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The Volume Preserving Mean Curvature Flow in a Compact Riemannian Manifold

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I, Mattia Miglioranza, confirm that
the work presented in this thesis is my own.
Where information has been derived from other sources,
I confirm that this has been indicated in the thesis.

Abstract

In this thesis we investigate the volume preserving mean curvature flow (VPMCF) of a closed and convex hypersurface M inside of a compact Riemannian manifold N. When the ambient manifold is the Euclidean space, long time existence and convergence of the solution to a sphere have been already proved. In the general Riemannian case, this approach cannot be readily generalised, because of the interaction between the evolving hypersurface and the geometry of the ambient space.
Alikakos and Freire overcome these difficulties, using although an infinite-dimensional dynamical systems approach and results from semigroup theory. In our work, instead, we offer a classical and more geometric outlook. We therefore exploit the isoperimetric nature of the flow: the hypersurface M is in fact moving inside N in a way to keep the volume of the region it encloses fixed, while its area is strictly decreasing. Thanks to this isoperimetric characteristic, we prove that, if the initial hypersurface is close enough to a small geodesic ball in N (a *bubble*), it keeps itself close even at the final existence time T (short time existence). The last fact, combined with good estimates of the major geometric quantities of M, allows us to extend the flow indefinitely for all times (*immortal flow*) and therefore to study its asymptotic behaviour. This is quite interesting, since, except for special cases, geodesic spheres are not equilibria for the VPMCF and, in general, the existence of time independent solutions is a non trivial issue.
We conclude our work by studying the asymptotic behaviour of a solution of the VPMCF. We prove that there exists at least a subsequence of times such that a subsequence of the family of bubbles converges to a limit surface of constant mean curvature.

Impact statement

Mean curvature flow has been a very productive and flourishing area of the Geometric Analysis over the past decades. It arises naturally in problems where a surface energy is relevant because of its property of being the gradient-like flow of the Area functional. It occurs, for example, in the description of the evolution of the interfaces in several multiphase physical models and, very recently, algorithms based on MCF have been developed extensively in the field of automatic treatment of digital data, in particular of images, due mostly to its parabolic nature.

From a more theoretical perspective, motivations to study the MCF come from geometric applications, in analogy with the Ricci flow: the techniques here developed have been widely used as a tool to obtain classification results for hypersurfaces with certain curvature conditions, to derive isoperimetric inequalities or to produce minimal surfaces. It is therefore clear that a research work in the mean curvature flow or, as in this present thesis, in the volume preserving mean curvature, it is relevant for both theoretical and applicative aspects.

Our research project offers results already known in the literature. However, these existing results have been proved by using techniques coming from very different areas of mathematics, based on centre manifold analysis, with tools from semigroup theory or infinite dimensional systems, and therefore far from the traditional approach. One of the main problems that comes by using these techniques is that it is pretty much unclear how the shape of the initial hypersurface affects the convergence of the flow.

It has been felt the necessity to present these results in a more classical way. Our work is very effective and geometric: it proceeds in an intrinsic fashion, by starting with natural conditions on the geometry of the initial hypersurface, deducing subsequently some important geometric properties and studying the evolution equation of the main geometric quantities.

In our work, the reader will immediately feel at ease, thanks to the maximum principle for parabolic equations and by exploiting the isoperimetric nature of the flow.

Observe that the traditional approach has never been attempted so far. The reason is simply because is not an easy task, since in the Riemannian setting the flow is a result of a complicated interaction between the geometry of the evolving hypersurface and the geometry of the ambient space, and, therefore, standard results like convexity properties may not be preserved if the initial surface is immersed in a general Riemannian manifold.

Our work can also be considered as a starting point for further developments and results.

In fact, it would be desirable to obtain, similarly to the Euclidean case, the fully convergence of the flow and the uniqueness of such limit. Moreover, it would be interesting to prove that the limit constant mean curvature sphere is a leaf of the local foliation around a critical point of the scalar curvature, which is assumed to be nondegenerate.

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It is time to go, a new story needs to be told,

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Contents

Introduction

The mean curvature flow is a nonlinear geometric evolution equation where a submanifold evolves in the direction of the gradient of the area functional, in order to decrease over time its area as rapidly as possible. Since during the evolution each point moves in the direction of the mean curvature vector with speed given by the mean curvature, we have that convex points move inwards, concave points move outwards, and the manifold moves faster where the curvature is larger. Therefore MCF can be seen as the negative gradient flow for the area functional and it is formally similar to the ordinary heat equation, with some important differences: MCF behaves like the heat equation only for a short time, making the solution smoother; however, after more time, the nonlinearities dominate and the solution becomes singular.
The mean curvature flow originated in the materials science literature, where for almost a century it has been used to model structures such as cell, grain and bubble growth. For example, around the 1950s, von Neumann studied soap foams whose interface tends to have constant mean curvature, whereas Mullins describes coarsening in metals. Partly as a consequence, Mullins [33] might have been the first to write down explicitly the MCF equation in general. Subsequently, MCF and related flows have been studied extensively in applied mathematics, image processing and other areas of science and engineering.
As we briefly explain in the second chapter, the simplest case is when the submanifold is a simple closed curve. A remarkable result of Grayson [15] from 1987, based on a joint work of Gage and Hamilton [12], shows that any simple closed curve in the plane remains smooth under the flow until it disappears in a point in a finite amount of time. Right before it shrinks to a point, the curve will be an almost round circle: by Grayson's theorem, the curve thus remains smooth until its length (area) becomes 0 and, as a corollary, one gets an exact formula for the lifespan of any curve. Therefore, in the case of curve shortening flow, as is called the MCF in the one dimensional case, each flow has only one singularity in all of space and time and the singularity looks just like a shrinking circle.
In higher dimensions, Huisken [18] proved in 1984 that closed convex hypersurfaces remain convex and flow smoothly until they become extinct at a point; in particular, they are almost round just before extinction. However, unlike the case of curves, there are many new types of singularities when the initial hypersurface is not convex. Therefore the analogue of Grayson's theorem does not hold for submanifolds of dimension $n \geq 2$. The main tool for analysing these different types of singularities is a blow-up method,

similar to tangent cone analysis for minimal varieties, that relies on Huisken's monotonicity formula. The flow once again has a self-similar structure near the singularities, but there are infinite families of different possible self-similar structures. Recent years have seen a great and flourishing activity in this area, with constructing examples, classifying the possibilities in certain cases, and understanding which types of singularities are generic.

In the present work, we want to investigate the volume preserving mean curvature flow (VPMCF) of a closed and convex hypersurface M inside of a compact Riemannian manifold N. We thus consider a family of immersions $F : M \times [0, T) \to N$ which satisfies

$$\frac{\partial}{\partial t} F(x,t) = [-H(x,t) + \phi(t)] \cdot \nu(x,t),$$

where

$$\phi(t) = \frac{1}{|M_t|} \int_M H d\mu_t,$$

and $\nu(x,t)$ is the unit normal vector to $M_t = F_t(M)$ in $x \in M_t$ and $|M_t|$ is the surface area of M_t at the time $t \in [0, T)$. Thus under VPMCF the family of immersions are evolving in a way to keep fixed the volume of the region enclosed by M_t.

Having in mind the crucial result of Huisken, where every compact convex hypersurface evolving by standard mean curvature shrinks to a point in a finite time and becomes spherical under rescaling, a behaviour which is usually called convergence "to a round point", one expects that under the volume preserving mean curvature flow the evolution of a convex hypersurface is defined for all times and converges, in the Euclidean space, to a sphere as $t \to \infty$.

When the ambient manifold N is indeed the Euclidean space, Huisken [20], for example, provided the expected result, by exploiting an initial condition on the principal curvatures. We have in fact that:

Theorem. *If the initial hypersurface $M_0^n \subset \mathbb{R}^{n+1}$, $n \geq 2$, is uniformly convex, then the VPMCF has a smooth solution M_t for all times $0 \leq t < \infty$ and the M_t converges to a round sphere enclosing the same volume as M_0 in the C^∞-topology as $t \to \infty$.*

However, in the general Riemannian case, the flow is a result of a complicated interaction between the geometry of the evolving hypersurface and the geometry of the ambient space, and the methods developed by Huisken himself cannot be readily generalised, since, in view of the term $\phi(t)$ in the evolution equation, the local evolution of M depends heavily on the global shape of the hypersurface inside N. Moreover, it can be shown that even the convexity properties of M_0 may not be preserved if M_0 is immersed in a general Riemannian manifold. The main reason is that the evolution law is non-local and thus the maximum principle for parabolic equations is much more subtle and difficult to apply: as a consequence, initially embedded hypersurfaces may develop self-interactions.

In this present work we nevertheless want to prove that if the initial hypersurface M_0 is "close enough" to a small geodesic sphere, the volume preserving mean curvature flow has a smooth convex solution and exists for all positive times.

The main ingredient in our work is the isoperimetric nature of this flow: the volume of the region enclosed initially by M_0 is preserved under the flow, while the area of M_t is strictly decreasing over time. Short time existence for these types of flows are already known in literature. However, by exploiting its isoperimetric nature, we are able to show that if our initial hypersurface M_0 is close enough to a small geodesic sphere in N, it keeps itself close even at the maximal existence time T. The last fact, combined with good estimates of the major geometric quantities of M_T, allows us to extend the flow indefinitely for all times and therefore to study its asymptotic behaviour. This is quite interesting, since, except for special cases, geodesic spheres are not equilibria for the VPMCF and, in general, the existence of time independent solutions is a nontrivial issue.

Our approach is very effective and geometric and proceeds in intrinsic fashion, by studying the evolution of the geometric quantities of M. In the first chapter we introduce some useful notions and definitions of Riemaniann geometry and we derive the evolution equations of the main geometric quantities which are going to be essential in our work, together with the maximum principle for parabolic equations, and we conclude the chapter with some important isoperimetric inequalities in both Euclidean and Riemannian settings.

We then continue with a review of the main existing results in the literature. After starting with Huisken's cornerstone work in MCF and the generalisations to other flows where the speed is represented by symmetric, positively homogeneous functions of the principal curvatures, we introduce the main results in the volume preserving mean curvature flow in the Euclidean ambient manifold and the further attempts to generalise the same results to the noneuclidean case. In particular, we dedicate time to introduce the approach of Alikakos and Freire who obtain our same results. Their work is interesting mainly because it is entirely based on semigroup theory and dynamical systems analysis, therefore far from the traditional approach which we do follow here.

In Chapters 3 and 4 we give a full, complete proof of our main result both in the Euclidean (Ch. 3) and Riemannian case (Ch.4). The reason to distinguish between the flat and non-flat case arises for two main purposes: firstly, even if the exact same results are already known in the literature, dealing with the Euclidean case allows us to explain well the ideas and the strategy which we are going to apply to the Riemannian case, given the fact that all the quantities and the evolution equations are much simpler and easy to treat, since the Riemann tensor is null. On the other hand, we are not able to replicate the full convergence to a limiting sphere and the exponential decay of the speed of the flow (and therefore of the other main quantities), if the ambient manifold is non-flat. In the Euclidean space, in fact, the full convergence is assured by the combination of three main facts: the volume-preserving nature of the flow, the use of the Alexandrov's Theorem (the only compact embedded hypersurfaces with constant mean curvature are the round spheres) and indeed the fact that the speed decays exponentially. In a general Riemannian manifold, the Alexandrov's Theorem does not hold and the average term of the mean curvature introduces a non-local effect to all the evolution equations, which are now much more complicated to treat. Therefore we are only able to guarantee the existence of a subsequence of surfaces converging to a surface of constant mean curvature. However, this leads to problematic situations, like the fact that the solution of the flow could converge to two (or more) different surfaces of the same constant mean curvature $\bar{H}$, or that the flow could move around the limit surface without ever reaching it. It would be interesting to overcome these problematic situations and prove full convergence also in the Riemannian case, but this needs the introduction of new ideas and techniques, which however goes beyond the scope of the present work.

We conclude this Introduction by summarizing the main results that we present in this thesis.
In both the Euclidean and the Riemannian case, we have proved the monotonicity of the isoperimetric ratio. When the ambient manifold is the flat Euclidean space, we have established the long time existence for this particular class of VPMCF, the exponential decay of the speed of the flow and of the main geometric quantities, which leads to the full convergence of the solution to the unit sphere at an exponential rate. In the Riemannian case, we have shown the long time existence of the flow and the subsequential convergence of the solution to a small bubble of constant mean curvature.

Chapter 1

Preliminary results

1.1 Hypersurfaces in Riemannian manifolds

Let $(N, \bar{g})$ be a compact Riemannian manifold of dimension $n+1$. We denote by a bar all the quantities on N. For example we write the metric with $\bar{g} = \bar{g}_{\alpha\beta}$ and $1 \leq \alpha, \beta \leq n+1$, the coordinates by $\bar{y} = \{\bar{y}^\alpha\}$, the Levi-Civita connection $\bar{\Gamma} = \{\bar{\Gamma}^\gamma_{\alpha\beta}\}$, by $\bar{\nabla}$ the covariant derivative and by $\bar{R}_{\alpha\beta\gamma\delta}$ the Riemann tensor. We always make use of the Einstein summation convention for the sum of repeated indices, unless otherwise specified. Therefore we write the Ricci tensor as $\overline{Ric} = \{\bar{R}_{\alpha\beta}\}$ with $\bar{R}_{\alpha\beta} = \bar{g}^{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta}$, and $\bar{R} = \bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta}$ the scalar curvature of $\bar{N}$, where $\bar{g}^{-1} = \bar{g}^{\alpha\beta}$ is the inverse metric of $\bar{g}$.

Let now $F : M \to N$ be a smooth hypersurface immersion, i.e. a smooth map such that its differential F_* is injective at each point and where M is a closed Riemaniann manifold of dimension n, therefore a compact topological space without boundary. We denote the induced metric on M by g and in local coordinates we have

$$g_{ij}(p) = \bar{g}\Big(\frac{\partial F}{\partial x_i}(p), \frac{\partial F}{\partial x_j}(p)\Big) = \bar{g}_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}(p)\frac{\partial F^\beta}{\partial x^j}(p),$$

for any $p \in M$. We furthermore denote without a bar the intrinsic geometry of the induced metric g on the hypersurface, i.e. $\{\Gamma^i_{jk}\}$, ∇ and R_{ijkl} with Latin indices i, j, k, l ranging from 1 to n.
If ν is a local choice of unit normal for $F(M)$, we often work in an adapted orthonormal frame $\{\nu, e_1, \ldots, e_n\}$ in a neighbourhood of $F(M)$ such that $e_1(p), \ldots, e_n(p) \in T_pM \subset T_pN$ and $g(e_i, e_j)(p) = \delta_{ij}$ for $p \in M$, $1 \leq i, j \leq n$.
Then the second fundamental form $A = h_{ij}$ as a bilinear form

$$A(p) : T_pM \times T_pM \to \mathbb{R},$$

and the Weingarten map $W = h^j_i = g^{ik}h_{kj}$ as an operator

$$W : T_pM \to T_pM,$$

are given by

$$h_{ij} = \bar{g}\,(\bar{\nabla}_{e_i}\nu, e_j) = -\bar{g}\,(\nu, \bar{\nabla}_{e_i}e_j).$$

Since W is a selfadjoint operator, we have that $A(p)$ is symmetric and its eigenvalues $k_1(p), \dots, k_n(p)$ are called the principal curvatures of $F(M)$ at $F(p)$. At any given point $p \in M$ it is always possible to choose normal coordinates and, possibly after a rotation, we can always arrange that

$$g_{ij} = \delta_{ij}, \qquad \nabla_{e_i}e_j = 0, \qquad h_i^j = \mathrm{diag}(k_1, \dots, k_n).$$

We also have that the classical scalar invariants of the second fundamental form are symmetric homogeneous polynomials in the principal curvatures. We can write the mean curvature as

$$H = g^{ij}h_{ij} = k_1 + \cdots + k_n,$$

and the total curvature as

$$|A|^2 = h_i^j h_j^i = k_1^2 + \cdots + k_n^2.$$

It is also important to recall the rules of computations involving the covariant derivatives, the second fundamental form of the hypersurface and the curvature of the ambient space. The commutator of the second derivatives of a vector field X on M is therefore given by

$$\nabla_i\nabla_j X^k - \nabla_j\nabla_i X^k = R_{ijlm}g^{kl}X^m,$$

and for a one-form ω on M by

$$\nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k = -R_{ij\ k}^{\ \ m}\omega_m = R_{ijkl}g^{lm}\omega_m.$$

Since we are dealing with hypersurfaces immersed in a Riemannian manifold, it is useful to recall the Gauss equations which relate the curvature of M with the one of N:

$$\begin{aligned} R_{ijkl} &= \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk} \qquad && 1 \le i,j,k,l \le n, \\ R_{ik} &= \bar{R}_{ik} - \bar{R}_{nink} + Hh_{ik} - h_{il}h_k^l && 1 \le i,k \le n, \\ R &= \bar{R} - 2\bar{R}_{nn} + H^2 - |A|^2, \end{aligned}$$

where with the index n we indicate the normal direction ν, so for example $\bar{R}_{nn} = \overline{Ric}(\nu,\nu)$. We also keep in mind the Codazzi-Mainardi equations:

$$\begin{aligned} \nabla_i h_{jk} - \nabla_k h_{ij} &= \bar{R}_{njki} \\ \nabla_i h_{ik} - \nabla_k H &= \bar{R}_{nk}. \end{aligned}$$

We want to introduce the following commutator identities for the second derivatives of the second fundamental form as in [21], which were first found by Simons [45]. They provide a crucial link between analytical methods and geometric properties of M and N. See also [39] for a derivation of the following facts from the structure equations.

Theorem 1.1. *The second derivatives of the second fundamental form A satisfies the following identity:*

$$\begin{aligned}\nabla_k\nabla_l h_{ij} &= \nabla_i\nabla_j h_{kl} + h_{kl}h_{im}h_{mj} - h_{km}h_{il}h_{mj} + h_{kj}h_{im}h_{ml} \\ &- h_{km}h_{ij}h_{ml} + \bar{R}_{kilm}h_{mj} + \bar{R}_{kijm}h_{ml} \\ &+ \bar{R}_{mjil}h_{km} + \bar{R}_{ninj}h_{kl} - \bar{R}_{nknl}h_{ij} + \bar{R}_{mljk}h_{im} \\ &+ \bar{\nabla}_k\bar{R}_{njil} + \bar{\nabla}_i\bar{R}_{nljk}.\end{aligned}$$

Proof. See for example [21]. □

If we trace the previous identity, we obtain the following result that plays an important role in the mean curvature theory.

Corollary 1.2. *The Laplacian of the second fundamental form A satisfies the identity*

$$\begin{aligned}\Delta h_{ij} &= \nabla_i\nabla_j H + Hh_{im}h_{mj} - h_{ij}|A|^2 + H\bar{R}_{ninj} \\ &- \bar{R}_{nn}h_{ij} + \bar{R}_{kikm}h_{mj} + \bar{R}_{kjkm}h_{im} \\ &- 2\bar{R}_{kimj}h_{km} + \bar{\nabla}_k\bar{R}_{njik} + \bar{\nabla}_i\bar{R}_{nkjk}.\end{aligned}$$

1.1.1 The Exponential Map

In our work we want to investigate how an immersed surface M moves inside a compact manifold N accordingly to a particular law that we will define later. In order to better understand the geometry involved in this process, we need to clarify some more ideas of Riemaniann geometry.
We first start with a general manifold $(N, \bar{g})$. Let p be a point in N and $V \in T_pN$ a tangent vector. We indicate with γ_V the unique geodesic such that $\gamma(0) = p$ and $\dot{\gamma}(0) = V$, and let $[0, l)$ be the nonnegative part of the maximal interval containing 0 on which γ is defined. For any $\alpha > 0$, we have the *rescaling property* $\gamma_{\alpha V}(t) = \gamma_V(\alpha t)$, with $t < l_\alpha$ (the maximal existence time for $\gamma_{\alpha V}$) and with $l_\alpha = \alpha^{-1}\, l$.
We therefore consider $O_p \subset T_pN$ as the set of vectors V such that $1 < l$, so that γ_V is defined on $[0, 1]$.

Definition 1.1. The *restricted exponential map* at p is the map $O_p \to N$ such that

$$\exp_p(V) = \gamma_V(1), \quad V \in O_p.$$

The restricted exponential map can be combined to form a smooth map $\exp : O \to N$, where $O = \bigcup O_p$ and where $\exp|_{O_p} = \exp_p$. This map is just called the *exponential map*.

We have the following important lemma, where we show that the exponential map is a local diffeomorphism.

Lemma 1.3. *(Normal Neighbourhood Lemma) For any $p \in N$, there is a neighbourhood $\mathcal{V}$ of the origin in T_pN and neighbourhood $\mathcal{U}$ of p in N such that $\exp_p : \mathcal{V} \to \mathcal{U}$ is diffeomorphism.*

Proof. See for example [27]. □

Any open neighbourhood $\mathcal{U}$ of $p \in N$ that is the diffeomorphic image under $\exp_p$ of a star-shaped open neighbourhood of $0 \in T_pN$ as in the preceding lemma is called a *normal neighbourhood* of p.

This lemma also justifies the following definitions.

Definition 1.2. The *injectivity radius* of $p \in N$ is the largest radius for which the exponential map at p is a diffeomorphism. If it exists, the *injectivity radius of* N, $\mathrm{inj}(N, \bar{g})$, is the infimum of the injectivity radii of all points of N.

Definition 1.3. Let ε be the injectivity radius of p and consider, $\forall r \leq \varepsilon$, the following subset of T_pN

$$B_r(0) = \{V \in T_pN : ||V||_{\bar{g}} < r\},$$

where the map $\exp_p : B_r(0) \to N$ is a diffeomorphism.
The image $\exp_p(B_r(0)) \subset N$ is called *geodesic ball* and $\exp_p(\partial \bar{B}_r(0)) \subset N$ is the *geodesic sphere*.

Definition 1.4. The *convexity radius* at $p \in N$ is the supremum of $r \in \mathbb{R}$ such that $\forall \rho < r$ the geodesic ball $\exp_p(B_\rho(0))$ is strongly geodesically convex, i.e. $\forall x, y$ in the ball, there exists a unique minimising geodesic connecting the two points inside the ball, except possibly the endpoints.
If it exists, the *convexity radius of* N, $\mathrm{conv}(N, \bar{g})$, is the infimum of the convexity radii at all points of N.

In this present thesis we assume that $(N, \bar{g})$ is a compact manifold, and therefore every closed bounded set in N is compact, i.e. N satisfies the *Heine-Borel property*. By *Hopf-Rinow* theorem, N is thus geodesically complete, i.e. all the geodesics are defined for all times (and, in particular, $\forall p \in N$ and $\forall V \in T_pN$, there exists a geodesic starting in p in the direction of V defined for all times), and it is complete as metric space. As a consequence, any two points in N can be joined by a segment, a curve such that its length is exactly equal to the distance between the two points.
In particular, given the compactness of N, it is possible to prove that both the injectivity radius and the convexity radius of N exist and they are strictly positive. Moreover, a nice result from Berger [4] shows that, if N is compact, the following estimate holds

$$0 < \mathrm{conv}(N, \bar{g}) \leq \frac{1}{2}\mathrm{inj}(N, \bar{g}).$$

For the reasons above, we indicate with $\varepsilon = \mathrm{inj}(N, \bar{g})$ the injectivity radius of the compact manifold N.

Let us consider a normal neighbourhood $\mathcal{U}$ of p and $\{e_i\}$ an orthonormal basis for T_pN. Note that such a basis gives an isomorphism $E : \mathbb{R}^{n+1} \to T_pN$ by $E(x^1, \cdots, x^{n+1}) = x^i e_i$. If $\mathcal{U}$ is a normal neighbourhood of p, we can combine this isomorphism with the exponential map to get a coordinate chart

$$\varphi := E^{-1} \circ \exp_p^{-1} : \mathcal{U} \to \mathbb{R}^n.$$

Any such coordinates are called *(Riemannian) normal coordinates* centred at p. Given $p \in N$ and a normal neighbourhood $\mathcal{U}$ of p, there is a one-to-one correspondence between normal coordinate charts and orthonormal bases at p.
In any normal coordinate chart centred at p, we can define the *radial distance function* $r = \psi(x) = d(p, x)$ as

$$r = \psi(x) = \Big(\sum_i (x^i)^2\Big)^{\frac{1}{2}},$$

and the *unit radial vector field* ∂_r

$$\partial_r = \nabla\psi = \frac{1}{r}\, x^i \partial_i = \frac{1}{\psi(x)}\, x^i \partial_i.$$

In the Euclidean space, $r = \psi(x)$ is the distance to the origin, while ∂_r is the unit vector tangent to the straight lines through the origin. By the next important Proposition, they also have some special geometric meaning for any metric in normal coordinates.

Proposition 1.4. *(Properties of Normal Coordinates). Let $(\mathcal{U}, (x^i))$ be any normal coordinate chart centred at p.*

(a) For any $V = V^i \partial_i \in T_pN$, the geodesic γ_V starting at p with initial velocity vector V is represented in normal coordinates by the radial line segment

$$\gamma_V = (tV^1, \ldots, tV^{n+1})$$

as long as γ_V stays within $\mathcal{U}$.

(b) The coordinates of p are $(0, \ldots, 0)$.

(c) The components of the metric at p are $g_{ij} = \delta_{ij}$.

(d) At any point $q \in \mathcal{U} - \{p\}$, ∂_r is the velocity vector of the unit speed geodesic from p to q, and therefore has unit length with respect to g.

(e) The first partial derivatives of g_{ij} and the Christoffel symbols vanish at p.

Proof. See for example [27]. □

We also have the following useful Lemma.

Lemma. *Let $p \in (N, \bar{g})$, with N compact, and ε its injectivity radius. Then for each $r \leq \varepsilon$ and $x \in N$ such that $B_r(x) \subset \exp_p(B_\varepsilon(0))$,*

$$\exp_x(B_r(0)) = B_r(x),$$

and moreover

$$\exp_x(\bar{B}_r(0)) = \bar{B}_r(x).$$

Proof. See for example [36]. □

That means that as far as $\exp_p$ is a diffeomorphism, any Riemannian ball is a geodesic ball, where as *Riemannian ball* we define as usual the open set in N

$$B_r(p) = \{x \in N : \tau(p, x) < r\},$$

and where $\tau(p, x)$ is the infimum over all the lengths $L(\gamma)$ of piecewise smooth curves connecting the two endpoints, i.e.

$$\tau(p, x) = \inf\{L(\gamma) \,|\, \gamma : [0, 1] \to N,\, \gamma(0) = p,\, \gamma(1) = x\}.$$

The exponential map plays an essential role in our work, as it will be clear in Chapter 4. Here below we want to illustrate another important consequence of the exponential map.
Let $B_r(x)$ be a Riemannian ball inside $U_p \subset N$ and consider $\partial B_r(x) = S$. Then we can define the *normal bundle* of S as the following space

$$TS^\perp = \{v \in T_pN : p \in S,\, v \in (T_pS)^\perp \subset T_pN\}.$$

Here $(T_pS)^\perp$ is the orthogonal complement of T_pS in T_pN. So for each $p \in S$, we have the orthogonal direct sum $T_pN = T_pS \oplus (T_pS)^\perp$.

Definition 1.5. The *normal exponential map* $\exp^\perp$ is the restriction of exp to

$$\exp^\perp : O \cap TS^\perp \to N.$$

By standard theory, the differential of $\exp^\perp$ is nonsingular at 0_p for a point $p \in S$ and it follows that there exists an open neighbourhood V of the zero section in $TS^\perp$ on which $\exp^\perp$ is a diffeomorphism onto its image in N. Such an image $\exp^\perp(V)$ is called a *tubular neighbourhood* of S in N, because intuitively it looks like a solid tube around S, containing S.
Note that in the tubular neighbourhood of $S = \partial B_r$ we can "build" a surface M in the following way. We first identify through the exponential map the Riemannian sphere S with the Euclidean sphere S^n_r of radius r. Then we define a "shape function" $f(q) : S^n_r \to \mathbb{R}$ for any q in S^n_r (therefore $\forall q \in S = \partial B_r \cong S^n_r$). In doing so, we can model the surface M as

$$M = \exp^\perp_q(f(q) \cdot \nu(q)), \tag{1.1.1}$$

for $\nu(q)$ the outer normal (unit) vector to $T_qS^n_r \cong T_qS$ in $(T_qS^n_r)^\perp \cong (T_qS)^\perp$.
Recall that for a given function $f : W \to \mathbb{R}$, for an open region $W \subset \mathbb{R}^n$, the C^k-norm

in $\bar{W}$ is defined as

$$||f||_{C^k(\bar{W})} = \sum_{|\alpha|\leq k} \sup_{x\in W} |D^\alpha f(x)|,$$

where we are using the *multiindex* notation, such that, given $|\alpha| = \alpha_1 + \cdots + \alpha_n$, for nonnegative integers α_i and k, we have

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} .$$

We are now able to define the following important concept.

Definition 1.6. Given a $\delta > 0$, the surface M, as defined in (1.1.1), is said to be $\delta-$close to $S = \partial B_r(x)$ in the C^2-norm if the function $\hat{f} = f(q)/r$ is such that

$$||\hat{f}||_{C^2(S_r^n)} = \sum_{|\alpha|\leq 2} \sup_{x\in S_r^n} |D^\alpha \hat{f}(x)| \leq \delta,$$

where $f : S_r^n \to \mathbb{R}$ is the shape function, $r > 0$ the radius of the Riemannian sphere S and D^i can be thought as the covariant derivative on the sphere induced by the standard metric.

We use the "corrected" function $\hat{f}$ because we want to have a "dimensionless measure" of the deviation of M from the sphere S compared to its size.

We conclude this paragraph with another important result. We will often work with normal coordinates in the open set $U_p = \exp(B_\varepsilon(0))$. It is therefore useful to remember that the metric $\bar{g}$ of N can be *Taylor expanded* at a point $q \in U_p$ (See for example Prop. 2.1 in [30], where the proof can be found in the Appendix). This expansion has the following form which depends on the curvature of N and on the radial distance:

$$\begin{aligned} \bar{g}_{ij}(q) &= \delta_{ij} + \frac{1}{3}\,\bar{R}_{kilj}|_p\, x^k x^l + \frac{1}{6}\,\nabla_k \bar{R}_{limj}|_p\, x^k x^l x^m + \cdots \\ &= \delta_{ij} + \frac{1}{3}\,\bar{R}_{kilj}|_p\, x^k x^l + O(r^3), \end{aligned} \tag{1.1.2}$$

where $\bar{R}_{ijkl}$ is the Riemann tensor of N computed in p, $r = \psi(x) = d(p,x)$ is the radial distance function defined above and q has coordinates $q = (x^1, x^2, \cdots x^{n+1})$.

1.2 Evolution equations

We are interested in studying one-parameter families of smooth hypersurface immersions $F : M \times [0,T) \to (N, \bar{g})$ into a compact Riemannian manifold, with $M_t = F(\cdot,t)(M)$, that satisfies an initial value problem

$$\begin{aligned} \frac{\partial}{\partial t}F(p,t) &= f(p,t)\cdot\nu, \qquad && p \in M, t \in [0,T), \\ F(p,0) &= F_0, && p \in M, \end{aligned} \tag{1.2.3}$$

where $\nu(p,t)$ is a choice of unit normal at $F(p,t)$, $f(p,t)$, called *speed*, is some smooth symmetric function of the principal curvatures k_i of the hypersurface at $F(p,t)$ and M_0 is a closed hypersurface.

Short time existence for this particular type of flows is already known in the literature, even when the ambient space is a general Riemannian manifold. One could for example check Section 7.5, in [21], where short time existence has been proved for a very broad class of immersions of the type

$$\frac{\partial}{\partial t}F = \big(-(\Delta)^p H + \psi(F, \nu, A, \nabla A, \nabla^2 A, \dots, \nabla^{2p-1}A)\big)\nu,$$

for any nonnegative integer p, and for an arbitrary ψ smooth in all its arguments. In particular, Theorem 7.17, [21], illustrates how the evolution equation may be reduced to strictly parabolic, quasilinear scalar equation, and hence that it has short-time solution under appropriate initial conditions.

Once short time existence is guaranteed for some class of flows, one is in general interested to investigate if the solution exists for all times and therefore to understand its asymptotic behaviour, both for large times and near the singularities. For this specific purpose it is essential to establish for all the relevant geometric quantities their evolution equations, in particular for the second fundamental form.

For a general speed f, we have the following result [21].

Theorem 1.5. *On any solution M_t of (1.2.3) the following evolution equations hold:*

(i) $\frac{\partial}{\partial t}g_{ij} = 2fh_{ij}$,

(ii) $\frac{\partial}{\partial t}d\mu = fHd\mu$,

(iii) $\frac{\partial}{\partial t}h_{ij} = -\nabla_i\nabla_j f - f(-h_{ik}h_j^k + \bar{R}_{ninj})$,

(iv) $\frac{\partial}{\partial t}h_i^j = -\nabla_i\nabla^j f - f(h_{ik}h^{kj} + \bar{R}_{in\ n}^{\ \ j})$,

(v) $\frac{\partial H}{\partial t} = -\Delta f - f\big(|A|^2 + \bar{Ric}(n,n)\big)$,

(vi) $\frac{\partial}{\partial t}|A|^2 = -2h^{ij}\nabla_i\nabla_j f - 2f(tr\,A^3 + h^{ij}\bar{R}_{ninj})$.

Here $d\mu$ is the induced measure on the hypersurface and Δ is the Laplace-Beltrami operator with respect to the time-dependent induced metric on the hypersurface.

In the present work we are interested in studying a particular class of initial value problems (1.2.3) called volume preserving mean curvature flows; we also set that the initial surface is strictly convex.

Definition 1.7. An hypersurface $M \subset N$ is said to be **strictly convex** if all the principal curvatures are strictly positive $k_i > 0$, $\forall i$ $1 \leq i \leq n$.

Definition 1.8. Let $(N, \bar{g})$ be a compact Riemannian manifold of dimension $n+1$ and F a family of hypersurface immersions $F : M \times [0, T) \to (N, \bar{g})$, with $F_0(M) = M_0$. Then the **volume preserving mean curvature flow** (VPMCF) of F_0 is such that the map $F : M \times [0, T) \to (N, \bar{g})$ is a solution of the following normal deformation problem

$$\frac{\partial}{\partial t} F(x, t) = [-H(x, t) + \phi(t)] \cdot \nu(x, t),$$

where $H(x, t)$ and $\nu(x, t)$ are respectively the mean curvature and the unit normal of the hypersurface $F_t(M) = M_t$ at the point $x \in M$, and where

$$\phi(t) = \frac{1}{|M_t|} \int_{M_t} H d\mu,$$

is called the average mean curvature.

We also suppose from now on that $M_0 = F_0(M)$ is closed and strictly convex.

In the particular setting of the VPMCF, thanks to Thm. 1.5, we have the following results.

Corollary 1.6. *Under the volume preserving mean curvature flow, for the following geometric quantities we have:*

$$\begin{aligned}
(i) \quad \frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} - 2H h_{im} h_{mj} + h_{ij} |A|^2 \\
&+ \bar{R}_{nn} h_{ij} - \bar{R}_{kikm} h_{mj} - \bar{R}_{kjkm} h_{im} \\
&+ 2\bar{R}_{kimj} h_{km} - \bar{\nabla}_k \bar{R}_{nijk} - \bar{\nabla}_j \bar{R}_{nkik} \\
&+ \phi(t) h_{ik} h_{kj} - \phi(t) \bar{R}_{ninj}. \\
(ii) \quad \frac{\partial}{\partial t} H &= \Delta H + \big(H - \phi\big)\big(|A|^2 + \overline{Ric}\,(n, n)\big). \\
(iii) \frac{\partial}{\partial t} |A|^2 &= \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2 (|A|^2 + \overline{Ric}(n, n)) \\
&- 4h_{ij} \big(\bar{R}_{kikm} h_{jm} - \bar{R}_{kimj} h_{km}\big) - 2h_{ij} \big(\bar{\nabla}_j \bar{R}_{nkik} - \bar{\nabla}_k \bar{R}_{nijk}\big) \\
&- 2\phi\big(tr\, A^3 + h^{ij} \bar{R}_{ninj}\big).
\end{aligned}$$

where the index n indicates the normal direction.

Proof. (i) Starting from Thm. 1.5 with the speed $f(x, t) = -H(x, t) + \phi(t)$,

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= -\nabla_i \nabla_j [-H + \phi] + [-H + \phi](h_{ik} h^k_j - \bar{R}_{ninj}) = \\
&= \nabla_i \nabla_j H + [-H + \phi](h_{ik} h^k_j - \bar{R}_{ninj}),
\end{aligned}$$

we then make use of Cor. 1.2 to get the desired result.

(ii) It is immediate, by Thm. 1.5.

(iii) From Thm. 1.5, part (v) with speed $f(x,t) = -H(x,t) + \phi(t)$, we have

$$\frac{\partial}{\partial t}|A|^2 = 2h^{ij}\nabla_i\nabla_j H + 2(H-\phi)(\operatorname{tr} A^3 + h^{ij}\bar{R}_{ninj}).$$

Using now Cor. 1.2, the symmetries of the Riemann tensor, and the following observations

(a) $2h^{ij}\Delta h_{ij} = \Delta(h_{ij}h^{ij}) - 2(\nabla_k h_{ij}\nabla^k h^{ij}) = \Delta|A|^2 - 2|\nabla A|^2$,

(b) $2h^{ij}h_{ij}|A|^2 = 2|A|^4$,

(c) $2h^{ij}h_{ij}\bar{R}_{nn} = 2|A|^2\overline{Ric}(n,n)$,

(d) $-2h^{ij}Hh_{im}h_{mj} + 2H\operatorname{tr}A^3 = -2Hh^i_j h^m_i h^j_m + 2Hh^j_i h^m_j h^i_m = 0$,

(e) $2h^{ij}\bar{R}_{kjkm}h_{im} = 2h^{ij}\bar{R}_{kikm}h_{jm}$,

we get the result. □

We introduce here the $*-$notation, a useful convention to write a product of two or more tensors. In particular, we will write $S * T$ for any linear combination of tensors formed by contraction on S and T by the metric g. This means that we start from the tensor field $S \otimes T$ and use the metric to switch the type of any number of components of the tangent bundle to components of the cotangent bundle, or vice versa (i.e. raising or lowering some indices) and take any number of contractions, and switch any number of components in the product. A very interesting property of this $*$-product is that allows us to write

$$|S * T| \leq K|S||T|$$

where the constant K depends only on the algebraic "structure" of $S * T$.
In the specific case of VPMCF, for example, the metric evolves like

$$\frac{\partial}{\partial t}g_{ij} = 2fh_{ij} = 2(\phi - H)h_{ij} = \phi\, A + A * A, \tag{1.2.4}$$

after plugging the speed $f = -H + \phi$ in Thm. 1.5.

We also prove the following Lemma.

Lemma 1.7. *We have the following evolution equation for the Christoffel symbols:*

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \phi\,\nabla A + A * \nabla A. \tag{1.2.5}$$

Proof. Using the definition of the Christoffel symbols in normal coordinates in a point p and the evolution equation for the metric (1.2.4), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^k_{ij} &= \frac{1}{2}g^{ij}\Big\{\nabla_j\Big(\frac{\partial}{\partial t}g_{kl}\Big)+\nabla_k\Big(\frac{\partial}{\partial t}g_{jl}\Big)-\nabla_l\Big(\frac{\partial}{\partial t}g_{jk}\Big)\Big\} = \\
&= g^{ij}\Big\{\nabla_j\Big[(\phi-H)h_{kl}\Big]+\nabla_k\Big[(\phi-H)h_{jl}\Big]-\nabla_l\Big[(\phi-H)h_{jk}\Big]\Big\} = \\
&= \phi\,\nabla A + A*\nabla A.
\end{aligned}$$

Since we want to study the evolution equation of the covariant derivative of any order of the second fundamental form, it is essential to introduce the following commutating formulas for a general tensor T.

Lemma 1.8. *The following formula for the interchange of time and covariant derivative of a tensor T under the VPMCF holds:*

$$\frac{\partial}{\partial t}\nabla T = \nabla\frac{\partial}{\partial t}T + \big(\phi\,\nabla A + A*\nabla A\big)*T. \tag{1.2.6}$$

Proof. W.l.o.g. we suppose that $T = T_{i_1\dots i_k}$ is a covariant tensor, since the general case is analogous, as it will be clear by the following computation. We thus have:

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_k} &= \frac{\partial}{\partial t}\Big(\frac{\partial T_{i_1\dots i_k}}{\partial x_j}-\sum_{s=1}^{k}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}i_k}\Big) = \\
&= \frac{\partial}{\partial x_j}\frac{\partial T_{i_1\dots i_k}}{\partial t}-\sum_{s=1}^{k}\Gamma^l_{ji_s}\frac{\partial T_{i_1\dots i_{s-1}li_{s+1}i_k}}{\partial t} \\
&\quad -\sum_{s=1}^{k}\frac{\partial}{\partial t}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}i_k} = \\
&= \nabla_j\frac{\partial T_{i_1\dots i_k}}{\partial t}-\sum_{s=1}^{k}\big(\phi\,\nabla A + A*\nabla A\big)^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}i_k},
\end{aligned}$$

which is exactly what we wanted to prove. □

Lemma 1.9. *Let T be a general tensor acting on (M,g). Then we have the following commutation formula:*

$$\Delta\nabla T - \nabla\Delta T = Rm*\nabla T + \nabla Rm * T, \tag{1.2.7}$$

where Rm is the Riemann tensor of M.

Proof. Let T be a general tensor. We also use the following convention for the third covariant derivative:

$$\nabla_i\nabla_j\nabla_k T = \nabla^3 T(\partial_i,\partial_j,\partial_k,\dots).$$

Then we have:

$$\begin{aligned}
\nabla_k \Delta T - \Delta \nabla_k T &= g^{ij}\big(\nabla_k \nabla_i \nabla_j T - \nabla_i \nabla_j \nabla_k T\big) = \\
&= g^{ij}\big([\nabla_k, \nabla_i]\nabla_j T + \nabla_i \nabla_k \nabla_j T - \nabla_i \nabla_j \nabla_k T\big) = \\
&= g^{ij}\big([\nabla_k, \nabla_i]\nabla_j T + \nabla_i([\nabla_k, \nabla_j]T)\big).
\end{aligned}$$

Using the $*-$notation we get:

$$\begin{aligned}
[\Delta, \nabla]\, T &= Rm * \nabla T + \nabla\big(Rm * T\big) = Rm * \nabla T + \nabla Rm * T + \nabla Rm * T = \\
&= Rm * \nabla T + \nabla Rm * T.
\end{aligned}$$

Remark 1. *It is convenient to recall the Gauss equation in terms of the $*-$notation, which relates the Riemann tensor of M with the one of N, and the relation between the covariant derivative $\bar{\nabla}$ of N with ∇ of M, for a general tensor T well defined. Therefore we have:*

$$\overline{Rm} = Rm + A * A, \tag{1.2.8}$$

and

$$\bar{\nabla} T = \nabla T + A * T. \tag{1.2.9}$$

Therefore for a general tensor T the (1.2.7) becomes

$$\begin{aligned}
\Delta \nabla T - \nabla \Delta T &= (\overline{Rm} + A * A) * \nabla T + \nabla(\,\overline{Rm} + A * A) * T = \\
&= \overline{Rm} * \nabla T + A * A * \nabla T + \nabla\, \overline{Rm} * T + A * \nabla A * T.
\end{aligned}$$

Example 1.1. If we consider the second fundamental form A, the formula for interchanging the Laplacian and the covariant derivative becomes:

$$\begin{aligned}
\Delta \nabla A - \nabla \Delta A &= \overline{Rm} * \nabla A + A * A * \nabla A + \nabla\, \overline{Rm} * A \\
&= \overline{Rm} * \nabla A + A * A * \nabla A + \bar{\nabla}\, \overline{Rm} * A + A * A * \overline{Rm}.
\end{aligned}$$

We rewrite the evolution equation of A in the $*-$notation as

$$\frac{\partial}{\partial t} A = \Delta A + A * A * A + A * \overline{Rm} + \bar{\nabla}\, \overline{Rm} + \phi\, A * A + \phi\, \overline{Rm}, \tag{1.2.10}$$

in order to prove the following Propositions.

Proposition 1.10. *Under the VPMCF, the evolution equation for the covariant derivative of A in the $*-$notation reads as*

$$\begin{aligned}
\frac{\partial}{\partial t} \nabla A &= \Delta \nabla A + A * A * \nabla A + \nabla A * \overline{Rm} + A * \bar{\nabla}\, \overline{Rm} + \bar{\nabla}^2\, \overline{Rm} + \\
&\quad + \phi\, A * \nabla A + \phi \bar{\nabla}\, \overline{Rm} + \phi A * \,\overline{Rm}.
\end{aligned} \tag{1.2.11}$$

Proof. Using the formula (1.2.6),

$$\frac{\partial}{\partial t}\nabla A = \nabla \frac{\partial}{\partial t} A + \phi\, \nabla A * A + A * A * \nabla A,$$

and substituting with the evolution equation for A, we obtain:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla A &= \nabla \Delta A + A * A * \nabla A + \nabla A * \overline{Rm} + A * \nabla\, \overline{Rm} + \nabla\bar{\nabla}\, \overline{Rm} + \\ &\quad +\phi A * \nabla A + \phi \nabla\, \overline{Rm}.\end{aligned}$$

We make now use of Ex. 1.1 for the interchange formula, and we have:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla A &= \Delta\nabla A + A * A * \nabla A + \overline{Rm} * \nabla A + \bar{\nabla}\, \overline{Rm} * A + \\ &= + A * A * \overline{Rm} + A * \nabla\, \overline{Rm} + \nabla\bar{\nabla}\, \overline{Rm} + \phi A * \nabla A + \phi\nabla\, \overline{Rm}.\end{aligned}$$

Note that:

$$\begin{aligned}A * \bar{\nabla}\, \overline{Rm} &= A * A * \overline{Rm} + A * \nabla\, \overline{Rm}; \\ \nabla\bar{\nabla}\, \overline{Rm} &= \bar{\nabla}^2\, \overline{Rm} + A * \bar{\nabla}\, \overline{Rm}; \\ \phi\nabla\, \overline{Rm} &= \phi\bar{\nabla}\, \overline{Rm} + \phi A * \overline{Rm}.\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla A &= \Delta\nabla A + A * A * \nabla A + \nabla A * \overline{Rm} + A * \bar{\nabla}\, \overline{Rm} + \bar{\nabla}^2\, \overline{Rm} + \\ &\quad +\phi\, A * \nabla A + \phi\bar{\nabla}\, \overline{Rm} + \phi A * \overline{Rm}.\end{aligned}$$

□

Proposition 1.11. *The following formula for the evolution equation of the higher derivatives of A holds:*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^m A &= \Delta\nabla^m A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \sum_{i+j=m} \nabla^i A * \bar{\nabla}^j\, \overline{Rm} + \\ &\quad + P(A, \nabla A, \overline{Rm}, \bar{\nabla}\, \overline{Rm}) + \bar{\nabla}^{m+1}\, \overline{Rm} + \phi\, \bar{\nabla}^m\, \overline{Rm} + \phi \sum_{i+j=m} \nabla^i A * \nabla^j A \\ &\quad + \phi \sum_{i+j=m-1} \nabla^i A * \bar{\nabla}^j\, \overline{Rm} + \phi\, Q(A, \nabla A, \overline{Rm}, \bar{\nabla}\, \overline{Rm}),\end{aligned} \tag{1.2.12}$$

where $P(A, \nabla A, \overline{Rm}, \bar{\nabla}\, \overline{Rm})$ and $Q(A, \nabla A, \overline{Rm}, \bar{\nabla}\, \overline{Rm})$ are the $*$-product of a term $\overline{Rm}$ or its $i-$covariant derivative $\bar{\nabla}^i\, \overline{Rm}$ of order at most $i \leq m-1$ with at most $m-$terms among A and $\nabla^i A$.

Proof. We proceed by induction on m. Note that we have already proved the zero case. We then have:

$$\frac{\partial}{\partial t}\nabla^{m+1} A = \frac{\partial}{\partial t}\nabla\big(\nabla^m A\big) = \nabla\frac{\partial}{\partial t}\big(\nabla^m A\big) + \big(\phi\nabla A + A * \nabla A\big) * \nabla^m A,$$

by the formula (1.2.6) of interchange of time and covariant derivative. Applying the inductive hypothesis we now get:

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^{m+1}A &= \nabla\Delta\nabla^m A + \nabla\Big[\sum_{i+j+k=m}\nabla^i A * \nabla^j A * \nabla^k A\Big] + \\
&+\nabla\Big[\sum_{i+j=m}\nabla^i A * \bar{\nabla}^j\,\overline{Rm}\Big] + \nabla\bar{\nabla}^{m+1}\,\overline{Rm} + \phi\nabla\,\bar{\nabla}^m\,\overline{Rm} + \\
&+\phi\,\nabla\Big[\sum_{i+j=m}\nabla^i A * \nabla^j A\Big] + \phi\nabla\Big[\sum_{i+j=m-1}\nabla^i A * \bar{\nabla}^j\,\overline{Rm}\Big] + \\
&+\nabla\big[P(A,\nabla A,\overline{Rm},\bar{\nabla}\,\overline{Rm}) + \phi\,Q(A,\nabla A,\overline{Rm},\bar{\nabla}\,\overline{Rm})\big] + \\
&+\phi\nabla A * \nabla^m A + A * \nabla A * \nabla^m A.
\end{aligned}$$

It is clear that the following terms behave as we would wish to:

$$\nabla\sum_{i+j+k=m}\nabla^i A * \nabla^j A * \nabla^k A = \sum_{i+j+k=m+1}\nabla^i A * \nabla^j A * \nabla^k A,$$

and

$$\phi\,\nabla\sum_{i+j=m}\nabla^i A * \nabla^j A = \phi\sum_{i+j=m+1}\nabla^i A * \nabla^j A,$$

and the terms $\phi\nabla A * \nabla^m A$ and $A * \nabla A * \nabla^m A$ are of the same type of the two above. Moreover, the $*-$products $\nabla P(\cdot)$ and $\phi\nabla Q(\cdot)$, are still of the same type: because of (1.2.8) and (1.2.9), we have for example that

$$\nabla\,P(A,\nabla A,\overline{Rm},\bar{\nabla}\,\overline{Rm}) = \bar{\nabla}P(\cdot) + A * P(\cdot) \simeq P(A,\nabla A,\overline{Rm},\bar{\nabla}\,\overline{Rm}).$$

For similar reasons, applying (1.2.9), the following terms behave as:

$$\begin{aligned}
\nabla\sum_{i+j=m}\nabla^i A * \bar{\nabla}^j\,\overline{Rm} &= \sum_{i+j=m+1}\nabla^i A * \bar{\nabla}^j\,\overline{Rm} + \sum_{i+j=m}\nabla^i A * \nabla\bar{\nabla}^j\,\overline{Rm} \\
&= \sum_{i+j=m+1}\nabla^i A * \bar{\nabla}^j\,\overline{Rm} + \sum_{i+j=m}\nabla^i A * \bar{\nabla}^j\,\overline{Rm} * A,
\end{aligned}$$

where the last addend above clearly belong to $P(\cdot)$. Analogous reasoning stands for the term $\phi\nabla\sum_{i+j=m-1}\nabla^i A * \bar{\nabla}^j\,\overline{Rm}$.
Observe now that, always by (1.2.9):

$$\nabla\bar{\nabla}^{m+1}\,\overline{Rm} = \bar{\nabla}^{m+2}\,\overline{Rm} + \bar{\nabla}^{m+1}\,\overline{Rm} * A,$$

and therefore the second addend clearly belongs to the term we previously treated. Similar reasoning for $\phi\nabla\,\bar{\nabla}^m\,\overline{Rm}$.

Finally, just observe that by Remark 1, (1.2.8) and (1.2.9), setting $T = \nabla^m A$, we obtain:

$$\begin{aligned}\nabla\Delta\nabla^m A &= \Delta\nabla^{m+1}A + \overline{Rm} * \nabla^{m+1}A + A * A * \nabla^{m+1}A + \nabla\,\overline{Rm} * \nabla^m A + \\ &\quad + A * \nabla A * \nabla^m A = \\ &= \Delta\nabla^{m+1}A + \overline{Rm} * \nabla^{m+1}A + A * A * \nabla^{m+1}A + \bar{\nabla}\,\overline{Rm} * \nabla^m A + \\ &\quad + \overline{Rm} * A * \nabla^m A + A + \nabla A * \nabla^m A.\end{aligned}$$

Just putting all together and we obtain the formula (1.2.12) we wanted to prove. $\square$

Proposition 1.12. *The following evolution equation holds:*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 &= \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A + \\ &\quad + \sum_{i+j=m} \nabla^i A * \nabla^m A * \bar{\nabla}^j\,\overline{Rm} + P(A, \nabla A, \nabla^m A, \overline{Rm}, \bar{\nabla}\,\overline{Rm}) + \\ &\quad + \nabla^m A * \bar{\nabla}^{m+1}\,\overline{Rm} + \phi\nabla^m A * \bar{\nabla}^m\,\overline{Rm} + \phi \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A + \\ &\quad + \phi \sum_{i+j=m-1} \nabla^m A * \nabla^i A * \bar{\nabla}^j\,\overline{Rm} + \phi\, Q(A, \nabla A, \nabla^m A, \overline{Rm}, \bar{\nabla}\,\overline{Rm}).\end{aligned} \tag{1.2.13}$$

Proof. Observe first that:

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = \frac{\partial}{\partial t}\Big[g\Big(\nabla^m A, \nabla^m A\Big)\Big] = \Big(\frac{\partial}{\partial t}g\Big)\Big(\nabla^m A, \nabla^m A\Big) + 2g\Big(\frac{\partial}{\partial t}\nabla^m A, \nabla^m A\Big).$$

Using (1.2.4), the first term on the RHS can be written as:

$$\begin{aligned}\Big(\frac{\partial}{\partial t}g\Big)\Big(\nabla^m A, \nabla^m A\Big) &= \big(A * A + \phi A\big) * \nabla^m A * \nabla^m A = \\ &= A * A * \nabla^m A * \nabla^m A + \phi A * \nabla^m A * \nabla^m A.\end{aligned}$$

For the second term, we plug the formula (1.2.12) and we recall the following formula:

$$2g\Big(\Delta\nabla^m A, \nabla^m A\Big) = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2.$$

Just also observe that the expressions $P(\cdot)$ and $Q(\cdot)$ are still defined as before, but now the $*-$product must contain the term $\nabla^m A$. $\square$

1.3 Maximum principle and consequences of the evolution equations

In studying the long term behaviour of solutions for parabolic equations, an important step is trying to obtain some a priori estimates. The main tool in order to get pointwise estimates is the **maximum principle**, in particular in the context of the mean curvature flow, and therefore specifically in the VPMCF one. We state in the Theorem here below both the *weak* and *strong* maximum principles for scalars, whose proofs can be found in [31].

Theorem 1.13. *Assume that g_t, for $t \in [0, T)$, is a family of Riemannian metrics on a manifold M, with a possible boundary ∂M, such that the dependence on t is smooth. Let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function satisfying*

$$\partial_t u \leq \Delta_{g_t} u + g_t\big(X(p, u, \nabla u, t), \nabla u\big) + b(u),$$

where X and b are respectively a continuous vector field and a locally Lipschitz function in their arguments.
Then, suppose that for every $t \in [0, T)$ there exists a value $\delta > 0$ and a compact subset $K \subset M \backslash \partial M$ such that at every time $t' \in (t - \delta, t + \delta) \cap [0, T)$ the maximum of $u(\cdot, t')$ is attained at least at one point of K (this is clearly true if M is compact without boundary).
Setting $u_{max}(t) = \max_{p \in M} u(p, t)$, we have that the function u_{max} is locally Lipschitz, hence differentiable at almost every time $t \in [0, T)$ and at every differentiability time,

$$\frac{d}{dt} u_{max}(t) \leq b(u_{max}(t)).$$

As a consequence, if $h : [0, T') \to \mathbb{R}$ is a solution of the ODE

$$\begin{cases} h'(t) & = b(h(t)) \\ h(0) & = u_{max}(0) \end{cases}$$

for $T' \leq T$, then $u \leq h$ in $M \times [0, T')$.
Moreover, if M is connected and at some time $\tau \in (0, T')$ we have $u_{max}(\tau) = h(\tau)$, then $u = h$ in $M \times [0, \tau]$, that is, $u(\cdot, t)$ is constant in space.

The relevance of the maximum principle can be seen in the proof of this new useful Theorem, which has been inspired by the works of [10], [18], [19] and [20].

Theorem 1.14. *Let assume that $|H(x, t)| < C_1$ and $|A(x, t)|^2 \leq C$, $\forall t \in [0, T']$, by positive constants C and C_1. Let also assume that there exists a small positive constant C_2 such that $|\overline{Rm}| < C_2$ and $|\bar{\nabla}\overline{Rm}| < C_2$, $\forall t \in [0, T']$. Then there exists a constant only depending on the dimension of M_t, C, C_1 and C_2, such that $\forall t \in (0, T']$ the covariant derivative of the second fundamental form stays bounded. In other words we have*

$$\sup_{x \in M_t} t|\nabla A|^2 \leq D,$$

$\forall t \in (0, T']$.

Proof. By direct computation, using (1.2.11), in the same spirit of (1.2.13), we have:

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla A|^2 &= \Delta|\nabla A|^2 - 2|\nabla^2 A|^2 + A * A * \nabla A * \nabla A + \nabla A * \nabla A * \overline{Rm} + \\ &+ A * \nabla A * \bar{\nabla}\,\overline{Rm} + \nabla A * \bar{\nabla}^2\,\overline{Rm} + \phi\, A * \nabla A * \nabla A + \\ &+ \phi \nabla A * \bar{\nabla}\,\overline{Rm} + \phi A * \nabla A *\, \overline{Rm}. \end{aligned} \tag{1.3.14}$$

W.l.o.g. we rename C_2 with C_1. Note also that by Thm. 1.5 and equation (1.2.13), by using the property of the $*$−product (and naming with C_1 the algebraic structural constant K), and the estimates on $\phi(t)$ (from the ones on H), the ones on the Riemann tensor and its covariant derivative, we have these following estimates:

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)|A|^2 &\leq -2|\nabla A|^2 + 2|A|^4 + C_1(|A| + |A|^2 + |A|^3), \\ \Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla A|^2 &\leq -2\,|\nabla^2 A|^2 + C_1|A|^2\,|\nabla A|^2 + C_1|\nabla A|^2 + C_1|A|\,|\nabla A| \\ &+ \;C_1\,|A|\,|\nabla A|^2 + C_1\,|\nabla A|. \end{aligned}$$

Using now the assumption of $|A|^2 \leq C$, we can manipulate the latter as

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla A|^2 &\leq -2\,|\nabla^2 A|^2 + C\,C_1\,|\nabla A|^2 + C_1|\nabla A|^2 + C\,C_1\,|\nabla A| \\ &+ \;C\,C_1\,|\nabla A|^2 + C_1\,|\nabla A| \\ &\leq -2\,|\nabla^2 A|^2 + K_1\,|\nabla A|^2 + K_1\,|\nabla A|, \end{aligned}$$

with $K_1 = (2CC_1 + C_1)$. Since we always have that:

$$|\nabla A| \leq \frac{1}{2}|\nabla A|^2 + \frac{1}{2},$$

we can rewrite the previous evolution equation as

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla A|^2 &\leq -2\,|\nabla^2 A|^2 + K_1\,|\nabla A|^2 + K_1\Big(\frac{|\nabla A|^2}{2} + \frac{1}{2}\Big) \\ &= -2\,|\nabla^2 A|^2 + K_2\,|\nabla A|^2 + K_3, \end{aligned}$$

with $K_2 = (K_1 + \frac{1}{2}K_1)$ and $K_3 = \frac{1}{2}K_1$.

Applying the estimates for the second fundamental form, and knowing that $|A| \leq \frac{1}{2}|A|^2 + \frac{1}{2}$, we can also rewrite its evolution equation as:

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)|A|^2 &\leq -2|\nabla A|^2 + 2|A|^4 + C_1(|A| + |A|^2 + |A|^3) \\ &\leq -2|\nabla A|^2 + 2|A|^4 + C_1\Big(2C + \frac{C^2}{2} + \frac{1}{2}\Big) \\ &\leq -2|\nabla A|^2 + K_4. \end{aligned}$$

with K_4 depending on C and C_1.

It is useful to recall that for any two smooth functions $B(x,t)$ and $C(x,t)$, we have that

$$\Big(\frac{d}{dt}-\Delta\Big)(B\,C)=\Big[\Big(\frac{d}{dt}-\Delta\Big)B\Big]C+B\Big[\Big(\frac{d}{dt}-\Delta\Big)C\Big]-2\nabla B\nabla C.$$

We can then compute:

$$\Big(\frac{d}{dt}-\Delta\Big)t\,|\nabla A|^2 \;=\; \Big[\Big(\frac{d}{dt}-\Delta\Big)t\Big]|\nabla A|^2+t\Big[\Big(\frac{d}{dt}-\Delta\Big)|\nabla A|^2\Big],$$

hence

$$\Big(\frac{d}{dt}-\Delta\Big)t\,|\nabla A|^2 \;\leq\; |\nabla A|^2-2\,t|\nabla^2 A|^2+t\,K_2|\nabla A|^2+t\,K_3.$$

We now study the following function, for positive constants Λ, L_1 and L_2 that we define later:

$$f(x,t)=t\,|\nabla A|^2\big(\Lambda+L_1|A|^2\big)+\frac{L_2}{2}|A|^2,$$

and we compute, line by line, its evolution behaviour

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)f(x,t).$$

Then we have, without using yet the estimates on $|A|^2$:

$$\begin{aligned}
\Big(\Big(\frac{\partial}{\partial t}-\Delta\Big)t|\nabla A|^2\Big)(\Lambda+L_1|A|^2)\leq&\\
&\leq (|\nabla A|^2-2\,t|\nabla^2 A|^2+t\,K_2|\nabla A|^2+t\,K_3)(\Lambda+L_1|A|^2)\\
&\leq |\nabla A|^2(\Lambda+L_1|A|^2)-2\,t|\nabla^2 A|^2(\Lambda+L_1|A|^2)\\
&+t\,K_2|\nabla A|^2(\Lambda+L_1|A|^2)+t\,K_3(\Lambda+L_1|A|^2),
\end{aligned}$$

And also

$$(t|\nabla A|^2)\Big(\frac{\partial}{\partial t}-\Delta\Big)(\Lambda+L_1|A|^2) \;\leq\; tL_1|\nabla A|^2(-2|\nabla A|^2+K_4).$$

We study the "gradient terms" as follows:

$$2\nabla(t|\nabla A|^2)\cdot\nabla(\Lambda+L_1|A|^2)=4t\,g(\nabla^2 A,\nabla A)\cdot 2L_1\,g(\nabla A,A).$$

Now we have, using the $*$-product notation:

(i) $4t\,g(\nabla^2 A,\nabla A)=4t\,\nabla^2 A*\nabla A\leq 4tK|\nabla^2 A||\nabla A|$,

(ii) $2L_1\,g(\nabla A,A)=2L_1\nabla A*A\leq 2L_1K|\nabla A||A|$, where K is the positive constant of the $*$-product. Putting all together, the estimate for the gradient term is:

$$\begin{aligned}
2\nabla(t|\nabla A|^2)\cdot\nabla(\Lambda + L_1|A|^2) &\leq 8L_1K^2\, t\,|\nabla^2 A|\,|\nabla A|^2\,|A| \\
&\leq 2t\,|\nabla^2 A|^2(\Lambda + L_1|A|^2) + 8L_1^2K^4\frac{t|A|^2}{\Lambda + L_1|A|^2}|\nabla A|^4,
\end{aligned}$$

where we have made use of the following Young inequality:

$$|ab| \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon},$$

with $\epsilon = (\Lambda + L_1 C)$, $a = 2\sqrt{t}|\nabla^2 A|$ and $b = 4L_1K^2\sqrt{t}|\nabla A|^2|A|$.
For the final term

$$\frac{L_2}{2}\Big(\frac{\partial}{\partial t} - \Delta\Big)|A|^2 \leq \frac{L_2}{2}(-2|\nabla A|^2 + 2|A|^4 + K_4) = -L_2\,|\nabla A|^2 + \frac{L_2}{2}C|A|^2 + K_5,$$

with $K_5 = \frac{1}{2}L_2K_4$ and we estimated for convenience and w.l.o.g. $2|A|^4 \leq C|A|^2$.
Putting all together, we finally get:

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)f(x,t) \;\; &\leq \;\; |\nabla A|^2(\Lambda + L_1|A|^2) - 2\,t|\nabla^2 A|^2(\Lambda + L_1|A|^2) \\
&+ \;\; t\,K_2|\nabla A|^2(\Lambda + L_1|A|^2) + t\,K_3(\Lambda + L_1|A|^2) \\
&- \;\; 2t\,L_1|\nabla A|^4 + t\,L_1K_4|\nabla A|^2 \\
&+ \;\; 2t|\nabla^2 A|^2(\Lambda + L_1|A|^2) + 8L_1^2K^4\frac{t|A|^2}{\Lambda + L_1|A|^2}|\nabla A|^4 \\
&- \;\; L_2\,|\nabla A|^2 + \frac{L_2}{2}C|A|^2 + K_5.
\end{aligned}$$

Reordering the terms as follow, we have:

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)f(x,t) \;\; &\leq \;\; |\nabla A|^2(\Lambda + L_1|A|^2) - L_2\,|\nabla A|^2 - 2\,t|\nabla^2 A|^2(\Lambda + L_1|A|^2) \\
&+ \;\; 2t|\nabla^2 A|^2(\Lambda + L_1|A|^2) - 2tL_1|\nabla A|^4 \\
&+ \;\; 8L_1^2K^4\frac{tC}{\Lambda + L_1 C}|\nabla A|^4 + t\,K_2|\nabla A|^2(\Lambda + L_1|A|^2) \\
&+ \;\; t\,L_1K_4|\nabla A|^2 + \frac{L_2}{2}C|A|^2 + t\,K_3(\Lambda + L_1|A|^2) + K_5.
\end{aligned}$$

We now simplify the two opposite terms $|\nabla^2 A|^2$ and we finally set the values for our constants. We choose $L_1 = \frac{1}{4K^4}$ and $L_2 = \Lambda + L_1 C$ (but we keep using L_1 and L_2 for the terms $|\nabla A|^2$ and $|A|^2$ for simplicity); note that $\Lambda + L_1 C \geq C + 1$ (therefore $\Lambda \geq C(1 - L_1) + 1 \geq 1$). We then obtain:

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)f(x,t) \;\; &\leq \;\; t\,K_2|\nabla A|^2(\Lambda + L_1|A|^2) + t\,L_1K_4|\nabla A|^2 + t\,K_3(\Lambda + L_1|A|^2) \\
&+\frac{L_2}{2}C|A|^2 + K_5.
\end{aligned}$$

Estimating the term $t\,K_3(\Lambda+L_1|A|^2)\leq t\,R$, with $R=K_3(\Lambda+L_1C)$, and factoring out, we obtain:

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)f(x,t)\ \leq\ t|\nabla A|^2\Big(\Lambda\Big(K_2+\frac{L_1K_4}{\Lambda}\Big)+K_2L_1|A|^2\Big)+\frac{L_2}{2}C|A|^2+t\,R+K_5.$$

Choosing now $L=\max\{K_2+\frac{L_1K_4}{\Lambda},C\}$, we get:

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)f(x,t)\ &\leq\ t|\nabla A|^2(\Lambda L+LL_1|A|^2)+L\frac{L_2}{2}|A|^2+t\,R+K_5\\ &\leq\ L\Big(t|\nabla A|^2(\Lambda+L_1|A|^2)+\frac{L_2}{2}|A|^2\Big)+t\,R+K_5\\ &\leq\ L\,f(x,t)+t\,R+K_5.\end{aligned}$$

We set $f_{\max}(t)=\max_{x\in M}f(x,t)$ and we apply the maximum principle. For this purpose we study the associated ODE:

$$h'(t)=Lh(t)+tR+K_5,\quad h(0)=f_{\max}(0).$$

As it is well known, this equation has solution:

$$h(t)=e^{Lt}h(0)+\int_0^t e^{L(t-s)}(sR+K_5)\,ds.$$

We then study:

$$\begin{aligned}e^{Lt}\int_0^t e^{-Ls}(sR+K_5)\,ds\ &=\ e^{Lt}\Big(\Big[-\frac{e^{-Ls}}{L}(sR+K_5)\Big]_0^t+\int_0^t\frac{e^{-Ls}}{L}R\,ds\Big)=\\ &=\ e^{Lt}\Big(-\frac{e^{-Lt}}{L}(tR+K_5)+\frac{K_5}{L}-\frac{R}{L^2}e^{-Lt}+\frac{R}{L^2}\Big)=\\ &=\ -\frac{1}{L}(tR+K_5)-\frac{R}{L^2}+e^{Lt}\,\frac{LK_5+R}{L^2}.\end{aligned}$$

And then:

$$h(t)=e^{Lt}h(0)-\frac{1}{L}(tR+K_5)-\frac{R}{L^2}+e^{Lt}\,\frac{LK_5+R}{L^2}.$$

Applying finally the maximum principle, we can say that

$$f_{\max}(t)\leq h(t),\quad\forall t\in[0,T'].$$

But $\forall t\in[0,T']$ it is also true that:

$$f(x,t)\leq f_{\max}(t)\leq h(t)\leq e^{LT'}\Big(h(0)+\frac{LK_5+R}{L^2}\Big)=D,$$

which implies that for $f(x,t)=t\,|\nabla A|^2\big(\Lambda+L_1|A|^2\big)+\frac{L_2}{2}|A|^2$, with $\Lambda\geq C(1-L_1)+1\geq 1$:

$$t\,|\nabla A|^2\leq t\,|\nabla A|^2\big(\Lambda+L_1|A|^2\big)+\frac{L_2}{2}|A|^2\leq D,$$

and finally we have proved that

$$\sup_{x\in M_t} |\nabla A| \leq \frac{D}{\sqrt{t}},$$

$\forall t \in [0, T']$. $\square$

We can extend the previous result to the higher covariant derivatives of A, assuming though, as it will be clear from the proof, that we also have an initial bound on $|\nabla A(x, 0)|$.

Theorem 1.15. *Let the mean curvature and the second fundamental form of M_t be bounded $\forall t \in [0, T]$. If for each $m \geq 1$ there is a small positive α such that $|\bar{\nabla}^m \overline{Rm}|^2 \leq \alpha$, then $\forall\, m \geq 1$ there is a positive constant C_m depending on M_0 such that*

$$|\nabla^m A|^2 \leq C_m,$$

uniformly on M_t for $0 \leq t \leq T < \infty$.

Proof. In view of (1.2.13), we have:

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^m A|^2 \;&=\; \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A + \\
&\quad + \sum_{i+j=m} \nabla^i A * \nabla^m A * \bar{\nabla}^j\, \overline{Rm} + P(A, \nabla A, \nabla^m A, \overline{Rm}, \bar{\nabla}\, \overline{Rm}) + \\
&\quad + \nabla^m A * \bar{\nabla}^{m+1}\, \overline{Rm} + \phi \nabla^m A * \bar{\nabla}^m\, \overline{Rm} + \phi \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A + \\
&\quad + \phi \sum_{i+j=m-1} \nabla^m A * \nabla^i A * \bar{\nabla}^j\, \overline{Rm} + \phi\, Q(A, \nabla A, \nabla^m A, \overline{Rm}, \bar{\nabla}\, \overline{Rm}).
\end{aligned} \tag{1.3.15}$$

Recall the following fact (*Young's inequality*) which we are going to use below:

$$|ab| \leq \frac{|a|^2}{2} + \frac{|b|^2}{2}.$$

Note also that we assumed that the second fundamental form is bounded, and therefore we have already proved the $m = 0$ case, since $|A|^2 = |\nabla^0 A|^2$. Moreover the mean curvature H is bounded as well, and this implies that the average term $\phi(t)$ for each t is bounded too.

Let now consider the following. If we look to the term $\nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A$, we ideally have two different cases:

(1) $i = j = 0$ and $k = m$ and therefore we estimate the term as $|A * A * \nabla^k A * \nabla^m A| \leq K|A|^2|\nabla^m A|^2$, using the properties of the $*-$product, with K the structural constant.

(2) all the $i, j, k < m$: we then estimate the same term as $|\nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A| \leq |\nabla^i A * \nabla^j A * \nabla^k A|^2/2 + |\nabla^m A|^2/2$ by the Young's inequality above.

It is clear that we can apply the above process to each term in the sum of (1.2.13). We thus estimate (1.2.13) as following:

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 &\leq \Delta|\nabla^m A|^2 + P_1\big(|A|,\dots|\nabla^{m-1}A|,|\bar\nabla\,\overline{Rm}|,\dots|\bar\nabla^{m+1}\,\overline{Rm}|,|\phi|)\big)|\nabla^m A|^2 + \\ &\quad + P_2\big(|A|,\dots|\nabla^{m-1}A|,|\bar\nabla\,\overline{Rm}|,\dots|\bar\nabla^{m+1}\,\overline{Rm}|,|\phi|)\big),\end{aligned}$$

where P_1 and P_2 are smooth functions independent of time (and actually they are polynomials in their arguments).
We now proceed by induction on m. Since the case $m = 0$ is true by hypothesis, we assume that all covariant derivatives of A up to order m are uniformly bounded by C_m. Since even the terms $|\phi|$ and $|\bar\nabla^m\,\overline{Rm}|$ are bounded $\forall m \in \mathbb{N}$, we also deduce that the functions P_1 and P_2 are bounded. Therefore we write for some C_2 depending on C_m and α:

$$\frac{\partial}{\partial t}|\nabla^{m+1}A|^2 \leq \Delta|\nabla^{m+1}A|^2 + C_2(|\nabla^{m+1}A|^2 + 1).$$

Choosing $N \geq C_2$ and $h = |\nabla^{m+1}A|^2 + N|\nabla^m A|^2$, we have:

$$\begin{aligned}\frac{\partial}{\partial t}h &= \frac{\partial}{\partial t}|\nabla^{m+1}A|^2 + N\frac{\partial}{\partial t}|\nabla^m A|^2 \leq \\ &\leq \Delta|\nabla^{m+1}A|^2 + C_2(|\nabla^{m+1}A|^2 + 1) + N\Delta|\nabla^m A|^2 - 2N|\nabla^{m+1}A|^2 + C_3 \\ &\leq \Delta h - N|\nabla^{m+1}A|^2 + C_3,\end{aligned}$$

applying the inductive hypothesis at the case m, with $C_3 = C_3(C_m, C_2, N, \alpha)$. But this implies that:

$$\frac{\partial}{\partial t}h \leq \Delta h - N\,h + C_3 + N^2\,C_m.$$

We have that a solution for the auxiliary problem

$$\frac{\partial}{\partial t}\varphi(t) = -N\varphi(t) + C_3 + N^2\,C_m, \quad \varphi(0) = h(0),$$

is:

$$\varphi(t) = \varphi(0)e^{-Nt} - \frac{C_3 + N^2\,C_m}{N}e^{-Nt} + \frac{C_3 + N^2\,C_m}{N},$$

which implies, substituting and by maximum principle and simple estimates,

$$|\nabla^{m+1}A(x,t)|^2 \leq \Big(|\nabla^{m+1}A(x,0)|^2 + N|\nabla^m A(x,0)|^2\Big)\cdot 1 + \frac{C_3 + N^2\,C_m}{N},$$

and finally

$$|\nabla^{m+1}A(x,t)|^2 \leq C_{m+1},$$

with C_{m+1} depending on C_m, C_3, N, α and $\max_{x\in M_0}|\nabla^{m+1}A(x,0)|^2$, since by induction $|\nabla^m A(x,0)|^2 \leq C_m$. This concludes the proof. □

1.4 Convergence of surfaces

The aim of this section is to introduce some important results about the convergence of surfaces. We will also introduce some isoperimetric inequalities both in the Euclidean and Riemannian setting, and, as a consequence, results about convergence.

1.4.1 Isoperimetric inequalities

Let us initially suppose to be in a $(n+1)-$dimensional Euclidean space endowed with its standard metric $(\mathbb{R}^{n+1}, \delta_{ij})$. If $\Omega \subset \mathbb{R}^{n+1}$ is an open bounded region with smooth boundary $\partial\Omega$, as in [37], the classical *isoperimetric inequality* reads

$$\frac{|\partial\Omega|^{n+1}}{|\Omega|^n} \geq \frac{(A(S^n))^{n+1}}{(V(B^{n+1}))^n},$$

where, with abuse of notation, we have denoted with $|\partial\Omega|$ the surface area of the boundary, with $|\Omega|$ the volume of the open region, with $A(S^n)$ the area of the n-dimensional unit sphere and by $V(B^{n+1})$ the volume of the $(n+1)$-dimensional unit ball. Note that the equality holds only if the region is the unit ball.

Observation 1. Note also that this ratio is independent of the radius of the ball. In fact, considering a ball of radius R with the same notation as before, we have:

$$\begin{aligned}\frac{(A(S^n(R)))^{n+1}}{(V(B^{n+1}(R)))^n} &= \Big(\frac{A(S^n(R))}{V(B^{n+1}(R))}\Big)^n \cdot A(S^n(R)) = \Big(\frac{n+1}{R}\Big)^n \cdot A(S^n)R^n = \\ &= (n+1)^n \cdot A(S^n) = C_e,\end{aligned}$$

calling with C_e the above ratio.
Since the area of the unit sphere in $\mathbb{R}^{n+1}$ can be written using the gamma function Γ, we have in this case that

$$C_e = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\Big(\frac{n+1}{2}\Big)} \cdot (n+1)^n.$$

Observation 2. The isoperimetric inequality above also shows that, amongst all regions with the same boundary area, the Euclidean balls have maximum volume. The inequality tells furthermore that, amongst all sets with the same volume, Euclidean balls have minimum boundary area.

If (N, g) is a smooth $(n+1)-$dimensional complete Riemannian manifold, we define the *isoperimetric profile* of N the function $I_N : (0, |N|) \to \mathbb{R}$, with

$$I_N(v) = \inf\{|\partial\Omega| : \Omega \subset\subset N \text{ has smooth boundary}, |\Omega| = v\},$$

for a region Ω of N. Note that, with an abuse of notation as before, $|\cdot|$ denotes both the area of the boundary and the volume of the region, i.e. the n and $(n+1)-$dimensional Riemannian measures.
The isoperimetric profile gives an isoperimetric inequality in N, since any region $\Omega \subset\subset$

N with smooth boundary satisfies

$$|\partial\Omega| \geq I_N(|\Omega|).$$

The isoperimetric inequality is optimal in the sense that, if some function I exists so that $|\partial\Omega| \geq I(|\Omega|)$ for any region $\Omega \subset\subset N$ with smooth boundary, one trivially has $I_N \geq I$.

Given a positive $v < |N|$, the *isoperimetric problem* consists in studying, among the compact hypersurfaces $\Sigma \subset N$ enclosing a region Ω of volume $\Omega = v$, those which minimize the area $|\Sigma|$.

Note that the following fundamental existence and regularity theorem holds. (See for a general review [32] for example).

Theorem 1.16. *If N is a compact $n-$dimensional manifold, then, for any v, $0 < v < |N|$, there exists a compact region $\Omega \subset N$ whose boundary $\Sigma = \partial\Omega$ minimizes the area among regions of volume v. Moreover, except for a closed singular set of Hausdorff dimension at most $n-8$, the boundary Σ of any minimizing region is a smooth hypersurface with constant mean curvature and, if $\partial N \cap \Sigma \neq \emptyset$, then ∂N and Σ meet orthogonally.*

In particular, if $n \leq 7$, Σ is smooth. In general a minimizing sequence $\{\Omega\}_{i\in\mathbb{N}}$ of sets with smooth boundary and volume v so that $|\partial\Omega_i| \to I_N(v)$ may converge, in a weak topology, to some set with non-smooth boundary. This motivates therefore the following definition:

Definition 1.9. An isoperimetric region of volume v in N is a finite perimeter set Ω_0 so that $|\Omega_0| = v$ and $\mathcal{P}(\Omega_0) = I_N(v)$, where the perimeter of a region Ω is defined as

$$\mathcal{P}(\Omega) = \sup\Big\{\int_\Omega \operatorname{div} X\, dN : |X| \leq 1\Big\},$$

for any smooth vector field X of N with compact support and where $|X|$ is the supremum norm $\sup\{|X|_p : p \in N\}$.

Note that the existence of isoperimetric regions are guaranteed, under some conditions, in compact manifolds, but not in non compact ones.

We are ready to prove the following Theorem.

Theorem 1.17. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a family of open sets of $(\mathbb{R}^{n+1}, \delta_{ij})$ endowed with g^k_{ij} Riemannian metrics defined on $B_4(0)$, such that $\forall k$*

$$\Omega_k \subset B_4(0),$$

with smooth boundaries $\partial\Omega_k$ and such that $g^k_{ij} \to \delta_{ij}$ smoothly to the Euclidean metric. Given two positive constants C and C', let us also suppose that the family of regions is such that $\forall k$:

(1) $|\Omega_k| = 1$ *and* $|\partial\Omega_k| \le C'$,

(2) $|A|_{\partial\Omega_k} \le C$ *and* $|\nabla A|_{\partial\Omega_k} \le C$,

(3) $\frac{|\partial\Omega_k|^{n+1}}{|\Omega_k|^n} \le C_e + \rho_k$,

with $\rho_k \to 0$, *and where* A *and* ∇A *denote respectively the second fundamental form of the boundary and its covariant derivative. Then we have that:*

$$\partial\Omega_k \to \partial B_r(x) \subset \overline{B}_4(0),$$

in C^2*-norm and such that* $|B_r(x)| = 1$.

Proof. Observe that for any k we have $\Omega_k \subset B_4(0)$, and therefore, by the Bolzano-Weierstrass Theorem, there exists an accumulation point $p \in \overline{B}_4(0)$ for the family $\{\Omega_k\}_k$. Note that the curvature estimates (2), combined with the Uniform Graph Lemma (see for example [35], Lemma 4.1.1), imply that there exist $R = R(p) > 0$ and disjoint graphs $U_k^i \subset \mathbb{R}^{n+1}$ of functions u_k^i defined over disks on the corresponding tangent planes $T_p\partial\Omega_k$, i.e. $B(p, 2R) \cap (p+ < \nu_k^i >^\perp)$, where ν_k^i are the corresponding unit normal vectors, $B(p, 2R) = B_{2R}(p)$ and with $1 \le i \le s = s(p, k)$, such that:

i) $\partial\Omega_k \cap B(p, R) = (U_k^1 \cup \cdots \cup U_k^s) \cap B(p, R)$.

ii) $|u_k^i|$, $|\nabla u_k^i|$ and $|\nabla^2 u_k^i|$ are uniformly bounded in the corresponding disk of radius $2R$, for all $k \in \mathbb{N}$ and $i = 1, \dots, s$.

Observe first that in fact the number s of such graphs (the so called *multiplicity* of p) doesn't depend on k, since we have an upper bound for the area of the surfaces $\partial\Omega_k$. Also note that we can take a convergent subsequence for the family of unit normals $\{\nu_k^i\}_k$, since they are "points" of the unit sphere, and therefore $\nu_k^i \to \nu^i$ and the tangent planes subconverge to a fix plane $\Pi_p = p+ < \nu^i >^\perp$. This also implies, by the Uniform Graph Lemma again, that there exists a subsequence of graphs $\{U_{k_l}^i\}_{k_l} \subset \{U_k^i\}_k$ over disks of radius $2R$ and centred in p in the plane Π_p, with the corresponding functions $u_{k_l}^i$ uniformly bounded up to the second derivative.
The uniform bounds for the graphs functions $u_{k_l}^i$ and its first two derivatives imply that these functions are equicontinuous on their domains and therefore, by the Ascoli-Arzelà Theorem, there exists a C^2-convergent subsequence of graph functions to the limit function u_∞^i.
Observe now the hypothesis (3) of this Theorem. By the isoperimetric inequality and the observations at the beginning of this paragraph, we have that

$$C_e \le \frac{|\partial\Omega_k|^{n+1}}{|\Omega_k|^n} \le C_e + \rho_k.$$

Since by hypothesis $\rho_k \to 0$, when $k \to \infty$, and by (1), $|\Omega_k| = 1$, $|\partial\Omega_k| \le C'$ for any k, the combination of these two facts gives a bound on the value of surface area and then forces the multiplicity i of p to be equal to 1.

If $p \in \overline{B}_4(0)$ is not an accumulation point for the family $\{\partial\Omega_k\}_{k\in\mathbb{N}}$, then we can choose a subsequence $\partial\Omega_{k_l}$ and $R > 0$ such that $\partial\Omega_{k_l} \cap B(\bar{p}, R) = \emptyset$.

Let us now take a countable dense set $\mathcal{A} = \{p_1, p_2, \dots\} \subset B_4(0)$. Applying the above process around p_1, we obtain a subsequence $\{\partial\Omega_{1,k_l}\}_{k_l} \subset \{\partial\Omega_k\}_k$ which converges in $B(p_1, R(p_1))$ to a limit graph. Applying again the process to $\{\partial\Omega_{1,k_l}\}_{k_l}$ around p_2 we obtain another subsequence $\{\partial\Omega_{2,k_l}\}_{k_l} \subset \{\partial\Omega_{1,k_l}\}_{k_l}$, which converges in $B(p_1, R(p_1)) \cup B(p_2, R(p_2))$ to a surface of multiplicity 1. Iterating the process and taking a diagonal subsequence, we obtain that $\{\partial\Omega_{k_l}\}_{k_l} \subset \{\partial\Omega_k\}_k$ converges to a surface $\partial\Omega_\infty$ in $B_4(0)$ of volume 1. Finally, combining (1), (3) and the isoperimetric inequality which states that in the Euclidean space the equality holds only for balls, we therefore have that the limit surface $\partial\Omega_\infty = \partial B_r(x) \subset \overline{B}_4(0)$, for some $r > 0$ and $x \in \overline{B}_4(0)$, and with volume $|B_r(x)| = 1$. $\square$

Recall the following Lemma previously introduced, that we are going to use in the proof of the Corollary below.

Lemma 1.18. *Let $p \in (N, g)$, with N compact and ε its injectivity radius. Then for each $r \leq \varepsilon$ and $x \in N$ such that $B_r(x) \subset \exp_p(B_\varepsilon(0))$,*

$$\exp_x(B_r(0)) = B_r(x),$$

and moreover

$$\exp_x(\bar{B}_r(0)) = \bar{B}_r(x).$$

In the Riemannian setting, we will make use of the following Corollary.

Corollary 1.19. *Let $(N, \bar{g})$ be a compact $(n+1)-$Riemannian manifold with injectivity radius ε and, for any $p \in N$, U_p the geodesic ball in p of radius ε. Let $\Omega \subset B_R(p) \subset U_p$, with $R < \varepsilon$, be an open region of N with smooth boundary $\partial\Omega$. Let $\tilde{C}$, C', C'' and $C = C(I_N(|\Omega|))$ positive constants. For any given $\delta > 0$, there exist $\rho > 0$ and $r_0 > 0$ such that if:*

(1) $r \leq r_0$,

(2) $|\Omega| = |B_r(p)|$, $|\partial\Omega| \leq \tilde{C} r^{n-1}$, $|A|_{\partial\Omega} \leq C' r^{-1}$ and $|\nabla A|_{\partial\Omega} \leq C'' r^{-2}$,

(3) $\frac{|\partial\Omega|^{n+1}}{|\Omega|^n} \leq C + \rho$,

then $\partial\Omega$ is δ-close in the C^2-norm to $\partial B_r(x)$, for some $x \in B_R(p)$, with $|B_r(x)| = |B_r(p)|$, and where $|B_r(p)|$ is the Euclidean volume of $B_r(p)$.

Proof. Note that in the normal neighbourhood $U_p = \exp_p(B_\varepsilon(0))$, where $B_\varepsilon(0)$ is the sphere in the tangent space T_pN of radius $\mathrm{inj}(N, g) = \varepsilon > 0$, we have seen that we can expand the metric g in a point $q \in U_p$ as

$$\begin{aligned} g_{ij}(q) &= \delta_{ij} + \frac{1}{3} R_{kilj}|_p\, x^k x^l + \frac{1}{6} \nabla_k R_{limj}|_p\, x^k x^l x^m + \cdots \\ &= \delta_{ij} + \frac{1}{3} R_{kilj}|_p\, x^k x^l + O(r^3), \end{aligned} \tag{1.4.16}$$

where R_{ijkl} is the Riemann tensor of N computed in p, $r = \psi(x) = d(p, x)$ is the radial distance function and q has coordinates $q = (x^1, x^2, \cdots x^{n+1})$.
Therefore we also have the following estimate

$$|g_{ij} - \delta_{ij}| = O(r^2) \leq r_0^2, \tag{1.4.17}$$

by hypothesis (1).
Since the region is such $\Omega \subset B_R(p) \subset U_p$, with $R < \varepsilon$, we can make use of normal coordinates,

$$\exp_p^{-1} : U_p \to T_pN \simeq \mathbb{R}^{n+1},$$

that are unique up to how we choose to identify T_pN with $\mathbb{R}^{n+1}$. This also implies that, through the above diffeomorphism, we can consider, with an abuse of notation, that $\Omega \subset B_R(0) \subset T_pN \simeq \mathbb{R}^{n+1}$.
We want to prove the Theorem by contradiction. We therefore assume that the region (Ω, g) satisfies (2) and (3), but it is not $\delta-$close in the C^2-norm to the sphere of radius r and centred in x, $\partial B_r(x)$, for $x \in B_R(0)$. But this is equivalent to assume that there thus exists a sequence of regions $\{\Omega_k\}_{k\in\mathbb{N}}$ endowed with Riemannian metrics g^k_{ij}, converging smoothly to the flat metric δ_{ij} by (1.4.17) and satisfying the following conditions

(2) $|\Omega_k| = |B_r(p)|$, $|\partial\Omega_k| \leq \tilde{C}r^{n-1}$, $|A|_{\partial\Omega_k} \leq C'r^{-1}$ and $|\nabla A|_{\partial\Omega_k} \leq C''r^{-2}$,

(3) $\frac{|\partial\Omega_k|^{n+1}}{|\Omega_k|^n} \leq C + \rho_k$, with $\rho_k \to 0$,

but not converging to the sphere $\partial B_r(x) \subset T_pN$.
It is clear that we are exactly in the situation of Theorem 1.17. Note in fact that the hypothesis (3), where the constant C depends on the isoperimetric profile of the constant volume of $|\Omega_k| = |B_r(p)|$ (Euclidean measure), can be rewritten using the Euclidean constant C_e plus a new parameter $\bar{\rho}_k$, that measures how much the metric g^k_{ij} fails to be flat. We can therefore write that:

$$C_e \leq \frac{|\partial\Omega_k|^{n+1}}{|\Omega_k|^n} \leq C_e + \bar{\rho}_k, \quad \bar{\rho}_k \to 0.$$

Finally note that the estimates (1) and (2) of Thm. 1.17 can be rescaled in order to fit the present setting.
Therefore we are in the situation of Thm. 1.17, and the family of surfaces $\{\Omega_k\}$ must converge to a sphere $\partial B_r(x) \subset T_pN$ of radius $r > 0$ and center $x \in B_R(0)$ of same volume. Using now the Lemma 1.18 and bringing everything back to the manifold N, we observe that the family converges to a Riemannian sphere $\partial B_r(x) \subset U_p \subset N$, of same fixed volume, arising therefore a contradiction.
This concludes the proof. $\square$

Chapter 2

Literature review

2.1 The existing results

Traditionally, differential geometry has been the study of curved spaces or shapes in which, for the most part, time did not play a role. However, in the last decades geometers have made a huge effort in understanding how shapes evolve in time. There are many ways by which a geometric object can evolve over time, but the most natural one is surely by the mean curvature flow.

The simplest case is that of a closed curve in the plane, where the flow is usually called "curve shortening flow". A remarkable result of Grayson [15] from 1987, using earlier work of Gage and Hamilton [12], shows that any simple closed curve in the plane remains smooth and eventually becomes convex under the flow, until it disappears in a point in a finite amount of time. As a corollary, one can get an exact formula for the lifespan of any curve. Therefore, in the case of curve shortening flow, each flow has only one singularity in all of space and time and the singularity looks just like a shrinking circle.

In higher dimensions, Huisken [18], inspired by Hamilton's paper [16], proved in 1984 that closed convex hypersurfaces remain convex and flow smoothly up until they become extinct at a point, and therefore no singularity will occur before the surface shrinks down to a point; in particular, they are almost round just before extinction. To better describe such "round sphere" behaviour, Huisken carries out a normalization and keeps fixed the area of the surface solution during the flow: the normalized equation has then a solution for any positive times and converges exponentially in any C^k-norm to a sphere of the same area.

However, unlike the case of curves, there are many new types of singularities when the initial hypersurface is not convex. Therefore the analogue of Grayson's theorem doesn't hold for submanifolds of dimension $n \geq 2$. Consider for example two spheres joined by a long thin tube. The spheres and the tube both shrink, but the mean curvature along the tube is much higher than on the spheres, so the middle of the tube collapses down to a point, forming a singularity. The surface then separates into two components, which eventually become convex and collapse to round points. Thus, differently from a curve, a surface can develop singularities before it shrinks away.

In 1986, Huisken [19] extends these results to the more general Riemannian case. He proves that if the initial hypersurface is "convex enough" in order to overcome the obstructions imposed by the geometry of the Riemannian ambient space, it first contracts to a small sphere and then to a single point. By convex enough, he means that the principal curvatures of the initial surface are bounded from below by a positive constant depending on the ambient manifold. If the same normalization in normal coordinates around the 0 point is carried out as in the Euclidean case in order to keep fixed the area, the solution assumes again a round behaviour and converges to a sphere of the same area. Note that the use of some pinching condition or other geometric assumptions on the initial surface is crucial when one is dealing with the mean curvature flow, to avoid that singularities may develop even before the volume goes to zero.
Since Huisken's work, many authors have investigated whether the same result holds for flows where the speed is given by a general symmetric, positively homogeneous function of the principal curvatures.
For example, flows with speed a power of the mean curvature H, the so called *standard* H^k-*flow*, have been first analysed by Schulze ([40], [41]), who later obtained a very interesting application of this evolution to isoperimetric inequalities in Euclidean and noneuclidean spaces [42]. Schulze observes that the H^k-flow of a smooth immersed mean convex hypersurface ($H(M_0) > 0$) in the Euclidean space has a unique smooth solution. Moreover, if the initial surface is strictly convex (when $0 < k < 1$) or weakly convex (when $k \geq 1$), the solution is strictly convex and contracts to a point in a finite amount of time. He also shows that the same result can be obtained if the initial surface satisfies a pinching condition instead (regarding the ratio of the Gauss curvature and the $n-$power of the mean curvature), and that the rescaled embeddings assume as usual the round sphere behaviour, with a $C^\infty-$topology convergence to a sphere.

The asymptotic round behaviour of the renormalised mean curvature flow inspired Huisken to investigate a flow where the volume enclosed by a compact immersed hypersurface without boundary is kept fixed instead of its area. In 1987, he proves, as expected, that the volume preserving mean curvature flow (VPMCF) of a strictly convex hypersurface immersed in the Euclidean space has a smooth solution which exists for all positive times and converges exponentially in the $C^\infty-$topology to a round sphere [20]. However, the $\phi(t)$ term in the flow equation, i.e. the average of the mean curvature, introduces a global term in all relevant evolution equations, making the analysis of this flow more complicated.
Cabezas-Rivas and Sinestrari [7] study the volume preserving curvature flow where the speed is given by a m-th power of the mean curvature, a particular symmetric homogeneous polynomial in the principal curvatures. They show that if a closed hypersurface immersed in the Euclidean space and satisfying a pinching condition on K, the Gauss curvature ($K > cH^n > 0$, for a positive constant c), the condition is preserved, the flow is immortal and the solution converges exponentially in the $C^\infty-$norm to a limiting hypersurface which is umbilical everywhere, and therefore is a sphere. Note also

that the pinching condition, preserved during the flow, implies that the surface solution maintains the principal curvatures strictly positive.
The study of the generalised H^k-flow in the volume-preserving (and area- preserving) setting has been first dealt by Sinestrari in [46]. Here, without any pinching condition or restriction on the dimensions, but just exploiting the isoperimetric nature of the flow (the area is decreasing if the flow runs keeping the volume fixed), a strictly convex hypersurface initially embedded in $\mathbb{R}^{n+1}$ converges asymptotically when $t \to \infty$ to a sphere, and the flow is then immortal. Note that if the law is area-preserving, the isoperimetric ratio is still decreasing since the area is fixed but the volume of the region is increasing. In 2018, the same result has been obtained by Bettini and Sinistrari [5] by studying the $\alpha-$power of $k-$th symmetric polynomial in the principal curvatures (E_k^α, $\alpha > 0$) without assuming any pinching condition, but only assuming that the initial hypersurface embedded in $\mathbb{R}^{n+1}$ is strictly convex.
The immortality of the VPMCF and the exponential convergence to a round sphere has been obtained also by H. Li in [28], without assuming a convexity property on the initial closed hypersruface immersed in the Euclidean space. Li instead assumes an integral condition on the traceless second fundamental form, a positive lower bound for the average mean curvature and an initial uniform bound for the second fundamental form. Observe that, if the initial surface is close enough to a sphere, the lower bound on the average term is a reasonable one, since it is always positive. In the second part of the paper, he weakens the constraints on the second fundamental form, and replace it with one on the mean curvature and use $\epsilon-$regularity theory, even though he gives a proof only in dimension $n + 1 = 3$. Note also, as we will see later, that both the results of Huisken and Yau [23] for large coordinate spheres in an asymptotically flat manifold and of Alikakos and Freire [1] for small geodesic balls in a Riemannian manifold, can be obtained using the same technique (he gives a proof only in the Euclidean case though).
Using the center manifold analysis from infinite-dimensional dynamical systems and semigroup theory, Escher and Simonett [11], Athanassenas [3] and Hartley [17] for the Euclidean case, and Cabezas-Rivas and Miquel [6] and Alikakos and Freire [1] in more general cases as we will see later, have studied as well the volume preserving mean curvature flow.
Escher and Simonett observe that if an initial surface is $h^{1+\beta}(S)-$close to a sphere S in the Euclidean space, the VPMCF is immortal and the solution converges exponentially to some sphere. Here the space $h^{1+\beta}(S)$ denotes the little Hölder space of order $1 + \beta$ (i.e. the limit sup of the Holder $\beta-$seminorm of the first derivative is going to zero), and it is in fact the closure of $C^\infty(S)$ functions in the usual Hölder norm of $C^{1+\beta}(S)$. The most interesting fact is that the initial surface does not need to be necessarily convex to assure the global existence of the flow, and then there exist non-convex surfaces which are solutions of the flow and converge exponentially to a sphere.
In Athanassenas' work [3], a smooth, compact, rotationally symmetric, initial hypersurface is immersed between two hyperplanes which intersects orthogonally (thus a surface with boundary) and encloses a fixed volume. Assuming an extra condition on the volume enclosed by M_0, in order to avoid that during the flow the surface pinches off, she

proves that the flow exists for all times and the surface converges to a cylinder of the same volume. In this special case, the uniform convexity is replaced by the rotational symmetric assumption, since she is dealing with a surface with boundary. A similar result is obtained by Hartley [17], where the rotationally symmetric condition is replaced by the fact that the hypersurface is close enough to a cylinder of radius R (the height function belongs to some little Hölder space) and the condition on the volume is replaced by a condition on this radius R, which still guarantees that the solution doesn't touch the axis of rotation along the flow. Note also that it converges to a limiting cylinder, which might not be the same initial one.

The methods developed by Huisken in [20] about the evolution of a convex hypersurface moving by volume preserving mean curvature flow cannot be readily generalised to the Riemannian case. In view of the average mean curvature term, the local evolution of the initial surface M_0 depends heavily on the global shape of the hypersurface immersed in the ambient manifold, and therefore introduces a global aspect in the evolution equations of all the relevant geometric quantities, making the application of the parabolic maximum principle, where possible, more subtle. Note also that even the convexity properties of M_0 may not be preserved if M_0 is immersed in a general Riemannian manifold: Huisken illustrates in fact that if M_0 is for example a convex hypersurface in the sphere S^{n+1} with a portion of M_0 being C^2-close to an equator of S^{n+1}, it has in this region its average term $\phi >> H_0$, such that initially the hypersurface is moving here onto the other side of the equator, changing the sign of the second fundamental form.
In a joint work with Yau [23] to define the center of mass in a isolated gravitational system, Huisken extends for the first time the techniques of the VPMCF to a Riemaniann manifold which is asymptotically flat, with the crucial assumption of a strictly positive (ADM) gravitational mass m. More precisely, for a radius large enough, an initial coordinate sphere moving by the VPMCF law evolves and converges to a constant mean curvature sphere when $t \to \infty$. The hypothesis of $m > 0$ guarantees that the initial coordinate sphere is strictly stable and, in particular, is essential to prevent that the surface drifts off to infinity during the evolution. Moreover, they prove the existence a stable constant mean curvature foliation which can be considered as the center of mass for an infinitely far observer.
The study of the VPMCF in a noneuclidean ambient space is also treated by Cabezas-Rivas and Miquel in [6]. The two authors consider the case of a compact hypersurface convex by horospheres (*h-convex*) immersed in a hyperbolic space of constant negative sectional curvature and moving by volume preserving mean curvature flow. As expected, the flow exists for all times, the convexity property is preserved and the solution is converging exponentially to a geodesic sphere. Using the method of Escher and Simonett [11] based on maximal regularity theory to prove the exponential convergence, the two authors can strength their results to a bigger class of hypersurfaces non necessarily h-convex, if the initial hypersurface is $h^{1+\beta}-$close to a geodesic sphere. They also point

out the difference with the Alikakos' and Freire's work, since Cabezas-Rivas and Miquel are working in the special case of an ambient manifold of constant scalar curvature.
As observed by H. Li [28], the studies of the VPMCF in the noneuclidean cases, as the hyperbolic space for Cabezas-Rival and Miquel from one side, and the work of Alikakos and Freire in a general Riemannian manifold on the other side, as we will see in the next paragraph, are massively based on the center manifold analysis and, therefore, leave unclear how the shape of the initial hypersurface affects the convergence of the flow. It would be interesting to have proofs of convergence with more natural conditions on the geometry of the initial hypersurface.

2.2 The approach of Alikakos and Freire

In 2003, Alikakos and Freire [1] investigate for the first time the evolution of a hypersurface, not necessarily convex but close enough to a geodesic sphere, inside a general compact Riemaniann manifold, that moves by a volume preserving mean curvature flow. The two authors leave the classical approach of studying the mean curvature flows in intrinsic fashion through the evolution equations of the main geometric quantities of the hypersurface, and they embrace methods and results from semigroup theory about maximal regularity and from infinite dimensional systems, in the same spirit of Escher and Simonett [11]. The idea is to "decouple" the effect of the ambient manifold from the effect of the geometry of the interface, and to describe how an appropriate "barycentre" moves inside the bigger manifold, if the hypersurface starts and remains close to a small geodesic sphere.

More precisely, with the same notation as in [1], let M be a n-dimensional compact Riemannian manifold and consider the submanifold $\mathcal{E}$ of the Banach manifold of the $C^{2+\alpha}$ "small quasispherical embeddings" $X : S \to M$, which are radial graphs over a small geodesic sphere in M with centre $\xi \in M$ and radius $R > 0$, and S is the unit sphere in the Euclidean space. To define such an embedding, Alikakos and Freire need a diffeomorphism $F : S \to S_\xi$ (the unit tangent sphere at ξ in $T_\xi M$) and a "shape function" $\psi : C^{2+\alpha} \to \mathbb{R}$, which it can be taken as a $C^{2+\alpha}$ function on S, with zero average on S. Finally, after introducing two positive parameters $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$, they consider $\mathcal{E} = \mathcal{E}_{\delta_0,\epsilon_0}$ as the space of embeddings which can be written in the form:

$$X_{(R,\xi,F,\psi)}(u) = \exp_\xi[\delta R(1 + \epsilon\psi(u))F(u)],$$

with $0 < R < 1$, $\|\psi\|_{C^{2+\alpha}} < 1$ and $\mathrm{ave}_S[\psi] = 0$. Note that the positive parameters δ_0 and ϵ_0 have been taken small enough that the open set $\mathrm{int}(X)$ bounded by $\Sigma = \mathrm{image}(X)$ (and containing ξ) is contained in a totally convex neighbourhood of ξ, and is uniformly convex.

In this approach, there are some difficulties that arise. Firstly, the same embedding $X \in \mathcal{E}$ can be written in the form $X_{(R,\xi,F,\psi)}$ in different ways, parametrised by $\xi \in \mathrm{int}(X)$. Therefore there is the need to find a choice of ξ that is as canonical as possible, given X. This leads to Lemma 1.1 and the definition of a barycentre. Note that, as in [26], the definition of a barycentre for a general Riemannian manifold already exists. However, in the setting of the center manifold analysis, the two authors have to find a more suitable definition of it, which they call *analytic barycentre*. As showed in Lemma 1.1, this analytic barycentre is a solution of a specific equation and the unique point $\xi \in \mathrm{int}(X)$ for which X may be written as $X_{(R,\xi,F,\psi)}$. In particular, given $\delta_1 \in (0, \delta_{0/2})$ and $\epsilon_1 \in (0, \epsilon_{0/2})$, the existence of the analytic barycentre allows to consider evolution equations on the submanifold $\mathcal{N}_{\mathrm{std}} \subset \mathcal{N}(\delta_1, \epsilon_1) \subset \mathcal{E}_{\delta_1,\epsilon_1}$. By $\mathcal{N}_{\mathrm{std}}$ it is intended always that the embeddings are the ones where the point ξ is the analytic barycentre for X and, secondly, that the function F defined above is in fact an isometry. Note that the space

$\mathcal{N}_{\text{std}}$ can be seen also as the image under a smooth injective map Φ of the manifold:

$$\mathcal{M}_0 = \mathcal{M}_0(\delta_1, \epsilon_1) = (0, \delta_1) \times FM \times K^{2+\alpha}_{\epsilon_1},$$

where FM represents the orthonormal frame bundle of M and the last factor is the ϵ_1-ball in $K^{2+\alpha} = C^{2+\alpha}(S) \cap C_0(S)$.

The following step is to find an evolution equation for $(R, \xi, F, \psi) \in \mathcal{M}_0$, the solution of which map under Φ to parametrised solutions of the VPMCF in $\mathcal{N}_{\text{std}}$. It is important to observe at this point that the above definition of analytic barycentre depends not just on the image of the hypersurface Σ, but also on the parametrization X. This implies that if one wants to find an equation of the motion for this barycentre, one must fix an evolution equation for the parametrisation $X(t)$ as well. As a first attempt, one might expect that the equations on $\mathcal{M}_0$ would be induced by:

$$X_t = \big(H^{\Sigma} - H\big)\hat{N}, \tag{2.2.1}$$

where $\hat{N}$ denotes the unit outward normal and H^{Σ} is the average mean curvature. However, it is possible to show that $\mathcal{N}_{\text{std}}$ is not invariant under (2.2.1), and therefore there is no $X(t)$ in $\mathcal{N}_{\text{std}}$ solving the equation (2.2.1). It is possible though to compute "a tangential correction" to (2.2.1) which does preserve $\mathcal{N}_{\text{std}}$. In Lemma 1.7, the two authors find a system on $\mathcal{M}_0$ whose solutions map to parametrised solutions of the VPMCF, and, conversely any motion of $\Sigma(t)$ of small bubbles by VPMCF can be parametrised by $X_{(R,\xi,F,\psi)} \in \mathcal{N}_{\text{std}}$, so that $(R, \xi, F, \psi)(t)$ is a solution of the system on $\mathcal{M}_0$. In particular, for any choice of δ, ϵ, the system on $\mathcal{M}_0$ is defined by:

$$\begin{aligned} \delta R_t &= \text{ave}_S[v_N - E], \\ \xi_t &= n\,\text{ave}_S[(v_N - E)F], \\ \nabla_{\xi_t} F &= 0, \\ \delta\epsilon\, R\psi_t &= (v_N - E)_K - (\delta\psi)\text{ave}_S[v_N - E]. \end{aligned}$$

Here $v_N = (H^{\Sigma} - H)\|N\|$ (where N is a particular normal vector to Σ) and $E = E(v_N)$ corresponds to the tangential correction previously mentioned, and both computed at $X_{(\delta R,\xi,F,\epsilon\psi)}$.

The main theorem presented by the authors is substantially divided in four parts: local existence, global existence, motion of the barycentre and asymptotic behaviour. Even though the local existence is a well-known result, an equivalent proof is given in the framework of semigroup theory, since this leads to the continuation criterion used for global existence. Before proving global existence, a standard approach of Taylor expansions for Jacobi fields in Riemannian normal coordinates is used to develop the asymptotic expansion of the equations inherent the geometric quantities involved in the parametrisation above. The global existence is thus proved with an argument involving the variation of constants representation formula and maximal regularity estimates.

The motion of the barycentre is a way to keep track of how the hypersurface is moving inside the ambient manifold. Recall first the isoperimetric nature of the flow: the volume of the region enclosed by Σ is preserved, while the area is strictly decreasing, unless H is constant. This leads, in Section 3, Alikakos and Freire to show that the evolution equation for the centre has as leading term the (negative) gradient of the scalar curvature of M, i.e.

$$\partial_t \xi \sim \frac{2n}{3(n+2)} R^2 \, \nabla \mathrm{Scal}(\xi) + \cdots .$$

By the above formula, the velocity of the centre of the bubble ξ is therefore given, by principal order, by the gradient of the scalar curvature and the immersed surface is therefore expected to move where the scalar curvature of M is bigger, i.e. $\xi(t)$ climbs towards peaks of maximal scalar curvature.
In the last section, the number 4, Alikakos and Freire show the asymptotic convergence to a constant mean curvature sphere: since the critical points of the scalar curvature function are assumed nondegenerate, a small constant mean curvature sphere near a critical point must be a leaf of the local foliation at that critical point, and there is only one of those enclosing a given volume. Recall that a *foliation* of dimension k of a $n-$dimensional manifold M is a collection of disjoint, connected, immersed k-dimensional submanifold of M (*leaves* of the foliation) and such that in a neighbourhood of each point $p \in M$ there is a smooth chart satisfying particular properties. Then, as proved in section 4, the limit solution of the flow is a constant mean curvature surface with a barycentre that corresponds to a critical point for Scal. Moreover, such limit surface is the unique leaf of the local constant mean curvature foliation at p enclosing the same volume of the initial surface (see also the work of Ye in [49]).

Chapter 3

Euclidean case

The evolution of an immersed surface under the normalised volume preserving mean curvature flow is the result of a complicated interaction between the geometry of the evolving surface M and the geometry of the ambient space N. In view of the average mean curvature term in the flow equation, the local evolution of M depends heavily on the global shape of the hypersurface and the convexity properties of the initial surface M_0 may not be preserved if M_0 is immersed in a general Riemannian manifold.
We thus decide to first study the flow in the simpler case $N = \mathbb{R}^{n+1}$, since in the Euclidean ambient space the evolution equations of the main geometric quantities of M_0 can be treated much more easily. We will proceed in a different fashion from Huisken [20] or Sinestrati [46], for example, because our intention is to explicitly show the method we are going to use in the more complex Riemannian case.

Let us therefore consider first $(\mathbb{R}^{n+1}, \delta_{ij})$ as ambient space. Let $F : M \times I \to \mathbb{R}^{n+1}$ be a family of immersions, with $F_0(M) = M_0$ an $n-$dimensional *closed* and *strictly convex* manifold immersed in $(\mathbb{R}^{n+1}, \delta_{ij})$, with $I = [0, T)$, satisfying the following normal deformation

$$\frac{\partial}{\partial t} F = [-H(x,t) + \phi(t)] \cdot \nu(x,t), \tag{3.0.1}$$

where $\nu(x,t)$ is the unit normal vector in $x \in M_t$ and

$$\phi(t) = \frac{1}{|M_t|} \int_M H d\mu_t.$$

Note that, as already mentioned, we have $T = T_{\max}$ in the interval of time above. In this context where the immersed surface M is moving inside a flat space, the main result we want to prove is the following.

Theorem 3.1. *Let $(\mathbb{R}^{n+1}, \delta_{ij})$ be the Euclidean space endowed with its standard metric. Let $F : M \times I \to \mathbb{R}^{n+1}$ a family of immersions such that $F_0(M) = M_0$ is a closed $n-$surface with all principal curvatures strictly positive and such that it encloses a region of volume equal to $|B_1(0)|$. Then there exists a constant $\delta > 0$, such that if $M_0 \subset B_4(0)$ is $\delta-$close in C^2-norm to the unit ball centred in the origin, then the volume preserving mean curvature flow of M_0 has a smooth strictly convex solution with maximal time* *interval* $I = [0, \infty)$. *In particular, the family of immersions* $F : M \times I \to \mathbb{R}^{n+1}$ *converges exponentially to a limit immersion* F_∞ *with image equal to the unit sphere.*

We also reformulate Theorem 1.17 about the convergence of hypersurfaces in the following equivalent way.

Proposition 3.2. *Let* $(\Omega, g) \subset B_4(0)$ *be an open region of* $(\mathbb{R}^{n+1}, \delta_{ij})$ *and with smooth boundary* $\partial\Omega$. *Let* $\tilde{C}$, C', C'' *and* C_e *positive constants. Given* $\delta > 0$, *there exists a* $\rho > 0$ *such that if:*

(1) $||g_{ij} - \delta_{ij}||_{C^4(\bar{\Omega})} \leq \rho$,

(2) $|\Omega| = |B_1(0)|$, $|\partial\Omega| \leq \tilde{C}$, $|A|_{\partial\Omega} \leq C'$ *and* $|\nabla A|_{\partial\Omega} \leq C''$,

(3) $\frac{|\partial\Omega|^{n+1}}{|\Omega|^n} \leq C_e + \rho$,

then $\partial\Omega$ *is* δ*-close in the* C^2*-norm to* $\partial B_1(x)$, *for some* $x \in B_4(0)$, *with* $|B_1(x)| = |B_1(0)|$ *and*

$$C_e = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot (n+1)^n.$$

3.1 Initial estimates

Let $F : M \times I \to \mathbb{R}^{n+1}$ be a family of immersions moving by volume preserving mean curvature flow as in the hypothesis of Thm. 3.1, with $F_t(M) = M_t$, $\forall t \in [0, T)$. We will consider $M_t = \partial\Omega_t$ to be the smooth boundary of an open region $\Omega_t \subset \mathbb{R}^{n+1}$, at the time $t \in [0, T)$.

Definition 3.1. If $\Omega_t \subset \mathbb{R}^{n+1}$ is an open region with smooth boundary, we define as *isoperimetric ratio* the following expression

$$\mathcal{I}(t) = \frac{|\partial\Omega_t|^{n+1}}{|\Omega_t|^n}, \tag{3.1.2}$$

with $t \in [0, T)$.

Remark 2. *The classical isoperimetric inequality in the Euclidean space thus states that $C_e \leq \mathcal{I}(t)$, $\forall t \in [0, T)$, with as usual*

$$C_e = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot (n+1)^n,$$

where the first factor is the surface area of the unit n-sphere in $\mathbb{R}^{n+1}$ and Γ the gamma function.

At the initial time $t = 0$, by Thm. 3.1, the closed and strictly convex surface M_0 is δ-close in the C^2-norm to the unit ball of principal curvatures $\tilde{k}_i = 1$, $\forall i = 1, 2, \ldots, n$, and it is contained in a ball of radius big enough, i.e. $M_0 = \partial\Omega_0 \subset B_4(0)$.
We then choose $\delta > 0$ such that the principal curvatures of M_0 are between

$$\frac{1}{2} \leq k_i(x, 0) \leq 2 \quad \forall i = 1, 2, \ldots, n \quad \forall x \in M_0.$$

Therefore $\forall x \in M_0$ we have the following estimates for the mean curvature and the second fundamental form:

(1) $|H(x, 0)| = \sum_{i=1}^n k_i \leq 2n$,

(2) $|A(x, 0)| = \sqrt{\sum k_i^2} \leq 2\sqrt{n}$.

By the hypothesis of Thm 3.1, we also have that the volume of the region Ω_0 enclosed by M_0 is equal to the one of the unit ball, i.e.

(3) $|\Omega_0| = |B_1(0)|$,

and, since $M_0 = \partial\Omega_0 \subset B_4(0)$, also that

$$|\partial\Omega_0| \leq |\partial B_4(0)|,$$

where

$$|\partial B_4(0)| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} 4^n.$$

We set $C' = 5n$ and $|\partial B_4(0)| = \tilde{C}$ and we also assume that $|\partial\Omega_0|$ satisfies the following hypothesis:

$$\frac{|\partial\Omega_0|^{n+1}}{|\Omega_0|^n} = \mathcal{I}(0) \leq C_e + \rho,$$

for a positive $\rho = \rho\,(\tilde{C}, C', C'', n, \delta)$, where this ρ is the one given by Prop. 3.2 and for a positive constant C'' we determine later.

We summarize below the estimates we have at the initial time $t = 0$:

(1) $|H(x,0)| \leq C'$,

(2) $|A(x,0)| \leq C'$,

(3) $|\Omega_0| = |B_1(0)|$,

(4) $|\partial\Omega_0|^{n+1} \cdot |\Omega_0|^{-n} = \mathcal{I}(0) \leq C_e + \rho$.

Observe that by (3) and (4), we also have the following estimate:

$$|\partial\Omega_0| \leq \left[(C_e + \rho)|\Omega_0|^n\right]^{\frac{1}{n+1}} = \tilde{C}.$$

3.2 Monotonicity

From the previous section, the isoperimetric ratio satisfies this inequality at the time $t = 0$:

$$\frac{|\partial\Omega_0|^{n+1}}{|\Omega_0|^n} = \mathcal{I}(0) \leq C_e + \rho. \tag{3.2.3}$$

We can furthermore prove the following.

Proposition 3.3. *The inequality (3.2.3) is preserved during the flow. In particular,*

$$C_e \leq \mathcal{I}(t) \leq \mathcal{I}(0) \leq C_e + \rho, \quad \forall t \in [0, T).$$

We then call this property the monotonicity of the isoperimetric ratio.

Proof. Setting as usual $M_t = \partial\Omega_t$, we have, indicating with $|M_t|$ the area of M_t, that

$$\begin{aligned} \frac{\partial}{\partial t}|M_t| &= \frac{\partial}{\partial t}|\partial\Omega_t| = \int_M [-H(x,t) + \phi(t)]H(x,t)\, d\mu_t = \\ &= -\int_M H^2(x,t)\, d\mu_t + \frac{1}{|M_t|}\int_M H(x,t)\, d\mu_t \int_M H(x,t)\, d\mu_t \leq 0, \end{aligned}$$

where we made use of the Jensen's inequality as in [46], i.e.

$$\begin{aligned} \frac{1}{|M_t|}\int_M H^{k+1}\, d\mu_t &= \left(\frac{1}{|M_t|}\int_M \left(H^k\right)^{\frac{k+1}{k}} d\mu_t\right)^{\frac{k}{k+1}} \left(\frac{1}{|M_t|}\int_M H^{k+1}\, d\mu_t\right)^{\frac{1}{k+1}} \\ &\geq \left(\frac{1}{|M_t|}\int_M H^k\, d\mu_t\right)\left(\frac{1}{|M_t|}\int_M H\, d\mu_t\right), \end{aligned}$$

$\forall k \geq 1$.

Moreover we expect that the volume of the open region enclosed by M_t, $|\Omega_t|$, is preserved during the flow:

$$\begin{aligned} \frac{\partial}{\partial t}|\Omega_t| &= -\int_M H(x,t)\, d\mu_t + \frac{1}{|M_t|}\int_M H(x,t)\, d\mu_t \int_M d\mu_t = \\ &= -\int_M H(x,t)\, d\mu_t + \int_M H(x,t)\, d\mu_t = 0. \end{aligned}$$

Combining these two computations, we thus obtain the *monotonicity* of the isoperimetric ratio (3.1.2), and therefore $\mathcal{I}(t)$ is non increasing during the flow.
Since by the isoperimetric inequality we have $\mathcal{I}(t) \geq C_e$ and by above $|\Omega_t| = |\Omega_0| = |B_1(0)|$, we deduce that:

$$|M_t| = \mathcal{I}(t)^{\frac{1}{n+1}}\, |\Omega_0|^{\frac{n}{n+1}} \geq C_e^{\frac{1}{n+1}}\, |\Omega_0|^{\frac{n}{n+1}} = \left(C_e\, |\Omega_0|^n\right)^{\frac{1}{n+1}} = M_*.$$

The monotonicity of $\mathcal{I}(t)$ then implies that during the flow

(1) $|\partial\Omega_0| = |M_0| \geq |M_t| \geq M_*$,

(2) if initially $C_e \leq \mathcal{I}(0) \leq C_e + \rho$, then:

$$C_e \leq \frac{|\partial\Omega_t|^{n+1}}{|\Omega_t|^n} = \mathcal{I}(t) \leq \mathcal{I}(0) \leq C_e + \rho, \quad \forall t \in [0, T).$$

$\square$

3.3 Long time existence

Let S be the following set of times:

$$S = \Big\{\tau \in [0, T) : \frac{1}{4} \leq k_i(x, t) \leq 4 \quad \forall i = 1, 2, \dots, n \quad \forall x \in M_t, \forall t \in [0, \tau]\Big\}.$$

Remember that the principal curvatures of M_0 are initially such that $1/2 \leq k_i(0) \leq 2$.

For any $t \in S$, M_t is strictly convex. Furthermore, we have that $|H(x,t)| = \sum_{i=1}^n k_i \leq 4n$ and $|A(x,t)| = \sqrt{\sum k_i^2} \leq 4\sqrt{n}$, and since we have set $C' = 5n$, we deduce:

(1) $|H(x,t)| < C'$,

(2) $|A(x,t)| < C'$,

(3) $\phi(t) < C'$,

for any $t \in S$.

We also set $S' = \sup S$ and we assume that $S' < \infty$.

The Weingarten map is a selfadjoint operator and we can thus write it in a diagonal form as $h^i_j = \mathrm{diag}(k_1, \dots, k_n)$. In the Euclidean setting, its evolution equation behaves as

$$\frac{\partial}{\partial t} h^i_j = \Delta h^i_j + |A|^2 h^i_j - \phi \left(h^k_j h^i_k\right). \tag{3.3.4}$$

Let us now define the following two functions:

(1) $k_{\min}(t) = \min_{x \in M_t} k_i(x,t)$;

(2) $k_{\max}(t) = \max_{x \in M_t} k_i(x,t)$.

To study their evolution equations, in order to understand how they behave over time, we need to find a way to overcome the fact that the functions as defined above might not be smooth. For this reason, in the same spirit of [40], we proceed in the following way. We first define, for a positive constant β, a smooth approximation u for the general function $\max(x_1, \dots, x_n)$:

$$\begin{aligned} u_2(x_1, x_2) &= \frac{x_1 + x_2}{2} + \sqrt{\big(\frac{x_1 - x_2}{2}\big)^2 + \beta^2}, \\ u_{n+1}(x_1, \dots, x_{n+1}) &= \frac{1}{n+1} \sum_{i=1}^{n+1} u_2(x_i, u_n(x_1, \dots, \hat{x}_i. \dots, x_{n+1})), n \geq 2. \end{aligned}$$

The approximation u has the properties stated here below:

Lemma 3.4. *For $\beta > 0$ and $n \geq 2$,*

(i) $u_n(x_1, \dots x_n)$ is smooth, symmetric, monotonically increasing and convex.

(ii) $\frac{\partial u_n}{\partial x_i} \leq 1$;

(iii) $\max(x_1, \dots, x_n) \leq u_n(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n) + (n-1)\beta$;

(iv) $u_n(x_1, \dots, x_n) - (n-1)\beta \leq \sum_{i=1}^{n} \frac{\partial u_n(x_1,\dots,x_n)}{\partial x_i} \cdot x_i \leq u_n(x_1, \dots, x_n)$;

(v) $\sum_{i=1} \frac{\partial u_n}{\partial x_i} = 1$.

Proof. Direct computation and induction. □

Theorem 3.5. *The supremum S' of the set of times S is bounded below by a positive dimensional constant.*

Proof. Let then be $u(h^i_j)$ the above approximation of $\max(k_1, \dots, k_n) = k_{\max}(t)$ and note that

$$\begin{aligned}
\frac{\partial}{\partial t} u &= \frac{\partial u}{\partial h^i_j}\Big(\frac{\partial}{\partial t} h^i_j\Big), \\
\Delta u &= \frac{\partial^2 u}{\partial h^p_q \, \partial h^l_m} \nabla^\nu h^p_q \, \nabla_\nu h^l_m + \frac{\partial u}{\partial h^i_j} \Delta h^i_j.
\end{aligned}$$

Using the evolution equation (3.3.4) above, we can now compute:

$$\begin{aligned}
\frac{\partial}{\partial t} u &= \frac{\partial u}{\partial h^i_j}\Big(\Delta h^i_j + |A|^2 h^i_j - \phi\left(h^k_j h^i_k\right)\Big) + \frac{\partial^2 u}{\partial h^p_q \, \partial h^l_m} \nabla^\nu h^p_q \, \nabla_\nu h^l_m + \\
&\quad - \frac{\partial^2 u}{\partial h^p_q \, \partial h^l_m} \nabla^\nu h^p_q \, \nabla_\nu h^l_m \\
&= \frac{\partial u}{\partial h^i_j} \Delta h^i_j + \frac{\partial^2 u}{\partial h^p_q \, \partial h^l_m} \nabla^\nu h^p_q \, \nabla_\nu h^l_m + \frac{\partial u}{\partial h^i_j} |A|^2 h^i_j + \\
&\quad - \frac{\partial u}{\partial h^i_j} \phi\left(h^k_j h^i_k\right) - \frac{\partial^2 u}{\partial h^p_q \, \partial h^l_m} \nabla^\nu h^p_q \, \nabla_\nu h^l_m.
\end{aligned}$$

By the properties of Lemma 3.4 and the estimate on $|A|$, we have by (i) that u is monotonically increasing and convex, and therefore the last line in the equation above is negative and it can be ignored. By (iv), we also have that

$$\frac{\partial u}{\partial h^i_j} |A|^2 h^i_j \leq |A|^2 u(t),$$

so we can finally deduce that

$$\frac{\partial}{\partial t} u(t) \leq \Delta u(t) + |A|^2 u(t) \leq \Delta u(t) + (C')^2 \, u(t),$$

which is always true when $t \leq S'$. Note also that the principal curvatures are initially $1/2 \leq k_i(x, 0) \leq 2$. Solving then the auxiliary equation

$$\frac{\partial}{\partial t} \varphi(t) = (C')^2 \varphi(t), \quad \varphi(0) = u(0),$$

we have

$$\int_{\varphi(0)}^{\varphi(t)} \varphi^{-1}\, d\varphi = \int_0^t (C')^2\, ds \Rightarrow \varphi(t) = \varphi(0)\, e^{(C')^2 t},$$

and by maximum principle,

$$u(t) \leq u(0)\, e^{(C')^2 t}.$$

Since u approximates the $\max(k_1, \ldots, k_n)$, we make use of Lemma 3.4 once again and let $\beta \to 0$, to write

$$k_{\max}(t) = \max_{x \in M_t} k_i(x,t) \leq k_{\max}(0)\, e^{(C')^2 t} \leq 2\, e^{(C')^2 t}.$$

We now impose the following inequality:

$$k_{\max}(t) \leq 2\, e^{(C')^2 t} \leq 4,$$

and solving by t, with $C' = 5n$, we obtain the first time T_1 for which the upper bound for the biggest principal curvature can hit 4:

$$T_1 = \frac{\ln 2}{25 n^2}.$$

For continuity reasons, we thus have that $k_{\max}(t) \leq 4$, $\forall t \in [0, \min\{T_1, S'\}]$.

Note that $\max(-x_1, \ldots, -x_n) = -\min(x_1, \ldots, x_n)$. Let then $u(\beta^i_j)$ be an approximation of the function $-\min(k_1, \ldots, k_n) = -k_{\min}(t)$, where $\beta^i_j = -h^i_j$. Given (3.3.4), the evolution equation for β^i_j is therefore:

$$\frac{\partial}{\partial t}\beta^i_j = \Delta \beta^i_j + |A|^2 \beta^i_j + \phi\left(\beta^k_j \beta^i_k\right). \tag{3.3.5}$$

As before, we also have:

$$\begin{aligned} \frac{\partial}{\partial t} u &= \frac{\partial u}{\partial \beta^i_j}\Big(\frac{\partial}{\partial t}\beta^i_j\Big), \\ \Delta u &= \frac{\partial^2 u}{\partial \beta^p_q\, \partial \beta^l_m} \nabla^\nu \beta^p_q\, \nabla_\nu \beta^l_m + \frac{\partial u}{\partial \beta^i_j}\Delta \beta^i_j. \end{aligned}$$

Using the equation (3.3.5), we get

$$\begin{aligned} \frac{\partial}{\partial t} u &= \frac{\partial u}{\partial \beta^i_j}\Big(\Delta \beta^i_j + |A|^2 \beta^i_j + \phi\left(\beta^k_j \beta^i_k\right)\Big) + \frac{\partial^2 u}{\partial \beta^p_q\, \partial \beta^l_m}\nabla^\nu \beta^p_q\, \nabla_\nu \beta^l_m + \\ &\quad - \frac{\partial^2 u}{\partial h^p_q\, \partial h^l_m}\nabla^\nu h^p_q\, \nabla_\nu h^l_m \\ &= \frac{\partial u}{\partial \beta^i_j}\Delta \beta^i_j + \frac{\partial^2 u}{\partial \beta^p_q\, \partial \beta^l_m}\nabla^\nu \beta^p_q\, \nabla_\nu \beta^l_m + \frac{\partial u}{\partial \beta^i_j}|A|^2 \beta^i_j + \\ &\quad + \frac{\partial u}{\partial \beta^i_j}\phi\left(\beta^k_j \beta^i_k\right) - \frac{\partial^2 u}{\partial \beta^p_q\, \partial \beta^l_m}\nabla^\nu \beta^p_q\, \nabla_\nu \beta^l_m \\ &\leq \Delta u(t) + |A|^2 u(t) + \phi\, \beta^i_j u(t), \end{aligned}$$

using the properties of Lemma 3.4 once again and the definition of Δu. Recall in fact that u is monotonically increasing and convex by (i), so the $\partial^2 u$-term can be ignored, and by (iv) we have that

$$\begin{aligned} \frac{\partial u}{\partial \beta^i_j}|A|^2\beta^i_j &\leq |A|^2 u(t), \\ \frac{\partial u}{\partial \beta^i_j}\phi\left(\beta^k_j\beta^i_k\right) &\leq \phi\, \beta^i_j u(t). \end{aligned}$$

Observe that in this situation the second term is negative and we also have $\phi \leq C'$. By definition, $\beta^i_j = -h^i_j = \mathrm{diag}(-k_1, \ldots, -k_n)$, and therefore $\beta^i_j \geq -k_{\max}(t)$. Recall that the function $u(\beta^i_j(t))$ approximates the function $-\min(k_1, \ldots, k_n) = -k_{\min}(t)$; therefore we have $\beta^i_j(-k_{\min}(t)) \leq -k_{\max}(t)(-k_{\min}(t))$, which implies $\phi\, \beta^i_j u(t) \leq -C'\, k_{\max}\, u(t)$. By also recalling the fact that u is negative, we can then write:

$$\frac{\partial}{\partial t}u(t) \leq \Delta u(t) + |A|^2 u(t) + \phi\, u(t)\beta^i_j \leq \Delta u(t) - C'\, k_{\max}\, u(t),$$

where, since $1/4 \leq k_{\max} \leq 4$, we call with $E = 4C'$, and we write:

$$\frac{\partial}{\partial t}u(t) \leq \Delta u(t) - E\, u(t).$$

Solving the auxiliary equation

$$\frac{\partial}{\partial t}\varphi(t) = -E\varphi(t), \quad \varphi(0) = u(0),$$

we have

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{\varphi} = -\int_0^t E\, ds \Rightarrow \varphi(t) = \varphi(0)e^{-Et},$$

and by maximum principle,

$$u(t) \leq u(0)\, e^{-Et}.$$

Note that u approximates smoothly the function $-\min(k_1, \ldots, k_n)$; we then make use of Lemma 3.4 once again and let the parameter $\beta \to 0$, to write

$$-k_{\min}(t) = -\min_{x \in M_t} k_i(x,t) \leq -k_{\min}(0)\, e^{-Et},$$

and since we have $k_{\min}(0) \geq 1/2$, we deduce:

$$k_{\min}(t) \geq k_{\min}(0)\, e^{-Et} \geq \frac{1}{2}\, e^{-Et}.$$

Imposing now the following inequality:

$$k_{\min}(t) \geq \frac{1}{2}e^{-Et} \geq \frac{1}{4}$$

and solving by t, we obtain the first time T_2 for which the lower bound for the smallest principal curvature can hit 1/4:

$$T_2 = -\frac{\ln 1/2}{E} = \frac{\ln 2}{E},$$

and we then have that $k_{\min}(t) \geq 1/4, \forall t \in [0, \min\{T_2, S'\}]$.

Observing that $C' = 5n$ and therefore $E = 20n$, we finally have that:

$$S' = \sup S \geq \min\{T_1, T_2\} = T_1 > 0.$$

□

If at the time $t = S'$ we are able to find a good control on the covariant derivative of the second fundamental form and an estimate on the diameter of $M_{S'}$, we would be able to apply Proposition 3.2 and to produce deeper considerations about the geometry of $M_{S'}$.
Fortunately we have the following Corollary, which is a direct consequence of Theorem 1.14.

Corollary 3.6. *There exists a constant only depending on the dimension of M_t and C' such that $\forall t \in (0, S']$ the covariant derivative of the second fundamental form stays bounded. In other words, as long as the hypersurface remains strictly convex and with $|A(x,t)| \leq C'$, we have*

$$\sup_{t \in M_t} |\nabla A(x,t)|^2 \leq \frac{B}{\min\{1, t\}},$$

$\forall t \in (0, S']$.

Proof. Since the surface is moving inside a flat Euclidean space, the Riemannian tensor and, consequently, its covariant derivative, are zero. We are therefore in the situation of Theorem 1.14 and the Corollary follows immediately. To be more accurate, note that:

(1) if $\min\{1, S'\} = S'$, we have $S' < 1$ and therefore

$$\sup_{x \in M_t} |\nabla A(x,t)|^2 \leq \frac{B}{t}, \quad \forall t \in (0, S'];$$

(2) if $\min\{1, S'\} = 1$, we have instead $1 < S'$ and we proceed as in (1), having then

$$\sup_{x \in M_t} |\nabla A(x,t)|^2 \leq \frac{B}{t}, \quad \forall t \in (0, 1].$$

The above inequality tells us that we have proved an upper bound for the derivative of A for a time interval of length one. We can therefore decide to start the flow from a time $\bar{t} > 0$ until a time $\bar{t} + 1$, to obtain:

$$\sup_{x \in M_t} |\nabla A(x,t)|^2 \leq B \quad \forall t \in [1, 1 + \bar{t}].$$

If we iterate this process again, as many times as we need, we can obtain an upper bound for ∇A until $t = S'$, as we actually desired.

Putting all these considerations together we finally have:

$$\sup_{x \in M_t} |\nabla A(x,t)|^2 \leq \frac{B}{\min\{1,t\}} \quad \forall t \in (0, S'].$$

□

Observation 3. Note that we made no assumptions on the behaviour of the covariant derivative near the initial time. Therefore, if we don't prove that this time S' is strictly positive, we cannot proceed any further, since the estimate could blows up very quickly, losing completely any control on ∇A. Thanks to Theorem 3.5, we have been able to show however that

$$S' \geq T_1 = \frac{\ln 2}{25n^2} > 0.$$

Remark 3. *Note that as previously observed* $\min\{1, S'\} \geq T_1$*: this implies that at the time* $t = S'$ *we have the following control*

$$|\nabla A(x, S')|^2 \leq \frac{B}{\min\{1, S'\}} \leq \frac{B}{T_1}.$$

We can finally determine the constant C'' *and we set*

$$C'' = \sqrt{\frac{B}{T_1}} \Longrightarrow |\nabla A(x, S')| \leq C''.$$

In the following, we make use of the Theorem below. See for example [36].

Theorem 3.7 (Myers-Synge, 1935)**.** *Suppose* (M, g) *is complete with sec* $\geq K > 0$*. Then* M *is compact and satisfies diam*$(M, g) \leq \pi/\sqrt{K} =$ *diam* $S^n(K)$*. In particular,* M *has finite fundamental group.*

Proposition 3.8. *The intrinsic diameter of the hypersurface* $M_{S'} \subset \mathbb{R}^{n+1}$ *is bounded. More precisely, there exist points* $x_t \in \mathbb{R}^{n+1}$ *such that*

$$M_t \subset B_4(x_t), \quad \forall t \in [0, S'].$$

Observation 4. Note that the *intrinsic* diameter of M_t is the one computed using the Riemannian distance on M induced by the immersion, in contrast with the *extrinsic* diameter of M_t, which is defined in terms of the standard distance of $\mathbb{R}^{n+1}$. Note also that in this case the extrinsic diameter of M_t is always controlled by its intrinsic one.

Proof. Let's consider a point $p \in M_{S'}$. Since the Weingarten map W_p is a selfadjoint operator, there exists an orthonormal frame in $T_pM_{S'}$ of eigenvectors $\{e_i\}$ with $i = 1, 2, \ldots, n$, i.e. the principal directions at p, with the relative eigenvalues k_i, i.e. the principal curvatures. Then for any two vectors X, Y in $T_p M_{S'}$, we have that

$$h(X,Y) = \sum_i^n k_i X^i Y^i.$$

Observe now that the Gauss equation for $M_{S'}$, an immersed hypersurface in $\mathbb{R}^{n+1}$, can be written as

$$Rm(X,Y,Z,W) = h(X,W)h(Y,Z) - h(X,Z)h(Y,W);$$

the sectional curvature $sec(X,Y)$ for two linearly independent vectors spanning a 2-plane in p is instead computed, by definition and the Gauss equation above, as

$$\begin{aligned}
sec(X,Y) &= \frac{Rm(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{h(X,X)h(Y,Y) - h(X,Y)^2}{g(X,X)g(Y,Y) - g(X,Y)^2} = \\
&= \frac{\left(\sum_i^n k_i (X^i)^2\right)\left(\sum_i^n k_i (Y^i)^2\right) - \left(\sum_i^n k_i X^i Y^i\right)^2}{\left(\sum_i^n (X^i)^2\right)\left(\sum_i^n (Y^i)^2\right) - \left(\sum_i^n X^i Y^i\right)^2} = \\
&= \frac{\sum_{i<j}^n k_i k_j \left(X^i Y^j - X^j Y^i\right)^2}{\sum_{i<j}^n \left(X^i Y^j - X^j Y^i\right)^2} \geq \\
&\geq \frac{1}{16} \frac{\sum_{i<j}^n \left(X^i Y^j - X^j Y^i\right)^2}{\sum_{i<j}^n \left(X^i Y^j - X^j Y^i\right)^2} = \frac{1}{16} = K,
\end{aligned}$$

since

$$\frac{1}{4} \leq k_i(p, S') \leq 4, \quad \forall i = 1, 2, \cdots, n,$$

and by the following fact that can be proved by induction:

$$\left(\sum_i^n k_i (X^i)^2\right)\left(\sum_i^n k_i (Y^i)^2\right) - \left(\sum_i^n k_i X^i Y^i\right)^2 = \sum_{i<j}^n k_i k_j \left(X^i Y^j - X^j Y^i\right)^2.$$

Since this is true for any 2-planes in $T_p M_{S'}$ and $\forall p \in M_{S'}$, we have obtained a lower uniform bound for the sectional curvature, i.e.

$$sec(X,Y) \geq K > 0, \quad \forall X, Y \in \mathcal{T}(M_{S'}).$$

By Hopf-Rinow theorem, $M_{S'}$ is a closed immersed hypersurface in the Euclidean ambient space and therefore is a complete metric space. Applying Theorem 3.7 to the surface $M_{S'}$, with $S^n(K)$ the $n-$sphere of constant curvature K, we finally get an estimate of its diameter:

$$\operatorname{diam}(M_{S'}) \leq \frac{\pi}{\sqrt{K}} = 4\pi = \operatorname{diam} S_4^n,$$

where $\operatorname{diam}(S_4^n)$ is the (intrinsic) diameter of the n-sphere of radius $R = 4$.
Since this argument can be repeated for any time t in the set S thanks to the bounds

on the principal curvatures, we have thus proved that there exist points $x_t \in \mathbb{R}^{n+1}$ such that:

$$M_t \subset B_4(x_t), \quad \forall t \in [0, S'].$$

$\square$

Remark 4. *We summarize here below the estimates at the time $t = S' < \infty$:*

(1) $M_{S'} \subset B_4(x_{S'})$, for a $x_{S'} \in \mathbb{R}^{n+1}$;

(2) $|\Omega_{S'}| = |\Omega_0| = |B_1(0)|$, $|M_{S'}| \leq |M_0| \leq \tilde{C}$;

(3) $|H(x, S')| \leq C'$;

(4) $|A(x, S')| \leq C'$;

(5) $|\nabla A(x, S')| \leq C''$.

And by the monotonicity of the isoperimetric ratio:

(6) $C_e \leq \mathcal{I}(S') \leq \mathcal{I}(0) \leq C_e + \rho$.

With the next theorem we finally establish the long time existence for the flow.

Theorem 3.9. *Let M_0 a smooth closed strictly convex immersed hypersurface in $\mathbb{R}^{n+1}$. Then the flow (3.0.1) has a smooth strictly convex solution which is defined for all times $t \in [0, \infty)$.*

Proof. The estimates $(1) - (6)$ proved and listed above in Remark 4 give a control on the main geometric quantities of interest, in particular for the *same* constants chosen at the initial time $t = 0$. In fact, at the time $t = 0$, we have the following estimates, which are summarized at the end of Paragraph 3.1:

(1) $|H(x, 0)| \leq C'$,

(2) $|A(x, 0)| \leq C'$,

(3) $|\Omega_0| = |B_1(0)|$,

(4) $|\partial\Omega_0| = |M_0| \leq \tilde{C}$,

with the same C' and $\tilde{C}$ of Remark 4, and with $M_0 \subset B_4(x_0)$, for some $x_0 \in \mathbb{R}^{n+1}$. This is a crucial step in the proof of this Theorem: we can in fact apply Proposition 3.2 again at $t = S'$ with the *same exact* $\delta > 0$ initially chosen at the time $t = 0$. This therefore implies that $M_{S'}$ is δ-close in the C^2-norm to the round sphere $\partial B_1(x)$ for some $x \in B_4(x_{S'})$ and, by this remarkable fact, we can conclude again that

$$\frac{1}{2} \leq k_i(x, S') \leq 2 \quad \forall i, \quad \forall x \in M_{S'}.$$

We have then proved that $\forall t \in S$, and in particular at $t = S'$, the surface M_t keeps itself δ-close to the round sphere in the C^2-norm, for an arbitrary positive small δ, and its

principal curvatures are very close to be optimal. Moreover, we have obtained an upper bound, independent of t, for H, A and ∇A, $\forall t \in [0, S']$, in particular at $t = S' = \sup S$. Clearly then $S' \neq \sup S$. We knew already that the flow exists at least for a short time; then we have begun the flow of M_0 being $\delta-$close to the unit ball centred in the origin and we have chosen "good" initial estimates; this allowed us to run the flow at least for a strictly positive time ($S' \geq T_1 > 0$) until the initial estimates hold, and to discover that even at the time S' we are still $\delta-$close to a unit ball, for the same δ. Since we can repeat again the same argument starting the flow this time at S', we cannot have $S' = \sup S$. But this also implies, for the same reasons, that $\sup S = +\infty$.
Therefore $T = +\infty$ and the flow exists $\forall t \in [0, +\infty)$. In other words, as it is called in the literature, the flow is *immortal*. $\square$

3.4 Asymptotic behaviour

In the previous section we have proved that the flow exists for all positive times. It is natural now to study its behaviour while t is approaching ∞.
A key ingredient is the monotonicity of the isoperimetric ratio. The area of M_t is in fact monotone decreasing during the flow. We first reformulate this fact in the Lemma below.

Lemma 3.10. *The evolution equation of the surface area $|M_t|$ assumes the following expression:*

$$\frac{\partial}{\partial t}|M_t| = \int_M [-H(x,t)+\phi(t)]H(x,t)\,d\mu_t = -\int_M (H(x,t)-\phi(t))^2\,d\mu_t. \tag{3.4.6}$$

Proof. Noting that $H(H-\phi) = H^2 - \phi H$ and $(H-\phi)^2 = H^2 - \phi H - \phi H + \phi^2$, to prove (3.4.6) we just need to show that

$$\int_M (\phi^2 - \phi H)\,d\mu_t = 0.$$

So:

$$\begin{aligned}
&\int_M \phi^2\,d\mu_t - \int_M \phi H\,d\mu_t = \\
= \;& \int_M \Big(\frac{1}{|M_t|}\int_M H\,d\mu_t\Big)^2 d\mu_t - \frac{1}{|M_t|}\int_M H\,d\mu_t \int_M H\,d\mu_t = \\
= \;& \frac{1}{|M_t|^2}\Big(\int_M H\,d\mu_t\Big)^2 \int_M d\mu_t - \frac{1}{|M_t|}\Big(\int_M H\,d\mu_t\Big)^2 = \\
= \;& \frac{1}{|M_t|}\Big(\int_M H\,d\mu_t\Big)^2 - \frac{1}{|M_t|}\Big(\int_M H\,d\mu_t\Big)^2 = 0.
\end{aligned}$$

And therefore we recover again the monotonicity of the total surface area:

$$\int_0^{+\infty}\int_M (H(x,t)-\phi(t))^2\,d\mu_t\,dt \le |M_0|.$$

□

We claim that M_t converges exponentially to a convex surface of constant mean curvature in $\mathbb{R}^{n+1}$, and therefore it must be the round sphere, by Alexandrov's theorem. We then proceed in the following way: we first find a uniform bound for the mean curvature H and we prove that it converges uniformly to its average value; finally, we show that the convergence of M_t happens exponentially and the limit surface M_∞ is a round sphere. We thus start proving this result:

Proposition 3.11. *The mean curvature $H(x,t)$ converges uniformly to its average value $\phi(t)$. In other words,*

$$\lim_{t\to+\infty}\max_{x\in M_t}|H(x,t)-\phi(t)| = 0.$$

Proof. Let us define $\mathcal{M} = \bigcup_{t\in[0,\infty)}(M_t \times \{t\}) \subset \mathbb{R}^{n+1} \times \mathbb{R}$ as the subset of the Euclidean space that contains all the points which belong to the moving surface M, for any positive time. In other words, we keep track of the evolution of the initial surface M_0 in $\mathbb{R}^{n+1}$. We indicate a point of $\mathcal{M}$ as a couple (p_1, t_1), meaning that we are considering the point p_1 which belongs to the particular surface M_{t_1} at the specific time t_1, and we define a neighbourhood of (p_1, t_1) as a *space-time neighbourhood.*
Let therefore (p_1, t_1) be this point: we want to find a uniform bound for the mean curvature H in a space-time neighbourhood of this point . Since $\nabla H(x,t) = \nabla(g^{ij}h_{ij}) = g^{ij}\nabla A(x,t)$, the uniform bound for the covariant derivative of the second fundamental form (see previous paragraph), gives a uniform bound for $\nabla H(x,t)$. But this implies that

$$|H(x,t) - H(y,t)| \leq D_1|(x,t) - (y,t)|$$

at any given $t \in [0,\infty)$, with D_1 a positive constant depending only on the bound of ∇A such that $|\nabla H| \leq D_1$. In other words, H is spatial Lipschitz.
The only thing left is to obtain a uniform control of the time derivative of H. Recall that its evolution equation behaves as

$$\frac{\partial}{\partial t}H = \Delta H + |A|^2 H - \phi|A|^2,$$

which can be estimated as

$$|\frac{\partial}{\partial t}H| = |\Delta H + |A|^2 H - \phi|A|^2| \leq |\Delta H| + C_1,$$

with C_1 a positive constant depending only on the estimates of H, A and ϕ. Then the problem reduces to obtain a uniform bound for the second covariant derivative of A, since $\Delta H = g^{ij}g^{kl}\nabla_i\nabla_j h_{kl}$. If we remind the $*$-product notation, the evolution equation for $|\nabla^m A|^2$ in the Euclidean setting becomes by Proposition 1.2.13

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 &= \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla A^m \\ &+ \phi \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A.\end{aligned}$$

But by Theorem 1.15 which we proved in the first Chapter, we know already that

$$|\nabla^m A(x,t)|^2 \leq C_m.$$

Let us summarize here below the two estimates we have just obtained:

(1) $|\nabla H| \leq D_1$,

(2) $|\nabla^2 A| \leq D_2$.

Consider $D = \max\{D_1, D_2\}$ and recall the expression (3.4.6)

$$-\frac{\partial}{\partial t}|M_t| = \int_M (H(x,t) - \phi(t))^2 \, d\mu_t > 0,$$

which implies that, for any $\eta > 0$,

$$\int_M (H(x,t) - \phi(t))^2 \, d\mu_t > \eta \Longrightarrow \frac{\partial}{\partial t}|M_t| < -\eta.$$

The previous estimates (1) and (2) give a uniform bound D for H in a neighbourhood of any point (p_1, t_1) of the space-time, since we have a uniform control of both the spatial and time derivatives of H and therefore a uniform control on the function $\int_M (H(x,t) - \phi(t))^2 \, d\mu_t$.
For this reason, as in [46], if at a certain point (p_1, t_1) in the space-time $|H - \phi| = c$ for some $c > 0$, then it remains larger than $c/2$ on a space-time neighbourhood of (p_1, t_1) of radius r for a uniform $r = r(c)$. Keeping into account also the bounds for $|M_t|$, we deduce that $d/dt(|M_t|) < -\eta$ for a $t \in [t_1 - r, t_1 + r]$ for some $\eta = \eta(c)$. Since $|M_t|$ is monotone decreasing and bounded from below ($|M_t| \geq M_*$), this can happen only for a finite number of intervals for any given $c > 0$.
This shows that $|H - \phi|$ tends to zero uniformly, i.e.

$$\lim_{t \to +\infty} \max_{x \in M_t} |H(x,t) - \phi(t)| = 0.$$

□

By Corollary 1.6 applied in the flat space $\mathbb{R}^{n+1}$, the following evolution equations simplify as

$$\frac{\partial}{\partial t} H = \Delta H + |A|^2 H - \phi |A|^2,$$

and

$$\frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2\phi Z_1,$$

with $Z_1 = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj} = \mathrm{tr} A^3$. We also compute:

$$\begin{aligned} \frac{\partial}{\partial t}\Big(|A|^2 - \frac{1}{n}H^2\Big) &= \Delta\Big(|A|^2 - \frac{1}{n}H^2\Big) - 2\Big(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\Big) \\ &+ 2|A|^2\Big(|A|^2 - \frac{1}{n}H^2\Big) + \frac{2}{n}\phi\big(H|A|^2 - nZ_1\big). \end{aligned}$$

Let us now introduce this useful definition below:

Definition 3.2. Let (M, g) be a Riemannian convex and closed manifold. We say that M satisfies a *pinching condition* if there exists a positive constant $c > 0$ such that

$$\frac{k_{\max}(x)}{k_{\min}(x)} \leq c,$$

$\forall x \in M$, i.e. if the ratio between the biggest and the smallest principal curvatures is bounded by a positive constant for any $x \in M$.

We have proved that for any $t \in [0, \infty)$, M_t is δ-close to a unit sphere in a way that:

$$\frac{1}{2} \leq k_i(x,t) \leq 2, \quad \forall i = 1, 2, \cdots, n.$$

Since this is true for any principal curvatures at any time t, during the volume preserving mean curvature flow M_t satisfies a pinching condition, i.e.

$$\frac{k_{\max}(x,t)}{k_{\min}(x,t)} \leq 4 = c.$$

By the fact that M_t is strictly convex $\forall t \in [0,\infty)$, we have $n\,k_{\min} \leq H = g^{ij}h_{ij} = \sum_i k_i \leq n\,k_{\max}$. Then, by using the pinching condition above, we have the following inequalities:

$$h_{ij} \geq k_{\min}\, g_{ij} \geq \frac{k_{\max}}{4}\, g_{ij} \geq \frac{H}{4n}\, g_{ij} = \frac{1}{c\,n} H\, g_{ij},$$

which is obviously preserved during the flow.

Similarly as in [20], let us consider the function:

$$f_0 = \frac{|A|^2 - \frac{1}{n}H^2}{H^2}.$$

We want to compute its evolution equation. Note that in view of the evolution equations of $|A|^2$ and H, we have:

$$\begin{aligned}\frac{\partial}{\partial t} f_0 &= \frac{\partial}{\partial t}\Big(\frac{|A|^2}{H^2} - \frac{1}{n}\Big)\\ &= \frac{H\Delta|A|^2 - 2|A|^2\Delta H}{H^3} - \frac{2|\nabla A|^2}{H^2} - \frac{2\phi\, Z_1}{H^2} + \frac{2\phi\,|A|^4}{H^3}.\end{aligned}$$

Furthermore

$$\nabla_i f_0 = \frac{H\,\nabla_i|A|^2 - 2|A|^2\,\nabla_i H}{H^3}$$

and

$$\Delta f_0 = \frac{H\Delta|A|^2 - 2|A|^2\Delta H}{H^3} + \frac{6|A|^2\,|\nabla H|^2}{H^4} - \frac{4}{H^3} < \nabla_i|A|^2,\, \nabla_i H > .$$

Now, using the identity

$$|\nabla_i h_{kl}\cdot H - \nabla_i H\cdot h_{kl}|^2 = H^2|\nabla A|^2 + |A|^2|\nabla H|^2 - < \nabla_i|A|^2,\, \nabla_i H > H,$$

and reordering the terms, we get

$$\begin{aligned}\frac{\partial}{\partial t} f_0 &= \Delta f_0 + \frac{2}{H} < \nabla_l H,\, \nabla_l f_0 > - \frac{2}{H^4}\,|\nabla_l h_{ij}\cdot H - \nabla_l H\cdot h_{ij}|^2\\ &+ \frac{2\phi}{H^3}\Big\{|A|^4 - H Z_1\Big\}.\end{aligned}$$

Our goal is to show that the solution M_t of the flow converges exponentially to a round sphere. To do that, we will make use of the following result (see [37] for example).

Theorem 3.12 (Alexandrov's Theorem). *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth compact embedded hypersurface with constant mean curvature. Then Σ must be a round sphere.*

We are finally ready to investigate the asymptotic behaviour of the flow.

Theorem 3.13. *The family of maps $F : M \times [0,\infty) \to \mathbb{R}^{n+1}$ converges exponentially to a limit map F_∞ with image equal to the unit sphere.*

Proof. Thanks to the pinching condition, we have that $h_{ij} \geq \frac{1}{c\,n} H\, g_{ij}$, and we can make use of the estimate below for the following term, as in [18] (Lemma 2.3):

$$|\nabla_i h_{kl}\, H - \nabla_i H\, h_{kl}|^2 \geq \frac{1}{2c^2} H^2\, |\nabla H|^2.$$

Setting $Z_2 = H\, Z_1 - |A|^4$, we also have:

$$\begin{aligned}
Z_2 &= H\, Z_1 - |A|^4 = \Big(\sum_{i=1}^n k_i\Big)\Big(\sum_{j=1}^n k_j^3\Big) - \Big(\sum_{i=1}^n k_i^2\Big)^2 = \\
&= \sum_{i<j}^n \Big(k_i k_j^3 + k_j k_i^3\Big) - \sum_{i<j}^n 2\, k_i^2\, k_j^2 = \\
&= \sum_{i<j}^n k_i k_j \big(k_i - k_j\big)^2 \geq \frac{1}{c^2} H^2 \sum_{i<j}^n \big(k_i - k_j\big)^2,
\end{aligned}$$

and since by definition

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j}^n \big(k_i - k_j\big)^2,$$

we conclude that

$$Z_2 \geq \frac{n}{c^2} H^2 \Big(|A|^2 - \frac{1}{n} H^2\Big).$$

We want to use this result to estimate the following term, i.e. the last of the zero terms in the evolution equation of f_0.

$$\begin{aligned}
-\frac{2\phi}{H^3} Z_2 &\leq -\frac{2\phi}{H^3} \frac{n}{c^2} H^2 \Big(|A|^2 - \frac{1}{n} H^2\Big) = -\frac{2n\phi}{c^2} H \Big(\frac{|A|^2 - \frac{1}{n} H^2}{H^2}\Big) = \\
&= -\frac{2n\phi}{c^2} H f_0.
\end{aligned}$$

By these results, the evolution equation of f_0 can be estimated from above as:

$$\frac{\partial}{\partial t} f_0 \leq \Delta f_0 + \frac{2}{H} < \nabla_l H, \nabla_l f_0 > - \frac{1}{c^2\, H^2} |\nabla H|^2 - \frac{2n\phi}{c^2} H f_0.$$

The pinching condition on the principal curvatures gives also a lower bound for both H and ϕ, that we call with the same constant $\theta > 0$. Then the evolution equation of f_0 is controlled by:

$$\frac{\partial}{\partial t} f_0 \leq \Delta f_0 + \frac{2}{H} < \nabla_l H, \nabla_l f_0 > - \frac{2\theta^2}{c^2} f_0.$$

Studying the associated problem for (max) f_0, with an abuse of notation,

$$\partial_t \varphi(t) = -\frac{2\theta^2}{c^2}\varphi(t), \quad \varphi(0) = f_0(0),$$

gives as solution

$$\int_0^t \frac{d\varphi(s)}{\varphi} ds = \int_0^t -\frac{2\theta^2}{c^2}\, ds \Rightarrow \varphi(t) = \varphi(0)e^{-\frac{2\theta^2}{c^2}t}.$$

By the maximum principle, this implies that, given the uniform bound for H, we have:

$$|A(x,t)|^2 - \frac{1}{n}|H(x,t)|^2 \leq \overline{C}e^{-\sigma_1 t},$$

for some positive constants $\overline{C}$ depending on the bounds for H and on the initial conditions, and σ_1 depending on the pinching condition, n, and the uniform bounds for H and ϕ.

Recalling that

$$|A(x,t)|^2 - \frac{1}{n}|H(x,t)|^2 = \frac{1}{n}\sum_{i<j}(k_i - k_j)^2,$$

then

$$\frac{1}{n}\sum_{i<j}(k_i - k_j)^2 \leq \overline{C}e^{-\sigma_1 t}.$$

If we define w_{ij} as in [41],

$$w_{ij} = h_{ij} - \frac{H}{n}g_{ij},$$

we have

$$\begin{aligned} |w|^2 &= w^{ij}w_{ij} = \left(h^{ij} - \frac{H}{n}g^{ij}\right)\left(h_{ij} - \frac{H}{n}g_{ij}\right) = \\ &= h^{ij}h_{ij} - \frac{H}{n}h^i_i - \frac{H}{n}h^i_i + \frac{H^2}{n^2}\delta^i_i = |A|^2 - \frac{H^2}{n}, \end{aligned}$$

and so by above:

$$|w|^2 \leq \overline{C}e^{-\sigma_1 t}.$$

We now make use of Lemma C.2 of [38], where the authors proved, by using inductively interpolation inequalities of the form

$$|\nabla u|^2 \leq C|u| \cdot \left(|\nabla^2 u| + |\nabla u|\right),$$

that if a smooth function $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$, with $\Omega \subset \mathbb{R}$ an open region, is such that $|\nabla^l u|^2 \leq C_l$, for C_l constants independent of t, and if there exist positive constants c and $\tilde{\lambda}$ such that

$$|u|^2 \leq c\, e^{-\tilde{\lambda}t},$$

then for any $0 < \lambda < \tilde{\lambda}$ there are positive constants c_l such that

$$|\nabla^l u|^2 \leq c_l\, e^{-\lambda t}.$$

Applying this Lemma to w_{ij}, we therefore obtain that

$$|\nabla_k w_{ij}| \leq \overline{C} e^{-\sigma_2 t},$$

for some positive $\sigma_2 \leq \sigma_1$ (See also [13] for a more general overview).
Assume we have $g_{ij} = \delta_{ij}$ at a point p, then

$$\nabla_k H = \sum_i \nabla_k h_{ii} = \sum_i \nabla_i h_{ki} = \frac{\nabla_k H}{n} + \sum_i \nabla_i w_{ki},$$

by the Codazzi equation and the definition of w_{ij}, which gives

$$|\nabla_k H| \leq \frac{n}{n-1} \sum_i |\nabla_i w_{ki}| \leq \overline{C} e^{-\sigma_2 t}.$$

Similarly one obtains

$$|\nabla A| \leq \overline{C} e^{-\sigma_3 t},$$

and all higher derivatives by interpolation.

We have previously proved that the mean curvature of M_t converges uniformly to its average value, i.e. $|H - \phi|$ tends to zero uniformly. In other words that

$$\lim_{t \to +\infty} \max_{x \in M_t} |H(x,t) - \phi(t)| = 0,$$

and this also means that the speed of the immersions tends to zero uniformly, since

$$|\frac{\partial}{\partial t} F(x,t)| = |H(x,t) - \phi(t)|.$$

In the proof of this proposition, we observed that we can obtain a uniform control for H in a neighbourhood of any point (p_1, t_1) of the space-time, since we can uniformly control $|\nabla H|$ (so H is spatial Lipschitz) and $|\nabla^2 A|$ (so we control uniformly the derivation of H in the time direction). By the previous estimates on the derivatives of H and A and all their higher derivatives, we obtained an exponential decay for all these quantities. Therefore, combining these results, we deduce that the speed of the immersions decays exponentially as well.
Alternatively, setting again

$$k_1(t) = \min_{x \in M_t} k_i(x,t)$$

and

$$k_n(t) = \max_{x \in M_t} k_i(x,t),$$

we have

$$|\frac{\partial}{\partial t} F(x,t)| = |H(x,t) - \phi(t)| \leq |n\big(k_n(t) - k_1(t)\big)| \leq C_1 e^{-\sigma_4 t},$$

for some positive constants C_1 depending on $\overline{C}$ and n, and σ_4 on σ_1, and because

$$|n\big(k_n(t) - k_1(t)\big)|^2 \leq n^2 \sum_{i<j}(k_i - k_j)^2 \leq \overline{C}e^{-\sigma_1 t}.$$

Note that the exponential decay of the speed implies that the immersed surface cannot "run away" in $\mathbb{R}^{n+1}$. In fact, for any given point $p \in M$, we have that, fixing an initial time t_0 and $\forall t \in [0, \infty)$:

$$\begin{aligned} |F(p,t) - F(p,t_0)| &= \Big|\int_{t_0}^{t} \frac{\partial}{\partial t} F(p,s) ds\Big| \leq \int_{t_0}^{t} \big|\frac{\partial}{\partial t} F(p,s)\big| ds \\ &= \int_{t_0}^{t} |H(p,s) - \phi(s)| ds \leq \int_{t_0}^{t} C_1 e^{-\sigma_3 s}\, ds, \end{aligned}$$

and then for $t \to +\infty$ we have that there exists a point $q_\infty \in \mathbb{R}^{n+1}$ such that

$$\lim_{t \to +\infty} F(p,t) = q_\infty,$$

for any $p \in M$. Therefore M_t converges exponentially to a limiting closed convex hypersurface M_∞ of constant mean curvature with principal curvatures all equal to $\bar{k}$ and enclosing the same volume of M_0 equal to $|B_1(0)|$. By Alexandrov's Theorem (Thm. 3.12), M_∞ must be the unit sphere $\partial B_1(x)$, with centre some point $x \in \mathbb{R}^{n+1}$. $\square$

Chapter 4

Riemannian case

Theorem 4.1. ***(Main Theorem)*** *Let $(N, \bar{g})$ be a compact Riemannian manifold of dimension $n+1$ and $F : M \times I \to N$ a family of immersions such that $F_0(M) = M_0$ is a closed $n-$surface with all principal curvatures strictly positive. Then there exists a $\delta = \delta(N, \bar{g}) > 0$ and a $r_0 = r_0(N, \bar{g}) > 0$ such that if M_0 is $\delta-$close in C^2-norm to a geodesic sphere of radius $r \leq r_0$, the volume preserving mean curvature flow of M_0 has a smooth convex solution with maximal time interval $I = [0, \infty)$. In particular, the solution converges subsequentially to a small bubble of constant mean curvature.*

This is the main result we prove in the present work, applying the techniques and ideas previously illustrated in the Euclidean chapter to the Riemannian case. In the general Riemannian setting, however, the flow is a result of a complicated interaction between the geometry of the moving hypersurface and the geometry of the ambient space, and, in view of the average mean curvature term $\phi(t)$ in the VPMCF equation, the local evolution of M depends heavily on the global shape of the hypersurface inside N. For these reasons, not only the time-equations of the principal geometric quantities are more insidious, but we will also be required to introduce some other "tricks" that will help to bring us back to the Euclidean space, although with some weaker results, as in the asymptotic behaviour of the flow solution.

Let then $(N, \bar{g})$ be a compact Riemannian manifold of dimension $n+1$ and consider a family of immersions $F : M \times I \to N$, where M is as before a closed and strictly convex hypersurface, such that $F_0(M) = M_0$ and $F_t(M) = M_t$, $\forall t \in I = [0, T)$. We again require that such family of immersions F_t is a solution of the volume preserving mean curvature flow, i.e. it satisfies the following normal deformation equation:

$$\frac{\partial}{\partial t} F(x, t) = [-H(x, t) + \phi(t)] \cdot \nu(x, t), \tag{4.0.1}$$

where

$$\phi(t) = \frac{1}{|M_t|} \int_M H d\mu_t,$$

$\nu(x, t)$ is the unit normal vector to M_t in $x \in M_t$ and $|M_t|$ is the surface area of M_t at the time $t \in [0, T)$, with $T = T_{\max}$, the maximum existence time of the flow. Recall in fact that, as we have already explained in Chapter 1, short time existence for this particular type of flows is already known in the literature, even when the ambient space is a general Riemannian manifold (see for example Section 7.5, and in particular Theorem 7.17, in [21]).

In the next section we are going to introduce some more definitions in order to enlighten and exploit another key characteristic of the volume preserving mean curvature flow, i.e. its *parabolic invariance*, which we will allow us to implement the ideas developed in the Euclidean setting to the Riemannian environment.

4.1 Equivalent flows

Definition 4.1. Two flows $F : M \times I \to N$ and $G : M \times I \to N$ are said *equivalent* if $F(M,t) = G(M,t), \forall t \in I$, or, alternatively, if there exists a diffeomorphism $\varphi_t : M \to M$ for each t such that $G(x,t) = F(\varphi(x,t),t)$.

In general, if there exists a diffeomorphism $\Psi : M \to M$ such that if $F : M \times I \to N$ is a flow and $F(p,t) = F(\Psi(p),t)$ is still the same flow, we say that *the flow is invariant under reparametrization.*

The volume preserving mean curvature flow in (4.0.1) is clearly invariant under reparametrization. It also satisfies another important property, i.e. it is *parabolic invariant* under rescaling.

Definition 4.2. Let $F : M \times I \to (N, \bar{g})$ be a family of immersions, with $I = [0,T)$. Then the *parabolic rescaling* of F is the smooth family of immersions $\tilde{F} : M \times \tilde{I} \to (N, \lambda^2 \bar{g})$, where $\tilde{I} = [0, \lambda^2 T)$ and $\lambda > 0$ and with $\tilde{F}(x,t) = F(x, \lambda^{-2} t)$.

Proposition 4.2. *The volume preserving mean curvature flow as defined in* (4.0.1) *is invariant under parabolic rescaling.*

Proof. Let $F : M \times I \to (N, \bar{g})$ be defined as in (4.0.1) and consider its parabolic rescaling by a parameter $\lambda > 0$ as defined in (4.2), i.e. $\tilde{F} : M \times \tilde{I} \to (N, \tilde{g})$, with $\tilde{g} = \lambda^2 \bar{g}$.
Let $\{\partial_i = \partial/\partial x^i\}$ be a *coordinate frame* at a point $p \in M$. For any given time t, the induced metric g on M from the rescaled $(N, \tilde{g})$ is:

$$\begin{aligned} g_{ij} &= \tilde{F}_t^* \, \tilde{g}_{ij} = \lambda^2 \bar{g}(\tilde{F}_{t*} \, \partial_i, \tilde{F}_{t*} \, \partial_j) = \lambda^2 \bar{g}(F_{\lambda^{-2}t*} \, \partial_i, F_{\lambda^{-2}t*} \, \partial_j) = \\ &= \lambda^2 \, F^*_{\lambda^{-2}t} \, \bar{g}_{ij}. \end{aligned}$$

For the same coordinate frame $\{\partial_i\}$, it is immediate to see that:

(1) the inverse metric rescales as $\tilde{g}^{ij} = \lambda^{-2} \, \bar{g}^{ij}$;

(2) the Christoffel symbols remain unchanged, i.e. $\tilde{\Gamma}^k_{ij} = \bar{\Gamma}^k_{ij}$, and so does the connection $\bar{\nabla}$.

However, if $\{\bar{e}_i\}$ is an o.n. basis for the tangent space of $(N, \bar{g})$ at a point $q \in N$, we have:

$$\delta_{ij} = \bar{g}(\bar{e}_i, \bar{e}_j) = \lambda^{-2} \tilde{g}(\bar{e}_i, \bar{e}_j) = \tilde{g}(\lambda^{-1} \bar{e}_i, \lambda^{-1} \bar{e}_j) = \tilde{g}(\tilde{e}_i, \tilde{e}_j),$$

and then $\{\tilde{e}_i = \lambda^{-1} \bar{e}_i\}$ is the corresponding o.n. basis for $(N, \tilde{g})$.

Observe now the following. Let h_{ij} and H be respectively the second fundamental form and the mean curvature of $M_t = F_t(M)$ in $(N, \bar{g})$, for a given t; let also consider $p \in M_t$ and $\{\bar{e}_i, \bar{\nu}\}$ an o.n. basis for $T_p N$, with $\bar{\nu}$ the unit outer vector in p for M_t.

Then, after the parabolic rescaling $\tilde{F}_t$, we have that $\tilde{h}_{ij}$ at $p \in \tilde{F}_t(M) \subset (N, \tilde{g})$ in the new o.n. basis $\{\tilde{e}_i, \tilde{\nu}\}$ changes as:

$$\tilde{h}_{ij} = -\tilde{g}(\bar{\nabla}_i \tilde{e}_j, \tilde{\nu}) = -\frac{1}{\lambda^3}\, \tilde{g}(\bar{\nabla}_i\, \bar{e}_j, \bar{\nu}) = -\frac{\lambda^2}{\lambda^3}\, \bar{g}(\bar{\nabla}_i\, \bar{e}_j, \bar{\nu}) = \lambda^{-1}\, h_{ij},$$

and the mean curvature $\tilde{H}$

$$\tilde{H} = \tilde{g}^{ij}\, \tilde{h}_{ij} = \bar{g}^{ij}\, \lambda^{-1}\, h_{ij} = \frac{1}{\lambda}\, H,$$

since $\tilde{g}^{-1}(\tilde{\varepsilon}_i, \tilde{\varepsilon}_j) = \bar{g}^{-1}(\bar{\varepsilon}_i, \bar{\varepsilon}_j)$ for the corresponding *dual basis*.

Finally, noting that

$$\tilde{\phi}(t) = \frac{\int_M \tilde{H} d\tilde{\mu}_t}{\int_M d\tilde{\mu}_t} = \frac{1}{\lambda} \frac{\int_M H d\mu_t}{\int_M d\mu_t} = \frac{1}{\lambda}\, \phi(t),$$

we thus have:

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{F}(x,t) &= \frac{\partial}{\partial t}\big(F(x, \lambda^{-2}\, t)\big) = \frac{1}{\lambda^2}\frac{\partial}{\partial t}F_{\lambda^{-2}\, t} = \frac{1}{\lambda^2}[-H + \phi]\cdot \bar{\nu} = \\
&= [-\frac{1}{\lambda}\, H(x, \lambda^{-2}t) + \frac{1}{\lambda}\, \phi(\lambda^{-2}t)] \cdot \frac{1}{\lambda}\bar{\nu}(x, \lambda^{-2}t) = \\
&= [-\tilde{H}(x,t) + \tilde{\phi}(t)] \cdot \tilde{\nu}(x,t).
\end{aligned}$$

This proves that $\tilde{F}(x,t)$ is a VPMCF. □

Let be $p \in N$ and consider the open neighbourhood $U_p = \exp(B_\varepsilon(0))$, with $\varepsilon = \mathrm{inj}(N, \bar{g})$ the injectivity radius of the manifold N as usual. As we have seen in Chapter 1 with the equation (1.1.2), the metric can be written as

$$\begin{aligned}
\bar{g}_{ij}(q) &= \delta_{ij} + \frac{1}{3}\, \bar{R}_{kilj}|_p\, x^k x^l + \frac{1}{6}\, \nabla_k \bar{R}_{limj}|_p\, x^k x^l x^m + \cdots \\
&= \delta_{ij} + \frac{1}{3}\, \bar{R}_{kilj}|_p\, x^k x^l + O(r^3),
\end{aligned}$$

at a point $q \in U_p$. If we now consider a (parabolic) rescaled flow $\tilde{F}_t$ by a parameter λ^2 as before, the curvature tensor of the rescaled flow $\tilde{R}^l{}_{ijk}$ in terms of the Christoffel symbols w.r.t the o.n. basis $\{\tilde{e}\}$ is defined as

$$\tilde{R}^l{}_{ijk} = \partial_j \tilde{\Gamma}^l_{ik} - \partial_k \tilde{\Gamma}^l_{ij} + \tilde{\Gamma}^l_{js}\tilde{\Gamma}^s_{ik} + \tilde{\Gamma}^l_{ks}\tilde{\Gamma}^s_{ij},$$

and since w.r.t the same o.n. basis we know that $\tilde{\Gamma}^i_{jk} = \bar{\Gamma}^i_{jk}$, we have that

$$\tilde{R}^l{}_{ijk} = \bar{R}^l{}_{ijk},$$

w.r.t $\{\tilde{e}\}$. Since the metric changes as $\tilde{g}_{ij} = \lambda^2 \bar{g}_{ij}$, the Riemann tensor changes as:

$$\tilde{R}_{ijkl} = \tilde{g}_{is}\tilde{R}^s{}_{jkl} = \lambda^2 \bar{g}_{is}\bar{R}^s{}_{jkl} = \lambda^2 \bar{R}_{ijkl},$$

w.r.t $\{\tilde{e}\}$. We also have $\tilde{\nabla}_m \tilde{R}_{ijkl} = \lambda^2 \bar{\nabla}_m \bar{R}_{ijkl}$, if the basis remains the same.
Let now $\{\bar{e}_i\}$ be an o.n. basis w.r.t. the metric $\bar{g}$ of N and recall that we have $\tilde{e}_i = \lambda^{-1}\bar{e}_i$. By changing the basis, indicating with the same Latin letters the two basis, we therefore obtain:

$$\begin{aligned}
\tilde{R}_{ijkl} &= \lambda^2 \bar{R}_{ijkl} = \frac{\lambda^2}{\lambda^4}\bar{R}_{ijkl} = \frac{1}{\lambda^2}\bar{R}_{ijkl} \\
\tilde{\nabla}_m \tilde{R}_{ijkl} &= \lambda^2 \bar{\nabla}_m \bar{R}_{ijkl} = \frac{\lambda^2}{\lambda^5}\bar{\nabla}_m \bar{R}_{ijkl} = \frac{1}{\lambda^3}\bar{\nabla}_m \bar{R}_{ijkl}.
\end{aligned}$$

Let now V be a vector field on N. Note that we must have $V = \tilde{V}^i \tilde{e}_i = \bar{V}^i \bar{e}_i$, from which we deduce that the coordinates change as $\tilde{V}_i = \lambda \bar{V}_i$. However, computed w.r.t. the different basis but w.r.t. the same metric, we have:

$$\|V\|^2 = \tilde{g}(\tilde{V}^i \tilde{e}_i, \tilde{V}^i \tilde{e}_i) = \bar{g}(\bar{V}^i \bar{e}_i, \bar{V}^i \bar{e}_i).$$

Keeping in mind these considerations, the Taylor expansion of $\tilde{g}(q)$ can be estimated as:

$$\begin{aligned}
\tilde{g}_{ij}(q) &= \delta_{ij} + \frac{1}{3}\,\tilde{R}_{kilj}|_p\,\tilde{x}^k\tilde{x}^l + \frac{1}{6}\,\tilde{\nabla}_k \tilde{R}_{limj}|_p\,\tilde{x}^k\tilde{x}^l\tilde{x}^m + \cdots \\
&= \delta_{ij} + \frac{1}{3\lambda^2}\,\bar{R}_{kilj}|_p \tilde{x}^k\tilde{x}^l + \frac{1}{6\lambda^3}\,\bar{\nabla}_k \bar{R}_{limj}|_p\,\tilde{x}^k\tilde{x}^l\tilde{x}^m + \cdots
\end{aligned}$$

and therefore:

$$|\tilde{g}_{ij}(q) - \delta_{ij}| \leq \frac{1}{3\lambda^2}\,|\bar{R}_{kilj}|_p|\,||x||^2 + \frac{|O(r^3)|}{\lambda^3}, \tag{4.1.2}$$

where the Riemann tensor is expressed in terms of the original metric, w.r.t. $\{\bar{e}_i\}$ and computed in p.
Look carefully at the equation (4.1.2): it estimates a control on the difference between the Euclidean flat metric δ_{ij} and the rescaled metric $\tilde{g}_{ij}$ of N computed in some $q \in U_p$ by the Riemann tensor computed in the original metric $\bar{g}$ of N, divided by the scaling factor with the same power of $||x||^2 = r^2$ and all other terms behaving as r^3. Therefore, since the term $|\bar{R}_{ijkl}|_p$ is independent on the choice of the scaling factor λ, we can choose the parameter $\bar{\lambda}$ in such a way that

$$\frac{1}{3\bar{\lambda}^2}\,|\bar{R}_{kilj}|_p|\,||x||^2 \leq \delta_0, \tag{4.1.3}$$

for some positive very small δ_0. Since the injectivity radius is changing as well as $\tilde{\varepsilon} = \lambda\bar{\varepsilon}$, we can assume that $\tilde{\varepsilon} \geq 4$ for example, therefore big enough in order to deal with the curvature terms as we would be in the flat Euclidean case.
This justifies the following definition, where $U_p = \exp(B_\varepsilon(0))$ is defined as usual.

Definition 4.3. Let $(N, \bar{g})$ be a compact Riemannian manifold. The metric $\bar{g}$ is said to be *nearly flat* in the neighbourhood U_p if there exists a parameter $\lambda > 0$ such that the rescaled metric $\tilde{g} = \lambda^2\bar{g}$ satisfies (4.1.2) and (4.1.3).

Note that since N is compact, this is true for any point $p \in N$ and normal neighbourhood U_p.

Since by Proposition 4.2 the flow $F : M \times I \to N$ defined in (4.0.1) is invariant under parabolic rescaling, we can translate the idea of finding a "big" scaling factor λ into considering instead a family $\{\tilde{F}_n\}_{n\in\mathbb{N}}$ of rescaled immersions by the relative positive parameters $\{\lambda_n\}_{n\in\mathbb{N}}$. More precisely, if we start with the initial original flow F, then we have for $\forall n, i, j \in \mathbb{N}$ and for $i < j$:

(1) $\tilde{F}_n(x,t) = F(x, \lambda_n^{-2} t)$, with $\lambda_n > 0$ and $\lambda_i < \lambda_j$;

(2) $\tilde{g}_n(x) = \lambda_n^2\, \bar{g}(x)$, $\forall x \in U_p$.

Therefore the manifold $(N, \bar{g})$ is nearly flat when we reach the first $\bar{N} \in \mathbb{N}$ such that $\forall n > \bar{N}$ the equation (4.1.3) holds.

4.2 Immortal flow

Let $F : M \times I \to N$ be the family of immersions as defined in (4.0.1) and consider the following parabolic rescaling $\tilde{F} : M \times \tilde{I} \to (N, \tilde{g})$ by a parameter λ_N big enough such that Definition 4.3 is satisfied. We are therefore considering the following flow:

$$\frac{\partial}{\partial t}\tilde{F} = [-\tilde{H}(x,t) + \tilde{\phi}(t)] \cdot \tilde{\nu}(x,t), \tag{4.2.4}$$

where as usual

$$\tilde{\phi}(t) = \frac{1}{|M_t|}\int_M \tilde{H}\, d\tilde{\mu}_t,$$

which is still a volume preserving mean curvature flow by Prop. 4.2. Observe that M is obviously unchanged, so it is still closed and strictly convex (and so it is $M_0 = \tilde{F}_0(M)$). Note also that if ε is the injectivity radius of $(N, \bar{g})$, then $\tilde{\varepsilon} = \lambda_N\, \varepsilon$ is the injectivity radius of $(N, \tilde{g})$. Let then now be $q \in U_p^N = \exp(B_{\tilde{\varepsilon}}(0))$ and observe that by (1.1.2) the rescaled metric in q is written as

$$\tilde{g}_{ij}(q) \quad = \quad \delta_{ij} + \frac{1}{3}\, \tilde{R}_{kilj}|_p\, x^k x^l + O(r^3),$$

and by the fact that the metric is nearly flat in U_p^N, the equation (4.1.2) holds, i.e.

$$|\tilde{g}_{ij}(q) - \delta_{ij}| \leq \frac{1}{3\lambda_N^2}\, |\bar{R}_{kilj}|_p|\, ||x||^2 + \frac{|O(r^3)|}{\lambda_N^3},$$

with the property of equation (4.1.3) also satisfied.

Observation 5. From now on we can reformulate the definition of a nearly flat metric (Def. 4.3) by simply writing that the (rescaled) Riemann tensor of $\tilde{g}$ and its covariant derivative are such that $|\tilde{R}_{kilj}| + |\bar{\nabla}\tilde{R}_{ijkl}| \leq \alpha$, by a very small positive α. If there is no risk of confusion, we also just use the simpler notation $|\widetilde{Rm}| \leq \alpha$.

We make use of the following notation:

(1) $g = g_{ij}^N = \lambda_N^2\ F^*_{\lambda_N^{-2}t}\bar{g}_{ij}$, the metric on M_t;

(2) $\tilde{g} = \lambda_N^2\, \bar{g}_{ij}$ the rescaled metric on N;

(3) $\tilde{\varepsilon} = \lambda_N\, \varepsilon$ the injectivity radius of $(N, \tilde{g})$;

(4) if $I = [0, T)$ for the immersion F, then $\tilde{I} = [0, \lambda_N^2\, T)$ for $\tilde{F}$.

To simplify the notation we identify $\lambda = \lambda_N$.

Suppose for a moment that $\tilde{F}_0(M) = M_0 = \partial\Omega_0$ is immersed inside a normal neighbourhood U_p^N of a point $p \in N$ and we initially have the following estimates:

(1) $|\tilde{H}(x,0)| \leq C' r^{-1}$,

(2) $|\tilde{A}(x,0)| \leq C' r^{-1}$,

(3) $|\nabla \tilde{A}(x,0)| \leq C'' r^{-2}$,

(4) $|\Omega_0| = |B_r(p)|$ (Euclidean measure),

(5) $|M_0| = |\partial\Omega_0| \leq \tilde{C} r^{n-1}$,

for $r \leq r_0 < \tilde{\varepsilon}$, that we determine later. If we recall the Gauss equation

$$R_{ijkl} = \tilde{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk} \qquad 1 \leq i,j,k,l \leq n,$$

and we define as $k_1 > 0$ the smallest principal curvature of M_0 (which is strictly convex), by the definition of the sectional curvature we obtain, for any two X, Y orthonormal vectors spanning a 2-plan in $T_q M_0, q \in M_0$, with $|\widetilde{Rm}| \leq \alpha$ (the metric is nearly flat):

$$\begin{aligned} sec(X,Y) &= Rm(X,Y,Y,X) \geq \\ &\geq -\alpha + h(X,X)h(Y,Y) - h(X,Y)^2 \geq \\ &\geq -\alpha + k_1^2 \geq \frac{k_1^2}{2} = K > 0, \end{aligned}$$

where $|\widetilde{Rm}| \leq \alpha$ is such a very small number compared to k_1, which allows us to proceed with similar computations as in Prop. 3.8. Therefore:

$$sec(X,Y) \geq K > 0.$$

By Myers-Synge Theorem (Thm. 3.7), we have that the intrinsic diameter of M_0 is controlled by the (intrinsic) diameter of $S^n(K)$, the sphere of radius $1/\sqrt{K}$ endowed by a rotationally symmetric metric of constant sectional curvature K. Since the extrinsic diameter of M_0 is always controlled by the intrinsic one, we thus have that

$$M_0 \subset B_R(y),$$

where $B_R(y)$ is the (Riemannian) ball of radius $R = 1/\sqrt{K} = \sqrt{2}/k_1$ centred in $y \in U_p^N$. Observe that we can still define at $t = 0$ the *isoperimetric ratio* as in the Euclidean case as

$$\mathcal{I}(0) = \frac{|\partial\Omega_0|^{n+1}}{|\Omega_0|^n},$$

and we can impose that for a $\delta > 0$

$$\mathcal{I}(0) = \frac{|\partial\Omega_0|^{n+1}}{|\Omega_0|^n} \leq C + \rho, \tag{4.2.5}$$

for a $C = C(I_N(|\Omega_0|))$, $r_0 > 0$ and $\rho > 0$ given by Cor. 1.19 and such that our initial r is $r \leq r_0 < \varepsilon$.

Observation 6. The following ratio between the (powers) of the surface area and the volume of a Riemannian ball can always be set in a way such that

$$\frac{|\partial B_R(y)|^{n+1}}{|B_R(y)|^n} \leq C_e + \bar{\rho},$$

where C_e is the same constant we have found in the Euclidean case,

$$C_e = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \cdot (n+1)^n.$$

In other words, with $\bar{\rho}$ we can measure the failure of the metric $\tilde{g}$ to be flat and it is only depending on the intrinsic geometric properties of $B_R(y)$. Therefore even $\mathcal{I}(0)$ in (4.2.5) could be rewritten by using the constant C_e instead of C, as we did already in the proof of Cor. 1.19.

It is now clear that, given the initial conditions $(1) - (5)$ above and the equation (4.2.5), we apply Cor. 1.19 and we deduce that M_0 is $\delta-$close in the C^2-norm to $\partial B_r(z)$ for some $r \leq R$ and some $z \in B_R(y)$. This fact mostly implies that we have an estimate on the principal curvatures k_i of M_0, i.e.

$$k_i \approx \frac{1}{r} + O(1), \tag{4.2.6}$$

with an error that we can control, and we can even obtain better estimates on the geometric quantities we are interested in, i.e. A, ∇A and H of M_0.

Remark 5. *It follows from the previous argument that we can equivalently start with the assumptions* $(1) - (5)$ *and equation* (4.2.5)*, and then deduce the* $\delta-$*closeness property of* M_0 *to a sphere of radius* r *small enough, or assuming, as in Thm. 4.1, that the initial surface is* $\delta-$*close to a small sphere and therefore jumps immediately to* (4.2.6)*.*

Observation 7. Note that the injectivity radius of $(N, \tilde{g})$ is $\tilde{\varepsilon} = \lambda_N \varepsilon$ for a very big $\lambda = \lambda_N$. Therefore, without loss of generality, we can suppose that $1 << \tilde{\varepsilon}$ and if we initially set $|\Omega_0| = |B_1(p)|$, $(r = 1)$, we can suppose, by the previous Remark, that M_0 is $\delta-$close to $\partial B_1(z)$ for some $z \in B_R(y)$, and therefore its principal curvatures are such that

$$k_i \approx 1 + O(r),\ \forall i \in \{1, 2, \cdots n\},$$

where $O(r)$ is controlled by the injectivity radius $\tilde{\varepsilon}$ of N.
We can then suppose that the principal curvatures of M_0 are initially in the following range:

$$\frac{1}{2} \leq k_i(x, 0) \leq 2,\ \forall i \in \{1, 2, \cdots n\},\ \forall x \in M_0.$$

Let us consider again the isoperimetric ratio

$$\mathcal{I}(t) = \frac{|\partial \Omega_t|^{n+1}}{|\Omega_t|^n},$$

with as usual $\partial\Omega_t = M_t$ and Ω_t the open region enclosed by M_t, $\forall t \in [0, \tilde{T})$, $\tilde{T} = \lambda^2 T$. We have the following property as in the Euclidean case.

Proposition 4.3. *The inequality $\mathcal{I}(0) \leq C + \rho$ is preserved during the flow. In particular,*

$$\mathcal{I}(t) \leq \mathcal{I}(0) \leq C + \rho, \quad \forall t \in [0, T).$$

We therefore call this property as monotonicity of the isoperimetric ratio.

Proof. It is immediate, just use Lemma 3.10 that can be generalised easily to the Riemannian setting and recall that the volume of the region is fixed. □

In particular we have:

$$|\partial\Omega_0| = |M_0| \geq |M_t| \geq M_*.$$

As we did in Chapter 3, let us consider the following set of times:

$$S = \left\{\tau \in [0, \tilde{T}) : \frac{1}{4} \leq k_i(x,t) \leq 4 \quad \forall i = 1, 2, \ldots, n \quad \forall x \in M_t, \forall t \in [0, \tau]\right\},$$

with $\tilde{T} = \lambda^2 T$ the rescaled time. We define $S' = \sup S$ and we want to prove the following crucial theorem.

Theorem 4.4. *Let M_0 be a smooth closed strictly convex hypersurface immersed in the normal neighbourhood $U_p^N \subset N$, and initial conditions for the flow defined as in the equation* (4.2.4)*. If M_0 is $\delta-$close to the unit sphere $\partial B_1(z) \subset U_p^N$, then the flow* (4.2.4) *has a smooth strictly convex solution which is defined for all times $t \in [0, \infty)$. In particular $S' = \infty$.*

Proof. As it will be clear in this proof, the idea used to prove the Theorem is exactly the same seen in the Euclidean case. The only complications arise from the fact that M_0 is moving inside the curved space N and therefore the evolution equations of the main geometric quantities, and in particular the one of the second fundamental form, are more complicated.
Note that by Observation 7, since M_0 is $\delta-$close to the unit (Riemannian) sphere, we can suppose that its principal curvatures are initially

$$\frac{1}{2} \leq k_i(x,0) \leq 2, \ \forall i \in \{1, 2, \cdots n\}, \ \forall x \in M_0.$$

Setting $C' = 5n$, with $R < \tilde{\varepsilon}$, we therefore have at the time $t = 0$ the following estimates:

(1) $M_0 \subset B_R(y), \quad |\Omega_0| = |B_1(p)|, \quad |M_0| \leq \tilde{C}$,

(2) $|\tilde{H}(x,0)| < C'$,

(3) $|\tilde{A}(x,0)| < C'$,

(4) $|\nabla\tilde{A}(x,0)| \leq C''$,

and the monotonicity of the isoperimetric ratio

(5) $\mathcal{I}(t) \leq \mathcal{I}(0) \leq C + \rho$.

As usual, the Weingarten map $\{h^i_j\}$ is a selfadjoint operator and it can be written in diagonal form. However, its evolution equation is much more complicated this time, since the curvature of N affects the shape of M_t. We have in fact:

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{h}^i_j &= \Delta \tilde{h}^i_j + \tilde{h}^i_j |\tilde{A}|^2 + \tilde{R}_{nn}\tilde{h}^i_j \\
&- \tilde{R}_{kjkm}\tilde{h}^i_m - \tilde{R}_k{}^i{}_{km}\tilde{h}_{jm} - 2\tilde{R}_m{}^i{}_{jk}\tilde{h}_{km} \\
&- \overline{\nabla}_k \tilde{R}_n{}^i{}_{jk} - \overline{\nabla}_j \tilde{R}_{nk}{}^i{}_k - \tilde{\phi}\left(\tilde{h}^k_j \tilde{h}^i_k + \tilde{R}_{jn}{}^i{}_n\right).
\end{aligned}$$

Observation 8. Recall that $M_0 \subset U^N_p$ and the metric is nearly flat, therefore by the previous considerations we have $|\widetilde{Rm}| \leq \alpha$, and, since α is a positive very small constant, also $C' >> \alpha$.

We define again the following two functions:

(1) $k_{\min}(t) = \min_{x \in M_t} k_i(x,t)$,

(2) $k_{\max}(t) = \max_{x \in M_t} k_i(x,t)$,

and we want to study their evolution. In order to derive their time-equation, we use the same trick that we adopted in the Euclidean case.
Let therefore $u(\tilde{h}^i_j)$ be again an approximation of $\max(k_1, \dots, k_n)$, with the properties as in Lemma 3.4. Remember also that:

$$\begin{aligned}
\frac{\partial}{\partial t}u &= \frac{\partial u}{\partial \tilde{h}^i_j}\Big(\frac{\partial}{\partial t}\tilde{h}^i_j\Big), \\
\Delta u &= \frac{\partial^2 u}{\partial \tilde{h}^p_q \partial \tilde{h}^l_m}\nabla^\nu \tilde{h}^p_q \nabla_\nu \tilde{h}^l_m + \frac{\partial u}{\partial \tilde{h}^i_j}\Delta \tilde{h}^i_j,
\end{aligned}$$

Using the evolution equation for $\tilde{h}^i_j$, the fact that in the o.n. basis $\tilde{h}^i_j$ has diagonal form with positive entries (the principal curvatures) since we are in the set of times S and the estimate $|\tilde{R}_{kilj}| + |\bar{\nabla}\tilde{R}_{ijkl}| \leq \alpha$ for a positive constant α small enough, w.l.o.g. we have:

$$\begin{aligned}
\frac{\partial}{\partial t}u &\leq \frac{\partial u}{\partial \tilde{h}^i_j}\Big(\Delta \tilde{h}^i_j + |\tilde{A}|^2 \tilde{h}^i_j + \alpha \tilde{h}^i_j + \alpha \delta^i_j - \tilde{\phi}\left(\tilde{h}^k_j \tilde{h}^i_k + \tilde{R}_{jn}{}^i{}_n\right)\Big) \\
&\quad + \frac{\partial^2 u}{\partial \tilde{h}^p_q \partial \tilde{h}^l_m}\nabla^\nu \tilde{h}^p_q \nabla_\nu \tilde{h}^l_m - \frac{\partial^2 u}{\partial \tilde{h}^p_q \partial \tilde{h}^l_m}\nabla^\nu \tilde{h}^p_q \nabla_\nu \tilde{h}^l_m \\
&= \frac{\partial u}{\partial \tilde{h}^i_j}\Delta \tilde{h}^i_j + \frac{\partial^2 u}{\partial \tilde{h}^p_q \partial \tilde{h}^l_m}\nabla^\nu \tilde{h}^p_q \nabla_\nu \tilde{h}^l_m + \frac{\partial u}{\partial \tilde{h}^i_j}|\tilde{A}|^2 \tilde{h}^i_j + \frac{\partial u}{\partial \tilde{h}^i_j}\alpha \tilde{h}^i_j + \frac{\partial u}{\partial \tilde{h}^i_j}\alpha \delta^i_j \\
&\quad - \frac{\partial u}{\partial \tilde{h}^i_j}\tilde{\phi}\left(\tilde{h}^k_j \tilde{h}^i_k\right) - \frac{\partial u}{\partial \tilde{h}^i_j}\tilde{\phi}\left(\tilde{R}_{jn}{}^i{}_n\right) - \frac{\partial^2 u}{\partial \tilde{h}^p_q \partial \tilde{h}^l_m}\nabla^\nu \tilde{h}^p_q \nabla_\nu \tilde{h}^l_m.
\end{aligned}$$

Recall now Lemma 3.4 and the estimates on $|\tilde{A}|$ and $|\widetilde{Rm}|$: since by (i) the function is monotonically increasing and convex, the last line in the equation above has only negative terms and it can be ignored. Moreover, by (ii) and (iv) of Lemma 3.4 and with $C' = 5n$, we have that:

$$\frac{\partial u}{\partial \tilde{h}^i_j}|\tilde{A}|^2\tilde{h}^i_j + \frac{\partial u}{\partial \tilde{h}^i_j}\alpha\tilde{h}^i_j + \frac{\partial u}{\partial \tilde{h}^i_j}\alpha\delta^i_j \leq |\tilde{A}|^2 u(t) + \alpha\, u(t) + C'\alpha.$$

By the definition of Δu and the estimate on $|\tilde{A}|$, we finally obtain:

$$\frac{\partial}{\partial t}u \leq \Delta u(t) + |\tilde{A}|^2 u(t) + \alpha\, u(t) + C'\alpha \leq \Delta u(t) + (C')^2\, u(t) + \alpha\, u(t) + C'\alpha,$$

and since $C' >> \alpha$, we then estimate the above equation as

$$\frac{\partial}{\partial t}u \leq \Delta u(t) + 2(C')^2\, u(t) + (C')^2.$$

Solving the auxiliary equation

$$\frac{\partial}{\partial t}\varphi(t) = 2(C')^2\,\varphi(t) + (C')^2 = (C')^2(2\,\varphi(t)+1), \qquad \varphi(0) = u(0),$$

we have

$$\int_{\varphi(0)}^{\varphi(t)} (2\varphi+1)^{-1} d\varphi = \int_0^t (C')^2 ds \Rightarrow \varphi(t) = \frac{2\varphi(0)+1}{2}e^{2(C')^2 t} - \frac{1}{2},$$

and by maximum principle

$$u(t) \leq \frac{2u(0)+1}{2}e^{2(C')^2 t}.$$

As in the previous chapter, u approximates the $\max(k_1, \cdots k_n)$ and therefore we make use of Lemma 3.4, with $\beta \to 0$, to get:

$$k_{\max}(t) = \max_{x\in M_t} k_i(x,t) \leq \frac{2k_{\max}(0)+1}{2}\, e^{2(C')^2 t} \leq \frac{5}{2}e^{2(C')^2 t},$$

since we know that initially $1/2 \leq k_i(x,0) \leq 2$.
If we now solve by t the following inequality

$$k_{\max}(t) \leq \frac{5}{2}\, e^{2(C')^2 t} \leq 4,$$

we obtain the first time T_1 for which the upper bound for the biggest principal curvature can hit 4, i.e.

$$T_1 = \frac{\ln\frac{8}{5}}{50n^2},$$

since $C' = 5n$. For continuity reasons, we thus have that $k_n(t) \leq 4$, $\forall t \in [0, \min\{T_1, S'\}]$.

As before, note that $\max(-k_1, \ldots, -k_n) = -\min(k_1, \ldots, k_n)$ and let $u(\beta^i_j)$ be an approximation of the function $-\min(k_1, \ldots, k_n)$, where $\beta^j_i = -\tilde{h}^i_j$. The evolution equation for β^i_j is therefore:

$$\begin{aligned}
\frac{\partial}{\partial t}\beta^i_j &= \Delta\beta^i_j + \beta^i_j|\tilde{A}|^2 + \tilde{R}_{nn}\beta^i_j \\
&- \tilde{R}_{kjkm}\beta^i_m - \tilde{R}_k{}^i{}_{km}\beta_{jm} - 2\tilde{R}_m{}^i{}_{jk}\beta_{km} \\
&+ \overline{\nabla}_k\tilde{R}_n{}^i{}_{jk} + \overline{\nabla}_j\tilde{R}_{nk}{}^i{}_k + \tilde{\phi}\left(\beta^k_j\beta^i_k + \tilde{R}_{jn}{}^i{}_n\right),
\end{aligned}$$

and we also have:

$$\begin{aligned}
\frac{\partial}{\partial t}u &= \frac{\partial u}{\partial\beta^i_j}\Big(\frac{\partial}{\partial t}\beta^i_j\Big), \\
\Delta u &= \frac{\partial^2 u}{\partial\beta^p_q\,\partial\beta^l_m}\nabla^\nu\beta^p_q\,\nabla_\nu\beta^l_m + \frac{\partial u}{\partial\beta^i_j}\Delta\beta^i_j.
\end{aligned}$$

Given the definition of β^i_j and $|\tilde{R}_{kilj}| + |\bar{\nabla}\tilde{R}_{ijkl}| \leq \alpha$, we estimate as:

$$\begin{aligned}
\frac{\partial}{\partial t}u &\leq \frac{\partial u}{\partial\beta^i_j}\Big(\Delta\beta^i_j - \tilde{R}_{kjkm}\beta^i_m - \tilde{R}_k{}^i{}_{km}\beta_{jm} - 2\tilde{R}_m{}^i{}_{jk}\beta_{km} + \alpha\delta^i_j + \tilde{\phi}\left(\beta^k_j\beta^i_k + \tilde{R}_{jn}{}^i{}_n\right)\Big) \\
&+\frac{\partial^2 u}{\partial\beta^p_q\,\partial\beta^l_m}\nabla^\nu\beta^p_q\,\nabla_\nu\beta^l_m - \frac{\partial^2 u}{\partial\beta^p_q\,\partial\beta^l_m}\nabla^\nu\beta^p_q\,\nabla_\nu\beta^l_m + C'\alpha.
\end{aligned}$$

Using now the properties of Lemma 3.4 and $|\tilde{R}_{kilj}| + |\bar{\nabla}\tilde{R}_{ijkl}| \leq \alpha$ once again, w.l.o.g. we can write:

$$\begin{aligned}
\frac{\partial}{\partial t}u &\leq \frac{\partial u}{\partial\beta^i_j}\Delta\beta^i_j + \frac{\partial^2 u}{\partial\beta^p_q\,\partial\beta^l_m}\nabla^\nu\beta^p_q\,\nabla_\nu\beta^l_m - \frac{\partial u}{\partial\beta^i_j}\alpha\,\beta^i_j + C'\alpha \\
&+\frac{\partial u}{\partial\beta^i_j}\tilde{\phi}\left(\beta^k_j\beta^i_k\right) + \frac{\partial u}{\partial\beta^i_j}\tilde{\phi}\left(\tilde{R}_{jn}{}^i{}_n\right) - \frac{\partial^2 u}{\partial\beta^p_q\,\partial\beta^l_m}\nabla^\nu\beta^p_q\,\nabla_\nu\beta^l_m \\
&\leq \Delta u(t) - \alpha\,u(t) + 2\phi\frac{\partial u}{\partial\beta^i_j}\left(\beta^k_j\beta^i_k\right) + C'\alpha,
\end{aligned}$$

where we approximated in the last line the Riemann tensor by the term $\beta^k_j\beta^i_k$, and we used once again the definition of Δu and the properties (i) and (iv) of u of Lemma 3.4; using again (iv) of Lemma 3.4, we have:

$$\frac{\partial}{\partial t}u \leq \Delta u(t) - \alpha\,u(t) + 2\phi\,u(t)\beta^i_j + C'\alpha,$$

Recall that, as in the Euclidean case, we can approximate the term $\phi\,u(t)\beta^i_j$ with $-C'k_{\max}u(t)$, in order to finally study the following equation:

$$\frac{\partial}{\partial t}u \leq \Delta u(t) - \alpha\,u(t) - 2Ck_{\max}u(t) + C'\alpha.$$

Calling by $E = \alpha + 2C'k_{\max}$ and noting that α is very small compared to $k_{\max}$, we have:

$$\frac{\partial}{\partial t}u \leq \Delta u(t) - Eu(t) + E \leq \Delta u(t) - E\big(u(t) - 1\big).$$

Solving the auxiliary equation

$$\frac{\partial}{\partial t}\varphi(t) = -E\big(\varphi(t)-1\big) \qquad \varphi(0)=u(0),$$

we obtain

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{\varphi-1} = -\int_0^t E ds \Rightarrow \varphi(t) = (\varphi(0)-1)e^{-Et}+1,$$

and by maximum principle,

$$u(t) \leq \big(u(0)-1\big)e^{-Et}+1.$$

Recall that u approximates smoothly the function $-\min(k_1,\dots,k_n)$; by Lemma 3.4, and letting the parameter $\beta \to 0$, we write

$$-k_{\min}(t) = -\min_{x\in M_t} k_i(x,t) \leq \big(-k_{\min}(0)-1\big)e^{-Et}+1,$$

and since we initially have $k_{\min}(0) \geq 1/2$, we deduce:

$$k_{\min}(t) \geq \big(k_{\min}(0)+1\big)e^{-Et}-1 \geq \Big(\frac{1}{2}+1\Big)e^{-Et}-1 \geq \frac{3}{2}e^{-Et}-1.$$

Imposing now the following inequality

$$k_{\min}(t) \geq \frac{3}{2}e^{-Et}-1 \geq \frac{1}{4},$$

and solving by t, we obtain the first time T_2 for which the lower bound for the smallest principal curvature can hit $1/4$:

$$T_2 = -\frac{\ln\frac{5}{6}}{E},$$

and we then have that $k_{\min}(t) \geq 1/4, \forall t \in [0, \min\{T_2, S'\}]$.

Since $E = \alpha + 2C'k_{\max}$, with $C' = 5n$ and $1/4 \leq k_{\max} \leq 4$, we clearly have:

$$S' = \sup S \geq \min\{T_1, T_2\} = T_1 > 0,$$

and therefore the supremum of the set of times S is strictly positive, i.e. there exists an interval of time of length $S' \geq T_1$ where the principal curvatures of M_t run between $1/4$ and 4.

To conclude the proof we need to show that $S' = \sup S = \infty$.

By Theorem 1.14 and exactly as in Corollary 3.6, we have the following bound for the covariant derivative of $\tilde{A}$:

$$\sup_{x\in M_t} |\nabla \tilde{A}(x,t)|^2 \leq \frac{B}{\min\{1,t\}},$$

$\forall t \in (0, S']$. Moreover, since $\min\{1, S'\} \geq T_1$, we have that at the time $t = S'$

$$|\nabla \tilde{A}(x, S')|^2 \leq \frac{B}{\min\{1, S'\}} \leq \frac{B}{T_1},$$

and therefore we set

$$C'' = \sqrt{\frac{B}{T_1}} \Longrightarrow |\nabla \tilde{A}(x, S')| \leq C''.$$

Once again, the fact that the principal curvatures are between $1/4$ and 4 also implies that we have a control on the sectional curvature $\forall t \in S$, in particular at the final time $t = S'$. In fact, since the metric is nearly flat (i.e. $|\widetilde{Rm}| \leq \alpha$, which is a very small number compared to k_1), we have, as before, for any two orthonormal vectors X and Y spanning a $2-$plane in $T_{p'}M_{S'}$, with $p' \in M_{S'}$, that

$$\begin{aligned} sec(X,Y) &= Rm(X,Y,Y,X) \geq \\ &\geq -\alpha + h(X,X)h(Y,Y) - h(X,Y)^2 \geq \\ &\geq -\alpha + k_1^2 \geq \frac{k_1^2}{2} = K > 0. \end{aligned}$$

Therefore:

$$sec(X,Y) \geq K > 0 \quad \forall X, Y \in \mathcal{T}(M_{S'}),$$

and by Myers-Synge Theorem (Thm. 3.7) once again, we have that the intrinsic diameter of $M_{S'}$ is controlled by the (intrinsic) diameter of $S^n(K)$, the sphere of radius $1/\sqrt{K} = R'$ endowed by a rotationally symmetric metric of constant sectional curvature K. Therefore there exists a point $y' \in N$ such that

$$M_{S'} \subset B_{R'}(y').$$

At the final time $t = S'$, we therefore have:

(1) $|\tilde{H}(x, S')| \leq C'$, $|\tilde{A}(x, S')| \leq C'$, $|\nabla \tilde{A}(x, S')| \leq C''$.

Moreover, by the monotonicity of the isoperimetric ratio as in Prop. 4.3:

(2) $|\Omega_{S'}| = |\Omega_0| = |B_1(0)|, \quad |M_{S'}| \leq |M_0| \leq \tilde{C}, \quad M_{S'} \subset B_4(y),$

(3) $\mathcal{I}(S') \leq \mathcal{I}(0) \leq C + \rho.$

Note that those are the same constants chosen at the initial time $t = 0$. Again by Cor. 1.19, $M_{S'}$ is $\delta-$close in the C^2-norm to the round (Riemannian) sphere $\partial B_1(z)$ for some $z \in B_{R'}(y')$, for the same $\delta > 0$ initially chosen. This also implies that

$$\frac{1}{2} \leq k_i(x, S') \leq 2 \quad \forall i \in \{1, \ldots, n\}, \quad \forall x \in M_{S'}.$$

Using now the same argument as in Thm. 3.9, we conclude that $S' = \infty$ and thus the flow is immortal. This concludes the proof of the Theorem and it also proves the first part of Thm. 4.1. $\square$

4.2.1 Asymptotic behaviour

We have just proved that the flow $F : M \times \mathbb{R}_{\geq 0} \to N$ satisfying the following equation

$$\frac{\partial}{\partial t}F(p,t) = [-H(p,t) + \phi(t)] \cdot \nu(p,t), \tag{4.2.7}$$

with the average mean curvature term

$$\phi(t) = \frac{1}{|M_t|}\int_M H d\mu_t,$$

is immortal. With the next Proposition, which we have already seen in the Euclidean case, we are able to say something more about the speed (4.2.7) and the asymptotic behaviour of the flow.

Proposition 4.5. *The mean curvature $H(x,t)$ converges uniformly to its average value $\phi(t)$. In other words,*

$$\lim_{t\to+\infty}\max_{x\in M_t}|H(x,t) - \phi(t)| = 0.$$

Proof. It easy to see how the proof of Prop. 3.11 can be generalised to the Riemannian case. After defining the space-time $\mathcal{M} = \bigcup_{t\in[0,\infty)}(M_t \times \{t\}) \subset N \times \mathbb{R}$, we want to find a uniform bound for the mean curvature H in a space-time neighbourhood of a point (p_1, t_1). As in the Euclidean case, since $\nabla H(x,t) = \nabla g^{ij}h_{ij} = g^{ij}\nabla A(x,t)$, the uniform bound for the covariant derivative gives a uniform control on $\nabla H(x,t)$, which makes H spatial Lipschitz.
Recall now the evolution equation for H from Thm. 1.5:

$$\frac{\partial H}{\partial t} = \Delta H + (H - \phi)\big(|A|^2 + \bar{Ric}\,(n,n)\big),$$

which can be estimated as:

$$\begin{aligned}\Big|\frac{\partial H}{\partial t}\Big| &\leq \Big|\Delta H + (H-\phi)\big(|A|^2 + \bar{Ric}\,(n,n)\big)\Big| \\ &\leq \big|\Delta H\big| + \big|H\,|A|^2\big| + \big|\phi\,|A|^2\big| + \big|H-\phi\big|\big|\bar{Ric}\,(n,n)\big|,\end{aligned}$$

and, given the estimates on H, A, ϕ and Rm, we can write as:

$$\Big|\frac{\partial H}{\partial t}\Big| \leq \big|\Delta H\big| + C_1, \tag{4.2.8}$$

with $C_1 = C_1(C', \alpha)$ a positive constant.
Since $\Delta H = g^{ij}g^{kl}\nabla_i\nabla_j h_{kl}$, the uniform bound for the time-derivative of H comes from the ones on $|\nabla^2 A|$, which are given by Prop. 1.2.13. The result now follows by continuing as in Prop. 3.11. □

Proposition 4.5 produces a uniform bound for the mean curvature H in a space-time neighbourhood of a point in $\mathcal{M}$ and therefore the speed (4.2.7) is (uniformly) bounded. The main consequence of this Proposition is although the fact that the volume preserving mean curvature flow of M_0, which is initially $\delta-$close to a small geodesic sphere of radius r, has a solution which subsequentially converges to a small bubble of constant mean curvature (CMC), where a *bubble* is exactly a surface which is close to a small geodesic ball. In fact, thanks to the previous Proposition, combined with the isoperimetric nature of the flow, there exists a subsequence of times $\{t_k\}$ for which the family of surfaces $\{M_{t_k}\}$ converges to a limit bubble with constant mean curvature $\bar{H}$, its average value. This result also concludes the proof of the second part of Thm 4.1.

However, this is a weaker result compared to the Euclidean case, where we have proved the full convergence to a limiting sphere. In fact, when the ambient manifold is flat, the evolution equations of the main geometric quantities are more easily to treat, which allowed us to show the exponential decay of these relevant geometric quantities and therefore the exponential decay of the speed of the flow. This fact, combined with the isoperimetric nature of the flow and the use of the Alexandrov's Theorem, which states that the only compact embedded hypersurfaces with constant mean curvature are the round spheres, prevents the moving initial hypersurface M_0 from running away in the non compact Euclidean space and forces the solution of the VPMCF to converge exponentially to a limit surface of constant mean curvature, which must be a sphere.

In a general Riemannian manifold, the Alexandrov's Theorem does not hold and, as we have seen, the average term of the mean curvature introduces a non-local effect to all the evolution equations, which become much more complicated to be treated. The subsequential convergence, which is however an important first result, opens to problematic situations, like the fact that the solution of the the flow could converge to two (or more) different surfaces of same constant mean curvature $\bar{H}$. Furthermore, it would be interesting to prove that the limit constant mean curvature surface is a leaf of the local foliation around a critical point of the scalar curvature, which is assumed to be nondegenerate, as in Alikakos and Freire [1], proof that however relies on center manifold analysis from infinite-dimensional dynamical systems and semigroup theory. To get therefore the same full convergence of the flow as in the flat space and to prove that the limit solution is a leaf of a CMC foliation, there is the need for some more powerful hypothesis, which although we do not explore in here.

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