}\right)\frac{dt}{t} \\ &= \lim_{k\to\infty}(-\log r_k)^{-1} \int_1^{1/\lambda} F\left(\frac{\mu_{x,\lambda t}}{(\lambda t)^\alpha}\right)\frac{dt}{t} + \lim_{k\to\infty}(-\log r_k)^{-1} \int_{r_k}^1 F\left(\frac{\mu_{x,\lambda t}}{(\lambda t)^\alpha}\right)\frac{dt}{t} \\ &\quad + \lim_{k\to\infty}(-\log r_k)^{-1} \int_{r_k/\lambda}^{r_k} F\left(\frac{\mu_{x,\lambda t}}{(\lambda t)^\alpha}\right)\frac{dt}{t} \\ &= \lim_{k\to\infty}(-\log r_k)^{-1} \int_{r_k}^1 F\left(T_\lambda\left(\frac{\mu_{x,t}}{t^\alpha}\right)\right)\frac{dt}{t} \\ &= \int F\,dP \circ T_\lambda^{-1}, \end{aligned}$$

using continuity of T_λ (see lemma 1.3.4). This proves the statement. ∎

More can be said about $\mathcal{P}^\alpha(\mu,x)$ in the case of measures with bounded upper densities.

Proposition 2.2.3 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $\overline{d}^\alpha(\mu,x) < \infty$. Then the following statements hold:*

1. *For every sequence (ε_k) such that $\varepsilon_k \downarrow 0$ there is a subsequence (r_k) such that $P^x_{r_k}$ converges weakly to a tangent measure distribution of μ at x. In particular $\mathcal{P}^\alpha(\mu,x)$ is non-empty. If additionally $\underline{d}^\alpha(\mu,x) > 0$, then for every $P \in \mathcal{P}^\alpha(\mu,x)$ we have $P(\{\phi\}) = 0$ and thus the set $\mathcal{P}^\alpha(\mu,x)$ contains only non-trivial distributions.*

2. *$\mathcal{P}^\alpha(\mu,x)$ is weakly compact and weakly connected.*

3. *P is a unique tangent measure distribution of μ at x if and only if*

$$\mathcal{P}^\alpha(\mu,x) = \{P\}\,.$$

4. *The average tangent measures are the barycentres of the tangent measure distributions, i.e. the set of all average tangent measures of μ at x is given by*

$$\left\{\int \nu \, dP(\nu) \,:\, P \in \mathcal{P}^\alpha(\mu,x)\right\}.$$

5. *We have*

$$\underline{D}^\alpha(\mu,x) = \inf\left\{\int \nu(B(0,1))\, dP(\nu) \,:\, P \in \mathcal{P}^\alpha(\mu,x)\right\}$$

and

$$\overline{D}^\alpha(\mu,x) = \sup\left\{\int \nu(B(0,1))\, dP(\nu) \,:\, P \in \mathcal{P}^\alpha(\mu,x)\right\}.$$

Proof

(1) Since $\overline{d}^\alpha(\mu,x) < \infty$ we know from proposition 1.3.9 that

$$\mathrm{cl}\left\{\frac{\mu_{x,t}}{t^\alpha} \,:\, t \in (0,1)\right\}$$

is a compact subset of $\mathcal{M}(\mathbb{R}^n)$. Since the distributions P^x_ε are supported by this set, lemma 1.2.8(3) implies that for every sequence (ε_k) there is a subsequence (r_k) such that

$\lim P^x_{r_k} = P$ exists. If $\underline{d}^\alpha(\mu, x) > 0$ then the zero-measure ϕ is not in this set and therefore $P(\{\phi\}) = 0$ for all tangent measures P of μ at x.

(2) Since every $P \in \mathcal{P}^\alpha(\mu, x)$ is supported by the compact set $\mathrm{Tan}_S(\mu, x)$ and $\mathcal{P}^\alpha(\mu, x)$ is weakly closed, $\mathcal{P}^\alpha(\mu, x)$ is weakly compact by lemma 1.2.8(3).

Suppose $\mathcal{P}^\alpha(\mu, x)$ is not connected. Then there are compact sets $D_1, D_2 \subseteq \mathcal{P}^\alpha(\mu, x)$ and $\varepsilon > 0$ such that $D_1 \cup D_2 = \mathcal{P}^\alpha(\mu, x)$ and $D(D_1, D_2) > \varepsilon$. Let

$$E_1 = \{t \in (0,1) : P^x_t \in B(D_1, \varepsilon/2)\} \quad \text{and} \quad E_2 = \{t \in (0,1) : P^x_t \in B(D_2, \varepsilon/2)\}.$$

There is $\delta > 0$ such that $(0, \delta) \subseteq E_1 \cup E_2$ and the union is disjoint. Since $t \mapsto P^x_t$ is continuous, the sets E_1, E_2 are closed in $(0,1)$ and this contradicts the connectedness of $(0, \delta)$. Hence $\mathcal{P}^\alpha(\mu, x)$ must be connected.

(3) Clearly we have $\mathcal{P}^\alpha(\mu, x) = \{P\}$ if P is a unique tangent measure distribution. On the other hand, if $\mathcal{P}^\alpha(\mu, x) = \{P\}$, then by (1) for every sequence (ε_k) there is a subsequence (r_k) such that $\lim P^x_{r_k} = P$. By the definition of weak convergence this implies $\lim_{\varepsilon \downarrow 0} P^x_\varepsilon = P$, hence P is a unique tangent measure distribution.

(4) Suppose first that $P = \lim P^x_{r_n} \in \mathcal{P}^\alpha(\mu, x)$. Let $f \in C_c(\mathbb{R}^n)$. Since $\mathrm{cl}\,\{\frac{\mu_{x,t}}{t^\alpha} : t \in (0,1)\}$ is compact the continuous map $\nu \mapsto \nu(f)$ is bounded on this set and thus there is $F \in C_b(\mathcal{M}(\mathbb{R}^n))$ such that $F(\nu) = \nu(f)$ for all $\nu \in \mathrm{cl}\,\{\frac{\mu_{x,t}}{t^\alpha} : t \in (0,1)\}$. Thus

$$\begin{aligned}
&\lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu_{x,t}(f)}{t^\alpha} \frac{dt}{t} \\
&= \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 F\left(\frac{\mu_{x,t}}{t^\alpha}\right) \frac{dt}{t} = \int F(\nu)\, dP(\nu) \\
&= \left[\int \nu\, dP(\nu)\right](f)
\end{aligned}$$

and hence $\int \nu\, dP(\nu)$ is an average tangent measure.

Suppose now that $\bar\nu$ is an average tangent measure and

$$\bar\nu(f) = \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu_{x,t}(f)}{t^\alpha} \frac{dt}{t}$$

for all $f \in C_c(\mathbb{R}^n)$. By (1) we can assume (by passing to a subsequence if necessary) that $P = \lim P^x_{r_n}$ exists and we have just seen that in this case

$$\bar\nu = \int \nu\, dP(\nu)$$

as required to finish the proof of (4).

(5) For any sequence $\varepsilon_n \downarrow 0$ we can assume (by passing to a subsequence if necessary) that there is $P = \lim P^x_{\varepsilon_n}$. We have seen in (4) that in this case $\lim \bar{\nu}^x_{\varepsilon_n} = \int \nu \, dP(\nu) =: \bar{\nu}$. We have $\int \nu(\partial B(0,1)) \, dP(\nu) = 0$, since otherwise, by lemma 2.2.2, $\bar{\nu}(\partial B(0,\lambda)) > 0$ for all $\lambda > 0$, contradicting the local finiteness of $\bar{\nu}$. By lemma 1.2.4 we thus get

$$\lim_{n\to\infty} (|\log \varepsilon_n|)^{-1} \int_{\varepsilon_n}^{1} \frac{\mu(B(x,t))}{t^\alpha} \frac{dt}{t} = \int \nu(B(0,1)) \, dP(\nu)$$

and this implies both statements. ∎

Corollary 2.2.4 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $\overline{d}^\alpha(\mu, x) < \infty$. For every average tangent measure $\bar{\nu}$ of μ at x and all $\lambda > 0$ we have $\bar{\nu} = T_\lambda \bar{\nu}$. Also, the support of every average tangent measure $\bar{\nu}$ of μ at x is a cone, i.e. whenever $u \in \operatorname{supp} \bar{\nu}$ and $\lambda \geq 0$ we have $\lambda u \in \operatorname{supp} \bar{\nu}$.*

Proof If $\overline{d}^\alpha(\mu, x) < C$ and $\bar{\nu}$ is an average tangent measure of μ at x, then, by proposition 2.2.3(4), there is a tangent measure distribution P of μ at x such that $\bar{\nu} = \int \nu \, dP(\nu)$. By proposition 2.2.2 we thus have

$$\bar{\nu} = \int \nu \, dP(\nu) = \int T_\lambda \nu \, dP(\nu) = T_\lambda \bar{\nu} \, .$$

If $u \in \operatorname{supp} \bar{\nu}$, $\lambda > 0$ then, for every $\delta > 0$, we have

$$\bar{\nu}(B(\lambda u, \delta)) = T_\lambda \bar{\nu}(B(u, \delta/\lambda)) \cdot \lambda^\alpha > 0 \, ,$$

and thus $\lambda u \in \operatorname{supp} \bar{\nu}$. ∎

Proposition 2.2.5 *Let μ, $\nu \in \mathcal{M}(\mathbb{R}^n)$. Suppose ν has finite upper densities ν-almost everywhere. If $\mu \ll \nu$ and $f = \frac{d\mu}{d\nu}$ is the Radon-Nikodym derivative then, for ν-almost every x,*

$$\mathcal{P}^\alpha(\mu, x) = \{P \circ M^{-1}_{f(x)} \, : \, P \in \mathcal{P}^\alpha(\nu, x)\} \, ,$$

where M_τ is defined as $M_\tau : \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathcal{M}(\mathbb{R}^n)$, $\nu \mapsto \tau \cdot \nu$ for $\tau \geq 0$.

Proof For ν-almost every x we have $f(x) < \infty$, $\overline{d}^{\alpha}(\nu, x) < \infty$ and, by lemma 1.2.2,

$$\lim_{t \downarrow 0} \frac{\int_{B(x,t)} |f(y) - f(x)| \, d\nu(y)}{\nu(B(x,t))} = 0 .$$

Fix such an $x \in \mathbb{R}^n$.

For every continuous $g : \mathbb{R}^n \longrightarrow [0, \infty)$ with supp $g \subseteq B(0, R)$ we get

$$\begin{aligned} & \left| \frac{\mu_{x,t}}{t^{\alpha}}(g) - \frac{\nu_{x,t}}{t^{\alpha}}(g) \cdot f(x) \right| \\ = \; & (1/t^{\alpha}) \left| \int g(\frac{y-x}{t}) \cdot (f(y) - f(x)) \, d\nu(y) \right| \\ \leq \; & \|g\|_{\sup} \cdot \frac{\int_{B(x,tR)} |f(y) - f(x)| \, d\nu(y)}{\nu(B(x,tR))} \cdot \frac{\nu(B(x,tR))}{t^{\alpha}} \stackrel{t \downarrow 0}{\longrightarrow} 0. \end{aligned}$$

Thus for every $\delta > 0$ there is $T > 0$ with

$$d\left(\frac{\mu_{x,t}}{t^{\alpha}}, f(x) \cdot \frac{\nu_{x,t}}{t^{\alpha}} \right) < \delta$$

for all $0 < t < T$. Let $F \in \mathcal{C}_b(\mathcal{M}(\mathbb{R}^n))$ and let $\varepsilon > 0$. Then F is uniformly continuous on the compact set

$$\mathrm{cl} \left\{ \frac{\mu_{x,t}}{t^{\alpha}}, f(x) \cdot \frac{\nu_{x,t}}{t^{\alpha}} \, : \, t \in (0,1) \right\},$$

and thus we can find a $T > 0$, such that

$$\left| F(\frac{\mu_{x,t}}{t^{\alpha}}) - F \circ M_{f(x)}(\frac{\nu_{x,t}}{t^{\alpha}}) \right| < \varepsilon/2$$

for all $0 < t < T$. Therefore

$$\begin{aligned} & \left| (|\log r|)^{-1} \int_r^1 F(\frac{\mu_{x,t}}{t^{\alpha}}) \frac{dt}{t} - (|\log r|)^{-1} \int_r^1 F \circ M_{f(x)}(\frac{\nu_{x,t}}{t^{\alpha}}) \frac{dt}{t} \right| \\ & \leq \varepsilon/2 + \|F\|_{\sup} \cdot (|\log r|)^{-1} \int_T^1 \frac{dt}{t} < \varepsilon \end{aligned}$$

for all sufficiently small $r > 0$. Hence whenever $P = \lim P^x_{r_n}$ is a tangent measure distribution of μ at x we have

$$(|\log r_n|)^{-1} \int_{r_n}^1 F \circ M_{f(x)}(\frac{\nu_{x,t}}{t^{\alpha}}) \frac{dt}{t} \stackrel{n \to \infty}{\longrightarrow} \int F \, dP ,$$

and therefore $P \circ M_{f(x)}$ is a tangent measure distribution of ν at x. Also whenever $P = \lim P^x_{r_n}$ is a tangent measure distribution of ν at x we have

$$(|\log r_n|)^{-1} \int_{r_n}^1 F(\frac{\mu_{x,t}}{t^{\alpha}}) \frac{dt}{t} \stackrel{n \to \infty}{\longrightarrow} \int F \circ M_{f(x)} \, dP ,$$

and therefore $P \circ M_{f(x)}^{-1}$ is a tangent measure distribution of μ at x. ∎

We conclude section 2.2 with an example of a fractal measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ with bounded 1-densities μ-almost everywhere, which has unique tangent measure distributions P μ-almost everywhere. In this example supp P is considerably smaller than $\mathrm{Tan}_S(\mu, x)$ and P conveys a good picture of the local geometry of μ at almost all points.

Example 2.2.6 Let (a_n) be an increasing sequence of integers. We construct a sequence $(I_n)_{n\in\mathbb{N}}$ of compact sets $I_n \subseteq \mathbb{R}^2$ as follows:

Let $I_0 = B(0,1)$. In the first step inscribe a_1 touching, closed balls of radius $1/a_1$ with centres on a vertical diameter in I_0. Denote the resulting set I_1.
In the next step inscribe a_2 touching, closed balls of radius $1/(a_1 a_2)$ with centres on a horizontal diameter in each ball of I_1. Denote the resulting set I_2.
Having constructed I_n, we inscribe a_{n+1} touching, closed balls of radius $(\prod_{i=1}^{n+1} a_i)^{-1}$ with centres on a horizontal diameter (if n odd) or on a vertical diameter (if n even) in each ball of I_n and denote the resulting set I_{n+1}.
Let

$$I := \bigcap_{n=0}^{\infty} I_n .$$

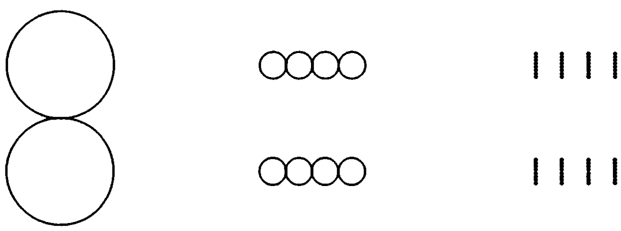

Fig.1 *Construction of* I_1, I_2, I_3 *for* $a_n = 2^n$.

Obviously I is compact. We define a codespace

$$\Sigma = \prod_{n=1}^{\infty} \{1, \ldots, a_n\}$$

and equip Σ with the "usual" metric structure

$$d(\sigma,\tau) = \begin{cases} 0 & \text{if } \sigma = \tau, \\ \left(\frac{1}{2}\right)^n & \text{if } n = \min\{n \in \mathbb{N} : \sigma_n \neq \tau_n\}\ . \end{cases}$$

We furthermore define a coding

$$\pi : I \longrightarrow \Sigma$$

in the natural way by numbering the balls of the n-th level inside a ball of $(n-1)$-th level by $1, \dots a_n$ and mapping a point which is successively in balls number b_n in the construction of I_n onto the sequence (b_n). π is a homeomorphism and thus preserves the Borel structure. We can define a measure $\bar{\mu}$ on the Borel σ-algebra of Σ by

$$\bar{\mu}(\{\sigma \in \Sigma : \sigma_1 = b_1, \dots, \sigma_n = b_n\}) = \left[\prod_{i=1}^{n} a_i\right]^{-1} \quad \text{for } b \in \Sigma,$$

and get a measure $\mu = 2 \cdot \bar{\mu} \circ \pi$ on the Borel σ-algebra of I. μ can be extended naturally to a measure on $\mathbb{R}^2$ and this measure, which we also denote μ, provides our example.

For a given $r > 0$ such that, say

$$k\left[\prod_{i=1}^{n} a_i\right]^{-1} \leq r \leq (k+1)\left[\prod_{i=1}^{n} a_i\right]^{-1}$$

with $1 \leq k < a_n$, we have for every $x \in I$

$$\frac{\mu(B(x,2r))}{2r} \geq \frac{2\,[\prod_{i=1}^{n} a_i]^{-1} \cdot k}{2\,[\prod_{i=1}^{n} a_i]^{-1} \cdot (k+1)} \geq 1/2$$

and

$$\frac{\mu(B(x,r))}{r} \leq \frac{2\,[\prod_{i=1}^{n} a_i]^{-1} \cdot (2k+1)}{[\prod_{i=1}^{n} a_i]^{-1} \cdot k} \leq 6\,,$$

and thus μ has positive and finite 1-densities.

We calculate the tangent measure distributions of μ for the sequence $a_n = 2^n$. Since we shall use the same construction in lemma 2.3.2(a) for a sequence (a_n) that is growing more quickly, we assume in claims 1 and 2 only $a_n \geq 2^n$.

Recall that

$$\varphi_\varepsilon(A) = (|\log\varepsilon|)^{-1} \cdot \int_\varepsilon^1 \mathbf{1}_A(t)\,\frac{dt}{t} \quad \text{for Borel sets } A \subseteq (0,1).$$

For $a \in \mathbb{R}$ let

$$h^a = \mathcal{H}^1|_{\{(x,0)\,:\,x>a\}}\,, \quad h_a = \mathcal{H}^1|_{\{(x,0)\,:\,x<a\}}\,,$$

and

$$v^a = \mathcal{H}^1|_{\{(0,x)\,:\,x>a\}}\,, \quad v_a = \mathcal{H}^1|_{\{(0,x)\,:\,x<a\}}\,.$$

Also let

$$h = \mathcal{H}^1|_{\{(x,0)\,:\,x\in\mathbb{R}\}}\,, \quad v = \mathcal{H}^1|_{\{(0,x)\,:\,x\in\mathbb{R}\}}\,,$$

and define (ϕ denoting the zero-measure)

$$H = \{\phi, h, h^a, h_a \,:\, a \in \mathbb{R}\} \text{ and } V = \{\phi, v, v^a, v_a \,:\, a \in \mathbb{R}\}\,.$$

H and V are closed sets in $\mathcal{M}(\mathbb{R}^2)$.

Claim 1 Given $\varepsilon > 0$ there are numbers $R > 1$, $1 > \delta > 0$ such that for

$$J_n = \Big[(4/\delta)\Big[\prod_{i=1}^n a_i\Big]^{-1}, (1/R)\Big[\prod_{i=1}^{n-1} a_i\Big]^{-1}\Big]$$

and

$$G_n = \Big((1/R)\Big(\prod_{i=1}^n a_i\Big)^{-1}, (4/\delta)\Big(\prod_{i=1}^n a_i\Big)^{-1}\Big)$$

we have, for every $x \in I$,

$$\Big\{\frac{\mu_{x,t}}{t} \,:\, t \in \bigcup_{k\in\mathbb{N}} J_{2k}\Big\} \subseteq B(H,\varepsilon) \text{ and } \Big\{\frac{\mu_{x,t}}{t} \,:\, t \in \bigcup_{k\in\mathbb{N}} J_{2k-1}\Big\} \subseteq B(V,\varepsilon)\,,$$

and

$$\varphi_r\Big(\bigcup_{k\in\mathbb{N}} G_k\Big) \xrightarrow{r\downarrow 0} 0\,.$$

Therefore, for every tangent measure distribution P of μ at x, we have $P(H \cup V) = 1$.

Proof For every $\varepsilon > 0$ we can find numbers $\delta > 0$ and $R > 1$ as in lemma 1.2.6 such that, for every measure $\nu \in \mathcal{M}(\mathbb{R}^2)$, the following geometric condition implies $d(\nu, H) \leq \varepsilon$: For some $0 < r < \delta/4$ the mass of ν is distributed in $B(0,R)$ in such a way that all the mass is inside a sequence of touching balls of radius r with centres on a horizontal line. The distance of this line to the abscissa is less or equal $2r$. One of these balls intersects the boundary of $B(0,R)$ and each ball has total mass $2r$.

Fix $x \in I$ and observe that $\mu_{x,t}/t$ is of this form if $t \in J_k$ for some even number $k \in \mathbb{N}$. Thus

$$\left\{\frac{\mu_{x,t}}{t} : t \in \bigcup_{k\in\mathbb{N}} J_{2k}\right\} \subseteq B(H,\varepsilon),$$

and the statement for V follows in an analogous manner. We have

$$\varphi_r(G_n) \le \frac{\log R - \log \delta/4}{|\log r|}$$

and thus if

$$(4/\delta)\Big[\prod_{i=1}^{n+1} a_i\Big]^{-1} \le r \le (4/\delta)\Big[\prod_{i=1}^{n} a_i\Big]^{-1}$$

we have

$$\begin{aligned}\varphi_r\left(\bigcup_{k\in\mathbb{N}} G_k\right) &\le \frac{n(\log R - \log\delta/4)}{|\log r|} \\ &\le \frac{n(\log R - \log\delta/4)}{\sum_{i=1}^n \log a_i + \log\delta/4},\end{aligned}$$

and the last term converges to 0 as $n \to \infty$ or $r \downarrow 0$. Since H and V are closed, we thus have for every tangent measure distribution of μ at x

$$P(H \cup V) = \inf_{\varepsilon>0} P(B(H,\varepsilon) \cup B(V,\varepsilon))$$

and

$$P(B(H,\varepsilon) \cup B(V,\varepsilon)) \ge \liminf_{r\downarrow 0} P_r^x(B(H,\varepsilon) \cup B(V,\varepsilon)) = 1$$

for all $\varepsilon > 0$. Thus $P(H \cup V) = 1$ as required to finish the proof of claim 1. □

Claim 2 For μ-almost every $x \in \mathbb{R}^2$, and every $P \in \mathcal{P}^1(\mu,x)$, we have

$$P(\{\phi, h^a, h_a, v^a, v_a : a \in \mathbb{R}\}) = 0.$$

Therefore we have supp $P \subseteq \{h, v\}$ for all tangent measure distributions P.

Proof Let P be a tangent measure distribution at some point $x \in I$ and observe that $P(\{\phi\}) = 0$ follows from the fact that the lower density of μ at x is positive. Observe that for any $a \in \mathbb{R}$ we have $T_\lambda(h^a) = h^{a/\lambda}$ and thus we have, by means of the scaling invariance property of P (lemma 2.2.2), for any bounded interval $I \subseteq \mathbb{R}$ such that $0 \notin I$, that $P(\{h^a : a \in I\}) = 0$. Using the analogous argument for h_a, v^a, v_a we get

$$P(\{h^a, v^a, h_a, v_a : a \neq 0\}) = 0 .$$

It remains to prove that $P(\{h^0, h_0, v^0, v_0\}) = 0$ for all $P \in \mathcal{P}^1(\mu, x)$ and μ-almost all x. For this purpose we choose a sequence $\varepsilon_i > 0$ such that $\sum_{i=1}^\infty \varepsilon_i < \infty$ and

$$\frac{n \cdot |\log \varepsilon_n|}{\sum_{i=1}^n \log a_i} \stackrel{n\to\infty}{\longrightarrow} 0 .$$

For example we can choose $\varepsilon_i = (1/i)^2$. We can use the Borel-Cantelli-lemma to see that for μ-almost every $x \in \mathbb{R}^2$ there is a number $K \in \mathbb{N}$ such that, for $\pi(x) = (x_1, x_2, x_3, \ldots)$,

$$\frac{x_i}{a_i} < (1 - \varepsilon_i) \text{ for all } i \geq K.$$

We fix such an x and a small $\varepsilon > 0$ such that $d(h_0, V) \geq 2\varepsilon$. Let R, δ, G_n and J_n as in claim 1. Whenever $t \in J_{2n-1}$ we have

$$d(\frac{\mu_{x,t}}{t}, h_0) \geq d(h_0, V) - d(\frac{\mu_{x,t}}{t}, V) \geq \varepsilon .$$

We can therefore concentrate on the $t \in J_{2n}$. For $2 \leq k < a_{2n} - 1$ we denote

$$J_{2n,k} := \Big[(2/R) \cdot k \cdot \Big[\prod_{i=1}^{2n} a_i\Big]^{-1}, (2/R) \cdot (k+1) \cdot \Big[\prod_{i=1}^{2n} a_i\Big]^{-1}\Big] .$$

Then

$$J_{2n} \subseteq \bigcup_{k=2}^{a_{2n}-1} J_{2n,k} .$$

Denote $B := B(0, R) \cap \{(x, y) : x \geq 0\}$. There is a $0 < \gamma < \varepsilon$ such that $\nu(B) < (1/3)$ whenever $d(\nu, h_0) \leq \gamma$. If $t \in J_{2n,k}$ then

$$\frac{\mu_{x,t}}{t}(B) \geq \begin{cases} (1/t) \cdot (k-1) \cdot 2[\prod_{i=1}^{2n} a_i]^{-1} & \text{if } k \leq a_{2n} - x_{2n}, \\ (1/t) \cdot (a_{2n} - x_{2n}) \cdot 2[\prod_{i=1}^{2n} a_i]^{-1} & \text{if } k > a_{2n} - x_{2n}. \end{cases}$$

Hence, if $t \in J_{2n,k}, k \leq a_{2n} - x_{2n}$, we have

$$\frac{\mu_{x,t}}{t}(B) \geq (1/t) \cdot (k-1) \cdot 2\Big[\prod_{i=1}^{2n} a_i\Big]^{-1} \geq R \cdot \frac{k-1}{k+1} \geq 1/3$$

and thus $d\Big(\frac{\mu_{x,t}}{t}, h_0\Big) > \gamma$. Finally, for $2n \geq K$,

$$J'_{2n} := \bigcup_{k=a_{2n}-x_{2n}}^{a_{2n}-1} J_{2n,k} \subseteq \Big[(2/R) \cdot a_{2n}\varepsilon_{2n} \cdot \Big[\prod_{i=1}^{2n} a_i\Big]^{-1}, (2/R) \cdot a_{2n} \cdot \Big[\prod_{i=1}^{2n} a_i\Big]^{-1}\Big],$$

and thus $\varphi_r(J'_{2n}) \leq (|\log r|)^{-1} \cdot \log(1/\varepsilon_{2n})$. If $r > 0$ is such that, say,

$$(2/R) \cdot \Big[\prod_{i=1}^{2m-1} a_i\Big]^{-1} \leq r \leq (2/R) \cdot \Big[\prod_{i=1}^{2m-2} a_i\Big]^{-1}$$

we have $\varphi_r(\bigcup_{n\in\mathbb{N}} J'_{2n}) \leq \sum_{n=1}^{m-1} \varphi_r(J'_{2n})$ and thus

$$\lim_{r\downarrow 0} \varphi_r\Big(\bigcup_{n\in\mathbb{N}} J'_{2n}\Big) \leq \lim_{m\to\infty} \frac{(m-1)|\log \varepsilon_{2m-2}|}{|\log(2/R)| - \sum_{k=1}^{2m-2} \log a_k} = 0\,.$$

Putting these arguments together we get, for all tangent measure distributions $P \in \mathcal{P}^1(\mu, x)$,

$$P(\{h_0\}) \leq \limsup_{r\downarrow 0} P_r^x(B(h_0, \gamma)) = 0\,,$$

and in the same way we find $P(\{h^0, v^0, v_0\}) = 0$. This finishes the proof of claim 2. □

Claim 3 Let $a_n = 2^n$. Then at μ-almost every x there is a unique tangent measure

$$P = \frac{1}{2} \cdot \delta_h + \frac{1}{2} \cdot \delta_v\,,$$

where δ_h, δ_v are the Dirac distributions with mass concentrated at h, v, respectively.

Proof Fix $\varepsilon > 0$. It remains to show that

$$\varphi_r\Big(\bigcup_{k\in\mathbb{N}} J_{2k}\Big), \varphi_r\Big(\bigcup_{k\in\mathbb{N}} J_{2k-1}\Big) \stackrel{r\downarrow 0}{\longrightarrow} \frac{1}{2}\,.$$

We concentrate on the first statement and look at an $r > 0$ such that, say,

$$(4/\delta)\Big[\prod_{i=1}^{n+1} a_i\Big]^{-1} \leq r \leq (4/\delta)\Big[\prod_{i=1}^{n} a_i\Big]^{-1}.$$

For $k \leq n$ we have

$$\varphi_r(J_k) \geq \frac{\log(\delta/(4R)) + \log a_k}{|\log r|},$$

and thus

$$\begin{aligned} \varphi_r\Big(\bigcup_{k\in\mathbb{N}} J_{2k}\Big) &\geq \sum_{k\leq n/2} \frac{\log(\delta/(4R)) + \log a_{2k}}{|\log r|} \\ &\geq \frac{(n/2)(\log\delta/4R) + \sum_{k\leq n/2}\log a_{2k}}{\sum_{i=1}^{n+1}\log a_i + |\log(\delta/4)|}, \end{aligned}$$

and, for $a_k = 2^k$, the last term converges to $1/2$. Thus we have for any tangent measure distribution P at x that $P(H) \geq 1/2$. An analogous calculation shows $P(V) \geq 1/2$ and this yields the statement of claim 3 and concludes the proof of the properties of μ. □

Observe that the set $\mathrm{Tan}_S(\mu, x)$ is considerably bigger than supp P : Recall that $\mathrm{Tan}_S(\mu, x)$ is a connected subset of $\mathcal{M}(\mathbb{R}^2)$ but supp P is not. Moreover, it is worth noting that the support of the unique average tangent measure is

$$\{(x,y) : \; x = 0 \text{ or } y = 0 \}$$

and hence a cone but not a linear subspace of $\mathbb{R}^2$. ■

2.3 Existence of Average Densities and Uniqueness of Tangent Measure Distributions

In this section we investigate the connection between the existence of average densities and the uniqueness of average tangent measures and tangent measure distributions. Proposition 2.3.1 gives one half of the solution of the problem:

Proposition 2.3.1 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be a measure with $\underline{d}^\alpha(\mu, x) < \infty$ μ-almost everywhere. Then the implications* (1) $\Rightarrow$ (2) *and* (2) $\Rightarrow$ (3) *hold.*

(1) *μ has a unique tangent measure distribution at μ-almost every point x.*

(2) *μ has a unique average tangent measure at μ-almost every point x.*

(3) *The average density of μ exists at μ-almost every point x.*

Moreover, if (1) holds, we have for the unique tangent measure distribution P_x and the unique average tangent measure $\bar{\nu}_x$ that

$$\bar{\nu}_x = \int \nu \, dP_x(\nu)$$

and if (2) holds we have for the unique average tangent measure $\bar{\nu}_x$ that

$$D^\alpha(\mu, x) = \bar{\nu}_x(B(0,1)).$$

Proof This was already stated in proposition 2.2.3(3),(4) and (5). ∎

The other half of the problem is probably more interesting. It turns out that the reversed implications in proposition 2.3.1 do not hold.

Proposition 2.3.2
(a) There is a measure μ with positive and finite α-densities μ-almost everywhere such that at μ-almost every point

- *the average α-density exists and*
- *the average tangent measures are not unique .*

(b) There is a measure μ with positive and finite α-densities μ-almost everywhere such that at μ-almost every point

- *there is a unique average tangent measure and*
- *the tangent measure distributions are not unique.*

T.C. O'Neil in [O'N94] was the first to give an example of type (a). His example is based on [Fed68, 3.3.19]. The example we construct below is simpler, but it is based on the same idea.

Construction of example (a)

We take the construction of the measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ with bounded 1-densities, which we used in example 2.2.6, but this time we pick a sequence (a_n) growing quickly enough such that $a_n \geq 2^n$ and

$$\frac{n \cdot \log a_n}{\log a_{n+1}} \longrightarrow 0 .$$

The behaviour of the tangent measure distributions has changed considerably. We have:

Claim 1 At μ-almost every x there are tangent measure distributions P_1 and P_2 such that $P_1(H) = 1$ and $P_2(V) = 1$.

Proof Recall the notation from example 2.2.6 and recall that 2.2.6[claim 1] is valid in our situation. Pick

$$r_n = \frac{4}{\delta} \cdot \Big(\prod_{i=1}^{n} a_i \Big)^{-1} .$$

Then

$$\varphi_{r_{2n}}\Big(\bigcup_{k \in \mathbb{N}} J_{2k} \Big) = \sum_{k=1}^{n} \frac{\log(\delta/(4R)) - \log a_{2k}}{|\log r_{2n}|}$$

and by the assumption on (a_k) we have

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{\log a_{2k}}{-\log r_{2n}} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \log a_{2k}}{\sum_{k=1}^{2n} \log a_k} = 1 .$$

This implies

$$\varphi_{r_{2n}}\Big(\bigcup_{k \in \mathbb{N}} J_{2k} \Big) \longrightarrow 1 .$$

We can pick a subsequence (s_n) of (r_{2n}) such that $P_1 = \lim P^x_{s_n}$ exists and get $P_1(H) = 1$. In an analogous manner we can find a tangent measure distribution P_2 with $P_2(V) = 1$.□

Claim 2 For μ-almost all points x the set of tangent measure distributions of μ at x is given by

$$\Lambda = \{ \lambda \cdot \delta_h + (1 - \lambda) \cdot \delta_v \,:\, \lambda \in [0, 1] \}$$

where δ_h, δ_v are Dirac measures with mass concentrated at h, v, respectively.

Proof We have already shown in 2.2.6[claim 2] that, for μ-almost all x and all tangent measure distributions $P \in \mathcal{P}^1(\mu, x)$,

$$P(\{h, v\}) = 1\,.$$

Therefore the statement of claim 2 follows from the connectedness of $\mathcal{P}^1(\mu, x)$ together with claim 1. □

Because μ has positive and finite densities, claim 2 implies that at μ-almost every point x the set of average tangent measures is given by

$$\Big\{ \int \nu \, dP(\nu) \, : \, P \in \mathcal{P}^1(\mu, x) \Big\} = \{\lambda \cdot h + (1 - \lambda) \cdot v \, : \, \lambda \in [0, 1]\}$$

and therefore μ does not have a unique average tangent measure at x. But for the average density we have, by proposition 2.2.3(5),

$$D^\alpha(\mu, x) = \lambda h(B(0,1)) + (1 - \lambda) v(B(0,1)) = 2$$

for μ-almost every x.

Observe that it is essential in the construction of this example that μ is defined on $\mathbb{R}^n$ with $n \geq 2$. The idea behind the construction of μ is based on the fact that there exist lines in more than one direction in the underlying Euclidean space.
The question, whether an example of the first type can exist on $\mathbb{R}^1$, will be answered (in the negative) in section 5.1. ■

The example of the second type is new. At this stage we can only give the construction of μ and show that the average densities of μ exist and the tangent measure distributions are not unique. We have to postpone the proof of the uniqueness of the average tangent measures until chapter 5.

Construction of example (b)

Fix a sequence (a_k) of integers with $a_0 = 0$ and $a_k \uparrow \infty$ such that

$$\frac{a_k}{a_{k+1}} \longrightarrow 0\,.$$

Define the codespace

$$\Sigma = \prod_{i=1}^{\infty}\{0,1,2\}$$

and define a measure $\bar{\mu}$ on Σ by

$$\bar{\mu}\left(\{(x_i)_{i\in\mathbb{N}} \,:\, x_1 = a_1, \ldots, x_n = a_n\}\right) = (1/3)^n \quad \text{for } a \in \Sigma.$$

Define sets I_1, I_2, I by

$$I_1 = \left\{x = \sum_{i=1}^{\infty}\frac{x_i}{7^i} \,:\, x_i \in \{0,2,6\}\right\}, \quad I_2 = \left\{x = \sum_{i=1}^{\infty}\frac{x_i}{7^i} \,:\, x_i \in \{0,4,6\}\right\}$$

and

$$I = \left\{x = \sum_{i=1}^{\infty}\frac{x_i}{7^i} \;:\; x_i \in \{0,2,6\} \text{ if } a_{2k} < i \le a_{2k+1} \text{ and } x_i \in \{0,4,6\} \text{ if } a_{2k+1} < i \le a_{2k+2}\right\},$$

and maps ϕ_1, ϕ_2, ϕ by

$$\phi_{1,2} : \Sigma \longrightarrow I_{1,2} \quad , \quad x \mapsto \sum_{i=1}^{\infty}\frac{\varphi_{1,2}(x_i)}{7^i}$$

and

$$\phi : \Sigma \longrightarrow I \quad , \quad x \mapsto \sum_{i=1}^{\infty}\frac{\varphi_3^i(x_i)}{7^i},$$

where

$$\varphi_1(x) = \begin{cases} 0 & \text{if } x = 0\ , \\ 2 & \text{if } x = 1\ , \\ 6 & \text{if } x = 2\ , \end{cases} \quad \text{and} \quad \varphi_2(x) = \begin{cases} 0 & \text{if } x = 0\ , \\ 4 & \text{if } x = 1\ , \\ 6 & \text{if } x = 2\ , \end{cases}$$

and

$$\varphi_3^i(x) = \begin{cases} 0 & \text{if } x = 0\ , \\ 2 & \text{if } x = 1 \text{ and } a_{2k} < i \le a_{2k+1}, \\ 4 & \text{if } x = 1 \text{ and } a_{2k+1} < i \le a_{2k+2}, \\ 6 & \text{if } x = 2\ . \end{cases}$$

Assuming the usual metric structure on Σ all maps ϕ, ϕ_1, ϕ_2 are bi-Lipschitz isomorphisms. Let $\mu = \bar{\mu}\circ\phi^{-1}$, $\mu_1 = \bar{\mu}\circ\phi_1^{-1}$ and $\mu_2 = \bar{\mu}\circ\phi_2^{-1}$. μ, μ_1 and μ_2 can be extended in a natural way to measures on $\mathbb{R}$.

Let $\alpha = \frac{\log 3}{\log 7}$. μ_1 and μ_2 are well-understood self-similar measures. We claim that μ is an example of type (b).

Claim 1 μ has positive and finite α-densities for all $x \in I$.

Proof If $0 < t < 1$ is given such that $(1/7)^k \geq t \geq (1/7)^{k+1}$ we have, for all $x \in I$,

$$\mu(B(x,t)) \leq (1/3)^k = 3[(1/7)^{k+1}]^\alpha \leq 3 \cdot t^\alpha ,$$

and

$$\mu(B(x,t)) \geq (1/3)^{k+1} = (1/3)[(1/7)^k]^\alpha \geq (1/3) \cdot t^\alpha .$$

This proves claim 1. □

As μ_1 and μ_2 are self-similar measures fulfilling the strong separation condition they have unique tangent measure distributions P_1, P_2 almost everywhere and, by [Gra93] or [AP94], we can describe P_1, P_2 by

$$P_{1,2}(E) = \frac{1}{\log 7} \int_{I_{1,2}} \int_{\eta/7}^{\eta} 1_E\Big(\frac{(\mu_{1,2})_{x,t}}{t^\alpha}\Big) \frac{dt}{t} \, d\mu_{1,2}(x) ,$$

where $E \in \mathcal{M}_b$, the σ-algebra on $\mathcal{M}(\mathbb{R}^n)$ generated by the mappings $\nu \mapsto \nu(B)$ for all Borel sets $B \subseteq B(0,b)$, and $\eta < (1/(7b))$.

Claim 2 $\displaystyle\int \nu(B(0,1)) \, dP_1(\nu) = \int \nu(B(0,1)) \, dP_2(\nu).$

Proof Define

$$\psi : I_1 \longrightarrow I_2 \quad , \quad \sum_{i=1}^{\infty} \frac{x_i}{7^i} \mapsto \sum_{i=1}^{\infty} \frac{6 - x_i}{7^i} .$$

As ψ is the reflection at the point $(1/2)$ it is a bi-Lipschitz isomorphism and

$$\mu_1 \circ \psi^{-1} = \mu_2 .$$

Observe that

$$\psi(B(x,r) \cap I_1) = B(\psi(x), r) \cap I_2$$

for all $x \in I_1$ and $r > 0$. Thus

$$\begin{aligned}\int \nu(B(0,1))\,dP_2(\nu) &= (\log 7)^{-1}\int_{I_2}\int_{\eta/7}^{\eta}\frac{\mu_2(B(x,t))}{t^\alpha}\frac{dt}{t}\,d\mu_2(x)\\ &= (\log 7)^{-1}\int_{I_1}\int_{\eta/7}^{\eta}\frac{\mu_2(B(\psi(x),t))}{t^\alpha}\frac{dt}{t}\,d\mu_1(x)\\ &= (\log 7)^{-1}\int_{I_1}\int_{\eta/7}^{\eta}\frac{\mu_1(B(x,t))}{t^\alpha}\frac{dt}{t}\,d\mu_1(x)\\ &= \int \nu(B(0,1))\,dP_1(\nu)\end{aligned}$$

and this proves claim 2. □

Claim 3 For μ-almost all points $x \in I$ the set of tangent measure distributions of μ at x is given by

$$\Lambda := \{\lambda P_1 + (1-\lambda)P_2 \, : \, \lambda \in [0,1]\}.$$

Before we give the proof of claim 3 we shall convince ourselves that this claim not only shows that μ has non-unique tangent measure distributions μ-almost everywhere but also implies that the average densities of μ exist at μ-almost all points. By claim 3 and claim 2 all tangent measure distributions P at x fulfill

$$\int \nu(B(0,1))\,dP(\nu) = \int \nu(B(0,1))\,dP_1(\nu)$$

and since μ has finite α-densities, we can use proposition 2.2.3(5) to see that this is the value of the average α-density at μ-almost every x. It remains to prove claim 3.

Proof of claim 3 Let $X_1 \subseteq I_1$ and $X_2 \subseteq I_2$ be the exceptional sets where we do not have the unique tangent measure distributions for μ_1, μ_2, respectively. Let $X = \phi(\phi_1^{-1}(X_1) \cup \phi_2^{-1}(X_2))$. Then $\mu(X) = \bar{\mu}(\phi_1^{-1}(X_1) \cup \phi_2^{-1}(X_2)) \le \mu_1(X_1) + \mu_2(X_2) = 0$ and it is sufficient to show that $\mathcal{P}^\alpha(\mu, x) = \Lambda$ for all $x \in I \setminus X$. Fix $x \in I \setminus X$.

First step: For every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that for all

$$(1/7)^{a_{2k+1}-n} \le t \le (1/7)^{a_{2k}+n}$$

we have

$$d\left(\frac{(\mu_1)_{\phi_1\phi^{-1}(x),t}}{t^\alpha}, \frac{\mu_{x,t}}{t^\alpha}\right) < \varepsilon$$

and for all

$$(1/7)^{a_{2k}-n} \le t \le (1/7)^{a_{2k-1}+n}$$

we have

$$d\left(\frac{(\mu_2)_{\phi_2\phi^{-1}(x),t}}{t^\alpha}, \frac{\mu_{x,t}}{t^\alpha}\right) < \varepsilon\,.$$

Proof of the first step: We concentrate on the first inequality and get, for a given $\varepsilon > 0$, numbers δ and R as in lemma 1.2.6 and $n \in \mathbb{N}$ such that $2 \cdot (1/7)^n < \min(\delta, 1/R)$. Let

$$(1/7)^{a_{2k+1}-n} \le t \le (1/7)^{a_{2k}+n}\,.$$

We want to apply lemma 1.2.6 to the collection $\mathcal{U}$ of sets

$$U_i = \left[\frac{2i-1}{t} \cdot (1/7)^{a_{2k+1}}, \frac{2i+1}{t} \cdot (1/7)^{a_{2k+1}}\right)$$

for those integers i which fulfill $|i| \le (1/4) \cdot 7^{a_{2k+1}-a_{2k}}$.
$\mathcal{U}$ is a disjoint cover of $B(0,R)$ since $0 \in U_0$ and

$$\left|\bigcup_{U\in\mathcal{U}} U\right| = (2/t) \cdot (1/7)^{a_{2k+1}}((1/2) \cdot 7^{a_{2k+1}-a_{2k}} + 1) \ge 7^n > 2R$$

Furthermore

$$|U_i| = \frac{2}{t}(1/7)^{a_{2k+1}} \le 2 \cdot (1/7)^n < \delta\,.$$

Since, for every $U \in \mathcal{U}$, we have

$$|tU| = 2 \cdot (1/7)^{a_{2k+1}}$$

and

$$\left|\bigcup_{U\in\mathcal{U}} tU\right| = ((1/2) \cdot 7^{a_{2k+1}-a_{2k}} + 1) \cdot 2 \cdot (1/7)^{a_{2k+1}} \le 2 \cdot (1/7)^{a_{2k}}\,,$$

every set $(\phi_1\phi^{-1}(x) + tU_i)$ either contains exactly one of the sets

$$\{y = \sum_{i=1}^{\infty} \frac{y_i}{7^i} \quad : \quad y_1 = z_1, \ldots, y_{a_{2k+1}} = z_{a_{2k+1}}\} \quad \text{for some } z \in \{0,2,6\}^{\mathbb{N}},$$

or it is disjoint from all of them. We are in the first case if and only if $(x+tU_i)$ contains exactly one of the sets

$$\{y=\sum_{i=1}^{\infty}\frac{y_i}{7^i}\,:\,y_1=z_1,\dots,y_{a_{2k+1}}=z_{a_{2k+1}}\}\quad\text{for some } z\in\{\varphi_3^i(x_i)\,:\,x\in\Sigma\},$$

and in the second case $(x+tU_i)$ is disjoint from all these sets. Accordingly

$$\frac{\mu_{x,t}}{t^\alpha}(U_i)=\frac{(\mu_1)_{\phi_1\phi^{-1}(x),t}}{t^\alpha}(U_i)$$

as required to use lemma 1.2.6. This finishes the proof of the first inequality in the first step. The second inequality is proved in an analogous way.

Define compact intervals

$$T_{1,k}^\varepsilon=\left[(1/7)^{a_{2k+1}-n},(1/7)^{a_{2k}+n}\right]\quad\text{and}\quad T_{2,k}^\varepsilon=\left[(1/7)^{a_{2k}-n},(1/7)^{a_{2k-1}+n}\right],$$

and a set

$$B_\varepsilon=\bigcup_{k=0}^{\infty}\left((1/7)^{a_k+n},(1/7)^{a_k-n}\right).$$

Second step : For every $\varepsilon>0$ we have that $\varphi_\delta(B_\varepsilon)\stackrel{\delta\downarrow 0}{\longrightarrow}0$.

Proof of the second step: Fix $\varepsilon>0$. Assume $(1/7)^{a_{k+1}+n}<\delta\le(1/7)^{a_k+n}$. Then, for all $j\in\mathbb{N}$,

$$\varphi_\delta\left(\left[(1/7)^{a_j+n},(1/7)^{a_j-n}\right]\right)\le\frac{2n}{a_k+n},$$

and thus

$$\varphi_\delta(B_\varepsilon)\le\sum_{j=0}^{k+1}\frac{2n}{a_k+n}\le\frac{(k+2)\cdot 2n}{a_k+n}\stackrel{k\to\infty}{\longrightarrow}0.$$

Third step: Let $\lambda\in[0,1]$ and $\varepsilon>0$. Then there is a $P^{(\varepsilon)}\in\mathcal{P}^\alpha(\mu,x)$ such that

$$|P^{(\varepsilon)}(F)-(\lambda\cdot P_1(F)+(1-\lambda)\cdot P_2(F)|\le\delta$$

for every bounded continuous function $F:\mathcal{M}(\mathbb{R})\to[0,\infty)$ such that $|F(\nu_1)-F(\nu_2)|\le\delta/2$ for all ν_1,ν_2 with $d(\nu_1,\nu_2)<\varepsilon$.

Proof of the third step: Apply the first step to ε in order to get a number $n \in \mathbb{N}$. Let

$$t_k = (1/7)^{\frac{a_{2k}+n}{1-\lambda}}.$$

Passing to a subsequence of t_k if necessary, we can assume that there is $P^{(\varepsilon)} = \lim P^x_{t_k}$. For sufficiently large k we have

$$[t_k, t_k^{1-\lambda}] \subseteq [(1/7)^{a_{2k+1}-n}, (1/7)^{a_{2k}+n}] = T^\varepsilon_{1,k}.$$

We thus have

$$\begin{aligned}
&\Big|(|\log t_k|)^{-1}\int_{t_k}^{t_k^{1-\lambda}} F\Big(\frac{\mu_{x,t}}{t^\alpha}\Big)\,\frac{dt}{t} - \lambda\cdot P_1(F)\Big| \\
&\leq\ \delta/2 + \Big|(|\log t_k|)^{-1}\int_{t_k}^{1} F\Big(\frac{(\mu_1)_{\phi_1\phi^{-1}(x),t}}{t^\alpha}\Big)\,\frac{dt}{t} \\
&\qquad - (1-\lambda)\cdot(|\log t_k^{1-\lambda}|)^{-1}\int_{t_k^{1-\lambda}}^{1} F\Big(\frac{(\mu_1)_{\phi_1\phi^{-1}(x),t}}{t^\alpha}\Big)\,\frac{dt}{t} - \lambda\cdot P_1(F)\Big|,
\end{aligned}$$

and this term converges to $\delta/2 + |P_1(F) - (1-\lambda)\cdot P_1(F) - \lambda\cdot P_1(F)| = \delta/2$. Moreover

$$\begin{aligned}
&\Big|(|\log t_k|)^{-1}\int_{(1/7)^{a_{2k}-n}}^{1} F\Big(\frac{\mu_{x,t}}{t^\alpha}\Big)\,\frac{dt}{t} - (1-\lambda)\cdot P_2(F)\Big| \\
&\leq\ \Big|(1-\lambda)\cdot\frac{a_{2k}-n}{a_{2k}+n}\cdot(|\log(1/7)^{a_{2k}-n}|)^{-1}\int_{(1/7)^{a_{2k}-n}}^{1} F\Big(\frac{(\mu_2)_{\phi_2\phi^{-1}(x),t}}{t^\alpha}\Big)\,\frac{dt}{t} \\
&\qquad - (1-\lambda)\cdot P_2(F)\Big| + \|F\|_{\sup}\cdot\varphi_{t_k}\Big(\big((1/7)^{a_{2k-1}+n},1\big)\Big) + \delta/2,
\end{aligned}$$

and since

$$\varphi_{t_k}\Big(\big((1/7)^{a_{2k-1}+n},1\big)\Big) \leq \frac{a_{2k-1}+n}{a_{2k}+n}(1-\lambda) \stackrel{k\to\infty}{\longrightarrow} 0$$

this term converges to $\delta/2$. Finally the second step implies that

$$\Big|(|\log t_k|)^{-1}\cdot\int_{t_k^{1-\lambda}}^{(1/7)^{a_{2k}-n}} F\Big(\frac{\mu_{x,t}}{t^\alpha}\Big)\,\frac{dt}{t}\Big| \stackrel{k\to\infty}{\longrightarrow} 0.$$

Therefore

$$\Big|P^{(\varepsilon)}(F) - \lambda\cdot P_1(F) - (1-\lambda)\cdot P_2(F)\Big| \leq \delta,$$

and thus $P^{(\varepsilon)}$ is as required to finish the third step.

By proposition 1.3.9(1) the set

$$M := \mathrm{cl}\Big\{\frac{\mu_{x,t}}{t^\alpha}, \frac{(\mu_1)_{\phi_1\phi^{-1}(x),t}}{t^\alpha}, \frac{(\mu_2)_{\phi_2\phi^{-1}(x),t}}{t^\alpha}\ :\ t\in(0,1)\Big\}$$

is compact and thus every continuous bounded function $F : \mathcal{M}(\mathbb{R}) \to [0,\infty)$ is uniformly continuous on M. Therefore, for every $\delta > 0$, we can pick $\varepsilon > 0$ such that for all $\nu_1, \nu_2 \in M$, $d(\nu_1,\nu_2) < \varepsilon$ we have $|F(\nu_1) - F(\nu_2)| < \delta/2$. Because $\mathcal{P}^\alpha(\mu,x)$ is compact we can pick a sequence $\varepsilon_n \downarrow 0$ such that $P = \lim P^{(\varepsilon_n)}$ exists. P fulfills, for all continuous bounded functions F, $|P(F) - (\lambda \cdot P_1(F) + (1-\lambda)\cdot P_2(F))| = 0$ and thus $\Lambda \subseteq \mathcal{P}^\alpha(\mu,x)$.

Fourth step: Let $P = \lim P^x_{s_k} \in \mathcal{P}^\alpha(\mu,x)$ and $\varepsilon > 0$. Then there is a $\lambda^{(\varepsilon)} \in [0,1]$ such that

$$|P(F) - \lambda^{(\varepsilon)} \cdot P_1(F) - (1-\lambda^{(\varepsilon)}) \cdot P_2(F)| \le \delta$$

for every bounded continuous function $F : \mathcal{M}(\mathbb{R}) \to [0,\infty)$ such that $|F(\nu_1) - F(\nu_2)| < \delta/2$ for all ν_1, ν_2 with $d(\nu_1,\nu_2) < \varepsilon$.

Proof of the fourth step: Fix $\varepsilon > 0$ and pick $n \in \mathbb{N}$ as in the first step. First assume that (s_k) is such that there is a subsequence (t_k) of (s_k) with $t_k \in T^\varepsilon_{1,n_k}$ for a sequence $n_k \uparrow \infty$. Since, for all sufficiently large $k \in \mathbb{N}$,

$$\lambda_k =: 1 - \frac{(a_{2n_k} + n)\cdot \log 7}{-\log t_k} \in [0,1],$$

we can assume that, by passing to a subsequence once more, $\lim \lambda_k =: \lambda^{(\varepsilon)} \in [0,1]$ exists. Using $t_k^{1-\lambda_k} = (1/7)^{a_{2n_k}+n}$ we can show, as in the third step, that

$$\left|P(F) - (\lambda^{(\varepsilon)} \cdot P_1(F) + (1-\lambda^{(\varepsilon)}) \cdot P_2(F))\right| \le \delta$$

for all bounded continuous functions F as in the statement.

With analogous estimates we get the same result if (s_k) is such that there is a subsequence (t_k) of (s_k) with $t_k \in T^\varepsilon_{2,n_k}$ for a sequence $n_k \uparrow \infty$. Also this result can be obtained if $t_k \in B_\varepsilon$ for all $k \in \mathbb{N}$ since, by the second step, in this case we have $P = \lim P^x_{t_k}$ with $t_k = (1/7)^{a_{n_k} - n}$ for some $n_k \uparrow \infty$.

Because $[0,1]$ is compact we can find a sequence $\varepsilon_n \downarrow 0$ such that $\lambda = \lim_{n\to\infty} \lambda^{(\varepsilon_n)}$ exists. Then $P = \lambda \cdot P_1 + (1-\lambda) \cdot P_2$ as required to show $\mathcal{P}^\alpha(\mu,x) \subseteq \Lambda$ and finish the proof. ■

Chapter 3

A Problem on One-Sided Average Densities

By a theorem of Besicovitch the lower one-sided α-densities of a measure μ on the real line vanish μ-almost everywhere if $0 < \alpha < 1$ and the circular α-densities of μ are finite and positive μ-almost everywhere.

The main result of this chapter is that the one-sided average densities show exactly the opposite behaviour: The one-sided average densities of a measure μ with positive and finite densities are positive μ-almost everywhere. Therefore the concept of one-sided average densities is able to reveal some of the local symmetry a measure with finite and positive densities necessarily possesses.

This result will be presented in section 3.1. For the proof we shall develop methods which will be extended considerably in chapter 5. The proof will be carried out in two steps: In section 3.2 we show some lemmas on the geometry of measures with bounded densities which are also of independent interest, and in section 3.3 we give the proof using these lemmas.

Much more can be said about one-sided average densities, but only in chapter 5, when we look at the problem from the point of view of tangent measure distributions, will we be able to prove this (see corollary 5.1.4).

3.1 One-Sided Average Densities do not Vanish

Let $0 < \alpha < 1$ and $\mu \in \mathcal{M}(\mathbb{R})$ be a measure such that

$$0 < \underline{d}^{\alpha}(\mu, x) \leq \overline{d}^{\alpha}(\mu, x) < \infty$$

μ-almost everywhere. It is a natural conjecture that such a measure cannot have arbitrarily bad asymmetry around almost all of its points. At a first glance, lower one-sided α-densities, which we define below, seem to be a good concept to study this phenomenon.

Definition

We define the *lower right-sided* and *lower left-sided α-densities* of μ at x by

$$\underline{d}^{\alpha}_{+}(\mu, x) = \liminf_{t \downarrow 0} \frac{\mu([x, x+t])}{t^{\alpha}} \quad \text{and} \quad \underline{d}^{\alpha}_{-}(\mu, x) = \liminf_{t \downarrow 0} \frac{\mu([x-t, x])}{t^{\alpha}}.$$

If now $\underline{d}^{\alpha}_{+}(\mu, x) > 0$ and $\underline{d}^{\alpha}_{-}(\mu, x) > 0$ this indicates that μ does not have complete asymmetry around x: At every scale there is mass on both sides of x. However, by the following theorem of Besicovitch this is not true:

Theorem 3.1.1 *For μ-almost every $x \in \mathbb{R}$ we have*

$$\underline{d}^{\alpha}_{+}(\mu, x) = 0 \quad \textit{and} \quad \underline{d}^{\alpha}_{-}(\mu, x) = 0\,.$$

Proof This is proved in [Bes29] and [Bes68] for Hausdorff measure on α-sets.
Supposing $\mu \in \mathcal{M}(\mathbb{R})$ and $0 < \overline{d}^{\alpha}(\mu, x) < \infty$ μ-almost everywhere we can use proposition 1.3.3: There is a family of disjoint α-sets $E_1, E_2, \ldots$ such that

$$\mu \ll \mathcal{H}^{\alpha}|_{\bigcup E_i} =: \nu\,.$$

In particular, we have $\mu|_{E_i} \ll \nu|_{E_i}$. Let $f = \frac{d\mu|_{E_i}}{d\nu|_{E_i}}$ be the Radon-Nikodym derivative. By lemma 1.2.2 we then have

$$\begin{aligned}
\underline{d}^{\alpha}_{+}(\mu, x) &\leq \liminf_{t \downarrow 0} (1/t^{\alpha}) \cdot \int (1_{[x,x+t]}(y) \cdot f(y) - 1_{[x,x+t]}(y) \cdot f(x))\, d\nu(y) \\
&\quad + \liminf_{t \downarrow 0} \frac{\nu([x, x+t])}{t^{\alpha}} \cdot f(x) \\
&\leq \liminf_{t \downarrow 0} \frac{\int_{B(x,t)} |f(y) - f(x)|\, d\nu(y)}{\nu(B(x,t))} \cdot \frac{\nu(B(x,t))}{t^{\alpha}} + \underline{d}^{\alpha}_{+}(\nu, x) \cdot f(x) \\
&= 0 \quad \text{for } \mu\text{-almost every } x \in E_i.
\end{aligned}$$

Therefore $\underline{d}_{+}^{\alpha}(\mu, x) = 0$ μ-almost everywhere, and, by an analogous argument, also $\underline{d}_{-}^{\alpha}(\mu, x) = 0$ μ-almost everywhere, as required. ■

This failure of the lower one-sided densities naturally leads to the question whether the lower one-sided average densities show the same behaviour or whether they are able to reveal some of the local symmetry of μ. Let us illustrate the situation by a simple example.

Example 3.1.2

Let

$$C = \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}, \, x_i \in \{0,2\}\}$$

be the ternary Cantor set. Furthermore let

$$\begin{array}{rccl} \pi : & C & \longrightarrow & \{0,2\}^{\mathbb{N}} \\ & x & \mapsto & (x_i)_{i \in \mathbb{N}} \end{array}$$

be the natural coding. Let $\alpha = \frac{\log 2}{\log 3}$ and μ be α-Hausdorff measure on the set C. It is easy to see that the measure μ has bounded α-densities and

$$\mu \circ \pi^{-1} = \bigotimes_{i \in \mathbb{N}} P \, ,$$

where $P(\{0\}) = 1/2 = P(\{2\})$.

The convenience of this example is that we can interpret x_i as independent, P-distributed random variables on C and use standard probability theory to get information on the one-sided lower densities and lower average densities of μ.

For this purpose note that, whenever $(x_{k+1}, \ldots, x_{k+l}) = (2, \ldots, 2)$ and $t = (1/3)^k$, we have

$$\frac{\mu([x, x+t])}{t^{\alpha}} \leq \frac{(1/2)^{k+l}}{(1/2)^k} = (1/2)^l .$$

By the strong law of large numbers we have for μ-almost every $x \in C$ a sequence $(k_l)_{l \in \mathbb{N}}$ of indices such that

$$(x_{k_l+1}, \ldots x_{k_l+l}) = (2, \ldots, 2)$$

and therefore

$$\liminf_{t \downarrow 0} \frac{\mu([x, x+t])}{t^{\alpha}} \leq \lim_{l \to \infty} \frac{\mu([x, x + (1/3)^{k_l}])}{(1/2)^{k_l}} = 0.$$

On the other hand we have, whenever $x_{k+1} = 0$ and $(1/3)^k \le t \le (1/3)^{k-1}$, that

$$\frac{\mu([x,x+t])}{t^\alpha} \ge \frac{(1/2)^{k+1}}{(1/2)^{k-1}} = 1/4 .$$

By the strong law of large numbers we have, for μ-almost every $x \in C$, that

$$\frac{1}{n}\sum_{k=1}^{n} 1_{\{x_k=0\}} \longrightarrow \frac{1}{2} .$$

Therefore we get, for every $\varepsilon \in (\frac{1}{3}^N, \frac{1}{3}^{N-1}]$,

$$\begin{aligned}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t} &\ge \frac{1}{N\cdot\log 3}\sum_{k=1}^{N-1}\int_{(1/3)^{N-k}}^{(1/3)^{N-k-1}} \frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t} \\ &\ge \frac{1}{N}\sum_{k=1}^{N-1}\frac{1}{4}\cdot 1_{\{x_{N-k+1}=0\}} ,\end{aligned}$$

and thus

$$\underline{D}_+^\alpha(\mu,x) = \liminf_{\varepsilon\downarrow 0}\,(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t} \ge \frac{1}{8}$$

μ-almost everywhere. ∎

We have seen in the example that, for Hausdorff measure on the Cantor set, at almost all points the one-sided lower average densities do not vanish. One can conjecture that this holds true for all measures μ on the line with finite and positive α-densities, and this turns out to be correct. In fact, the one-sided lower average densities of μ at μ-almost all x are bounded from below by a constant depending only on α and the upper and lower density of μ at x, but not on the particular geometry of μ. This is the statement of theorem 3.1.3.

Theorem 3.1.3 *Let $\mu \in \mathcal{M}(\mathbb{R})$ be a measure such that for $0 < \alpha < 1$ we have*

$$0 < \underline{d}^\alpha(\mu,x) \le \overline{d}^\alpha(\mu,x) < \infty$$

μ-almost everywhere. Then for μ-almost every x there is a number $\tau > 0$ such that

$$\underline{D}_-^\alpha(\mu,x), \underline{D}_+^\alpha(\mu,x) \ge \tau.$$

τ is a function of $\underline{d}^\alpha(\mu,x)$, $\overline{d}^\alpha(\mu,x)$ and α, and is otherwise independent of μ and x.

Observe that the statement of theorem 3.1.3 holds trivially in the cases $\alpha = 0, 1$.
The following corollary provides information on the one-sided upper average densities:

Corollary 3.1.4 *Let $\mu \in \mathcal{M}(\mathbb{R})$ be a measure such that for $0 < \alpha < 1$ we have*

$$0 < \underline{d}^{\alpha}(\mu, x) \leq \overline{d}^{\alpha}(\mu, x) < \infty$$

μ-almost everywhere. Then

$$\overline{D}^{\alpha}_{-}(\mu, x), \overline{D}^{\alpha}_{+}(\mu, x) \leq \overline{D}^{\alpha}(\mu, x) - \tau$$

μ-almost everywhere, where τ is as in theorem 3.1.3.

Proof The statement follows from

$$\overline{D}^{\alpha}_{+}(\mu, x) + \tau \leq \overline{D}^{\alpha}_{+}(\mu, x) + \underline{D}^{\alpha}_{-}(\mu, x) \leq \overline{D}^{\alpha}(\mu, x)$$

μ-almost everywhere, and the analogous statement for the left-sided densities. ■

Theorem 3.1.3 leaves a lot of questions about one-sided average densities open.
For example:

- If the average density exists, does this imply that the left-sided and right-sided average densities exist?
- Do the values of the left-sided and right-sided lower (or upper) average densities always agree?
- Is the value of the one-sided lower (or upper) average density determined by the value of the lower (or upper) average density?

All these questions will be answered in chapter 5.

3.2 The Geometry of Measures with Bounded Densities

Let $0 < \alpha < 1$ and $\mu \in \mathcal{M}(\mathbb{R})$ be a measure with support contained in a compact interval D_μ with $|D_\mu| \leq 1$. Suppose there is a compact set $E \subseteq D_\mu$ with $\mu(E) > 0$ such that there are $0 < c \leq C < \infty$ and $\varepsilon_0 > 0$ with

$$\mu([x - r, x + r]) \leq Cr^\alpha \tag{3.1}$$

for all $x \in E$ and $r \geq 0$, and

$$\mu([x - r, x + r]) \geq cr^\alpha \tag{3.2}$$

for all $x \in E$ and $0 \leq r \leq \varepsilon_0$. We shall assume $\varepsilon_0 < (1/e)$.
In this section we study the geometry of the set E. This constitutes an important part of the proof of theorem 3.1.3 and will also be of use in chapter 5.

In order to avoid confusion about the exact dependence of the constants in the following lemmas (and also in section 3.3) we will stick to the following convention: "Constants" may depend on the measure μ (and in particular on c or C) and are named $\mathcal{C}$ with a subscript. "Absolute constants" may only depend on α and are named $\mathcal{D}$ with a subscript.

Lemma 3.2.1 *E is an α-set.*

Proof As a compact set, E is clearly $\mathcal{H}^\alpha$-measurable.
For every $\varepsilon_0 > \varepsilon > 0$ we can cover E with a family $\mathcal{U} = \{(x - \varepsilon, x + \varepsilon) : x \in S\}$ of intervals such that $S \subseteq E$ and every $y \in \mathbb{R}$ is contained in at most two sets $U \in \mathcal{U}$. Then

$$\sum_{U \in \mathcal{U}} |U|^\alpha \leq (2^\alpha/c) \cdot \sum_{U \in \mathcal{U}} \mu(U) \leq 2(2^\alpha/c) \cdot \mu(E) < \infty$$

and thus $\mathcal{H}^\alpha(E) < \infty$.
Now let $\varepsilon > 0$ and let $\mathcal{U}$ be an arbitrary cover of E such that $|U| \leq \varepsilon$ and $U \cap E \neq \emptyset$ for all $U \in \mathcal{U}$. Then

$$\sum_{U \in \mathcal{U}} |U|^\alpha \geq (1/C) \sum_{U \in \mathcal{U}} \mu(U) \geq \mu(E)/C$$

and thus $\mathcal{H}^\alpha(E) > 0$. ∎

We can write

$$D_\mu \setminus E = \bigcup_{I \in \mathcal{A}} I ,$$

where $\mathcal{A}$ is the collection of connected components of $D_\mu \setminus E$. $\mathcal{A}$ is a collection of disjoint intervals open in D_μ. We can also write

$$D_\mu \setminus \bigcup_{\substack{I \in \mathcal{A} \\ |I| \geq \varepsilon}} I = \bigcup_{K \in \mathcal{N}_\varepsilon} K ,$$

where $\mathcal{N}_\varepsilon$ is the collection of connected components of the set $D_\mu \setminus \bigcup_{\substack{I \in \mathcal{A} \\ |I| \geq \varepsilon}} I$. $\mathcal{N}_\varepsilon$ is a collection of disjoint compact intervals.

Before we give an upper bound to the length of the intervals in $\mathcal{N}_\varepsilon$, let us introduce some useful notation. For every interval $I \subseteq \mathbb{R}$ and every $\kappa \geq 0$ let

$$I^-(\kappa) = \{x \in \mathbb{R} : \text{ there is } y \in I \text{ such that } 0 \leq y - x \leq \kappa \cdot |I|\} \setminus I ,$$

and

$$I^+(\kappa) = \{x \in \mathbb{R} : \text{ there is } y \in I \text{ such that } 0 \leq x - y \leq \kappa \cdot |I|\} \setminus I ,$$

and also

$$I^0(\kappa) = I^-(\kappa) \cup I \cup I^+(\kappa) .$$

Lemma 3.2.2 *There is a constant $C_1 > 1$ such that, for all $0 < \varepsilon \leq \varepsilon_0$ and all $K \in \mathcal{N}_\varepsilon$, we have*

$$|K| < C_1 \cdot \varepsilon .$$

Proof For $0 < \varepsilon \leq \varepsilon_0$ denote $r = r(\varepsilon) = \max\{|N| : N \in \mathcal{N}_\varepsilon\}$ and pick $N \in \mathcal{N}_\varepsilon$ such that $|N| = r$. Let $\tilde{N} = N^-(1) \cup N$, in other words $\tilde{N}$ is the closed interval of diameter $2r$ with centre at the left endpoint of N. Then, by (3.1),

$$\mu(\tilde{N}) \leq C \cdot r^\alpha .$$

Look at the intervals $I_1, I_2, I_3, \ldots \in \mathcal{A}$ that fulfill $I_i \subseteq N$. Obviously $|I_i| \leq \varepsilon$ and thus

$$\varepsilon^{\alpha-1} \cdot |I_i| \leq |I_i|^\alpha .$$

Define $\tilde{I}_i = I_i \cup I_i^-(1) \subseteq \tilde{N}$. E is a set of Lebesgue measure 0 by lemma 3.2.1, and therefore the sets $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \ldots$ cover almost all of N in the sense of Lebesgue measure. By Vitali's covering theorem (see for example [Mat95, theorem 2.1]) we can pick a disjoint subsequence

$$\tilde{I}_{k_1}, \tilde{I}_{k_2}, \tilde{I}_{k_3}, \ldots$$

covering at least 1/10 of the length of N. Now we can use (3.2) to see

$$C \geq \frac{\mu(\tilde{N})}{r^\alpha} \geq \frac{1}{r^\alpha}\sum_{i=1}^{\infty}\mu(\tilde{I}_{k_i}) \geq \frac{c}{r^\alpha}\sum_{i=1}^{\infty}|I_{k_i}|^\alpha \geq \frac{c}{r^\alpha}\varepsilon^{\alpha-1}\sum_{i=1}^{\infty}|I_{k_i}| \geq \frac{c}{10}\left(\frac{\varepsilon}{r}\right)^{\alpha-1}$$

and, defining $C_1 = 2(10 \cdot C/c)^{1/(1-\alpha)}$, we have $r(\varepsilon) < C_1\varepsilon$, as required. ∎

Lemma 3.2.3 *There is a constant $C_2 > 1$ such that, for all $0 < \varepsilon \leq \varepsilon_0$ and every $K \in \mathcal{N}_\delta$, $\delta > 0$, we have*

$$\sum_{N \in \mathcal{N}_\varepsilon, N \subseteq K} |N|^\alpha \leq C_2 \cdot |K|^\alpha,$$

and also

$$\sum_{N \in \mathcal{N}_\varepsilon} |N|^\alpha \leq C_2.$$

Proof We fix a $K \in \mathcal{N}_\delta$ and $0 < \varepsilon \leq \varepsilon_0$. We first show that for all $N \in \mathcal{N}_\varepsilon$ we have

$$|N|^\alpha \leq (C_1^\alpha/c) \cdot \mu(N^*),$$

where $N^* = N^-(1/C_1) \cup N$. For this purpose look at the closed interval B of diameter $2|N|/C_1$ centred at the left endpoint of N. We have $B \subseteq N^*$ and thus we get, using (3.2),

$$\mu(N^*) \geq \mu(B) \geq c\,(|N|/C_1)^\alpha .$$

Because, by lemma 3.2.2, the intervals I separating the $N \in \mathcal{N}_\varepsilon$ fulfill $|I| \geq \varepsilon > |N|/C_1$, the collection

$$\{N^* : N \in \mathcal{N}_\varepsilon \text{ and } N \subseteq K\}$$

is disjoint. Also $N^* \subseteq K^*$ for all N^* in the collection. Therefore

$$\sum_{N \in \mathcal{N}_\varepsilon, N \subseteq K} |N|^\alpha \leq (C_1^\alpha/c) \cdot \sum_{N \in \mathcal{N}_\varepsilon, N \subseteq K} \mu(N^*) \leq (C_1/c)^\alpha \cdot \mu(K^*).$$

By (3.1) we have $\mu(K^*) \leq C|K|^\alpha$ and therefore we can put $C_2 = (C_1/c)^\alpha \cdot C$ and get the first inequality. The second inequality follows by picking δ sufficiently large to ensure $D_\mu \in \mathcal{N}_\delta$. ∎

Lemma 3.2.4

1. *There is a constant* $C_3 > 1$ *such that, for all* $0 < \varepsilon \leq \delta \leq \varepsilon_0$ *and all* $K \in \mathcal{N}_\delta$, *we have*
$$\sum |I|^\alpha \leq C_3 \cdot |K|^\alpha \cdot |\log \varepsilon| \,,$$
where the sum extends over all $I \in \mathcal{A}$ *such that* $|I| \geq \varepsilon$ *and* $I \subseteq K$.

2. *There is a constant* $C_4 > 1$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$ *we have*
$$\sum |I|^\alpha \leq C_4 \cdot |\log \varepsilon| \,,$$
where the sum extends over all $I \in \mathcal{A}$ *such that* $|I| \geq \varepsilon$.

Proof Fix $K \in \mathcal{N}_\delta$ and denote $\tilde{K} = K^-(1) \cup K$. Observe that by (3.1) $\mu(\tilde{K}) \leq C|K|^\alpha$. Similarly, for $I \in \mathcal{A}$ with $|I| \geq \varepsilon$, $I \subseteq K$, we define $\tilde{I} = I^-(1) \cup I$ and observe that $\tilde{I} \subseteq \tilde{K}$ and by (3.2) $\mu(\tilde{I}) \geq c|I|^\alpha$. For $x \in \tilde{K}$ denote by
$$I_1 < I_2 < \ldots < I_n$$
the collection of intervals $I \in \mathcal{A}$ such that $|I| \geq \varepsilon$, $I \subseteq K$ and $x \in \tilde{I}$ in their natural order. For $3 \leq k \leq n$ we have
$$|I_k| \geq |I_{k-1}| + |I_{k-2}| + \ldots + |I_2|$$
and thus (provided $n \geq 3$)
$$|I_n| \geq \varepsilon 2^{n-3} \,.$$
Since $|I_n| \leq |D_\mu|$ we get an upper bound for n, namely
$$n \leq 3 + |\log \varepsilon| / \log 2 \leq (3 + 1/\log 2) \cdot |\log \varepsilon| \,,$$
observing $\varepsilon \leq \varepsilon_0 < (1/e)$. For the indicator functions $1_{\tilde{I}}$ we get
$$\sum 1_{\tilde{I}} \leq n \leq (3 + 1/\log 2) \cdot 1_{\tilde{K}} \cdot |\log \varepsilon| \,, \tag{3.3}$$

where the sum extends over all $I \in \mathcal{A}$ such that $|I| \geq \varepsilon$ and $I \subseteq K$. Let $\mathcal{C}_3 = (3+1/\log 2)\cdot C/c$. Integrating inequality (3.3) with respect to μ yields

$$\sum |I|^\alpha \leq \sum (1/c)\cdot\mu(\tilde{I}) \leq ((3+1/\log 2)/c)\cdot|\log\varepsilon|\cdot\mu(\tilde{K}) \leq \mathcal{C}_3\cdot|\log\varepsilon|\cdot|K|^\alpha$$

as required.

To prove the second statement let $\mathcal{C}_4 = \sum_{\substack{I\in\mathcal{A}\\|I|\geq\varepsilon_0}} |I|^\alpha + \mathcal{C}_2\mathcal{C}_3$ and observe

$$\sum_{\substack{I\in\mathcal{A}\\|I|\geq\varepsilon}} |I|^\alpha \leq \sum_{\substack{I\in\mathcal{A}\\|I|\geq\varepsilon_0}} |I|^\alpha + \sum_{K\in\mathcal{N}_{\varepsilon_0}} \sum_{\substack{I\subseteq K\\|I|\geq\varepsilon}} |I|^\alpha \leq \mathcal{C}_4\cdot|\log\varepsilon|,$$

using lemma 3.2.4 and the first part. ■

Define measures ψ_ε by

$$\psi_\varepsilon(A) = (|\log\varepsilon|)^{-1}\int_\varepsilon^1 (\mathbf{1}_A(t)+\mathbf{1}_A(-t))\,\frac{dt}{t}$$

for all Borel sets $A \subseteq \mathbb{R}$ and measures ψ_ε^x for $x \in \mathbb{R}$ by

$$\psi_\varepsilon^x(A) = \psi_\varepsilon(A-x)$$

for Borel sets $A \subseteq \mathbb{R}$. Note that the total mass of each of the measures ψ_ε and ψ_ε^x is 2.

Lemma 3.2.5 *There are absolute constants $\mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8 \geq 1$ such that, for all intervals $I \subseteq \mathbb{R}$ with endpoints in E, and all $\varepsilon > 0$, $\kappa > 0$, the following estimates hold:*

(a) $\displaystyle\int_{I^-(\kappa)\cup I^+(\kappa)} \psi_\varepsilon^x(I)\,d\mu(x) \leq \mathcal{D}_5\cdot C\cdot\frac{|I|^\alpha}{|\log\varepsilon|}\cdot\left(\log\left(\frac{\kappa+1}{\kappa}\right)\cdot\kappa^\alpha\right),$

(b) $\displaystyle\int_{(I^o(\kappa)^c)} \psi_\varepsilon^x(I)\,d\mu(x) \leq \mathcal{D}_6\cdot C\cdot\frac{|I|^\alpha}{|\log\varepsilon|}\cdot\left(\frac{1}{\kappa}\right)^{1-\alpha},$

(c) $\displaystyle\int_{I^c} \psi_\varepsilon^x(I)\,d\mu(x) \leq \mathcal{D}_7\cdot C\cdot\frac{|I|^\alpha}{|\log\varepsilon|},$

(d) $\displaystyle\int_{I^c} (\psi_\varepsilon^x(I))^2\,d\mu(x) \leq \mathcal{D}_8\cdot C\cdot\frac{|I|^\alpha}{|\log\varepsilon|^2}.$

Proof (a) Denote the left endpoint of I by a and let $R(t) = \mu([a-t,a])$. By (3.1) we have $R(t) \leq Ct^\alpha$. We use integration by parts to see

$$\begin{aligned}
\int_{I^-(\kappa)} \psi_\varepsilon^x(I)\,d\mu(x) &\leq (|\log\varepsilon|)^{-1}\cdot\int_0^{\kappa|I|}\log\left(\frac{t+|I|}{t}\right)dR(t)\\
&\leq (|\log\varepsilon|)^{-1}\cdot\int_0^{\kappa}\log\left(\frac{t+1}{t}\right)dR(t|I|)\\
&\leq (|\log\varepsilon|)^{-1}\cdot\left[\log\left(\frac{\kappa+1}{\kappa}\right)\cdot R(\kappa\cdot|I|)+\int_0^{\kappa}\frac{R(t|I|)}{(t+1)t}\,dt\right]\\
&\leq \frac{C|I|^\alpha}{|\log\varepsilon|}\cdot\left[\log\left(\frac{\kappa+1}{\kappa}\right)\cdot\kappa^\alpha+\int_0^{\kappa}\frac{dt}{t^{1-\alpha}}\right]\\
&\leq (\mathcal{D}_5/2)\cdot C\cdot\frac{|I|^\alpha}{|\log\varepsilon|}\cdot\left(\log\left(\frac{\kappa+1}{\kappa}\right)\cdot\kappa^\alpha\right)
\end{aligned}$$

for $\mathcal{D}_5 = 2(1+1/\alpha)$. An analogous calculation can be performed for $I^+(\kappa)$ and thus (a) follows.

(b) Observe that by Fubini's theorem

$$\int_{(I^0(\kappa))^c}\psi_\varepsilon^x(I)\,d\mu(x) = \int_0^1 \mu(\{x\notin I^0(\kappa)\,:\,\psi_\varepsilon^x(I)>t\})\,dt\,.$$

$\psi_\varepsilon^x(I) > t$ implies

$$(|\log\varepsilon|)^{-1}\cdot\log\left(\frac{d(x,I)+|I|}{d(x,I)}\right) > t\,,$$

and thus $d(x,I) < |I|\cdot(\varepsilon^t/(1-\varepsilon^t))$. Therefore the set

$$\{x\notin I^0(\kappa)\,:\,\psi_\varepsilon^x(I)>t\}\subseteq\{x\,:\,\kappa|I|\leq d(x,I)<(\varepsilon^t/(1-\varepsilon^t))|I|\}$$

is empty if $\kappa \geq \varepsilon^t/(1-\varepsilon^t)$, which is equivalent to $t \geq (|\log\varepsilon|)^{-1}\cdot\log(1+1/\kappa)$. Otherwise,

$$\begin{aligned}
\mu(\{x\notin I^0(\kappa)\,:\,\psi_\varepsilon^x(I)>t\}) &\leq \mu(\{x\notin I\,:\,d(x,I)<|I|\cdot(\varepsilon^t/(1-\varepsilon^t))\})\\
&\leq 2C\cdot|I|^\alpha\cdot(\varepsilon^t/(1-\varepsilon^t))^\alpha.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_0^1 \mu(\{x\notin I^0(\kappa)\,:\,d(x,I)<|I|\cdot\varepsilon^t/(1-\varepsilon^t)\})\,dt\\
&\leq 2C\cdot|I|^\alpha\cdot\int_0^{\frac{\log(1+1/\kappa)}{|\log\varepsilon|}}(\varepsilon^t/(1-\varepsilon^t))^\alpha\,dt
\end{aligned}$$

and

$$\int_0^{\frac{\log(1+1/\kappa)}{|\log\varepsilon|}}(\varepsilon^t/(1-\varepsilon^t))^\alpha\,dt = (|\log\varepsilon|)^{-1}\cdot\int_1^{1+1/\kappa}\frac{d\tau}{(\tau-1)^\alpha\tau}$$

$$\leq (|\log\varepsilon|)^{-1}\cdot\int_0^{1/\kappa}\frac{d\tau}{\tau^\alpha}\leq 1/(1-\alpha)\cdot(|\log\varepsilon|)^{-1}\cdot(1/\kappa)^{1-\alpha}$$

which finishes the proof of (b) with $\mathcal{D}_6 = 2/(1-\alpha)$.

(c) follows by adding inequalities (a) and (b) for $\kappa = 1$, and putting $\mathcal{D}_7 = \log 2\cdot\mathcal{D}_5 + \mathcal{D}_6$.

(d) is proved in a manner similar to (b). We have

$$\int_{I^-(\kappa)}(\psi_\varepsilon^x(I))^2\,d\mu(x) = \int_0^1 \mu(\{x\in I^-(\kappa)\,:\,\psi_\varepsilon^x(I) > \sqrt{t}\})\,dt\,.$$

$\psi_\varepsilon^x(I) > \sqrt{t}$ implies $d(x,I) < |I|\cdot(\varepsilon^{\sqrt{t}}/(1-\varepsilon^{\sqrt{t}}))$ and hence we have

$$\begin{aligned}\mu(\{x\in I^-(\kappa)\,:\,\psi_\varepsilon^x(I)>\sqrt{t}\}) &\leq \mu(\{x\in I^-(\kappa)\,:\,d(x,I)<|I|\cdot(\varepsilon^{\sqrt{t}}/(1-\varepsilon^{\sqrt{t}}))\})\\ &\leq C\cdot|I|^\alpha\cdot(\varepsilon^{\sqrt{t}}/(1-\varepsilon^{\sqrt{t}}))^\alpha\,,\end{aligned}$$

and thus

$$\begin{aligned}\int_{I^-(\kappa)}(\psi_\varepsilon^x(I))^2\,d\mu(x) &\leq C\cdot|I|^\alpha\cdot\int_0^1(\varepsilon^t/(1-\varepsilon^t))^\alpha\,(2t)\,dt\\ &\leq 2C\cdot|I|^\alpha\cdot(|\log\varepsilon|)^{-2}\int_1^\infty\frac{\log\tau\,d\tau}{(\tau-1)^\alpha\tau}\,.\end{aligned}$$

Together with the analogous calculation for $I^+(\kappa)$ this completes the proof of inequality (b) with

$$\mathcal{D}_8 = 4\cdot\int_1^\infty\frac{\log\tau\,d\tau}{(\tau-1)^\alpha\tau} < \infty\,.$$

■

Note that all intervals $I\in\mathcal{A}$ and all intervals $I\in\mathcal{N}_\varepsilon$, $\varepsilon>0$, have endpoints in E and thus lemma 3.2.5 applies to these intervals.

We finish this section with a useful lemma of approximation.

Lemma 3.2.6 *For every fixed $\gamma\geq 1$ there are $C_9, C_{10} > 1$ depending on γ such that, for all $0<\delta\leq\varepsilon_0$, $\kappa>1$ and all $\varepsilon>0$ with $\gamma\varepsilon\leq\varepsilon_0$, and for every $K\in\mathcal{N}_\delta$, we have*

$$\int_E\Big(\psi_\varepsilon^x(K)-\sum\psi_\varepsilon^x(I)\Big)\,d\mu(x)\leq C_9\cdot|K|^\alpha\cdot\left(\left(\frac{1}{\kappa}\right)^{1-\alpha}+\frac{1}{|\log\varepsilon|}\right),$$

where the sum extends over all $I\in\mathcal{A}$ such that $|I|\geq\gamma\varepsilon$, $I\subseteq K$ and $x\in I^0(\kappa)$; and also

$$\int_E\Big(\psi_\varepsilon^x(\mathbb{R})-\sum\psi_\varepsilon^x(I)\Big)\,d\mu(x)\leq C_{10}\cdot\left(\left(\frac{1}{\kappa}\right)^{1-\alpha}+\frac{1}{|\log\varepsilon|}\right),$$

where the sum extends over all $I\in\mathcal{A}$ such that $|I|\geq\gamma\varepsilon$ and $x\in I^0(\kappa)$.

Proof We start with the first inequality. For all $x \in \mathbb{R}$ we have

$$\psi_\varepsilon^x(K) - \sum \psi_\varepsilon^x(I) = \sum_{N \in \mathcal{N}_{\gamma\varepsilon}, N \subseteq K} \psi_\varepsilon^x(N) + \sum_{\substack{I \in \mathcal{A}, |I| \geq \gamma\varepsilon \\ x \notin I^0(\kappa), I \subseteq K}} \psi_\varepsilon^x(I)\,.$$

Therefore we have to give appropriate estimates for the expressions

$$\sum_{N \in \mathcal{N}_{\gamma\varepsilon}, N \subseteq K} \int \psi_\varepsilon^x(N)\, d\mu(x) \quad \text{and} \quad \sum_{\substack{I \in \mathcal{A}, |I| \geq \gamma\varepsilon \\ I \subseteq K}} \int_{(I^0(\kappa))^c} \psi_\varepsilon^x(I)\, d\mu(x)\,.$$

To estimate the first expression, we use $|N| \leq C_1\gamma\varepsilon$ to see

$$\int_N \psi_\varepsilon^x(N)\, d\mu(x) \leq \frac{\log C_1\gamma}{|\log\varepsilon|} \cdot \mu(N) \leq C \cdot \log C_1\gamma \cdot \frac{|N|^\alpha}{|\log\varepsilon|}\,.$$

By lemma 3.2.5(c) we know that

$$\int_{N^c} \psi_\varepsilon^x(N)\, d\mu(x) \leq \mathcal{D}_7 \cdot C \cdot \frac{|N|^\alpha}{|\log\varepsilon|}$$

holds. These two estimates together with lemma 3.2.3 give

$$\begin{aligned} \sum \int \psi_\varepsilon^x(N)\, d\mu(x) &\leq (C \cdot \log C_1\gamma + C \cdot \mathcal{D}_7) \cdot \sum \frac{|N|^\alpha}{|\log\varepsilon|} \\ &\leq \mathcal{C} \cdot \frac{|K|^\alpha}{|\log\varepsilon|} \end{aligned}$$

with $\mathcal{C} = C_2 \cdot (C \cdot \log C_1\gamma + C \cdot \mathcal{D}_7)$.

For the second expression we use lemmas 3.2.5(b) and 3.2.4 to see

$$\begin{aligned} \sum_{\substack{I \in \mathcal{A}, |I| \geq \gamma\varepsilon \\ I \subseteq K}} \int_{(I^0(\kappa))^c} \psi_\varepsilon^x(I)\, d\mu(x) &\leq \mathcal{D}_6 \cdot C \cdot \sum_{\substack{I \in \mathcal{A}, |I| \geq \varepsilon \\ I \subseteq K}} \frac{|I|^\alpha}{|\log\varepsilon|} \cdot \left(\frac{1}{\kappa}\right)^{1-\alpha} \\ &\leq C\mathcal{D}_6 C_3 \cdot |K|^\alpha \cdot \left(\frac{1}{\kappa}\right)^{1-\alpha}. \end{aligned}$$

This gives the first inequality with $C_9 = \mathcal{C} + C\mathcal{D}_6 C_3$.

To prove the second inequality, observe that

$$\int \left(\psi_\varepsilon^x(D_\mu) - \sum \psi_\varepsilon^x(I)\right) d\mu(x) \leq (\mathcal{C} + C\mathcal{D}_6 C_4) \cdot \left(\left(\frac{1}{\kappa}\right)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\right),$$

which follows in the same manner as above, replacing K by D_μ and using lemma 3.2.4(2) instead of lemma 3.2.4(1) in the final step. Denote $a = \min(E)$, $b = \max(E)$ and use lemma 3.2.5(c) to see

$$\begin{aligned} \int_E (\psi_\varepsilon^x(\mathbb{R}) - \psi_\varepsilon^x(D_\mu))\, d\mu(x) &\leq \int_a^b \psi_\varepsilon^x([a-1, a])\, d\mu(x) + \int_a^b \psi_\varepsilon^x([b, b+1])\, d\mu(x) \\ &\leq \frac{2C\mathcal{D}_7}{|\log\varepsilon|}\,. \end{aligned}$$

Thus the second inequality holds with $\mathcal{C}_{10} = \mathcal{C} + C\mathcal{D}_7\mathcal{C}_4 + 2C\mathcal{D}_7$. ■

3.3 Proof of Theorem 3.1.3

The proof will be done according to the following plan:
First we show that it suffices to study the average densities of measures μ at almost all points x of a set E, where μ and E fulfill the conditions assumed in section 3.2. We then introduce a family of functions (φ_I), the sum of which approximates, roughly speaking, a modification of the difference of the left and right average densities. We show in the main step that the set of points where the approximating function has large modulus has small measure (lemma 3.3.5). Finally, we conclude that this implies that neither right- nor left-sided average density can vanish.

We suppose that $0 < \alpha < 1$ and a measure $\mu \in \mathcal{M}(\mathbb{R})$ is given such that

$$0 < \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) < \infty$$

μ-almost everywhere. Without losing generality, if necessary by restricting μ to countably many open sets, we can assume that μ is a finite measure. Due to the symmetry of the problem, we can concentrate our effort on the investigation of right-sided average densities.

For every pair of integers n, k let

$$E_n^k = \{x \,:\, (1/2)^n < \underline{d}^\alpha(\mu, x) \leq (1/2)^{n-1} \text{ and } 2^k > \overline{d}^\alpha(\mu, x) \geq 2^{k-1}\}.$$

The sets E_n^k are Borel sets. Moreover, by our assumption, μ-almost every x is contained in one E_n^k. By proposition 1.3.1 we have, for μ-almost every $x \in E_n^k$, that

$$(1/2)^{n-1} \geq \underline{d}^\alpha(\mu|_{E_n^k}, x) = \underline{d}^\alpha(\mu, x) > (1/2)^n \tag{3.4}$$

and

$$2^k > \overline{d}^\alpha(\mu|_{E_n^k}, x) = \overline{d}^\alpha(\mu, x) \geq 2^{k-1}, \tag{3.5}$$

and by proposition 2.2.5

$$\overline{D}^{\alpha}_{+}(\mu, x) = \overline{D}^{\alpha}_{+}(\mu|_{E_n^k}, x).$$

Therefore it is sufficient to show the statement of our theorem for the restricted measures $\mu|_{E_n^k}$. Let us fix such a measure and also denote it by μ. Let $c = (1/2)^n$ and $C = 2^k$.

Lemma 3.3.1 *Let $\mu \in \mathcal{M}(\mathbb{R})$ be a finite measure such that*

$$c < \underline{d}^{\alpha}(\mu, x) \leq \overline{d}^{\alpha}(\mu, x) < C$$

for μ-almost every x. Then, for every $1 > \delta > 0$, there is a family of disjoint Borel sets $(B_i)_{i\in\mathbb{N}}$ with $|B_i| < 1$ and a family of compact sets $(E_i)_{i\in\mathbb{N}}$ with $E_i \subseteq B_i$, such that

$$\mu\Big(\mathbb{R} \setminus \bigcup_{i=1}^{\infty} E_i\Big) < \delta\,, \tag{3.6}$$

and for every $i \in \mathbb{N}$ there is a number $0 < \varepsilon_0(i) < (1/e)$ such that, for all $x \in E_i$,

$$\mu([x-t, x+t] \cap B_i) \geq ct^{\alpha} \quad \text{if } 0 \leq t \leq \varepsilon_0(i),$$

and, for all $x \in B_i$,

$$\mu([x-t, x+t] \cap B_i) \leq Ct^{\alpha} \quad \text{if } t \geq 0.$$

Proof Pick a countable dense subset $Q \subseteq (0,1)$. The functions

$$s(x) = \sup_{\varepsilon > t > 0} \frac{\mu([x-t, x+t])}{t^{\alpha}} = \sup_{\substack{\varepsilon > t > 0 \\ t \in Q}} \frac{\mu([x-t, x+t])}{t^{\alpha}}$$

and

$$i(x) = \inf_{\varepsilon > t > 0} \frac{\mu([x-t, x+t])}{t^{\alpha}} = \inf_{\substack{\varepsilon > t > 0 \\ t \in Q}} \frac{\mu([x-t, x+t])}{t^{\alpha}}$$

are Borel-measurable for every $\varepsilon > 0$. Given an arbitrarily small $\delta > 0$ we can use the boundedness of the upper densities to find a number $0 < \varepsilon < 1$ and a Borel set $B \subseteq \mathbb{R}$, such that $\mu(\mathbb{R} \setminus B) < \delta/2$ and, for all $x \in B$ and $0 \leq t \leq \varepsilon$, we have $\mu([x-t, x+t]) \leq Ct^{\alpha}$. Write B as the union of pairwise disjoint Borel sets $B_1, B_2, B_3, \ldots \subseteq B$, such that $|B_i| \leq \varepsilon$. Then we have, for all $x \in B_i$ and $t > 0$,

$$\mu([x-t, x+t] \cap B_i) \leq Ct^{\alpha}\,.$$

By lemma 1.3.1 we have, for μ-almost every $x \in B_i$,

$$\underline{d}^\alpha(\mu|_{B_i}, x) = \underline{d}^\alpha(\mu, x) > c\,.$$

Using this and the inner regularity of μ we can find compact sets $E_i \subseteq B_i$ and numbers $0 < \varepsilon_0(i) < (1/e)$ such that $\mu(B_i \setminus E_i) < (\delta/2)\cdot(1/2)^i$ and, for all $x \in E_i$ and $0 \le t \le \varepsilon_0(i)$,

$$\mu([x-t, x+t] \cap B_i) \ge ct^\alpha\,.$$

Finally observe that $\mu\left(\mathbb{R} \setminus \bigcup_{i=1}^\infty E_i\right) < \delta$ to conclude the proof. ■

We can apply lemma 3.3.1 to our measure μ. By (3.6) and because we have

$$\overline{D}_+^\alpha(\mu, x) = \overline{D}_+^\alpha(\mu|_{B_i}, x)$$

for μ-almost every $x \in E_i$, it suffices to prove the inequality of theorem 3.1.3 for the restricted measures $\mu|_{B_i}$ and μ-almost every point $x \in E_i$.

Fix such a measure $\mu|_{B_i}$ and let us also denote it by μ. Let $E = E_i$ and $\varepsilon_0 = \varepsilon_0(i)$. We now have a measure μ with support contained in a compact interval of length less than one and a compact set E such that, without loss of generality, $\mu(E) > 0$ and such that, for all $x \in E$,

$$\mu([x-r, x+r]) \le Cr^\alpha \quad \text{if } r \ge 0, \tag{3.7}$$

and

$$\mu([x-r, x+r]) \ge cr^\alpha \quad \text{if } 0 \le r \le \varepsilon_0. \tag{3.8}$$

Thus the results on the geometry of E, as formulated in section 3.2, hold.

Let us proceed to the second step of the proof. Recall the definition of the collection $\mathcal{A}$ of intervals and the measures ψ_ε^x from the previous section. For every $\varepsilon > 0$ and $I \in \mathcal{A}$ define a function $\tilde{\varphi}_I$ describing the influence of the scales in the interval $I - x$ on the one-sided average densities by

$$\tilde{\varphi}_I(x,\varepsilon) = \begin{cases} \displaystyle\int_I \frac{\mu([x,y])}{|y-x|^\alpha}\, d\psi_\varepsilon^x(y) & \text{if } x < I\ , \\ \displaystyle\int_I \frac{\mu([y,x])}{|y-x|^\alpha}\, d\psi_\varepsilon^x(y) & \text{if } x > I\ , \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the functions $\tilde{\varphi}_I$ fulfill

$$|\tilde{\varphi}_I(x,\varepsilon)| \le C \cdot \psi_\varepsilon^x(I). \tag{3.9}$$

Therefore we have, using lemma 3.2.5(c), that

$$\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\,d\mu(x)\,,\ \int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\,d\mu(x) \le \mathcal{D}_7 \cdot C^2 \cdot \frac{|I|^\alpha}{|\log\varepsilon|} \tag{3.10}$$

for all $I \in \mathcal{A}$ and $\varepsilon > 0$. We can use (3.8) to get a converse inequality, at least for many I.

Lemma 3.3.2 *There is an absolute constant $\mathcal{D}_{11} > 0$ and there is a constant $1 > \lambda > 0$, such that, for all $\varepsilon > 0$, $\kappa > 1$ and all $I \in \mathcal{A}$ with $\varepsilon_0 \ge |I| \ge \varepsilon/\lambda$ and $\mu(I) \le (c/4)\cdot|I|^\alpha$, we have*

$$\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\,d\mu(x)\,,\ \int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\,d\mu(x) \ge \mathcal{D}_{11} \cdot \frac{c^3}{C} \cdot \frac{|I|^\alpha}{|\log\varepsilon|}\,.$$

Proof Let $\lambda = \left(c/(4C)\right)^{1/\alpha}$ and $\mathcal{D}_{11} = (1/16)\cdot\log(4/3)$.
Suppose I is an interval as in the hypothesis and denote its left endpoint by a. For all $x \in I^-(1) \setminus I^-(\lambda)$ and $y \in I$ such that $y - a \ge \lambda|I|$ we get

$$\frac{\mu([x,y])}{(y-x)^\alpha} \ge \frac{\mu(B(a,\lambda|I|))}{(2|I|)^\alpha} \ge \frac{c\lambda^\alpha|I|^\alpha}{(2|I|)^\alpha} \ge \frac{c^2}{8C}\,.$$

Denote $R(t) = \mu([a-t,a])$ and observe that our assumptions imply, for $0 < t \le 1$,

$$|I|^\alpha C t^\alpha \ge R(t|I|) \ge |I|^\alpha(ct^\alpha - (c/4))\,.$$

These inequalities yield

$$\begin{aligned}
\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\,d\mu(x) &\ge \frac{c^2}{8C}\cdot\int_{I^-(1)\setminus I^-(\lambda)} \psi_\varepsilon^x(\{y\in I\,:\,y-a\ge\lambda|I|\})\,d\mu(x)\\
&\ge \frac{c^2}{8C}\cdot\frac{1}{|\log\varepsilon|}\cdot\int_\lambda^1 \log\left(\frac{t+1}{t+\lambda}\right)\,dR(t|I|)\\
&\ge \frac{c^2}{8C}\cdot\frac{1}{|\log\varepsilon|}\cdot\log\left(\frac{2}{1+\lambda}\right)\cdot\Big(R(|I|)-R(\lambda|I|)\Big)\\
&\ge \frac{c^2}{8C}\cdot\log(4/3)\cdot\frac{|I|^\alpha}{|\log\varepsilon|}\cdot(c-(c/4)-C\lambda^\alpha)\\
&= (1/16)\cdot\log(4/3)\cdot\left(c^3/C\right)\cdot\frac{|I|^\alpha}{|\log\varepsilon|}.
\end{aligned}$$

This completes the proof for $I^-(\kappa)$. For $I^+(\kappa)$ the proof is analogous. ∎

We are now able to define $\tau > 0$ as

$$\tau = \frac{c^4}{C^3} \cdot \frac{\mathcal{D}_{11}}{8 \cdot \mathcal{D}_7}.$$

Note that τ depends only on C, c and α and that C, c are functions of $\underline{d}^\alpha(\mu, x)$ and $\overline{d}^\alpha(\mu, x)$. We fix the value of κ depending on ε as

$$\kappa = \kappa(\varepsilon) = (\log|\log \varepsilon|)^{\frac{2}{1-\alpha}}.$$

For $\varepsilon > 0$ let

$$\mathcal{G}_\varepsilon = \{I \in \mathcal{A} : \varepsilon_0 \geq |I| \geq \varepsilon/\lambda \text{ and } \mu(I) \leq (c/4) \cdot |I|^\alpha\},$$

the collection of those intervals to which lemma 3.3.2 can be applied. Also let

$$\mathcal{B}_\varepsilon = \{I \in \mathcal{A}, |I| \geq \varepsilon/\lambda \text{ and } I \notin \mathcal{G}_\varepsilon\}.$$

For $I \in \mathcal{G}_\varepsilon$ we have, by (3.10) and lemma 3.3.2, that

$$\mathcal{D}_{11} \cdot \frac{c^3}{C} \cdot \frac{|I|^\alpha}{|\log \varepsilon|} \leq \int_{I^-(\kappa)} \tilde{\varphi}_I(x, \varepsilon)\, d\mu(x) \leq \mathcal{D}_7 \cdot C^2 \cdot \frac{|I|^\alpha}{|\log \varepsilon|}$$

and

$$\mathcal{D}_{11} \cdot \frac{c^3}{C} \cdot \frac{|I|^\alpha}{|\log \varepsilon|} \leq \int_{I^+(\kappa)} \tilde{\varphi}_I(x, \varepsilon)\, d\mu(x) \leq \mathcal{D}_7 \cdot C^2 \cdot \frac{|I|^\alpha}{|\log \varepsilon|}.$$

Thus the numbers

$$\eta_I(\varepsilon) := \frac{\int_{I^-(\kappa)} \tilde{\varphi}_I(x, \varepsilon)\, d\mu(x)}{\int_{I^+(\kappa)} \tilde{\varphi}_I(x, \varepsilon)\, d\mu(x)}$$

fulfill

$$\frac{8\tau}{c} \leq \eta_I(\varepsilon) \leq \frac{c}{8\tau}. \tag{3.11}$$

For $I \in \mathcal{B}_\varepsilon$ let $\eta_I(\varepsilon) = 1$. For $\varepsilon > 0$ and $I \in \mathcal{A}$ with $|I| \geq \varepsilon/\lambda$ define the function φ_I by

$$\varphi_I(x, \varepsilon) = \begin{cases} -\eta_I(\varepsilon) \cdot \tilde{\varphi}_I(x, \varepsilon) & \text{if } x \in I^+(\kappa(\varepsilon)), \\ \tilde{\varphi}_I(x, \varepsilon) & \text{if } x \in I^-(\kappa(\varepsilon)), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.3.3 *The functions φ_I have the following properties:*

1. *There is a constant $\mathcal{C}_{12} > 0$ such that, for all $x \in \mathbb{R}$ and $\varepsilon > 0$,*

$$|\varphi_I(x, \varepsilon)| \leq \mathcal{C}_{12} \cdot \psi_\varepsilon^x(I).$$

2. $\varphi_I(\cdot,\varepsilon)$ *is supported by* $I^-(\kappa(\varepsilon)) \cup I^+(\kappa(\varepsilon))$.

3. *For* $I \in \mathcal{G}_\varepsilon$ *we have* $\int \varphi_I(x,\varepsilon)\, d\mu(x) = 0$.

Proof Define $C_{12} = Cc/(8\tau)$. Then for all $x \in \mathbb{R}$

$$|\varphi_I(x,\varepsilon)| \le \max\{|\eta_I|, 1\} \cdot |\tilde{\varphi}_I(x,\varepsilon)| \le C_{12} \cdot \psi_\varepsilon^x(I),$$

by (3.11) and (3.9). This proves the first statement.

The second statement is immediate from the definition of φ_I.

To prove the third statement, let $I \in \mathcal{G}_\varepsilon$. Then, by definition of η_I,

$$\begin{aligned}
\int \varphi_I(x,\varepsilon)\, d\mu(x) &= \int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) - \eta_I(\varepsilon) \cdot \int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) \\
&= \int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) - \frac{\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x)}{\int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x)} \cdot \int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) \\
&= 0\,.
\end{aligned}$$

■

We now proceed to the main step of the proof and show that the sum

$$\sum_{I \in \mathcal{A}, |I| \ge \varepsilon/\lambda} \varphi_I(x,\varepsilon)$$

has small mean square (with respect to μ). We start by showing that, for most x, the summands in the sum above are small.

Lemma 3.3.4 *Define the set* $S(\varepsilon)$ *as the set of all* $x \in \mathbb{R}$ *such that there is* $J \in \mathcal{A}$ *with* $|J| \ge \varepsilon/\lambda$ *and*

$$|\varphi_J(x,\varepsilon)| \ge \left(\frac{2C_{12}}{\alpha}\right) \frac{\log|\log\varepsilon|}{|\log\varepsilon|}\,.$$

Then there is a constant $C_{13} > 1$ *such that, for all sufficiently small* $\varepsilon > 0$,

$$\mu(S(\varepsilon)) \le C_{13} \cdot (|\log\varepsilon|)^{-1}.$$

Proof Denote

$$\delta = \frac{2 \cdot \log|\log\varepsilon|}{|\log\varepsilon^\alpha|}$$

and observe

$$\mu(S(\varepsilon)) \leq \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} \mu(\{x \in \mathbb{R}\setminus J \,:\, (|\log\varepsilon|)^{-1}\cdot\log\left(\frac{d(x,J)+|J|}{d(x,J)}\right) \geq \delta\}).$$

The condition

$$(|\log\varepsilon|)^{-1}\cdot\log\left(\frac{d(x,J)+|J|}{d(x,J)}\right) \geq \frac{2\cdot\log|\log\varepsilon|}{|\log\varepsilon^{\alpha}|}$$

implies

$$\frac{d(x,J)+|J|}{d(x,J)} \geq |\log\varepsilon|^{2/\alpha}$$

and therefore, if ε is sufficiently small, $d(x,J) \leq 2|J|\cdot|\log\varepsilon|^{-2/\alpha}$. Thus, denoting $C_{13} = 4C\cdot C_4$, we have, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\mu(S(\varepsilon)) &\leq \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} \mu(\{x\in\mathbb{R}\setminus J \,:\, d(x,J)\leq 2|J|\cdot|\log\varepsilon|^{-2/\alpha}\})\\ &\leq 2C\cdot 2^{\alpha}\cdot\sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}}\frac{|J|^{\alpha}}{|\log\varepsilon|^{2}} \leq \frac{C_{13}}{|\log\varepsilon|}.\end{aligned}$$

■

We can now formulate the main lemma of the proof:

Lemma 3.3.5 *Denote*

$$B_\varepsilon = \{x\in E \,:\, |\sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon/\lambda}} \varphi_I(x,\varepsilon)| \geq \tau\}.$$

Then there is a constant $C_{14} > 1$ such that, for all sufficiently small $\varepsilon > 0$,

$$\mu(B_\varepsilon) \leq \frac{C_{14}}{\sqrt{|\log\varepsilon|}}.$$

Proof We fix $0 < \varepsilon \leq \varepsilon_0$, sufficiently small in the sense of the preceding lemma, and sufficiently small to fulfill several computational conditions, specified as they appear in the proof. We begin the proof by estimating $\mu(B_\varepsilon)$ by means of suitable sums of integrals

$$\begin{aligned}\mu(B_\varepsilon)\cdot\tau^2 &\leq \int\Big(\sum_{I\in\mathcal{A},|I|\geq\varepsilon/\lambda}\varphi_I(x,\varepsilon)\Big)^2\,d\mu(x)\\ &= \sum_{I\in\mathcal{B}_\varepsilon \text{ or } J\in\mathcal{B}_\varepsilon}\sum\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x) \qquad (3.12)\\ &\quad+\sum_{I\in\mathcal{G}_\varepsilon}\sum_{J\in\mathcal{G}_\varepsilon}\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x). \qquad (3.13)\end{aligned}$$

In order to give an estimate of (3.12), we define $\mathcal{C}_{15} = (4/c) + \sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon_0}} |I|^\alpha$ and observe

$$\sum_{I\in\mathcal{B}_\varepsilon} |I|^\alpha \leq \sum_{\substack{I\in\mathcal{A}\\ \mu(I)>(c/4)|I|^\alpha}} |I|^\alpha + \sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon_0}} |I|^\alpha \leq (4/c)\cdot\mu(\mathbb{R}\setminus E) + \sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon_0}} |I|^\alpha \leq \mathcal{C}_{15}.$$

Therefore we can use lemmas 3.3.3 and 3.2.5(c) to show

$$\begin{aligned}
&\sum_{I\in\mathcal{B}_\varepsilon \text{ or } J\in\mathcal{B}_\varepsilon} \int \varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\, d\mu(x) \\
&\quad\leq 2\sum_{I\in\mathcal{B}_\varepsilon} \int \Big(|\varphi_I(x,\varepsilon)| \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} |\varphi_J(x,\varepsilon)|\Big)\, d\mu(x) \\
&\quad\leq 2\mathcal{C}_{12}^2\cdot \sum_{I\in\mathcal{B}_\varepsilon} \int_{I^c} \Big(\psi_\varepsilon^x(I)\cdot \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} \psi_\varepsilon^x(J)\Big)\, d\mu(x) \\
&\quad\leq 4\mathcal{C}_{12}^2\cdot \sum_{I\in\mathcal{B}_\varepsilon} \int_{I^c} \psi_\varepsilon^x(I)\, d\mu(x) \\
&\quad\leq 4C\mathcal{D}_7\mathcal{C}_{12}^2\cdot \sum_{I\in\mathcal{B}_\varepsilon} \frac{|I|^\alpha}{|\log\varepsilon|} \\
&\quad\leq 4C\mathcal{D}_7\mathcal{C}_{12}^2\mathcal{C}_{15}\cdot \frac{1}{|\log\varepsilon|} \leq \frac{1}{\sqrt{|\log\varepsilon|}}
\end{aligned}$$

for sufficiently small $\varepsilon > 0$, as required to estimate (3.12).

For an estimate of (3.13) we have to work harder. We split the sum again

$$\sum_{I\in\mathcal{G}_\varepsilon}\sum_{J\in\mathcal{G}_\varepsilon} \int \varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\, d\mu(x)$$

$$\leq \sum_{I\in\mathcal{G}_\varepsilon} \int \varphi_I^2(x,\varepsilon)\, d\mu(x) \tag{3.14}$$

$$+2\sum_{I,J\in\mathcal{G}_\varepsilon,\, I<J} \int \varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\, d\mu(x). \tag{3.15}$$

In order to estimate (3.14), we can use lemmas 3.2.5(d) and 3.2.4 and get

$$\sum_{I\in\mathcal{G}_\varepsilon} \int \varphi_I^2(x,\varepsilon)\, d\mu(x) \leq C\mathcal{D}_8\mathcal{C}_{12}^2\cdot \sum_{I\in\mathcal{G}_\varepsilon} \frac{|I|^\alpha}{|\log\varepsilon|^2} \leq C\mathcal{D}_8\mathcal{C}_{12}^2\mathcal{C}_4\cdot\frac{1}{|\log\varepsilon|} \leq \frac{1}{\sqrt{|\log\varepsilon|}}$$

for sufficiently small $\varepsilon > 0$, as required.

The investigation of (3.15) constitutes the main part of the proof. Observe that

$$\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon) \leq 0$$

unless $x \in I^-(\kappa)\cap J^-(\kappa)$ or $x \in I^+(\kappa)\cap J^+(\kappa)$. We concentrate our effort on those pairs (I,J) that fulfill $I^-(\kappa)\cap J^-(\kappa) \neq \emptyset$.

We fix an interval $I = (a,b) \in \mathcal{G}_\varepsilon$ and look at the family of intervals $J \in \mathcal{G}_\varepsilon$ that fulfill $I < J$ and $I^-(\kappa) \cap J^-(\kappa) \neq \emptyset$. We order them from left to right

$$I < J_1 < J_2 < \ldots < J_N$$

and denote $J_i = (c_i, d_i)$ for $1 \leq i \leq N$ and $c_{N+1} = \infty$. Because $I^-(\kappa) \cap J_i^-(\kappa) \neq \emptyset$, we have $c_i - a \leq \kappa(d_i - c_i)$, and thus

$$c_i - a \leq \frac{\kappa}{\kappa+1}(d_i - a) \leq \frac{\kappa}{\kappa+1}(c_{i+1} - a).$$

Observing that $c_1 - a \geq |I|$, we therefore get by induction, for all $1 \leq i \leq N$

$$c_i - a \geq |I| \cdot \left(\frac{\kappa+1}{\kappa}\right)^{i-1}.$$

Let $\xi = \sup(I^+(\kappa))$ and denote by k the smallest integer such that

$$c_k - \xi \geq |\log\varepsilon|^{1/\alpha} \cdot |I|. \tag{3.16}$$

We have

$$|I|\left(\frac{\kappa+1}{\kappa}\right)^{k-2} \leq (c_{k-1} - \xi) + (\xi - a) \leq |\log\varepsilon|^{1/\alpha}|I| + (\kappa+1)|I|,$$

and therefore we can put $\beta = 3/(1-\alpha)$ and get, provided ε is small enough,

$$k \leq (2/\alpha) \cdot \frac{\log|\log\varepsilon|}{\log((\kappa+1)/\kappa)} \leq (\log|\log\varepsilon|)^\beta. \tag{3.17}$$

We now give an upper bound for the influence of $J_1, J_2, \ldots, J_k$ by means of lemma 3.3.4. We first use $\mu(S(\varepsilon)) \leq C_{13} \cdot (|\log\varepsilon|)^{-1}$ to get

$$\begin{aligned}
&\sum_{I \in \mathcal{G}_\varepsilon} \sum_{i=1}^{k} \int_{S(\varepsilon)} \varphi_I(x,\varepsilon)\varphi_{J_i}(x,\varepsilon)\, d\mu(x) \\
\leq\ & C_{12} \sum_{I \in \mathcal{G}_\varepsilon} \int_{S(\varepsilon)} \Big(|\varphi_I(x,\varepsilon)| \cdot \psi_\varepsilon^x(\bigcup_{i=1}^{k} J_i)\Big)\, d\mu(x) \\
\leq\ & 2C_{12} \cdot \int_{S(\varepsilon)} \sum_{I \in \mathcal{G}_\varepsilon} |\varphi_I(x,\varepsilon)|\, d\mu(x) \\
\leq\ & 4C_{12}^2 \cdot \mu(S(\varepsilon)) \leq \frac{4C_{12}^2 C_{13}}{|\log\varepsilon|}.
\end{aligned}$$

Now we use the moderate growth of k and the fact that outside the set $S(\varepsilon)$ the $|\varphi_{J_i}(x,\varepsilon)|$ are small, to see

$$\begin{aligned}
&\sum_{I\in\mathcal{G}_\varepsilon}\sum_{i=1}^{k}\int_{S(\varepsilon)^c}\varphi_I(x,\varepsilon)\varphi_{J_i}(x,\varepsilon)\,d\mu(x)\\
\le\ &\sum_{I\in\mathcal{G}_\varepsilon}\int_{S(\varepsilon)^c}\Big(|\varphi_I(x,\varepsilon)|(\sum_{i=1}^{k}|\varphi_{J_i}(x,\varepsilon)|)\Big)\,d\mu(x)\\
\le\ &\sum_{I\in\mathcal{G}_\varepsilon}\int_{S(\varepsilon)^c}(|\varphi_I(x,\varepsilon)|\cdot k\cdot \max_{i=1}^{k}\{|\varphi_{J_i}(x,\varepsilon)|\})\,d\mu(x)\\
\le\ &(2C_{12}/\alpha)\cdot\frac{(\log|\log\varepsilon|)^{1+\beta}}{|\log\varepsilon|}\cdot\int\sum_{I\in\mathcal{G}_\varepsilon}|\varphi_I(x,\varepsilon)|\,d\mu(x)\\
\le\ &4C_{12}^2\cdot\frac{(\log|\log\varepsilon|)^{1+\beta}}{\alpha|\log\varepsilon|}\cdot\mu(\mathbb{R}),
\end{aligned}$$

and summing these two estimates we get, for sufficiently small $\varepsilon>0$,

$$\sum_{I\in\mathcal{G}_\varepsilon}\sum_{i=1}^{k}\int\varphi_I(x,\varepsilon)\varphi_{J_i}(x,\varepsilon)\,d\mu(x)\le\frac{1}{\sqrt{|\log\varepsilon|}}$$

as required.

In order to give an upper bound for the influence ot the intervals $J_{k+1},\dots,J_N$ we first calculate that on the domain of $\varphi_I(\cdot,\varepsilon)$ the variation of the function $\tilde\varphi_{J_i}(\cdot,\varepsilon)$ is small, namely there is a constant $C_{17}>0$ such that, for

$$c(I,J_i)=\sup_{x,y\in I^-(\kappa)\cup I^+(\kappa)}|\tilde\varphi_{J_i}(x,\varepsilon)-\tilde\varphi_{J_i}(y,\varepsilon)|,$$

we have

$$c(I,J_i)\le C_{17}\cdot\psi_\varepsilon^\xi(J_i)\cdot\kappa^{2\alpha}\cdot\left(\frac{|I|}{c_{i-1}-\xi}\right)^\alpha. \tag{3.18}$$

In order to prove (3.18) we pick $x,y\in I^-(\kappa)\cup I^+(\kappa)$ with $x<y$. For $z\in J_i$ we get

$$1\le\frac{z-x}{z-y}\le 1+\frac{y-x}{z-y}\le 1+\frac{(2\kappa+1)|I|}{c_i-\xi},$$

and, if ε is sufficiently small to ensure

$$\frac{(2\kappa+1)|I|}{c_i-\xi}\le\frac{(2\kappa+1)}{|\log\varepsilon|^{1/\alpha}}<1,$$

we get

$$\left|1-\left(\frac{z-x}{z-y}\right)^{1+\alpha}\right|\le\left(1+\frac{(2\kappa+1)|I|}{c_i-\xi}\right)^{1+\alpha}-1\le 3\left(\frac{(2\kappa+1)|I|}{c_i-\xi}\right)^\alpha. \tag{3.19}$$

Because $c_i - c_{i-1} \geq (c_{i-1} - \xi)/\kappa$ we have $[y, c_i] \supseteq B(c_{i-1}, (c_{i-1} - \xi)/\kappa)$. If ε is small enough to ensure $1/\kappa(\varepsilon) < \varepsilon_0$ we have

$$\mu([y, c_i]) \geq \mu(B(c_{i-1}, \frac{c_{i-1} - \xi}{\kappa})) \geq (c/\kappa^\alpha)(c_{i-1} - \xi)^\alpha .$$

Thus, for all $z \in J_i$,

$$1 \leq \frac{\mu([x, z])}{\mu([y, z])} \leq 1 + \frac{\mu([x, y])}{\mu([y, c_i])} \leq 1 + \frac{\kappa^\alpha C|I|^\alpha (2\kappa + 1)^\alpha}{c(c_{i-1} - \xi)^\alpha} . \tag{3.20}$$

Let $C_{17} = 12(C^2/c)$. (3.19) and (3.20) together imply

$$\begin{aligned} &\left| \frac{\mu([x, z])}{(z - x)^{1+\alpha}} - \frac{\mu([y, z])}{(z - y)^{1+\alpha}} \right| \\ \leq\ & \frac{\mu([x, z])}{(z - x)^{1+\alpha}} \cdot \left| 1 - \left(\frac{z - x}{z - y} \right)^{1+\alpha} \right| + \frac{\mu([y, z])}{(z - y)^{1+\alpha}} \cdot \left| 1 - \frac{\mu([x, z])}{\mu([y, z])} \right| \\ \leq\ & C_{17} \cdot \frac{1}{z - \xi} \cdot \kappa^{2\alpha} \left(\frac{|I|}{c_{i-1} - \xi} \right)^\alpha , \end{aligned}$$

and thus

$$\begin{aligned} |\tilde{\varphi}_{J_i}(x, \varepsilon) - \tilde{\varphi}_{J_i}(y, \varepsilon)| &\leq (|\log \varepsilon|)^{-1} \int_{J_i} \left| \frac{\mu([x, z])}{(z - x)^{1+\alpha}} - \frac{\mu([y, z])}{(z - y)^{1+\alpha}} \right| dz \\ &\leq C_{17} \cdot \psi_\varepsilon^\xi(J_i) \cdot \kappa^{2\alpha} \left(\frac{|I|}{c_{i-1} - \xi} \right)^\alpha , \end{aligned}$$

which proves (3.18).

We can now use lemma 3.3.3(3) to see that, for an arbitrarily chosen fixed $y \in I^-(\kappa) \cup I^+(\kappa)$, we have

$$\int \varphi_I(x, \varepsilon) \tilde{\varphi}_{J_i}(y, \varepsilon) \, d\mu(x) = 0,$$

and use this together with (3.18) to estimate

$$\begin{aligned} \sum_{i=k+1}^{N} \int \varphi_I \varphi_{J_i} \, d\mu &\leq \sum_{i=k+1}^{N} \int \varphi_I(x, \varepsilon) \tilde{\varphi}_{J_i}(x, \varepsilon) \, d\mu(x) \\ &= \sum_{i=k+1}^{N} \int \varphi_I(x, \varepsilon) \left(\tilde{\varphi}_{J_i}(x, \varepsilon) - \tilde{\varphi}_{J_i}(y, \varepsilon) \right) d\mu(x) \\ &\leq \sum_{i=k+1}^{N} c(I, J_i) \cdot \int |\varphi_I(x, \varepsilon)| \, d\mu(x) \\ &\leq C C_{17} C_{12} \mathcal{D}_7 \cdot \sum_{i=k+1}^{N} \psi_\varepsilon^\xi(J_i) \cdot \kappa^{2\alpha} \cdot \left(\frac{|I|}{c_{i-1} - \xi} \right)^\alpha \cdot \frac{|I|^\alpha}{|\log \varepsilon|} \end{aligned}$$

$$\overset{(3.16)}{\leq} \quad CC_{17}C_{12}\mathcal{D}_7 \cdot \frac{1}{|\log\varepsilon|} \sum_{i=k+1}^{N} \psi_\varepsilon^\xi(J_i) \cdot \frac{|I|^\alpha}{|\log\varepsilon|} \cdot \kappa^{2\alpha}$$
$$\leq \quad CC_{17}C_{12}\mathcal{D}_7 \cdot \frac{|I|^\alpha}{|\log\varepsilon|^2} \cdot \kappa^{2\alpha},$$

and finally, summing over all $I \in \mathcal{G}_\varepsilon$ and using lemma 3.2.4(2),

$$\sum_{I\in\mathcal{G}_\varepsilon} \sum_{i=k+1}^{N} \int \varphi_I(x,\varepsilon)\varphi_{J_i}(x,\varepsilon)\, d\mu(x) \leq CC_{17}C_{12}\mathcal{D}_7C_4 \cdot \frac{\kappa^{2\alpha}}{|\log\varepsilon|} \leq \frac{1}{\sqrt{|\log\varepsilon|}}$$

for sufficiently small $\varepsilon > 0$.

In the same manner we get estimates for intervals I, J with $I^+(\kappa) \cap J^+(\kappa) \neq \emptyset$.

This finishes the proof of our main lemma. ∎

We now proceed to the final step in the proof of theorem 3.1.3.

The function $\sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon/\lambda}} \varphi_I(x,\varepsilon)$ is an approximation of

$$D(x,\varepsilon) = (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x,x+t]) - \eta(x-t,\varepsilon)\cdot\mu([x-t,x])}{t^\alpha}\frac{dt}{t},$$

where

$$\eta(x,\varepsilon) = \begin{cases} \eta_I(\varepsilon) & \text{if } x \in I \text{ for some } I \in \mathcal{G}_\varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.3.6 *Define the approximation error*

$$E(x,\varepsilon) = \left| D(x,\varepsilon) - \sum_{I\in\mathcal{A},|I|\geq\varepsilon/\lambda} \varphi_I(x,\varepsilon) \right|.$$

There is a constant $C_{16} > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$*, we have*

$$\mu(\{x \in E : E(x,\varepsilon) > \tau\}) \leq \frac{C_{16}}{(\log|\log\varepsilon|)^2}.$$

Proof We have for all $x \in E$, using (3.11),

$$E(x,\varepsilon) \leq \frac{cC}{8\tau} \cdot \Big(\psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A},|I|\geq\varepsilon/\lambda \\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big).$$

Putting $\gamma = (1/\lambda)$ in the second inequality of the approximation lemma 3.2.6 yields

$$\int_E \Big(\psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A},|I|\geq\varepsilon/\lambda \\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big)\, d\mu(x) \leq C_{10} \cdot \left(\left(\frac{1}{\kappa(\varepsilon)}\right)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\right) \leq \frac{C_{10}}{(\log|\log\varepsilon|)^2}.$$

Thus, defining $\mathcal{C}_{16} = 2cC\mathcal{C}_{10}/(8\tau^2)$,

$$\begin{aligned}
&\mu(\{x \in E : E(x,\varepsilon) > \tau\}) \\
&\leq \frac{1}{\tau}\int_E E(x,\varepsilon)\,d\mu(x) \\
&\leq cC/(8\tau^2)\cdot\int_E \Big(\psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A},|I|\geq\varepsilon/\lambda \\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big)\,d\mu(x) \\
&\leq \frac{\mathcal{C}_{16}}{(\log|\log\varepsilon|)^2}.
\end{aligned}$$

■

We can now finish the proof of theorem 3.1.3.

Observe that

$$|D(x,\varepsilon)| \leq |E(x,\varepsilon)| + \Big|\sum_{\substack{I\in\mathcal{A} \\ |I|\geq\varepsilon/\lambda}} \varphi_I(x,\varepsilon)\Big|.$$

We use lemmas 3.3.5 and 3.3.6 to see that, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&\mu(\{x \in E : |D(x,\varepsilon)| > 2\tau\}) \\
&\leq \mu(\{x \in E : |E(x,\varepsilon)| > \tau\}) + \mu(\{x \in E : |\sum_{I\in\mathcal{A},|I|\geq\varepsilon/\lambda} \varphi_I(x,\varepsilon)| > \tau\}) \\
&\leq \frac{\mathcal{C}_{16}+\mathcal{C}_{14}}{(\log|\log\varepsilon|)^2}.
\end{aligned}$$

Define the sequence (δ_k) by $\delta_k = \exp(-\exp k)$. Then

$$\mu(\{x \in E : |D(x,\delta_k)| > 2\tau\}) \leq \frac{\mathcal{C}_{16}+\mathcal{C}_{14}}{k^2}$$

and, since $\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$, we have, by the Borel-Cantelli-lemma,

$$\mu(\{x \in E : \limsup_{k\to\infty} |D(x,\delta_k)| > 2\tau\}) = 0\,.$$

We have

$$\begin{aligned}
&(|\log\delta_k|)^{-1}\int_{\delta_k}^1 \frac{\mu([x,x+t])}{t^\alpha}\,\frac{dt}{t} \\
&= \frac{1}{2}\left((|\log\delta_k|)^{-1}\int_{\delta_k}^1 \frac{\mu([x,x+t]) + \eta(x-t,\varepsilon)\cdot\mu([x-t,x])}{t^\alpha}\,\frac{dt}{t} + D(x,\delta_k)\right),
\end{aligned}$$

and

$$\int_{\delta_k}^1 \frac{\mu([x,x+t]) + \eta(x-t,\varepsilon)\cdot\mu([x-t,x])}{t^\alpha}\,\frac{dt}{t} \geq (8\tau/c)\cdot\int_{\delta_k}^1 \frac{\mu([x-t,x+t])}{t^\alpha}\,\frac{dt}{t}\,.$$

Therefore, for μ-almost every point $x \in E$,

$$\liminf_{k\to\infty}(|\log\delta_k|)^{-1}\int_{\delta_k}^{1}\frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t}\geq(1/2)\cdot((8\tau/c)\cdot c-2\tau)>e\cdot\tau.$$

For $\delta_{k+1}<\varepsilon\leq\delta_k$ we have

$$(|\log\varepsilon|)^{-1}\cdot\int_{\varepsilon}^{1}\frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t}\geq\frac{1}{e}\cdot(|\log\delta_k|)^{-1}\int_{\delta_k}^{1}\frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t},$$

and thus we finally get, for μ-almost every $x \in E$,

$$\begin{aligned}&\liminf_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\cdot\int_{\varepsilon}^{1}\frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t}\\ \geq\ &\frac{1}{e}\cdot\liminf_{k\to\infty}(|\log\delta_k|)^{-1}\int_{\delta_k}^{1}\frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t}\\ >\ &(1/e)\cdot e\tau=\tau,\end{aligned}$$

as required to finish the proof.

Chapter 4

Measures with Unique Tangent Measure Distributions

Tangent measure distributions at almost all points of a rectifiable measure are Dirac distributions with mass concentrated at a constant multiple of Hausdorff measure on a line, plane or a higher-dimensional approximate tangent space depending on the dimension of the measure. The governing question of this chapter is the following: What is the shared feature of the unique tangent measure distributions of a measure and the tangent spaces? What types of distributions can occur as unique tangent measure distributions? The answer is surprisingly elegant: The shared feature is statistical self-similarity. Unique tangent measure distributions of α-dimensional measures are (in a certain sense) α-self-similar random measures. On the other hand every α-self-similar random measure that fulfills an ergodicity condition appears as unique tangent measure distribution at almost every point of a suitably constructed measure. The key to this self-similarity property is the notion of a Palm distribution, which originates from stochastic geometry, and which was introduced into fractal geometry by U. Zähle.

In section 4.1 we give a short introduction into U. Zähle's approach to self-similar random measures and introduce the notion of a Palm distribution. This section closely follows [Zäh88]. Section 4.2 contains the main result of the chapter: At almost all points of a measure with positive and finite α-densities the unique tangent measure distribution, provided it exists, is a Palm distribution, and therefore it is an α-self-similar random measure in the sense of U. Zähle. Some applications of this result are given. In section 4.3 we give the proof of the main theorem.

4.1 Self-Similar Random Measures and Palm Distributions

The main idea behind U.Zähle's axiomatic approach to statistical self-similarity is, roughly speaking, the following: a random measure could be called statistically self-similar if it is statistically scale-invariant with respect to any centre chosen at random according to that measure. The important point, and also the difficulty, is that the scale invariance should hold with respect to a "typical point" of the random measure and not with respect to every point which would be too restrictive, see [Zäh88, 1.5]. To make the idea of a "typical point" of a random measure precise we have to introduce the notion of Palm distribution (see [Zäh88], [Mec67] or [Kal83]).

Starting from a stationary quasi-distribution, we derive by means of a "conditioning process" a distribution, which has the origin as a "typical mass point of its realizations". The precise method is as follows (see e.g. [Mec67, chapter 2]):

Suppose Q is a σ-finite measure on the Borel sets of $\mathcal{M}(\mathbb{R}^n)$, a so-called quasi-distribution, which is stationary, i.e. invariant with respect to all shifts. The barycentre of Q is a measure on the Borel sets of $\mathbb{R}^n$, which is not necessarily locally finite, defined by

$$\Lambda_Q(A) = \int \nu(A)\, dQ(\nu).$$

Λ_Q is called the *intensity measure* of Q.

Because Q is stationary the intensity measure Λ_Q is shift-invariant, and therefore, for Borel sets $B \subseteq \mathbb{R}^n$ with finite Lebesgue-measure $0 < \mathcal{L}^n(B) < \infty$, the ratio

$$\lambda_Q := \frac{\Lambda_Q(B)}{\mathcal{L}^n(B)} \in [0, \infty]$$

does not depend on the choice of B. λ_Q is called the *intensity* of Q.

For a stationary quasi-distribution Q with finite and positive intensity $0 < \lambda_Q < \infty$ and a Borel set $B \subseteq \mathbb{R}^n$ with $0 < \mathcal{L}^n(B) < \infty$ we can define a probability measure Q_0 on $\mathcal{M}(\mathbb{R}^n)$ by

$$Q_0(M) = \frac{1}{\Lambda_Q(B)} \int \int_B 1_M(T^x \nu)\, d\nu(x)\, dQ(\nu),$$

for all Borel sets $M \subseteq \mathcal{M}(\mathbb{R}^n)$.
Using the stationarity of Q it is easy to see that Q_0 is independent of the choice of the Borel set B. Q_0 is called the *Palm distribution* of Q.

The quasi-distribution Q can be reconstructed from Q_0 outside the zero-measure up to a constant multiple (see [Mec67, 2.4]). In the special case of a stationary point process Q (i.e. Q is the distribution of a stationary random measure, which is the countable sum of Dirac-measures at random points), the Palm distribution Q_0 can be interpreted as the conditional distribution of Q given that the origin is a mass point (see [Kal83, theorem 12.8]).

We call a probability measure P on $\mathcal{M}(\mathbb{R}^n)$ a *Palm distribution*, or say that P *has the Palm property*, if there is a stationary quasi-distribution Q with finite and positive intensity such that $P = Q_0$. A well known theorem of Mecke characterizes Palm distributions:

Lemma 4.1.1 *A probability measure P on $\mathcal{M}(\mathbb{R}^n)$ is a Palm distribution if and only if $P(\{\phi\}) = 0$, where ϕ is the zero-measure, and the following Palm formula holds:*

$$\int \int G(T^x \nu, -x)\, d\nu(x)\, dP(\nu) = \int \int G(\nu, x)\, d\nu(x)\, dP(\nu) \tag{4.1}$$

for all Borel functions $G : \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow [0, \infty)$.

Proof The proof can be found in [Mec67]. ■

In order to explain why the origin is a "typical mass point" of the realizations of a Palm distribution, let us define what we mean by a statement $\mathcal{A}(\nu, x)$ about a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ at a point $x \in \mathbb{R}^n$.

Definition

A statement $\mathcal{A}(\nu, x)$ is called a *statement about* ν *at* x, if and only if the set $\{(\nu, x) \in \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n : \mathcal{A}(\nu, x)\}$ is a Borel set and $\mathcal{A}(\nu, x)$ and $\mathcal{A}(T^x\nu, 0)$ are equivalent.

Lemma 4.1.2 *If P is a Palm distribution and $\mathcal{A}(\nu, x)$ is a statement about ν at x, then the following statements are equivalent:*

(1) $\mathcal{A}(\nu, 0)$ *for P-almost all ν.*

(2) $\mathcal{A}(\nu, x)$ *for ν-almost all $x \in \mathbb{R}^n$ for P-almost all $\nu \in \mathcal{M}(\mathbb{R}^n)$.*

Proof Denote $A_x = \{\nu \in \mathcal{M}(\mathbb{R}^n) : \mathcal{A}(\nu, x)\}$ and note that A_x is Borel set for all $x \in \mathbb{R}^n$. Assume first that $\mathcal{A}(\nu, 0)$ for P-almost all ν. We use the Palm formula (4.1) to get

$$0 = \int\int 1_{A_0^c}(\nu)\, d\nu(x)\, dP(\nu) = \int\int 1_{A_0^c}(T^x\nu)\, d\nu(x)\, dP(\nu).$$

Therefore for ν-almost all x we have $T^x\nu \in A_0$, for P-almost all ν. But $T^x\nu \in A_0$ is equivalent to $\mathcal{A}(T^x\nu, 0)$, which again is equivalent to $\mathcal{A}(\nu, x)$.
Now assume that for ν-almost all x we have $\mathcal{A}(\nu, x)$ for P-almost all ν. Then by the Palm formula (4.1) we get

$$0 = \int\int 1_{A_x^c}(\nu)\, d\nu(x)\, dP(\nu) = \int\int 1_{A_{-x}^c}(T^x\nu)\, d\nu(x)\, dP(\nu).$$

Therefore we have $T^x\nu \in A_{-x}$ for ν-almost all x, for P-almost all ν. Note that, since $P(\{\phi\}) = 0$, for P-almost every $\nu \in \mathcal{M}(\mathbb{R}^n)$ there are $x \in \mathbb{R}^n$ such that $T^x\nu \in A_{-x}$. This is equivalent to $\mathcal{A}(T^x\nu, -x)$, which on the other hand is equivalent to $\mathcal{A}(\nu, 0)$ and this finishes the proof. ■

We can interpret lemma 4.1.2 in the sense that the origin is a typical point of the realizations of a Palm distribution P. If the Palm distribution P is the distribution of a random measure which is α-scale invariant, i.e. $P = P \circ T_\lambda$ for all T_λ in the rescaling group, then P is α-scale invariant in a typical point, and therefore the random measure is statistically self-similar in the axiomatic sense of U. Zähle.

Definition

A probability distribution P on $\mathcal{M}(\mathbb{R}^n)$ is the distribution of an *α-self similar random measure* if P is an α-scale invariant Palm distribution.

Let us now close this excursion into the theory of random measures and go back to the study of unique tangent measure distributions.

4.2 Unique Tangent Measure Distributions are Palm Distributions

Let us look at a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ with positive and finite α-densities almost everywhere which has unique tangent measure distributions. If μ is a rectifiable measure then α is an integer and at μ-almost every x there is a linear space $T \subseteq \mathbb{R}^n$ of dimension α such that the unique tangent measure distribution of μ at x is a Dirac distribution concentrated at the point

$$(1/2)^\alpha d^\alpha(\mu, x) \cdot \mathcal{H}^\alpha|_T \in \mathcal{M}(\mathbb{R}^n),$$

where T is the approximate tangent space to μ at x (see theorem 1.3.12). If μ is a fractal measure which has a unique tangent measure distribution at almost all points it is a natural question to ask which general feature of a tangent space holds for the tangent measure distributions or, in other words, in which sense the tangent measure distributions possess a higher degree of regularity than the original measure. Of course we cannot expect the tangent measure distributions to be deterministic, but we have to expect a "statistical" property. Neither can we expect the "random tangents" to be a linear space in some sense, in particular, of course, if α is not an integer. But one characteristic feature of linear spaces which can be formulated for random measures of non-integer dimension is statistical self-similarity.

Looking at unique tangent measure distributions from this point of view, and keeping in mind the examples we have seen so far, it is not too far-fetched to conjecture that the unique tangent measure distributions at μ-almost every point are statistically self-similar random measures. It turns out that U. Zähle's notion of an α-self-similar random measure is the notion of statistical self-similarity which makes this conjecture true.

We have already seen in lemma 2.2.2 that the tangent measure distributions are α-scale invariant. The remaining problem therefore is to prove the following theorem:

Theorem 4.2.1 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and suppose there is a $0 \le \alpha \le n$ such that*

$$0 < \underline{d}^{\alpha}(\mu, x) \le \overline{d}^{\alpha}(\mu, x) < \infty$$

for μ-almost every x. Then for μ-almost every x the following statement holds: If there is a unique tangent measure distribution P of μ at x, then P is a Palm distribution.

Theorem 4.2.1 will be proved in section 4.3. Theorems 4.2.1 and 2.2.2 together immediately imply:

Corollary 4.2.2 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and suppose there is an $0 < \alpha \le n$ such that*

$$0 < \underline{d}^{\alpha}(\mu, x) \le \overline{d}^{\alpha}(\mu, x) < \infty$$

for μ-almost every x. Then for μ-almost every x the following statement holds: If there is a unique tangent measure distribution P of μ at x, then P is an α-self similar random measure.

It is an interesting question whether the statements of theorem 4.2.1 and corollary 4.2.2 also hold for non-unique tangent measure distributions. For measures μ on the line the question will be answered in the affirmative in chapter 5. In higher dimensions it remains open (see chapter 7).

We state some consequences of theorem 4.2.1 and corollary 4.2.2.
Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and suppose there is a $0 \leq \alpha \leq n$ such that, for μ-almost every x

$$0 < \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) < \infty .$$

Let us look at Palm distributions from a different point of view, namely as distributions with a shift-invariance property. The fact that the origin is a "typical point" for the realizations of the unique tangent measure distributions can be interpreted as an analogue to theorem 1.3.10. Theorem 1.3.10 is frequently used in the following equivalent form (see for example the proof of Marstrand's theorem in [Mat95]):

Let $\mathcal{A}(\nu, u)$ be a statement about ν at u. Then for μ-almost all x, the statement $\mathcal{A}(\nu, 0)$ holds for all $\nu \in \text{Tan}(\mu, x)$, if and only if $\mathcal{A}(\nu, u)$ holds for all $u \in \text{supp } \nu$ and all $\nu \in \text{Tan}(\mu, x)$.

In this form it becomes clear by means of lemma 4.1.2 how to interpret theorem 4.2.1 as a shift-invariance property. This can also be formulated directly in terms of the support of P. Recall the definition of the shift–operator T from section 1.2.

Corollary 4.2.3 *For μ-almost every $x \in \mathbb{R}^n$ the following property holds:*
If there is a unique tangent measure distribution P of μ at x, $\nu \in$ supp P and $u \in$ supp ν, then $T^u\nu \in$ supp P.

Proof Suppose P is a unique tangent measure distribution which is a Palm distribution. Look at the statement "$T^u\nu \in$ supp P". We have $T^0\nu \in$ supp P for P-almost every ν and hence (by lemma 4.1.2) that $T^u\nu \in$ supp P for ν-almost every u for P-almost every ν.
If $\nu \in$ supp P and $u \in$ supp ν we thus have sequences $\nu_k \to \nu$ and $u_k \to u$ such that $T^{u_k}\nu_k \in$ supp P. By lemma 1.3.4 this implies $T^u\nu \in$ supp P. ∎

What happens if we iterate the procedure of taking tangent measure distributions?
For the case of unique tangent measure distributions the answer is provided by proposition 4.2.4 below, because proposition 4.2.4 applies in particular to the random measures defined by the unique tangent measure distributions of μ at μ-almost every point.

Proposition 4.2.4 *Let P be the distribution of an α-self similar random measure. Then P-almost every measure ν has a unique tangent measure distribution Q_x^ν at ν-almost all points x and at $x = 0$.*
Furthermore let $\mathcal{A}$ be the σ-algebra of all Borel sets of $\mathcal{M}(\mathbb{R}^n)$ that are invariant with respect to the action of the rescaling-group $(T_\lambda)_{\lambda>0}$ and let

$$\begin{array}{rccc} P_{\mathcal{A}}: & \mathcal{M}(\mathbb{R}^n) & \longrightarrow & \mathcal{P} \\ & \nu & \mapsto & P_{\mathcal{A}}[\nu], \end{array}$$

a conditional distribution of P given $\mathcal{A}$. Then $Q_x^\nu = P_{\mathcal{A}}[T^x\nu]$ at ν-almost every x and at $x = 0$, for P-almost every ν.
In particular, if P is ergodic with respect to the action of the rescaling group, then for P-almost every ν, we have $Q_x^\nu = P$ at ν-almost every x and at $x = 0$.

Proof We first look at the origin and use Birkhoff's ergodic theorem to calculate, for $F : \mathcal{M}(\mathbb{R}^n) \longrightarrow [0, \infty)$ continuous and bounded,

$$\begin{aligned} \lim_{r\downarrow 0}(-\log r)^{-1} \int_r^1 F\left(\frac{\nu_{0,t}}{t^\alpha}\right)\frac{dt}{t} &= \lim_{s\uparrow\infty} 1/s \int_0^s F\left(\frac{\nu_{0,e^{-\tau}}}{e^{-\tau\alpha}}\right) d\tau \\ &= \lim_{s\uparrow\infty} 1/s \int_0^s F \circ T_{e^{-\tau}}(\nu)\, d\tau \\ &= \int F\, dP_{\mathcal{A}}[\nu] \end{aligned}$$

for P-almost every ν. Therefore

$$\lim_{r\downarrow 0} P_r^0 = P_{\mathcal{A}}[\nu]$$

for P-almost every ν, since by lemma 1.2.8(1) the convergence has only to be checked on a countable set of continuous and bounded functions F. This proves the statement for the origin. For the other points we only have to observe that

$$\mathcal{A}(\nu, x) = \text{“ the unique tangent measure distribution of } \nu \text{ at } x \text{ equals } P_{\mathcal{A}}[T^x\nu] \text{ ”}$$

is a statement about ν at x. Then apply lemma 4.1.2.
In the ergodic case, finally, the constant kernel P is itself a conditional distribution of P given $\mathcal{A}$. Therefore the first part applied to this particular conditional distribution yields the statement. ∎

In corollary 4.2.5 we describe random measures with a particularly beautiful self-similarity property:

Corollary 4.2.5 *Let $P \in \mathcal{P}$ be the distribution of a random measure ν such that $0 < \underline{d}_\alpha(\nu, x) \leq \overline{d}_\alpha(\nu, x) < \infty$ for ν-almost all x almost surely.*
Then the following statements are equivalent:

(1) ν has a unique tangent measure distribution equal to P at almost every point almost surely.

(2) P is α-self similar and ergodic with respect to the action of the rescaling group.

Proof Suppose (1) holds for P and let $A \in \mathcal{A}$. P-almost every ν has unique tangent measure distribution equal to P at ν-almost every point. By corollary 4.2.2 P is α-self similar. In particular P is a Palm distribution, and by lemma 4.1.2 we get that P-almost every ν has unique tangent measure distribution equal to P at the origin. Hence by Birkhoff's ergodic theorem, for P-almost all ν,

$$\int F(\bar{\nu})\, dP_{\mathcal{A}}[\nu](\bar{\nu}) = \lim_{s\uparrow\infty} \frac{1}{s} \int_0^s F\Big(\frac{\nu_{0,e^{-\tau}}}{e^{-\tau\alpha}}\Big)\, d\tau = \int F(\bar{\nu})\, dP(\bar{\nu}),$$

for $F : \mathcal{M}(\mathbb{R}^n) \to [0, \infty)$ continuous and bounded. Therefore we have $P_{\mathcal{A}}[\nu] = P$ for P-almost all ν. Consequently

$$P(A) = \int_A P_{\mathcal{A}}[\nu](A)\, dP(\nu) = \int_A P(A)\, dP(\nu) = P(A)^2,$$

and thus $P(A) = 1$ or $P(A) = 0$. Hence P is ergodic.
Suppose (2) holds. Use Birkhoff's ergodic theorem as before to see

$$\lim_{s\uparrow\infty} \frac{1}{s} \int_0^s F\Big(\frac{\nu_{0,e^{-\tau}}}{e^{-\tau\alpha}}\Big)\, d\tau = \int F(\bar{\nu})\, dP(\bar{\nu})$$

for P-almost all ν. Therefore P is the unique tangent measure distribution of ν at 0 for P-almost all ν and we can use lemma 4.1.2 to see that, for P-almost all ν, P is the unique tangent measure distribution of ν at ν-almost every x. ■

Finally we note that corollary 4.2.4 provides a proof of the fact that self-similar sets (or statistically self-similar random sets in the constructive sense of Falconer, Graf, Mauldin and Williams) fulfilling an open-set condition have unique tangent measure distributions almost everywhere almost surely. The method is based on the construction of [PZ90] or [PZ94]. First we randomize the self-similar set, turning it into an α-self-similar random measure as described in the papers mentioned above. By proposition 4.2.4 the random measure has unique tangent measure distribution at almost every point almost surely. We can then use the argument used in [PZ92, section 3] to see that this property carries over to the original measure. This method has been used in [PZ92] to show that statistically self-similar random sets in the constructive sense have average densities almost everywhere almost surely.

4.3 Proof of Theorem 4.2.1

Let $0 \leq \alpha \leq 1$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$ be such that

$$\overline{d}^{\alpha}(\mu, x) < \infty$$

for μ-almost every x. The main step in the proof of theorem 4.2.1 is the following lemma.

Lemma 4.3.1 *Let*

$$G : \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow [0, \infty)$$

be a continuous function such that there are $a, b > 0$ with

$$G(\nu, u) \leq \frac{b}{1 + \nu(B(0, 2a))} \cdot \mathbf{1}_{B(0,a)}(u)$$

for all $\nu \in \mathcal{M}(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$.
Then for μ-almost all $x \in \mathbb{R}^n$ the following statement holds:
If a unique tangent measure distribution P of μ at x exists, then the Palm formula

$$\int\int G(T^u\nu, -u)\, d\nu(u)\, dP(\nu) = \int\int G(\nu, u)\, d\nu(u)\, dP(\nu)$$

holds.

Proof Let

$$E := \{x \in \text{supp } \mu \,:\, P_x := \lim_{r \downarrow 0} P_r^x \text{ exists}\}$$

and note that, since $\mathcal{P}$ is a separable metric space by lemma 1.2.8(2), E is a Borel set and therefore μ-measurable. Define functions g_1, g_2 and G_1, G_2 associated with G by

$$\begin{array}{rccc} g_1 : & \mathcal{M}(\mathbb{R}^n) & \longrightarrow & [0,\infty) \\ & \nu & \mapsto & \int G(\nu,u)\,d\nu(u) \quad ; \\ g_2 : & \mathcal{M}(\mathbb{R}^n) & \longrightarrow & [0,\infty) \\ & \nu & \mapsto & \int G(T^u\nu,-u)\,d\nu(u) \quad ; \end{array}$$

and

$$\begin{array}{rccc} G_1 : & E & \longrightarrow & [0,\infty) \\ & y & \mapsto & \int g_1(\nu)\,dP_y(\nu) \quad ; \\ G_2 : & E & \longrightarrow & [0,\infty) \\ & y & \mapsto & \int g_2(\nu)\,dP_y(\nu) \quad . \end{array}$$

Observe that

$$g_1(\nu) \le \frac{b\cdot\nu(B(0,a))}{1+\nu(B(0,2a))} \le b$$

and

$$g_2(\nu) \le \int \frac{b\cdot 1_{B(0,a)}(u)}{1+\nu(B(u,2a))}\,d\nu(u) \le \frac{b\cdot\nu(B(0,a))}{1+\nu(B(0,a))} \le b$$

for all ν. G is continuous and $\{y\ :\ \text{there is } \nu \text{ such that } G(\nu,y)\neq 0\ \}\subseteq B(0,a)$. Hence g_1 and g_2 are bounded and continuous by lemma 1.2.7. Furthermore G_1 and G_2 are Borel measurable and bounded.

We recall that the measure φ_ε on $(0,1)$ was defined by

$$\varphi_\varepsilon(A) = (|\log\varepsilon|)^{-1}\int_\varepsilon^1 1_A(t)\,\frac{dt}{t} \quad \text{for all Borel sets } A\subseteq(0,1).$$

For every open ball $\tilde{B}\subseteq\mathbb{R}^n$ denote $B := \tilde{B}\cap E$. B is μ-measurable. Let $\bar{\mu} := \mu|_B$. Using lemma 2.2.5 in (4.2) and (4.7), Lebesgue's dominated convergence theorem in (4.3) and (4.6), and Fubini's theorem in (4.4), we get

$$\begin{aligned} \int_B G_1(y)\,d\mu(y) &= \int G_1(y)\,d\bar{\mu}(y) \\ &= \int \lim_{r\downarrow 0}\int g_1\Big(\frac{\mu_{y,t}}{t^\alpha}\Big)\,d\varphi_r(t)\,d\bar{\mu}(y) \\ &= \int \lim_{r\downarrow 0}\int g_1\Big(\frac{\bar{\mu}_{y,t}}{t^\alpha}\Big)\,d\varphi_r(t)\,d\bar{\mu}(y) \qquad (4.2) \end{aligned}$$

$$= \lim_{r\downarrow 0} \int\int\int G(\frac{\bar\mu_{y,t}}{t^\alpha}, x)\, \frac{d\bar\mu_{y,t}(x)}{t^\alpha}\, d\bar\mu(y)\, d\varphi_r(t) \tag{4.3}$$

$$= \lim_{r\downarrow 0} \int\int\int G(\frac{\bar\mu_{y,t}}{t^\alpha}, \frac{x-y}{t})\, \frac{d\bar\mu(x)}{t^\alpha}\, d\bar\mu(y)\, d\varphi_r(t)$$

$$= \lim_{r\downarrow 0} \int\int\int G(\frac{\bar\mu_{x,t}}{t^\alpha}, \frac{y-x}{t})\, \frac{d\bar\mu(x)}{t^\alpha}\, d\bar\mu(y)\, d\varphi_r(t) \tag{4.4}$$

$$= \lim_{r\downarrow 0} \int\int\int G(\frac{\bar\mu_{y+tx,t}}{t^\alpha}, -x)\, \frac{d\bar\mu_{y,t}(x)}{t^\alpha}\, d\bar\mu(y)\, d\varphi_r(t)$$

$$= \lim_{r\downarrow 0} \int\int\int G(T^x(\frac{\bar\mu_{y,t}}{t^\alpha}), -x)\, \frac{d\bar\mu_{y,t}(x)}{t^\alpha}\, d\bar\mu(y)\, d\varphi_r(t) \tag{4.5}$$

$$= \int \lim_{r\downarrow 0} \int g_2(\frac{\bar\mu_{y,t}}{t^\alpha})\, d\varphi_r(t)\, d\bar\mu(y) \tag{4.6}$$

$$= \int \lim_{r\downarrow 0} \int g_2(\frac{\mu_{y,t}}{t^\alpha})\, d\varphi_r(t)\, d\bar\mu(y) \tag{4.7}$$

$$= \int G_2(y)\, d\bar\mu(y)$$

$$= \int_B G_2(y)\, d\mu(y).$$

By differentiation of measures (see lemma 1.2.2) we now get

$$G_1(y) = G_2(y)\,,$$

for μ-almost all $y \in E$. This is the statement of the lemma. ∎

Lemma 4.3.2 *For μ-almost all $x \in \mathbb{R}^n$ the following statement holds:*
If P is the unique tangent measure distribution of μ at x, then the Palm formula

$$\int\int G(T^x\nu, -x)\, d\nu(x)\, dP(\nu) = \int\int G(\nu, x)\, d\nu(x)\, dP(\nu)$$

holds for all Borel measurable functions $G : \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow [0,\infty)$.

Proof The proof is merely technical. We first extend the result of lemma 4.3.1 to more general G. Let

$$G : \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow [0,1]$$

be continuous. For every $a \in \mathbb{N}$ define functions G_a by

$$G_a(\nu, u) = \min\{\inf_{x \notin B(0,a)} |x-u|, 1\} \cdot G(\nu, u)\,.$$

The G_a are continuous and bounded. Also fix $f \in C_c(\mathbb{R}^n)$ with $f(x) > 1$ for all $x \in B(0, 2a)$. For every $b \in \mathbb{N}$ define the function $G_{a,b}$ by

$$G_{a,b}(\nu, u) = \min\left\{\frac{b}{1+\nu(f)}, 1\right\} \cdot G_a(\nu, u).$$

We have

$$G_{a,b}(\nu, u) \leq \frac{b}{1+\nu(B(0,2a))} \cdot 1_{B(0,a)}(u),$$

and thus the functions $G_{a,b}$ fulfill the requirements of lemma 4.3.1. Hence the Palm formula holds μ-almost everywhere for all functions $G_{a,b}$ with $a, b \in \mathbb{N}$. Since

$$G_a = \lim_{b\to\infty} G_{a,b} \quad \text{and} \quad G = \lim_{a\to\infty} G_a,$$

and both limits are monotone, the Palm formula holds μ-almost everywhere for G. By monotone approximation from below we thus get the Palm formula for every indicator function 1_U for open sets $U \subseteq \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n$.

Because $\mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n$ is separable, we can find a countable basis $\mathcal{O}$ of the topology. Let

$$E := \{x \in \text{supp}\,\mu \ :\ P_x := \lim_{r\downarrow 0} P_r^x \text{ exists }\}$$

and

$$A := \{x \in E \ :\ \text{the Palm formula holds for } P_x \text{ and all functions } 1_O,\, O \in \mathcal{O}\}.$$

We have seen so far that $\mu(A) = \mu(E)$. Now fix $x \in A$. Let $r > 0$ and define $\mathcal{S}(r)$ to be the collection of all Borel subsets $B \subseteq \mathcal{M}(\mathbb{R}^n) \times U(0, r)$ such that

$$\int\int 1_B(\nu, y)\, d\nu(y)\, dP_x(\nu) = \int\int 1_B(T^y\nu, -y)\, d\nu(y)\, dP_x(\nu).$$

$\mathcal{S}(s)$ contains $\mathcal{M}(\mathbb{R}^n) \times U(0,r)$ and is closed under proper differences and, by the monotone convergence theorem, under non-decreasing limits. We have

$$\mathcal{S}(r) \supseteq \mathcal{O}(r) = \{O \in \mathcal{O} \ :\ O \subseteq \mathcal{M}(\mathbb{R}^n) \times U(0, r)\}.$$

$\mathcal{O}(r)$ is closed under finite intersection and generates the Borel-σ-algebra on $\mathcal{M}(\mathbb{R}^n) \times U(0, r)$. Hence, by the monotone class theorem (as formulated for example in [Kal83, 15.2.1]), $\mathcal{S}(r)$ equals the Borel-σ-algebra on $\mathcal{M}(\mathbb{R}^n) \times U(0,r)$. Thus the Palm-formula holds for all Borel step-functions and, by monotone approximation from below and the monotone convergence theorem, we can conclude that, for all $x \in A$,

$$\int\int G(\nu, y)\, d\nu(y)\, dP_x(\nu) = \int\int G(T^y\nu, -y)\, d\nu(y)\, dP_x(\nu)\,,$$

for all Borel functions $G : \mathcal{M}(\mathbb{R}^n) \times \mathbb{R}^n \longrightarrow [0, \infty)$. This finishes the proof of the lemma.

■

To finish the proof of theorem 4.2.1 we additionally require that

$$\underline{d}^\alpha(\mu, x) > 0$$

μ-almost everywhere. For μ-almost every x and every $\nu \in \mathrm{Tan}_{\mathcal{S}}(\mu, x)$ we then have

$$\nu(B(0,1)) \geq \liminf_{t \downarrow 0} \frac{\mu(B(x,t))}{t^\alpha} > 0\,.$$

Therefore every tangent measure distribution P at x fulfills

$$P(\{\phi\}) = 0\,,$$

where ϕ is the zero-measure. Therefore the two conditions in the characterization theorem for Palm distributions (lemma 4.1.1) are fulfilled for the unique tangent measure distribution of μ at x (if it exists) and hence it is a Palm distribution.

Chapter 5

Tangent Measures Distributions of Measures on the Line

We have seen in the previous chapter that, for measures with positive and finite α-densities almost everywhere, at almost all points the unique tangent measure distribution, if it exists, is a Palm distribution. The drawback of this result, of course, is the requirement of the uniqueness of the tangent measure distribution. The existence of unique tangent measure distributions has been established for certain classes of self-similar measures (see for example [Gra93], [AP94], [Kri95]), but results which hold for more general measures are of greater interest in geometric measure theory.

The main result of this chapter is that, for measures μ on the real line with positive and finite α-densities μ-almost everywhere, at μ-almost every point every tangent measure distribution is a Palm distribution, even if it is not unique. This result is formulated together with some interesting consequences in section 5.1. These consequences comprise a local symmetry principle (see theorem 5.1.3) and a complete description of the one-sided average densities of the measure in terms of its average densities (see corollary 5.1.4). In particular some questions left open in chapters 2 and 3 can be answered using the Palm property. The proof of the result combines new ideas and methods from the proofs of theorems 3.1.3 and 4.2.1 and is carried out in section 5.2.

5.1 The Palm Property and Some of its Consequences

Let us formulate the main result of this thesis without further delay:

Theorem 5.1.1 *Let μ be a measure on the real line such that, for some $0 < \alpha < 1$,*

$$0 < \underline{d}^{\alpha}(\mu, x) \leq \overline{d}^{\alpha}(\mu, x) < \infty$$

μ-almost everywhere. Then for μ-almost every x every tangent measure distribution $P \in \mathcal{P}^{\alpha}(\mu, x)$ is a Palm distribution.

Observe that the statement of theorem 5.1.1 holds trivially in the cases $\alpha = 0, 1$.
Let us look at some consequences of theorem 5.1.1. For this purpose fix a measure μ on the real line such that, for some $0 < \alpha < 1$,

$$0 < \underline{d}^{\alpha}(\mu, x) \leq \overline{d}^{\alpha}(\mu, x) < \infty$$

μ-almost everywhere.

Corollary 5.1.2 *For μ-almost every x every tangent measure distribution $P \in \mathcal{P}^{\alpha}(\mu, x)$ is the distribution of an α-self similar random measure.*

Proof Combine theorem 5.1.1 and proposition 2.2.2. ∎

A very remarkable geometric consequence of theorem 5.1.1 is the following theorem.

Theorem 5.1.3 *For μ-almost all x*

$$\lim_{\varepsilon \downarrow 0} (|\log \varepsilon|)^{-1} \int_{\varepsilon}^{1} \frac{\mu([x - t, x]) - \mu([x, x + t])}{t^{\alpha}} \frac{dt}{t} = 0\,.$$

Proof Let x be such that $\overline{d}^{\alpha}(\mu, x) < \infty$ and every tangent measure distribution of μ at x is a Palm distribution. Suppose $\varepsilon_n \downarrow 0$ is given. By lemma 2.2.3(1) there is a subsequence

(r_n) of (ε_n) such that there is a tangent measure distribution $P = \lim_{n\to\infty} P^x_{r_n}$.
Define $G(\nu, x) = \mathbf{1}_{[0,1]}(x)$. Observe that

$$\int G(\nu, y)\, d\nu(y) = \nu([0,1])$$

and

$$\int G(T^y\nu, -y)\, d\nu(y) = \nu([-1,0])\,.$$

If $\bar{\nu}$ is the barycentre of P, then $\bar{\nu}(\{0,1\}) = \bar{\nu}(\{-1,0\}) = 0$, and therefore we have

$$\begin{aligned}
\lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x, x+t])}{t^\alpha} \frac{dt}{t} &= \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu_{x,t}}{t^\alpha}(\mathbf{1}_{[0,1]}) \frac{dt}{t} \\
&= \lim_{n\to\infty} \int\int G(\nu, y)\, d\nu(y)\, dP^x_{r_n}(\nu) \\
&= \int\int G(\nu, y)\, d\nu(y)\, dP(\nu)\,,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x-t, x])}{t^\alpha} \frac{dt}{t} &= \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu_{x,t}}{t^\alpha}(\mathbf{1}_{[-1,0]}) \frac{dt}{t} \\
&= \lim_{n\to\infty} \int\int G(T^y\nu, -y)\, d\nu(y)\, dP^x_{r_n}(\nu) \\
&= \int\int G(T^y\nu, -y)\, d\nu(y)\, dP(\nu)\,,
\end{aligned}$$

and this implies, by means of the Palm formula,

$$\lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x, x+t])}{t^\alpha} \frac{dt}{t} = \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x-t, x])}{t^\alpha} \frac{dt}{t}\,,$$

which implies the statement. ∎

Remark (Example 3.1.2 revisited)

We have seen in example 3.1.2 that for Hausdorff measure μ on the ternary Cantor set C the map

$$t \mapsto \frac{\mu([x-t, x]) - \mu([x, x+t])}{t^\alpha}$$

oscillates as $t \to 0$. Therefore the convergence of the averages in corollary 5.1.3 can have essentially two possible reasons:

1. Only a small number of scales $t \in (0,1)$ are responsible for the oscillation. In this case one should expect a stronger statement, namely

$$\lim_{\varepsilon \downarrow 0} (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \left|\frac{\mu([x-t,x]) - \mu([x,x+t])}{t^\alpha}\right| \frac{dt}{t} = 0\,, \tag{5.1}$$

   to hold.

2. The oscillations to the positive and negative side cancel in the logarithmic average.

We are going to rule out the first reason by showing that (5.1) does not hold in our example.

Suppose (5.1) holds. Then by lemma 2.1.3 the set

$$S := \{t \in (0,1) : \left|\frac{\mu([x-t,x]) - \mu([x,x+t])}{t^\alpha}\right| \geq (1/4)\}$$

fulfills

$$\lim_{\varepsilon \downarrow 0} (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_S(t)\, \frac{dt}{t} = 0\,. \tag{5.2}$$

Recall the notation from example 3.1.2. By the strong law of large numbers we have, for μ-almost all $x \in C$,

$$\frac{1}{N}\#\{i \in \{1,\dots N\} : (x_{3i}, x_{3i+1}, x_{3i+2}) = (0,0,0)\} \longrightarrow (1/8)\,.$$

Look at such a point x. Whenever $(x_{3i}, x_{3i+1}, x_{3i+2}) = (0,0,0)$ and $t \in (1/3^{3i}, 1/3^{3i-1})$ we have

$$\mu([x,x+t]) - \mu([x-t,x]) \geq 2 \cdot (1/2)^{3i+2}$$

and thus

$$\left|\frac{\mu([x-t,x]) - \mu([x,x+t])}{t^\alpha}\right| \geq (1/4)\,.$$

Accordingly

$$S \supseteq \bigcup_{i \in \mathbb{N}} \{(1/3^{3i}, 1/3^{3i-1}) : (x_{3i}, x_{3i+1}, x_{3i+2}) = (0,0,0)\}\,.$$

Suppose $\varepsilon \in [(1/3)^N, (1/3)^{N-1}]$. Then

$$\frac{1}{|\log \varepsilon|} \int_\varepsilon^1 1_S(t)\, \frac{dt}{t} \geq \frac{1}{N}\#\{i \in \{1,\dots N-1\} : (x_{3i}, x_{3i+1}, x_{3i+2}) = (0,0,0)\}$$

and thus

$$\liminf_{\varepsilon\downarrow 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 1_S(t)\,\frac{dt}{t} \ge (1/8)\,,$$

and this contradicts (5.2) and thus (5.1). □

The following two corollaries answer questions left open in chapter 3 and 2.

Recall our investigation of one-sided average densities in chapter 3. With the help of theorem 5.1.3 we can formulate a substantial improvement of theorem 3.1.3 answering the questions at the end of section 3.1.

Corollary 5.1.4 *At μ-almost every point x we have the following equations for the one-sided average densities*

$$\underline{D}^\alpha_-(\mu,x) = \underline{D}^\alpha_+(\mu,x) = (1/2)\cdot \underline{D}^\alpha(\mu,x)$$

and

$$\overline{D}^\alpha_-(\mu,x) = \overline{D}^\alpha_+(\mu,x) = (1/2)\cdot \overline{D}^\alpha(\mu,x)\,.$$

In particular, the one-sided average densities exist if and only if the average density exists, and in this case

$$D^\alpha_-(\mu,x) = D^\alpha_+(\mu,x) = (1/2)\cdot D^\alpha(\mu,x)\,.$$

Proof Let x be such that $\overline{d}^\alpha(\mu,x) < \infty$ and the statement of theorem 5.1.3 holds.
For any sequence (ε_n) with $\varepsilon_n \downarrow 0$ we have (provided either the first or the last limit in the following equation exists),

$$\begin{aligned}
&\lim_{n\to\infty} (|\log\varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 \frac{\mu([x-t,x])}{t^\alpha}\frac{dt}{t} \\
&= \lim_{n\to\infty} (|\log\varepsilon_n|)^{-1} \left[\int_{\varepsilon_n}^1 \frac{\mu([x,x+t])}{t^\alpha}\,\frac{dt}{t} + \int_{\varepsilon_n}^1 \frac{\mu([x-t,x]) - \mu([x,x+t])}{t^\alpha}\,\frac{dt}{t}\right] \\
&= \lim_{n\to\infty} (|\log\varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 \frac{\mu([x,x+t])}{t^\alpha}\,\frac{dt}{t}
\end{aligned}$$

and this implies the statement. ■

Another remarkable fact is that on the real line the average tangent measures are completely determined by the average densities. Note that by the example in lemma 2.3.2(a) this is different in higher dimensions.

Corollary 5.1.5

(a) *For μ-almost every x all average tangent measures $\bar{\nu}$ of μ at x are symmetric around the origin.*

(b) *Suppose the measure μ has average densities μ-almost everywhere. Then μ has unique average tangent measures $\bar{\nu}^x$ at μ-almost every x. Moreover, $\bar{\nu}^x$ is given by*

$$\bar{\nu}^x(A) = (1/2) \cdot D^\alpha(\mu, x) \cdot \int_A \alpha |t|^{\alpha-1}\, dt$$

for every Borel set $A \subseteq \mathbb{R}$.

Proof Let x be such that $\overline{d}^\alpha(\mu, x) < \infty$ and every tangent measure distribution at x is a Palm distribution. If $\bar{\nu}$ is an average tangent measure at x, then there is a tangent measure distribution P such that $\bar{\nu} = \int \nu \, dP(\nu)$. Using the Palm formula for the function $G(\nu, y) = \mathbf{1}_A(y)$ we get, for every Borel set $A \subseteq \mathbb{R}$,

$$\bar{\nu}(A) = \int \nu(A)\, dP(\nu) = \int \nu(-A)\, dP(\nu) = \bar{\nu}(-A)\,,$$

which is the first statement.

Suppose now that the average density at x exists. For $\lambda > 0$ and any half-open interval $[0, \lambda)$ we have, by lemma 2.2.2,

$$\bar{\nu}([0, \lambda)) = \lambda^\alpha \cdot \bar{\nu}([0, 1))\,,$$

and using the symmetry and $\bar{\nu}(\{0\}) = 0$ we have

$$\bar{\nu}([0, 1)) = (1/2) \cdot \bar{\nu}((-1, 1)) = (1/2) \cdot D^\alpha(\mu, x)\,,$$

and similarly for intervals $[-\lambda, 0)$. Therefore the measure ψ defined by

$$\psi(A) = (1/2) \cdot D^\alpha(\mu, x) \cdot \int_A \alpha |t|^{\alpha-1}\, dt$$

and the measure $\bar{\nu}$ agree on all (right-)half-open intervals and hence they are identical. This implies the uniqueness of the average tangent measures as well as the formula stated in the corollary. ■

Recall lemma 2.3.2 now. Corollary 5.1.5(b) closes the gap in the proof of lemma 2.3.2(b): The measure $\mu \in \mathcal{M}(\mathbb{R})$, which was shown to have average densities μ-almost everywhere, automatically has unique average tangent measures. Moreover the corollary shows that examples as in lemma 2.3.2(a) can only exist in Euclidean spaces of dimension at least 2.

In the next corollary we formulate a consequence of theorem 5.1.3 in the language of singular integrals. For $0 < \alpha \leq 1$ consider the kernel

$$\begin{aligned} K_\alpha : \ \mathbb{R} \setminus \{0\} &\longrightarrow \mathbb{R} \\ x &\mapsto \frac{\text{sign }(x)}{|x|^\alpha} . \end{aligned}$$

K_α is a natural generalization of the kernel $1/x$ of the classical Hilbert transform

$$\begin{aligned} Hf(x) &= \lim_{\varepsilon \downarrow 0} \int_{\{y:|x-y|>\varepsilon\}} \frac{f(t)}{t-x}\, dt \\ &= \lim_{\varepsilon \downarrow 0} \int_{\{y:|x-y|>\varepsilon\}} K_1(t-x) f(t)\, dt . \end{aligned}$$

The question whether for $0 < \alpha < 1$ the limit

$$\lim_{\varepsilon \downarrow 0} \int_{\{y:|x-y|>\varepsilon\}} K_\alpha(y-x)\, d\mu(y)$$

exists on a set of positive measure has been answered in the negative by P.Mattila and D.Preiss in [MP95] (see also [Mat95]). Theorem 5.1.3 implies the following statement:

Corollary 5.1.6 *For μ-almost all x we have*

$$\lim_{\varepsilon \downarrow 0} (|\log \varepsilon|)^{-1} \int_{\{y:|x-y|>\varepsilon\}} K_\alpha(y-x)\, d\mu(y) = 0 .$$

Proof We can assume without loss of generality that μ is finite. Fix x such that $\overline{d}^\alpha(\mu, x) < \infty$ and the statement of theorem 5.1.3 holds. Integration by parts yields

$$\int_{\{y:\, y-x>\varepsilon\}} \frac{1}{(y-x)^\alpha}\, d\mu(y) = \left[-\frac{\mu([x, x+\varepsilon])}{\varepsilon^\alpha} + \alpha \cdot \int_\varepsilon^\infty \frac{\mu([x, x+t])}{t^{\alpha+1}}\, dt \right] ,$$

and thus, for some constant $C > 0$,

$$-\frac{C}{|\log\varepsilon|} + \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t} \le (|\log\varepsilon|)^{-1}\int_{\{y:\,y-x>\varepsilon\}} \frac{1}{(y-x)^\alpha}\,d\mu(y)$$
$$\le \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x,x+t])}{t^{\alpha+1}}\,dt\,,$$

and analogously we get

$$-\frac{C}{|\log\varepsilon|} + \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x-t,x])}{t^\alpha}\frac{dt}{t} \le (|\log\varepsilon|)^{-1}\int_{\{y:\,x-y>\varepsilon\}} \frac{1}{(x-y)^\alpha}\,d\mu(y)$$
$$\le \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x-t,x])}{t^{\alpha+1}}\,dt\,.$$

As $\varepsilon \downarrow 0$ we thus have

$$\lim_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_{\{y:|x-y|>\varepsilon\}} K_\alpha(y-x)\,d\mu(y)$$
$$= \alpha\cdot\lim_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \frac{\mu([x,x+t])-\mu([x-t,x])}{t^\alpha}\frac{dt}{t} = 0.$$

■

5.2 Proof of Theorem 5.1.1

The proof consists of the following steps: In the first step we show that it suffices to study the tangent measure distributions of measures μ at μ-almost all points $x \in E$, where μ and E fulfill the conditions assumed in section 3.2. In the second step we fix a suitably chosen function $G : \mathcal{M}(\mathbb{R}) \times \mathbb{R} \to [0,\infty)$ and show that the Palm formula holds for G and all tangent measure distributions at μ-almost all points $x \in E$. For this purpose we introduce a family (φ_I) of functions, the sum of which approximates the difference of the two sides of the Palm formula (lemma 5.2.2) and show that the set of points where the approximating function has large modulus has small measure (lemma 5.2.3). In the final step we extend the result of the second step to the full statement.

We suppose that $0 < \alpha < 1$ and $\mu \in \mathcal{M}(\mathbb{R})$ is given with

$$0 < \underline{d}^\alpha(\mu,x) \le \overline{d}^\alpha(\mu,x) < \infty$$

μ-almost everywhere for some $0 < \alpha < 1$. Without losing generality, if necessary by restricting μ to countably many open sets, we can assume that $\mu(\mathbb{R}) < 1$.

Given $\delta > 0$ we can find a Borel set $B \subseteq \mathbb{R}$ and constants $0 < c \leq C < \infty$ such that $\mu(\mathbb{R} \setminus B) < \delta$ and, using proposition 1.3.1,

$$c < \underline{d}^\alpha(\mu|_B, x) = \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu|_B, x) < C$$

for μ-almost every $x \in B$. We can now apply lemma 3.3.1 to the measure $\mu|_B$. Because, by proposition 2.2.5,

$$\mathcal{P}^\alpha(\mu, x) = \mathcal{P}^\alpha(\mu|_{B\cap B_i}, x)$$

for μ-almost every $x \in E_i$, it suffices to prove the Palm formula for the restricted measures $\mu|_{B\cap B_i}$ and μ-almost every point $x \in E_i$.

Fix such a measure $\mu|_{B\cap B_i}$ and let us also denote it by μ. Let $E = E_i$ and $\varepsilon_0 = \varepsilon_0(i)$. We now have a measure μ with support contained in a compact interval of length less than one and a compact set E such that, without loss of generality, $\mu(E) > 0$ and such that, for all $x \in E$,

$$\mu([x - r, x + r]) \leq Cr^\alpha \quad \text{if } r \geq 0, \tag{5.3}$$

and

$$\mu([x - r, x + r]) \geq cr^\alpha \quad \text{if } 0 \leq r \leq \varepsilon_0. \tag{5.4}$$

Denote $\bar{\mu} = \mu|_E$, the restriction of μ to the compact set E. Observe that by (5.3) we have, for every $B \subseteq \mathbb{R}$,

$$\bar{\mu}(B) \leq C \cdot |B|^\alpha . \tag{5.5}$$

The results on the geometry of E, as formulated in section 3.2, hold.

For every $x \in E$ and every $\nu \in \mathrm{Tan}_S(\mu, x)$ we have by (5.4)

$$\nu(B(0,1)) \geq \liminf_{t \downarrow 0} \frac{\mu(B(x,t))}{t^\alpha} \geq c\,.$$

Therefore every tangent measure distribution P at x fulfills $P(\{\phi\}) = 0$, where ϕ is the zero-measure. By theorem 4.1.1 it remains to prove the Palm formula (4.1) for μ-almost every $x \in E$.

We now proceed to the second step, which constitutes the main part of the proof. At μ-almost every $x \in E$ the tangent measure distributions of μ and $\bar{\mu}$ coincide by proposition 2.2.5. Therefore we can work with the tangent measure distributions of $\bar{\mu}$ instead of μ. We fix a continuous function

$$G : \mathcal{M}(\mathbb{R}) \times \mathbb{R} \longrightarrow [0, \infty)$$

of the form

$$G(\nu, y) = g(y) \cdot h(\nu(f)), \tag{5.6}$$

where

$$f : \mathbb{R} \longrightarrow [0, \infty), \quad g : \mathbb{R} \longrightarrow [0, \infty)$$

are Lipschitz functions with compact support and

$$h : [0, \infty) \longrightarrow [0, 1]$$

is a Lipschitz function. We denote the Lipschitz constants of f, g, h by $L(f)$, $L(g)$, $L(h)$. By $R(f)$, $R(g)$ we denote the smallest integers such that supp $f \subseteq B(0, R(f))$ and supp $g \subseteq B(0, R(g))$. In the following, we allow the constants $C_{17}, C_{18}, \ldots$ to depend on the choice of G.

Our aim is to prove the Palm-formula

$$\int \int G(\nu, y)\, d\nu(y)\, dP(\nu) = \int \int G(T^y \nu, -y)\, d\nu(y)\, dP(\nu) \tag{5.7}$$

for every tangent measure distribution P of $\bar{\mu}$ at μ-almost every $x \in E$. For this purpose define $G_1, G_2 : \mathcal{M}(\mathbb{R}) \longrightarrow [0, \infty)$ by

$$\begin{aligned} G_1(\nu) &= \int G(\nu, y)\, d\nu(y), \\ G_2(\nu) &= \int G(T^y \nu, -y)\, d\nu(y). \end{aligned}$$

Lemma 5.2.1 *G_1 and G_2 are continuous and there is $C_{17} > 0$ such that, for all $x \in \mathbb{R}$ and $t > 0$,*

$$G_1\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}\Big), G_2\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}\Big) < C_{17}.$$

Proof G_1 is obviously continuous. Continuity of G_2 follows from continuity of $(\nu, y) \mapsto G(T^y\nu, -y)$ by means of lemma 1.2.7. Let $C_{17} = C \cdot \|g\|_{\sup} \cdot [2R(g)]^\alpha$. Then, for all $x \in \mathbb{R}$ and $t > 0$, we have by (5.5)

$$G_1\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}\Big), G_2\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}\Big) \le \|g\|_{\sup} \cdot \frac{\bar{\mu}_{x,t}(B(0,R(g)))}{t^\alpha} \le C_{17}.$$

■

Recall the definition of the measures ψ^x_ε from section 3.2. Fix

$$\kappa = \kappa(\varepsilon) = (\log|\log\varepsilon|)^{6/(1-\alpha)}.$$

For every interval $I \subseteq \mathbb{R}$ we define functions $\tilde{\varphi}_I$ and φ_I by

$$\tilde{\varphi}_I(x,\varepsilon) = \int_{I\cap[x,\infty)} G_1\left(\frac{\bar{\mu}_{x,z-x}}{(z-x)^\alpha}\right) d\psi^x_\varepsilon(z) - \int_{I\cap(-\infty,x]} G_2\left(\frac{\bar{\mu}_{x,x-z}}{(x-z)^\alpha}\right) d\psi^x_\varepsilon(z),$$

and

$$\varphi_I(x,\varepsilon) = \begin{cases} \tilde{\varphi}_I(x,\varepsilon) & \text{if } x \in I^-(\kappa) \cup I^+(\kappa), \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{R}$ and $\varepsilon > 0$. Observe that for all intervals $I \subseteq \mathbb{R}$, $\varepsilon > 0$ and for all $x \in \mathbb{R}$,

$$|\varphi_I(x,\varepsilon)| \le |\tilde{\varphi}_I(x,\varepsilon)| \le C_{17} \cdot \psi^x_\varepsilon(I), \tag{5.8}$$

using the boundedness of G_1, G_2 (see lemma 5.2.1).

Recall the definition of $\mathcal{A}$ and $\mathcal{N}_\varepsilon$ from section 3.2 and denote

$$\mathcal{A}_\varepsilon = \{I \in \mathcal{A} : |I| \ge \varepsilon\}.$$

For small $\varepsilon > 0$ the function $\sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)$ is a good approximation of

$$(|\log\varepsilon|)^{-1} \int_\varepsilon^1 G_1(\frac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\frac{\bar{\mu}_{x,t}}{t^\alpha})\,\frac{dt}{t},$$

as the following lemma shows.

Lemma 5.2.2 *There is a constant $C_{18} > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$ and every $\sigma > 0$, we have*

$$\mu\Big(\Big\{x \in E : \Big|(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\tfrac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\tfrac{\bar{\mu}_{x,t}}{t^\alpha})\Big)\frac{dt}{t} - \sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)\Big| > \sigma\Big\}\Big)$$

$$\leq \frac{C_{18}}{\sigma\cdot(\log|\log\varepsilon|)^2}.$$

Proof We use lemma 5.2.1 and the approximation lemma 3.2.6 for $\gamma = 1$ to get

$$\begin{aligned}
&\sigma\cdot\mu\Big(\Big\{x \in E : |(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\tfrac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\tfrac{\bar{\mu}_{x,t}}{t^\alpha})\Big)\frac{dt}{t} - \sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)| > \sigma\Big\}\Big)\\
\leq\ & \int_E\Big|(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\tfrac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\tfrac{\bar{\mu}_{x,t}}{t^\alpha})\Big)\frac{dt}{t} - \sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)\Big|\,d\mu(x)\\
\leq\ & C_{17}\cdot\int_E\Big(\psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A}_\varepsilon\\ x\in I^0(\kappa)}}\psi_\varepsilon^x(I)\Big)\,d\mu(x)\\
\leq\ & C_{17}C_{10}\cdot\Big(\Big(\frac{1}{\kappa(\varepsilon)}\Big)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\Big)\\
\leq\ & \frac{2C_{17}C_{10}}{(\log|\log\varepsilon|)^2}.
\end{aligned}$$

The statement follows with $C_{18} = 2C_{17}C_{10}$. ■

We now show that the set of points $x \in E$ where the function

$$\sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)$$

has large modulus is small. This is the main step in the proof.

Lemma 5.2.3 *For $\sigma > 0$ and $\varepsilon > 0$ denote*

$$B_\varepsilon = \{x \in E : |\sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)| \geq \sigma\}.$$

Then there is a constant $C_{19} > 0$ such that, for every $\sigma > 0$ and all sufficiently small $\varepsilon > 0$,

$$\mu(B_\varepsilon) \leq \frac{C_{19}}{\sigma^2\cdot(\log|\log\varepsilon|)^2}.$$

Proof Define a sequence $(p_k)_{k\in\mathbb{N}}$ by

$$p_k = (1 + (1/k)^{3/4}).$$

For $k \in \mathbb{N}$ let

$$\varepsilon_k = \exp\Big(- \prod_{i=1}^{k} p_i\Big).$$

For $0 < \varepsilon < \varepsilon_0$ define $p = p(\varepsilon) \in \mathbb{N}$ such that

$$\varepsilon_{p-1} > \varepsilon \geq \varepsilon_p ,$$

and define $\lambda = \lambda(\varepsilon)$ as the largest integer such that

$$\lambda(\varepsilon) \leq (\log|\log\varepsilon|)^2 .$$

We now establish some, in many cases very crude, numerical estimates, marked (a) to (g), which hold for all sufficiently small $\varepsilon > 0$. We can later refer to these estimates without having to interrupt the flow of the proof for standard calculations.

We have, by definition of p,

$$p_1 \cdots p_p \geq |\log\varepsilon|$$

and

$$\log(p_1 \cdots p_p) \leq \sum_{k=1}^{p} (1/k)^{3/4} \leq \int_0^p \Big(\frac{1}{x}\Big)^{3/4} dx = 4\sqrt[4]{p}.$$

Therefore

$$p \geq \Big(\frac{1}{4} \cdot \log|\log\varepsilon|\Big)^4 .$$

We have for all $k \leq l$

$$\begin{aligned} \frac{|\log\varepsilon_k|}{|\log\varepsilon_l|} &= \Big(\prod_{j=k+1}^{l} p_j\Big)^{-1} = \Big(\prod_{j=k+1}^{l} (1 + (1/j)^{3/4})\Big)^{-1} \\ &\leq \Big(\prod_{j=k+1}^{l} \frac{j+1}{j}\Big)^{-1} = \frac{k+1}{l+1}, \end{aligned}$$

and thus

$$\begin{aligned} \frac{|\log\varepsilon_\lambda|}{|\log\varepsilon|} &\leq \frac{|\log\varepsilon_\lambda|}{|\log\varepsilon_{p-1}|} \leq \frac{\lambda+1}{p} \leq \frac{2(\log|\log\varepsilon|)^2}{[1/4 \cdot \log|\log\varepsilon|]^4} \\ &= \frac{512}{(\log|\log\varepsilon|)^2} . \end{aligned} \qquad (a)$$

Because

$$\frac{\log \frac{\varepsilon_{k-1}}{\varepsilon_k}}{\log \frac{\varepsilon_k}{\varepsilon_{k+1}}} = \frac{(1/k)^{3/4}}{p_k \cdot (1/(k+1))^{3/4}} = \frac{(k+1)^{3/4}}{k^{3/4}+1} \leq 1\,,$$

the sequence $(\frac{\varepsilon_{k-1}}{\varepsilon_k})$ is monotonically increasing. For all $k \in \{\lambda, \ldots, p\}$,

$$\begin{aligned}\frac{\log \frac{\varepsilon_{k-2}}{\varepsilon_k}}{|\log \varepsilon|} &\leq \frac{\log \varepsilon_{p-2} - \log \varepsilon_p}{|\log \varepsilon_{p-1}|} = \frac{p_p \cdot p_{p-1} - 1}{p_{p-1}} = (1/p)^{3/4} + \frac{1}{(p-1)^{3/4}+1} \\ &\leq 2(1/p)^{3/4} \leq \frac{128}{(\log|\log \varepsilon|)^3}\,,\end{aligned}$$

and thus

$$\max_{k=\lambda}^{p} \log\left((\kappa(\varepsilon)+1) \cdot \frac{\varepsilon_{k-2}}{\varepsilon_k}\right) \cdot \frac{1}{|\log \varepsilon|} \leq \frac{1}{(\log|\log \varepsilon|)^2}\,. \qquad (b)$$

We have $p_1 \cdots p_{k-1} \geq \prod_{i=1}^{k-1}(1+1/i) = k$, and hence

$$\log\left(\frac{\varepsilon_{k-1}}{\varepsilon_k}\right) = p_1 \cdots p_k - p_1 \cdots p_{k-1} = p_1 \cdots p_{k-1} \cdot (1/k)^{3/4} \geq k^{1/4}.$$

Therefore we have

$$\frac{\varepsilon_\lambda}{\varepsilon_{\lambda+1}} \geq \exp\left(\sqrt{\log|\log \varepsilon|}\,\right).$$

In particular, we get

$$\frac{\varepsilon_k}{\varepsilon_{k+1}} \geq \kappa(\varepsilon) \text{ for all } k \geq \lambda. \qquad (c)$$

Let $\delta = C_1(1+\kappa(\varepsilon))\frac{\varepsilon_{\lambda+2}}{\varepsilon_{\lambda+1}}$. Since

$$\delta \leq \frac{2C_1 \cdot \sqrt{\log|\log \varepsilon|}^{(12/1-\alpha)}}{\exp(\sqrt{\log|\log \varepsilon|})} \leq \frac{1}{(\log|\log \varepsilon|)^{(3/\alpha)}}$$

we get

$$\log\left(\frac{\delta+1}{\delta}\right) \cdot \delta^\alpha \leq \frac{1}{(\log|\log \varepsilon|)^2}\,. \qquad (d)$$

We also have, for all $k \geq \lambda$,

$$\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^\alpha \cdot \log^2\left(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}} \cdot \kappa(\varepsilon) + C_1\right) \leq 2 \cdot \left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^\alpha \cdot \log^2\left(\left(\frac{\varepsilon_k}{\varepsilon_{k+1}}\right)^3\right) \leq \left(\sqrt{\frac{\varepsilon_{k+1}}{\varepsilon_k}}\right)^\alpha,$$

and thus

$$\begin{aligned}
&\max_{k=\lambda}^{p} \Big(\frac{\varepsilon_{k+1}}{\varepsilon_k}\Big)^{\alpha} \cdot \log^2\Big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}} \cdot \kappa(\varepsilon) + C_1\Big) \cdot (\kappa(\varepsilon)+1) \\
&\leq 2\Big[\exp\big(-\sqrt{\log|\log\varepsilon|}\big)\Big]^{\alpha/2} \cdot (\log|\log\varepsilon|)^{6/(1-\alpha)} \\
&\leq \frac{1}{(\log|\log\varepsilon|)^2}\,. \qquad (e)
\end{aligned}$$

By definition of $p(\varepsilon)$ we have

$$\begin{aligned}
\log|\log\varepsilon| &\geq \sum_{k=1}^{p-1} \log\big(1+(1/k)^{3/4}\big) \geq (\log 2)\cdot \sum_{k=1}^{p-1}(1/k)^{3/4} \\
&\geq (\log 2)\cdot \int_1^p (1/x)^{3/4}\,dx \geq (\log 2)\cdot(\sqrt[4]{p}-1)
\end{aligned}$$

and therefore

$$p(\varepsilon) \leq \Big(\frac{2}{\log 2}\Big)^4 \cdot (\log|\log\varepsilon|)^4\,. \qquad (f)$$

Finally, we also have

$$\frac{(\kappa(\varepsilon)+1)^{\alpha}}{|\log\varepsilon|} \cdot p(\varepsilon) \leq \frac{1}{(\log|\log\varepsilon|)^2}\,. \qquad (g)$$

Now fix a $0<\varepsilon<\varepsilon_0$ which is small enough such that (a) to (g) hold.

Define

$$\mathcal{I}_1 = \{I \in \mathcal{A}_\varepsilon \,:\, |I| \geq \varepsilon_1\}$$

and, for all $k>1$,

$$\mathcal{I}_k = \{I \in \mathcal{A}_\varepsilon \,:\, \varepsilon_{k-1} > |I| \geq \varepsilon_k\}\,.$$

Then

$$\mathcal{A}_\varepsilon = \bigcup_{k=1}^{p} \mathcal{I}_k\,.$$

We estimate $\mu(B_\varepsilon)$ by means of the mean square of $\sum_{I\in\mathcal{A}_\varepsilon}\varphi_I$ as follows

$$\mu(B_\varepsilon)\cdot\sigma^2 \leq \int\Big(\sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)\Big)^2\,d\bar{\mu}(x)$$

$$= \int \Big(\sum_{k=1}^{p} \sum_{I \in \mathcal{I}_k} \varphi_I(x,\varepsilon) \Big)^2 \, d\bar{\mu}(x) \tag{5.9}$$

$$\leq 2 \cdot \sum_{k=1}^{\lambda} \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \sum_{J \in \mathcal{I}_k} \int \Big| \varphi_I(x,\varepsilon) \varphi_J(x,\varepsilon) \Big| \, d\bar{\mu}(x) \tag{5.10}$$

$$+ \sum_{k=\lambda+1}^{p} \sum_{j=\lambda+1}^{p} \sum_{I \in \mathcal{I}_j} \sum_{J \in \mathcal{I}_k} \int \Big(\varphi_I(x,\varepsilon) \varphi_J(x,\varepsilon) \Big) \, d\bar{\mu}(x) \,. \tag{5.11}$$

We can give an estimate for (5.10). Observe that by (5.8)

$$\sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \Big| \varphi_I(x,\varepsilon) \Big| \leq 2\mathcal{C}_{17} \,,$$

and therefore we have, using lemmas 3.2.5(c) and 3.2.4(2),

$$\begin{aligned}
& \sum_{k=1}^{\lambda} \sum_{J \in \mathcal{I}_k} \int \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \Big| \varphi_I(x,\varepsilon) \cdot \varphi_J(x,\varepsilon) \Big| \, d\bar{\mu}(x) \\
& \leq 2\mathcal{C}_{17} \cdot \sum_{k=1}^{\lambda} \sum_{J \in \mathcal{I}_k} \int |\varphi_J(x,\varepsilon)| \, d\bar{\mu}(x) \\
& \leq 2\mathcal{C}_{17}^2 \mathcal{D}_7 \cdot C \cdot \sum_{k=1}^{\lambda} \sum_{J \in \mathcal{I}_k} \frac{|J|^\alpha}{|\log \varepsilon|} \\
& \leq 2C\mathcal{D}_7 \mathcal{C}_{17}^2 \mathcal{C}_4 \cdot \frac{|\log \varepsilon_\lambda|}{|\log \varepsilon|} \\
& \leq \frac{2C\mathcal{D}_7 \mathcal{C}_{17}^2 \mathcal{C}_4 \cdot 512}{(\log |\log \varepsilon|)^2} \,,
\end{aligned}$$

using (a). This finishes the estimate of (5.10).

We now split (5.11) as follows

$$\begin{aligned}
& \sum_{k=\lambda+1}^{p} \sum_{j=\lambda+1}^{p} \sum_{I \in \mathcal{I}_j} \sum_{J \in \mathcal{I}_k} \int \Big(\varphi_I(x,\varepsilon) \varphi_J(x,\varepsilon) \Big) \, d\bar{\mu}(x) \\
& = \int \sum_{k=\lambda+1}^{p} \Big(\sum_{I \in \mathcal{I}_k} \sum_{J \in \mathcal{I}_k} \varphi_I(x,\varepsilon) \varphi_J(x,\varepsilon) \Big) \, d\bar{\mu}(x) && (5.12) \\
& \quad + 2 \cdot \int \sum_{k=\lambda+1}^{p-1} \Big(\sum_{I \in \mathcal{I}_{k+1}} \sum_{J \in \mathcal{I}_k} \varphi_I(x,\varepsilon) \varphi_J(x,\varepsilon) \Big) \, d\bar{\mu}(x) && (5.13) \\
& \quad + 2 \cdot \int \sum_{k=\lambda+1}^{p-2} \sum_{j=k+2}^{p} \Big(\sum_{I \in \mathcal{I}_j} \sum_{J \in \mathcal{I}_k} \varphi_I(x,\varepsilon) \varphi_J(x,\varepsilon) \Big) \, d\bar{\mu}(x) \,. && (5.14)
\end{aligned}$$

It is not hard to give an estimate for the terms (5.12) and (5.13). Observe that

$$\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon) \le 0$$

unless $x \in I^-(\kappa) \cap J^-(\kappa)$ or $x \in I^+(\kappa) \cap J^+(\kappa)$. Therefore

$$\begin{aligned}
&\int \sum_{k=\lambda+1}^{p} \Big(\sum_{I\in\mathcal{I}_k} \sum_{J\in\mathcal{I}_k} \varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon) + 2\cdot \sum_{I\in\mathcal{I}_{k+1}} \sum_{J\in\mathcal{I}_k} \varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x) \\
&\le 2\cdot\Big(\sum_{k=\lambda+1}^{p} \sum_{I\in\mathcal{I}_k\cup\mathcal{I}_{k+1}} \sum_{J\in\mathcal{I}_k} \int_{I^-(\kappa)\cap J^-(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)|\, d\bar{\mu}(x)\Big) \\
&\quad + 2\cdot\Big(\sum_{k=\lambda+1}^{p} \sum_{I\in\mathcal{I}_k\cup\mathcal{I}_{k+1}} \sum_{J\in\mathcal{I}_k} \int_{I^+(\kappa)\cap J^+(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)|\, d\bar{\mu}(x)\Big).
\end{aligned}$$

We can restrict our attention to the first sum, i.e. to the case of intervals I, J with $I^-(\kappa) \cap J^-(\kappa) \neq \emptyset$, since the second sum can be treated in exactly the same manner. Splitting this sum again we can write

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p} \sum_{I\in\mathcal{I}_k\cup\mathcal{I}_{k+1}} \sum_{J\in\mathcal{I}_k} \int_{I^-(\kappa)\cap J^-(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)|\, d\bar{\mu}(x) \\
&\le 2\cdot \sum_{k=\lambda+1}^{p-1} \sum_{I\in\mathcal{I}_k\cup\mathcal{I}_{k+1}} \sum_{\substack{J\in\mathcal{I}_k\cup\mathcal{I}_{k+1}\\ I<J}} \int_{I^-(\kappa)\cap J^-(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)|\, d\bar{\mu}(x) \qquad (5.15)\\
&\quad + \sum_{k=\lambda+1}^{p} \sum_{I\in\mathcal{I}_k} \int_{I^-(\kappa)} |\varphi_I(x,\varepsilon)|^2\, d\bar{\mu}(x)\,. \qquad (5.16)
\end{aligned}$$

Let us look at (5.15) and fix an interval $I \in \mathcal{I}_k \cup \mathcal{I}_{k+1}$. Denote its left and right endpoint by a and b. If $J \in \mathcal{I}_k \cup \mathcal{I}_{k+1}$ with $I < J$ and $I^-(\kappa) \cap J^-(\kappa) \neq \emptyset$, then $|J| < \varepsilon_{k-1}$ and thus

$$J \subseteq [b, a + (\kappa+1)\cdot\varepsilon_{k-1}]\,.$$

For all $x \in I^-(\kappa)$ we thus get, using $|I| \ge \varepsilon_{k+1}$,

$$\begin{aligned}
\sum |\varphi_J(x,\varepsilon)| &\le C_{17}\cdot \psi_\varepsilon^x([a+\varepsilon_{k+1}, a+(\kappa+1)\cdot\varepsilon_{k-1}]) \\
&\le C_{17}\cdot \frac{\log(\kappa+1) + \log\varepsilon_{k-1} - \log\varepsilon_{k+1}}{|\log\varepsilon|},
\end{aligned}$$

where the sum extends over all $J \in \mathcal{I}_k \cup \mathcal{I}_{k+1}$ such that $I < J$ and $I^-(\kappa) \cap J^-(\kappa) \neq \emptyset$. We use this equation and

$$\sum_{I\in\mathcal{A}_\varepsilon} |\varphi_I(x,\varepsilon)| \le 2C_{17}\,,$$

to estimate

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-1}\sum_{I\in\mathcal{I}_k\cup\mathcal{I}_{k+1}}\sum_{\substack{J\in\mathcal{I}_k\cup\mathcal{I}_{k+1}\\ I<J}}\int_{I^-(\kappa)\cap J^-(\kappa)}|\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)|\,d\bar{\mu}(x)\\
\le\ & C_{17}\cdot\max_{k=\lambda+2}^{p}\frac{\log(\kappa+1)+\log\varepsilon_{k-2}-\log\varepsilon_k}{|\log\varepsilon|}\cdot\int 2\cdot\sum_{I\in\mathcal{A}_\varepsilon}|\varphi_I(x,\varepsilon)|\,d\bar{\mu}(x)\\
\le\ & 4C_{17}^2\cdot\max_{k=\lambda}^{p}\log\left((\kappa(\varepsilon)+1)\cdot\frac{\varepsilon_{k-2}}{\varepsilon_k}\right)\cdot\frac{1}{|\log\varepsilon|}\\
\le\ & \frac{4C_{17}^2}{(\log|\log\varepsilon|)^2},
\end{aligned}$$

by (b), finishing the estimate of term (5.15).

Let us now look at (5.16), and estimate using lemma 3.2.5(d),

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p}\sum_{I\in\mathcal{I}_k}\int_{I^-}|\varphi_I(x,\varepsilon)|^2\,d\bar{\mu}(x)\\
\le\ & C_{17}^2\mathcal{D}_8\cdot C\cdot\sum_{I\in\mathcal{A}_\varepsilon}\frac{|I|^\alpha}{|\log\varepsilon|^2}\le C_{17}^2C_4\mathcal{D}_8\cdot C\cdot\frac{1}{|\log\varepsilon|}\\
\le\ & C_{17}^2C_4\mathcal{D}_8\cdot C\cdot\frac{1}{(\log|\log\varepsilon|)^2},
\end{aligned}$$

finishing the estimate of term (5.16) and thus of the terms (5.12) and (5.13).

We now look at term (5.14). Given $J\in\mathcal{I}_k$ we denote by $\mathcal{K}_J^-$, respectively $\mathcal{K}_J^+$, the collection of all $K\in\mathcal{N}_{\varepsilon_{k+1}}$ such that

$$K\cap J^-(\kappa)\neq\emptyset\ \text{, respectively }\ K\cap J^+(\kappa)\neq\emptyset.$$

Recall again that $\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\le 0$ unless $x\in I^-(\kappa)\cap J^-(\kappa)$ or $x\in I^+(\kappa)\cap J^+(\kappa)$. We have picked $\varepsilon>0$ sufficiently small to ensure

$$\frac{\varepsilon_k}{\varepsilon_{k+1}}\ge\kappa(\varepsilon)\tag{5.17}$$

for all $k\ge\lambda$ (see (c)). Consequently, whenever $k\ge\lambda$, $J\in\mathcal{I}_k$ and $I\in\mathcal{I}_j$, $j\ge k+2$ and $I^-(\kappa)\cap J^-(\kappa)\neq\emptyset$, there is a $K\in\mathcal{K}_J^-$ such that $I\subseteq K$.

To see this we suppose the contrary. Since I is contained in some $K\in\mathcal{N}_{\varepsilon_{k+1}}$ we then must have $I>J$. Hence

$$\varepsilon_k\le|J|<\kappa(\varepsilon)|I|\le\kappa(\varepsilon)\cdot\varepsilon_{k+1}.$$

This contradicts (5.17) and therefore our statement holds.

Also, by the analogous argument, if $J \in \mathcal{I}_k$, $I \in \mathcal{I}_j$ and $I^+(\kappa) \cap J^+(\kappa) \neq \emptyset$, there is $K \in \mathcal{K}_J^+$ such that $I \subseteq K$. Therefore we have

$$\sum_{k=\lambda+1}^{p-2} \sum_{j=k+2}^{p} \sum_{J \in \mathcal{I}_k} \sum_{I \in \mathcal{I}_j} \int \Big(\varphi_I(x,\varepsilon) \cdot \varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x)$$

$$\leq \sum_{k=\lambda+1}^{p-2} \sum_{j=k+2}^{p} \sum_{J \in \mathcal{I}_k} \sum_{K \in \mathcal{K}_J^-} \sum_{\substack{I \in \mathcal{I}_j \\ I \subseteq K}} \int \Big(\varphi_I(x,\varepsilon) \cdot \varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x) \tag{5.18}$$

$$+ \sum_{k=\lambda+1}^{p-2} \sum_{j=k+2}^{p} \sum_{J \in \mathcal{I}_k} \sum_{K \in \mathcal{K}_J^+} \sum_{\substack{I \in \mathcal{I}_j \\ I \subseteq K}} \int \Big(\varphi_I(x,\varepsilon) \cdot \varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x)\,. \tag{5.19}$$

We can concentrate our investigation on one of these expressions, say the first, since the other one can be treated analogously. We write

$$\mathcal{K}_J^- = \{K_1, \ldots, K_N\}\,,$$

where

$$K_N < \ldots < K_1 < J\,.$$

Denote the right endpoint of K_i by ζ_i. Also denote

$$\tilde{K}_i = \Big(K_i^-(\kappa) \cup K_i\Big) \cap J^-(\kappa)\,.$$

For $J \in \mathcal{I}_k$ and $K \in \mathcal{K}_J^-$ define

$$\varphi_K'(x,\varepsilon) = \sum_{j=k+2}^{p} \sum_{\substack{I \in \mathcal{I}_j \\ I \subseteq K}} \varphi_I(x,\varepsilon)\,.$$

Since $\varphi_{K_i}'(x,\varepsilon) \cdot \varphi_J(x,\varepsilon) \leq 0$ for all $x \in \mathbb{R} \setminus \tilde{K}_i$, we have

$$\sum_{k=\lambda+1}^{p-2} \sum_{j=k+2}^{p} \sum_{J \in \mathcal{I}_k} \sum_{K \in \mathcal{K}_J^-} \sum_{\substack{I \in \mathcal{I}_j \\ I \subseteq K}} \int \Big(\varphi_I(x,\varepsilon) \cdot \varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x)$$

$$\leq \sum_{k=\lambda+1}^{p-2} \sum_{J \in \mathcal{I}_k} \sum_{i=1}^{N} \int_{\tilde{K}_i} \Big(\varphi_{K_i}'(x,\varepsilon)\varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x)\,.$$

We can split this term in the following way,

$$\sum_{k=\lambda+1}^{p-2} \sum_{J \in \mathcal{I}_k} \sum_{i=1}^{N} \int_{\tilde{K}_i} \Big(\varphi_{K_i}'(x,\varepsilon)\varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x)$$

$$\leq \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \int_{\tilde{K}_1} \left|\varphi'_{K_1}(x,\varepsilon)\varphi_J(x,\varepsilon)\right| d\bar{\mu}(x) \tag{5.20}$$

$$+ \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} \int_{\tilde{K}_i} \left(\varphi'_{K_i}(x,\varepsilon)\varphi_J(x,\varepsilon)\right) d\bar{\mu}(x). \tag{5.21}$$

Let us give the estimate for (5.20) first. Observe that, by lemma 3.2.2,

$$|\tilde{K}_1| \leq (\kappa+1)\cdot \mathcal{C}_1 \cdot \varepsilon_{k+1}$$

and therefore, recalling that ζ_1 is the right endpoint of K_1,

$$\tilde{K}_1 \subseteq [\zeta_1 - (\mathcal{C}_1\varepsilon_{k+1}(1+\kappa)), \zeta_1] \subseteq [\zeta_1 - \delta|J|, \zeta_1],$$

where

$$\delta = \mathcal{C}_1(1+\kappa(\varepsilon))\frac{\varepsilon_{\lambda+2}}{\varepsilon_{\lambda+1}} \geq \mathcal{C}_1(1+\kappa(\varepsilon))\cdot\frac{\varepsilon_{k+1}}{|J|}.$$

Using lemma 3.2.5(a), we get

$$\begin{aligned}\int_{\tilde{K}_1} \left|\varphi_J(x,\varepsilon)\right| d\bar{\mu}(x) &\leq \mathcal{C}_{17}\int_{J^-(\delta)} \psi^x_\varepsilon(J)\, d\bar{\mu}(x)\\ &\leq \mathcal{C}_{17}\cdot C\mathcal{D}_5\cdot\frac{|J|^\alpha}{|\log\varepsilon|}\log\left(\frac{\delta+1}{\delta}\right)\cdot\delta^\alpha.\end{aligned}$$

Since $|\varphi'_{K_1}(x,\varepsilon)| \leq 2\mathcal{C}_{17}$, we get using lemma 3.2.4(2)

$$\begin{aligned}\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \int_{\tilde{K}_1} \left|\varphi'_{K_1}(x,\varepsilon)\varphi_J(x,\varepsilon)\right| d\bar{\mu}(x) &\leq 2C\mathcal{D}_5\mathcal{C}_{17}^2\cdot\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k}\frac{|J|^\alpha}{|\log\varepsilon|}\log\left(\frac{\delta+1}{\delta}\right)\cdot\delta^\alpha\\ &\leq 2C\mathcal{D}_5\mathcal{C}_{17}^2\mathcal{C}_4\cdot\frac{1}{(\log|\log\varepsilon|)^2},\end{aligned}$$

by (d), finishing the estimate of term (5.20).

It remains to investigate (5.21). This is the crucial part. In order to carry out the estimate we shall formulate two claims, which constitute the core of our proof. The first claim is a "local version" of the key argument in the proof of theorem 4.2.1, namely the transformation of the integral carried out in equation (4.3) to (4.5). The second claim is an adaptation of the "variation argument"(3.18), which was vital in the proof of theorem 3.1.3.

Claim 1: There is a constant $\mathcal{C}_{20} > 0$ such that, for all $K \in \mathcal{K}_J^-$,

$$\left|\int_{\tilde{K}} \tilde{\varphi}_K(x,\varepsilon)\, d\bar{\mu}(x)\right| \le \mathcal{C}_{20} \cdot \frac{(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|} \cdot |K|^\alpha .$$

Proof Recall that

$$G_1\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}\Big) = \int G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, y\Big)\, d\frac{\bar{\mu}_{x,t}}{t^\alpha}(y) = \int G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}(y),$$

and

$$G_2\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}\Big) = \int G\Big(T^y\frac{\bar{\mu}_{x,t}}{t^\alpha}, -y\Big)\, d\frac{\bar{\mu}_{x,t}}{t^\alpha}(y) = \int G\Big(\frac{\bar{\mu}_{y,t}}{t^\alpha}, \frac{x-y}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}(y).$$

Therefore

$$\begin{aligned}
\tilde{\varphi}_K(x,\varepsilon) &= \int_{K\cap[x,\infty)} G_1\Big(\frac{\bar{\mu}_{x,z-x}}{(z-x)^\alpha}\Big)\, d\psi_\varepsilon^x(z) - \int_{K\cap(-\infty,x]} G_2\Big(\frac{\bar{\mu}_{x,x-z}}{(x-z)^\alpha}\Big)\, d\psi_\varepsilon^x(z) \\
&= \int_{K\cap[x,\infty)} \int G\Big(\frac{\bar{\mu}_{x,z-x}}{(z-x)^\alpha}, \frac{y-x}{z-x}\Big)\, d\bar{\mu}(y)\, \frac{d\psi_\varepsilon^x(z)}{(z-x)^\alpha} \\
&\quad - \int_{K\cap(-\infty,x]} \int G\Big(\frac{\bar{\mu}_{y,x-z}}{(x-z)^\alpha}, \frac{x-y}{x-z}\Big)\, d\bar{\mu}(y)\, \frac{d\psi_\varepsilon^x(z)}{(x-z)^\alpha}.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_{\tilde{K}} \tilde{\varphi}_K(x,\varepsilon)\, d\bar{\mu}(x) \\
&= (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \int_{\tilde{K}\cap(K-t)} \Big(\int G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}(y)\Big)\, d\bar{\mu}(x) \\
&\quad - \int_{\tilde{K}\cap(K+t)} \Big(\int G\Big(\frac{\bar{\mu}_{y,t}}{t^\alpha}, \frac{x-y}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}(y)\Big)\, d\bar{\mu}(x)\, \frac{dt}{t} \\
&= (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \int_{K^1} G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}^2(x,y) \\
&\quad - \int_{K^2} G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}^2(x,y)\, \frac{dt}{t},
\end{aligned}$$

where

$$K^1 = \{(x,y) \in E^2 \,:\, x \in (K-t) \cap \tilde{K},\, \frac{y-x}{t} \in B(0,R(g))\}$$

and

$$K^2 = \{(x,y) \in E^2 \,:\, y \in (K+t) \cap \tilde{K},\, \frac{y-x}{t} \in B(0,R(g))\},$$

using $G(\nu,x) = 0$ for all ν if $x \notin B(0,R(g))$. Thus we can use the cancellation and get

$$\begin{aligned}
&\int_{\tilde{K}} \tilde{\varphi}_K(x,\varepsilon)\, d\bar{\mu}(x) \\
&\le (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \int_{K^1\setminus K^2} G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}^2(x,y)\, \frac{dt}{t} \\
&\le (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \|g\|_{\sup} \cdot \bar{\mu}^2(K^1 \setminus K^2)\, \frac{dt}{t^{1+\alpha}},
\end{aligned}$$

recalling that G is bounded by $\|g\|_{\sup}$. We take a closer look at the set

$$K^1 \setminus K^2 = \{(x,y) \in E^2 \,:\, x \in (K-t) \cap \tilde{K}\,,\, \frac{y-x}{t} \in B(0, R(g))\,, y \notin (K+t) \cap \tilde{K}\}\,.$$

First observe that if $t > |K|(\kappa(\varepsilon)+1)$ we have $(K-t) \cap \tilde{K} = \emptyset$ and thus $K^1 \setminus K^2 = \emptyset$. Otherwise if $(x,y) \in K^1 \setminus K^2$ then

$$\begin{aligned} y \in Y &:= B((K-t) \cap \tilde{K}, R(g)t) \setminus ((K+t) \cap \tilde{K}) \\ &\subseteq \Big(B((K-t) \cap \tilde{K}, R(g)t) \setminus (K+t)\Big) \cup \Big(B((K-t) \cap \tilde{K}, R(g)t) \setminus \tilde{K}\Big)\,. \end{aligned}$$

We have

$$\begin{aligned} B((K-t) \cap \tilde{K}, R(g)t) \setminus (K+t) &\subseteq B((K-t), R(g)t) \setminus (K+t) \\ &\subseteq B((K+t), R(g)t + 2t) \setminus (K+t)\,, \end{aligned}$$

and by (5.5)

$$\bar{\mu}\Big(B(K+t, R(g)t+2t) \setminus (K+t)\Big) \leq 2C(R(g)+2)^\alpha \cdot t^\alpha.$$

Also $B((K-t) \cap \tilde{K}, R(g)t) \setminus \tilde{K} \subseteq B(\tilde{K}, R(g)t) \setminus \tilde{K}$, and thus

$$\bar{\mu}(B(\tilde{K}, R(g)t) \setminus \tilde{K}) \leq 2CR(g)^\alpha \cdot t^\alpha.$$

Therefore $\bar{\mu}(Y) \leq 2C(R(g)^\alpha + (R(g)+2)^\alpha) \cdot t^\alpha$, and thus

$$\begin{aligned} \bar{\mu}^2(K^1 \setminus K^2) &\leq \int_Y \bar{\mu}(B(y, tR(g)))\, d\bar{\mu}(y) \\ &\leq 2C^2 R(g)^\alpha (R(g)^\alpha + (R(g)+2)^\alpha) \cdot t^{2\alpha}\,. \end{aligned}$$

Let $C_{20} = 4/\alpha \cdot \|g\|_{\sup} \cdot \Big(C^2 R(g)^\alpha (R(g)^\alpha + (R(g)+2)^\alpha)\Big)$. Then

$$\begin{aligned} &\int_{\tilde{K}} \tilde{\varphi}_K(x,\varepsilon)\, d\bar{\mu}(x) \\ &\leq (|\log\varepsilon|)^{-1} \cdot \int_\varepsilon^{|K|(\kappa+1)} \bar{\mu}^2(K^1 \setminus K^2) \cdot \|g\|_{\sup} \frac{dt}{t^{1+\alpha}} \\ &\leq (|\log\varepsilon|)^{-1} \cdot (\alpha/2) \cdot C_{20} \cdot \int_\varepsilon^{|K|(\kappa+1)} t^{\alpha-1}\, dt \\ &\leq (1/2) \cdot C_{20} \cdot \frac{|K|^\alpha \cdot (\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|}\,. \end{aligned}$$

We also have

$$-\int_{\tilde{K}} \tilde{\varphi}_K(x,\varepsilon)\, d\bar{\mu}(x) \le (|\log\varepsilon|)^{-1} \int_{\varepsilon}^{|K|(\kappa+1)} \|g\|_{\sup} \cdot \bar{\mu}^2(K^2 \setminus K^1)\, \frac{dt}{t^{1+\alpha}},$$

and for $\bar{\mu}^2(K^2 \setminus K^1)$ we can derive the same estimate as for $\bar{\mu}^2(K^1 \setminus K^2)$.

This finishes the proof of claim 1. ∎

Claim 2 There is a constant $C_{21} > 0$ such that the following holds: For every $J \in \mathcal{I}_k$ and any interval $K_i \in \mathcal{K}_J^-$ such that $d_i = d(J, K_i) > 0$, denote

$$c(K_i, J) = \sup_{x,y\in\tilde{K}_i} |\varphi_J(x,\varepsilon) - \varphi_J(y,\varepsilon)|,$$

and denote the right endpoint of K_i by ζ_i. Then we have

$$c(K_i, J) \le C_{21} \cdot \psi_\varepsilon^{\zeta_i}(J) \cdot \frac{|K_i|(\kappa(\varepsilon)+1)}{d_i}.$$

Proof Let $J \in \mathcal{I}_k$ and let $K_i \in \mathcal{K}_J^-$. For $x \in \tilde{K}_i$ we have

$$\varphi_J(x,\varepsilon) = (|\log\varepsilon|)^{-1} \int_J \int g(\frac{y-x}{z-x}) \cdot \frac{d\bar{\mu}(y)}{(z-x)^\alpha} \cdot \left(h\Big[\int f(\frac{w-x}{z-x})\, \frac{d\bar{\mu}(w)}{(z-x)^\alpha}\Big]\right) \frac{dz}{z-x}.$$

We show that φ_J is a Lipschitz function in x and determine a Lipschitz constant.

For this purpose we fix $z \in J$ and let l be a nonnegative Lipschitz function with Lipschitz constant $L(l)$, and compact support contained in $B(0, R(l))$. We investigate the function $\psi_l : \tilde{K}_i \to \mathbb{R}$ defined by

$$\psi_l(x) = \int l\left(\frac{y-x}{z-x}\right) \cdot \frac{d\bar{\mu}(y)}{|z-x|^\alpha}.$$

ψ_l is bounded, since by (5.5)

$$\psi_l(x) \le \|l\|_{\sup} \cdot \frac{\bar{\mu}(B(x, R(l)\cdot|z-x|))}{|z-x|^\alpha} \le \|l\|_{\sup} \cdot C(2R(l))^\alpha.$$

We show that ψ_l is Lipschitz on the domain $\tilde{K}_i$.

Let $x_1, x_2 \in \tilde{K}_i$ with $x_1 < x_2$. We have

$$\begin{aligned}
&|\psi_l(x_1) - \psi_l(x_2)| \\
&\le L(l) \cdot \int_{B(x_1, R(l)|z-x_1|)} \left|\frac{y-x_1}{z-x_1} - \frac{y-x_2}{z-x_2}\right| \frac{d\bar{\mu}(y)}{|z-x_1|^\alpha} \\
&\quad + L(l) \cdot \int_{B(x_2, R(l)|z-x_2|)} \left|\frac{y-x_1}{z-x_1} - \frac{y-x_2}{z-x_2}\right| \frac{d\bar{\mu}(y)}{|z-x_1|^\alpha} \\
&\quad + \int l\left(\frac{y-x_2}{z-x_2}\right) \left|\frac{1}{|z-x_1|^\alpha} - \frac{1}{|z-x_2|^\alpha}\right| d\bar{\mu}(y).
\end{aligned}$$

We use the estimates

$$\left|\frac{1}{(z-x_1)^\alpha}-\frac{1}{(z-x_2)^\alpha}\right| \le \frac{1}{(z-x_2)^\alpha}\cdot\left|\frac{z-x_2}{z-x_1}-1\right| \le \frac{1}{(z-x_2)^\alpha}\cdot\frac{|x_1-x_2|}{d_i},$$

and

$$\begin{aligned}\left|\frac{y-x_1}{z-x_1}-\frac{y-x_2}{z-x_2}\right| &\le \left|\frac{y-x_1}{z-x_1}\right|\cdot\left|\frac{z-x_1}{z-x_2}-1\right|+\left|\frac{x_2-x_1}{z-x_2}\right| \\ &\le \left|\frac{y-x_1}{z-x_1}\right|\cdot\frac{|x_1-x_2|}{d_i}+\frac{|x_1-x_2|}{d_i},\end{aligned}$$

to see

$$|\psi_l(x_1)-\psi_l(x_2)| \le |x_1-x_2|\cdot(1/d_i)\cdot\Big(C(2R(l))^\alpha\cdot(2L(l)(R(l)+1)+\|l\|_{\text{sup}})\Big).$$

Hence for every l as above there is a constant $C(l)>0$, depending on l, such that ψ_l is bounded and Lipschitz with constant $C(l)/d_i$. Therefore there is a constant $C'_{21}>0$ such that for every $z\in J$ the function

$$\psi'(x)=\int g(\tfrac{y-x}{z-x})\cdot\tfrac{d\bar\mu(y)}{(z-x)^\alpha}\cdot\Big(h\Big[\int f(\tfrac{w-x}{z-x})\,\tfrac{d\bar\mu(w)}{(z-x)^\alpha}\Big]\Big)$$

is Lipschitz with Lipschitz constant C'_{21}/d_i. To find the Lipschitz constant for $\varphi_J(x,\varepsilon)$ we use

$$\left|\frac{1}{z-x_1}-\frac{1}{z-x_2}\right| \le \frac{1}{z-x_1}\cdot\left|\frac{z-x_1}{z-x_2}-1\right| \le \frac{1}{z-\zeta_i}\cdot\frac{|x_1-x_2|}{d_i},$$

and estimate as follows

$$\begin{aligned}&|\varphi_J(x_1,\varepsilon)-\varphi_J(x_2,\varepsilon)| \\ &\le (|\log\varepsilon|)^{-1}\cdot\int_J|\psi'(x_1)-\psi'(x_2)|\cdot\frac{dz}{|z-x_1|} \\ &\quad+(|\log\varepsilon|)^{-1}\cdot\int_J\psi'(x_2)\cdot\left|\frac{1}{z-x_1}-\frac{1}{z-x_2}\right|dz \\ &\le |x_1-x_2|\cdot\Big(\frac{C'_{21}}{d_i}\cdot\psi_\varepsilon^{\zeta_i}(J)+\frac{\|g\|_{\text{sup}}\cdot C(2R(g))^\alpha}{d_i}\cdot\psi_\varepsilon^{\zeta_i}(J)\Big).\end{aligned}$$

Let $C_{21}=C'_{21}+\|g\|_{\text{sup}}\cdot C(2R(g))^\alpha$. Then φ_J has Lipschitz constant

$$\frac{C_{21}}{d_i}\cdot\psi_\varepsilon^{\zeta_i}(J),$$

and this, together with the observation $|x_1-x_2|\le|\tilde K_i|\le|K_i|(\kappa+1)$, yields the statement.

■

We can split (5.21) again;

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} \int_{\tilde{K}_i} \Big(\varphi'_{K_i}(x,\varepsilon)\varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x) \\
&\leq \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} c(K_i,J)\cdot \int_{\tilde{K}_i} |\varphi'_{K_i}(x,\varepsilon)|\, d\bar{\mu}(x) \qquad (5.22)\\
&\quad + \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} |\varphi_J(\zeta_i,\varepsilon)|\cdot \int_{\tilde{K}_i} |\tilde{\varphi}_{K_i}(x,\varepsilon) - \varphi'_{K_i}(x,\varepsilon)|\, d\bar{\mu}(x) \qquad (5.23)\\
&\quad + \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} |\varphi_J(\zeta_i,\varepsilon)|\cdot \Big|\int_{\tilde{K}_i} \tilde{\varphi}_{K_i}(x,\varepsilon)\, d\bar{\mu}(x)\Big|. \qquad (5.24)
\end{aligned}$$

To finish the proof we have to give estimates for (5.22) to (5.23).

Let us start by looking at (5.22). Using lemmas 3.2.5(c), 3.2.4(1) and 3.2.1 we get

$$\begin{aligned}
\int_{\tilde{K}_i} |\varphi'_{K_i}(x,\varepsilon)|\, d\bar{\mu}(x) &\leq \mathcal{C}_{17}\cdot \sum_{j=k+2}^{p} \sum_{\substack{I\in\mathcal{I}_j\\ I\subseteq K_i}} \int_{I^0(\kappa)} \psi_\varepsilon^x(I)\, d\bar{\mu}(x) \\
&\leq 2\mathcal{C}_{17}\mathcal{D}_7 C\cdot \sum_{\substack{|I|\geq\varepsilon\\ I\subseteq K_i}} \frac{|I|^\alpha}{|\log\varepsilon|} \\
&\leq 2C\mathcal{C}_{17}\mathcal{D}_7\mathcal{C}_3\cdot |K_i|^\alpha \\
&\leq (2C\mathcal{D}_7\mathcal{C}_{17}\mathcal{C}_3\mathcal{C}_1^\alpha)\cdot \varepsilon_{k+1}^\alpha .
\end{aligned}$$

By claim 2 we have

$$c(K_i,J) \leq \mathcal{C}_{21}\cdot \psi_\varepsilon^{\zeta_i}(J)\cdot \frac{(\kappa+1)|K_i|}{d_i},$$

and we observe, using $d_i \geq \varepsilon_{k+1}$ for all $i \geq 2$,

$$\begin{aligned}
\psi_\varepsilon^{\zeta_i}(J) &\leq (|\log\varepsilon|)^{-1}\cdot \log\Big(\frac{d_i + |J|}{d_i}\Big) \\
&\leq \frac{1}{|\log\varepsilon|}\cdot \log\Big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}} + 1\Big) .
\end{aligned}$$

Using $|K_i| \leq \mathcal{C}_1\cdot\varepsilon_{k+1} \leq \mathcal{C}_1\cdot d_i$, we get

$$\begin{aligned}
\sum_{i=2}^{N} \frac{|K_i|}{d_i} &= \sum_{i=2}^{N} \frac{|K_i| + d_i}{d_i}\cdot \frac{|K_i|}{|K_i| + d_i} \\
&\leq (1+\mathcal{C}_1)\cdot \int \frac{dt}{t},
\end{aligned}$$

where the integral is taken with respect to the domain

$$\{t\,:\,\varepsilon_{k+1} \leq t \leq (\kappa|J| + \mathcal{C}_1\varepsilon_{k+1})\}.$$

This integral is bounded by

$$\int \frac{dt}{t} \leq \log \Big(\kappa \cdot \big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}\big) + C_1\Big).$$

We can now put all these ingredients together and get

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} c(K_i,J) \cdot \int_{\tilde{K}_i} |\varphi'_{K_i}(x,\varepsilon)|\, d\bar{\mu}(x) \\
\leq\ & 2C_{21}(1+C_1)C_{17}C\mathcal{D}_7C_3C_1^\alpha \\
&\times \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \frac{(\kappa+1)\cdot\varepsilon_{k+1}^\alpha}{|\log\varepsilon|} \log\big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}+1\big)\cdot \log\big((\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}})\cdot\kappa + C_1\big) \\
\leq\ & 2C_{21}(1+C_1)C_{17}C\mathcal{D}_7C_3C_1^\alpha \cdot \Big(\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k} \frac{|J|^\alpha}{|\log\varepsilon|}\Big) \\
&\times \max_{k=\lambda}^{p} \Big(\big(\frac{\varepsilon_{k+1}}{\varepsilon_k}\big)^\alpha \cdot \log^2\big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}\cdot\kappa(\varepsilon)+C_1\big)\cdot(\kappa(\varepsilon)+1)\Big) \\
\leq\ & 2C_{21}(1+C_1)C_{17}C\mathcal{D}_7C_3C_1^\alpha C_4 \cdot \frac{1}{(\log|\log\varepsilon|)^2},
\end{aligned}$$

by (e), and this finishes the estimate of term (5.22).

Let us give an estimate for (5.23). We use the first inequality of the approximation lemma 3.2.6 with $\gamma = 1$ to see

$$\begin{aligned}
&\int_{\tilde{K}_i} |\varphi_{K_i}(x,\varepsilon) - \varphi'_{K_i}(x,\varepsilon)|\, d\bar{\mu}(x) \\
\leq\ & C_{17}\cdot \int \Big(\psi_\varepsilon^x(K_i) - \sum_{\substack{|I|\geq\varepsilon \\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big)\, d\bar{\mu}(x) \\
\leq\ & C_{17}C_9\cdot|K_i|^\alpha\cdot\left(\Big(\frac{1}{\kappa(\varepsilon)}\Big)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\right) \\
\leq\ & 2C_{17}C_9\cdot\frac{|K_i|^\alpha}{(\log|\log\varepsilon|)^6}.
\end{aligned}$$

Recall $\sum_{J\in\mathcal{I}_k} |\varphi_J(\zeta_i,\varepsilon)| \leq 2C_{17}$. For (5.23) we get, using lemma 3.2.3,

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\sum_{i=2}^{N} |\varphi_J(\zeta_i,\varepsilon)|\cdot\int_{\tilde{K}_i} |\tilde{\varphi}_{K_i}(x,\varepsilon) - \varphi'_{K_i}(x,\varepsilon)|\, d\bar{\mu}(x) \\
\leq\ & \sum_{k=\lambda+1}^{p-2}\sum_{K\in\mathcal{N}_{\varepsilon_{k+1}}} (2C_{17})\cdot\Big(2C_{17}C_9\cdot\frac{|K|^\alpha}{(\log|\log\varepsilon|)^6}\Big) \\
\leq\ & (4C_{17}^2C_9C_2)\cdot\frac{p(\varepsilon)}{(\log|\log\varepsilon|)^6} \\
\leq\ & (4C_{17}^2C_9C_2)\cdot\Big(\frac{2}{\log 2}\Big)^4\cdot\frac{1}{(\log|\log\varepsilon|)^2},
\end{aligned}$$

by (f). This finishes the estimate of (5.23).

Finally look at (5.24). Recall that by claim 1,

$$\left|\int_{\bar{K}_i} \tilde{\varphi}_{K_i}(x,\varepsilon)\, d\bar{\mu}(x)\right| \le C_{20} \cdot \frac{(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|} \cdot |K_i|^\alpha .$$

Thus we get for (5.24), using lemma 3.2.3,

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{i=2}^{N} |\varphi_J(\zeta_i,\varepsilon)| \cdot \left|\int_{\bar{K}_i} \tilde{\varphi}_{K_i}(x,\varepsilon)\, d\bar{\mu}(x)\right| \\
&\le \sum_{k=\lambda+1}^{p-2} \sum_{K\in\mathcal{N}_{\varepsilon_{k+1}}} (2C_{17}) \cdot \left(C_{20} \cdot \frac{(\kappa+1)^\alpha}{|\log\varepsilon|} |K|^\alpha\right) \\
&\le (2C_{20}C_{17}C_2) \cdot \frac{(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|} \cdot p(\varepsilon) \\
&\le (2C_{20}C_{17}C_2) \cdot \frac{1}{(\log|\log\varepsilon|)^2},
\end{aligned}$$

by (g), finishing the estimate of (5.24).

We have thus finished the proof of lemma 5.2.3 by showing that all the sums, in which we have split the original expression (5.9), are bounded by a constant multiple of $1/(\log|\log\varepsilon|)^2$. ■

Now we have done most of the work to finish the second step.

Lemma 5.2.4 *For any function G defined as in (5.6) we have*

$$\mu\left(\left\{x \in E : \int\int G(\nu,y)\, d\nu(y)\, dP(\nu) = \int\int G(T^y\nu,-y)\, d\nu(y)\, dP(\nu) \text{ for all } P \in \mathcal{P}^\alpha(\mu,x)\right\}\right) = \mu(E).$$

Proof To begin with, fix $s > 1$ and let $\delta_k = \exp(-s^k)$. Let $1 \ge \sigma > 0$. We have

$$\begin{aligned}
&\left|(|\log\delta_n|)^{-1} \int_{\delta_n}^1 \left(G_1(\tfrac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\tfrac{\bar{\mu}_{x,t}}{t^\alpha})\right) \frac{dt}{t}\right| \\
&\le \left|(|\log\delta_n|)^{-1} \int_{\delta_n}^1 \left(G_1(\tfrac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\tfrac{\bar{\mu}_{x,t}}{t^\alpha})\right) \frac{dt}{t} - \sum_{I\in\mathcal{A}_{\delta_n}} \varphi_I(x,\delta_n)\right| \\
&\quad + \left|\sum_{I\in\mathcal{A}_{\delta_n}} \varphi_I(x,\delta_n)\right| .
\end{aligned}$$

Lemma 5.2.2 and lemma 5.2.3 therefore give

$$\mu\left(\left\{x\in E:\left|(|\log\delta_n|)^{-1}\int_{\delta_n}^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\right|>2\sigma\right\}\right)$$

$$\leq\frac{C_{19}+C_{18}}{\sigma^2\cdot(\log|\log\delta_n|)^2}=\frac{C_{19}+C_{18}}{\sigma^2\cdot(\log s)^2\cdot n^2}.$$

Since $\sum_{n=1}^\infty(1/n)^2<\infty$, the Borel-Cantelli-lemma yields

$$\mu\left(\left\{x\in E:\limsup_{n\to\infty}\left|(|\log\delta_n|)^{-1}\int_{\delta_n}^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\right|>2\sigma\right\}\right)=0.$$

This holds for all $\sigma>0$, and thus

$$\mu\left(\left\{x\in E:\limsup_{n\to\infty}\left|(|\log\delta_n|)^{-1}\int_{\delta_n}^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\right|>0\right\}\right)=0. \qquad (5.25)$$

For every $\delta_n<\varepsilon\leq\delta_{n-1}$ we have

$$\begin{aligned}&(|\log\varepsilon|)^{-1}\int_\varepsilon^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\\ =\;&(|\log\varepsilon|)^{-1}\int_\varepsilon^{\delta_n}\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\\ &+\frac{|\log\delta_n|}{|\log\varepsilon|}\cdot(|\log\delta_n|)^{-1}\int_{\delta_n}^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\end{aligned}$$

Now $|\log\delta_n|/|\log\varepsilon|\leq s$ and thus

$$\left|(|\log\varepsilon|)^{-1}\int_\varepsilon^{\delta_n}\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\right|\leq C_{17}\cdot\left|\frac{\log(\delta_n/\varepsilon)}{\log\varepsilon}\right|\leq C_{17}\cdot(s-1).$$

This and (5.25) together imply, for μ-almost every $x\in E$,

$$\limsup_{\varepsilon\downarrow0}\left|(|\log\varepsilon|)^{-1}\int_\varepsilon^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}\right|\leq C_{17}\cdot(s-1).$$

Since this holds for all $s>1$, we get

$$\lim_{\varepsilon\downarrow0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1\left(G_1(\frac{\bar\mu_{x,t}}{t^\alpha})-G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\right)\frac{dt}{t}=0$$

for μ-almost all $x\in E$.

By (5.5) the closure of the set $\{\frac{\bar\mu_{x,t}}{t^\alpha}:t\in(0,1)\}$ is compact. Therefore we can find continuous functions H_1 and H_2, bounded on $\mathcal{M}(\mathbb{R})$, which agree with G_1 and G_2 on the closure of $\{\frac{\bar\mu_{x,t}}{t^\alpha}:t\in(0,1)\}$. Hence, for μ-almost every $x\in E$, every tangent measure distribution $P = \lim_{n\to\infty} P_{\varepsilon_n}$ of μ at x fulfills

$$\begin{aligned}
&\int\int G(\nu, y)\, d\nu(y)\, dP(\nu) \\
= &\int H_1(\nu)\, dP(\nu) = \lim_{n\to\infty} (|\log\varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 H_1(\frac{\bar\mu_{x,t}}{t^\alpha})\,\frac{dt}{t} \\
= &\lim_{n\to\infty} (|\log\varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t} + \lim_{n\to\infty} (|\log\varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 H_2(\frac{\bar\mu_{x,t}}{t^\alpha})\,\frac{dt}{t} \\
= &\int H_2(\nu)\, dP(\nu) \\
= &\int\int G(T^y\nu, -y)\, d\nu(y)\, dP(\nu)
\end{aligned}$$

as required. ■

To finish the proof it remains to show in the third step that the set of all $x \in E$ where the Palm formula holds for *all* Borel measurable functions

$$G : \mathcal{M}(\mathbb{R}) \times \mathbb{R} \longrightarrow [0,\infty)$$

has full measure. For this purpose we work with Fourier transform. Define functions

$$\begin{aligned}
h_1(x) &= \sin^+(x) \quad , \quad h_2(x) = \sin^-(x)\,, \\
h_3(x) &= \cos^+(x) \quad , \quad h_4(x) = \cos^-(x)\,,
\end{aligned}$$

and for $i,j \in \{1,\ldots,4\}$ define

$$h_{ij}(x) = h_i(x)h_j(x)\,.$$

Choose a sequence $(f_i)_{i\in\mathbb{N}}$ of Lipschitz functions $f_i \geq 0$ with compact support such that, whenever $f \in C_c(\mathbb{R})$ and $f \geq 0$ with supp $f \subseteq B(0,R)$, $R \in \mathbb{N}$ and $\varepsilon > 0$, there is an $i \in \mathbb{N}$ such that supp $f_i \subseteq B(0,R+1)$ and $\|f - f_i\|_{\sup} < \varepsilon$. See for example lemma 1.2.5(2) for the construction of such a sequence. Let

$$G_{i,j,k,l}(\nu, x) = f_i(x) \cdot h_{jk}(\nu(f_l))\,,$$

and

$$A = \{x \in E : \text{ the Palm formula holds for all } G_{i,j,k,l} \text{ and all } P \in \mathcal{P}^\alpha(\mu, x)\}\,.$$

We know by lemma 5.2.4 that $\mu(A) = \mu(E)$.

Lemma 5.2.5 *Let $x \in A$ and suppose that $G : \mathcal{M}(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{C}$ has the form*

$$G(\nu, x) = g(x) \cdot \exp(i\nu(f))$$

for $g, f \in \mathcal{C}_c(\mathbb{R})$ with $g \geq 0$. Then the Palm formula holds for G and all tangent measure distributions P of μ at x.

Proof We fix a function G given, as in the statement, by functions $g, f \in \mathcal{C}_c(\mathbb{R})$ and argue by approximation. Suppose $\varepsilon > 0$ is given. Let $R > 0$ be such that $\operatorname{supp} f, \operatorname{supp} g \subseteq B(0, R)$, and $S > 0$ be such that $\|g\|_{\sup} < S$. We find f_i, f_j, f_k, with supports contained in $B(0, R+1)$ and

$$\|(f_i - f_j) - f\|_{\sup} < \eta_1 := \frac{\varepsilon}{8SC^2(2R+1)^{2\alpha}},$$

$$\|f_k - g\|_{\sup} < \eta_2 := \frac{\varepsilon}{4C(R+1)^{\alpha}}.$$

For all $\nu \in \operatorname{Tan}(\mu, x)$, the estimate (5.3) implies $\nu(B(0, R+1)) \leq C \cdot (R+1)^{\alpha}$, and therefore

$$\int\int G(\nu, y)\, d\nu(y)\, dP(\nu) = \int\int f_k(y) \cdot \exp(i\nu(f_i - f_j))\, d\nu(y)\, dP(\nu) + \eta,$$

with

$$|\eta| \leq \eta_2 \cdot C(R+1)^{\alpha} + \eta_1 \cdot 2SC^2(R+1)^{2\alpha} \leq \varepsilon/2.$$

Analogously

$$\int\int G(T^y\nu, -y)\, d\nu(y)\, dP(\nu) = \int\int f_k(-y) \cdot \exp(iT^y\nu(f_i - f_j))\, d\nu(y)\, dP(\nu) + \eta',$$

with

$$|\eta'| \leq \eta_2 \cdot C(R+1)^{\alpha} + \eta_1 \cdot 2SC^2(2R+1)^{2\alpha} \leq \varepsilon/2.$$

By definition of the set A the Palm formula holds for the functions

$$(\nu, y) \mapsto f_k(y) \cdot \exp(i\nu(f_i - f_j))$$

and thus the Palm formula holds for G. ∎

Now fix a function $g \in \mathcal{C}_c(\mathbb{R})$ with $g \geq 0$. Define two finite measures Λ_1, Λ_2 on the Borel σ-algebra of $\mathcal{M}(\mathbb{R})$ by means of

$$\begin{aligned}\Lambda_1(M) &= \int\int g(y)1_M(\nu)\,d\nu(y)dP(\nu),\\ \Lambda_2(M) &= \int\int g(-y)1_M(T^y\nu)\,d\nu(y)dP(\nu)\end{aligned}$$

for all Borel sets $M \subseteq \mathcal{M}(\mathbb{R})$. We know that

$$\Lambda_1(\exp(i\nu(f)) = \Lambda_2(\exp(i\nu(f))$$

for all $f \in \mathcal{C}_c(\mathbb{R})$, and this means that the Fourier transforms of Λ_1 and Λ_2 coincide. Thus Λ_1 and Λ_2 coincide (see for example [Kal83, theorem 3.1]) and hence the Palm formula holds for all bounded functions G of the form

$$G(\nu, y) = g(y) \cdot F(\nu)$$

for Borel functions $F : \mathcal{M}(\mathbb{R}) \to [0, \infty)$ and $g \in \mathcal{C}_c(\mathbb{R})$.

Fix $x \in A$. For every $r > 0$ let $\mathcal{S}(r)$ be the collection of all Borel subsets $B \subseteq \mathcal{M}(\mathbb{R}) \times U(0, r)$ such that

$$\int\int 1_B(\nu, y)\,d\nu(y)\,dP(\nu) = \int\int 1_B(T^y\nu, -y)\,d\nu(y)\,dP(\nu)$$

for all tangent measure distributions $P \in \mathcal{P}^\alpha(\mu, x)$. $\mathcal{S}(r)$ contains $\mathcal{M}(\mathbb{R}) \times U(0, r)$ and is closed under proper differences and, by the monotone convergence theorem, under non-decreasing limits. Using monotone approximation from below and the monotone convergence theorem, we see that $\mathcal{S}(r)$ comprises the collection

$$\mathcal{G} = \{M \times I \,:\, M \subseteq \mathcal{M}(\mathbb{R}) \text{ Borel, and } I \subseteq U(0, r) \text{ open }\}.$$

$\mathcal{G}$ is stable under finite intersection and generates the Borel-σ-algebra on $\mathcal{M}(\mathbb{R}) \times U(0, r)$. Hence, by the monotone class theorem (as formulated for example in [Kal83, 15.2.1]), $\mathcal{S}(r)$ equals the Borel-σ-algebra on $\mathcal{M}(\mathbb{R}) \times U(0, r)$. Thus the Palm-formula holds for all Borel step-functions and, by monotone approximation from below and the monotone convergence theorem, we can conclude that, for all $x \in A$; the Palm formula holds for all nonnegative Borel functions G and all tangent measure distributions P of μ at x. This finishes the proof of theorem 5.1.1.

Chapter 6

Normalized Tangent Measure Distributions

Standardized tangent measures have turned out to be an approporiate tool for the investigation of measures with positive and finite α-densities. However, if the behaviour of $\mu(B(x,t))$ as t tends to 0 is not comparable to a law of the type t^α but depends substantially on x, it does not seem appropriate to study μ via distributions on the set of standardized tangent measures. In these cases it seems more suitable to compare $\mu_{x,t}$ directly to $\mu_{x,t}(\Delta)$ for a suitably chosen normalizing function $\Delta : \mathbb{R}^n \longrightarrow [0,\infty)$ and define tangent measure distributions on the set $\mathrm{Tan}^\Delta(\mu,x)$ of normalized tangent measures, the Δ-normalized tangent measure distributions or short Δ-tangent measure distributions.

In section 6.1 we first illustrate the limitations of standardized tangent measure distributions by means of an example of a nonzero measure μ that has no non-trivial standardized tangent measure distributions μ-almost everywhere. We then define the normalized tangent measure distributions and give some their basic properties. In section 6.2 we prove an existence theorem for normalized tangent measure distributions, and in section 6.3 we prove a shift-invariance theorem for unique normalized tangent measure distributions.

6.1 Definition and Basic Properties of Normalized Tangent Measure Distributions

We start this section with an example of a nonzero measure $\mu \in \mathcal{M}(\mathbb{R}^n)$, which has no nontrivial α-standardized tangent measure distributions μ-almost everywhere. The example has been studied in a more general form in [Gra93].

Example 6.1.1 Let C be the ternary Cantor set and $x \mapsto (x_i)_{i\in\mathbb{N}}$ the canonical coding of C in the codespace $\prod_{i\in\mathbb{N}}\{0,2\}$. For a given $x \in C$ and $n \in \mathbb{N}$ let

$$I_n(x) = \{a \in C : a_1 = x_1, \dots, a_n = x_n\}$$

and define the mappings $S_1, S_2 : C \longrightarrow C$ by $S_1(x) = y$ with $y_1 = 0$ and $y_{i+1} = x_i$ and similarly $S_2(x) = y$ with $y_1 = 2$ and $y_{i+1} = x_i$.

Let $p_0, p_2 \in (0,1)$ with $p_0 + p_2 = 1$.
We shall study measures μ with $\mu(C) = 1$ which are *self-similar* in the sense that

$$\mu(A) = p_0 \cdot \mu(S_1^{-1}(A)) + p_2 \cdot \mu(S_2^{-1}(A)) \tag{6.1}$$

for all $A \in \mathcal{B}(C)$. For every pair (p_0, p_2) there is exactly one measure $\mu \in \mathcal{M}(\mathbb{R})$ with $\mu(C) = 1$ such that (6.1) holds. For $p_0 = p_2$ we know that μ is Hausdorff measure on C. Assume $p_0 \neq p_2$. The *similarity dimension* of μ is given by

$$\beta = \frac{p_0 \cdot \log p_0 + p_2 \cdot \log p_2}{-\log 3}.$$

Claim 1 *For $\alpha \neq \beta$ there are no non-zero α-standardized tangent measures.*

Proof It suffices to pick $\alpha \neq \beta$ and show that for every $R > 0$ we have

$$\left|\log \frac{\mu(B(x,tR))}{t^\alpha}\right| \stackrel{t\downarrow 0}{\longrightarrow} \infty \quad \text{for } \mu\text{-almost every } x.$$

For this purpose observe that

$$X_i(x) = \log \frac{p_{x_i}}{(1/3)^\alpha}$$

are independent identically distributed random variables on the probability space $(C, \mathcal{B}(C), \mu)$, which take on only two values. We have

$$EX_i = p_0 \cdot \log p_0 + p_2 \cdot \log p_2 + \alpha \cdot \log 3 = (\alpha - \beta) \cdot \log 3 .$$

Hence $EX_i = 0$ if and only if $\alpha = \beta$. In our case, $EX_i \neq 0$ and therefore by the law of large numbers

$$\sum_{i=1}^{n} X_i \longrightarrow \pm\infty \quad \mu\text{-almost everywhere.} \tag{6.2}$$

Suppose now $(1/3)^{n+1} \leq tR \leq (1/3)^n$. Then, for all $x \in C$,

$$\frac{\mu(B(x,tR))}{t^\alpha} \leq \frac{\mu(I_n(x))}{t^\alpha} \leq \prod_{i=1}^{n} \frac{p_{x_i}}{(1/3)^\alpha} \cdot (3R)^\alpha ,$$

and taking the logarithm

$$\log \frac{\mu(B(x,tR))}{t^\alpha} \leq \sum_{i=1}^{n} X_i + \alpha \cdot \log 3 + \alpha \cdot \log R .$$

Analogously we get

$$\log \frac{\mu(B(x,tR))}{t^\alpha} \geq \sum_{i=1}^{n+1} X_i - \alpha \cdot \log 3 + \alpha \cdot \log R ,$$

and thus by (6.2)

$$\left| \log \frac{\mu(B(x,tR))}{t^\alpha} \right| \xrightarrow{t \downarrow 0} \infty \quad \mu\text{-almost everywhere}$$

as required to prove claim 1. □

Claim 2 $\underline{D}_\beta(\mu, x) = \infty$ *and hence there are no β-standardized average tangent measures at μ-almost every point.*

Proof This was shown by S. Graf, see [Gra93]. □

The question whether there is a unique β-standardized tangent measure distribution is left open in [Gra93], but, in fact, we can prove that:

Claim 3 *There are no nontrivial β-standardized tangent measure distributions μ-almost everywhere.*

Proof The proof uses the following lemma from probability theory:

Suppose X_i are independent identically distributed random variables with $EX_i = 0$, $0 < \sigma^2(X_i) < \infty$ and

$$S_n = X_1 + \ldots + X_n .$$

Then for all $\zeta > 0$

$$\frac{1}{N}\sum_{i=1}^{N} 1_{\{|S_i|<\zeta\}} \longrightarrow 0 \quad \textit{almost surely.} \tag{6.3}$$

The following proof of the lemma was suggested by H. v.Weizsäcker:
Let $S_0 = 0$. Fix $\varepsilon > 0$ and denote $I = (-\varepsilon - \zeta, \varepsilon + \zeta)$. First observe that, by the central limit theorem, for every $x \in \mathbb{R}$

$$E\Big(\frac{1}{N}\sum_{i=1}^{N} 1_{\{x+S_i \in I\}}\Big) = \frac{1}{N}\sum_{i=1}^{N} P(\{|x + S_i|/\sqrt{i} < (\varepsilon + \zeta)/\sqrt{i}\}) \overset{N\to\infty}{\longrightarrow} 0 . \tag{6.4}$$

Consider $\Omega = \mathbb{R}^{\mathbf{N}_0}$, $\mathcal{B} = \bigotimes_{n\in\mathbf{N}_0} \mathcal{B}(\mathbb{R})$. Let P_x be the distribution of $(S_n + x)_{n\in\mathbf{N}_0}$ and $\nu = \int P_x \, dx$. Let $\tilde{P}$ be the distribution of X_1.

ν is σ-finite, since $\nu(\Omega) = \sum \nu([k, k+1) \times \mathbb{R}^{\mathbf{N}})$ where the sum extends over all integers k, and

$$\nu([k, k+1) \times \mathbb{R}^{\mathbf{N}}) = \int_k^{k+1} P_x(\Omega) \, dx = 1 .$$

We show that ν is invariant with respect to the left shift T on Ω. For this purpose denote $A = A_0 \times A_1 \times \cdots$ and calculate

$$\begin{aligned}
\nu(T^{-1}A) &= \int P_x(\mathbb{R} \times A_0 \times A_1 \times \cdots) \, dx \\
&= \int P_0(x + S_1 \in A_0, x + S_2 \in A_1, \ldots) \, dx \\
&= \int\int P_0(x + y \in A_0, x + y + X_2 \in A_1, \ldots) \, d\tilde{P}(y) \, dx
\end{aligned}$$

$$\begin{aligned} &= \int\int P_{x+y}(A_0 \times A_1 \times \cdots)\, d\tilde{P}(y)\, dx \\ &= \int\int P_x(A)\, dx\, d\tilde{P}(y) \\ &= \nu(A). \end{aligned}$$

Define $f : \Omega \longrightarrow \{0,1\}$, by $f((\omega_n)) = 1_I(\omega_0)$. Then $f \in L^1(\nu)$, since

$$\int |f|\, d\nu = \int\int 1_I(x)\, dP_x(y)\, dx = 2(\varepsilon + \zeta) < \infty .$$

By the ergodic theorem, applied to $(\Omega, \mathcal{B}, \nu)$, $T : \Omega \to \Omega$ and f, we get that

$$\frac{1}{N} \sum_{i=0}^{N-1} f(T^i\omega) \quad \text{converges } \nu\text{-almost everywhere.}$$

This means that, for Lebesgue-almost every x,

$$\frac{1}{N} \sum_{i=0}^{N-1} 1_{\{x+S_i \in I\}} \quad \text{converges } P_0\text{-almost surely.}$$

Pick such an $x \in U(0,\varepsilon)$. Recalling (6.4) we get

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} 1_{\{|S_i|<\zeta\}} \leq \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} 1_{\{x+S_i\in I\}} = 0$$

for P_0-almost every sequence (S_n). This finishes the proof of the lemma.

Let us return to the proof of claim 3. As before let $X_i(x) = \log \frac{p_{x_i}}{(1/3)^\beta}$ and observe that the random variables X_i on $(C, \mathcal{B}(C), \mu)$ fulfill the requirements of the lemma. Let $x \in C$ be such that (6.3) holds.

Let $\varepsilon_k \downarrow 0$. Suppose there is a tangent measure distribution $P = \lim P^x_{\varepsilon_k} \in \mathcal{P}^\beta(\mu, x)$. By Prohorov's theorem (lemma 1.2.8(3)) the family $(P^x_{\varepsilon_k})_{k\in\mathbb{N}}$ is uniformly tight. This implies, by lemma 1.2.5(1), that for every $\varepsilon > 0$ there is $M > 0$ such that

$$\varphi_{\varepsilon_k}(\{t > 0 \ :\ \tfrac{\mu(B(x,t))}{t^\beta} < M\}) > 1 - \varepsilon \tag{6.5}$$

for all $k \in \mathbb{N}$.

Pick an arbitrarily small $m > 0$. Suppose $(1/3)^{n+1} \le t \le (1/3)^n$. Then

$$\frac{\mu(B(x,t))}{t^\beta} \ge \frac{\mu(I_{n+1}(x))}{t^\beta} \ge \prod_{i=1}^{n+1} \frac{p_{x_i}}{(1/3)^\beta} \cdot (1/3)^\beta$$

and

$$\frac{\mu(B(x,t))}{t^\beta} \le \frac{\mu(I_n(x))}{t^\beta} \le \prod_{i=1}^{n} \frac{p_{x_i}}{(1/3)^\beta} \cdot 3^\beta .$$

If $m \le \frac{\mu(B(x,t))}{t^\beta} < M$ we therefore have

$$\sum_{i=1}^{n+1} X_i \le \log M + \beta \log 3$$

and

$$\sum_{i=1}^{n} X_i \ge \log m - \beta \log 3 .$$

Hence, for a suitable $\zeta > 0$,

$$\{t > 0 \,:\, m \le \tfrac{\mu(B(x,t))}{t^\beta} < M\} \subseteq \bigcup_{j \ge 0} \{[(1/3)^{j+1}, (1/3)^j] \,:\, |\sum_{i=1}^{j} X_i| < \zeta\}.$$

Thus if $(1/3)^{n+1} \le \varepsilon \le (1/3)^n$ we have

$$\begin{aligned} \varphi_\varepsilon(\{t > 0 \,:\, m \le \tfrac{\mu(B(x,t))}{t^\beta} < M\}) &\le \sum_{j=0}^{n} (|\log(1/3)^n|)^{-1} \cdot \log 3 \cdot 1_{\{|\sum_{i=1}^{j} X_i| < \zeta\}} \\ &\le \frac{1}{n} \sum_{j=0}^{n} 1_{\{|S_j| < \zeta\}} , \end{aligned}$$

and since the last term tends to 0 as $n \to \infty$ by (6.3), we have for every $m > 0$, using (6.5),

$$\varphi_{\varepsilon_k}(\{t > 0 \,:\, \tfrac{\mu(B(x,t))}{t^\beta} < m\}) \stackrel{k \to \infty}{\longrightarrow} 1 .$$

By lemma 2.1.3 there is a set $Z \subseteq (0,1)$ such that $\varphi_{\varepsilon_k}(Z) \longrightarrow 0$ and

$$\lim_{\substack{t \downarrow 0 \\ t \notin Z}} \frac{\mu(B(x,t))}{t^\beta} = 0. \tag{6.6}$$

P is concentrated on the set

$$\left\{ \nu = \lim_{n \to \infty} \frac{\mu_{x,t_n}}{t_n^\beta} \,:\, t_n \notin Z, t_n \downarrow 0 \right\},$$

and therefore, by (6.6), on the set $\{\nu \in \mathcal{M}(\mathbb{R}^n) \,:\, \nu(B(0,1)) = 0\}$. Since P is scaling invariant, by proposition 2.2.2, this implies $P(\{\phi\}) = 1$, where ϕ is the zero-measure.

Hence P is the trivial distribution. □

Finally, note that μ fulfills a doubling condition at every $x \in C$ since, for $(1/3)^{n+1} \le t \le (1/3)^n$, we have

$$\frac{\mu(B(x,2t))}{\mu(B(x,t))} \le \frac{\mu(I_{n-1}(x))}{\mu(I_{n+1}(x))} \le (p_{x_{n+1}} p_{x_n})^{-1} \le \frac{1}{p_0^2 p_2^2} \, .$$

This proves the doubling-condition at $x \in C$.

Claims 1 to 3 show that standardized tangent measure distributions in the present form are unsuitable for the investigation of this class of self-similar measures. ■

In order to study measures as in the previous example, C. Bandt has suggested to look at distributions on the set of normalized tangent measure distributions. As in section 1.3 we consider normalizing functions $\Delta : \mathbb{R}^n \longrightarrow [0,\infty)$, which are continuous with bounded support and positive values on a neighbourhood of the origin. The definitions of normalized average tangent measures and tangent measure distributions are now completely analogous to those of standardized average tangent measures and tangent measure distributions.

Definition

Let Δ be a normalizing function and recall

$$\begin{aligned} M^\Delta : \ \mathcal{M}(\mathbb{R}^n) & \longrightarrow \mathcal{M}(\mathbb{R}^n) \, , \\ \nu & \mapsto \begin{cases} \frac{\nu}{\nu(\Delta)} & \text{if } \nu(\Delta) > 0 \, , \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. Define distributions $P_\varepsilon^{\Delta,x}$ by

$$P_\varepsilon^{\Delta,x}(M) := (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_M\Big(M^\Delta(\mu_{x,t})\Big) \, \frac{dt}{t}$$

for Borel sets $M \subseteq \mathcal{M}(\mathbb{R}^n)$. Denote the set of all weak limit points of $(P_\varepsilon^{\Delta,x})_{\varepsilon>0}$ as $\varepsilon \downarrow 0$ by $\mathcal{P}^\Delta(\mu,x)$. The elements of $\mathcal{P}^\Delta(\mu,x)$ are called the *Δ-normalized tangent measure distributions* or *Δ-tangent measure distributions* of μ at x. If the limit

$$P = \lim_{\varepsilon \downarrow 0} P_\varepsilon^{\Delta,x}$$

exists in the weak topology we say that P is the *unique Δ-tangent measure distribution of μ at x*. Define $\bar{\nu}_\varepsilon^{\Delta,x} \in \mathcal{M}(\mathbb{R}^n)$ by

$$\bar{\nu}_\varepsilon^{\Delta,x}(A) := (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu_{x,t}(A)}{\mu_{x,t}(\Delta)} \frac{dt}{t}$$

for Borel sets $A \subseteq \mathbb{R}^n$. The *Δ-normalized average tangent measures* or *Δ-average tangent measures* of μ at x are the limit points with respect to the vague topology of the measures $\bar{\nu}_\varepsilon^{\Delta,x}$ as $\varepsilon \downarrow 0$. If the limit

$$\bar{\nu} = \lim_{\varepsilon \downarrow 0} \bar{\nu}_\varepsilon^{\Delta,x}$$

exists, we say that $\bar{\nu}$ is the *unique Δ-average tangent measure of μ at x*.

Let us state some properties of normalized tangent measure distributions:

Proposition 6.1.2 *For every $x \in \mathbb{R}^n$ the set $\mathcal{P}^\Delta(\mu, x)$ is a weakly closed subset of $\mathcal{P}$ and every Δ-tangent measure distribution P of μ at x fulfills* supp $P \subseteq \mathrm{Tan}^\Delta(\mu, x)$. *In particular, if $x \in$ supp μ every $\nu \in$ supp P fulfills $\nu(\Delta) = 1$ and hence P is nontrivial.*

Proof The proof of the first two statements is completely analogous to the proof in the case of standardized tangent measure distributions (see lemma 2.2.1). If $x \in$ supp μ then $\mu_{x,t}(\Delta) > 0$ for all $t > 0$ and thus, for every tangent measure $\nu = \lim M^\Delta(\mu_{x,t_n}) \in \mathrm{Tan}^\Delta(\mu, x)$, we have $\nu(\Delta) = \lim M^\Delta(\mu_{x,t_n})(\Delta) = 1$. ■

In the following proposition we discuss the connection between different normalized tangent measure distributions and standardized tangent measure distributions.

Proposition 6.1.3

1. *Suppose μ fulfills a doubling condition at x and Δ_1, Δ_2 are two normalizing functions. Then $P \mapsto P \circ (M^{\Delta_2})^{-1}$ is a bijection of $\mathcal{P}^{\Delta_1}(\mu, x)$ onto $\mathcal{P}^{\Delta_2}(\mu, x)$.*

2. *Suppose μ has positive and finite α-densities at x. Then $P \mapsto P \circ (M^\Delta)^{-1}$ is a surjection of $\mathcal{P}^\alpha(\mu, x)$ onto $\mathcal{P}^\Delta(\mu, x)$. In particular, if P is a unique α-standardized*

tangent measure distribution of μ at x, then $P \circ (M^{\Delta})^{-1}$ is a unique Δ-tangent measure distribution of μ at x.

Proof (1) Suppose μ fulfills a doubling condition at x and $P = \lim_{j\to\infty} P^{\Delta,x}_{r_j} \in \mathcal{P}^{\Delta_1}(\mu, x)$. Let $F \in \mathcal{C}_b(\mathcal{M}(\mathbb{R}^n))$. $F \circ M^{\Delta_2}$ is bounded and continuous on the open set $\{\nu \in \mathcal{M}(\mathbb{R}^n) : \nu(\Delta_2) > 0\}$ and, by proposition 1.3.6(3),

$$\mathrm{cl}\,\{M^{\Delta_1}(\mu_{x,t}) \,:\, t \in (0,1)\} \subseteq \{\nu \in \mathcal{M}(\mathbb{R}^n) \,:\, \nu(\Delta_2) > 0\}\,.$$

Hence we also have

$$(|\log r_j|)^{-1} \int_{r_j}^{1} F \circ M^{\Delta_2}(\mu_{x,t})\,\frac{dt}{t} \longrightarrow \int F \circ M^{\Delta_2}\,dP = \int F\,dP \circ (M^{\Delta_2})^{-1}\,.$$

Therefore $P \circ (M^{\Delta_2})^{-1}$ is a Δ_2-tangent measure distribution.
For every Δ_1-tangent measure distribution P we have $P = P \circ (M^{\Delta_2})^{-1} \circ (M^{\Delta_1})^{-1}$. Hence $P \mapsto P \circ M^{\Delta_2}$ is one–to–one. Reversing the rôles of Δ_1 and Δ_2 in the above arguments we see that for every $P \in \mathcal{P}^{\Delta_2}(\mu, x)$ we have $P \circ (M^{\Delta_1})^{-1} \in \mathcal{P}^{\Delta_1}(\mu, x)$ and $P = P \circ (M^{\Delta_1})^{-1} \circ (M^{\Delta_2})^{-1}$. Thus $P \mapsto P \circ M^{\Delta_2}$ is also onto.
(2) Suppose now that μ has positive and finite densities at x and $P = \lim P^{x}_{r_j} \in \mathcal{P}^{\alpha}(\mu, x)$. Let $F \in \mathcal{C}_b(\mathcal{M}(\mathbb{R}^n))$. Then $F \circ M^{\Delta}$ is continuous and bounded on the open set $\{\nu : \nu(\Delta) > 0\}$ and thus

$$\begin{aligned}\lim_{j\to\infty} (|\log r_j|)^{-1} \int_{r_j}^{1} F(M^{\Delta}(\mu_{x,t}))\,\frac{dt}{t} &= \lim_{j\to\infty} (|\log r_j|)^{-1} \int_{r_j}^{1} F \circ M^{\Delta}\Big(\frac{\mu_{x,t}}{t^{\alpha}}\Big)\,\frac{dt}{t} \\ &= \int F\,dP \circ (M^{\Delta})^{-1}\,.\end{aligned}$$

Thus $P \circ (M^{\Delta})^{-1} \in \mathcal{P}^{\Delta}(\mu, x)$.
Suppose now $P = \lim_{j\to\infty} P^{\Delta,x}_{\varepsilon_j} \in \mathcal{P}^{\Delta}(\mu, x)$. By proposition 2.2.3(1) there is a subsequence (r_j) of (ε_j) such that $\tilde{P} = \lim_{j\to\infty} P^{x}_{r_j} \in \mathcal{P}^{\alpha}(\mu, x)$ exists. Then $P = \tilde{P} \circ (M^{\Delta})^{-1}$ and the mapping $P \mapsto P \circ (M^{\Delta})^{-1}$ is onto. ∎

Let us now address the question of existence of Δ-tangent measure distributions and Δ-average tangent measures. If μ fulfills a doubling condition the situation is easy:

Proposition 6.1.4 *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ fulfills a doubling condition at x then*

1. *For every sequence* $(\varepsilon_j)_{j\in\mathbb{N}}$ *with* $\varepsilon_j \downarrow 0$ *there is a subsequence* $(r_j)_{j\in\mathbb{N}}$ *such that* $(P_{r_j}^{\Delta,x})_{j\in\mathbb{N}}$ *converges weakly to a* Δ*-tangent measure distribution of* μ *at* x*.*

2. $\mathcal{P}^\Delta(\mu, x)$ *is weakly closed and weakly connected.*

3. P *is a unique* Δ*-tangent measure distribution of* μ *at* x *if and only if* $\mathcal{P}^\Delta(\mu, x) = \{P\}$*.*

4. *The* Δ*-average tangent measures of* μ *at* x *are the barycentres of the* Δ*-tangent measure distributions of* μ *at* x*, i.e. the set of all* Δ*-average tangent measures of* μ *at* x *is given by*
$$\left\{ \int \nu \, dP(\nu) \,:\, P \in \mathcal{P}^\Delta(\mu, x) \right\}.$$

Proof (1) Because μ fulfills a doubling condition at x we have, by lemma 1.3.6(1), that $\mathrm{cl}\{\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)} : t \in (0,1)\}$ is compact. The measures $(P_\varepsilon^{\Delta,x})_{\varepsilon>0}$ are concentrated on this set, and therefore (1) and the weak compactness of $\mathcal{P}^\Delta(\mu, x)$ are consequences of lemma 1.2.8(3).
(2) As in proposition 2.2.3(2) the weak connectedness follows from the fact that $\mathcal{P}^\Delta(\mu, x) \subseteq \mathcal{P}$ is compact and $\varepsilon \mapsto P_\varepsilon^{\Delta,x}$ is continuous.
(3) If P is a unique Δ-tangent measure distribution then obviously $\mathcal{P}^\Delta(\mu, x) = \{P\}$. If $\mathcal{P}^\Delta(\mu, x) = \{P\}$, then by (1) for every $\varepsilon_k \downarrow 0$ there is a subsequence (r_k) of (ε_k) such that $\lim_{k\to\infty} P_{r_k}^{\Delta,x} = P$. Thus $P = \lim_{\varepsilon\downarrow 0} P_\varepsilon^{\Delta,x}$.
(4) For every $f \in C_c(\mathbb{R}^n)$ the continuous evaluation map $\nu \mapsto \nu(f)$ is bounded on the compact set $\mathrm{cl}\left\{\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)} : t \in (0,1)\right\}$. Therefore, for every tangent measure distribution $P = \lim_{j\to\infty} P_{\varepsilon_j}^{\Delta,x}$, we have
$$\lim_{j\to\infty} (|\log \varepsilon_j|)^{-1} \int_{\varepsilon_j}^1 M^\Delta(\mu_{x,t})(f) \, \frac{dt}{t} = \int \nu(f) \, dP(\nu)$$
and thus the barycentre of P is a Δ-average tangent measure.
On the other hand, given a Δ-average tangent measure $\bar{\nu} = \lim_{j\to\infty} \bar{\nu}_{\varepsilon_j}^{\Delta,x}$ we can use (1) to find a subsequence (r_j) of (ε_j) such that there is a Δ-tangent measure distribution $P = \lim_{j\to\infty} P_{r_j}^{\Delta,x}$. Then the barycentre of P equals $\bar{\nu}$. ∎

But even if μ does not fulfill a doubling condition an existence theorem for normalized tangent measure distributions and average tangent measures holds. This theorem will be stated and proved in section 6.2. Let us now give an analogue to proposition 2.2.5.

Proposition 6.1.5 *Let μ, $\nu \in \mathcal{M}(\mathbb{R}^n)$. Suppose ν fulfills a doubling-condition ν-almost everywhere. If $\mu \ll \nu$ then, for μ-almost every x,*

$$\mathcal{P}^{\Delta}(\mu, x) = \mathcal{P}^{\Delta}(\nu, x) .$$

Proof Let $f = \frac{d\mu}{d\nu}$ be the Radon-Nikodym derivative. For μ-almost every x we have $0 < f(x) < \infty$, $\limsup_{r\to 0} \frac{\nu(B(x,2r))}{\nu(B(x,r))} < \infty$ and, by lemma 1.2.2,

$$\lim_{t\downarrow 0} \frac{\int_{B(x,t)} |f(y) - f(x)|\, d\nu(y)}{\nu(B(x,t))} = 0 .$$

Fix such an $x \in \mathbb{R}^n$. For every $g : \mathbb{R}^n \longrightarrow [0,\infty)$ from $\mathcal{C}_c(\mathbb{R}^n)$, say with supp $g \subseteq B(0,R)$, we get

$$\begin{aligned}
&\left|\frac{\mu_{x,t}}{\nu(B(x,t))}(g) - f(x)\cdot\frac{\nu_{x,t}}{\nu(B(x,t))}(g)\right| \\
&\le \Big(1/\nu(B(x,t))\Big) \left|\int g(\frac{y-x}{t})\cdot(f(y) - f(x))\, d\nu(y)\right| \\
&\le \frac{\int_{B(x,tR)} |f(y) - f(x)|\, d\nu(y)}{\nu(B(x,tR))} \cdot \|g\|_{\sup} \cdot \frac{\nu(B(x,tR))}{\nu(B(x,t))} .
\end{aligned}$$

The first factor in this expression tends to 0 and the last factor remains bounded by the doubling condition. Thus for every $\delta > 0$ there is $T > 0$ with

$$d\left(\frac{\mu_{x,t}}{\nu(B(x,t))}, f(x)\cdot\frac{\nu_{x,t}}{\nu(B(x,t))}\right) < \delta$$

for all $0 < t < T$. It is easy to see that the set

$$E := \mathrm{cl}\left\{\frac{\mu_{x,t}}{\nu(B(x,t))}, f(x)\cdot\frac{\nu_{x,t}}{\nu(B(x,t))}\ :\ t \in (0,1)\right\}$$

is a compact subset of the open set $\{\nu\ :\ \nu(\Delta) > 0\}$. Let $F \in \mathcal{C}_b(\mathcal{M}(\mathbb{R}^n))$ and $\varepsilon > 0$. $F \circ M^{\Delta}$ is uniformly continuous on this subset and thus we can find for every $\varepsilon > 0$ some $T > 0$, such that

$$\left|F\circ M^{\Delta}(\tfrac{\mu_{x,t}}{\nu(B(x,t))}) - F\circ M^{\Delta}(f(x)\cdot\tfrac{\nu_{x,t}}{\nu(B(x,t))})\right| = \left|F(M^{\Delta}(\mu_{x,t})) - F(M^{\Delta}(\nu_{x,t}))\right| < \varepsilon/2$$

for all $0 < t < T$. Therefore, for all sufficiently small $r > 0$,

$$\left|(|\log r|)^{-1}\int_r^1 F(M^{\Delta}(\mu_{x,t}))\,\frac{dt}{t} - (|\log r|)^{-1}\int_r^1 F(M^{\Delta}(\nu_{x,t}))\,\frac{dt}{t}\right|$$

$$\leq \varepsilon/2 + \|F\|_{\sup} \cdot (|\log r|)^{-1} \int_T^1 \frac{dt}{t} < \varepsilon\,.$$

Hence P is a Δ-tangent measure distribution of ν at x if and only if P is a tangent measure distribution of μ at x. ■

We now formulate an analogue to the scale-invariance property 2.2.2 for normalized tangent measure distributions of measures. Define a family of operators (S_λ) on $\mathcal{M}(\mathbb{R}^n)$ by

$$S_\lambda(\nu) = M^\Delta(\nu_{0,\lambda})\,.$$

S_λ is continuous on the set $\{\nu : \nu_{0,\lambda}(\Delta) > 0\}$.

Proposition 6.1.6 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ fulfills a doubling condition at x and let $\lambda > 0$. Then $P = P \circ S_\lambda^{-1}$ for every $P \in \mathcal{P}^\Delta(\mu, x)$.*

Proof Suppose $P = \lim_{j\to\infty} P_{\varepsilon_j}^{\Delta,x}$. S_λ is continuous on the open set $\{\nu : \nu_{0,\lambda}(\Delta) > 0\}$ and we have seen in lemma 2.2.3(2) that the doubling condition implies that this set contains the compact set $\mathrm{cl}\,\{\mu_{x,t}/t^\alpha : t \in (0,1)\}$. For every $F \in \mathcal{C}_b(\mathcal{M}(\mathbb{R}^n))$ we thus get

$$\begin{aligned}
\int F\,dP &= \lim_{j\to\infty} (|\log\varepsilon_j|)^{-1} \int_{\varepsilon_j}^1 F \circ M^\Delta(\mu_{x,t})\,\frac{dt}{t} \\
&= \lim_{j\to\infty} (|\log\varepsilon_j|)^{-1} \int_{\varepsilon_j/\lambda}^{1/\lambda} F \circ M^\Delta(\mu_{x,\lambda t})\,\frac{dt}{t} \\
&= \lim_{j\to\infty} (|\log\varepsilon_j|)^{-1} \int_{\varepsilon_j}^1 F \circ S_\lambda(M^\Delta(\mu_{x,t}))\,\frac{dt}{t} \\
&= \int F\,dP \circ S_\lambda^{-1}\,.
\end{aligned}$$

This proves the statement. ■

Before we finish section 6.1 we recall the example which came at the beginning of this section (example 6.1.1). S. Graf has shown in [Gra93] that self-similar measures fulfilling the open-set-condition (as in example 6.1.1) have unique normalized average tangent measure and unique normalized tangent measure distribution almost everywhere. He also gives an explicit formula for the normalized tangent measure distributions of such a measure.

6.2 An Existence Theorem for Normalized Tangent Measure Distributions

In the previous section we studied an example of a measure μ that did not have a standardized average tangent measure or a nontrivial standardized tangent measure distribution at μ-almost all points. In this section we shall see that this cannot occur for normalized average tangent measures or tangent measure distributions, even if the doubling condition does not hold. The existence theorem below is analogous to the existence theorem for tangent measures, which is proved in [Pre87, Theorem 2.5].

Theorem 6.2.1 *Suppose $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $(\varepsilon_j)_{j\in\mathbb{N}}$ fulfills $\varepsilon_j \downarrow 0$.*
Then, for μ-almost every $x \in \mathbb{R}^n$, there is a subsequence $(\delta_j)_{j\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $(P_{\delta_j}^{\Delta,x})_{j\in\mathbb{N}}$ converges to a Δ-tangent measure distribution of μ at x and $(\bar{\nu}_{\delta_j}^{\Delta,x})_{j\in\mathbb{N}}$ converges to a Δ-average tangent measure of μ at x.

The proof of theorem 6.2.1 requires two lemmas:

Lemma 6.2.2 *For $\mu \in \mathcal{M}(\mathbb{R}^n)$, $B = B(0,s) \subseteq \mathbb{R}^n$ and $R \geq 1 > r > 0$, there is a $C > 0$ such that*

$$\mu(\{x \in B : \int_0^1 \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\, d\varphi_\varepsilon(t) > c\}) \leq c^{-1} \cdot C$$

for all $\varepsilon \in (0,1)$ and $c > 0$.

Proof To begin with fix $0 < t < 1$ and let

$$D = \{(x,y) \in B \times \mathbb{R}^n : |x-y| \leq rt/2\}.$$

Note that, for all $x \in B(y, rt/2) \cap \operatorname{supp} \mu$, we get

$$\frac{\mu(B(x,Rt))}{\mu(B(x,rt))} \leq \frac{\mu(B(y,(R+1)t))}{\mu(B(y,rt/2))}.$$

Using this and Fubini's theorem we get

$$\begin{aligned} &\mathcal{L}^n(B(0,rt/2)) \cdot \int_B \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\, d\mu(x) \\ = &\int_D \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\, d\mu \otimes \mathcal{L}^n(x,y) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n}\int_{B(y,rt/2)\cap B} \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\,d\mu(x)\,d\mathcal{L}^n(y)\\
&\leq \int_{B(B,rt/2)} \mu(B(y,(R+1)t)\,d\mathcal{L}^n(y)\\
&= \int \mathcal{L}^n(B(B,rt/2)\cap B(x,(R+1)t))\,d\mu(x)\\
&\leq \mu(B(B,(R+2)t))\cdot\mathcal{L}^n(B(0,(R+1)t)).
\end{aligned}$$

Thus

$$\int_B \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\,d\mu(x) \leq \mu(B(0,s+R+2))\cdot[2(R+1)/r]^n =: C,$$

and finally

$$\begin{aligned}
&\mu(\{x\in B : \int_0^1 \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\,d\varphi_\varepsilon(t) > c\})\\
&\leq c^{-1}\int_B\int_0^1 \frac{\mu(B(x,Rt))}{\mu(B(x,rt))}\,d\varphi_\varepsilon(t)d\mu(x) \leq c^{-1}\cdot C
\end{aligned}$$

as required. ■

Lemma 6.2.3 *For μ-almost every $x\in\mathbb{R}^n$ there is a subsequence $(r_j)_{j\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$, such that for all $R\in\mathbb{N}$ there is $C_R>0$, such that for all $j\in\mathbb{N}$*

$$\int_0^1 \frac{\mu(B(x,Rt))}{\mu_{x,t}(\Delta)}\,d\varphi_{r_j}(t) \leq C_R.$$

Proof Fix a ball $B=B(0,s)\subseteq\mathbb{R}^n$ and $\delta>0$. Let $r,\eta>0$ be such that $B(0,r)\subseteq\{x : \Delta(x)>\eta\}$. For $R\in\mathbb{N}$ let $C_R=(2^RC)/(\delta\eta)$ where C is chosen according to lemma 6.2.2. From this lemma we know that, for all $\varepsilon\in(0,1)$,

$$\mu(\{x\in B : \int_0^1 \frac{\mu(B(x,Rt))}{\mu_{x,t}(\Delta)}\,d\varphi_\varepsilon > C_R\}) \leq \delta/2^R.$$

Let

$$A_\varepsilon = \{x\in B : \int_0^1 \frac{\mu(B(x,Rt))}{\mu_{x,t}(\Delta)}\,d\varphi_\varepsilon > C_R \text{ for some } R\in\mathbb{N}\ \}$$

and

$$A=\bigcup_{j=1}^\infty\bigcap_{k=j}^\infty A_{\varepsilon_k}.$$

Then $\mu(A_\varepsilon)\leq\delta$ and $\mu(A)\leq\delta$. If $x\in B\setminus A$ there is a subsequence (r_j) of (ε_j) such that $x\notin A_{r_j}$ and this means that, for all $R\in\mathbb{N}$ and $j\in\mathbb{N}$,

$$\int_0^1 \frac{\mu(B(x,Rt))}{\mu_{x,t}(\Delta)}\,d\varphi_{r_j}(t) \le C_R$$

as required. ■

Proof of 6.2.1 From lemma 6.2.3 we see that, for μ-almost all $x \in \mathbb{R}^n$ and all $j \in \mathbb{N}$,

$$\left(\int_0^1 \frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}\,d\varphi_{r_j}(t)\right)(B(0,R)) \le C_R.$$

Let us look at average tangent measures first. By lemma 1.2.5(1) there is a subsequence $(\delta_j)_{j\in\mathbb{N}}$ of $(r_j)_{j\in\mathbb{N}}$ such that $\int_0^1 \frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}\,d\varphi_{\delta_j}(t)$ converges to a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$, which by definition is a Δ-average tangent measure.
Now let us now look at tangent measure distributions. In order to show that $(P_{\delta_j}^{\Delta,x})_{j\in\mathbb{N}}$ has a convergent subsequence it suffices to show, by Prohorov's theorem (see lemma 1.2.8(3)), that for every $\delta > 0$ there is a compact set $E \subseteq \mathcal{M}(\mathbb{R}^n)$ such that $P_{\delta_j}^{\Delta,x}(E) \ge 1-\delta$ for all $j \in \mathbb{N}$. For this purpose let $\delta > 0$ be given. We define $c_R = C_R/(\delta 2^{-R})$ and

$$T = \{t \in (0,1) \,:\, \frac{\mu_{x,t}(B(0,R))}{\mu_{x,t}(\Delta)} \le c_R \text{ for all } R \in \mathbb{N}\,\}\,,$$

and observe that by lemma 1.2.5(1) the set

$$E = \mathrm{cl}\left\{\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)} \,:\, t \in T\right\}$$

is compact. Since

$$\begin{aligned}\varphi_{\delta_j}(\{t \in (0,1) : \frac{\mu_{x,t}(B(0,R))}{\mu_{x,t}(\Delta)} > c_R\}) &\le (c_R)^{-1}\cdot \int_0^1 \frac{\mu_{x,t}(B(0,R))}{\mu_{x,t}(\Delta)}\,d\varphi_{\delta_j}(t)\\ &\le C_R/c_R = \delta\cdot 2^{-R},\end{aligned}$$

we conclude

$$P_{\delta_j}^{\Delta,x}(E^c) \le \varphi_{\delta_j}(T^c) \le \sum_{R\in\mathbb{N}} \varphi_{\delta_j}(\{t \in (0,1) : \frac{\mu_{x,t}(B(0,R))}{\mu_{x,t}(\Delta)} > c_R\}) \le \delta$$

as required for the use of Prohorov's theorem. ■

A final remark should be made: Similarly to the situation with tangent measures (see [O'N95]) this existence result does not necessarily say that normalized tangent measure distributions always provide useful information on the geometry of the measure, although, by proposition 6.1.2, they are nontrivial almost everywhere.

6.3 A Shift–Invariance Theorem for Unique Normalized Tangent Measure Distributions

Fix a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ and a normalizing function Δ. In this section we prove a shift–invarance statement for unique tangent measure distributions, which is analogous to corollary 4.2.3 for standardized tangent measure distributions and to theorem 1.3.10 for tangent measures. We define a *shift–operator* S as follows:
Fix $\delta > 0$ such that $U(0,\delta) \subseteq \{x : \Delta(x) > 0\}$ and let

$$A_\delta = \{(u,\nu) \in \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n) \, : \, U(u,\delta) \cap \operatorname{supp} \nu \neq \emptyset\}.$$

Then S is defined by

$$\begin{array}{rccc} S: & A_\delta & \longrightarrow & \mathcal{M}(\mathbb{R}^n) \\ & (u,\nu) & \mapsto & S^u\nu = \frac{\nu_{u,1}}{\nu_{u,1}(\Delta)}. \end{array}$$

We formulate a shift-invariance theorem for the supports of unique Δ-tangent measure distributions.

Theorem 6.3.1 *For μ-almost every $x \in \mathbb{R}^n$ the following property holds:*
If P is the unique Δ-tangent measure distribution of μ at x, $\nu \in \operatorname{supp} P$ and $u \in \operatorname{supp} \nu$, then $S^u\nu \in \operatorname{supp} P$.

The analogous statement for unique standardized tangent measure distributions has been derived from the Palm formula (see 4.2.3). The proof of theorem 6.3.1 requires two lemmas:

Lemma 6.3.2 *The set A_δ is open in $\mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n)$ and the shift-operator S is continuous on A_δ.*

Proof The map

$$\begin{array}{rccc} S': & \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n) & \longrightarrow & \mathcal{M}(\mathbb{R}^n) \\ & (x,\nu) & \mapsto & \nu_{x,1}\,. \end{array}$$

is continuous by lemma 1.3.4. Hence the set $A_\delta = \{(x,\nu) \in \mathbb{R}^n \times \mathcal{M}(\mathbb{R}^n) \, : \, \nu_{x,1}(U(0,\delta)) > 0\}$ is open. Also S is continuous since $S = M^\Delta \circ S'$ and M^Δ is continuous on the set $S'(A_\delta)$. ∎

Lemma 6.3.3

(a) If $f : (0,1) \longrightarrow [0, C]$ *is a Borel mapping and*

$$\liminf_{r \downarrow 0} \int_0^1 f(t)\, d\varphi_r(t) \geq \lambda C\,,$$

then for $\lambda > \tau > 0$ *we have*

$$\liminf_{r \downarrow 0} \varphi_r(\{t \in (0,1) \,:\, f(t) > \tau \cdot C\}) \geq \frac{\lambda - \tau}{1 - \tau}.$$

(b) Let $E \subseteq \mathbb{R}^n$ *be a Borel set with* $0 < \mu(E) < \infty$*, and* $A \subseteq E \times (0,1)$ *be a Borel set with*

$$\liminf_{r \downarrow 0} \varphi_r(\{t \in (0,1) \,:\, (x,t) \in A\}) \geq \lambda$$

for μ*-almost every* $x \in E$*. Then for all* $\lambda > \tau > 0$ *we have*

$$\liminf_{r \downarrow 0} \varphi_r\Big(\Big\{t \in (0,1) \,:\, \mu(\{x \in E : (x,t) \in A\}) > \tau \cdot \mu(E)\Big\}\Big) \geq \frac{\lambda - \tau}{1 - \tau}.$$

Proof

(a) Suppose for a sequence (s_n) with $s_n \downarrow 0$ we had a constant η such that

$$\varphi_{s_n}(\{t \,:\, f(t) \leq \tau \cdot C\}) \geq \eta > 1 - \frac{\lambda - \tau}{1 - \tau} = \frac{1 - \lambda}{1 - \tau}$$

for all $n \in \mathbb{N}$. Then

$$\int_0^1 f(t)\, d\varphi_{s_n}(t) \leq \eta(\tau \cdot C) + (1 - \eta) C < \lambda C,$$

which is a contradiction to the hypothesis.

(b) By Fubini's theorem we have

$$\int_E \varphi_r(\{t : (x,t) \in A\})\, d\mu(x) = \mu \otimes \varphi_r(A) = \int_0^1 \mu(\{x \in E : (x,t) \in A\})\, d\varphi_r(t).$$

Applying Fatou's lemma to the left-hand side yields

$$\begin{aligned} \liminf_{r \downarrow 0} \int_0^1 \mu(\{x \in E : (x,t) \in A\})\, d\varphi_r(t) &\geq \int_E \liminf_{r \to 0} \varphi_r(\{t : (x,t) \in A\})\, d\mu(x) \\ &\geq \lambda \cdot \mu(E). \end{aligned}$$

Now (a) can be applied with $f(t) = \mu(\{x \in E \,:\, (x,t) \in A\})$ to give the result. ∎

Proof of 6.3.1 Observe that, since $\mathcal{P}$ is a separable metric space by lemma 1.2.8(2), the set

$$E_0 := \{x \in \text{supp } \mu \,:\, \lim_{r\downarrow 0} P_r^x =: P_x \text{ exists } \}$$

is a Borel set and

$$\begin{array}{cccc} P: & E_0 & \longrightarrow & \mathcal{P}(\mathcal{M}(\mathbb{R}^n)) \\ & x & \mapsto & P_x \end{array}$$

is a Borel mapping. The first step in the proof is to achieve a suitable Borel decomposition of the set $B \subseteq E_0$ of all "bad points", i.e. the points where our statement fails.

We can pick dense sequences $(u_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$ and, by lemma 1.2.5(3), $(\nu_i)_{i\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^n)$ and observe that $\mathcal{U} = \{U(u_i, 1/p) : p, i \in \mathbb{N}, 1/p < \delta\}$ and $\mathcal{V} = \{U(\nu_i, 1/p) : p, i \in \mathbb{N}\}$ are countable bases of the topologies of $\mathbb{R}^n$ and $\mathcal{M}(\mathbb{R}^n)$, respectively.
Suppose $x \in E_0$ is a "bad point". Then there is $\nu \in \text{supp } P_x$ and $u \in \text{supp } \nu$ such that $S^u\nu \notin \text{supp } P_x$. By lemma 6.3.2 we can even find a rational number $\varepsilon > 0$ and sets $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that

- $(u, v) \in A_\delta$ for all $u \in U$, $\nu \in V$,
- $d(\text{supp } P_x, S^u\nu) > \varepsilon$ for all $u \in U$, $\nu \in V$,
- $\text{supp } P_x \cap V \neq \emptyset$ and $\text{supp } \nu \cap U \neq \emptyset$ for all $\nu \in V$.

Pick $\nu \in \text{supp } P_x \cap V$ and $u \in \text{supp } \nu \cap U$. By lemma 6.3.2 we can find $U' \in \mathcal{U}$ with $u \in U' \subseteq U$ and $V' \in \mathcal{V}$ with $\nu \in V' \subseteq V$ such that, whenever $\varrho \in V'$ and $y \in U'$,

$$d(S^u\nu, S^y\varrho) \leq \varepsilon/2\,.$$

Say $U' = U(u_j, 1/(p+1))$ with $(1/p) < \delta$ and denote $\lambda := |u_j| + 1/p$.
Using lemma 1.2.4 we can find a set $V'' \in \mathcal{V}$ with $\nu \in V'' \subseteq V'$ such that, for all $\varrho_1, \varrho_2 \in V''$,

(a) $\varrho_1(U(u_j, 1/(p+1))) \geq \nu(U(u_j, 1/(p+1)) \cdot (1/\sqrt{2})$,

(b) $\varrho_1(U(0,\lambda)) \leq \varrho_2(B(0, 2\lambda)) \cdot 2$,

(c) $\varrho_1(B(u_j, 1/p)) \geq \varrho_2(U(u_j, 1/(p+1))) \cdot (1/2)$.

Now say $V'' = U(\nu_k, 1/q)$. Observe that $|u_j - u| < 1/(p+1)$ and $d(\nu_k, \nu) < 1/q$. By (a) and because $u \in \text{supp } \nu$ we have $(u_j, \nu_k) \in A_{1/(p+1)}$. Because $\nu \in \text{supp } P_x$ there is a rational $\eta > 0$ such that $P_x(U(\nu_k, 1/q)) > \eta$.

Let us assume that the set B of "bad points" has $\mu(B) > 0$. We have seen so far that B is contained in the union of countably many Borel sets E_i, each of which is characterized by a family $(\varepsilon, \eta, p, q, u, \nu)$ of parameters such that $\varepsilon, \eta > 0$ are rational numbers, $p, q \in \mathbb{N}$ with $1/p < \delta$, $u \in \{u_i : i \in \mathbb{N}\}$ and $\nu \in \{\nu_i : i \in \mathbb{N}\}$ such that

1. $(u, \nu) \in A_{1/(p+1)}$,
2. for all $y \in \mathbb{R}^n$, $\varrho \in \mathcal{M}(\mathbb{R}^n)$ with $|y - u| \leq 1/p$ and $d(\varrho, \nu) \leq 1/q$ we have $d(S^u\nu, S^y\varrho) \leq \varepsilon/2$,
3. for all $\varrho_1, \varrho_2 \in \mathcal{M}(\mathbb{R}^n)$ with $d(\varrho_i, \nu) \leq 1/q$ we have $\varrho_1(B(u, 1/p)) \geq (1/2) \cdot \varrho_2(U(u, 1/(p+1)))$ and $\varrho_1(U(0, \lambda)) \leq 2 \cdot \varrho_2(B(0, 2\lambda))$ for $\lambda = |u| + (1/p)$,

and E_i is the Borel set of all x fulfilling

4. $P_x(U(\nu, 1/q)) > \eta$,
5. $d(\text{supp } P_x, S^u\nu) > \varepsilon$.

By our assumption one of the sets E_i must have $\mu(E_i) > 0$, and without losing generality we can assume that $\mu(E_i) < \infty$. We denote this set E and its characterizing parameters $(\varepsilon, \eta, p, q, u, \nu)$. Define

$$\kappa = \frac{\nu(U(u, 1/(p+1)))}{8 \cdot \nu(B(0, 2\lambda))}.$$

Since $U(u, 1/(p+1)) \subseteq B(0, 2\lambda)$ and, by (1), $\nu(U(u, 1/(p+1))) > 0$, κ is well-defined and we have $0 < \kappa \leq 1/8$.

The second step in the proof is to show that, for μ-almost all $x \in E$, we have

$$\liminf_{r\downarrow 0} \varphi_r\Big(\Big\{t \in (0,1) : \mu(B(x+tu, t/p) \cap E) \geq \kappa \cdot \mu(B(x, \lambda t)) \text{ and }$$

$$d(\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \nu) < 1/q\Big\}\Big) \geq \eta. \tag{6.7}$$

For the proof we observe that, by lemma 1.2.3, there is a μ-density point x of E. If t is such that $d(\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \nu) < 1/q$ we get, using (3),

$$\begin{aligned} \mu_{x,t}(B(u, 1/p)) &\geq \mu_{x,t}(\Delta) \cdot (1/2) \cdot \nu(U(u, 1/(p+1))) \\ &= \mu_{x,t}(\Delta) \cdot 4\kappa \cdot \nu(B(0, 2\lambda)) \\ &\geq 2\kappa \cdot \mu_{x,t}(B(0,\lambda)), \end{aligned}$$

and using (4)

$$\liminf_{r\downarrow 0} \varphi_r(\{t \in (0,1) : d(\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \nu) < 1/q\}) \geq P_x(U(\nu, 1/q)) > \eta.$$

If (6.7) did not hold, we could find a sequence $t_n \downarrow 0$ such that

$$\mu(B(x + t_n u, t_n/p) \cap E) < \kappa \cdot \mu(B(x, \lambda t_n)) \quad \text{and} \quad d(\frac{\mu_{x,t_n}}{\mu_{x,t_n}(\Delta)}, \nu) < 1/q.$$

Then

$$\begin{aligned} 2\kappa &\leq \frac{\mu_{x,t_n}(B(u,1/p))}{\mu_{x,t_n}(B(0,\lambda))} = \frac{\mu(B(x+t_n u, t_n/p))}{\mu(B(x,\lambda t_n))} \\ &= \frac{\mu(B(x+t_n u, t_n/p) \cap E)}{\mu(B(x,\lambda t_n))} + \frac{\mu(B(x+t_n u, t_n/p) \setminus E)}{\mu(B(x,\lambda t_n))} \\ &< \kappa + \frac{\mu(B(x,\lambda t_n) \setminus E)}{\mu(B(x,\lambda t_n))} \end{aligned}$$

and, since the last summand tends to 0 as n tends to ∞, this is a contradiction. Thus (6.7) is proved.

Now apply lemma 6.3.3(b) to get, for all $\eta > \tau > 0$,

$$\liminf_{r\downarrow 0} \varphi_r\Big(\Big\{t : \mu(\{x \in E : \mu(B(x+tu, t/p) \cap E) \geq \kappa \cdot \mu(B(x, \lambda t)) \text{ and }$$

$$d(\tfrac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \nu) < 1/q\}) \geq \tau \cdot \mu(E)\Big\}\Big) \geq \frac{\eta - \tau}{1 - \tau}. \tag{6.8}$$

For every $x \in E$ let

$$T(x) = \{t \in (0,1) : d(\tfrac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \text{supp } P_x) < \varepsilon/2\}.$$

Since

$$\liminf_{r \downarrow 0} \varphi_r(T(x)) \geq P_x(\{\nu \in \mathcal{M}(\mathbb{R}^n) : d(\nu, \text{supp } P_x) < \varepsilon/2\}) = 1\,,$$

we can apply lemma 6.3.3(b) again and get, for all $0 < \tau < 1$,

$$\lim_{r \downarrow 0} \varphi_r\Big(\Big\{t \in (0,1) : \mu(\{y \in E : t \in T(y)\}) > \tau \cdot \mu(E)\Big\}\Big) = 1. \tag{6.9}$$

Using (6.8) and (6.9) we can now finish the proof as follows:

Let $N \in \mathbb{N}$ be the constant appearing in Besicovitch's covering theorem, see lemma 1.2.1, and denote $c = (1/4)\kappa\eta/N > 0$. Using (6.8) and (6.9), we can pick a $t \in (0,1)$, such that there are Borel sets $A_1, A_2 \subseteq E$ such that $\mu(A_1) \geq (\eta/2)\cdot\mu(E)$ and $\mu(A_2) \geq (1-c)\cdot\mu(E)$, and such that

6. $\mu(B(x+tu, t/p) \cap E) \geq \kappa \cdot \mu(B(x, \lambda t))$ and $d(\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \nu) < 1/q$ for all $x \in A_1$,

7. $t \in T(x)$ for all $x \in A_2$.

Cover A_1 with the family

$$\mathcal{B} := \{B(x, \lambda t) : x \in A_1\}$$

of closed balls of fixed radius.

By Besicovitch's covering theorem we can find a disjoint subfamily

$$\mathcal{B}' = \{B(x_i, \lambda t) : i \in I\} \subseteq \mathcal{B}$$

such that

$$\mu(\bigcup_{i \in I} B(x_i, \lambda t) \cap A_1) \geq \frac{1}{N} \cdot \mu(A_1).$$

Since $\mathcal{B}'$ is disjoint I is at most countable. Let

$$B := \bigcup_{i \in I} B(x_i + tu, t/p) \cap E.$$

This union is disjoint since, by the definition of λ, $B(x_i + tu, t/p) \cap E \subseteq B(x_i, \lambda t)$. Using (6) in the second step we get

$$\begin{aligned}
\mu(B) &= \sum_{i \in I} \mu(B(x_i + tu, t/p) \cap E) \\
&\geq \sum_{i \in I} \kappa \cdot \mu(B(x_i, \lambda t) \cap E) \\
&\geq \kappa \cdot \mu(\bigcup_{i \in I} B(x_i, \lambda t) \cap A_1) \\
&\geq \frac{\kappa}{N} \cdot \mu(A_1) \geq \frac{\kappa}{2N} \cdot \eta \cdot \mu(E) > c \cdot \mu(E).
\end{aligned}$$

Therefore we can find $y \in A_2 \cap B$, which implies, by definition of B, that there is $x \in A_1$ such that $|y - (x + tu)| \leq t/p$ and thus

$$\Big|\frac{y - x}{t} - u\Big| \leq 1/p\,.$$

Moreover, by (6) we have

$$d\Big(\frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}, \nu\Big) < 1/q\,.$$

Using also (2) these facts imply that

$$d\Big(S^u \nu, \frac{\mu_{y,t}}{\mu_{y,t}(\Delta)}\Big) = d\Big(S^u \nu, S^{(y-x)/t} \frac{\mu_{x,t}}{\mu_{x,t}(\Delta)}\Big) \leq \frac{\varepsilon}{2}.$$

Since $y \in A_2$ we have, by (7),

$$d(\text{supp } P_y, \frac{\mu_{y,t}}{\mu_{y,t}(\Delta)}) < \frac{\varepsilon}{2}\,,$$

and hence, using (5),

$$\begin{aligned}
\varepsilon &< d(\text{supp } P_y, S^u \nu) \\
&\leq d(\text{supp } P_y, \frac{\mu_{y,t}}{\mu_{y,t}(\Delta)}) + d(\frac{\mu_{y,t}}{\mu_{y,t}(\Delta)}, S^u \nu) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,
\end{aligned}$$

a contradiction, which proves the statement. ■

Chapter 7

Summary and Outlook

In the preceding chapters of this thesis we have investigated sets and measures in Euclidean spaces by means of their average densities, average tangent measures and tangent measure distributions. In this chapter we give a short summary of the thesis, indicating the main results and pointing out some questions which remain open.

In the introductory chapter we introduce the objects of study: Sets and measures in Euclidean spaces. A survey of some important definitions and known results in geometric measure theory is given. These results show how the notions of densities and tangent measures reflect the fundamental difference between the behaviour of rectifiable measures on the one hand and non-rectifiable or fractal measures on the other hand. They also motivate the introduction of average densities and tangent measure distributions as tools for the study of non-rectifiable measures.

In the second chapter we give definitions and some basic results on average densites, standardized average tangent measures and standardized tangent measure distributions. We study several examples. The most important new contribution is the construction of an example of a measure with positive and finite densities which has unique average tangent measures but non-unique tangent measure distributions almost everywhere (see proposition 2.3.2(b)).

In the third chapter we approach the description of one-sided average densities of measures on the line by showing that they exhibit a completely different behaviour than one-sided ordinary densities: We prove that for a measure with bounded α-densities almost everywhere the one-sided lower average α-densities can only vanish on a set of measure zero (see theorem 3.1.3). In the course of the proof we provide lemmas on the geometry of the measure which are interesting in their own right and which will also be of use in the proof of the main result.

In the fourth chapter we study measures with unique tangent measure distributions. We prove that, for a measure with positive and finite α-densities almost everywhere, at almost every point the unique tangent measure distribution, if it exists, is a Palm distribution (see theorem 4.2.1). This result yields an interesting connection to the theory of self-similar random measures: The unique tangent measure distributions define α-self similar random measures in the axiomatic sense of U. Zähle. We give some applications of the result.

In the fifth chapter we prove the main result of this thesis: We investigate tangent measure distributions of measures on the line without imposing any uniqueness conditions. We prove that, for a measure on the line with positive and finite α-densities almost everywhere, at almost all points all tangent measure distributions are Palm distributions and therefore define α-self similar random measures (see theorem 5.1.1). This result has a couple of interesting consequences, like a local symmetry principle (see theorem 5.1.3), or a complete description of the one-sided average densities of a measure in terms of its lower and upper average densities (see corollary 5.1.4).

In the sixth chapter we introduce the normalized tangent measure distributions. We give an example of a measure which has a unique normalized tangent measure distribution but no non-trivial standardized tangent measure distributions at almost every point (see example 6.1.1). We prove an existence theorem for normalized tangent measure distributions (see theorem 6.2.1) and a shift-invariance theorem for unique normalized tangent measure distributions (see theorem 6.3.1).

Here are some natural questions one could approach in future research:

- *Can the condition of positive lower density be weakened in theorem 3.1.3 and theorem 5.1.1 ?*

  Recall from section 4.3 that the proof of the Palm formula (4.1) does not require the lower densities to be positive almost everywhere. However, in the proof of theorem 3.1.3 and theorem 5.1.1 this condition is needed (it is required for lemma 3.2.4).

- *Can the statement of theorem 5.1.1 be generalized to measures in higher dimensions?*
  This is subject of current research.

- *Is there an analogue of the Palm property for normalized tangent measure distributions?*

  Observe that theorem 6.3.1 is an analogue of corollary 4.2.3, which was an immediate consequence of the Palm property.

- *To what extent can order-two notions like average densities and tangent measure distributions be used to solve problems from geometric measure theory (e.g. regularity problems)?*

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