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Doctoral Thesis

Nonlinear transmission problems for the Laplace operator: a functional analytic approach

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Summary of Thesis

Department of Mathematics, Doctoral Thesis

Nonlinear transmission problems for the Laplace operator: a functional analytic approach

by Riccardo Molinarolo

This dissertation is devoted to the study of two nonlinear nonautonomous transmission boundary value problems for the Laplace operator in perturbed domains.

From a geometrical point of view, two configurations will be considered: singularly perturbed domains and regularly perturbed domains. The former are obtained by removing from a given bounded open set a portion whose size is proportional to a positive parameter ϵ close to 0, the latter are obtained by removing a portion whose form is shaped by a suitable diffeomorphism ϕ_ϵ, which depends regularly on ϵ. Adopting a functional analytic approach, we prove real analyticity theorems for the dependence of the solutions upon the parameter that describes the singular or regular perturbation, and a local uniqueness theorem for the solutions of the singularly perturbed boundary value problem: this last, in particular, is an improvement of the uniqueness results for families of solutions typically obtained in this framework.

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Contents

Introduction

This dissertation is devoted to the study of two nonlinear nonautonomous transmission boundary value problems for the Laplace operator in perturbed domains.

From a geometrical point of view, two configurations will be considered: singularly perturbed domains and regularly perturbed domains. The former are obtained by removing from a given bounded open set a portion whose size is proportional to a positive parameter ϵ close to 0, the latter are obtained by removing a portion whose form is shaped by a suitable diffeomorphism ϕ_ϵ. The main results consist in real analyticity theorems for the dependence of the solutions upon the parameter that describes the regular or singular perturbation, and a local uniqueness theorem for the solutions of the singularly perturbed boundary value problem: this last in particular is a considerable improvement of uniqueness results for the families of solutions typically obtained in this framework.

The study of the behaviour of the solutions of boundary value problems in domain with small holes or inclusions has attracted the attention of several pure and applied mathematicians and it is impossible to provide a complete list of contributions. From an application point of view, boundary value problems in domains with small holes or inclusions can be the mathematical model of the heat conduction in bodies with small cavities and impurities and thus they are extensively studied in the theory of dilute composite and porous materials (cf. Movchan, Movchan, and Poulton [65]). In particular, transmission conditions like the ones that we have studied can be analytically derived in the case of a thin reactive heat conducting interphase situated between two different materials (see the works of Mishuris, Miszuris and Öchsner [58], [59], and of Miszuris and Öchsner [62] and the references therein). Moreover, we point out that nonlinear transmission conditions arise also in the framework of elasto-plastic material (see e.g. Miszuris and Öchsner [61] and Mishuris, Miszuris, Öchsner, and Piccolroaz [60]) and in the framework of articular cartilage problems (cf. Vitucci, Argatov, and Mishuris [77]).

We briefly note that the computational analysis of structures consisting of components with very different lengths or dimensions (such analysis appears, for example, in continuum mechanics, composite materials, meta-materials, biological fluids, cellular lattice and the above mention frameworks) often leads to numerical inaccuracy and instability. Thus, an analytic-mathematical treatment of perturbed boundary value problems, which provides existence and uniqueness results and possibly real analytic dependence of the solutions upon the perturbation parameters, is extremely important in order to obtain consistent numerical methods.

In literature, existence and uniqueness of solutions of nonlinear boundary value problems have been largely investigated by means of variational techniques (see, e.g., the monographs of Nečas [67] and of Roubíček [72] and the references therein). Moreover, potential theoretic techniques have been widely exploited to study nonlinear boundary value problems with transmission conditions by Berger, Warnecke, and Wendland [10], by Costabel and

Stephan [21], by Gatica and Hsiao [34], and by Barrenechea and Gatica [9], and that boundary integral methods have been applied also by Mityushev and Rogosin for the analysis of transmission problems in the two dimensional plane (cf. [63, Chap. 5]).

For regularly perturbed boundary value problems, in particular, the dependence of the solution upon the domain perturbation has been considered: here we mention the works of Keldysh [44], Sokolowski and Zolésio [75], Henry [36] and references therein. On the other hand, boundary value problems in singularly perturbed domains are usually studied by expansion methods of Asymptotic Analysis. In particular we mention the works of Ammari and collaborators [2, 3, 4, 5, 6, 7], Maz'ya, Movchan, and Nieves [54], Nieves [68], Novotny and Sokołowski [69], the methods of matching inner and outer expansions (cf., e.g., Il'in [39, 40]) and the multiscale expansion method (as in Maz'ya, Nazarov, and Plamenenvskii [55]), and, in particular, concerning nonlinear problems, Iguernane, Nazarov, Roche, Sokołowski, and Szulc [38]. Moderately close holes have been also considered in the works of Bonnaillie-Noël and Dambrine [11], Bonnaillie-Noël, Dambrine, and Lacave [14], Bonnaillie-Noël, Dambrine, Tordeux, and Vial [15], and Dalla Riva and Musolino [28, 29]; holes approaching to the boundary (cf. Bonnaillie-Noël, Dalla Riva, Dambrine, and Musolino [13]), and perturbations close to the vertex of a sector (cf. Costabel, Dalla Riva, Dauge, and Musolino [20]) have also been considered.

Moreover, functional equation methods for the analysis of linear and nonlinear transmission problems in domains with circular inclusions have been applied, for example, in Castro, Kapanadze, and Pesetskaya [16], Kapanadze, Mishuris, and Pesetskaya [41, 42], Kapanadze, Miszuris, and Pesetskaya [43].

Finally, we point out that problems with small holes or inclusions have been analysed also from the numerical point of view, for example in the works of Chesnel and Claeys [17] and of Babuška, Soane, and Suri [12]. We also mention the works of Mishuris, Miszuris and Öchsner [58], [59], and of Miszuris and Öchsner [62], in which transmission conditions are numerically tested with simulation based on the finite element method.

In order to explain the method of Asymptotic Analysis, the questions and results one usually expects in this framework, we introduce a model problem. For the sake of simplicity of the exposition and to avoid technical complexity, we assume that the dimension of the space for the model problem is

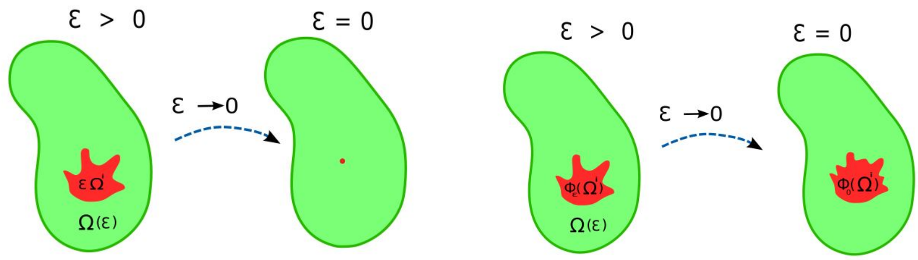
$$n \geq 3.$$

So, let $\Omega(\epsilon)$ be a perturbed domain obtained from a given bounded open set of $\mathbb{R}^n$ or by removing a portion whose size is proportional to a small positive parameter $\epsilon \in]0, \epsilon_0[$, with $\epsilon_0 > 0$, or by removing a portion whose form is shaped by a diffeomorphisms ϕ_ϵ (which we think as a point in a suitable Banach space), belonging to an appropriate family of admissible diffeomorphism $\{\phi_\epsilon\}_{\epsilon \in]0,\epsilon_0[}$, which depends real analytically on the parameter ϵ. Then we are is interested in the following two cases:

C1. the parameter ϵ tends to 0, i.e. the hole shrinks to a point (singularly perturbed);

C2. the family of diffeomorphisms $\{\phi_\epsilon\}_{\epsilon \in]0,\epsilon_0[}$ tends, in a sense which will be explained below, to a fixed diffeomorphism ϕ_0 as $\epsilon \to 0^+$(regularly perturbed).

Then one considers a boundary value problem (for the Laplace operator or, in principle, also for others differential operators) for each small positive ϵ in



(a) (b)

Figure 1: Singularly and regularly perturbed domains

$\Omega(\epsilon)$ and one denotes by u_ϵ the solution of the singularly perturbed problem and by u_{ϕ_ϵ} the solution of the regularly perturbed problem (see Figure 1).[1]

Then one can pick a point $\bar{x} \in \Omega(\epsilon)$ and pose the following questions:

Q1. what can be said on the map $\epsilon \mapsto u_\epsilon(\bar{x})$ when $\epsilon > 0$ is close to 0? (corresponding to **C1**)

Q2. what can be said on the map $\epsilon \mapsto u_{\phi_\epsilon}(\bar{x})$ when $\epsilon > 0$ is close to 0? (corresponding to **C2**)

The possible answers that one may obtain depend on the approach adopted. By the Asymptotic Analysis, and for some specific problem, one can hope to implement the following strategy:

1. first, one has to formulate an "ansatz" on the expected expansion. We note that this of course is in general not an easy task. For example, especially if a real analytic dependence of the solutions upon the perturbation parameter is expected, one may try expansions of the following types

$$u_\epsilon(\bar{x}) = \sum_{|\alpha| \leq r} c_{\alpha,1}(\bar{x})\epsilon^\alpha + R_1(\epsilon) \quad \text{as } \epsilon \to 0^+$$

[1]The author is indebted to Prof. Dalla Riva and Dr. Musolino for the sample of Figure 1, which has been modified, with their approval, by the author. For sake of simplicity, we have drawn the 2-d version of the model.

for the case **C1** or

$$u_{\phi_\epsilon}(\bar{x}) = \sum_{|\alpha|\leq r} d_\alpha\left(\bar{x}, \{\phi_\epsilon\}_{\epsilon\in]0,\epsilon_0[}\right)\epsilon^\alpha + R_2(\epsilon) \quad \text{as } \epsilon \to 0^+$$

for the case **C2**, where $\{c_\alpha(\bar{x})\}_{|\alpha|\leq r}$ and $\left\{d_\alpha\left(\bar{x}, \{\phi_\epsilon\}_{\epsilon\in]0,\epsilon_0[}\right)\right\}_{|\alpha|\leq r}$ are two families of coefficients (the former depends only on the fixed point $\bar{x}$, the latter on the fixed point $\bar{x}$ and the family of perturbation parameter $\{\phi_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$), and $R_1(\epsilon)$ and $R_2(\epsilon)$ are two error functions;

2. one has to compute the family of coefficients

$$\{c_\alpha(\bar{x})\}_{|\alpha|\leq r} \text{ or } \left\{d_\alpha\left(\bar{x}, \{\phi_\epsilon\}_{\epsilon\in]0,\epsilon_0[}\right)\right\}_{|\alpha|\leq r};$$

3. finally one has to estimate the error functions $R_1(\epsilon)$ or $R_2(\epsilon)$.

We notice that, *a priori*, one cannot expect in general that the two power series

$$\sum_{\alpha\in\mathbb{N}} c_\alpha(\bar{x})\epsilon^\alpha \quad \text{and} \quad \sum_{\alpha\in\mathbb{N}} d_\alpha\left(\bar{x}, \{\phi_\epsilon\}_{\epsilon\in]0,\epsilon_0[}\right)\epsilon^\alpha$$

associated to the "ansatz" expansions converge to $u_\epsilon(\bar{x})$ and $u_{\phi_\epsilon}(\bar{x})$, respectively. Moreover, in some particular cases (for example if the dimension of the space is $n = 2$), then one would have to add some terms in the ansatz expansion, possibly singular at $\epsilon = 0$ (for example $\epsilon\log(\epsilon)$ or $\log(\epsilon)^{(-1)}$).

Moreover, we point out that, in particular for nonlinear problems, the Asymptotic Approach may be hard to implement and to the best of our knowledge, nonlinear boundary value problems in domains with small hole or inclusion have been addressed by the techniques of Asymptotic Analysis only in few papers.

In this dissertation instead we have adopted an approach which has revealed to be an extremely powerful tool to analyse perturbed nonlinear boundary value problems: the Functional Analytic Approach. As we will explain, this method uses analytic functions in order to describe the effect of the perturbation's parameters on the solutions of the problem, without having to guess, *a priori*, the form of the expansion of the solution. From this point of view, the Functional Analytic Approach can be considered as alternative to the Asymptotic Analysis.

This method is based on functional analysis and potential theory and exploits techniques of analytic functions theory, regularity theory, fixed point theory, harmonic analysis, and superposition operator theory. In general, the aim is to represent the dependence of the solution of a boundary value problem upon the perturbations of the domains in terms of

- real analytic functions defined in a whole neighborhood of $\epsilon = 0$ (these are usually sufficient in dimension $n \geq 3$, e.g, for the model problem);
- possibly singular but completely known functions of ϵ, such as, for example, $\epsilon \log(\epsilon)$ or $\log(\epsilon)^{(-1)}$ (likely needed in dimension $n = 2$).

This method has been first applied to investigate perturbation problems for the conformal representation, for the Schwarz problem, and for boundary value problems for the Laplace and Poisson equations in bounded domain with a small hole (cf. Lanza de Cristoforis [46, 47, 49, 50], Dalla Riva and Musolino [28, 29], Preciso and Rogosin [71]). Later on the approach has been extended to nonlinear traction problems in elastostatics (cf. Dalla Riva and Lanza de Cristoforis [24]), to the Stokes's flow (cf. Dalla Riva [22]) and to the case of an infinite periodically perforated domains (cf. Dalla Riva and Musolino [27] and Musolino [66]).

For a basic model problem, we now briefly outline the strategy of the

Functional Analytic Approach:

S1. for each ϵ small and positive we consider a boundary value problem, called $(\mathrm{BVP})_\epsilon$, defined on an ϵ-dependent domain $\Omega(\epsilon)$, which tends to a limiting configuration for $\epsilon = 0$;

S2. by potential theory, using a suitable integral representation, we transform $(\mathrm{BVP})_\epsilon$ into equivalent integral equations defined on $\partial\Omega(\epsilon)$;

S3. we get rid of the dependence of the domain on ϵ obtaining equivalent integral equations on a fixed domain;

S4. we analyse the solutions of the integral equations around the case $\epsilon = 0$ by means of the Implicit Function Theorem;

S5. using the suitable integral representation chosen, we prove real analyticity properties of the solutions.

In this dissertation, by adopting the Functional Analytic Approach, we analyse two nonlinear boundary value problems for the Laplace operator: the first one will present a singularly perturbed domain, the second one a regularly perturbed domain.

We now describe in details the content of each chapter.

Chapter 1. The first chapter is devoted to the presentation of classical notion of Potential Theory, which will be widely used in the sequel. The author does not take any credit for the results exposed in this chapter: references can be found therein. In section 1.1 we introduce harmonic functions and the definition of fundamental solution for the Laplace operator. Section 1.2 is devoted to the presentation of Green's Identities, from where Potential

Theory stems. In section 1.3, for an open subset Ω of $\mathbb{R}^n$ of class $C^{1,\alpha}$, we introduce the single layer potential v_Ω and the double layer potential w_Ω, and the boundary integral operators $W_{\partial\Omega}$, $W^*_{\partial\Omega}$ and $V_{\partial\Omega}$, which derive from the analysis of the behaviour of the single and double layer potentials on the boundary of Ω. Classical properties of those objects are presented, such as jump formulas, regularity results and mapping properties. Moreover in Section 1.4 we briefly present the Fredholm method for the Dirichlet and Neumann boundary value problems for harmonic functions, which we consider as the prototype of the nonlinear transmission problems studied in Chapters 2 and 3. Uniqueness theorems for the above mentioned problems are stated in subsection 1.4.1. In subsection 1.4.2 we present the existence theorems and we also analyse Fredholm operators arising in the framework of classical Potential Theory and the kernel of important operators deriving from the jump formulas. All this results will play a central role in the succeeding chapters.

Chapter 2. In order to describe the results stated in the second chapter and the problem we have analysed, we begin by presenting the geometric framework. We fix once for all a natural number

$$n \geq 3$$

that will be the dimension of the space $\mathbb{R}^n$ we are going to work in and a parameter

$$\alpha \in]0,1[$$

which we use to define the regularity of our sets and functions. We remark that the case of dimension $n = 2$ requires specific techniques and it is not treated in this dissertation (the analysis for $n = 3$ and for $n \geq 3$ is instead very similar).

Then, we introduce two sets Ω^o and Ω^i that satisfy the following conditions:

Ω^o, Ω^i are bounded open connected subsets of $\mathbb{R}^n$ of class $C^{1,\alpha}$,

their exteriors $\mathbb{R}^n \setminus \overline{\Omega^o}$ and $\mathbb{R}^n \setminus \overline{\Omega^i}$ are connected,

and the origin 0 of $\mathbb{R}^n$ belongs both to Ω^o and to Ω^i.

Here the superscript "o" stands for "outer domain" whereas the superscript "i" stands for "inner domain". We take

$$\epsilon_0 \equiv \sup\{\theta \in]0, +\infty[\colon \epsilon\overline{\Omega^i} \subseteq \Omega^o, \ \forall \epsilon \in] - \theta, \theta[\},$$

and we define the perforated domain $\Omega(\epsilon)$ by setting

$$\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$$

for all $\epsilon \in] - \epsilon_0, \epsilon_0[$. Then we fix three functions

$$F, G \colon] - \epsilon_0, \epsilon_0[\times \partial\Omega^i \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad f^o \in C^{1,\alpha}(\partial\Omega^o).$$

We mention that, in general F and G would be nonlinear functions and they will depend on the size ϵ of the inclusion, and on the position on the boundary of the inclusion $\partial\Omega^i$. This latter fact is stressed with the term nonautonomous, in contrast with the case in which the functions F and G do not depend on the position on the boundary of the inclusion $\partial\Omega^i$ (also called autonomous case).

Then, for $\epsilon \in]0, \epsilon_0[$, we consider the following nonlinear nonautonomous transmission problem in the perforated domain $\Omega(\epsilon)$ for a pair of functions

$(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\overline{\epsilon\Omega^i})$:

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega(\epsilon), \\ \Delta u^i = 0 & \text{in } \epsilon\Omega^i, \\ u^o(x) = f^o(x) & \forall x \in \partial\Omega^o, \\ u^o(x) = F\left(\epsilon, \frac{x}{\epsilon}, u^i(x)\right) & \forall x \in \epsilon\partial\Omega^i, \\ \nu_{\epsilon\Omega^i} \cdot \nabla u^o(x) - \nu_{\epsilon\Omega^i} \cdot \nabla u^i(x) = G\left(\epsilon, \frac{x}{\epsilon}, u^i(x)\right) & \forall x \in \epsilon\partial\Omega^i. \end{cases} \tag{1}$$

Here $\nu_{\epsilon\Omega^i}$ denotes the outer exterior normal to $\epsilon\Omega^i$. Since problem (1) is nonlinear, one cannot, *a priori*, claim that it has a solution. Moreover, we mention that a similar problem, but with homogeneous contact conditions, i.e. the autonomous case, has been studied by Lanza de Cristoforis in [50] for a bounded domain with a small hole and in Lanza de Cristoforis and Musolino [52] in the periodic setting.

Chapter 2 is devoted to prove the following two main results:

R1. Possibly shrinking ϵ_0, problem (1) has a solution $(u^o_\epsilon, u^i_\epsilon) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\overline{\epsilon\Omega^i})$ for all $\epsilon \in]0, \epsilon_0[$ (cf. Theorem 2.7.2).

R2. Possibly shrinking ϵ_0, the map which takes $\epsilon \in]0, \epsilon_0[$ to a suitable restrictions of the family of solutions $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon_0[}$ can be represented in terms of real analytic functions (cf. Theorem 2.7.3).

Going more into details of the contents of Chapter 2, in section 2.1 we represent harmonic functions in $\overline{\Omega(\epsilon)}$ and $\overline{\epsilon\Omega^i}$ in terms of u^o_0 (unique solution of the Dirichet problem in Ω^o with boundary data f^o), double layer potentials with appropriate densities, and a suitable restriction of the fundamental solution S_n. Section 2.2 is devoted to the proof of two Taylor expansion's lemmas. In section 2.3 we provide a formulation of problem (1) in terms of integral equations. In section 2.4 we prove existence and uniqueness results for the integral system obtained by letting $\epsilon \to 0$. Section 2.5 we present a result on integral operators: in the framework of Pettis integral, we prove real analyticity of the map from a suitable subspace U of a Banach space X to a Banach space Y, which takes

$$w \in U \mapsto \int_0^1 f(\tau) A(\tau w)\, d\tau \in Y$$

with $f \in L^1([0,1])$ and $A : U \mapsto Y$ real analytic. Then we apply this result in a particular case for the remainder of the Taylor expansion of F. In section 2.6 we rewrite our integral system depending on ϵ obtained in section 2.3 into an equation for an auxiliary map $M :]-\epsilon_0, \epsilon_0[\times X \to Y$ (with X and Y suitable Banach spaces), namely we will prove that if $\epsilon \in]-\epsilon_0, \epsilon_0[$, then

$$M[\epsilon, \mu] = 0 \quad \text{ with } \mu \in X \text{ and } 0 \in Y$$

if and only if the element μ solves the integral system for that specific ϵ (cf. Proposition 2.6.1). Then we prove that M is real analytic (cf. Proposition 2.6.2) and the differential with respect to the variable $\mu \in X$ evaluated at the point $(0, \mu_0)$ (with μ_0 the unique solution of the limiting system provided by section 2.4) is an isomorphism (cf. Proposition 2.6.4). Hence we apply the Implicit Function Theorem and find the densities as implicit functions (cf. Theorem 2.6.5). Finally in section 2.7, using again the integral representation provided in section 2.1, we show the existence, for ϵ_0 small enough, of a family of solutions $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon \in]0,\epsilon_0[}$ of problem (1) and we prove that it can be represented in terms of real analytic functions (cf. Theorem 2.7.3).

Chapter 3. This chapter is devoted to prove uniqueness of the solutions provided in Chapter 2 for the problem 1. More precisely, the aim of this chapter is to prove that each of such solutions $(u^o_\epsilon, u^i_\epsilon)$ is locally unique, i.e. for $\epsilon > 0$ smaller than a certain $\epsilon^* \in]0, \epsilon_0[$, any solution (v^o, v^i) of problem (1) that is "close enough" to the pair $(u^o_\epsilon, u^i_\epsilon)$ has to coincide with $(u^o_\epsilon, u^i_\epsilon)$. We will see that the "distance" from the solution $(u^o_\epsilon, u^i_\epsilon)$ can be measured solely in terms of the $C^{1,\alpha}$-norm of the trace of the rescaled function $v^i(\epsilon\cdot)$ on $\partial\Omega^i$.

More precisely, we will prove that there exists $\delta^* > 0$ such that, if $\epsilon \in]0, \epsilon^*]$, (v^o, v^i) is a solution of (1), and

$$\left\| v^i(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} < \epsilon\delta^*,$$

then

$$(v^o, v^i) = (u^o_\epsilon, u^i_\epsilon)$$

(cf. Theorem 3.4.1). We emphasize that in general one cannot expect for nonlinear boundary value problems the solution to be locally unique (see, e.g., [25] where it has been shown that for a "big" inclusion, i.e. for $\epsilon > 0$ fixed but not small, problem (1) may have solutions that are not locally unique).

We also observe that uniqueness results are not new in the applications of the Functional Analytic Approach to nonlinear boundary value problems (see, e.g., the above mentioned papers [23, 47, 50]). However, the results so far presented concern the uniqueness of the entire family of solutions rather than the uniqueness of a single solution for $\epsilon > 0$ fixed. For our specific problem (1), a uniqueness result for the family $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ would consist in proving that if $\{(v^o_\epsilon, v^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ is another family of solutions which satisfies a certain "limiting condition", for example that

$$\lim_{\epsilon\to 0} \epsilon^{-1} \left\| v^i_\epsilon(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} = 0,$$

then

$$(v^o_\epsilon, v^i_\epsilon) = (u^o_\epsilon, u^i_\epsilon)$$

for ϵ small enough.

One can verify that the local uniqueness of a single solution stated in Theorem 3.4.1 implies the uniqueness of the family of solutions $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon_0[}$ in the sense described here above (see Corollary 3.5.1). From this point of view, we can say that the uniqueness result presented in this Chapter 2 strengthen the uniqueness result for families which is typically obtained in the application of the functional analytic approach.

Going more into details of the contents of Chapter 3, in section 3.1 we prove Theorem 3.1.2, which is a weaker version of our main result Theorem 3.4.1. We mention that Theorem 3.1.2 follows from the Implicit Function Theorem argument used to obtain the family $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$. The statement of Theorem 3.1.2 is similar to that of Theorem 3.4.1, but the assumptions are much stronger. In particular, together with the aforementioned condition

$$\left\| v^i(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} < \epsilon\delta^*,$$

we have to require other two conditions, namely that

$$\|v^o - u^o_\epsilon\|_{C^{1,\alpha}(\partial\Omega^o)} < \epsilon\delta^* \quad \text{and} \quad \|v^o(\epsilon\cdot) - u^o_\epsilon(\epsilon\cdot)\|_{C^{1,\alpha}(\partial\Omega^i)} < \epsilon\delta^*,$$

in order to prove that $(v^o, v^i) = (u^o_\epsilon, u^i_\epsilon)$. In our main Theorem 3.4.1 we will see that those last two conditions can be dropped.
Section 3.2 is devoted to present some results on composition operators in Schauder spaces which will play an important role in the proof of Theorem 3.4.1: Lemmas 3.2.5 and 3.2.6, in particular, provide uniform bounds for

the $C^{m,\alpha}$-norm (with $m \in \{0,1\}$) of specific classes of composition operators generated by functions $A :]-\epsilon_0, \epsilon_0[\times \overline{B_{n-1}(0,1)} \times \mathbb{R} \to \mathbb{R}$ or $B :]-\epsilon_0, \epsilon_0[\times \partial\Omega^i \times \mathbb{R} \to \mathbb{R}$, respectively. In section 3.3 we introduce and analyse two auxiliary maps N and S that, under suitable conditions, allowed us for ϵ small enough to rewrite the equation $M(\epsilon, \mu) = 0$ with $\mu \in X$ into a fixed point equation, namely

$$\mu = N[\epsilon, \cdot]^{(-1)} [S[\epsilon, \mu]] \quad \text{with } \mu \in X.$$

Finally section 3.4 is devoted to the proof of our main result Theorem 3.4.1.

Chapter 4. In order to describe the results stated in the fourth chapter and the problem we have analysed, we begin by presenting the geometric framework. We fix again once for all a natural number

$$n \geq 2$$

and a parameter

$$\alpha \in]0,1[.$$

Then, we introduce two sets Ω^o and Ω^i that satisfy the following conditions:

Ω^o, Ω^i are bounded open connected subsets of $\mathbb{R}^n$ of class $C^{1,\alpha}$,

their exteriors $\mathbb{R}^n \setminus \overline{\Omega^o}$ and $\mathbb{R}^n \setminus \overline{\Omega^i}$ are connected,

the origin 0 of $\mathbb{R}^n$ belongs both to Ω^o and to Ω^i,

and $\Omega^i \subset \Omega^o$.

Then we fix three functions

$$F_1, F_2 \in C^0(\partial\Omega^i \times \mathbb{R} \times \mathbb{R}) \text{ and } f^o \in C^{0,\alpha}(\partial\Omega^o).$$

We introduce a transmission problem in the pair of domains consisting of $\Omega^o \setminus \overline{\Omega^i}$ and Ω^i. The functions F_1 and F_2 determine the transmission conditions on the (inner) boundary $\partial\Omega^i$. Instead, f^o plays the role of the Neumann datum on the (outer) boundary $\partial\Omega^o$. We are now ready to introduce the following nonlinear non-autonomous transmission boundary value problem for a pair of functions $(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$:

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega^o \setminus \overline{\Omega^i}, \\ \Delta u^i = 0 & \text{in } \Omega^i, \\ \nu_{\Omega^o} \cdot \nabla u^o(x) = f^o(x) & \forall x \in \partial\Omega^o, \\ \nu_{\Omega^i} \cdot \nabla u^o(x) = F_1(x, u^o(x), u^i(x)) & \forall x \in \partial\Omega^i, \\ \nu_{\Omega^i} \cdot \nabla u^i(x) = F_2(x, u^o(x), u^i(x)) & \forall x \in \partial\Omega^i. \end{cases} \tag{2}$$

We note that, *a priori*, it is not clear why problem (2) should admit a classical solution. We prove that under suitable conditions on F_1 and F_2, problem (4.3) has at least a solution $(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$.

Then we introduce a regularly perturbed variant of problem (2). We fix the external domain Ω^o and we assume that the boundary of the internal domain is of the form $\phi(\partial\Omega^i)$, where ϕ is a diffeomorphism of $\partial\Omega^i$ into $\mathbb{R}^n$ and belongs to the class

$$\mathcal{A}_{\partial\Omega^i} \equiv \left\{ \phi \in C^1(\partial\Omega^i, \mathbb{R}^n) : \phi \text{ injective and } d\phi(y) \text{ injective for all } y \in \partial\Omega^i \right\} .$$

Clearly the canonical injection $\mathrm{id}_{\partial\Omega^i}$ of $\partial\Omega^i$ into $\mathbb{R}^n$ belongs to the class $\mathcal{A}_{\partial\Omega^i}$, and, for convenience, we set

$$\phi_0 \equiv \mathrm{id}_{\partial\Omega^i}.$$

Then by the Jordan Leray Separation Theorem, $\mathbb{R}^n \setminus \phi(\partial\Omega^i)$ has exactly two open connected components for all $\phi \in \mathcal{A}_{\partial\Omega^i}$, and we define $\mathbb{I}[\phi]$ to be the unique bounded open connected component of $\mathbb{R}^n \setminus \phi(\partial\Omega^i)$. Finally we set

$$\mathcal{A}_{\partial\Omega^i}^{\Omega^o} \equiv \left\{ \phi \in \mathcal{A}_{\partial\Omega^i} : \overline{\mathbb{I}[\phi]} \subset \Omega^o \right\}.$$

Now let $\phi \in \mathcal{A}_{\partial\Omega^i}^{\Omega^o}$. We consider the following nonlinear non-autonomous trasmission problem in the perforated domain $\Omega^o \setminus \overline{\mathbb{I}[\phi]}$ for a pair of functions $(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \mathbb{I}[\phi]) \times C^{1,\alpha}(\overline{\mathbb{I}[\phi]})$:

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega^o \setminus \overline{\mathbb{I}[\phi]}, \\ \Delta u^i = 0 & \text{in } \mathbb{I}[\phi], \\ \nu_{\Omega^o} \cdot \nabla u^o(x) = f^o(x) & \forall x \in \partial\Omega^o, \\ \nu_{\mathbb{I}[\phi]} \cdot \nabla u^o(x) = F_1(\phi^{(-1)}(x), u^o(x), u^i(x)) & \forall x \in \phi(\partial\Omega^i), \\ \nu_{\mathbb{I}[\phi]} \cdot \nabla u^i(x) = F_2(\phi^{(-1)}(x), u^o(x), u^i(x)) & \forall x \in \phi(\partial\Omega^i). \end{cases} \tag{3}$$

Going into details of the contents of Chapter 4, in section 4.1, we first represent harmonic functions in $\overline{\Omega^o} \setminus \Omega^i$ and $\overline{\Omega^i}$ in terms of single layer potentials with appropriate densities and constants. Moreover we prove an uniqueness result in $C^{1,\alpha}(\Omega^o \setminus \overline{\Omega^i}) \times C^{1,\alpha}(\overline{\Omega^i})$ for an homogeneous linear transmission problem and we analyse an auxiliary boundary operator arising from the integral formulation of that problem (cf. Lemma 4.1.2 and Proposition 4.1.3). In section 4.2 we provide a formulation of problem (3) in terms of integral equations. Section 4.3 is devoted to prove an existence result for the integral system obtained by choosing $\phi = \phi_0$ in the integral equations obtained in section 4.3. In particular, the limiting system is solved by means of a fixed point theorem, namely the Leray-Schauder Theorem (cf. Proposition 4.3.3). We observe that the limiting system is linked with the integral formulation of problem (2): hence we obtain, under suitable conditions on the functions F_1

and F_2, an existence results in $C^{1,\alpha}(\Omega^o \setminus \overline{\Omega^i}) \times C^{1,\alpha}(\overline{\Omega^i})$ for problem (2) (cf. Proposition 4.3.4). In section 4.4 we rewrite our integral system depending on the diffeomorphism ϕ obtained in section 4.2 into an equation for an auxiliary map $M : \mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times X \to Y$ (with X and Y suitable Banach spaces), namely, we will prove that if $\phi \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$, then

$$M[\phi, \mu] = 0 \quad \text{with } \mu \in X \text{ and } 0 \in Y$$

if and only if the element μ solves the integral system for that specific ϕ (cf. Proposition 4.4.1). Then by means of real analytic results for the dependence of single and double layer potential upon the perturbation fo the support (see, e.g., Lanza de Cristoforis and Rossi [53]), we prove that M is real analytic (cf. Proposition 4.4.3). Moreover the differential with respect to the variable $\mu \in X$ evaluated at the point (ϕ_0, μ_0) (with μ_0 the solution of the limiting system provided by section 4.3) is an isomorphism (cf. Proposition 4.4.4). Hence we apply the Implicit Function Theorem and we find the densities as real analytic implicit functions (cf. Theorem 4.4.5). Finally in section 4.5, using again the integral representation provided in section 4.1, we show the existence, for ϕ in a neighborhood Q_0 of ϕ_0, of a family of solutions $\{(u^o_\phi, u^i_\phi)\}_{\phi \in Q_0}$ of problem (3) and we prove it can be represented in terms of real analytic functions (cf. Theorem 4.5.3).

Notation

We denote by $\mathbb{N}$ the set of natural numbers including 0. We denote the norm of a real normed space X by $\|\cdot\|_X$. We denote by I_X the identity operator from X to itself and we omit the subscript X where no ambiguity can occur. For $x \in X$ and $R > 0$, we denote the ball in X of centre x and radius R by

$$B_X(x,R) \equiv \{y \in X : \|y - x\|_X < R\}.$$

When $X = \mathbb{R}^d$, $d \in \mathbb{N} \setminus \{0,1\}$, we simply write $B_d(x,R)$ and when $X = \mathbb{R}$ we write $B(x,R)$. If X and Y are normed spaces we endow the product space $X \times Y$ with the norm defined by $\|(x,y)\|_{X\times Y} \equiv \|x\|_X + \|y\|_Y$ for all $(x,y) \in X \times Y$, while we use the Euclidean norm for $\mathbb{R}^d$, $d \in \mathbb{N} \setminus \{0,1\}$. We denote by $\mathcal{L}(X,Y)$ the space of linear and continuous map of X to Y, equipped with its usual norm of the uniform convergence on the unit sphere of X. If U is an open subset of X, and $F : U \to Y$ is a Fréchet-differentiable map in U, we denote the differential of F by dF. Higher order differentials are denoted by d^mF, $m \in \mathbb{N}\setminus\{0,1\}$. The inverse function of an invertible function f is denoted by $f^{(-1)}$, while the reciprocal of a non-zero scalar function g or the inverse of an invertible matrix A are denoted by g^{-1} and A^{-1} respectively. Let $\Omega \subseteq \mathbb{R}^n$. Then $\overline{\Omega}$ denotes the closure of Ω in $\mathbb{R}^n$, $\partial\Omega$ denotes the boundary of Ω, and ν_Ω denotes the outward unit normal to $\partial\Omega$. For $x \in \mathbb{R}^d$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^d$.

Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{N} \setminus \{0\}$. The space of m times continuously differentiable real-valued function on Ω is denoted by $C^m(\Omega, \mathbb{R})$ or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^r$. The s-th component of f is denoted by f_s and the gradient of f_s is denoted by ∇f_s. Let $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$ and $|\eta| = \eta_1 + \cdots + \eta_n$. Then $D^\eta f \equiv \frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1}, \dots, \partial x_n^{\eta_n}}$. We retain the standard notion for the space $C^\infty(\Omega)$ and its subspace $C_c^\infty(\Omega)$ of functions with compact support.

The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega})$. We denote by $C_b^m(\overline{\Omega})$ the space of functions of $C^m(\overline{\Omega})$ such that $D^\eta f$ is bounded for $|\eta| \leq m$. Then the space $C_b^m(\overline{\Omega})$ equipped with the usual norm

$$\|f\|_{C_b^m(\overline{\Omega})} \equiv \sum_{|\eta| \leq m} \sup_{\overline{\Omega}} |D^\eta f|$$

is well known to be a Banach space.

Let $f \in C^0(\overline{\Omega})$. Then we define its Hölder constant as

$$|f : \Omega|_\alpha \equiv \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in \overline{\Omega}, x \neq y \right\}.$$

Then we can define the subspace of $C^0(\overline{\Omega})$ of Hölder continuous function with exponent $\alpha \in]0, 1[$ by

$$C^{0,\alpha}(\overline{\Omega}) \equiv \{f \in C^0(\overline{\Omega}) : |f : \Omega|_\alpha < \infty\}$$

Similarly, the subspace of $C^m(\overline{\Omega})$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0, 1[$ is denoted $C^{m,\alpha}(\overline{\Omega})$. Then the space

$$C_b^{m,\alpha}(\overline{\Omega}) \equiv C^{m,\alpha}(\overline{\Omega}) \cap C_b^m(\overline{\Omega}),$$

equipped with its usual norm

$$\|f\|_{C_b^{m,\alpha}(\overline{\Omega})} \equiv \|f\|_{C_b^m(\overline{\Omega})} + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha,$$

is well known to be a Banach space. If Ω is bounded, then $C_b^{m,\alpha}(\overline{\Omega}) = C^{m,\alpha}(\overline{\Omega})$, and we omit the subscript b. We denote by $C_{\text{loc}}^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ the space of functions on $\mathbb{R}^n \setminus \Omega$ whose restriction to $\overline{U}$ belongs to $C^{m,\alpha}(\overline{U})$ for all open bounded subsets U of $\mathbb{R}^n \setminus \Omega$. On $C_{\text{loc}}^{m,\alpha}(\mathbb{R}^n \setminus \Omega)$ we consider the natural structure of Fréchet space. Finally if Ω is bounded, we set

$$C_{\text{harm}}^{m,\alpha}(\overline{\Omega}) \equiv \{u \in C^{m,\alpha}(\overline{\Omega}) \cap C^2(\Omega) : \Delta u = 0 \text{ in } \Omega\},$$

and if Ω is unbounded we set

$$C_{\text{harm}}^{m,\alpha}(\overline{\Omega}) \equiv \{u \in C^{m,\alpha}(\overline{\Omega}) \cap C^2(\Omega) : \Delta u = 0 \text{ in } \Omega, \text{ u harmonic at infinity}\}.$$

We say that a bounded open subset of $\mathbb{R}^n$ is of class $C^{m,\alpha}$ if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class $C^{m,\alpha}$. In particular if Ω is a $C^{1,\alpha}$ subset of $\mathbb{R}^n$, then $\partial\Omega$ is a $C^{1,\alpha}$ sub-manifold of $\mathbb{R}^n$ of co-dimension 1. If M is a $C^{m,\alpha}$ sub-manifold of $\mathbb{R}^n$ of dimension $d \geq 1$, we define the space $C^{m,\alpha}(M)$ by exploiting a finite local parametrization. Namely, we take a finite open covering $\mathcal{U}_1, \ldots, \mathcal{U}_k$ of M and $C^{m,\alpha}$ local parametrization maps $\gamma_l : \overline{B_d(0,1)} \to \overline{\mathcal{U}_l}$ with $l = 1, \ldots, k$ and we say that $\phi \in C^{m,\alpha}(M)$ if and only if $\phi \circ \gamma_l \in C^{m,\alpha}(\overline{B_d(0,1)})$ for all $l = 1, \ldots, k$. Then for all $\phi \in C^{m,\alpha}(M)$ we define

$$\|\phi\|_{C^{m,\alpha}(M)} \equiv \sum_{l=1}^{k} \|\phi \circ \gamma_l\|_{C^{m,\alpha}(\overline{B_d(0,1)})} .$$

One verifies that different $C^{m,\alpha}$ finite atlases define the same space $C^{m,\alpha}(M)$ and equivalent norms on it. We retain the standard notion for the Lebesgue spaces L^p, $p \geq 1$. If $\partial\Omega$ is a $C^{1,\alpha}$ sub-manifold of $\mathbb{R}^n$, then we denote by $d\sigma$ the area element on $\partial\Omega$. If Z is a subspace of $L^1(\partial\Omega)$, then we set

$$Z_0 \equiv \left\{ f \in Z : \int_{\partial\Omega} f \, d\sigma = 0 \right\} .$$

CHAPTER 1

Classical Potential Theory results

In this chapter we present a summary of the main results of classical Potential Theory used in this thesis. We begin by introducing the notions of harmonic functions and fundamental solution of the Laplace operator. Then we state Green's Identities (see section 1.2), from where the definitions of single and double layer potential stem (see section 1.3). Finally, in section 1.4, we briefly summarize results for the interior and exterior boundary value problem for the Laplace equation with Dirichlet or Neumann condition. We state uniqueness results, which follow essentially from Maximum Principle and standard energy arguments for harmonic functions, and existence results, which are obtained via Fredholm method. We also present key results on some particular Fredholm operators which will play an essential role in the solution of the transmission problems we will deal with in the next chapters. We point out that the tools introduced to treat Dirichlet and Neumann boundary value problems are, in a sense, the basis for the techniques that we will use to study the transmission conditions that are the focus of this thesis. For this reason, the existence results presented in section 1.4 have been considered by the author worth mentioning.

1.1 Harmonic functions and fundamental solution for the Laplace operator

In this chapter we fix once for all

$$n \in \mathbb{N} \setminus \{0, 1\},$$

which will denote the dimension of the space $\mathbb{R}^n$ we will work in. We start introducing the notion of harmonic functions.

Definition 1.1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^2(\Omega)$. We define the Laplace operator by*

$$\Delta u \equiv \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Moreover, we say that u is harmonic in Ω if $\Delta u = 0$ in Ω.

Among the numerous properties that harmonic functions satisfies, we mention a variant of a result of Kobe that ensures that if Ω is an open subset of $\mathbb{R}^n$ and if $u \in C^2_{\mathrm{harm}}(\Omega)$, then $u \in C^\infty(\Omega)$ (even a stronger result can be proven: u is real analytic in Ω). Then we introduce the notion of fundamental solution for the Laplace operator (see, e.g., Folland [33, Chap. 1]).

Definition 1.1.2. *We say that a function $f \in L^1_{loc}(\mathbb{R}^n)$ is a fundamental solution for the Laplace operator Δ in $\mathbb{R}^n$ if*

$$\int_{\mathbb{R}^n} f \Delta\phi \, dx = \phi(0) \quad \forall \phi \in C^\infty_c(\mathbb{R}^n),$$

i.e. $\Delta f = \delta_0$ *in the distributional sense.*

Then, we have the following result (for a proof of Theorem 1.1.3 we refer to Gilbarg and Trudinger [35, Section 2.4]).

Theorem 1.1.3. *We denote by* S_n *the function from* $\mathbb{R}^n \setminus \{0\}$ *to* $\mathbb{R}$ *defined by*

$$S_n(x) = \begin{cases} \frac{1}{s_n} \log |x| & \forall x \in \mathbb{R}^n \setminus \{0\} \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\} \quad \text{if } n \geq 3. \end{cases}$$

where s_n *denotes the* $(n-1)$*-dimensional measure of* $\partial B_n(0,1)$*. Then,* S_n *is a fundamental solution of the Laplace operator in* $\mathbb{R}^n$*.*

By a simple computation one can verify that

$$\begin{aligned} \frac{\partial}{\partial x_j} S_n(x) &= \frac{1}{s_n} \frac{x_j}{|x|^n} \qquad \forall j \in \{1, \dots, n\}, \\ \nabla S_n(x) &= \frac{1}{s_n} \frac{x}{|x|^n} \qquad |\nabla S_n(x)| \leq \frac{1}{s_n} \frac{1}{|x|^{n-1}}, \\ \left| \frac{\partial^2 S_n}{\partial x_i \partial x_j} \right| &\leq \frac{1}{w_n} \frac{1}{|x|^n} \qquad \forall i, j \in \{1, \dots, n\}, \end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

Then we consider the behaviour at infinity of an harmonic function. We have the following characterisation (see, e.g., Folland [33, Prop. 2.74]).

Definition 1.1.4. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *such that there exists a compact subset* K *of* $\mathbb{R}^n$ *with* $\mathbb{R}^n \setminus K \subseteq \Omega$*. Let* u *be an harmonic function from* Ω *to* $\mathbb{R}$*. Then* u *is harmonic at infinity if and only if* $u(x) = O(|x|^{2-n})$ *as* x *tends to* ∞*. In particular, if* $n \geq 3$*, then*

$$\lim_{x \to \infty} u(x) = 0,$$

and if $n = 2$*, then*

$$u(x) = o(\log|x|) \text{ as } x \text{ tends to } \infty.$$

Remark 1.1.5. *Classically the definition of harmonicity at infinity is done using the notion of removable singularity and the Kelvin transform. We decide to avoid this complication and use the above characterisation.*

Notice that the condition $u(x) = O(|x|^{2-n})$ *as* x *tends to* ∞ *can be written as*

$$\sup_{|x|>R} |x|^{n-2}|u(x)| < +\infty$$

for some $R > 0$ *such that* $\mathbb{R}^n \setminus \Omega \subset B_n(0, R)$*.*

1.2 Green's Identities

This section is devoted to the presentation of the Green's Identities and the Green's Representation Formula which play an important role in Potential Theory, in the Fredholm Method and in the analysis of boundary value problems for the Laplace operator. The Green's Identities can be derived from the Divergence Theorem which, roughly speaking, establishes a link between the interior and the boundary of a region on which differential operators act. In the following theorem we present the classical versions of the First Green's Identity (for a proof see, e.g., Gilbarg and Trudinger [35, Section 2.4]).

Theorem 1.2.1 (First Green's Identity). *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *of class* C^1*. Let* $u, v \in C^2(\overline{\Omega})$*. Then the following First Green's Identity holds*

$$\int_\Omega v\Delta u + \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} v\,(\nu_\Omega \cdot \nabla u)\, d\sigma. \tag{1.1}$$

Then interchanging u and v in (1.1) and subtracting side by side, one obtains the Second Green's Identity.

Theorem 1.2.2 (Second Green's Identity). *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. Let $u \in C^2(\overline{\Omega})$. Then the following identity holds*

$$\int_\Omega v\Delta u - u\Delta v\, dx = \int_{\partial\Omega} (\nu_\Omega \cdot \nabla u)\, v - u\, (\nu_\Omega \cdot \nabla v)\, d\sigma.$$

Then using the fundamental solution S_n of the Laplace operator for the function v, one can obtain the Third Green's Identity (for a proof see, e.g., Gilbarg and Trudinger [35, Section 2.4]).

Theorem 1.2.3 (Third Green's Identity). *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. Let $u \in C^2(\overline{\Omega})$. Then the following Green's Representation Formula holds*

$$\begin{aligned} u(x) = &\int_\Omega \Delta u(y) S_n(x-y)\, dy \\ &+ \int_{\partial\Omega} u(y)\, (\nu_\Omega(y) \cdot \nabla S_n(x-y)) - (\nu_\Omega(y) \cdot \nabla u(y))\, S_n(x-y)\, d\sigma_y \end{aligned} \tag{1.2}$$

for all $x \in \Omega$.

We want to underline the importance of (1.2). In fact, if one chooses as u a function that is harmonic in Ω, then the Third Green's Identity actually provides a formula for representing u in terms only of its boundary value and the boundary value of the normal derivative of u. This result is presented in the following corollary.

Corollary 1.2.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. Let*

$u \in C^2(\overline{\Omega})$ be harmonic in Ω. Then

$$\int_{\partial\Omega} u(y)\,(\nu_\Omega(y)\cdot\nabla S_n(x-y)) - (\nu_\Omega(y)\cdot\nabla u(y))\,S_n(x-y)\,d\sigma_y = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

In particular, by taking $u \equiv 1$ on Ω, we get

$$\int_{\partial\Omega} \nu_\Omega(y)\cdot\nabla S_n(x-y)\,d\sigma_y = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Now a remark has to be done.

Remark 1.2.5. *In our work we will deal with functions that exhibit a C^2 regularity (actually they will be even more regular, analytic) inside the domain of definition, but only a $C^{1,\alpha}$ regularity up to the boundary. Hence, a priori, the results of this section do not apply to our situation. A refined version of the statements presented in this section ensure that the First and Second Green's Identity actually hold for functions $u, v \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$, just adding conditions on the integrability of the functions $v\Delta u$, $\nabla u \cdot \nabla v$ and $u\Delta v$. In particular, in the situations we will deal with, the above mentioned conditions will be always satisfied and so we can apply all the results of this section. For a reference on this argument, see Dautray and Lions [30, pp. 226-229].*

1.3 Single and double layer potential

In this section we present the key objects in Potential Theory: the single and double layer potentials. The introduction of such boundary integral operators is well motivated by the Third Green's Identity (see also Corollary 1.2.4). Moreover, for connection of single and double layer potentials with the theory of partial differential equations, we refer to Gilbarg and Trudinger [35]. In particular, for the method of layer potentials for the Dirichlet and Neumann problems for the Laplace operator we refer to Folland [33, Chap. 3]. See also section 1.4.

Definition 1.3.1. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$ *for some* $\alpha \in]0,1[$*. If* $\mu \in L^2(\partial\Omega)$*, we denote by* $v_\Omega[\mu]$ *and* $w_\Omega[\mu]$ *the single and the double layer potentials with density* μ *respectively given by*

$$v_\Omega[\mu](x) = \int_{\partial\Omega} S_n(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n, \tag{1.3}$$

and

$$w_\Omega[\mu](x) = -\int_{\partial\Omega} \nu_\Omega(y) \cdot \nabla S_n(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n. \tag{1.4}$$

Then, considering (1.3) and (1.4) restricted to $\partial\Omega$, the following boundary integral operators arise.

Definition 1.3.2. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$ *for some* $\alpha \in]0,1[$*. If* $\mu \in L^2(\partial\Omega)$*, we denote by* $W_{\partial\Omega}[\mu]$ *and* $W^*_{\partial\Omega}[\mu]$ *the boundary integral operators given by*

$$W_{\partial\Omega}[\mu](x) = -\int_{\partial\Omega} \nu_\Omega(y) \cdot \nabla S_n(x-y)\mu(y)\, d\sigma_y \qquad \text{for a.a. } x \in \partial\Omega$$

and

$$W^*_{\partial\Omega}[\mu](x) = -\int_{\partial\Omega} \nu_\Omega(x) \cdot \nabla S_n(x-y)\mu(y)\, d\sigma_y \qquad \text{for a.a. } x \in \partial\Omega.$$

Moreover, we denoted by $V_{\partial\Omega}$ *the operator from* $L^2(\partial\Omega)$ *to itself which takes*

μ to the function $V_{\partial\Omega}[\mu]$ defined by

$$V_{\partial\Omega}[\mu] \equiv v_\Omega[\mu]_{|\partial\Omega},$$

where the restriction operator is understood as a trace.

In the following Theorem 1.3.3 we summarize some of the most important classical results on the single layer potential. They will be widely used in the sequel. For the proof of the results stated, we refer to Cialdea [18], Miranda [56], and Lanza [53, Thm 3.1].

Theorem 1.3.3 (Property of the single layer potential). *Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$. Then the following statements hold.*

(i) *Let $\mu \in C^0(\partial\Omega)$. Then the function $v_\Omega[\mu]$ from $\mathbb{R}^n$ to $\mathbb{R}$ is continuous in $\mathbb{R}^n$ and harmonic in $\mathbb{R}^n \backslash \partial\Omega$. Let $v^+_\Omega[\mu] = v_\Omega[\mu]_{|\overline{\Omega}}$ and $v^-_\Omega[\mu] = v_\Omega[\mu]_{|\mathbb{R}^n\backslash\Omega}$.*
If $n \geq 3$, then the function $v^-_\Omega[\mu]$ is harmonic at infinity.
If $n = 2$, then the function $v^-_\Omega[\mu]$ is harmonic at infinity if and only if $\int_{\partial\Omega} \mu \, d\sigma = 0$. In that case $\lim_{x\to\infty} v^-_\Omega[\mu](x) = 0$.

(ii) *If $\mu \in C^{0,\alpha}(\partial\Omega)$, then $v^+_\Omega[\mu] \in C^{1,\alpha}(\overline{\Omega})$ and the map from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega})$ which takes μ to $v^+_\Omega[\mu]$ is linear and continuous.*

(iii) *Let $\mu \in C^{0,\alpha}(\partial\Omega)$. If $n \geq 3$, then the function $v^-_\Omega[\mu] \in C^{1,\alpha}_b(\mathbb{R}^n \setminus \Omega)$.*
If $n = 2$ and $\int_{\partial\Omega} \mu \, d\sigma = 0$, then the function $v^-_\Omega[\mu] \in C^{1,\alpha}_b(\mathbb{R}^n \setminus \Omega)$.
Moreover, the map from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ which takes μ to $v^-_\Omega[\mu]$ is linear and continuous.

(iv) *If $\mu \in C^{0,\alpha}(\partial\Omega)$, then we have following jump relations*

$$\nu_\Omega \cdot \nabla v^\pm_\Omega[\mu](x) = \mp\frac{1}{2}\mu(x) + W^*_{\partial\Omega}[\mu](x) \qquad \forall x \in \partial\Omega.$$

(v) *The operator* $V_{\partial\Omega}$ *from* $C^{0,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\partial\Omega)$ *is linear and continuous. If* $n \geq 3$*, it is an isomorphism.*

(vi) *The map from* $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes* (μ, ρ) *to* $V_{\partial\Omega}[\mu] + \rho$ *is an isomorphism.*

Then we have the following result for the double layer potential. As before, we refer to Cialdea [18], Miranda [56], and Lanza [53, Thm 3.1] for a proof.

Theorem 1.3.4 (**Property of double layer potential**)**.** *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *of class* $C^{1,\alpha}$ *for some* $\alpha \in]0,1[$*. Then the following statements hold.*

(i) *Let* $\mu \in C^0(\partial\Omega)$*. Then the function* $w_\Omega[\mu]$ *from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *is harmonic in* $\mathbb{R}^n \setminus \partial\Omega$*. Moreover the restriction* $w_\Omega[\mu]_{|\Omega}$ *can be extended uniquely to a continuous function* $w_\Omega^+[\mu]$ *of* $\overline{\Omega}$ *to* $\mathbb{R}$*. The restriction* $w_\Omega[\mu]_{|\mathbb{R}^n \setminus \overline{\Omega}}$ *can be extended uniquely to a continuous function* $w_\Omega^-[\mu]$ *from* $\mathbb{R}^n \setminus \Omega$ *to* $\mathbb{R}$ *which is harmonic at infinity.*
Finally we have the following jump relations

$$w_\Omega^\pm[\mu](x) = \pm\frac{1}{2}\mu(x) + W_{\partial\Omega}[\mu](x) \qquad \forall x \in \partial\Omega.$$

(ii) *Let* $\mu \in C^{1,\alpha}(\partial\Omega)$*. Then* $w_\Omega^+[\mu] \in C^{1,\alpha}(\overline{\Omega})$ *and* $w_\Omega^-[\mu] \in C_b^{1,\alpha}(\mathbb{R}^n \setminus \Omega)$ *and we have*

$$\nu_\Omega \cdot \nabla w_\Omega^+[\mu] - \nu_\Omega \cdot \nabla w_\Omega^-[\mu] = 0 \qquad \text{on } \partial\Omega.$$

(iii) *The map from* $C^{1,\alpha}(\partial\Omega)$ *to* $C^{1,\alpha}(\overline{\Omega})$ *which takes* μ *to* $w_\Omega^+[\mu]$ *is linear and continuous.*
The map from $C^{1,\alpha}(\partial\Omega)$ *to* $C_b^{1,\alpha}(\mathbb{R}^n \setminus \Omega)$ *which takes* μ *to* $w_\Omega^-[\mu]$ *is linear and continuous.*

We finally describe some classical results of Schauder, about regularity and compactness properties of the boundary integral operators $W_{\partial\Omega}$ and $W^*_{\partial\Omega}$. We first point out the following well known property of Hölder spaces:

$$\text{If } \alpha,\beta \in]0,1[,\ \alpha<\beta,\ \text{ then } C^{0,\beta}(\partial\Omega) \text{ is compactely embedded in } C^{0,\alpha}(\partial\Omega)$$

Then, by the weak singularity of the kernels and by mapping properties of the operators $W_{\partial\Omega}$ and $W^*_{\partial\Omega}$, we can summarize in the following form the compactness results of Schauder. For a proof of the statements we refer to Schauder [73, 74].

Theorem 1.3.5. *Let $\alpha \in]0,1[$ and let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following holds.*

(i) $W_{\partial\Omega} : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ and $W^*_{\partial\Omega} : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ are compact operators, adjoint one to the other.

(ii) Let $\beta \in]0,1]$. Then the map which takes μ to $W_{\partial\Omega}[\mu]$ is continuous from $C^0(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ and from $C^{1,\beta}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$. The map which takes μ to $W^*_{\partial\Omega}[\mu]$ is continuous from $C^{0,\beta}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$.

(iii) The map which takes μ to $W_{\partial\Omega}[\mu]$ is compact from $C^0(\partial\Omega)$ to itself, from $C^{0,\alpha}(\partial\Omega)$ to itself, and from $C^{1,\alpha}(\partial\Omega)$ to itself. The map which takes μ to $W^*_{\partial\Omega}[\mu]$ is compact from $C^{0,\alpha}(\partial\Omega)$ to itself.

1.4 Boundary value problems for harmonic functions

Let us consider $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in]0, 1[$ and an open bounded subset Ω of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We set

$$\Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega}.$$

Then, by a compactness argument, it can be easily seen that the number of connected components of Ω and Ω^- are finite, i.e. there exist $m, m^- \in \mathbb{N} \setminus \{0\}$ such that

$\{\Omega_1, \dots, \Omega_m\}$ are the bounded connected components of Ω,

$\{\Omega_0^-, \Omega_1^-, \dots, \Omega_{m^-}^-\}$ are the connected components of Ω^-

with one and only one of such components unbounded,

and

$$\Omega = \bigcup_{j=1}^{m} \Omega_j\,, \quad \Omega^- = \bigcup_{j=1}^{m^-} \Omega_j^-\,.$$

Let us introduce the following basic boundary value problems for the Laplace operator.

The interior Dirichlet boundary value problem. Given $g \in C^0(\partial\Omega)$, find $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\textbf{(ID)} \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The exterior Dirichlet boundary value problem. Given $g \in C^0(\partial\Omega)$, find $u \in C^0(\overline{\Omega^-}) \cap C^2(\Omega^-)$ such that

$$\textbf{(ED)} \quad \begin{cases} \Delta u = 0 \quad \text{in } \Omega^-, \\ u = g \quad \text{on } \partial\Omega. \end{cases}$$

The interior Neumann boundary value problem. Given $g \in C^0(\partial\Omega)$, find $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\textbf{(IN)} \quad \begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \nu_\Omega \cdot \nabla u = g \quad \text{on } \partial\Omega. \end{cases}$$

The exterior Neumann boundary value problem. Given $g \in C^0(\partial\Omega)$, find $u \in C^1(\overline{\Omega^-}) \cap C^2(\Omega^-)$ such that

$$\textbf{(EN)} \quad \begin{cases} \Delta u = 0 \quad \text{in } \Omega^-, \\ \nu_\Omega \cdot \nabla u = g \quad \text{on } \partial\Omega. \end{cases}$$

1.4.1 Uniqueness Theorems

It is well known that uniqueness theorems for the Dirichlet and Neumann boundary value problems described above can be deduced by classical Maximum Principle and by standard energy argument for harmonic functions. More precisely the following theorems hold (see Folland [33, Prop. 3.1-3.4]).

Theorem 1.4.1 (Uniqueness for the interior boundary value problem).

(i) The interior Dirichlet problem has at most one solution in $C^0(\overline{\Omega}) \cap C^2(\Omega)$.

(ii) Two solutions in $C^1(\overline{\Omega}) \cap C^2(\Omega)$ *of the interior Neumann problem differ by a function which is constant on each connected component of* Ω.

Theorem 1.4.2 (Uniqueness for the exterior boundary value problem).

(i) The exterior Dirichlet problem has at most one solution in $C^0(\overline{\Omega^-}) \cap C^2(\Omega^-)$ *harmonic at infinity.*

(ii) Two solutions in $C^1(\overline{\Omega^-}) \cap C^2(\Omega^-)$ *harmonic at infinity of the exterior Neumann problem differ by a function which is constant on each connected component of* Ω^-. *If* $n \geq 3$, *such a constant is* 0 *on the unbounded connected component* Ω_0^- *of* Ω^-.

1.4.2 Existence Theorems

In view of the Green's Identities (see Theorems 1.2.1-1.2.3) and of the jump relations (see Theorem 1.3.3 (v) and Theorem 1.3.4 (i)), it is natural to study of the operators $\pm\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}$ and $\pm\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}$ to obtain existence result for the Dirichlet and Neumann boundary value problems. In particular, we will see that the Fredholm Alternative ensures the existence of solutions in $L^2(\partial\Omega)$ for integral equations related to the Dirichelet and Neumann boundary value problems with compatible data. In order to obtain classical solutions, one needs regularization theorems and to study the mapping properties of the operators $\pm\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}$ and $\pm\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}$. A central role will be played by the characterization of the kernels of the operators mentioned above. Hence, we set

$$\begin{aligned}
\ker\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right) &\equiv \left\{\mu\in L^2(\partial\Omega):\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)[\mu]=0\right\},\\
\ker\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right) &\equiv \left\{\mu\in L^2(\partial\Omega):\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)[\mu]=0\right\},\\
\ker\left(\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right) &\equiv \left\{\mu\in L^2(\partial\Omega):\left(\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)[\mu]=0\right\},\\
\ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right) &\equiv \left\{\mu\in L^2(\partial\Omega):\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)[\mu]=0\right\}.
\end{aligned}$$

We now proceed presenting the Fredholm method. The main idea consists in searching for solutions of the form

$$u=w^+_\Omega[\mu]+v^+_\Omega[\xi]+c \quad \text{on } \overline{\Omega},$$

or

$$u=w^-_\Omega[\mu]+v^-_\Omega[\xi]+c \quad \text{on } \overline{\Omega^-},$$

for some unknown functions $\mu,\xi\in L^2(\partial\Omega)$ and constant $c\in\mathbb{R}$ to be determined by imposing that u satisfies the boundary condition and also the condition of harmonicity at infinity in the exterior case. In both situations one has to deal with Fredholm integral equations of the second type, more precisely we get:

$$\begin{aligned}
\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)[\mu]=g &\qquad \text{corresponding to problem } \textbf{(ID)},\\
\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)[\mu]=g &\qquad \text{corresponding to problem } \textbf{(ED)},\\
\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)[\mu]=g &\qquad \text{corresponding to problem } \textbf{(IN)},\\
\left(\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)[\mu]=g &\qquad \text{corresponding to problem } \textbf{(EN)},
\end{aligned}$$

for suitable function $\mu\in L^2(\partial\Omega)$. Then we have the following definition.

Definition 1.4.3. *We say that a bounded linear operator L from a Banach space X to a Banach space Y is Fredholm if the following conditions hold:*

(i) The null space $KerL$ has finite dimension;

(ii) The co-kernel $Y/RanL$ has finite dimension.

If L is a Fredholm operator, then the index of L is the integer given by

$$indexL \equiv dim\ KerL - dim\ (Y/RanL).$$

A well known result ensures that the sum of an isomophism and a compact operator generates a Fredholm operator of index 0. Moreover, the index of a composition of Fredholm operators is additive, i.e. if F_1 and F_2 are Fredholm operators, then $F = F_1 \circ F_2$ is a Fredholm operator of index

$$\text{index}F = \text{index}F_1 + \text{index}F_2 \,.$$

Thus, by Theorem 1.3.5, one deduces the following important result.

Theorem 1.4.4. *The following holds:*

(i) The operators $\pm\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}$ are Fredholm of index 0 from $L^2(\partial\Omega)$ to itself, from $C^0(\partial\Omega)$ to itself, from $C^{0,\alpha}(\partial\Omega)$ to itself, and from $C^{1,\alpha}(\partial\Omega)$ to itself.

(ii) The operators $\pm\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}$ are Fredholm of index 0 from $L^2(\partial\Omega)$ to itself and from $C^{0,\alpha}(\partial\Omega)$ to itself.

In particular we underline that

$$\dim\ \ker\left(-\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right) = \dim\ \ker\left(-\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right),$$

$$\dim \ker \left(\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right) = \dim \ker \left(\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right).$$

Moreover the following Schauder regularity result holds (see Miranda [56, Chap. II, §14] and Dalla Riva and Mishuris [25, Lemma 3.3]).

Theorem 1.4.5. *The following holds.*

(i) If $g \in C^{1,\alpha}(\partial\Omega)$ and if $\mu \in L^2(\partial\Omega)$ satisfies either equation

$$\left(\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right)[\mu] = g \quad \text{or} \quad \left(-\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right)[\mu] = g,$$

then $\mu \in C^{1,\alpha}(\partial\Omega)$. In particular we have

$$ker\left(\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right) \subseteq C^{1,\alpha}(\partial\Omega),\ ker\left(-\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right) \subseteq C^{1,\alpha}(\partial\Omega).$$

(ii) If $g \in C^{0,\alpha}(\partial\Omega)$ and if $\mu \in L^2(\partial\Omega)$ satisfies either equation

$$\left(\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right)[\mu] = g \quad \text{or} \quad \left(-\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right)[\mu] = g,$$

then $\mu \in C^{0,\alpha}(\partial\Omega)$. In particular we have

$$ker\left(-\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right) \subseteq C^{0,\alpha}(\partial\Omega),\ ker\left(\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right) \subseteq C^{0,\alpha}(\partial\Omega).$$

By Fredholm Alternative Theorem in the Hilbert space $L^2(\partial\Omega)$, we obtain the following.

Theorem 1.4.6. *The following statements hold.*

(i) $Im\left(\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right) \oplus_\perp ker\left(\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right) = L^2(\partial\Omega)$ (corresponding to the problem **(ID)***).*

(ii) $Im\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)\oplus_{\perp} ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)=L^2(\partial\Omega)$ *(corresponding to the problem* **(ED)***).*

(iii) $Im\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)\oplus_{\perp} ker\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)=L^2(\partial\Omega)$ *(corresponding to the problem* **(IN)***).*

(iv) $Im\left(\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)\oplus_{\perp} ker\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)=L^2(\partial\Omega)$ *(corresponding to the problem* **(EN)***).*

Here $\oplus_{\perp}$ *denotes the symbol of orthogonal direct sum and Im denotes the image of an operator.*

The next step in the Fredholm method is to understand for which boundary data g the integral equations are solvable. We do so by mean of the following characterization (see Folland [33, Prop. 3.34, 3.36 and 3.37]).

Theorem 1.4.7. *The following statements hold.*

(i) The space $ker\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)$ *has finite dimension* m *and it is generated by* $\{\chi_{\partial\Omega_j}\}_{j=1}^m$.

(ii) Let $n\geq 3$. *Then* $V_{\partial\Omega}$ *induces an isomorphism of* $ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)$ *onto* $ker\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)$. *Moreover, if* $\mu\in ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)$, *then* $v^-_{\Omega}[\mu]$ *solves the exterior Dirichlet problem*

$$\begin{cases}\Delta u=0 & \textit{in } \Omega^-,\\ u=V_{\partial\Omega}[\mu] & \textit{on } \partial\Omega^-,\\ \lim_{x\to\infty}u(x)=0. & \end{cases}$$

(iii) Let $n=2$. *Then* $V_{\partial\Omega}$ *is injective on* $\left(ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)\right)_0$ *and we*

have

$$V_{\partial\Omega}\left(\left(ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)\right)_0\right)\oplus <\chi_{\partial\Omega}> = ker\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right).$$

Moreover, if $\mu \in \left(ker\left(-\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)\right)_0$, *then* $v^-_\Omega[\mu] \in C^{1,\alpha}_b(\overline{\Omega^-})$ *and solves the exterior Dirichlet problem*

$$\begin{cases}\Delta u = 0 & \text{in } \Omega^-,\\ u = V_{\partial\Omega}[\mu] & \text{on } \partial\Omega^-.\\ u \text{ is harmonic at infinity.}\end{cases}$$

(iv) The space $ker\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)$ *has finite dimension* m^- *and it is generated by* $\{\chi_{\partial\Omega^-_j}\}_{j=1}^{m^-}$ *and* $V_{\partial\Omega}$ *induces an isomorphism of* $ker\left(\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)$ *onto* $ker\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)$. *Moreover, if* $\mu \in ker\left(\frac{1}{2}I_{\partial\Omega}+W^*_{\partial\Omega}\right)$, *then* $v^+_\Omega[\mu]$ *solves the interior Dirichlet problem*

$$\begin{cases}\Delta u = 0 & \text{in } \Omega,\\ u = V_{\partial\Omega}[\mu] & \text{on } \partial\Omega,\end{cases}$$

and $v^-_\Omega[\mu]$ *is constant on each* Ω^-_j *for all* $j = 1,\ldots,m^-$, *and equals* 0 *on* Ω^-_0 *and on* $\partial\Omega^-_0$.

In order to prove an existence theorem for the boundary value problems considered, we have to relax the orthogonal condition in the decomposition of the space $L^2(\partial\Omega)$. Then we have the following (see Folland [33, Cor. 3.39]).

Theorem 1.4.8. *The following statements hold.*

(i) $Im\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)\oplus ker\left(\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right) = L^2(\partial\Omega)$.

(ii) $Im\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right)\oplus ker\left(-\frac{1}{2}I_{\partial\Omega}+W_{\partial\Omega}\right) = L^2(\partial\Omega)$.

We are now ready to state existence theorems for the considered boundary value problems (the uniqueness of the solutions has been stated in Theorems 1.4.1 and 1.4.2). For the interior and the exterior Dirichlet problem we have the following two results (see Folland [33, Thm. 3.40 (a)-(b)]).

Theorem 1.4.9 (**Interior Dirichlet problem**). *Let $g \in C^{1,\alpha}(\partial\Omega)$. Then there exists a unique solution $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ of the interior Dirichlet problem* (**ID**)*. In particular, there exists a unique $\xi \in ker\left(\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right)$ such that the integral equation*

$$\left(\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right)[\mu] + V[\xi] = g$$

has at least a solution $\mu \in C^{1,\alpha}(\partial\Omega)$.
The affine space of such solutions μ has dimension m^- and we have

$$u = w^+_\Omega[\mu] + v^+_\Omega[\xi] \quad in\ \overline{\Omega}.$$

Theorem 1.4.10 (**Exterior Dirichlet problem**). *Let $g \in C^{1,\alpha}(\partial\Omega)$. Then there exists a unique solution $u \in C^{1,\alpha}(\overline{\Omega^-}) \cap C^2(\Omega^-)$ harmonic at infinity of the exterior Dirichlet problem* (**ID**)*. Moreover, the following statements hold.*

(i) Let $n \geq 3$. Then there exists a unique $\xi \in ker\left(-\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right)$ such that the integral equation

$$\left(-\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right)[\mu] + V[\xi] = g$$

has at least a solution $\mu \in C^{1,\alpha}(\partial\Omega)$.
The affine space of such solutions μ has dimension m and we have

$$u = w^-_\Omega[\mu] + v^-_\Omega[\xi] \quad in\ \overline{\Omega^-}.$$

(ii) Let $n = 2$. *Then there exists a unique* $\xi \in \left(ker\left(-\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right)\right)_0$ *and a unique* $c \in \mathbb{R}$ *such that the integral equation*

$$\left(-\frac{1}{2}I_{\partial\Omega} + W_{\partial\Omega}\right)[\mu] + V[\xi] + c = g$$

has at least a solution $\mu \in C^{1,\alpha}(\partial\Omega)$.

The affine space of such solutions μ *has dimension* m *and we have*

$$u = w^-_\Omega[\mu] + v^-_\Omega[\xi] + c \quad \text{in } \overline{\Omega^-}.$$

For the interior and the exterior Neumann Problem we have the following existence theorems (see Folland [33, Thm. 3.40 (c)-(d)]).

Theorem 1.4.11 (For the Interior Neumann problem). *Let* $g \in C^{0,\alpha}(\partial\Omega)$ *satisfy the compatibility condition*

$$\int_{\partial\Omega_j} g\, d\sigma = 0 \quad \forall j \in \{1, \ldots, m\}.$$

Then the integral equation

$$\left(-\frac{1}{2}I_{\partial\Omega} + W^*_{\partial\Omega}\right)[\mu] = g$$

has at least a solution $\mu \in C^{0,\alpha}(\partial\Omega)$ *and* $u = v^+_\Omega[\mu] \in C^{1,\alpha}(\overline{\Omega})$ *and solves the interior Neumann problem* **(IN)**. *All other solutions of* **(IN)** *in* $C^{1,\alpha}(\overline{\Omega})$ *can be obtained by adding to* u *an arbitrary function constant on each* Ω_j *for* $j = 1, \ldots, m$. *The affine space of such solutions has dimension* m.

Theorem 1.4.12 (For the Exterior Neumann problem). *Let* $g \in C^{0,\alpha}(\partial\Omega)$. *The following statements hold.*

(i) Let $n \geq 3$. If the compatibility conditions

$$\int_{\partial\Omega_j^-} g \, d\sigma = 0 \quad \forall j \in \{0, \ldots, m^-\}$$

are satisfied, then the integral equation

$$\left(\frac{1}{2} I_{\partial\Omega} + W_{\partial\Omega}^*\right)[\mu] = g$$

has at least a solution $\mu \in C^{0,\alpha}(\partial\Omega)$ and $u = v_\Omega^-[\mu] \in C_b^{1,\alpha}(\overline{\Omega^-})$ and solves the exterior Neumann problem **(EN)** *and is harmonic at infinity. All other solutions of* **(EN)** *in $C_b^{1,\alpha}(\overline{\Omega^-})$ and harmonic at infinity can be obtained by adding to u an arbitrary function constant on each Ω_j^- for $j = 1, \ldots, m^-$ and which equals 0 on. Ω_0^-. The affine space of such solutions has dimension m^-.*

(ii) Let $n = 2$. If the compatibility conditions

$$\int_{\partial\Omega_j^-} g \, d\sigma = 0 \quad \forall j \in \{0, \ldots, m^-\}$$

are satisfied, then the integral equation

$$\left(\frac{1}{2} I_{\partial\Omega} + W_{\partial\Omega}^*\right)[\mu] = g$$

has at least a solution $\mu \in C^{0,\alpha}(\partial\Omega)_0$ and $u = v_\Omega^-[\mu] \in C_b^{1,\alpha}(\overline{\Omega^-})$ and solves the exterior Neumann problem **(EN)** *and is harmonic at infinity. All other solutions of* **(EN)** *in $C_b^{1,\alpha}(\overline{\Omega^-})$ and harmonic at infinity can be obtained by adding to u an arbitrary function constant on each Ω_j^- for $j = 0, \ldots, m^-$. The affine space of such solutions has dimension $m^- + 1$.*

CHAPTER 2

Existence results for the nonlinear transmission problem (1)

This chapter is mainly devoted to prove the existence of a specific family of solutions of a boundary value problem for the Laplace equation with nonlinear non-autonomous transmission conditions on the boundary of a small inclusion of size ϵ. Moreover we analyse the dependence of that specific family of solutions upon the parameter ϵ. The results presented in this chapter are mainly based on a published article by the author [64].

For the sake of exposition, we recall the geometric framework of our problem already briefly described in the Introduction.

We fix once for all a natural number

$$n \geq 3$$

that will be the dimension of the space $\mathbb{R}^n$ we are going to work in and a parameter

$$\alpha \in]0,1[$$

which we use to define the regularity of our sets and functions. We remark that the case of dimension $n = 2$ requires specific techniques and it is not treated in this dissertation (the analysis for $n = 3$ and for $n \geq 3$ is instead very similar).

Then, we introduce two sets Ω^o and Ω^i that satisfy the following conditions:

Ω^o, Ω^i are bounded open connected subsets of $\mathbb{R}^n$ of class $C^{1,\alpha}$,

their exteriors $\mathbb{R}^n \setminus \overline{\Omega^o}$ and $\mathbb{R}^n \setminus \overline{\Omega^i}$ are connected,

and the origin 0 of $\mathbb{R}^n$ belongs both to Ω^o and to Ω^i.

Here the superscript "o" stands for "outer domain" whereas the superscript "i" stands for "inner domain". We take

$$\epsilon_0 \equiv \sup\{\theta \in]0,+\infty[: \epsilon\overline{\Omega^i} \subseteq \Omega^o, \ \forall \epsilon \in]-\theta,\theta[\},$$

and we define the perforated domain $\Omega(\epsilon)$ by setting

$$\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$$

for all $\epsilon \in]-\epsilon_0,\epsilon_0[$. Then we fix three functions

$$F, G \colon]-\epsilon_0,\epsilon_0[\times \partial\Omega^i \times \mathbb{R} \to \mathbb{R}\,, \text{ and } f^o \in C^{1,\alpha}(\partial\Omega^o) \tag{2.1}$$

and, for $\epsilon \in]0,\epsilon_0[$, we consider the following nonlinear nonautonomous transmission problem in the perforated domain $\Omega(\epsilon)$ for a pair of functions

$(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\overline{\epsilon\Omega^i})$:

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega(\epsilon), \\ \Delta u^i = 0 & \text{in } \epsilon\Omega^i, \\ u^o(x) = f^o(x) & \forall x \in \partial\Omega^o, \\ u^o(x) = F\left(\epsilon, \frac{x}{\epsilon}, u^i(x)\right) & \forall x \in \epsilon\partial\Omega^i, \\ \nu_{\epsilon\Omega^i} \cdot \nabla u^o(x) - \nu_{\epsilon\Omega^i} \cdot \nabla u^i(x) = G\left(\epsilon, \frac{x}{\epsilon}, u^i(x)\right) & \forall x \in \epsilon\partial\Omega^i. \end{cases} \tag{2.2}$$

Here $\nu_{\epsilon\Omega^i}$ denotes the outer exterior normal to $\epsilon\Omega^i$. Since problem (2.2) is nonlinear and degenerate for $\epsilon = 0$, one cannot, *a priori*, claim that it has a solution.

We briefly summarize our strategy and the contents of this chapter. We first introduce a suitable representation of harmonic functions in $\Omega(\epsilon)$ and $\epsilon\Omega^i$ (i.e. functions which satisfy the first and the second equation of problem (2.2)) in terms of layer potentials with unknown densities (cf. Proposition 2.1.3). Then, by an appropriate change of variables and by exploiting the Taylor expansion of certain terms, we convert problem (2.2) into a system of nonlinear integral equations on the boundaries of Ω^o and Ω^i (cf. Proposition 2.3.1). The new system is constructed in such a way that we can use the Implicit Function Theorem to analyse its solutions around the degenerate case when $\epsilon = 0$. In such a way, we find the unknown densities as implicit functions and we deduce that they depend real analytically on ϵ (cf. Theorem 2.6.5). Finally, exploiting again the suitable integral representation of the harmonic functions in $\Omega(\epsilon)$ and $\epsilon\Omega^i$, we prove the existence of u^o_ϵ and u^i_ϵ and analyse their dependence on ϵ (cf. Theorems 2.7.2 and 2.7.3).

2.1 Representation Results

Let Ω^h be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ with $\mathbb{R}^n \setminus \overline{\Omega^h}$ connected, $0 \in \Omega^h$ and $\overline{\Omega^h} \subseteq \Omega^o$. Here the superscript "h" stands for "hole". In the sequel we will exploit the inequality

$$\int_{\partial\Omega^h} S_n \, d\sigma < 0 \tag{2.3}$$

which follows by the fact that $S_n(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Let us define

$$\Omega \equiv \Omega^o \setminus \overline{\Omega^h}.$$

Then we have the following representation result for harmonic function on $\overline{\Omega}$.

Lemma 2.1.1. *Let $\rho \in \mathbb{R} \setminus \{0\}$. The map from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)_0 \times \mathbb{R}$ to $C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ which takes (μ^o, μ^h, ξ) to the function*

$$u[\mu^o, \mu^h, \xi] \equiv (w^+_{\Omega^o}[\mu^o] + w^-_{\Omega^h}[\mu^h] + \rho\, \xi\, S_n)_{|\overline{\Omega}}$$

is an isomorphism.

Proof. The map from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)$ to $C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ which takes a pair (ϕ^o, ϕ^i) to the unique solution $u[\phi^o, \phi^h]$ of the Dirichlet problem with boundary data ϕ^o and ϕ^h on $\partial\Omega^o$ and $\partial\Omega^h$, respectively, is well known to be an isomorphism (cf. Theorems 1.4.1 and 1.4.9). Then we consider the operator $L = (L_1, L_2)$ from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)_0 \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)$

which takes (μ^o, μ^h, ξ) to

$$L_1[\mu^o, \mu^h, \xi] \equiv \left(\frac{1}{2}I + W_{\partial\Omega^o}\right)[\mu^o] + w^-_{\Omega^h}[\mu^h]_{|\partial\Omega^o} + \rho\, \xi\, S_{n|\partial\Omega^o},$$
$$L_2[\mu^o, \mu^h, \xi] \equiv \left(-\frac{1}{2}I + W_{\partial\Omega^h}\right)[\mu^h] + w^+_{\Omega^o}[\mu^o]_{|\partial\Omega^h} + \rho\, \xi\, S_{n|\partial\Omega^h}\,.$$

We observe that we can rewrite L as $L = \hat{L} + \tilde{L}$ where $\hat{L}$ and $\tilde{L}$ are the operators from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)_0 \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)$ defined by

$$\hat{L}[\mu^o, \mu^h, \xi] \equiv \left(\frac{1}{2}\mu^o, -\frac{1}{2}\mu^h + \rho\, \xi\, S_{n|\partial\Omega^h}\right),$$
$$\tilde{L}[\mu^o, \mu^h, \xi] \equiv \left(W_{\partial\Omega^o}[\mu^o] + w^-_{\Omega^h}[\mu^h]_{|\partial\Omega^o} + \rho\, \xi\, S_{n|\partial\Omega^o}, W_{\partial\Omega^h}[\mu^h] + w^+_{\Omega^o}[\mu^o]_{|\partial\Omega^h}\right).$$

We observe that, for all $(\phi^o, \phi^h) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)$, we have $\hat{L}[\mu^o, \mu^h, \xi] = (\phi^o, \phi^h)$ if and only if

$$\mu^o = 2\phi^o\,, \quad \xi = \frac{\int_{\partial\Omega^h} \phi^h\, d\sigma}{\rho \int_{\partial\Omega^h} S_n\, d\sigma}\,, \quad \mu^h = -2f^h + 2\frac{S_{n|\partial\Omega^h}}{\int_{\partial\Omega^h} S_n\, d\sigma}\int_{\partial\Omega^h} \phi^h\, d\sigma$$

(cf. (2.3)). Hence, one can exhibit a bounded inverse $\hat{L}^{(-1)}$ of $\hat{L}$ from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^h)_0 \times \mathbb{R}$ and as a consequence one deduces that $\hat{L}$ is an isomorphism. Next we observe that $\tilde{L}$ is compact. In fact, by Theorem 1.3.5 (iii), the map which takes μ^o to $W_{\partial\Omega^o}[\mu^o]$ is compact from $C^{1,\alpha}(\partial\Omega^o)$ to itself and the map which takes μ^h to $W_{\partial\Omega^h}[\mu^h]$ is compact from $C^{1,\alpha}(\partial\Omega^h)_0$ to $C^{1,\alpha}(\partial\Omega^h)$. Moreover the map which takes μ^h to $w^-_{\Omega^h}[\mu^h]_{|\partial\Omega^o}$ is compact from $C^{1,\alpha}(\partial\Omega^h)_0$ to $C^{1,\alpha}(\partial\Omega^o)$ and the map which takes μ^o to $w^+_{\Omega^o}[\mu^o]_{|\partial\Omega^h}$ is compact from $C^{1,\alpha}(\partial\Omega^o)$ to $C^{1,\alpha}(\partial\Omega^h)$, because $\Omega^h \subset \Omega^0$ and the integrals involved display no singularities (cf. Theorem A.2.1 (ii) in the Appendix with $G(\psi(x), \phi(y), z) = \nu_{\Omega^o}(y) \cdot \nabla S_n(x - y)$ and $f(y) = \mu^o(y)$ and the compactness of the embedding $C^{m,\alpha}(\partial\Omega^h) \subseteq C^{m,\beta}(\partial\Omega^h)$

for $m \in \mathbb{N}$ and $0 \leq \beta < \alpha \leq 1$). Finally the map which takes ξ to $\rho S_{n|\partial\Omega^o}\xi$ is compact from $\mathbb{R}$ into $C^{1,\alpha}(\partial\Omega^o)$, because it has a finite dimensional range. So $L = \hat{L} + \tilde{L}$ is a compact perturbation of an isomorphism and henceforth a Fredholm operator of index 0. Accordingly, in order to prove that L is an isomorphism, it suffices to show that it is injective. Thus we assume that $L[\mu^o, \mu^h, \xi] = 0$ and we prove that $(\mu^o, \mu^h, \xi) = (0, 0, 0)$. If $L[\mu^o, \mu^h, \xi] = 0$, then by the jump relations of Theorem 1.3.4 (i) and by the uniqueness of the solution of the Dirichlet problem in Ω (cf. Theorem 1.4.1 (i)) we have

$$(w^+_{\Omega^o}[\mu^o] + w^-_{\Omega^h}[\mu^h] + \rho\,\xi\, S_n)_{|\overline{\Omega}} = 0. \tag{2.4}$$

Hence

$$\int_{\partial\Omega^h} \nu_{\Omega^h} \cdot \nabla(w^+_{\Omega^o}[\mu^o] + w^-_{\Omega^h}[\mu^h] + \rho\,\xi\, S_n)\, d\sigma = 0. \tag{2.5}$$

By a standard argument based on the Divergence Theorem, one shows that

$$\int_{\partial\Omega^h} \nu_{\Omega^h} \cdot \nabla w^+_{\Omega^o}[\mu^o]\, d\sigma = 0 \tag{2.6}$$

and by the jump relation of Theorem 1.3.4 (ii) we get

$$\int_{\partial\Omega^h} \nu_{\Omega^h} \cdot \nabla w^-_{\Omega^h}[\mu^h]\, d\sigma = \int_{\partial\Omega^h} \nu_{\Omega^h} \cdot \nabla w^+_{\Omega^h}[\mu^h]\, d\sigma = 0. \tag{2.7}$$

Finally, by the definition of the double layer potential and by Corollary 1.2.4, we have

$$\int_{\partial\Omega^h} \nu_{\Omega^h} \cdot \nabla(\rho S_n \xi)\, d\sigma = \rho\,\xi \int_{\partial\Omega^h} \nu_{\Omega^h} \cdot \nabla S_n\, d\sigma = \rho\,\xi\, w^+_{\Omega^h}[1](0) = \rho\,\xi. \tag{2.8}$$

Hence by (2.5)-(2.8), we deduce that $\xi = 0$. Then by (2.4) we have

$$(w^+_{\Omega^o}[\mu^o] + w^-_{\Omega^h}[\mu^h])_{|\overline{\Omega}} = 0. \tag{2.9}$$

Now we consider the function $\mu \in C^{1,\alpha}(\partial\Omega)$ defined by

$$\mu(x) \equiv \begin{cases} \mu^o(x) & \text{if } x \in \partial\Omega^o, \\ -\mu^h(x) & \text{if } x \in \partial\Omega^h. \end{cases}$$

Equality (2.9), the jump relations of Theorem 1.3.4 (i), and the fact that

$$\nu_\Omega(x) = -\nu_{\Omega^h}(x) \qquad \forall x \in \partial\Omega^h,$$

imply that $\left(\frac{1}{2}I + W_{\partial\Omega}\right)[\mu] = 0$. Then, by Theorem 1.4.7 (iv), we obtain that $\mu^o = 0$ on $\partial\Omega^o$ and μ^h is constant on $\partial\Omega^h$. Since $\mu^h \in C^{1,\alpha}(\partial\Omega^h)_0$, it follows that $\mu^h = 0$ and we conclude that $(\mu^o, \mu^h, \xi) = (0, 0, 0)$. Hence L is injective and our proof is complete. □

Lemma 2.1.2. *The map from $C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (μ, ξ) to the function*

$$J[\mu, \xi] \equiv \left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\mu] + \xi\, S_{n|\partial\Omega^i}$$

is an isomorphism.

Proof. We write

$$J[\mu, \xi] = \left(-\frac{1}{2}\mu + \xi\, S_{n|\partial\Omega^i}\right) + W_{\partial\Omega^i}[\mu].$$

Then we observe that the map which takes $(\mu, \xi) \in C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ to

$-\frac{1}{2}\mu + \xi\, S_{n|\partial\Omega^i} \in C^{1,\alpha}(\partial\Omega^i)$ is an isomorphism with inverse given by

$$h \mapsto \left(-2\left(h - \frac{\int_{\partial\Omega^i} h\, d\sigma}{\int_{\partial\Omega^i} S_n\, d\sigma} S_{n|\partial\Omega^i} \right) , \frac{\int_{\partial\Omega^i} h\, d\sigma}{\int_{\partial\Omega^i} S_n\, d\sigma} \right)$$

(cf. (2.3)). Moreover, $W_{\partial\Omega^i}$ is compact from $C^{1,\alpha}(\partial\Omega^i)_0$ to $C^{1,\alpha}(\partial\Omega^i)$ by Theorem 1.3.5 (iii). Hence, J is a compact perturbation of an isomorphism and therefore a Fredholm operator of index 0. Accordingly, to prove that it is an isomorphism it suffices to show that it is injective. Let $(\mu,\xi) \in C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ be such that

$$J[\mu,\xi] = \left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\mu] + \xi\, S_{n|\partial\Omega^i} = 0. \tag{2.10}$$

Then by the jump relation of Theorem 1.3.4 (i) we have that $w^-_{\Omega^i}[\mu]_{|\partial\Omega^i} = -\xi\, S_{n|\partial\Omega^i}$ and, by the uniqueness of the solution of the exterior Dirichlet problem in Ω^{i-} (cf. Theorem 1.4.2 (i) and note that S_n and $w^-_{\Omega^i}[\mu]$ are both harmonic at infinity), we deduce that

$$w^-_{\Omega^i}[\mu](x) = -\xi\, S_n(x) \quad \forall x \in \overline{\Omega^{i-}}. \tag{2.11}$$

Then we observe that

$$\lim_{|x|\to+\infty} (n-2)s_n|x|^{n-2} w^-_{\Omega^i}[\mu](x) = 0, \qquad \lim_{|x|\to+\infty} (n-2)s_n|x|^{n-2}(-\xi\, S_n(x)) = -\xi$$

(see decay inequalities for ∇S_n after Theorem 1.1.3 and recall that here $n \geq 3$). Hence $\xi = 0$ by (2.11). Then, $\left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\mu] = 0$ by (2.10). Finally, by Theorem 1.4.7 (i), and by the membership of μ in $C^{1,\alpha}(\partial\Omega^i)_0$, we also have $\mu = 0$. Hence $(\mu,\xi) = (0,0)$ and the proof is completed. □

In the sequel we denote by u_0^o the unique solution in $C^{1,\alpha}(\overline{\Omega^o})$ of the interior Dirichlet problem in Ω^o with boundary datum f^o, namely

$$\begin{cases} \Delta u_0^o = 0 & \text{in } \Omega^o\,, \\ u_0^o = f^o & \text{on } \partial\Omega^o\,. \end{cases} \tag{2.12}$$

We mention that this is *a posteriori* notation, in accordance with the results obtained in Theorem 2.7.3 (ii)-(iii).

We indicate by $\partial_\epsilon F$ and $\partial_\zeta F$ the partial derivative of F with respect to the first the last argument, respectively. We shall exploit the following assumption:

- There exists $\zeta^i \in \mathbb{R}$ such that $F(0,\cdot,\zeta^i) = u_0^o(0)$ and $(\partial_\zeta F)(0,\cdot,\zeta^i)$ is constant and positive. (2.13)

Then we have the following Proposition 2.1.3, where we represent harmonic functions in $\overline{\Omega(\epsilon)}$ and $\epsilon\overline{\Omega^i}$ in terms of u_0^o, double layer potentials with appropriate densities, and a suitable restriction of the fundamental solution S_n.

Proposition 2.1.3. *Let* $\epsilon \in]0,\epsilon_0[$. *The map* $(U_\epsilon^o[\cdot,\cdot,\cdot,\cdot], U_\epsilon^i[\cdot,\cdot,\cdot,\cdot])$ *from* $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ *to* $C^{1,\alpha}_{\text{harm}}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}_{\text{harm}}(\epsilon\overline{\Omega^i})$ *which takes* $(\phi^o,\phi^i,\zeta,\psi^i)$ *to the pair of functions*

$$(U_\epsilon^o[\phi^o,\phi^i,\zeta,\psi^i], U_\epsilon^i[\phi^o,\phi^i,\zeta,\psi^i])$$

defined by

$$
\begin{aligned}
U^o_\epsilon[\phi^o,\phi^i,\zeta,\psi^i](x) &\equiv u^o_0(x)+\epsilon w^+_{\Omega^o}[\phi^o](x)\\
&\quad+\epsilon w^-_{\epsilon\Omega^i}\left[\phi^i\left(\frac{\cdot}{\epsilon}\right)\right](x)+\epsilon^{n-1}\zeta\, S_n(x) && \forall x\in\overline{\Omega(\epsilon)},\\
U^i_\epsilon[\phi^o,\phi^i,\zeta,\psi^i](x) &\equiv \epsilon w^+_{\epsilon\Omega^i}\left[\psi^i\left(\frac{\cdot}{\epsilon}\right)\right](x)+\zeta^i && \forall x\in\epsilon\overline{\Omega^i},
\end{aligned}
\tag{2.14}
$$

is bijective.

Proof. Let $\epsilon\in]0,\epsilon_0[$ and $(v^o,v^i)\in C^{1,\alpha}_{\text{harm}}(\overline{\Omega^o}\setminus\epsilon\Omega^i)\times C^{1,\alpha}_{\text{harm}}(\epsilon\overline{\Omega^i})$. We prove that there exists a unique quadruple $(\phi^o,\phi^i,\zeta,\psi^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ such that

$$
(U^o_\epsilon[\phi^o,\phi^i,\zeta,\psi^i],U^i_\epsilon[\phi^o,\phi^i,\zeta,\psi^i])=(v^o,v^i)\,. \tag{2.15}
$$

Indeed, (2.15) is equivalent to

$$
w^+_{\Omega^o}[\phi^o](x)+w^-_{\epsilon\Omega^i}\left[\phi^i\left(\frac{\cdot}{\epsilon}\right)\right](x)+\epsilon^{n-2}\zeta\, S_n(x)=\frac{1}{\epsilon}(v^o(x)-u^o_0(x))\quad\forall x\in\overline{\Omega(\epsilon)}, \tag{2.16}
$$

$$
w^+_{\epsilon\Omega^i}\left[\psi^i\left(\frac{\cdot}{\epsilon}\right)\right](x)=\frac{1}{\epsilon}(v^i(x)-\zeta^i)\qquad\forall x\in\epsilon\overline{\Omega^i}. \tag{2.17}
$$

Since $\frac{1}{\epsilon}(v^o-u^o_0)\in C^{1,\alpha}_{\text{harm}}(\overline{\Omega^o}\setminus\epsilon\Omega^i)$, the existence and uniqueness of $(\phi^o,\phi^i,\xi)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ which satisfies (2.16) follow from Lemma 2.1.1 (with $\rho=\epsilon^{n-2}$). By Theorem 1.3.4 (i) and by the uniqueness of the solution of the interior Dirichlet problem (cf. Theorem 1.4.1 (i)), equation (2.17) is equivalent to

$$
\left(\frac{1}{2}I+W_{\epsilon\partial\Omega^i}\right)\left[\psi^i\left(\frac{\cdot}{\epsilon}\right)\right]=\frac{1}{\epsilon}(v^i-\zeta^i)_{|\epsilon\partial\Omega^i}.
$$

By Theorems 1.4.4 (i) and 1.4.7 (iv) one verifies that $\frac{1}{2}I + W_{\epsilon\partial\Omega^i}$ is an isomorphism from $C^{1,\alpha}(\epsilon\partial\Omega^i)$ to itself. Hence there exists a unique $\psi^i \in C^{1,\alpha}(\partial\Omega^i)$ solution of (2.17). □

2.2 Taylor expansion lemmas for F and u_0^o

In this section we will present two results on the Taylor expansions of the functions F and u_0^o, which will play a key role on the conversion of problem (2.2) into a system of integral equations.

First we consider the Taylor expansion of the function F with remainder in integral form. In addition, we assume that:

- For all $t \in \partial\Omega^i$ fixed, the map from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ to $\mathbb{R}$ which takes (ϵ, ζ) to $F(\epsilon, t, \zeta)$ is of class C^2. (2.18)

Then we have the following lemma.

Lemma 2.2.1. *Let* (2.18) *hold true. Let $a, b \in \mathbb{R}$. Then*

$$F(\epsilon, t, a + \epsilon b) = F(0, t, a) + \epsilon(\partial_\epsilon F)(0, t, a) + \epsilon b(\partial_\zeta F)(0, t, a) + \epsilon^2 \tilde{F}(\epsilon, t, a, b),$$

for all $(\epsilon, t) \in]-\epsilon_0, \epsilon_0[\times\partial\Omega^i$, where

$$\begin{aligned}\tilde{F}(\epsilon, t, a, b) \equiv \int_0^1 (1-\tau)\{(\partial_\epsilon^2 F)(\tau\epsilon, t, a + \tau\epsilon b) + 2b(\partial_\epsilon\partial_\zeta F)(\tau\epsilon, t, a + \tau\epsilon b) \\ + b^2(\partial_\zeta^2 F)(\tau\epsilon, t, a + \tau\epsilon b)\}\, d\tau.\end{aligned} \tag{2.19}$$

Proof. It suffices to consider the following identities:

$$d_\epsilon(F(\epsilon, t, a + \epsilon b)) = \partial_\epsilon F(\epsilon, t, a + \epsilon b) + b\partial_\zeta F(\epsilon, t, a + \epsilon b)$$

and

$$d_\epsilon^2(F(\epsilon,t,a+\epsilon b)) = (\partial_\epsilon^2 F)(\epsilon,t,a+\epsilon b) + 2b(\partial_\epsilon\partial_\zeta F)(\epsilon,t,a+\epsilon b) \\ + b^2(\partial_\zeta^2 F)(\epsilon,t,a+\epsilon b), \tag{2.20}$$

and to take the Taylor expansion of $F(\epsilon,t,a+\epsilon b)$ with respect to ϵ and with remainder in integral form. □

Then, under assumption (2.13), we are able to prove the following technical Lemma 2.2.2 for the Taylor expansion of the function u_0^o.

Lemma 2.2.2. *Let* (2.13) *hold true. Then*

$$u_0^o(\epsilon t) - F(0,t,\zeta^i) = \epsilon\, t \cdot \nabla u_0^o(0) + \epsilon^2\, \tilde{u}^o(\epsilon,t) \qquad \forall \epsilon \in]-\epsilon_0,\epsilon_0[\,,\ t \in \partial\Omega^i \tag{2.21}$$

with

$$\tilde{u}^o(\epsilon,t) \equiv \int_0^1 (1-\tau) \sum_{i,j=1}^n t_i\, t_j\, (\partial_{x_i}\partial_{x_j} u_0^o)(\tau\epsilon t)\, d\tau\,. \tag{2.22}$$

Moreover, the map from $]-\epsilon_0,\epsilon_0[$ *to* $C^{1,\alpha}(\partial\Omega^i)$ *which takes* ϵ *to* $\tilde{u}^o(\epsilon,\cdot)$ *is real analytic.*

Proof. To prove (2.21) and (2.22) it suffices to take the Taylor expansion of $u_0^o(\epsilon t)$ with respect to ϵ and with remainder in integral form (see also (2.13)). Then we observe that the map from $]-\epsilon_0,\epsilon_0[$ to $(C^{1,\alpha}([0,1]\times\overline{\Omega^i}))^n$ which takes ϵ to the function $\epsilon\tau t$ of the variable (τ,t) is real analytic. Moreover we have $\epsilon\tau t \in \Omega^o$ for all $\epsilon \in]-\epsilon_0,\epsilon_0[$ and all $(\tau,t)\in[0,1]\times\overline{\Omega^i}$. Then, by the real analyticity of $\partial_{x_i}\partial_{x_j}u_0^o$ in Ω^o and by known results on composition operators (cf. Valent [76, Thm. 5.2, p. 44]), one verifies that the map from

$$\{h \in (C^{1,\alpha}([0,1]\times\overline{\Omega^i}))^n : h([0,1]\times\overline{\Omega^i}) \subset \Omega^o\}$$

to $C^{1,\alpha}([0,1]\times\overline{\Omega^i})$ which takes a function $h(\cdot,\cdot)$ to $\partial_{x_i}\partial_{x_j}u_0^o(h(\cdot,\cdot))$ is real analytic. Since the sum and the pointwise product in $C^{1,\alpha}([0,1]\times\overline{\Omega^i})$ are bilinear and continuous, the map from $]-\epsilon_0,\epsilon_0[$ to $C^{1,\alpha}([0,1]\times\overline{\Omega^i})$ which takes ϵ to the function

$$(1-\tau)\sum_{i,j=1}^{n} t_i\, t_j\, (\partial_{x_i}\partial_{x_j}u_0^o)(\tau\epsilon t)$$

of the variable (τ,t) is real analytic. Then, since the map from $C^{1,\alpha}([0,1]\times\overline{\Omega^i})$ to $C^{1,\alpha}(\overline{\Omega^i})$ which takes a function $g(\cdot,\cdot)$ to $\int_0^1 g(\tau,\cdot)d\,\tau$ is linear and continuous and since the restriction operator is linear and continuous from $C^{1,\alpha}(\overline{\Omega^i})$ to $C^{1,\alpha}(\partial\Omega^i)$, we conclude that the map from $]-\epsilon_0,\epsilon_0[$ to $C^{1,\alpha}(\partial\Omega^i)$ that takes ϵ to $\tilde{u}^o(\epsilon,\cdot)$ is real analytic. □

2.3 Conversion of problem (1) into system of integral equations

We are now ready to provide a formulation of problem (2.2) in terms of integral equations. As before, let u_0^o be defined by (2.12).

Proposition 2.3.1. *Let assumptions* (2.13) *and* (2.18) *hold true. Let* $\epsilon\in\,]0,\epsilon_0[$ *and* $(\phi^o,\phi^i,\zeta,\psi^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$. *Then the pair of functions*

$$(U_\epsilon^o[\phi^o,\phi^i,\zeta,\psi^i],U_\epsilon^i[\phi^o,\phi^i,\zeta,\psi^i])$$

defined by (2.14) *is a solution of* (2.2) *if and only if*

$$\left(\frac{1}{2}I+W_{\partial\Omega^o}\right)[\phi^o](x)-\epsilon^{n-1}\int_{\partial\Omega^i}\nu_{\Omega^i}(y)\cdot\nabla S_n(x-\epsilon y)\phi^i(y)\,d\sigma_y$$
$$+\epsilon^{n-2}\zeta\, S_n(x)=0 \qquad \forall x\in\partial\Omega^o\,, \tag{2.23}$$

$$
\begin{aligned}
&t \cdot \nabla u_0^o(0) + \epsilon \tilde{u}^o(\epsilon, t) + \left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\phi^i](t) + \zeta\, S_n(t) + w^+_{\Omega^o}[\phi^o](\epsilon t) \\
&\qquad = (\partial_\epsilon F)(0, t, \zeta^i) + (\partial_\zeta F)(0, t, \zeta^i)\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) \\
&\qquad + \epsilon \tilde{F}\left(\epsilon, t, \zeta^i, \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t)\right) \qquad \forall t \in \partial\Omega^i, \qquad (2.24) \\
&\nu_{\Omega^i}(t) \cdot \left(\nabla u_0^o(\epsilon t) + \epsilon \nabla w^+_{\Omega^o}[\phi^o](\epsilon t) + \nabla w^-_{\Omega^i}[\phi^i](t) + \zeta\, \nabla S_n(t) - \nabla w^+_{\Omega^i}[\psi^i](t)\right) \\
&\qquad = G\left(\epsilon, t, \epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) + \zeta^i\right) \qquad \forall t \in \partial\Omega^i\,. \qquad (2.25)
\end{aligned}
$$

Proof. The assertion can be deduced by definition (2.14), by the jump relations of Theorem 1.3.4 (i), by changing the variable x with ϵt in the integral equations on $\epsilon\partial\Omega^i$ and in the integrals over $\epsilon\partial\Omega^i$, by Lemma 2.2.1 with $a = \zeta^i$ and $b = \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](x)$, and by Lemma 2.2.2. □

Incidentally we observe that, by integrating (2.25) over $\partial\Omega^i$, one shows that

$$
\zeta = \int_{\partial\Omega^i} G\left(\epsilon, t, \epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) + \zeta^i\right) d\sigma_t
$$

for all $\epsilon \in]0, \epsilon_0[$ (see also Corollary 1.2.4).

2.4 Limiting system

In this section we prove an existence and uniqueness theorem for the limiting system, i.e. for the system of integral equations obtained by letting $\epsilon \to 0^+$ in (2.23), (2.24) and (2.25). It consists of the following three equations in the unknowns $(\phi^o, \phi^i, \zeta, \psi^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$:

$$
\begin{aligned}
&\left(\frac{1}{2}I + W_{\partial\Omega^o}\right)[\phi^o](x) = 0 \qquad \forall x \in \partial\Omega^o, \\
&t \cdot \nabla u_0^o(0) + \left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\phi^i](t) + \zeta\, S_n(t) + w^+_{\Omega^o}[\phi^o](0)
\end{aligned}
$$

$$
\begin{aligned}
&= (\partial_\epsilon F)(0,t,\zeta^i) + (\partial_\zeta F)(0,t,\zeta^i)\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) \qquad \forall t \in \partial\Omega^i, \\
&\nu_{\Omega^i}(t)\cdot\Big(\nabla u_0^o(0) + \nabla w_{\Omega^i}^-[\phi^i](t) + \zeta\,\nabla S_n(t) - \nabla w_{\Omega^i}^+[\psi^i](t)\Big) \\
&= G(0,t,\zeta^i) \qquad \forall t \in \partial\Omega^i.
\end{aligned}
\tag{2.26}
$$

We begin with an existence and uniqueness result for an auxiliary exterior transmission problem.

Lemma 2.4.1. *Let $\lambda > 0$. Let $f_1 \in C^{1,\alpha}(\partial\Omega^i)$ and $f_2 \in C^{0,\alpha}(\partial\Omega^i)$. Then there exists a unique solution $(u^-, u^+) \in C^{1,\alpha}(\mathbb{R}^n \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$ of*

$$
\begin{cases}
\Delta u^- = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega^i}, \\
\Delta u^+ = 0 & \text{in } \Omega^i, \\
u^- = \lambda u^+ + f_1 & \text{on } \partial\Omega^i, \\
\nu_{\Omega^i}\cdot\nabla u^- - \nu_{\Omega^i}\cdot\nabla u^+ = f_2 & \text{on } \partial\Omega^i, \\
\lim_{x\to\infty} u^-(x) = 0.
\end{cases}
\tag{2.27}
$$

Proof. We first prove the uniqueness. By linearity it suffices to show that the only solution with $f_1 = f_2 = 0$ is $(u^-, u^+) = (0,0)$. If $f_1 = f_2 = 0$, then by the Divergence Theorem and by the harmonicity at infinity of u^-, we compute

$$
\begin{aligned}
0 \le \int_{\Omega^i} |\nabla u^+|^2\,dx &= -\int_{\Omega^i} u^+\Delta u^+\,dx + \int_{\partial\Omega^i} u^+\,\nu_{\Omega^i}\cdot\nabla u^+\,d\sigma \\
&= \int_{\partial\Omega^i} \frac{1}{\lambda}u^-\,\nu_{\Omega^i}\cdot\nabla u^-\,d\sigma = -\frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\overline{\Omega^i}} |\nabla u^-|^2\,dx \le 0.
\end{aligned}
$$

It follows that u^- and u^+ are constant functions (note that $\lambda > 0$), hence the fifth condition of (2.27) implies that $u^- = 0$ and, in turn, the third condition (with $f_1 = 0$) implies that $u^+ = 0$.

Now we show that the solution of (2.27) exists for any $f_1 \in C^{1,\alpha}(\partial\Omega^i)$

and $f_2 \in C^{0,\alpha}(\partial\Omega^i)$ fixed. To do so, we prove that there exists a (unique) pair $(\phi, \mu) \in C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)$ such that the pair of functions $(u^-, u^+) \in C^{1,\alpha}_{\text{harm}}(\mathbb{R}^n \setminus \Omega^i) \times C^{1,\alpha}_{\text{harm}}(\overline{\Omega^i})$ defined by

$$
\begin{aligned}
u^- &\equiv \lambda v^-_{\Omega^i}[\phi] && \text{on } \mathbb{R}^n \setminus \Omega^i, \\
u^+ &\equiv v^+_{\Omega^i}[\phi + \mu] && \text{on } \overline{\Omega^i},
\end{aligned}
$$

is a solution of (2.27). Indeed, since $v^+_{\Omega^i}[\phi] = v^-_{\Omega^i}[\phi]$ on $\partial\Omega^i$, the third equation of (2.27) is equivalent to

$$
V_{\partial\Omega^i}[\mu] = \frac{1}{\lambda} f_1 \qquad \text{on } \partial\Omega^i,
$$

and then the existence (and uniqueness) of $\mu \in C^{0,\alpha}(\partial\Omega^i)$ follows by Theorem 1.3.3 (v). Moreover, by the jump relations for the normal derivative of the single layer potential (see Theorem 1.3.3 (iv)), we deduce that the fourth equation of (2.27) is equivalent to

$$
\lambda \left(\frac{1}{2} I + W^*_{\partial\Omega} \right) [\phi] - \left(-\frac{1}{2} I + W^*_{\partial\Omega} \right) [\phi + \mu] = f_2 \qquad \text{on } \partial\Omega^i.
$$

By a straightforward computation we obtain

$$
\left(\frac{1}{2} I + \frac{\lambda - 1}{\lambda + 1} W^*_{\partial\Omega} \right) [\phi] = \frac{1}{\lambda + 1} \left(f_2 + \left(-\frac{1}{2} I + W^*_{\partial\Omega} \right) [\mu] \right) \qquad \text{on } \partial\Omega^i,
$$

and the existence (and uniqueness) of $\phi \in C^{0,\alpha}(\partial\Omega^i)$ comes from [25, Lemma 3.5] (note that $\left| \frac{\lambda-1}{\lambda+1} \right| < 1$). Finally, we observe that

$$
\lim_{x \to \infty} v^-_{\Omega^i}[\phi](x) = 0
$$

and thus u^- satisfies also the last equation of (2.27). □

We now get back to the analysis of (2.26) and, in the following theorem, we prove an existence and uniqueness result for the limiting system.

Theorem 2.4.2. *Let assumptions* (2.13) *and* (2.18) *hold true. Then, the quadruple* $(\phi_0^o, \phi_0^i, \zeta_0, \psi_0^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ *is a solution of* (2.26) *if and only if*

$$\phi_0^o = 0$$

and the pair of functions $(u^-, u^+) \in C^{1,\alpha}(\mathbb{R}^n \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$ *defined by*

$$\begin{aligned} u^- &\equiv w^-_{\Omega^i}[\phi_0^i] + \zeta_0 \, S_n && \text{on } \mathbb{R}^n \setminus \Omega^i, \\ u^+ &\equiv w^+_{\Omega^i}[\psi_0^i] && \text{on } \overline{\Omega^i}, \end{aligned} \tag{2.28}$$

is a solution of

$$\begin{cases} \Delta u^- = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega^i}, \\ \Delta u^+ = 0 & \text{in } \Omega^i, \\ u^-(t) = (\partial_\zeta F)(0, t, \zeta^i) u^+(t) + (\partial_\epsilon F)(0, t, \zeta^i) - t \cdot \nabla u_0^o(0) & \forall t \in \partial\Omega^i, \\ \nu_{\Omega^i}(t) \cdot \nabla u^-(t) - \nu_{\Omega^i}(t) \cdot \nabla u^+(t) = G(0, t, \zeta^i) - \nu_{\Omega^i}(t) \cdot \nabla u_0^o(0) & \forall t \in \partial\Omega^i, \\ \lim_{t \to \infty} u^-(t) = 0. \end{cases} \tag{2.29}$$

In particular, there exist a unique solution $(u^-, u^+) \in C^{1,\alpha}(\mathbb{R}^n \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$ *of* (2.29) *and a unique solution* $(\phi_0^o, \phi_0^i, \zeta_0, \psi_0^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ *of* (2.26).

Proof. By Theorem 1.4.4 (i) and by Theorem 1.4.7 (iv) the only solution of the first equation of (2.26) is $\phi_0^o = 0$. Then, by Theorem 1.3.4 (i), one verifies that the triple $(\phi_0^i, \zeta_0, \psi_0^i) \in C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ is a solution of the last two equations of (2.26) if and only if the pair (u^-, u^+) defined by (2.28) is a solution of (2.29). In addition, Lemma 2.4.1 implies that (2.29) has a unique solution $(u^-, u^+) \in C^{1,\alpha}_{\mathrm{harm}}(\mathbb{R}^n \setminus \Omega^i) \times C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega^i})$. Then the existence and uniqueness of $(\phi^i_0, \zeta_0) \in C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ follow by the uniqueness of the solution of the exterior Dirichlet problem (cf. Theorem 1.4.2 (i)), by the jump relations of Theorem 1.3.4 (i), and by Lemma 2.1.2. Finally, the existence and uniqueness of $\psi^i_0 \in C^{1,\alpha}(\partial\Omega^i)$ can be deduced by the uniqueness of the solution of the Dirichlet problem (cf. Theorem 1.4.1 (i)), by Theorem 1.3.4 (i), by Theorem 1.4.4 (i) and by Theorem 1.4.7 (iv). □

We incidentally observe that by integrating the third equation of (2.26) over $\partial\Omega^i$ we get

$$\zeta_0 = \int_{\partial\Omega^i} G(0, t, \zeta^i)\, d\sigma_t$$

(cf. Corollary 1.2.4).

2.5 Real analyticity results for integral operators

In this section we prove a real analyticity result for a specific type of integral operators. More specifically we introduce the definition of Pettis integral in the case of maps from a bounded interval of $\mathbb{R}$ to a Banach space X (see, for example, Pettis [70]) and in that framework we will prove Theorem 2.5.2 below. Then we will present an application of that result, namely Lemma 2.5.5 below.

Definition 2.5.1. *Let X be a Banach space and $a, b \in \mathbb{R}$. A function F from $]a, b[$ to X is said to be Pettis integrable over $]a, b[$ if there exists an element $x \in X$ such that*

$$L[x] = \int_a^b L[F(\tau)]\, d\tau \qquad \forall L \in X',$$

where the integral on the right hand side is the standard Lebesgue integral on $\mathbb{R}$*. Then we define*

$$\int_a^b F(\tau)\, d\tau \equiv x.$$

Our aim now is to prove the following theorem (see also Appendix A.1 for notation and definition of real analytic maps between Banach spaces).

Theorem 2.5.2. *Let* X, Y *be Banach spaces. Let* U *be an open star-shaped subset of* X *and let* A *be a real analytic map from* U *to* Y*. Let* $f \in L^1([0,1])$*. Then, for all* $w \in U$ *the integral*

$$\int_0^1 f(\tau) A(\tau w)\, d\tau \tag{2.30}$$

exists in the sense of Pettis and the map from U *to* Y *which takes* w *to* (2.30) *is real analytic.*

Proof. Since real analyticity is a local property, it suffices to prove the statement in a neighborhood of a fixed point w^* of U. In the first part of the proof we introduce a suitable neighborhood.

Step 1. We begin by observing that, being U open and star-shaped, for every $\bar{\tau} \in [0,1]$ there exist $\delta_{\bar{\tau}} \in]0, +\infty[$ and an open neighborhood $U_{\bar{\tau}}(w^*) \subset U$ of w^* such that

$$\tau w \in U \qquad \forall (\tau, w) \in]\bar{\tau} - \delta_{\bar{\tau}}, \bar{\tau} + \delta_{\bar{\tau}}[\times U_{\bar{\tau}}(w^*). \tag{2.31}$$

Then, by the compactness of $[0,1]$, there exist $\tau_1, \dots, \tau_k \in [0,1]$ such that

$$[0,1] \subset \bigcup_{j=1}^{k}]\tau_j - \delta_{\tau_j}, \tau_j + \delta_{\tau_j}[$$

with δ_{τ_j} as in (2.31) for all $j \in \{1, \dots, k\}$. If we now define

$$U(w^*) \equiv \bigcap_{j=1}^{k} U_{\tau_j}(w^*) \quad \text{and} \quad I(w^*) \equiv \bigcup_{j=1}^{k}]\tau_j - \delta_{\tau_j}, \tau_j + \delta_{\tau_j}[\,,$$

then we have that $U(w^*)$ and $I(w^*)$ are open, that

$$[0,1] \subset I(w^*)\,, \tag{2.32}$$

and that

$$\tau w \in U \qquad \forall (\tau, w) \in I(w^*) \times U(w^*)\,.$$

As a consequence, the map from $I(w^*) \times U(w^*)$ to U which takes (τ, w) to τw is well defined and, being bilinear and continuous, it is also real analytic. It follows that the map from $I(w^*) \times U(w^*)$ to Y which takes (τ, w) to $A(\tau w)$ is real analytic, being the composition of real analytic maps. By Definition A.1.1 of real analytic maps in Appendix A.1, we deduce that, for all fixed $\tau' \in I(w^*)$, there exist positive real numbers $M(\tau', w^*)$ and $\rho(\tau', w^*)$ and a family of multilinear maps $\{a_{ij}(\tau', w^*)\}_{i,j \in \mathbb{N}} \subset \mathcal{L}^{i,j}(\mathbb{R}, X; Y)$ such that

$$\|a_{ij}(\tau', w^*)\|_{\mathcal{L}^{i,j}(\mathbb{R},X;Y)} \leq M(\tau', w^*) \left(\frac{1}{\rho(\tau', w^*)} \right)^{i+j} \qquad \forall i, j \in \mathbb{N},$$

and such that

$$A(\tau w) = \sum_{i,j=0}^{\infty} a_{ij}(\tau', w^*)[(\tau - \tau')^{(i)}, (w - w^*)^{(j)}]$$

for all $(\tau, w) \in]\tau' - \rho(\tau', w^*), \tau' + \rho(\tau', w^*)[\, \times B_X(w^*, \rho(\tau', w^*))$ (see also (A.1) in Appendix A.1). Moreover, since the first i arguments of the $a_{i,j}(\tau', w^*)$'s are real, one verifies that there are multilinear maps $b_{i,j}(\tau', w^*) \in \mathcal{L}^j(X; Y)$

such that

$$a_{i,j}(\tau', w^*)[(\tau - \tau')^{(i)}, (w - w^*)^{(j)}] = (\tau - \tau')^i \, b_{i,j}(\tau', w^*)[(w - w^*)^{(j)}]$$

for all $i, j \in \mathbb{N}$. Then we have

$$\|b_{ij}(\tau', w^*)\|_{\mathcal{L}^j(X;Y)} \leq M(\tau', w^*) \left(\frac{1}{\rho(\tau', w^*)} \right)^{i+j} \qquad \forall i, j \in \mathbb{N} \tag{2.33}$$

and

$$A(\tau w) = \sum_{i,j=0}^{\infty} (\tau - \tau')^i \, b_{ij}(\tau', w^*)[(w - w^*)^{(j)}] \tag{2.34}$$

where the series converges absolutely and uniformly for (τ, w) in $]\tau' - \rho(\tau', w^*), \tau' + \rho(\tau', w^*)[\, \times B_X(w^*, \rho(\tau', w^*))$. We now observe that the set $\{B(\tau', \rho(\tau', w^*)/2) \; : \; \tau' \in I(w^*)\}$ is an open covering of $[0,1]$ (cf. (2.32)). Then, by a standard compactness argument it follows that there exist $\tau'_1, \ldots, \tau'_h \in [0,1]$ and disjoint intervals $I_1, \ldots, I_h \subset [0,1]$ such that $I_1 \cup \cdots \cup I_h = [0,1]$ and

$$I_l \subset B\left(\tau'_l, \frac{\rho(\tau'_l, w^*)}{2}\right) \qquad \forall l \in \{1, \ldots, h\} \tag{2.35}$$

(some of the I_l's might be empty). Finally, we define

$$\rho(w^*) \equiv \min_{l \in 1, \ldots, h} \rho(\tau'_l, w^*)\,. \tag{2.36}$$

In the next step of the proof we show that the statement of the theorem holds in $B\left(w^*, \frac{\rho(w^*)}{2}\right)$. To do so, we also find convenient to set

$$M(w^*) \equiv \max_{l=1, \ldots, h} M(\tau'_l, w^*). \tag{2.37}$$

Step 2. We claim that for all $w \in B\left(w^*, \frac{\rho(w^*)}{2}\right)$ the Pettis integral

$$\int_0^1 f(\tau)A(\tau w)\, d\tau$$

is given by the sum

$$\sum_{l=1}^{h} \sum_{i,j=0}^{\infty} \left(\int_{I_l} f(\tau)(\tau - \tau_l')^i \, d\tau \right) b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}] \,. \tag{2.38}$$

To prove it, we first verify that (2.38) defines an element of Y. Indeed, if $w \in B\left(w^*, \frac{\rho(w^*)}{2}\right)$, then (2.33), (2.36), and (2.37) imply that

$$\begin{aligned} \left\| b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}] \right\|_Y &\leq M(\tau_l', w^*) \left(\frac{1}{\rho(\tau_l', w^*)} \right)^{i+j} \left(\frac{\rho(w^*)}{2} \right)^j \\ &\leq \left(\frac{1}{2}\right)^j M(\tau_l', w^*) \left(\frac{1}{\rho(\tau_l', w^*)} \right)^i \left(\frac{\rho(w^*)}{\rho(\tau_l', w^*)} \right)^j \\ &\leq \left(\frac{1}{2}\right)^j M(w^*) \left(\frac{1}{\rho(\tau_l', w^*)} \right)^i \end{aligned} \tag{2.39}$$

for all $i, j \in \mathbb{N}$. Hence, by (2.35) and (2.39) we have

$$\begin{aligned} &\left\| \left(\int_{I_l} f(\tau)(\tau - \tau_l')^i \, d\tau \right) b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}] \right\|_Y \\ &\qquad \leq \|f\|_{L^1([0,1])} \left(\frac{\rho(\tau_l', w^*)}{2} \right)^i \left\| b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}] \right\|_Y \\ &\qquad \leq \left(\frac{1}{2}\right)^{i+j} \|f\|_{L^1([0,1])} M(w^*). \end{aligned}$$

The last inequality readily implies the convergence in Y of the series in (2.38).

In view of Definition 2.5.1 of Pettis integral, we now consider a functional $L \in Y'$ and we observe that for all fixed $w \in B\left(w^*, \frac{\rho(w^*)}{2}\right)$ the function which takes $\tau \in]0, 1[$ to $L[A(\tau w)]$ is continuous. Since $f \in L^1([0, 1])$, it follows that the function which takes $\tau \in]0,1[$ to

$$L[f(\tau)A(\tau w)] = f(\tau)L[A(\tau w)]$$

belongs to $L^1([0,1])$. Then, by splitting the integral on $\tau \in]0,1[$ over the partition $I_1,\ldots,I_h$, by the uniform convergence of the series in (2.34), and by (2.35) we obtain that

$$\begin{aligned}
\int_0^1 L[f(\tau)A(\tau w)]\, d\tau &= \sum_{l=1}^{h} \int_{I_l} L[f(\tau)A(\tau w)]\, d\tau \\
&= \sum_{l=1}^{h} \int_{I_l} L\left[f(\tau) \sum_{i,j=0}^{\infty} (\tau - \tau_l')^i\, b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}]\right] d\tau \\
&= \sum_{l=1}^{h} \int_{I_l} \sum_{i,j=0}^{\infty} L\left[f(\tau)(\tau - \tau_l')^i\, b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}]\right] d\tau \\
&= \sum_{l=1}^{h} \int_{I_l} \sum_{i,j=0}^{\infty} f(\tau)(\tau - \tau_l')^i L\left[b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}]\right] d\tau\,.
\end{aligned} \tag{2.40}$$

To verify that the Pettis integral of $f(\tau)A(\tau w)$ on $[0,1]$ is given by (2.38), it remains to show that we can change the order of the integration over I_l and of the summation on i,j in (2.40). By a classical corollary of the Dominated Convergence Theorem it suffices to prove that

$$\sum_{i,j=0}^{\infty} \int_{I_l} \left| f(\tau)(\tau - \tau_l')^i L\left[b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}]\right]\right| d\tau$$

is a convergent series. This latter fact can be deduced by noting that, as a consequence of (2.39), we have

$$\begin{aligned}
&\left|(\tau - \tau_l')^i L\left[b_{ij}(\tau_l', w^*)[(w - w^*)^{(j)}]\right]\right| \\
&\quad \leq \left(\frac{\rho(\tau_l', w^*)}{2}\right)^i \|L\|_{Y'} \left(\frac{1}{2}\right)^j M(w^*) \left(\frac{1}{\rho(\tau_l', w^*)}\right)^i \leq \left(\frac{1}{2}\right)^{i+j} \|L\|_{Y'}\, M(w^*)
\end{aligned}$$

for all $i, j \in \mathbb{N}$, $l \in \{1, \dots, h\}$, and $\tau \in I_l$.

Now that we know that the integral $\int_0^1 f(\tau)A(\tau w)\, d\tau$ is given by (2.38), the real analyticity of the map that takes w to $\int_0^1 f(\tau)A(\tau w)\, d\tau$ is a direct consequence of Definition A.1.1 of real analytic maps in Appendix A.1. $\square$

We now wish to apply Theorem 2.5.2 to obtain a result of real analyticity of the map from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which take a pair (ϵ, ψ^i) to the function

$$\tilde{F}\left(\epsilon, t, \zeta^i, \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t)\right) \qquad \forall t \in \partial\Omega^i$$

(cf. Lemma 2.2.1 and Proposition 2.3.1). This result will be crucial in section 2.6 in order to apply the Implicit Function Theorem for real analytic maps (see Proposition 2.6.2).

We begin by introducing some notation.

Definition 2.5.3. *If H is a measurable function from $]-\epsilon_0, \epsilon_0[\times\partial\Omega^i \times \mathbb{R}$ to $\mathbb{R}$, then we denote by $\mathcal{N}_H$ the (nonlinear non-autonomous) superposition operator which takes a pair (ϵ, v) consisting of a real number $\epsilon \in]-\epsilon_0, \epsilon_0[$ and of a measurable function v from $\partial\Omega^i$ to $\mathbb{R}$ to the function $\mathcal{N}_H(\epsilon, v)$ defined by*

$$\mathcal{N}_H(\epsilon, v)(t) \equiv H(\epsilon, t, v(t)) \qquad \forall t \in \partial\Omega^i \,.$$

Here the letter "$\mathcal{N}$" stands for "Nemytskii operator".

Remark 2.5.4. *If H is a measurable function from $]-\epsilon_0, \epsilon_0[\times\partial\Omega^i \times \mathbb{R}$ to $\mathbb{R}$ such that the superposition operator $\mathcal{N}_H$ is real analytic from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$, then for every $(\epsilon, \overline{v}) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ we have*

$$d_v\mathcal{N}_H(\epsilon, \overline{v}).\tilde{v} = \mathcal{N}_{(\partial_\zeta H)}(\epsilon, \overline{v})\tilde{v} \qquad \forall \tilde{v} \in C^{1,\alpha}(\partial\Omega^i). \tag{2.41}$$

The same result holds replacing the domain and the target space of the operator $\mathcal{N}_H$ *with* $]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega^i)$ *and* $C^{0,\alpha}(\partial\Omega^i)$ *respectively and using functions* $\overline{v}, \tilde{v} \in C^{0,\alpha}(\partial\Omega^i)$ *in* (2.41)*.*

The proof of Remark 2.5.4 is a straightforward modification of the corresponding argument of Lanza de Cristoforis [47, Prop. 6.3]. Moreover, for examples and assumptions which imply the real analyticity of a Nemytskii type-operator generated by a measurable function H, we refer to section 3.2 and references therein (in particular Valent [76, Chap. II]). See also Lanza De Cristoforis [50, Section 8] for a concrete example in dimension 2 of problem (2.2).

In the sequel we will exploit the following assumption:

- For all $(\epsilon, v) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ we have $\mathcal{N}_F(\epsilon, v) \in C^{1,\alpha}(\partial\Omega^i)$. Moreover, the superposition operator $\mathcal{N}_F$ is real analytic from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$. (2.42)

Then we have the following technical Lemma 2.5.5.

Lemma 2.5.5. *Let assumptions* (2.13), (2.18)*, and* (2.42) *hold true. Then, the map from* $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ *to* $C^{1,\alpha}(\partial\Omega^i)$ *which takes* (ϵ, ψ^i) *to the function*

$$\tilde{F}\left(\epsilon, t, \zeta^i, \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t)\right) \qquad \forall t \in \partial\Omega^i$$

is real analytic (see also (2.19)*).*

Proof. We plan to exploit Theorem 2.5.2. We begin by observing that, by the definition of $\tilde{F}$ (cf. (2.19)) and by equalities (2.20) and (2.41), we have

$$
\begin{aligned}
&\tilde{F}\left(\epsilon,t,\zeta^i,\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\right)\\
&=\int_0^1(1-\tau)\left\{(\partial_\epsilon^2 F)\left(\tau\epsilon,t,\zeta^i+\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\right)\right.\\
&\qquad+2\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\,(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,t,\zeta^i+\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\right)\\
&\qquad\left.+\left(\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]\right)^2(t)\,(\partial_\zeta^2 F)\left(\tau\epsilon,t,\zeta^i+\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\right)\right\}d\tau\\
&=\int_0^1(1-\tau)\,d_\epsilon^2\mathcal{N}_F\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right).(1,1)(t)\,d\tau\\
&\qquad+2\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\\
&\qquad\times\int_0^1(1-\tau)\,d_\epsilon d_v\mathcal{N}_F\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right).(1,1_{\partial\Omega^i})(t)\,d\tau\\
&\qquad+\left(\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]\right)^2(t)\\
&\qquad\times\int_0^1(1-\tau)\,d_v^2\mathcal{N}_F\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right).(1_{\partial\Omega^i},1_{\partial\Omega^i})(t)\,d\tau
\end{aligned}
$$

for all $(\epsilon,t,\psi^i)\in]-\epsilon_0,\epsilon_0[\times\partial\Omega^i\times C^{1,\alpha}(\partial\Omega^i)$ (cf. Remark 2.5.4). Here above $1_{\partial\Omega^i}$ denotes the constant function identically equal to 1 on $\partial\Omega^i$.

Now let A be the map from $]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes a pair $h=(h_1,h_2)$ to the function $A(h)$ defined by

$$
A(h)(t)\equiv d_\epsilon^2\mathcal{N}_F\left(h_1,h_2+\zeta^i\right).(1,1)(t)\qquad\forall t\in\partial\Omega^i\,.
$$

By assumption (2.42) one deduces that A is real analytic and thus Theorem 2.5.2 implies that the map from $]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes h to the function

$$
\int_0^1(1-\tau)A(\tau h)\,d\tau
$$

is also real analytic. Then we set

$$h[\epsilon, \psi^i] = (h_1[\epsilon, \psi^i], h_2[\epsilon, \psi^i]) \equiv \left(\epsilon, \epsilon \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i]\right)$$

for all $(\epsilon, \psi^i) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ and we observe that the map from the space $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to itself which takes (ϵ, ψ^i) to $h[\epsilon, \psi^i]$ is real analytic (because the first component is linear and continuous and the second one is bilinear and continuous). Since the composition of real analytic maps is real analytic, it follows that the map from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (ϵ, ψ^i) to the function

$$\int_0^1 (1-\tau)\, d_\epsilon^2 \mathcal{N}_F \left(\tau\epsilon, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right).(1,1)(t)\, d\tau$$

is real analytic. In a similar way, one can prove that the map from the space $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (ϵ, ψ^i) to the function

$$\int_0^1 (1-\tau)\, d_\epsilon d_v \mathcal{N}_F \left(\tau\epsilon, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right).(1, 1_{\partial\Omega^i})(t)\, d\tau$$

and the map from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (ϵ, ψ^i) to the function

$$\int_0^1 (1-\tau)\, d_v^2 \mathcal{N}_F \left(\tau\epsilon, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right).(1_{\partial\Omega^i}, 1_{\partial\Omega^i})(t)\, d\tau$$

are real analytic. The map from $C^{1,\alpha}(\partial\Omega^i)$ to itself which takes ψ^i to the function $\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i]$ is linear and continuous, hence real analytic. Since the product of real analytic maps is real analytic, the map from $C^{1,\alpha}(\partial\Omega^i)$ to itself which takes ψ^i to the function $\left(\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i]\right)^2$ is real analytic. Finally, since the sum of real analytic maps is real analytic, we conclude that the map from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (ϵ, ψ^i) to the function

$$\tilde{F}\left(\epsilon, t, \zeta^i, \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t)\right) \qquad \forall t \in \partial\Omega^i$$

is real analytic. The lemma is now proved. □

2.6 Application of the Implicit Function Theorem

In view of the equivalence of problem (2.2) and equations (2.23), (2.24), and (2.25), we now introduce the auxiliary map $M = (M_1, M_2, M_3)$ from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)$ defined by

$$\begin{aligned}
M_1[\epsilon, \phi^o, \phi^i, \zeta, \psi^i](x) \equiv & \left(\frac{1}{2}I + W_{\partial\Omega^o}\right)[\phi^o](x) \\
& - \epsilon^{n-1} \int_{\partial\Omega^i} \nu_{\Omega^i}(y) \cdot \nabla S_n(x - \epsilon y)\phi^i(y)\, d\sigma_y + \epsilon^{n-2}\zeta\, S_n(x) \quad \forall x \in \partial\Omega^o, \\
M_2[\epsilon, \phi^o, \phi^i, \zeta, \psi^i](t) \equiv & \ t \cdot \nabla u_0^o(0) + \epsilon \tilde{u}^o(\epsilon, t) + \left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\phi^i](t) \\
& + \zeta\, S_n(t) + w^+_{\Omega^o}[\phi^o](\epsilon t) - (\partial_\epsilon F)(0, t, \zeta^i) - (\partial_\zeta F)(0, t, \zeta^i) \\
& \times \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) - \epsilon\tilde{F}\left(\epsilon, t, \zeta^i, \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t)\right) \quad \forall t \in \partial\Omega^i, \\
M_3[\epsilon, \phi^o, \phi^i, \zeta, \psi^i](t) \equiv & \ \nu_{\Omega^i}(t) \cdot \Big(\nabla u_0^o(\epsilon t) + \epsilon \nabla w^+_{\Omega^o}[\phi^o](\epsilon t) \\
& + \nabla w^-_{\Omega^i}[\phi^i](t) + \nabla S_n(t)\zeta - \nabla w^+_{\Omega^i}[\psi^i](t)\Big) \\
& - G\left(\epsilon, t, \epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) + \zeta^i\right) \quad \forall t \in \partial\Omega^i,
\end{aligned}$$

for all $(\epsilon, \phi^o, \phi^i, \zeta, \psi^i) \in]-\epsilon_0, \epsilon_0[\times\mathbb{R} \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$. Then one readily verifies the following.

Proposition 2.6.1. *Let assumptions* (2.13) *and* (2.18) *hold true. Let* $\epsilon \in]0, \epsilon_0[$. *Then the system consisting of equations* (2.23), (2.24), *and* (2.25) *is* *equivalent to*

$$M[\epsilon, \phi^o, \phi^i, \zeta, \psi^i] = (0,0,0)\,. \tag{2.43}$$

We now wish to apply the Implicit Function Theorem for real analytic functions (see, for example, Deimling [31, Thm. 15.3]) to equation (2.43) around the degenerate value $\epsilon = 0$. As a first step we have to analyse the regularity of the map M.

To prove that M is real analytic we will exploit the following assumption:

- For all $(\epsilon, v) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ we have $\mathcal{N}_G(\epsilon, v) \in C^{0,\alpha}(\partial\Omega^i)$. Moreover, the superposition operator $\mathcal{N}_G$ is real analytic from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$. (2.44)

We now show that M is real analytic.

Proposition 2.6.2. *Let assumptions* (2.13)*,* (2.18)*,* (2.42) *and* (2.44) *hold true. Then, the map M is real analytic from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times \mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)$.*

Proof. We first show that M_1 is real analytic from $]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o)$. To do so, we analyse M_1 term by term. The map from $C^{1,\alpha}(\partial\Omega^o)$ to $C^{1,\alpha}(\partial\Omega^o)$ which takes ϕ^o to the function $\left(\frac{1}{2}I + W_{\partial\Omega^o}\right)[\phi^o]$ is linear and continuous, so real analytic. The second term can be treated in this way: one considers the integral operator from $]-\epsilon', \epsilon'[\times L^1(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o)$ which takes the pair (ϵ, f) to $\int_{\partial\Omega^i}\nu_{\Omega^i}(y)\cdot\nabla S_n(\cdot-\epsilon y)f(y)\,d\sigma_y$. By the real analyticity of S_n on $\mathbb{R}^n\setminus\{0\}$, by the fact that the integral kernel does not display singularities (see also Theorem A.2.2 (ii) in Appendix A.2) and since $C^{1,\alpha}(\partial\Omega^i)_0$ is linearly and continuously imbedded in $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon', \epsilon'[\times C^{1,\alpha}(\partial\Omega^i)_0$ to $C^{1,\alpha}(\partial\Omega^o)$ which takes the pair (ϵ, ϕ^i) to $\int_{\partial\Omega^i}\nu_{\Omega^i}(y)\cdot\nabla S_n(\cdot-\epsilon y)\phi^i(y)\,d\sigma_y$

is real analytic. Finally, one easily verifies that the map from $]-\epsilon',\epsilon'[\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^o)$ which takes (ϵ,ζ) to $\epsilon^{n-2}S_n(\cdot)\zeta$ is real analytic.

We now analyse M_2. For the first term there is nothing to say, because it does not depend on $(\epsilon,\phi^o,\phi^i,\zeta,\psi^i)$. For the second term, we invoke Lemma 2.2.2. The map from $C^{1,\alpha}(\partial\Omega^i)_0$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes ϕ^i to $\left(-\frac{1}{2}I+W_{\partial\Omega^i}\right)[\phi^i]$ is linear and continuous, so real analytic. Since continuous linear maps are real analytic, the map from $\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes ζ to $\zeta\, S_n(\cdot)$ is real analytic. The map from $]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega^o)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes (ϵ,ϕ^o) to $w^+_{\Omega^o}[\phi^o](\epsilon\cdot)$ can be proven to be real analytic by the properties of integral operators with real analytic kernels (see Theorem A.2.2 (ii) in the Appendix A.2). For the sixth term there is nothing to say, because it does not depend on $(\epsilon,\phi^o,\phi^i,\zeta,\psi^i)$. For the seventh term, the map from $C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^i)$ which takes ψ^i to $(\partial_\zeta F)(0,\cdot,\zeta^i)\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]$ is linear and continuous and hence real analytic. Finally, for the eighth term, we invoke Lemma 2.5.5.

Then we pass to consider M_3. The map from $]-\epsilon_0,\epsilon_0[$ to $(C^{0,\alpha}(\overline{\Omega^i}))^n$ which takes ϵ to the function ϵt of the variable t is real analytic. Moreover we have $\epsilon t\in\Omega^o$ for all $\epsilon\in]-\epsilon_0,\epsilon_0[$ and all $t\in\overline{\Omega^i}$. Then, by the real analyticity of $\nu_{\Omega^i}\cdot\nabla u^o_0$ in Ω^o and by known results on composition operators (cf. Valent [76, Thm. 5.2, p. 44]), one verifies that the map from

$$\{h\in(C^{0,\alpha}(\overline{\Omega^i}))^n : h(\overline{\Omega^i})\subset\Omega^o\}$$

to $C^{0,\alpha}(\overline{\Omega^i})$ which takes a function h to $\nu_{\Omega^i}\cdot\nabla u^o_0(h(\cdot))$ is real analytic. Since the restriction operator is linear and continuous from $C^{0,\alpha}(\overline{\Omega^i})$ to $C^{0,\alpha}(\partial\Omega^i)$, we conclude that the map from $]-\epsilon',\epsilon'[$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes ϵ to $\nu_{\Omega^i}\cdot\nabla u^o_0(\epsilon\cdot)$ is real analytic. Since continuous linear maps are real analytic, the map from $\mathbb{R}$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes ζ to $\nu_{\Omega^i}\cdot\nabla S_n\zeta$ is real analytic. By the properties of integral operators with real analytic kernels (see Theorem A.2.2 (ii) in the Appendix A.2)) it follows that the map from $]-\epsilon', \epsilon'[\times C^{1,\alpha}(\partial\Omega^o)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϵ, ϕ^o) to $\nu_{\Omega^i} \cdot \epsilon\nabla w^+_{\Omega^o}[\phi^o](\epsilon\cdot)$ is real analytic. Since linear and continuous map are real analytic, the map from $C^{1,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes ϕ^i to $\nu_{\Omega^i} \cdot \nabla w^-_{\Omega^i}[\phi^i]$, the map from $C^{1,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes ψ^i to $\nu_{\Omega^i} \cdot \nabla w^+_{\Omega^i}[\psi^i]$, and the map from $C^{1,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes ψ^i to $\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i]$ are real analytic. Since product of real analytic functions is real analytic, the map from $]-\epsilon', \epsilon[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϵ, ψ^i) to $\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i$ is real analytic. Finally using hypothesis (2.44) and again the fact that the composition of real analytic functions is real analytic, we conclude that the map from $]-\epsilon', \epsilon[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϵ, ψ^i) to

$$G\left(\epsilon, \cdot, \epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](\cdot) + \zeta^i\right) = \mathcal{N}_G\left(\epsilon, \epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right)$$

is real analytic.

The proof of the proposition is now complete. □

In order to analyse problem (2.2) for $\epsilon > 0$ close to 0, and thus equation (2.43) for $\epsilon > 0$ close to 0, we need to consider (2.43) at the singular value $\epsilon = 0$. Then, by the definition of M, by a straightforward computation, and by Theorem 2.4.2, we deduce the following.

Proposition 2.6.3. *Let assumptions* (2.13)*,* (2.18)*,* (2.42) *and* (2.44) *hold true. Then, equation*

$$M[0, \phi^o, \phi^i, \zeta, \psi^i] = (0, 0, 0)$$

is equivalent to the limiting system (2.26) *and has one and only one solution*

$$(\phi_0^o, \phi_0^i, \zeta_0, \psi_0^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i).$$

Finally, we have the following Lemma 2.6.4 concerning the partial differential of M with respect to $(\phi^o, \phi^i, \zeta, \psi^i)$ evaluated at $(0, \phi_0^o, \phi_0^i, \zeta_0, \psi_0^i)$.

Lemma 2.6.4. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Then, the partial differential of M with respect to $(\phi^o, \phi^i, \zeta, \psi^i)$ evaluated at $(0, \phi_0^o, \phi_0^i, \zeta_0, \psi_0^i)$, which we denote by*

$$\partial_{(\phi^o,\phi^i,\zeta,\psi^i)} M[0, \phi_0^o, \phi_0^i, \zeta_0, \psi_0^i]\,, \tag{2.45}$$

is an isomorphism from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)$.

Proof. By standard calculus in Banach spaces one verifies that the partial differential (2.45) is the linear and continuous operator delivered by

$$\begin{aligned}
&\partial_{(\phi^o,\phi^i,\zeta,\psi^i)} M_1[0, \phi_0^o, \phi_0^i, \zeta_0, \psi_0^i].(\tilde{\phi}^o, \tilde{\phi}^i, \tilde{\zeta}, \tilde{\psi}^i)(x) \\
&\qquad = \left(\frac{1}{2} I + W_{\partial\Omega^o}\right)[\tilde{\phi}^o](x) && \forall x \in \partial\Omega^o, \\
&\partial_{(\phi^o,\phi^i,\zeta,\psi^i)} M_2[0, \phi_0^o, \phi_0^i, \zeta_0, \psi_0^i].(\tilde{\phi}^o, \tilde{\phi}^i, \tilde{\zeta}, \tilde{\psi}^i)(t) \\
&\qquad = \left(-\frac{1}{2} I + W_{\partial\Omega^i}\right)[\tilde{\phi}^i](t) + \tilde{\zeta}\, S_n(t) + w^+_{\Omega^o}[\tilde{\phi}^o](0) \\
&\qquad\quad - (\partial_\zeta F)(0, t, \zeta^i)\left(\frac{1}{2} I + W_{\partial\Omega^i}\right)[\tilde{\psi}^i](t) && \forall t \in \partial\Omega^i\,, \\
&\partial_{(\phi^o,\phi^i,\zeta,\psi^i)} M_3[0, \phi_0^o, \phi_0^i, \zeta_0, \psi_0^i].(\tilde{\phi}^o, \tilde{\phi}^i, \tilde{\zeta}, \tilde{\psi}^i)(t) \\
&\qquad = \nu_{\Omega^i}(t)\left(\nabla w^-_{\Omega^i}[\tilde{\phi}^i](t) + \tilde{\zeta}\, \nabla S_n(t) - \nabla w^+_{\Omega^i}[\tilde{\psi}^i](t)\right) && \forall t \in \partial\Omega^i,
\end{aligned}$$

for all $(\tilde{\phi}^o, \tilde{\phi}^i, \tilde{\zeta}, \tilde{\psi}^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$. Then, to prove that $\partial_{(\phi^o,\phi^i,\zeta,\psi^i)}M[0,\phi^o_0,\phi^i_0,\zeta_0,\psi^i_0]$ is an isomorphism of Banach spaces it will suffice to prove that it is a bijection and then apply the Open Mapping Theorem. So let $(g^i,h^i,h^o)\in C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^o)$. We have to prove that there exists a unique quadruple $(\bar{\phi^o},\bar{\phi^i},\bar{\zeta},\bar{\psi^i})\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ such that

$$\partial_{(\phi^o,\phi^i,\zeta,\psi^i)}M[0,\phi^o_0,\phi^i_0,\zeta_0,\psi^i_0].(\bar{\phi^o},\bar{\phi^i},\bar{\zeta},\bar{\psi^i})=(g^i,h^i,h^o). \tag{2.46}$$

The last two equations of (2.46) written in full are

$$\begin{aligned}
&\left(-\frac{1}{2}I+W_{\partial\Omega^i}\right)[\bar{\phi^i}](t)+\bar{\zeta}\,S_n(t)\\
&+w^+_{\Omega^o}[\bar{\phi^o}](0)-(\partial_\zeta F)(0,t,\zeta^i)\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\bar{\psi^i}](t)=g^i(t)\,,\\
&\nu_{\Omega^i}(t)\cdot\left(\nabla w^-_{\Omega^i}[\bar{\phi^i}](t)+\bar{\zeta}\,\nabla S_n(t)-\nabla w^+_{\Omega^i}[\bar{\psi^i}](t)\right)=h^i(t)\,,
\end{aligned} \tag{2.47}$$

for all $t\in\partial\Omega^i$. Then, by Theorem 1.3.4 (i), one verifies that the triple $(\bar{\phi^i},\bar{\zeta},\bar{\psi^i})\in C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ is a solution of system (2.47) if and only if the pair (u^-,u^+) defined by

$$\begin{aligned}
u^-&\equiv w^-_{\Omega^i}[\bar{\phi^i}]+\bar{\zeta}\,S_{n|\mathbb{R}^n\setminus\Omega^i} &&\text{in } \mathbb{R}^n\setminus\Omega^i,\\
u^+&\equiv w^+_{\Omega^i}[\bar{\psi^i}] &&\text{in } \overline{\Omega^i},
\end{aligned} \tag{2.48}$$

is a solution of the transmission problem

$$\begin{cases} \Delta u^- = 0 & \text{in } \mathbb{R}^n \setminus \Omega^i, \\ \Delta u^+ = 0 & \text{in } \Omega^i, \\ u^- = (\partial_\zeta F)(0,\cdot,\zeta^i)\, u^+ - w^+_{\Omega^o}[\tilde{\phi^o}](0) + g^i & \text{on } \partial\Omega^i, \\ \nu_{\Omega^i}\cdot\nabla u^- - \nu_{\Omega^i}\cdot\nabla u^+ = h^i & \text{on } \partial\Omega^i, \\ \lim_{x\to\infty} u^-(x) = 0\,. \end{cases} \tag{2.49}$$

By assumption (2.13) and by Lemma 2.4.1, the solution (u^-, u^+) of problem (2.49) exists and is unique. Then the existence and uniqueness of $(\bar{\phi}^i, \bar{\zeta}) \in C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ follow by the first equation of (2.48), by the uniqueness of the solution of the exterior Dirichlet problem (cf. Theorem 1.4.2 (i)), by the jump relations of Theorem 1.3.4 (i), and by Lemma 2.1.2. The existence and uniqueness of $\bar{\psi}^i \in C^{1,\alpha}(\partial\Omega^i)$ can be deduced by the second equation of (2.48), by the uniqueness of the solution of the internal Dirichlet problem (cf. Theorem 1.4.1 (i)), by Theorem 1.3.4 (i), by Theorem 1.4.4 (i) and by Theorem 1.4.7 (iv). Finally, to prove that $\bar{\phi}^o$ exists and is unique we observe that the first equation of (2.46) is

$$\left(\frac{1}{2}I + W_{\partial\Omega^o}\right)[\bar{\phi}^o] = h^o$$

and by Theorem 1.4.4 (i) and by Theorem 1.4.7 (iv) the operator $\frac{1}{2}I + W_{\partial\Omega^o}$ is invertible from $C^{1,\alpha}(\partial\Omega^o)$ into itself. □

We are now ready to show that there is a real analytic family of solutions of equation (2.43).

Theorem 2.6.5. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Then there exist* $\epsilon' \in]0, \epsilon_0[$, *an open neighborhood* U_0 *of* $(\phi^o_0, \phi^i_0, \zeta_0, \psi^i_0)$ *in*

$C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$, *and a real analytic map*

$$(\Phi^o[\cdot], \Phi^i[\cdot], Z[\cdot], \Psi^i[\cdot]) :]-\epsilon', \epsilon'[\to U_0$$

such that the set of zeros of M *in* $]-\epsilon', \epsilon'[\times U_0$ *coincides with the graph of the function* $(\Phi^o[\cdot], \Phi^i[\cdot], Z[\cdot], \Psi^i[\cdot])$. *In particular,*

$$(\Phi^o[0], \Phi^i[0], Z[0], \Psi^i[0]) = (\phi_0^o, \phi_0^i, \zeta_0, \psi_0^i). \tag{2.50}$$

Proof. It follows by Proposition 2.6.2, by Lemma 2.6.4, and by the Implicit Function Theorem for real analytic maps (see Theorem A.1.2 in Appendix A.1). The validity of (2.50) is a consequence of Proposition 2.6.3. □

2.7 Real analytic representation of the family of solutions of problem (1)

We are now ready to exhibit a family of solutions of problem (2.2) for ϵ sufficiently small and describe its asymptotic behaviour in terms of real analytic functions of ϵ.

Definition 2.7.1. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Let* ϵ' *and* $(\Phi^o[\cdot], \Phi^i[\cdot], Z[\cdot], \Psi^i[\cdot])$ *be as in Theorem 2.6.5. Then, for all* $\epsilon \in]0, \epsilon'[$ *we set*

$$u_\epsilon^o(x) \equiv U_\epsilon^o[\Phi^o[\epsilon], \Phi^i[\epsilon], Z[\epsilon], \Psi^i[\epsilon]](x) \quad \forall x \in \overline{\Omega(\epsilon)}\,,$$
$$u_\epsilon^i(x) \equiv U_\epsilon^i[\Phi^o[\epsilon], \Phi^i[\epsilon], Z[\epsilon], \Psi^i[\epsilon]](x) \quad \forall x \in \epsilon\overline{\Omega^i}\,,$$

with $U_\epsilon^o[\cdot,\cdot,\cdot,\cdot]$ *and* $U_\epsilon^i[\cdot,\cdot,\cdot,\cdot]$ *defined as in* (2.14).

As a consequence of Propositions 2.3.1 and 2.6.1 and of Theorem 2.6.5 we have the following.

Theorem 2.7.2. *Under assumptions* (2.13), (2.18), (2.42) *and* (2.44), *the pair of functions*

$$(u_\epsilon^o, u_\epsilon^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\overline{\epsilon\Omega^i})$$

is a solution of (2.2) *for all* $\epsilon \in]0, \epsilon'[$.

We now verify that the map which takes ϵ to (suitable restrictions of) the pair of functions $(u_\epsilon^o, u_\epsilon^i)$ admits a real analytic continuation in a neighborhood of $\epsilon = 0$.

Theorem 2.7.3. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Then the following statements hold.*

(i) There exists a real analytic map

$$U_m^i :]-\epsilon', \epsilon'[\to C^{1,\alpha}(\overline{\Omega^i})$$

such that

$$u_\epsilon^i(\epsilon t) = \zeta^i + \epsilon U_m^i[\epsilon](t) \qquad \forall t \in \overline{\Omega^i}$$

for all $\epsilon \in]0, \epsilon'[$.

(ii) Let Ω_M *be a bounded open subset of* $\Omega^o \setminus \{0\}$ *such that* $0 \notin \overline{\Omega_M}$. *Let* $\epsilon_M \in]0, \epsilon'[$ *be such that*

$$\overline{\Omega_M} \cap \epsilon\overline{\Omega^i} = \emptyset \qquad \forall \epsilon \in]-\epsilon_M, \epsilon_M[\,.$$

Then there exists a real analytic map

$$U_M^o :]-\epsilon_M, \epsilon_M[\to C^{1,\alpha}(\overline{\Omega_M})$$

such that

$$u^o_\epsilon(x) = u^o_0(x) + \epsilon U^o_M[\epsilon](x) \qquad \forall x \in \overline{\Omega_M}$$

for all $\epsilon \in]0, \epsilon_M[$.

(iii) Let Ω_m *be a bounded open subset of* $\mathbb{R}^n \setminus \overline{\Omega^i}$. *Let* $\epsilon_m \in]0, \epsilon'[$ *be such that*

$$\epsilon\overline{\Omega_m} \subseteq \Omega^o \qquad \forall \epsilon \in] - \epsilon_m, \epsilon_m[.$$

Then there exists a real analytic map

$$U^o_m :] - \epsilon_m, \epsilon_m[\to C^{1,\alpha}(\overline{\Omega_m})$$

such that

$$u^o_\epsilon(\epsilon t) = u^o_0(0) + \epsilon U^o_m[\epsilon](t) \qquad \forall t \in \overline{\Omega_m}$$

for all $\epsilon \in]0, \epsilon_m[$.

Proof. We first prove statement (i). By (2.14) and by Definition 2.7.1 we have

$$u^i_\epsilon(x) = \epsilon w^+_{\epsilon\Omega^i}\left[\Psi^i[\epsilon]\left(\frac{\cdot}{\epsilon}\right)\right](x) + \zeta^i \qquad \forall x \in \epsilon\overline{\Omega^i},$$

for all $\epsilon \in]0, \epsilon'[$. Then, by a computation based on the theorem of change of variable in integrals and on the homogeneity of ∇S_n we obtain that

$$\begin{aligned} u^i_\epsilon(\epsilon t) &= \epsilon w^+_{\epsilon\Omega^i}\left[\Psi^i[\epsilon]\left(\frac{\cdot}{\epsilon}\right)\right](\epsilon t) + \zeta^i \\ &= -\epsilon\, \epsilon^{n-1} \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(\epsilon t - \epsilon s)\Psi^i[\epsilon](s)\, d\sigma_s + \zeta^i \\ &= -\epsilon \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(t - s)\Psi^i[\epsilon](s)\, d\sigma_s + \zeta^i \\ &= \epsilon w^+_{\Omega^i}[\Psi^i[\epsilon]](t) + \zeta^i \qquad \forall t \in \Omega^i, \end{aligned}$$

for all $\epsilon \in]0, \epsilon'[$. Then it is natural to take

$$U^i_m[\epsilon] \equiv w^+_{\Omega^i}[\Psi^i[\epsilon]] \qquad \forall \epsilon \in]-\epsilon', \epsilon'[\,.$$

Since $w^+_{\Omega^i}[\cdot]$ is linear and continuous from $C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\overline{\Omega^i})$ (cf. Theorem 1.3.4 (iii)) and $\Psi^i[\cdot]$ is real analytic (cf. Theorem 2.6.5), we conclude that the map U^i_m is real analytic. The validity of (i) is proved.

We now proceed with statement (ii). By (2.14) and by Definition 2.7.1 we have

$$u^o_\epsilon(x) = u^o_0(x) + \epsilon w^+_{\Omega^o}[\Phi^o[\epsilon]](x) + \epsilon w^-_{\epsilon\Omega^i}\left[\Phi^i[\epsilon]\left(\frac{\cdot}{\epsilon}\right)\right](x) + \epsilon^{n-1} Z[\epsilon]\, S_n(x) \qquad \forall x \in \overline{\Omega(\epsilon)}$$

for all $\epsilon \in]0, \epsilon'[$. Then, by changing the variable of integration over $\epsilon\partial\Omega^i$ we obtain

$$u^o_\epsilon(x) = u^o_0(x) + \epsilon w^+_{\Omega^o}[\Phi^o[\epsilon]](x) - \epsilon \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(x - \epsilon s)\Phi^i[\epsilon](s)\, \epsilon^{n-1}\, d\sigma_s + \epsilon^{n-1} Z[\epsilon]\, S_n(x) \quad \forall x \in \overline{\Omega(\epsilon)}$$

for all $\epsilon \in]0, \epsilon'[$. Then it is natural to define

$$U^o_M[\epsilon](x) \equiv w^+_{\Omega^o}[\Phi^o[\epsilon]](x) - \epsilon^{n-1} \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(x - \epsilon s)\Phi^i[\epsilon](s)\, d\sigma_s + \epsilon^{n-2} Z[\epsilon]\, S_n(x) \quad \forall x \in \overline{\Omega_M}$$

for all $\epsilon \in]-\epsilon_M, \epsilon_M[$.

Since $\Phi^o[\cdot]$ is real analytic (cf. Theorem 2.6.5), since $w^+_{\Omega^o}[\cdot]$ is linear and continuous from $C^{1,\alpha}(\partial\Omega^o)$ to $C^{1,\alpha}(\overline{\Omega^o})$ (cf. Theorem 1.3.4 (iii)), and since the restriction operator from $C^{1,\alpha}(\overline{\Omega^o})$ to $C^{1,\alpha}(\overline{\Omega_M})$ is linear and continuous, then the map from $]-\epsilon_M,\epsilon_M[$ to $C^{1,\alpha}(\overline{\Omega_M})$ which takes ϵ to $w^+_{\Omega^o}[\Phi^o[\epsilon]]$ is real analytic. Then, one considers the operator from $]-\epsilon_M,\epsilon_M[\times L^1(\partial\Omega^i)$ to $C^{1,\alpha}(\overline{\Omega_M})$ which takes the pair (ϵ, f) to the function

$$\int_{\partial\Omega^i} \nu_{\Omega^i}(s)\cdot\nabla S_n(\cdot-\epsilon s)f(s)\,d\sigma_s.$$

By the real analyticity of S_n on $\mathbb{R}^n\setminus\{0\}$, by the fact that the integral does not display singularities (by hypothesis $\overline{\Omega_M}\cap\epsilon\overline{\Omega^i}=\emptyset$ for all $\epsilon\in]-\epsilon_M,\epsilon_M[$), by the real analyticity of the map from $]-\epsilon_M,\epsilon_M[$ to $C^{1,\alpha}(\partial\Omega^i)_0$ which takes ϵ to $\Phi^i[\epsilon]$ (cf. Theorem 2.6.5) and since $C^{1,\alpha}(\partial\Omega^i)_0$ is linearly and continuously imbedded in $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon_M,\epsilon_M[$ to $C^{1,\alpha}(\overline{\Omega_M})$ which takes ϵ to $\epsilon^{n-1}\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla S_n(\cdot-\epsilon s)\Phi^i[\epsilon](s)\,d\sigma_s$ is real analytic (cf. Theorem A.2.2 (ii) in Appendix A.2). Finally, by the real analyticity of $Z[\cdot]$ (cf. Theorem 2.6.5), one verifies that the map from $]-\epsilon_M,\epsilon_M[$ to $C^{1,\alpha}(\overline{\Omega_M})$ which takes ϵ to $\epsilon^{n-2}Z[\epsilon]\,S_n$ is real analytic. Hence, one deduces the validity of (ii).

Finally we prove statement (iii). By (2.14), by Definition 2.7.1, by exploiting the homogeneity properties of S_n and ∇S_n, and by adding and subtracting the term $u^o_0(0)$, we obtain that

$$\begin{aligned} u^o_\epsilon(\epsilon t) &= u^o_0(0)+u^o_0(\epsilon t)-u^o_0(0)+\epsilon w^+_{\Omega^o}[\Phi^o[\epsilon]](\epsilon t)\\ &\quad -\epsilon\int_{\partial\Omega^i}\nu_{\Omega^i}(s)\cdot\nabla S_n(t-s)\Phi^i[\epsilon](s)\,d\sigma_s+\epsilon Z[\epsilon]\,S_n(t)\quad \forall t\in\overline{\Omega_m}\end{aligned}$$

for all $\epsilon\in]0,\epsilon'[$. Then one observes that the map from $]-\epsilon_m,\epsilon_m[$ to $(C^{1,\alpha}(\overline{\Omega_m}))^n$ which takes ϵ to the function ϵt of the variable t is real analytic. Moreover, we have $\epsilon t\in\Omega^o$ for all $\epsilon\in]-\epsilon_m,\epsilon_m[$ and all $t\in\overline{\Omega_m}$. Then, by the real analyticity of u^o_0 in Ω^o and known results on composition operators (cf. Valent

[76, Thm. 5.2, p. 44]), one verifies that the map from

$$\{h \in (C^{1,\alpha}(\overline{\Omega_m}))^n : h(\overline{\Omega_m}) \subset \Omega^o\}$$

to $C^{1,\alpha}(\overline{\Omega_m})$ which takes a function h to $u_0^o(h(\cdot))$ is real analytic. Hence, the map from $]-\epsilon_m, \epsilon_m[$ to $C^{1,\alpha}(\overline{\Omega_m})$ which takes ϵ to $u_0^o(\epsilon\cdot)-u_0^o(0)$ is real analytic and equal to 0 for $\epsilon = 0$. This implies that the map from $]-\epsilon_m, \epsilon_m[\setminus\{0\}$ to $C^{1,\alpha}(\overline{\Omega_m})$ which takes ϵ to $\frac{u_0^o(\epsilon\cdot)-u_0^o(0)}{\epsilon}$ has a real analytic continuation to $]-\epsilon_m, \epsilon_m[$. Then it is natural to define

$$\begin{aligned} U_m^o[\epsilon](t) \equiv & \frac{u_0^o(\epsilon\cdot) - u_0^o(0)}{\epsilon} + w^+_{\Omega^o}[\Phi^o[\epsilon]](\epsilon t) \\ & - \int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(t-s)\Phi^i[\epsilon](s)\, d\sigma_s + Z[\epsilon]\, S_n(t) \quad \forall t \in \overline{\Omega_m} \end{aligned}$$

for all $\epsilon \in]-\epsilon_m, \epsilon_m[$.

By the real analyticity of $\Phi^o[\cdot]$ (cf. Theorem 2.6.5) and by the properties of integral operators with real analytic kernels (see Theorem A.2.2 (ii) in Appendix A.2), it follows that the map from $]-\epsilon_m, \epsilon_m[$ to $C^{1,\alpha}(\overline{\Omega_m})$ which takes ϵ to $w^+_{\Omega^o}[\Phi^o[\epsilon]](\epsilon\cdot)$ is real analytic. Then, one considers the operator from $L^1(\partial\Omega^i)$ to $C^{1,\alpha}(\overline{\Omega_m})$ which takes f to $\int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(\cdot - s) f(s)\, d\sigma_s$. By the real analyticity of S_n on $\mathbb{R}^n \setminus \{0\}$, by the fact that the integral does not display singularities (by hypothesis $\Omega_m \subseteq \mathbb{R}^n \setminus \overline{\Omega^i}$), by the real analyticity of the map from $]-\epsilon_m, \epsilon_m[$ to $C^{1,\alpha}(\partial\Omega^i)_0$ which takes ϵ to $\Phi^i[\epsilon]$ (cf. Theorem 2.6.5) and since $C^{1,\alpha}(\partial\Omega^i)_0$ is linearly and continuously imbedded in $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon_m, \epsilon_m[$ to $C^{1,\alpha}(\overline{\Omega_m})$ which takes ϵ to $\int_{\partial\Omega^i} \nu_{\Omega^i}(s) \cdot \nabla S_n(\cdot - s)\Phi^i[\epsilon](s)\, d\sigma_s$ is real analytic (see Theorem A.2.2 (ii) in Appendix A.2). Finally, by the real analyticity of $Z[\cdot]$ (cf. Theorem 2.6.5), the map from $]-\epsilon_m, \epsilon_m[$ to $C^{1,\alpha}(\overline{\Omega_m})$ which takes ϵ to $Z[\epsilon]\, S_n$ is real analytic. Hence, one deduces the validity of (iii). □

CHAPTER 3

Uniqueness result for the nonlinear transmission problem (1)

In this chapter we study uniqueness properties of the family of solution $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ of problem (2.2) (cf. Theorem 2.7.2). In particular, we first prove a local uniqueness result (cf. Theorem 3.1.2) which is, in a sense, a consequence of the argument based on the Implicit Function Theorem used in Chapter 2 to prove the existence of such family of solutions. Then, thanks to a precise analysis of the nonlinear operators involved (cf. section 3.2), we are able to weaken the assumptions of Theorem 3.1.2 and obtain a much stronger result (cf. Theorem 3.4.1). The results presented in this chapter are mainly based on a submitted article by Dalla Riva, Musolino, and the author [26].

3.1 A first local uniqueness result for the solution $(u^o_\epsilon, u^i_\epsilon)$

In this section we prove a first local uniqueness result for the family of solutions $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ of Theorem 2.7.2. We will denote by $\mathcal{B}_{0,r}$ the ball in the product space $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ of radius $r>0$ and centered in the 4-tuple $(\phi^o_0,\phi^i_0,\zeta_0,\psi^i_0)$ of Proposition 2.6.3. Namely, we set

$$\mathcal{B}_{0,r}\equiv\Big\{(\phi^o,\phi^i,\zeta,\psi^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)\ :\ \|\phi^o-\phi^o_0\|_{C^{1,\alpha}(\partial\Omega^o)}+\|\phi^i-\phi^i_0\|_{C^{1,\alpha}(\partial\Omega^i)}+|\zeta-\zeta_0|+\|\psi^i-\psi^i_0\|_{C^{1,\alpha}(\partial\Omega^i)}<r\Big\} \tag{3.1}$$

for all $r>0$. Then, for ϵ' as in Theorem 2.6.5, we denote by $\Lambda=(\Lambda_1,\Lambda_2)$ the map from $]-\epsilon',\epsilon'[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$ defined by

$$\begin{aligned}\Lambda_1[\epsilon,\phi^o,\phi^i,\zeta](x)\equiv&\left(\frac{1}{2}I+W_{\partial\Omega^o}\right)[\phi^o](x)\\&-\epsilon^{n-1}\int_{\partial\Omega^i}\nu_{\Omega^i}(y)\cdot\nabla S_n(x-\epsilon y)\phi^i(y)\,d\sigma_y\\&+\epsilon^{n-2}S_n(x)\zeta \qquad\qquad \forall x\in\partial\Omega^o,\\ \Lambda_2[\epsilon,\phi^o,\phi^i,\zeta](t)\equiv&\left(-\frac{1}{2}I+W_{\partial\Omega^i}\right)[\phi^i](t)+w^+_{\Omega^o}[\phi^o](\epsilon t)+S_n(t)\zeta\quad \forall t\in\partial\Omega^i,\end{aligned} \tag{3.2}$$

for all $(\epsilon,\phi^o,\phi^i,\zeta)\in]-\epsilon',\epsilon'[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$. We now prove the following proposition which provides a uniform bound for the operator norm of $\Lambda[\epsilon,\cdot,\cdot,\cdot]^{(-1)}$.

Proposition 3.1.1. *There exist $\epsilon''\in]0,\epsilon'[$ and $C\in]0,+\infty[$ such that the* *operator* $\Lambda[\epsilon,\cdot,\cdot,\cdot]$ *from* $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ *to* $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$ *is linear continuous and invertible for all* $\epsilon\in]-\epsilon'',\epsilon''[$ *fixed and such that*

$$\|\Lambda[\epsilon,\cdot,\cdot,\cdot]^{(-1)}\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R})}\leq C$$

uniformly for $\epsilon\in]-\epsilon'',\epsilon''[$.

Proof. By the mapping properties of the double layer potential (cf. Theorem 1.3.4 (iii) and Theorem 1.3.5 (ii)) and of integral operators with real analytic kernels (cf. Theorem A.2.2 in Appendix A.2) one verifies that the map from $]-\epsilon',\epsilon'[$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}, C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i))$ which takes ϵ to $\Lambda[\epsilon,\cdot,\cdot,\cdot]$ is continuous. Since the set of invertible operators is open in the space $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}, C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i))$, to complete the proof it suffices to show that for $\epsilon=0$ the map which takes $(\phi^o,\phi^i,\zeta)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ to the pair

$$\Lambda[0,\phi^o,\phi^i,\zeta]=\left(\left(\frac{1}{2}I+W_{\partial\Omega^o}\right)[\phi^o],\left(-\frac{1}{2}I+W_{\partial\Omega^i}\right)[\phi^i]+w^+_{\Omega^o}[\phi^o](0)+S_{n|\partial\Omega^i}\zeta\right)$$

belonging to $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$ is invertible. To prove it, we verify that it is a bijection and then we exploit the Open Mapping Theorem. So let $(h^o,h^i)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$. We claim that there exists a unique $(\phi^o,\phi^i,\zeta)\in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ such that

$$\Lambda[0,\phi^o,\phi^i,\zeta]=(h^o,h^i). \tag{3.3}$$

Indeed, by Theorem 1.4.4 (i) and Theorem 1.4.7 (iv), $\frac{1}{2}I+W_{\partial\Omega^o}$ is an isomorphism from $C^{1,\alpha}(\partial\Omega^o)$ into itself and there exists a unique $\phi^o\in C^{1,\alpha}(\partial\Omega^o)$ that satisfies the first equation of (3.3). Moreover, by Lemma 2.1.1, the map from $C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^i)$ that takes (ϕ^i,ζ) to $\left(-\frac{1}{2}I+W_{\partial\Omega^i}\right)[\phi^i]+S_{n|\partial\Omega^i}\zeta$, is an isomorphism. Hence, there exists a unique $(\phi^i,\zeta)\in C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ such that

$$\left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\phi^i] + S_{n|\partial\Omega^i}\zeta = h^i - w^+_{\Omega^o}[\phi^o](0).$$

Accordingly, there exists a unique $(\phi^i, \zeta) \in C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ that satisfies the second equation of (3.3). Thus $\Lambda[0, \cdot, \cdot, \cdot]$ is an isomorphism from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ and the proof is complete. □

We are now ready to state our first local uniqueness result for the solution $(u^o_\epsilon, u^i_\epsilon)$. Theorem 3.1.2 here below is, in a sense, a consequence of an argument based on the Implicit Function Theorem for real analytic maps (see Theorem A.1.2 in Appendix A.1) that has been used in Chapter 2 to prove the existence of such solution. More precisely, for ϵ small enough, given $(v^o, v^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\epsilon\overline{\Omega^i})$ another solution of problem (2.2) which is closed enough to the solution $(u^o_\epsilon, u^i_\epsilon)$ with respect to the trace norm on $\partial\Omega^o$ and $\partial\Omega^i$ (cf. conditions (3.4)-(3.6)), then we are able to estimate the distance of the densities that describe the two pair of solutions (cf. Proposition 2.1.3) and to prove that they coincide.

We shall see in the following section 3.4 that the statement of Theorem 3.1.2 holds under much weaker assumptions.

Theorem 3.1.2. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Let* $\epsilon' \in]0, \epsilon_0[$ *be as in Theorem 2.6.5. Let* $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ *be as in Theorem 2.7.2. Then there exist* $\epsilon^* \in]0, \epsilon'[$ *and* $\delta^* \in]0, +\infty[$ *such that the following property holds:*

If $\epsilon \in]0, \epsilon^*[$ *and* $(v^o, v^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\epsilon\overline{\Omega^i})$ *is a solution of problem* (2.2) *with*

$$\|v^o - u^o_\epsilon\|_{C^{1,\alpha}(\partial\Omega^o)} \leq \epsilon\delta^*, \tag{3.4}$$

$$\|v^o(\epsilon\cdot) - u^o_\epsilon(\epsilon\cdot)\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \epsilon\delta^*, \tag{3.5}$$

$$\left\| v^i(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \epsilon\delta^*, \tag{3.6}$$

then

$$(v^o, v^i) = (u^o_\epsilon, u^i_\epsilon)\,.$$

Proof. Let U_0 be the open neighborhood of $(\phi^o_0, \phi^i_0, \zeta_0, \psi^i_0)$ in $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ introduced in Theorem 2.6.5. We take $K > 0$ such that

$$\overline{\mathcal{B}_{0,K}} \subseteq U_0\,.$$

Since $(\Phi^o[\cdot], \Phi^i[\cdot], Z[\cdot], \Psi^i[\cdot])$ is continuous (indeed real analytic) from $]-\epsilon', \epsilon'[$ to U_0, then there exists $\epsilon'_* \in]0, \epsilon'[$ such that

$$(\Phi^o[\eta], \Phi^i[\eta], Z[\eta], \Psi^i[\eta]) \in \mathcal{B}_{0,K/2} \qquad \forall \eta \in]0, \epsilon'_*[\,. \tag{3.7}$$

Let ϵ'' be as in Proposition 3.1.1 and let

$$\epsilon^* \equiv \min\{\epsilon'_*, \epsilon''\}.$$

Let $\epsilon \in]0, \epsilon^*[$ be fixed and let $(v^o, v^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\epsilon\overline{\Omega^i})$ be a solution of problem (2.2) that satisfies (3.4), (3.5), and (3.6) for a certain $\delta^* \in]0, +\infty[$. We show that for δ^* sufficiently small $(v^o, v^i) = (u^o_\epsilon, u^i_\epsilon)$. By Proposition 2.1.3, there exists a unique quadruple $(\phi^o, \phi^i, \zeta, \psi^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ such that

$$v^o = U^o_\epsilon[\phi^o, \phi^i, \zeta, \psi^i] \qquad \text{in } \overline{\Omega(\epsilon)}, \tag{3.8}$$

$$v^i = U^i_\epsilon[\phi^o, \phi^i, \zeta, \psi^i] \qquad \text{in } \epsilon\overline{\Omega^i}. \tag{3.9}$$

By (3.6) and by (3.9), we have

$$
\begin{aligned}
\delta^* &\geq \left\| \frac{v^i(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot)}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&= \left\| \frac{U^i_\epsilon[\phi^o, \phi^i, \zeta, \psi^i](\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot)}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&= \left\| \frac{\epsilon w^+_{\epsilon\Omega^i}\left[\psi^i\left(\frac{\cdot}{\epsilon}\right)\right](\epsilon\cdot) + \zeta^i - \epsilon w^+_{\epsilon\Omega^i}\left[\Psi^i[\epsilon]\left(\frac{\cdot}{\epsilon}\right)\right](\epsilon\cdot) - \zeta^i}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&= \left\| w^+_{\Omega^i}[\psi^i] - w^+_{\Omega^i}[\Psi^i[\epsilon]] \right\|_{C^{1,\alpha}(\partial\Omega^i)} .
\end{aligned}
\tag{3.10}
$$

By the jump relation in Theorem 1.3.4 (i), we obtain

$$
\left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] - \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] \right\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \delta^*. \tag{3.11}
$$

By Theorem 1.4.4 (i) and Theorem 1.4.7 (iv), the operator $\frac{1}{2}I + W_{\partial\Omega^i}$ is a linear isomorphism from $C^{1,\alpha}(\partial\Omega^i)$ to itself. Then, if we denote by D the norm of its inverse, namely we set

$$
D \equiv \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)^{(-1)} \right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^i))} ,
$$

we obtain, by (3.6) and by (3.11), that

$$
\begin{aligned}
\|\psi^i - \Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)} \leq & \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)^{(-1)} \right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^i))} \\
& \times \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] - \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] \right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
\leq & \, D\delta^*.
\end{aligned}
\tag{3.12}
$$

By (3.4) and (3.8) we have

$$\begin{aligned}
\delta^* &\geq \left\| \frac{v^o - u^o_\epsilon}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^o)} = \left\| \frac{U^o_\epsilon[\phi^o, \phi^i, \zeta, \psi^i] - u^o_\epsilon}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^o)} \\
&= \left\| \frac{\epsilon w^+_{\Omega^o}[\phi^o - \Phi^o[\epsilon]] + \epsilon w^-_{\epsilon\Omega^i}\left[\phi^i\left(\frac{\cdot}{\epsilon}\right) - \Phi^i[\epsilon]\left(\frac{\cdot}{\epsilon}\right)\right] + \epsilon^{n-1}(\zeta - Z[\epsilon])\, S_n}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^o)} \\
&= \left\| w^+_{\Omega^o}[\phi^o - \Phi^o[\epsilon]] + w^-_{\epsilon\Omega^i}\left[\phi^i\left(\frac{\cdot}{\epsilon}\right) - \Phi^i[\epsilon]\left(\frac{\cdot}{\epsilon}\right)\right] + \epsilon^{n-2}(\zeta - Z[\epsilon])\, S_n \right\|_{C^{1,\alpha}(\partial\Omega^o)}.
\end{aligned} \tag{3.13}$$

Similarly, (3.5) and (3.8) yield

$$\begin{aligned}
\delta^* &\geq \left\| \frac{v^o(\epsilon\cdot) - u^o_\epsilon(\epsilon\cdot)}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^i)} = \left\| \frac{U^o_\epsilon[\phi^o, \phi^i, \zeta, \psi^i](\epsilon\cdot) - u^o_\epsilon(\epsilon\cdot)}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&= \left\| w^-_{\Omega^i}\left[\phi^i - \Phi^i[\epsilon]\right] + w^+_{\Omega^o}[\phi^o - \Phi^o[\epsilon]](\epsilon\cdot) + (\zeta - Z[\epsilon])\, S_n \right\|_{C^{1,\alpha}(\partial\Omega^i)}.
\end{aligned} \tag{3.14}$$

Then, by (3.13) and (3.14) and by the definition of the operator Λ in (3.2), we deduce that

$$\left\| \Lambda\left[\epsilon, \phi^o - \Phi^o[\epsilon], \phi^i - \Psi^i[\epsilon], \zeta - Z[\epsilon]\right] \right\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)} \leq 2\delta^* \tag{3.15}$$

(see also the jump relations for the double layer potential in Theorem 1.3.4 (i)). Now let $C > 0$ as in the statement of Proposition 3.1.1. Then, by the membership of ϵ in $]0, \epsilon^*[$, we have

$$\begin{aligned}
&\left(\phi^o - \Phi^o[\epsilon], \phi^i - \Psi^i[\epsilon], \zeta - Z[\epsilon]\right) \\
&\qquad = \Lambda[\epsilon, \cdot, \cdot, \cdot]^{(-1)} \Lambda\left[\epsilon, \phi^o - \Phi^o[\epsilon], \phi^i - \Psi^i[\epsilon], \zeta - Z[\epsilon]\right]
\end{aligned} \tag{3.16}$$

and, by (3.15) and (3.16), we obtain

$$\begin{aligned}
&\left\|\left(\phi^o-\Phi^o[\epsilon],\phi^i-\Psi^i[\epsilon],\zeta-Z[\epsilon]\right)\right\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}}\\
&\quad\le \|\Lambda[\epsilon,\cdot,\cdot,\cdot]^{(-1)}\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R})}\\
&\qquad\times\left\|\Lambda\left[\epsilon,\phi^o-\Phi^o[\epsilon],\phi^i-\Psi^i[\epsilon],\zeta-Z[\epsilon]\right]\right\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}}\\
&\quad\le 2C\delta^*\,.
\end{aligned}$$

The latter inequality, combined with (3.12), yields

$$\begin{aligned}
\left\|\left(\phi^o-\Phi^o[\epsilon],\phi^i-\Psi^i[\epsilon],\zeta-Z[\epsilon],\psi^i-\Psi[\epsilon]\right)\right\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)}&\\
\le (2C+D)\delta^*\,.&
\end{aligned}\tag{3.17}$$

Hence, by (3.7) and (3.17) and by a standard computation based on the triangle inequality one sees that

$$\begin{aligned}
\|(\phi^o,\phi^i,\zeta,\psi^i)-(\phi^o_0,\phi^i_0,\zeta_0,\psi^i_0)\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)}&\\
\le (2C+D)\delta^*+\frac{K}{2}.&
\end{aligned}$$

Accordingly, in order to have $(\phi^o,\phi^i,\zeta,\psi^i)\in\mathcal{B}_{0,K}$, it suffices to take

$$\delta^*<\frac{K}{2(2C+D)}$$

in inequalities (3.4), (3.5), and (3.6). Then, by the inclusion $\overline{\mathcal{B}_{0,K}}\subseteq U_0$ and by Theorem 2.6.5, we deduce that for such choice of δ^* we have

$$(\phi^o,\phi^i,\zeta,\psi^i)=\left(\Phi^o[\epsilon],\Psi^i[\epsilon],Z[\epsilon],\Psi[\epsilon]\right)$$

and thus $(v^o,v^i)=(u^o_\epsilon,u^i_\epsilon)$ (cf. Definition 2.7.1). □

3.2 Some preliminary results on composition operators in Schauder spaces

In this section we prove some results on composition operators in Schauder spaces which will play an important role in the proof of the main result of this chapter, namely Theorem 3.4.1. In fact this results will be used in the aforementioned theorem in order to obtain an uniform bound on the growth of the nonlinear map S which will be defined in section 3.3 (cf. (3.44)-(3.46)).

Moreover we mention that results of the type presented in this section (in particular Proposition 3.2.4 and Lemmas 3.2.5 and 3.2.6) are not new at all: the literature of superposition operators, which include the study of boundedness, continuity and differentiability properties in different Banach spaces (e.g. Lebesgue space, Sobolev spaces and Schauder spaces) is vast. For a reference, we refer to the monograph of Appell and Zabrejko [8]. In particular for composition operators on Schauder spaces see also Valent [76, Chap. II].

We begin with some possibly known elementary results for product of functions in Schauder spaces (cf. Lanza [45]).

Lemma 3.2.1. *Let $d \in \mathbb{N} \setminus \{0\}$. Let Ω be an open bounded convex subset of $\mathbb{R}^d$. Then the following statements hold.*

(i) $\|uv\|_{C^{0,\alpha}(\overline{\Omega})} \leq \|u\|_{C^{0,\alpha}(\overline{\Omega}))}\|v\|_{C^{0,\alpha}(\overline{\Omega})} \qquad \forall u, v \in C^{0,\alpha}(\overline{\Omega}).$

(ii) $\|uv\|_{C^{1,\alpha}(\overline{\Omega})} \leq 2\,\|u\|_{C^{1,\alpha}(\overline{\Omega})}\,\|v\|_{C^{1,\alpha}(\overline{\Omega})} \qquad \forall u, v \in C^{1,\alpha}(\overline{\Omega}).$

Proof. We first prove statement (i). Let $u, v \in C^{0,\alpha}(\overline{\Omega})$. Then

$$\|uv\|_{C^{0,\alpha}(\overline{\Omega})} \leq \|u\|_{C^{0}(\overline{\Omega})}\|v\|_{C^{0}(\overline{\Omega})} + |uv : \overline{\Omega}|_{\alpha}.$$

By triangle inequality we have that

$$|uv : \overline{\Omega}|_\alpha \le \|u\|_{C^0(\overline{\Omega})} \, |v : \overline{\Omega}|_\alpha + \|v\|_{C^0(\overline{\Omega})} \, |u : \overline{\Omega}|_\alpha.$$

Hence we obtain

$$\begin{aligned}
\|uv\|_{C^{0,\alpha}(\overline{\Omega})} &\le \|u\|_{C^0(\overline{\Omega})} \|v\|_{C^0(\overline{\Omega})} + \|u\|_{C^0(\overline{\Omega})} \, |v : \overline{\Omega}|_\alpha + \|v\|_{C^0(\overline{\Omega})} \, |u : \overline{\Omega}|_\alpha \\
&\le \|u\|_{C^0(\overline{\Omega})} \left\{ \|v\|_{C^0(\overline{\Omega})} + |v : \overline{\Omega}|_\alpha \right\} + \|v\|_{C^0(\overline{\Omega})} \, |u : \overline{\Omega}|_\alpha \\
&\le \|u\|_{C^0(\overline{\Omega})} \, \|v\|_{C^{0,\alpha}(\overline{\Omega})} + \|v\|_{C^0(\overline{\Omega})} \, |u : \overline{\Omega}|_\alpha \\
&\le \|v\|_{C^{0,\alpha}(\overline{\Omega})} \left\{ \|u\|_{C^0(\overline{\Omega})} + |u : \overline{\Omega}|_\alpha \right\} \\
&\le \|u\|_{C^{0,\alpha}(\overline{\Omega})} \|v\|_{C^{0,\alpha}(\overline{\Omega})},
\end{aligned}$$

and the proof of point (i) is complete. We now prove statement (ii). By definition of $C^{1,\alpha}$ norm and by triangle inequality we have that

$$\begin{aligned}
\|uv\|_{C^{1,\alpha}(\overline{\Omega})} &\le \|u\|_{C^0(\overline{\Omega})} \|v\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{d} \|\partial_j (uv)\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\le \|u\|_{C^0(\overline{\Omega})} \|v\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{d} \|v \, \partial_j u + u \, \partial_j v\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\le \|u\|_{C^0(\overline{\Omega})} \|v\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{d} \|v \, \partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} + \sum_{j=1}^{d} \|u \, \partial_j v\|_{C^{0,\alpha}(\overline{\Omega})} \, .
\end{aligned}$$

Moreover, by point (i), it follows that

$$\begin{aligned}
\|v \, \partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} &\le \|v\|_{C^{0,\alpha}(\overline{\Omega})} \|\partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} \\
\|u \, \partial_j v\|_{C^{0,\alpha}(\overline{\Omega})} &\le \|u\|_{C^{0,\alpha}(\overline{\Omega})} \|\partial_j v\|_{C^{0,\alpha}(\overline{\Omega})}
\end{aligned}$$

for all $j \in \{1, \dots, d\}$. Hence we obtain that

$$\begin{aligned}
\|uv\|_{C^{1,\alpha}(\overline{\Omega})} &\leq \|u\|_{C^0(\overline{\Omega})}\|v\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{d} \|v\|_{C^{0,\alpha}(\overline{\Omega})}\|\partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\quad + \sum_{j=1}^{d} \|u\|_{C^{0,\alpha}(\overline{\Omega})}\|\partial_j v\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \left\{ \|v\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{d} \|\partial_j v\|_{C^{0,\alpha}(\overline{\Omega})} \right\} \\
&\quad + \sum_{j=1}^{d} \|v\|_{C^{0,\alpha}(\overline{\Omega})}\|\partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \, \|v\|_{C^{1,\alpha}(\overline{\Omega})} + \sum_{j=1}^{d} \|v\|_{C^{0,\alpha}(\overline{\Omega})}\|\partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} \, .
\end{aligned}$$

Finally, using the facts that $|u : \overline{\Omega}|_\alpha \leq \|u\|_{C^{1,\alpha}(\overline{\Omega})}$ and so $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq \|u\|_{C^{1,\alpha}(\overline{\Omega})}$ for all $u \in C^{1,\alpha}(\overline{\Omega})$ (see Lanza [45, Lemma 2.4]), we conclude that

$$\begin{aligned}
\|uv\|_{C^{1,\alpha}(\overline{\Omega})} &\leq \|v\|_{C^{1,\alpha}(\overline{\Omega})} \left\{ \|u\|_{C^0(\overline{\Omega})} + |u : \overline{\Omega}|_\alpha + \sum_{j=1}^{d} \|\partial_j u\|_{C^{0,\alpha}(\overline{\Omega})} \right\} \\
&\leq \|v\|_{C^{1,\alpha}(\overline{\Omega})} \left\{ \|u\|_{C^1(\overline{\Omega})} + |u : \overline{\Omega}|_\alpha \right\} \\
&\leq 2\|u\|_{C^{1,\alpha}(\overline{\Omega})} \, \|v\|_{C^{1,\alpha}(\overline{\Omega})} \, .
\end{aligned}$$

□

Then, we have the following immediate consequence of Lemma 3.2.1.

Lemma 3.2.2. *Let $n \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) $\|uv\|_{C^{0,\alpha}(\partial\Omega)} \leq \|u\|_{C^{0,\alpha}(\partial\Omega)} \, \|v\|_{C^{0,\alpha}(\partial\Omega)} \qquad \forall u, v \in C^{0,\alpha}(\partial\Omega).$

(ii) $\|uv\|_{C^{1,\alpha}(\partial\Omega)} \leq 2 \, \|u\|_{C^{1,\alpha}(\partial\Omega)} \, \|v\|_{C^{1,\alpha}(\partial\Omega)} \qquad \forall u, v \in C^{1,\alpha}(\partial\Omega).$

Proof. By exploiting a finite $C^{1,\alpha}$ local parametrization $\gamma_1, \ldots, \gamma_k$ for $\partial\Omega$ (see the definition of $C^{1,\alpha}$ sub-manifold of $\mathbb{R}^n$ in the Notation) and by using Lemma 3.2.1 (i) for the open convex unit ball of $\mathbb{R}^{n-1}$, we obtain that

$$\begin{aligned}\|uv\|_{C^{0,\alpha}(\partial\Omega)} &= \sum_{l=1}^{k} \|(uv)\circ\gamma_l\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})}\\ &\leq \sum_{l=1}^{k} \|u\circ\gamma_l\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})}\,\|v\circ\gamma_l\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})}\\ &\leq \|u\|_{C^{0,\alpha}(\partial\Omega)} \sum_{l=1}^{k} \|v\circ\gamma_l\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})} \leq \|u\|_{C^{0,\alpha}(\partial\Omega)}\,\|v\|_{C^{0,\alpha}(\partial\Omega)}.\end{aligned}$$

Arguing in the same way and using instead Lemma 3.2.1 (ii), one can prove the second inequality in the statement. □

We now present a result on composition of a $C^{m,\alpha}$ function, with $m \in \{0,1\}$, with a $C^{1,\alpha}$ function.

Lemma 3.2.3. *Let $n, d \in \mathbb{N}\setminus\{0\}$ and $\alpha \in]0,1]$. Let Ω_1 be an open bounded convex subset of $\mathbb{R}^n$ and Ω_2 be an open bounded convex subset of $\mathbb{R}^d$. Let $v = (v_1, \ldots, v_n) \in (C^{1,\alpha}(\overline{\Omega_2}))^n$ such that $v(\overline{\Omega_2}) \subset \overline{\Omega_1}$. Then the following statements hold.*

(i) If $u \in C^{0,\alpha}(\overline{\Omega_1})$, then

$$\|u(v(\cdot))\|_{C^{0,\alpha}(\overline{\Omega_2})} \leq \|u\|_{C^{0,\alpha}(\overline{\Omega_1})}\left(1 + \|v\|^{\alpha}_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right).$$

(ii) If $u \in C^{1,\alpha}(\overline{\Omega_1})$, then

$$\|u(v(\cdot))\|_{C^{1,\alpha}(\overline{\Omega_2})} \leq (1+nd)^2 \|u\|_{C^{1,\alpha}(\overline{\Omega_1})}\left(1 + \|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right)^2.$$

Proof. We first prove statement (i). We observe that

$$\frac{|u(v(t)) - u(v(t'))|}{|t-t'|^\alpha} \leq \frac{|u(v(t)) - u(v(t'))|}{|v(t)-v(t')|^\alpha} \frac{|v(t)-v(t')|^\alpha}{|t-t'|^\alpha}$$

for all $t, t' \in \Omega_2$ such that $v(t) \neq v(t')$. Then for such $t, t' \in \Omega_2$ we have

$$\frac{|u(v(t)) - u(v(t'))|}{|t-t'|^\alpha} \leq \|u\|_{C^{0,\alpha}(\overline{\Omega_1})} \|v\|^\alpha_{(C^{1,\alpha}(\overline{\Omega_2}))^n}.$$

If instead $v(t) = v(t')$, then $|u(v(t)) - u(v(t'))| = 0$ and the inequality here above is readily verified. Hence

$$|u(v(\cdot)) : \Omega_2|_\alpha \leq \|u\|_{C^{0,\alpha}(\overline{\Omega_1})} \|v\|^\alpha_{(C^{1,\alpha}(\overline{\Omega_2}))^n}$$

and the proof of point (i) follows. We now prove statement (ii). By definition of $C^{1,\alpha}$ norm and by Lemma 3.2.1 (i) we have that

$$\|u(v(\cdot))\|_{C^{1,\alpha}(\overline{\Omega_2})} \leq \|u\|_{C^0(\overline{\Omega_1})} + \sum_{i=1}^d \sum_{j=1}^n \|\partial_j u(v(\cdot))\|_{C^{0,\alpha}(\overline{\Omega_2})} \|\partial_i v_j\|_{C^{0,\alpha}(\overline{\Omega_2})}.$$

Moreover, by hypothesis, $u \in C^{1,\alpha}(\overline{\Omega_1})$, hence, for every $j \in \{1, \dots, n\}$, we have that $\partial_j u \in C^{0,\alpha}(\overline{\Omega_1})$. Then, by point (i), the following estimate holds:

$$\begin{aligned}\|\partial_j u(v(\cdot))\|_{C^{0,\alpha}(\overline{\Omega_2})} &\leq \|\partial_j u\|_{C^{0,\alpha}(\overline{\Omega_1})} \left(1 + \|v\|^\alpha_{(C^{0,\alpha}(\overline{\Omega_2}))^n}\right) \\ &\leq \|u\|_{C^{1,\alpha}(\overline{\Omega_1})} \left(1 + \|v\|^\alpha_{(C^{0,\alpha}(\overline{\Omega_2}))^n}\right).\end{aligned}$$

Hence, keeping in mind that $1+\alpha \leq 2$, $nd \leq (nd)^2$ and that

$$\left(1 + nd\, \|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right)^2 \leq (1+nd)^2 \left(1 + \ \|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right)^2,$$

we conclude that

$$\begin{aligned}
\|u(v(\cdot))\|_{C^{1,\alpha}(\overline{\Omega_2})} &\leq \|u\|_{C^0(\overline{\Omega_1})} + nd\,\|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\left(\|u\|_{C^{1,\alpha}(\overline{\Omega_1})}\left(1+\|v\|^{\alpha}_{(C^{0,\alpha}(\overline{\Omega_2}))^n}\right)\right)\\
&\leq \|u\|_{C^{1,\alpha}(\overline{\Omega_1})}\left(1+nd\,\|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\left(1+\|v\|^{\alpha}_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right)\right)\\
&\leq \|u\|_{C^{1,\alpha}(\overline{\Omega_1})}\left(1+nd\,\|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right)^2\\
&\leq (1+nd)^2\,\|u\|_{C^{1,\alpha}(\overline{\Omega_1})}\left(1+\,\|v\|_{(C^{1,\alpha}(\overline{\Omega_2}))^n}\right)^2.
\end{aligned}$$

□

In the sequel we will exploit Schauder spaces over suitable subsets of $\partial\Omega \times \mathbb{R}$, with Ω an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. We observe indeed that for all open bounded intervals $\mathcal{J}$ of $\mathbb{R}$, the product $\partial\Omega \times \overline{\mathcal{J}}$ is a compact sub-manifold (with boundary) of co-dimension 1 in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and accordingly, we can define the spaces $C^{0,\alpha}(\partial\Omega \times \overline{\mathcal{J}})$ and $C^{1,\alpha}(\partial\Omega \times \overline{\mathcal{J}})$ by exploiting a finite atlas.

Then, by Lemma 3.2.3, we deduce the following Proposition 3.2.4.

Proposition 3.2.4. *Let $n \in \mathbb{N} \setminus \{0\}$. Let $\alpha \in]0,1]$. Let Ω be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $R > 0$. Then the following holds.*

(i) There exists $c_0 > 0$ such that

$$\|u(\cdot, v(\cdot))\|_{C^{0,\alpha}(\partial\Omega)} \leq c_0 \|u\|_{C^{0,\alpha}(\partial\Omega\times[-R,R])}\left(1+\|v\|^{\alpha}_{C^{1,\alpha}(\partial\Omega)}\right).$$

for all $u \in C^{0,\alpha}(\partial\Omega \times \mathbb{R})$ and for all $v \in C^{1,\alpha}(\partial\Omega)$ such that $v(\partial\Omega) \subset [-R, R]$.

(ii) There exists $c_1 > 0$ such that

$$\|u(\cdot, v(\cdot))\|_{C^{1,\alpha}(\partial\Omega)} \leq c_1 \|u\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}\left(1+\|v\|_{C^{1,\alpha}(\partial\Omega)}\right)^2.$$

for all $u \in C^{1,\alpha}(\partial\Omega \times \mathbb{R})$ *and for all* $v \in C^{1,\alpha}(\partial\Omega)$ *such that* $v(\partial\Omega) \subset [-R, R]$.

Proof. We prove only statement (ii). The proof of statement (i) can be obtained adapting the one of point (ii) and using Lemma 3.2.3 (i) instead of Lemma 3.2.3 (ii). More precisely, one proceeds in the same way until equation (3.18) where the inequality provided by Lemma 3.2.3 (i) would be used.
Since $\partial\Omega$ is compact and of class $C^{1,\alpha}$, a standard argument shows that there are a finite cover of $\partial\Omega$ consisting of open subsets $\mathcal{U}_1, \dots, \mathcal{U}_k$ of $\partial\Omega$ and for each $j \in \{1, \dots, k\}$ a $C^{1,\alpha}$ diffeomorphism γ_j from $\overline{B_{n-1}(0,1)}$ to the closure of $\mathcal{U}_j$ in $\partial\Omega$. Then, for a fixed $j \in \{1, \dots, k\}$ we define the functions $\tilde{u}^j : \overline{B_{n-1}(0,1)} \times \mathbb{R} \to \mathbb{R}$, $\tilde{v}^j : \overline{B_{n-1}(0,1)} \to \mathbb{R}$, and $\tilde{w}^j : \overline{B_{n-1}(0,1)} \to \overline{B_{n-1}(0,1)} \times \mathbb{R}$ by setting

$$
\begin{aligned}
&\tilde{u}^j(t', s) \equiv u(\gamma_j(t'), s) && \forall (t', s) \in \overline{B_{n-1}(0,1)} \times \mathbb{R}, \\
&\tilde{v}^j(t') \equiv v(\gamma_j(t')) && \forall t' \in \overline{B_{n-1}(0,1)}, \\
&\tilde{w}^j(t') \equiv (t', \tilde{v}^j(t')) && \forall t' \in \overline{B_{n-1}(0,1)}.
\end{aligned}
$$

Since $v(\partial\Omega) \subset [-R, R]$, it follows that $\tilde{w}^j(\overline{B_{n-1}(0,1)}) \subset \overline{B_{n-1}(0,1)} \times [-R, R]$. Moreover, we can see that there exists $d_j > 0$ such that

$$
\|\tilde{w}^j\|_{(C^{1,\alpha}(\overline{B_{n-1}(0,1)}))^n} \leq d_j \|\tilde{v}^j\|_{\left(C^{1,\alpha}(\overline{B_{n-1}(0,1)})\right)}.
$$

Then Lemma 3.2.3 (ii) implies that there exists $c_j > 0$ such that

$$
\begin{aligned}
\|\tilde{u}^j(\cdot, \tilde{v}^j(\cdot))\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})} &= \|\tilde{u}^j(\tilde{w}^j(\cdot))\|_{(C^{1,\alpha}(\overline{B_{n-1}(0,1)}))^n} \\
&\leq c_j \|\tilde{u}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)} \times [-R,R])} \left(1 + \|\tilde{w}^j\|_{(C^{1,\alpha}(\overline{B_{n-1}(0,1)}))^n}\right)^2 \\
&\leq c_j \|\tilde{u}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)} \times [-R,R])} \left(1 + d_j \|\tilde{v}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})}\right)^2 .
\end{aligned}
\tag{3.18}
$$

Without loss of generality, we can now assume that the norm of $C^{1,\alpha}(\partial\Omega)$ is defined on the atlas $\{(\mathcal{U}_j,\gamma_j)\}_{j\in\{1,\dots,k\}}$. Then by (3.18) we have

$$\begin{aligned}
&\|u(\cdot,v(\cdot))\|_{C^{1,\alpha}(\partial\Omega)}\\
&\quad=\sum_{j=1}^{k}\|u(\gamma_j(\cdot),v(\gamma_j(\cdot)))\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})}=\sum_{j=1}^{k}\|\tilde{u}^j(\cdot,\tilde{v}^j(\cdot))\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})}\\
&\quad\le\sum_{j=1}^{k}c_j\|\tilde{u}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)}\times[-R,R])}\left(1+d_j\|\tilde{v}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})}\right)^2 .
\end{aligned}\tag{3.19}$$

Moreover,

$$\|\tilde{u}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)}\times[-R,R])}\le\|u\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}$$

and

$$\begin{aligned}
\left(1+d_j\|\tilde{v}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})}\right)^2&\le(1+d_j)^2\left(1+\|\tilde{v}^j\|_{C^{1,\alpha}(\overline{B_{n-1}(0,1)})}\right)^2\\
&\le(1+d_j)^2\left(1+\|v\|_{C^{1,\alpha}(\partial\Omega)}\right)^2 .
\end{aligned}$$

Hence, (3.19) implies that

$$\begin{aligned}
&\|u(\cdot,v(\cdot))\|_{C^{1,\alpha}(\partial\Omega)}\\
&\quad\le k\,\max\{c_1(1+d_1)^2,\dots,c_k(1+d_k)^2\}\|u\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}\left(1+\|v\|_{C^{1,\alpha}(\partial\Omega)}\right)^2
\end{aligned}$$

and the proposition is proved. □

Then we proceed with the following Lemma 3.2.5, which provides an uniform bound for the $C^{m,\alpha}$-norm, $m\in\{0,1\}$, of a specific class of composition operators generated by a function A from $]-\epsilon_0,\epsilon_0[\times\overline{B_{n-1}(0,1)}\times\mathbb{R}$ to $\mathbb{R}$.

Lemma 3.2.5. *Let A be a function from $]-\epsilon_0,\epsilon_0[\times\overline{B_{n-1}(0,1)}\times\mathbb{R}$ to $\mathbb{R}$. Let $\mathcal{M}_A$ be the map which takes a pair $(\epsilon,\zeta)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to the function*

$\mathcal{M}_A(\epsilon,\zeta)$ *defined by*

$$\mathcal{M}_A(\epsilon,\zeta)(z) \equiv A(\epsilon,z,\zeta) \qquad \forall z \in \overline{B_{n-1}(0,1)}. \tag{3.20}$$

Let $m \in \{0,1\}$. *If* $\mathcal{M}_A(\epsilon,\zeta) \in C^{m,\alpha}(\overline{B_{n-1}(0,1)})$ *for all* $(\epsilon,\zeta) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ *and if the map* $\mathcal{M}_A$ *is real analytic from* $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ *to* $C^{m,\alpha}(\overline{B_{n-1}(0,1)})$, *then for every open bounded interval* $\mathcal{J}$ *of* $\mathbb{R}$ *and every compact subset* $\mathcal{E}$ *of* $]-\epsilon_0,\epsilon_0[$ *there exists* $C>0$ *such that*

$$\sup_{\epsilon\in\mathcal{E}} \|A(\epsilon,\cdot,\cdot)\|_{C^{m,\alpha}(\overline{B_{n-1}(0,1)}\times\overline{\mathcal{J}})} \leq C. \tag{3.21}$$

Proof. We first prove the statement of Lemma 3.2.5 for $m=0$. If $\mathcal{M}_A$ is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{0,\alpha}(\overline{B_{n-1}(0,1)})$, then for every $(\tilde{\epsilon},\tilde{\zeta}) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ there exist $M \in]0,+\infty[$, $\rho \in]0,1[$, and a family of coefficients $\{a_{jk}\}_{j,k\in\mathbb{N}} \subset C^{0,\alpha}(\overline{B_{n-1}(0,1)})$ such that

$$\|a_{jk}\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})} \leq M\left(\frac{1}{\rho}\right)^{k+j} \qquad \forall j,k\in\mathbb{N}, \tag{3.22}$$

and

$$\mathcal{M}_A(\epsilon,\zeta)(\cdot) = \sum_{j,k=0}^{\infty} a_{jk}(\cdot)(\epsilon-\tilde{\epsilon})^k(\zeta-\tilde{\zeta})^j \quad \forall(\epsilon,\zeta)\in]\tilde{\epsilon}-\rho,\tilde{\epsilon}+\rho[\times]\tilde{\zeta}-\rho,\tilde{\zeta}+\rho[\,, \tag{3.23}$$

where ρ is less than or equal to the radius of convergence of the series in (3.23). Now let $\mathcal{J}\subset\mathbb{R}$ be open and bounded and $\mathcal{E}\subset]-\epsilon_0,\epsilon_0[$ be compact. Since the product $\overline{\mathcal{J}}\times\mathcal{E}$ is compact, a standard finite covering argument shows that in order to prove (3.21) for $m=0$ it suffices to find a uniform upper bound (independent of $\tilde{\epsilon}$ and $\tilde{\zeta}$) for the quantity

$$\sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \|A(\epsilon,\cdot,\cdot)\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)}\times[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}])}.$$

By (3.20), (3.22), and (3.23) we have

$$\begin{aligned}
&\sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \|A(\epsilon,\cdot,\cdot)\|_{C^{0}(\overline{B_{n-1}(0,1)}\times[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}])} \\
&\qquad \leq \sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \sum_{j,k=0}^{\infty} \|a_{jk}(\cdot)(\epsilon-\tilde{\epsilon})^k(\cdot-\tilde{\zeta})^j\|_{C^{0}(\overline{B_{n-1}(0,1)}\times[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}])} \\
&\qquad \leq \sum_{j,k=0}^{\infty} M\left(\frac{1}{\rho}\right)^{j+k}\left(\frac{\rho}{2}\right)^{k}\left(\frac{\rho}{4}\right)^{j} = \sum_{j,k=0}^{\infty} M\left(\frac{1}{2}\right)^{k}\left(\frac{1}{4}\right)^{j} = \frac{8}{3}M
\end{aligned} \tag{3.24}$$

for all $l = 1, \ldots, m$. Then inequality (3.24) yields an estimate of the C^0 norm of A. To complete the proof of (3.21) for $m = 0$ we have now to study the Hölder constant of $A(\epsilon,\cdot,\cdot)$ on $\overline{B_{n-1}(0,1)} \times [\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}]$. To do so, we take $z', z'' \in \overline{B_{n-1}(0,1)}$, $\zeta',\zeta'' \in [\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}]$, and $\epsilon \in [\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]$, and we consider the difference

$$|a_{jk}(z')(\epsilon-\tilde{\epsilon})^k(\zeta'-\tilde{\zeta})^j - a_{jk}(z'')(\epsilon-\tilde{\epsilon})^k(\zeta''-\tilde{\zeta})^j|. \tag{3.25}$$

For $j \geq 1$ and $k \geq 0$ we argue as follow: we add and subtract the term $a_{jk}(z'')(\epsilon-\tilde{\epsilon})^k(\zeta'-\tilde{\zeta})^j$ inside the absolute value in (3.25), we use the triangle inequality to split the difference in two terms and then we exploit the membership of a_{jk} in $C^{0,\alpha}(\overline{B_{n-1}(0,1)})$ and an argument based on the Taylor expansion at the first order for the function from $[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}]$ to $\mathbb{R}$ that takes

ζ to $(\zeta - \tilde{\zeta})^j$. Doing so we show that (3.25) is less than or equal to

$$\begin{aligned}|a_{jk}(z') - a_{jk}(z'')||\epsilon - \tilde{\epsilon}|^k|\zeta' - \tilde{\zeta}|^j + |a_{jk}(z'')||\epsilon - \tilde{\epsilon}|^k|(\zeta' - \tilde{\zeta})^j - (\zeta'' - \tilde{\zeta})^j| \\ \leq \|a_{jk}\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})}|z' - z''|^\alpha|\epsilon - \tilde{\epsilon}|^k|\zeta' - \tilde{\zeta}|^j \\ + \|a_{jk}\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})}|\epsilon - \tilde{\epsilon}|^k\left(j|\overline{\zeta} - \tilde{\zeta}|^{j-1}|\zeta' - \zeta''|\right)\end{aligned} \tag{3.26}$$

for a suitable $\overline{\zeta} \in [\tilde{\zeta} - \frac{\rho}{4}, \tilde{\zeta} + \frac{\rho}{4}]$. Then by (3.22), by inequalities $|\epsilon - \tilde{\epsilon}| \leq \frac{\rho}{2}$, $|\zeta' - \tilde{\zeta}| \leq \frac{\rho}{4}$, and $|\overline{\zeta} - \tilde{\zeta}| \leq \frac{\rho}{4}$, and by a straightforward computation we see that the right hand side of (3.26) is less than or equal to

$$\begin{aligned}M\left(\frac{1}{\rho}\right)^{j+k}|z' - z''|^\alpha\left(\frac{\rho}{2}\right)^k\left(\frac{\rho}{4}\right)^j + M\left(\frac{1}{\rho}\right)^{j+k}\left(\frac{\rho}{2}\right)^k j\left(\frac{\rho}{4}\right)^{j-1}|\zeta' - \zeta''| \\ = M\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)^{j+k}|z' - z''|^\alpha + 4M\rho^{-1}\frac{j}{2^j}\left(\frac{1}{2}\right)^{j+k}|\zeta' - \zeta''|^{1-\alpha}|\zeta' - \zeta''|^\alpha.\end{aligned} \tag{3.27}$$

Now, since ζ' and ζ'' are taken in the interval $[\tilde{\zeta}-\frac{\rho}{4}, \tilde{\zeta}+\frac{\rho}{4}]$ we have $|\zeta'-\zeta''|^{1-\alpha} \leq (\rho/2)^{1-\alpha}$ and since $\rho \in]0,1[$ and $\alpha \in]0,1[$, we deduce that $|\zeta' - \zeta''|^{1-\alpha} \leq 1$. Moreover, since $j \geq 1$, we have $j/2^j \leq 1$ and $(1/2)^j < 1$. It follows that the right hand side of (3.27) is less than or equal to

$$\begin{aligned}M\left(\frac{1}{2}\right)^{j+k}|z' - z''|^\alpha + 4M\rho^{-1}\left(\frac{1}{2}\right)^{j+k}|\zeta' - \zeta''|^\alpha \\ \leq 4M\rho^{-1}\left(\frac{1}{2}\right)^{j+k}\left(|z' - z''|^\alpha + |\zeta' - \zeta''|^\alpha\right)\end{aligned} \tag{3.28}$$

(also note that $\rho^{-1} > 1$). Finally, by inequality

$$a^\alpha + b^\alpha \leq 2^{1-\frac{\alpha}{2}}(a^2 + b^2)^{\frac{\alpha}{2}}\,,$$

which holds for all $a, b > 0$, we deduce that the right hand side of (3.28) is less than or equal to

$$2^{3-\frac{\alpha}{2}} M \rho^{-1} \left(\frac{1}{2}\right)^{j+k} |(z', \zeta') - (z'', \zeta'')|^{\alpha}, \tag{3.29}$$

where $|(z', \zeta') - (z'', \zeta'')|$ denotes the Euclidean norm of the vector $(z', \zeta') - (z'', \zeta'')$ in $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$. Then, by (3.26)–(3.29), we obtain that

$$\begin{aligned} &|a_{jk}(z')(\epsilon - \tilde{\epsilon})^k (\zeta' - \tilde{\zeta})^j - a_{jk}(z'')(\epsilon - \tilde{\epsilon})^k (\zeta'' - \tilde{\zeta})^j| \\ &\qquad \leq 2^{3-\frac{\alpha}{2}} M \rho^{-1} \left(\frac{1}{2}\right)^{j+k} |(z', \zeta') - (z'', \zeta'')|^{\alpha} \end{aligned} \tag{3.30}$$

for all $j \geq 1$, $k \geq 0$, and $\epsilon \in [\tilde{\epsilon} - \frac{\rho}{2}, \tilde{\epsilon} + \frac{\rho}{2}]$. Now, for every $\epsilon \in [\tilde{\epsilon} - \frac{\rho}{2}, \tilde{\epsilon} + \frac{\rho}{2}]$ we denote by $\tilde{a}_{jk,\epsilon}$ the function

$$\begin{aligned} \tilde{a}_{jk,\epsilon} \,:\, &\overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta} - \frac{\rho}{4}, \tilde{\zeta} + \frac{\rho}{4}\right] \to \mathbb{R} \\ &\qquad (z, \zeta) \mapsto \tilde{a}_{jk,\epsilon}(z, \zeta) \equiv a_{jk}(z)(\epsilon - \tilde{\epsilon})^k (\zeta - \tilde{\zeta})^j \,. \end{aligned} \tag{3.31}$$

Then inequality (3.30) readily implies that

$$\begin{aligned} &\left|\tilde{a}_{jk,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta} - \frac{\rho}{4}, \tilde{\zeta} + \frac{\rho}{4}\right]\right|_{\alpha} \leq 2^{3-\frac{\alpha}{2}} M \rho^{-1} \left(\frac{1}{2}\right)^{j+k} \\ &\qquad \forall j \geq 1\,,\; k \geq 0\,,\; \epsilon \in \left[\tilde{\epsilon} - \frac{\rho}{2}, \tilde{\epsilon} + \frac{\rho}{2}\right]\,, \end{aligned}$$

which in turn implies that

$$\sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \sum_{j=1,k=0}^{\infty} \left|\tilde{a}_{jk,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \leq \sum_{j=1,k=0}^{\infty} 2^{3-\frac{\alpha}{2}} M\rho^{-1}\left(\frac{1}{2}\right)^{j+k} \leq 2^{4-\frac{\alpha}{2}} M\rho^{-1}. \tag{3.32}$$

We now turn to consider (3.25) in the case where $j=0$ and $k\geq 0$. In such case, one verifies that the quantity in (3.25) is less than or equal to

$$\|a_{0k}\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)})}|\epsilon-\tilde{\epsilon}|^k|z'-z''|^{\alpha},$$

which, by (3.22) and by inequality $|\epsilon-\tilde{\epsilon}|\leq\frac{\rho}{2}$, is less than or equal to

$$M\left(\frac{1}{\rho}\right)^k\left(\frac{\rho}{2}\right)^k|z'-z''|^{\alpha} = M\left(\frac{1}{2}\right)^k|z'-z''|^{\alpha}.$$

Hence, for $\tilde{a}_{0k,\epsilon}$ defined as in (3.31) (with $j=0$) we have

$$\left|\tilde{a}_{0k,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \leq M\left(\frac{1}{2}\right)^k$$

for all $k\geq 0$ and $\epsilon\in\left[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}\right]$, which implies that

$$\sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \sum_{k=0}^{\infty} \left|\tilde{a}_{0k,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}+\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \leq \sum_{k=0}^{\infty} M\left(\frac{1}{2}\right)^k = 2M. \tag{3.33}$$

Finally, by (3.23), (3.24), (3.32), and (3.33) we obtain

$$\begin{aligned}
&\sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \|A(\epsilon,\cdot,\cdot)\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)}\times[\tilde{\zeta}+\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}])} \\
&\qquad = \sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \|A(\epsilon,\cdot,\cdot)\|_{C^{0}(\overline{B_{n-1}(0,1)}\times[\tilde{\zeta}+\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}])} \\
&\qquad\quad + \sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \left|A(\epsilon,\cdot,\cdot) : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \\
&\qquad \leq \frac{8}{3}M + \sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \sum_{j,k=0}^{\infty} \left|\tilde{a}_{jk,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \\
&\qquad = \frac{8}{3}M + \sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \sum_{j=1,k=0}^{\infty} \left|\tilde{a}_{jk,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \\
&\qquad\quad + \sup_{\epsilon\in[\tilde{\epsilon}-\frac{\rho}{2},\tilde{\epsilon}+\frac{\rho}{2}]} \sum_{k=0}^{\infty} \left|\tilde{a}_{0k,\epsilon} : \overline{B_{n-1}(0,1)} \times \left[\tilde{\zeta}-\frac{\rho}{4},\tilde{\zeta}+\frac{\rho}{4}\right]\right|_{\alpha} \\
&\qquad \leq \frac{8}{3}M + 2^{4-\frac{\alpha}{2}}M\rho^{-1} + 2M\,.
\end{aligned}$$

We deduce that (3.21) for $m=0$ holds with $C=\frac{14}{3}M+2^{4-\frac{\alpha}{2}}M\rho^{-1}$.

We now assume that the map $\mathcal{M}_A$ is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{1,\alpha}(\overline{B_{n-1}(0,1)})$ and we prove (3.21) for $m=1$. To do so we will exploit the (just proved) statement of Lemma 3.2.5 for $m=0$. We begin by observing that, since the embedding of $C^{1,\alpha}(\overline{B_{n-1}(0,1)})$ into $C^{0,\alpha}(\overline{B_{n-1}(0,1)})$ is linear and continuous, the map $\mathcal{M}_A$ is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{0,\alpha}(\overline{B_{n-1}(0,1)})$. Hence, by Lemma 3.2.5 for $m=0$ and by the continuity of the imbedding of $C^{0,\alpha}(\overline{B_{n-1}(0,1)}\times\overline{\mathcal{J}})$ into $C^{0}(\overline{B_{n-1}(0,1)}\times\overline{\mathcal{J}})$ we deduce that

$$\sup_{\epsilon\in\mathcal{E}} \|A(\epsilon,\cdot,\cdot)\|_{C^{0}(\overline{B_{n-1}(0,1)}\times\overline{\mathcal{J}})} \leq C_1\,. \tag{3.34}$$

Moreover, since differentials of real analytic maps are real analytic, we have that the map $\mathcal{M}_{\partial_\zeta A}=\partial_\zeta\mathcal{M}_A$ which takes (ϵ,ζ) to $\partial_\zeta A(\epsilon,\cdot,\zeta)$ is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{1,\alpha}(\overline{B_{n-1}(0,1)})$, and thus from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to

$C^{0,\alpha}(\overline{B_{n-1}(0,1)})$. By Lemma 3.2.5 for $m = 0$ it follows that

$$\sup_{\epsilon \in \mathcal{E}} \|\partial_\zeta A(\epsilon, \cdot, \cdot)\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)} \times \overline{\mathcal{J}})} \leq C_2, \tag{3.35}$$

for some $C_2 > 0$. Finally, we observe that the map ∂_z from $C^{1,\alpha}(\overline{B_{n-1}(0,1)})$ to $C^{0,\alpha}(\overline{B_{n-1}(0,1)})$ that takes a function f to $\partial_z f$ is linear and continuous. Then, the map $\mathcal{M}_{\partial_z A}$ which takes (ϵ, ζ) to the function

$$\partial_z A(\epsilon, z, \zeta) \qquad \forall z \in \overline{B_{n-1}(0,1)}$$

is the composition of $\mathcal{M}_A$ and ∂_z. Namely, we can write

$$\mathcal{M}_{\partial_z A} = \partial_z \circ \mathcal{M}_A \,.$$

Since $\mathcal{M}_A$ is real analytic from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ to $C^{1,\alpha}(\overline{B_{n-1}(0,1)})$, it follows that $\mathcal{M}_{\partial_z A}$ is real analytic from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ to $C^{0,\alpha}(\overline{B_{n-1}(0,1)})$. Hence Lemma 3.2.5 for $m = 0$ implies that there exists $C_3 > 0$ such that

$$\sup_{\epsilon \in \mathcal{E}} \|\partial_z A(\epsilon, \cdot, \cdot)\|_{C^{0,\alpha}(\overline{B_{n-1}(0,1)} \times \overline{\mathcal{J}})} \leq C_3 \,. \tag{3.36}$$

Now, the validity of (3.21) for $m = 1$ is a consequence of (3.34), (3.35), and (3.36). □

Then, by Lemma 3.2.5, we deduce the following Lemma 3.2.6.

Lemma 3.2.6. *Let B be a function from $]-\epsilon_0, \epsilon_0[\times\partial\Omega^i \times \mathbb{R}$ to $\mathbb{R}$. Let $\tilde{\mathcal{N}}_B$ be the map which takes a pair $(\epsilon, \zeta) \in]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ to the function $\tilde{\mathcal{N}}_B(\epsilon, \zeta)$ defined by*

$$\tilde{\mathcal{N}}_B(\epsilon, \zeta)(t) \equiv B(\epsilon, t, \zeta) \qquad \forall t \in \partial\Omega^i \,.$$

Let $m \in \{0, 1\}$. If $\tilde{\mathcal{N}}_B(\epsilon, \zeta) \in C^{m,\alpha}(\partial\Omega^i)$ for all $(\epsilon, \zeta) \in]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ and the *map* $\tilde{\mathcal{N}}_B$ *is real analytic from* $]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ *to* $C^{m,\alpha}(\partial\Omega^i)$*, then for every open bounded interval* $\mathcal{J}$ *of* $\mathbb{R}$ *and every compact subset* $\mathcal{E}$ *of* $]-\epsilon_0, \epsilon_0[$ *there exists* $C > 0$ *such that*

$$\sup_{\epsilon\in\mathcal{E}} \|B(\epsilon,\cdot,\cdot)\|_{C^{m,\alpha}(\partial\Omega^i\times\overline{\mathcal{J}})} \leq C\,. \tag{3.37}$$

Proof. Since $\partial\Omega^i$ is a compact sub-manifold of class $C^{1,\alpha}$ in $\mathbb{R}^n$, there exist a finite open covering $\mathcal{U}_1, \ldots, \mathcal{U}_k$ of $\partial\Omega^i$ and $C^{1,\alpha}$ local parametrization maps $\gamma_l\colon \overline{B_{n-1}(0,1)} \to \overline{\mathcal{U}_l}$ with $l = 1,\ldots,k$. Moreover, we can assume without loss of generality that the norm of $C^{m,\alpha}(\partial\Omega^i)$ is defined on the atlas $\{(\overline{\mathcal{U}_l}, \gamma_l^{(-1)})\}_{l=1,\ldots,k}$ and the norm of $C^{m,\alpha}(\partial\Omega^i\times\overline{\mathcal{J}})$ is defined on the atlas $\{(\overline{\mathcal{U}_l}\times\overline{\mathcal{J}}, (\gamma_l^{(-1)}, \mathrm{id}_{\overline{\mathcal{J}}}))\}_{l=1,\ldots,k}$, where $\mathrm{id}_{\overline{\mathcal{J}}}$ is the identity map from $\overline{\mathcal{J}}$ to itself. Then, in order to prove (3.37) it suffices to show that

$$\sup_{\epsilon\in\mathcal{E}} \|B(\epsilon,\gamma_l(\cdot),\cdot)\|_{C^{m,\alpha}(\overline{B_{n-1}(0,1)}\times\overline{\mathcal{J}})} \leq C \qquad \forall l\in\{1,\ldots,k\} \tag{3.38}$$

for some $C > 0$. Let $l \in \{1,\ldots,k\}$ and let A be the map from $]-\epsilon_0, \epsilon_0[\times\overline{B_{n-1}(0,1)}\times\mathbb{R}$ to $\mathbb{R}$ defined by

$$A(\epsilon,z,\zeta) = B(\epsilon,\gamma_l(z),\zeta) \qquad \forall(\epsilon,z,\zeta)\in]-\epsilon_0,\epsilon_0[\times\overline{B_{n-1}(0,1)}\times\mathbb{R}\,. \tag{3.39}$$

Then, with the notation of Lemma 3.2.5, we have

$$\mathcal{M}_A(\epsilon,\zeta) = \gamma_l^*\left(\tilde{\mathcal{N}}_B(\epsilon,\zeta)_{|\overline{\mathcal{U}_l}}\right),$$

where $\gamma_l^*\left(\tilde{\mathcal{N}}_B(\epsilon,\zeta)_{|\overline{\mathcal{U}_l}}\right)$ is the pull back of the restriction $\tilde{\mathcal{N}}_B(\epsilon,\zeta)_{|\overline{\mathcal{U}_l}}$ by the parametrization γ_l. Since the restriction map from $C^{m,\alpha}(\partial\Omega^i)$ to $C^{m,\alpha}(\overline{\mathcal{U}_l})$ and the pullback map γ_l^* from $C^{m,\alpha}(\overline{\mathcal{U}_l})$ to $C^{m,\alpha}(\overline{B_{n-1}(0,1)})$ are linear and continuous and since $\mathcal{N}_B$ is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega^i)$, it follows that the map $\mathcal{M}_A$ is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{m,\alpha}(\overline{B_{n-1}(0,1)})$.

Then Lemma 3.2.5 implies that

$$\sup_{\epsilon\in\mathcal{E}} \|A(\epsilon,\cdot,\cdot)\|_{C^{m,\alpha}(\overline{B_{n-1}(0,1)}\times\overline{\mathcal{J}})} \leq C \tag{3.40}$$

for some $C > 0$. Now the validity of (3.38) follows by (3.39) and (3.40). The proof is complete. □

3.3 The auxiliary maps N and S

In the proof of our main Theorem 3.4.1 we will exploit two auxiliary maps, which we denote by N and S and are defined as follows. Let ϵ' be as in Theorem 2.6.5. We denote by $N = (N_1, N_2, N_3)$ the map from $]-\epsilon',\epsilon'[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)$ defined by

$$\begin{aligned} N_1[\epsilon,\phi^o,\phi^i,\zeta,\psi^i](x) \equiv & \left(\frac{1}{2}I + W_{\partial\Omega^o}\right)[\phi^o](x) \\ & - \epsilon^{n-1}\int_{\partial\Omega^i}\nu_{\Omega^i}(y)\cdot\nabla S_n(x-\epsilon y)\phi^i(y)\,d\sigma_y \\ & + \epsilon^{n-2}\zeta S_n(x) \qquad \forall x\in\partial\Omega^o, \end{aligned} \tag{3.41}$$

$$\begin{aligned} N_2[\epsilon,\phi^o,\phi^i,\zeta,\psi^i](t) \equiv & \left(-\frac{1}{2}I + W_{\partial\Omega^i}\right)[\phi^i](t) \\ & + \zeta\, S_n(t) + w^+_{\Omega^o}[\phi^o](\epsilon t) \\ & - (\partial_\zeta F)(0,t,\zeta^i)\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) \qquad \forall t\in\partial\Omega^i, \end{aligned} \tag{3.42}$$

$$\begin{aligned} N_3[\epsilon,\phi^o,\phi^i,\zeta,\psi^i](t) \equiv & \nu_{\Omega^i}(t)\cdot\Big(\epsilon\nabla w^+_{\Omega^o}[\phi^o](\epsilon t) + \nabla w^-_{\Omega^i}[\phi^i](t) \\ & + \zeta\nabla S_n(t) - \nabla w^+_{\Omega^i}[\psi^i](t)\Big) \qquad \forall t\in\partial\Omega^i, \end{aligned} \tag{3.43}$$

for all $(\epsilon, \phi^o, \phi^i, \zeta, \psi^i) \in]-\epsilon', \epsilon'[\times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ and we denote by $S = (S_1, S_2, S_3)$ the map from $]-\epsilon', \epsilon'[\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)$ defined by

$$S_1[\epsilon, \psi^i](x) \equiv 0 \qquad \forall x \in \partial\Omega^o, \tag{3.44}$$

$$\begin{aligned} S_2[\epsilon, \psi^i](t) \equiv & -t \cdot \nabla u^o(0) - \epsilon u^o(\epsilon t) + (\partial_\epsilon F)(0, t, \zeta^i) \\ & + \epsilon \tilde{F}\left(\epsilon, t, \zeta^i, \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t)\right) \qquad \forall t \in \partial\Omega^i, \end{aligned} \tag{3.45}$$

$$\begin{aligned} S_3[\epsilon, \psi^i](t) \equiv & -\nu_{\Omega^i}(t) \cdot \nabla u^o(\epsilon t) \\ & + G\left(\epsilon, t, \epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) + \zeta^i\right) \qquad \forall t \in \partial\Omega^i, \end{aligned} \tag{3.46}$$

for all $(\epsilon, \psi^i) \in]-\epsilon', \epsilon'[\times C^{1,\alpha}(\partial\Omega^i)$.

For the maps N and S we have the following result.

Proposition 3.3.1. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Then there exists $\epsilon'' \in]0, \epsilon'[$ such that the following statements hold.*

(i) For all fixed $\epsilon \in]-\epsilon'', \epsilon''[$ the operator $N[\epsilon, \cdot, \cdot, \cdot, \cdot]$ is a linear homeomorphism from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)$;

(ii) The map from $]-\epsilon'', \epsilon''[$ to

$$\mathcal{L}(C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i))$$

which takes ϵ to $N[\epsilon, \cdot, \cdot, \cdot, \cdot]^{(-1)}$ is real analytic;

(iii) Equation (2.43) *is equivalent to*

$$(\phi^o, \phi^i, \zeta, \psi^i) = N[\epsilon, \cdot, \cdot, \cdot, \cdot]^{(-1)}[S[\epsilon, \psi^i]] \tag{3.47}$$

for all $(\epsilon, \phi^o, \phi^i, \zeta, \psi^i) \in]-\epsilon'', \epsilon''[\times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$.

Proof. By the definition of N (cf. (3.41)–(3.43)), by the mapping properties of the double layer potential (cf. Theorem 1.3.4 (iii) and Theorem 1.3.5 (ii)) and of integral operators with real analytic kernels and no singularity (see Theorem A.2.1 (ii) in Appendix A.2), by assumption (2.42) (which implies that $(\partial_\zeta F)(0,\cdot,\zeta^i)$ belongs to $C^{1,\alpha}(\partial\Omega^i)$), and by standard calculus in Banach spaces, one verifies that the map from $]-\epsilon',\epsilon'[$ to

$$\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)\,,\ C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i))$$

which takes ϵ to $N[\epsilon,\cdot,\cdot,\cdot,\cdot]$ is real analytic. Then one observes that

$$N[0,\phi^o,\phi^i,\zeta,\psi^i]=\partial_{(\phi^o,\phi^i,\zeta,\psi^i)}M[0,\phi^o_0,\phi^i_0,\zeta_0,\psi^i_0].(\phi^o,\phi^i,\zeta,\psi^i)$$

and thus Lemma 2.6.4 implies that $N[0,\cdot,\cdot,\cdot,\cdot]$ is an isomorphism from $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^i)$. Since the set of invertible operators is open in $\mathcal{L}(C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i))$ and since the map which takes a linear invertible operator to its inverse is real analytic (cf. Hille and Phillips [37]), we deduce the validity of (i) and (ii). To prove (iii) we observe that, by the definition of N in (3.41)–(3.43) and by the definition of S in (3.44)–(3.46), it readily follows that (2.43) is equivalent to

$$N[\epsilon,\phi^o,\phi^i,\zeta,\psi^i]=S[\epsilon,\psi^i]\,.$$

Then the validity of (iii) is a consequence of (i). □

3.4 A stronger local uniqueness result for the solution $(u^o_\epsilon, u^i_\epsilon)$

In this section we will prove our main Theorem 3.4.1 on the local uniqueness of the solution $(u^o_\epsilon, u^i_\epsilon)$ provided by Theorem 2.7.2. In particular, we will prove that the local uniqueness of the solution can be achieved weakening the assumptions of Theorem 3.1.2: only one condition on the trace of the function $v^i(\epsilon\cdot)$ on $\partial\Omega^i$ is required, instead of the three conditions used in Theorem 3.1.2.

We are now ready to state and prove our main result of this chapter.

Theorem 3.4.1. *Let assumptions* (2.13)*,* (2.18)*,* (2.42) *and* (2.44) *hold true. Let* $\epsilon' \in]0, \epsilon_0[$ *be as in Theorem 2.6.5. Let* $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ *be as in Theorem 2.7.2. Then there exist* $\epsilon^* \in]0, \epsilon'[$ *and* $\delta^* \in]0, +\infty[$ *such that the following property holds:*

If $\epsilon \in]0, \epsilon^*[$ *and* $(v^o, v^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\epsilon\overline{\Omega^i})$ *is a solution of problem* (2.2) *with*

$$\left\| v^i(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} < \epsilon\delta^*,$$

then

$$(v^o, v^i) = (u^o_\epsilon, u^i_\epsilon).$$

Proof. • *Step 1: Fixing* ϵ^**.*

Let $\epsilon'' \in]0, \epsilon'[$ be as in Proposition 3.3.1 and let $\epsilon''' \in]0, \epsilon''[$ be fixed. By the compactness of $[-\epsilon''', \epsilon''']$ and by the continuity of the norm in $\mathcal{L}(C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o), C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i))$, there exists a real number $C_1 > 0$ such that

$$\|N[\epsilon,\cdot,\cdot,\cdot,\cdot]^{(-1)}\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^o),C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i))} \leq C_1 \tag{3.48}$$

for all $\epsilon \in [-\epsilon''', \epsilon''']$ (see also Proposition 3.3.1 (ii)). Let U_0 be the open neighborhood of $(\phi^o_0, \phi^i_0, \zeta_0, \psi^i_0)$ in $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ introduced in Theorem 2.6.5. Then we take $K > 0$ such that

$$\overline{\mathcal{B}_{0,K}} \subseteq U_0$$

(see (3.1) for the definition of $\mathcal{B}_{0,K}$). Since $(\Phi^o[\cdot], \Phi^i[\cdot], Z[\cdot], \Psi^i[\cdot])$ is continuous (indeed real analytic) from $]-\epsilon', \epsilon'[$ to U_0, there exists $\epsilon^* \in]0, \epsilon'''[$ such that

$$(\Phi^o[\eta], \Phi^i[\eta], Z[\eta], \Psi^i[\eta]) \in \mathcal{B}_{0,K/2} \subset U_0 \qquad \forall \eta \in]0, \epsilon^*[\,. \tag{3.49}$$

Moreover, we assume that

$$\epsilon^* < 1.$$

We will prove that the theorem holds for such choice of ϵ^*. We observe that the condition $\epsilon^* < 1$ is not really needed in the proof but simplifies many computations.

- *Step 2: Planning our strategy.*

We suppose that there exists a pair of functions $(v^o, v^i) \in C^{1,\alpha}(\overline{\Omega(\epsilon)}) \times C^{1,\alpha}(\epsilon\overline{\Omega^i})$ that is a solution of problem (2.2) for a certain $\epsilon \in]0, \epsilon^*[$ (fixed) and such that

$$\left\| \frac{v^i(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot)}{\epsilon} \right\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \delta^*\,, \tag{3.50}$$

for some $\delta^* \in]0,+\infty[$. Then, by Proposition 2.1.3, there exists a unique quadruple $(\phi^o, \phi^i, \zeta, \psi^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ such that

$$\begin{aligned} v^o &= U^o_\epsilon[\phi^o, \phi^i, \zeta, \psi^i] \quad \text{in } \overline{\Omega(\epsilon)}, \\ v^i &= U^i_\epsilon[\phi^o, \phi^i, \zeta, \psi^i] \quad \text{in } \epsilon\overline{\Omega^i}. \end{aligned} \tag{3.51}$$

We shall show that for δ^* small enough we have

$$(\phi^o, \phi^i, \zeta, \psi^i) = (\Phi^o[\epsilon], \Phi^i[\epsilon], Z[\epsilon], \Psi^i[\epsilon]) \,. \tag{3.52}$$

Indeed, if we have (3.52), then Definition 2.7.1 would imply that

$$(v^o, v^i) = (u^o_\epsilon, v^i_\epsilon),$$

and our proof would be completed. Moreover, to prove (3.52) it suffices to show that

$$(\phi^o, \phi^i, \zeta, \psi^i) \in \mathcal{B}_{0,K} \subset U_0 \,. \tag{3.53}$$

In fact, in that case, both $(\epsilon, \phi^o, \phi^i, \zeta, \psi^i)$ and $(\epsilon, \Phi^o[\epsilon], \Phi^i[\epsilon], Z[\epsilon], \Psi^i[\epsilon])$ would stay in the zero set of M (cf. Proposition 2.6.1 and Theorem 2.6.5) and thus (3.53) together with (3.49) and Theorem 2.6.5 would imply (3.52).

So, our aim is now to prove that (3.53) holds true for a suitable choice of $\delta^* > 0$. It will be also convenient to restrict our search to

$$0 < \delta^* < 1.$$

As for the condition $\epsilon^* < 1$, this condition on δ^* is not really needed, but simplifies our computations. Then to find δ^* and prove (3.53) we will proceed as follows. First we obtain an estimate for ψ^i and $\Psi^i[\epsilon]$ with a bound that does not depend on ϵ and δ^*. Then we use such estimate to show that

$$\|S[\epsilon, \psi^i] - S[\epsilon, \Psi^i[\epsilon]]\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)}$$

is smaller than a constant times δ^*, with a constant that does not depend on ϵ and δ^*. We will split the analysis for S_1, S_2, and S_3 and we find convenient to study S_3 before S_2. Indeed, the computations for S_2 and S_3 are very similar but those for S_3 are much shorter and can serve better to illustrate the techniques employed. We also observe that the analysis for S_2 requires the study of other auxiliary functions T_1, T_2, and T_3 that we will introduce. Finally, we will exploit the estimate for $\|S[\epsilon, \psi^i] - S[\epsilon, \Psi^i[\epsilon]]\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)}$ to determine δ^* and prove (3.53).

- *Step 3: Estimate for ψ and $\Psi[\epsilon]$.*

By condition (3.50), by the second equality in (3.51), by Definition 2.7.1, and by arguing as in (3.10) and (3.12) in Theorem 3.1.2, we obtain

$$\left\|\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] - \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \delta^* \tag{3.54}$$

and

$$\begin{aligned}\|\psi^i - \Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)} \leq & \left\|\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)^{(-1)}\right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))} \\ & \times \left\|\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] - \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)} \\ \leq & \, C_2\delta^*,\end{aligned} \tag{3.55}$$

where

$$C_2 \equiv \left\| \left(\frac{1}{2} I + W_{\partial\Omega^i} \right)^{(-1)} \right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^i))}.$$

By (3.49) we have

$$\|\psi_0^i - \Psi^i[\eta]\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \frac{K}{2} \qquad \forall \eta \in]0, \epsilon^*[. \tag{3.56}$$

Then, by (3.55) and (3.56), and by the triangle inequality, we see that

$$\begin{aligned} \|\psi^i\|_{C^{1,\alpha}(\partial\Omega^i)} &\leq \|\psi_0^i\|_{C^{1,\alpha}(\partial\Omega^i)} + \|\psi^i - \Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)} + \|\Psi^i[\epsilon] - \psi_0^i\|_{C^{1,\alpha}(\partial\Omega^i)} \\ &\leq \|\psi_0^i\|_{C^{1,\alpha}(\partial\Omega^i)} + C_2\, \delta^* + \frac{K}{2}, \\ \|\Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)} &\leq \|\psi_0^i\|_{C^{1,\alpha}(\partial\Omega^i)} + \frac{K}{2}. \end{aligned}$$

Then, by taking $R_1 \equiv \|\psi_0^i\|_{C^{1,\alpha}(\partial\Omega^i)} + C_2 + \frac{K}{2}$ and $R_2 \equiv \|\psi_0^i\|_{C^{1,\alpha}(\partial\Omega^i)} + \frac{K}{2}$ (and recalling that $\delta^* \in]0,1[$), one verifies that

$$\|\psi^i\|_{C^{1,\alpha}(\partial\Omega^i)} \leq R_1 \qquad \text{and} \qquad \|\Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)} \leq R_2\,. \tag{3.57}$$

We note here that both R_1 and R_2 do not depend on ϵ and δ^* as long they belong to $]0, \epsilon^*[$ and $]0,1[$, respectively.

- *Step 4: Estimate for* S_1.

We now pass to estimate the norm

$$\|S[\epsilon, \psi^i] - S[\epsilon, \Psi^i[\epsilon]]\|_{C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)}.$$

To do so we consider separately S_1, S_2, and S_3. Since $S_1 = 0$ (cf. definition (3.44)), we readily obtain that

$$\|S_1[\epsilon, \psi^i] - S_1[\epsilon, \Psi^i[\epsilon]]\|_{C^{1,\alpha}(\partial\Omega^o)} = 0. \tag{3.58}$$

- *Step 5: Estimate for* S_3.

We consider S_3 before S_2 because its treatment is simpler and more illustrative of the techniques used. By (3.46) and by the Mean Value Theorem in Banach space (see, e.g., Ambrosetti and Prodi [1, Thm. 1.8]), we compute that

$$
\begin{aligned}
&\|S_3[\epsilon,\psi^i] - S_3[\epsilon,\Psi^i[\epsilon]]\|_{C^{0,\alpha}(\partial\Omega^i)} \\
&\qquad = \Big\| G\left(\epsilon,\cdot,\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) \\
&\qquad\quad - G\left(\epsilon,\cdot,\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right)\Big\|_{C^{0,\alpha}(\partial\Omega^i)} \\
&\qquad = \Big\| \mathcal{N}_G\left(\epsilon,\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) \\
&\qquad\quad - \mathcal{N}_G\left(\epsilon,\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right)\Big\|_{C^{0,\alpha}(\partial\Omega^i)} \\
&\qquad \leq \left\| d_v\mathcal{N}_G(\epsilon,\tilde{\psi}^i)\right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{0,\alpha}(\partial\Omega^i))} \\
&\qquad\quad \times \epsilon \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] - \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{0,\alpha}(\partial\Omega^i)},
\end{aligned}
\tag{3.59}
$$

where

$$
\tilde{\psi}^i = \theta\left(\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) + (1-\theta)\left(\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right),
$$

for some $\theta \in]0,1[$. Then, by the membership of ϵ and θ in $]0,1[$ we have

$$
\begin{aligned}
\|\tilde{\psi}^i\|_{C^{1,\alpha}(\partial\Omega^i)} \leq & \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
& + \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right\|_{C^{1,\alpha}(\partial\Omega^i)}
\end{aligned}
$$

and, by setting

$$
C_3 \equiv \left\| \frac{1}{2}I + W_{\partial\Omega^i}\right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))},
$$

we obtain

$$\|\tilde{\psi}^i\|_{C^{1,\alpha}(\partial\Omega^i)} \leq C_3\|\psi^i\|_{C^{1,\alpha}(\partial\Omega^i)} + C_3\|\Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)} + 2|\zeta^i| \leq R, \qquad (3.60)$$

with

$$R \equiv C_3(R_1 + R_2) + 2|\zeta^i| \qquad (3.61)$$

which does not depend on ϵ. We wish now to estimate the operator norm

$$\left\| d_v \mathcal{N}_G(\eta, \tilde{\psi}^i) \right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^i))}$$

uniformly for $\eta \in]0, \epsilon^*[$. However, we cannot exploit a compactness argument on $[0, \epsilon^*] \times \overline{B_{C^{1,\alpha}(\partial\Omega^i)}(0, R)}$, because $\overline{B_{C^{1,\alpha}(\partial\Omega^i)}(0, R)}$ is not compact in the infinite dimension space $C^{1,\alpha}(\partial\Omega^i)$. Then we argue as follows. We observe that, by assumption (2.44), the partial derivative $\partial_\zeta G(\eta, t, \zeta)$ exists for all $(\eta, t, \zeta) \in] - \epsilon_0, \epsilon_0[\times \partial\Omega^i \times \mathbb{R}$ and, by Remark 2.5.4 and Lemma 3.2.2 (i), we obtain that

$$\begin{aligned} \|d_v \mathcal{N}_G(\eta, \tilde{\psi}^i)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^i))} &\leq \|\mathcal{N}_{\partial_\zeta G}(\eta, \tilde{\psi}^i)\|_{C^{0,\alpha}(\partial\Omega^i)} \\ &\leq \|\partial_\zeta G(\eta, \cdot, \tilde{\psi}^i(\cdot))\|_{C^{0,\alpha}(\partial\Omega^i)} \end{aligned} \qquad (3.62)$$

for all $\eta \in]0, \epsilon^*[$. By Proposition 3.2.4 (i), there exists $C_4 > 0$ such that

$$\|\partial_\zeta G(\eta, \cdot, \tilde{\psi}^i(\cdot))\|_{C^{0,\alpha}(\partial\Omega^i)} \leq C_4 \|\partial_\zeta G(\eta, \cdot, \cdot)\|_{C^{0,\alpha}(\partial\Omega \times [-R,R])} \left(1 + \|\tilde{\psi}^i\|^\alpha_{C^{1,\alpha}(\partial\Omega^i)}\right) \quad \forall \eta \in]0, \epsilon^*[. \qquad (3.63)$$

Moreover, by assumption (2.44) one deduces that the map $\tilde{\mathcal{N}}_G$ defined as in Lemma 3.2.6 (with $B = G$) is real analytic from $] - \epsilon_0, \epsilon_0[\times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega^i)$ and, by Remark 2.5.4, one has that $\partial_\zeta \tilde{\mathcal{N}}_G = \tilde{\mathcal{N}}_{\partial_\zeta G}$. Hence, by Lemma 3.2.6

(with $m = 0$), there exists $C_5 > 0$ (which does not depend on $\epsilon \in]0, \epsilon^*[$ and $\delta^* \in]0, 1[$) such that

$$\sup_{\eta \in [-\epsilon^*, \epsilon^*]} \|\partial_\zeta G(\eta, \cdot, \cdot)\|_{C^{0,\alpha}(\partial\Omega^i \times [-R,R])} \leq C_5. \tag{3.64}$$

Hence, by (3.60), (3.62), (3.63) and (3.64), we deduce that

$$\|d_v \mathcal{N}_G(\epsilon, \tilde{\psi}^i)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{0,\alpha}(\partial\Omega^i))} \leq C_4\, C_5\, (1 + R^\alpha). \tag{3.65}$$

By (3.54), (3.59), and (3.65), and by the membership of ϵ in $]0, \epsilon^*[\subset]0, 1[$, we obtain that

$$\|S_3[\epsilon, \psi^i] - S_3[\epsilon, \Psi^i[\epsilon]]\|_{C^{0,\alpha}(\partial\Omega^i)} \leq C_4\, C_5\, (1 + R^\alpha)\, \delta^*. \tag{3.66}$$

- *Step 6: Estimate for S_2.*

Finally, we consider S_2. By (3.45) and by the fact that $\epsilon \in]0, 1[$, we have

$$\begin{aligned}
&\|S_2[\epsilon, \psi^i] - S_2[\epsilon, \Psi^i[\epsilon]]\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \leq \epsilon \left\| \tilde{F}\left(\epsilon, \cdot, \zeta^i, \left(\frac{1}{2} I + W_{\partial\Omega^i}\right)[\psi^i]\right) \right. \\
&\qquad\quad \left. - \tilde{F}\left(\epsilon, \cdot, \zeta^i, \left(\frac{1}{2} I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right) \right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \leq \left\| \int_0^1 (1-\tau) \Big\{ T_1[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) \right. \\
&\qquad\quad \left. + 2\, T_2[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) + T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) \Big\}\, d\tau \right\|_{C^{1,\alpha}(\partial\Omega^i)},
\end{aligned} \tag{3.67}$$

where $T_1[\epsilon, \psi^i, \Psi^i[\epsilon]]$, $T_2[\epsilon, \psi^i, \Psi^i[\epsilon]]$, and $T_3[\epsilon, \psi^i, \Psi^i[\epsilon]]$ are the functions from

$]0,1[\times\partial\Omega^i$ to $\mathbb{R}$ defined by

$$\begin{aligned}
&T_1[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,t)\\
&\equiv (\partial_\epsilon^2 F)\left(\tau\epsilon,t,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)+\zeta^i\right)\\
&\quad-(\partial_\epsilon^2 F)\left(\tau\epsilon,t,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]](t)+\zeta^i\right), \qquad (3.68)\\
&T_2[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,t)\\
&\equiv \left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)\,(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,t,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)+\zeta^i\right)\\
&\quad-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]](t)(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,t,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]](t)+\zeta^i\right), \qquad (3.69)\\
&T_3[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,t)\\
&\equiv \left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]^2(t)\,(\partial_\zeta^2 F)\left(\tau\epsilon,t,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i](t)+\zeta^i\right)\\
&\quad-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2(t)(\partial_\zeta^2 F)\left(\tau\epsilon,t,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]](t)+\zeta^i\right) \qquad (3.70)
\end{aligned}$$

for every $(\tau,t)\in]0,1[\times\partial\Omega^i$.

We now want to bound the $C^{1,\alpha}$ norm with respect to the variable $t\in\partial\Omega^i$ of (3.68), (3.69), and (3.70) uniformly with respect to $\tau\in]0,1[$. By doing that, we will obtain an estimate for the norm

$$\left\|T_1[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)+2\,T_2[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)+T_3[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)\,d\tau\right\|_{C^{1,\alpha}(\partial\Omega^i)},$$

henceforth a bound for (3.67).

- *Step 6.1: Estimate for T_1.*

First we consider

$$T_1[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot).$$

By the Mean Value Theorem in Banach space (see, e.g., Ambrosetti and Prodi [1, Thm. 1.8]), we can estimate the $C^{1,\alpha}(\partial\Omega^i)$ norm of $T_1[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot)$ (cf. (3.68)) as follows:

$$\begin{aligned}
&\|T_1[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot)\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad = \Big\| \mathcal{N}_{\partial^2_\epsilon F}\left(\tau\epsilon, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) \\
&\qquad\quad - \mathcal{N}_{\partial^2_\epsilon F}\left(\tau\epsilon, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right)\Big\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \leq \left\| d_v\mathcal{N}_{\partial^2_\epsilon F}(\tau\epsilon, \tilde{\psi}^i_1)\right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i), C^{1,\alpha}(\partial\Omega^i))} \\
&\qquad\quad \times \tau\epsilon \left\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] - \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)}
\end{aligned} \tag{3.71}$$

where

$$\tilde{\psi}^i_1 = \theta_1\left(\tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) + (1-\theta_1)\left(\tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right),$$

for some $\theta_1 \in]0, 1[$.

- *Step 6.2: Estimate for T_2.*

We now consider

$$T_2[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot).$$

Adding and subtracting

$$\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\,(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon, t, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) + \zeta^i\right)$$

in the right hand side of (3.69) and by using the triangle inequality, we obtain

$$
\begin{aligned}
&\|T_2[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\leq \left\| \left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]\,(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right.\\
&\left. -\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\,(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&+\left\| \left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\,(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right.\\
&\left. -\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]](\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}.
\end{aligned}
\tag{3.72}
$$

By Lemma 3.2.2 (ii) and by the Mean Value Theorem in Banach space (see, e.g., Ambrosetti and Prodi [1, Thm. 1.8]), we can estimate the $C^{1,\alpha}(\partial\Omega^i)$ norm of $T_2[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)$ (cf. (3.69) and (3.72)) as follows:

$$
\begin{aligned}
&\|T_2[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\leq 2\left\|(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\quad\times\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\quad+2\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)}\left\|\mathcal{N}_{\partial_\epsilon\partial_\zeta F}\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right.\\
&\quad\left.-\mathcal{N}_{\partial_\epsilon\partial_\zeta F}\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\leq 2\left\|(\partial_\epsilon\partial_\zeta F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\quad\times\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\quad+2\,C_3\,\|\Psi^i[\epsilon]\|_{C^{1,\alpha}(\partial\Omega^i)}\left\|d_v\mathcal{N}_{\partial_\epsilon\partial_\zeta F}(\tau\epsilon,\tilde{\psi}^i_2)\right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))}\\
&\quad\times\tau\epsilon\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)},
\end{aligned}
\tag{3.73}
$$

where

$$\tilde{\psi}^i_{,2} = \theta_2 \left(\tau\epsilon \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) + (1-\theta_2)\left(\tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right),$$

for some $\theta_2 \in]0,1[$.

- *Step 6.3: Estimate for T_3.*

Finally we consider

$$T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot).$$

Adding and subtracting the term

$$\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2 \, (\partial^2_\zeta F)\left(\tau\epsilon, t, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i](t) + \zeta^i\right)$$

in the right hand side of (3.70) and using the triangle inequality, we obtain

$$\begin{aligned}
&\|T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot)\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\leq \Bigg\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i]^2 \, (\partial^2_\zeta F)\left(\tau\epsilon, \cdot, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) \\
&- \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2 \, (\partial^2_\zeta F)\left(\tau\epsilon, \cdot, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right)\Bigg\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&+ \Bigg\| \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2 \, (\partial^2_\zeta F)\left(\tau\epsilon, \cdot, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right) \\
&- \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2 (\partial^2_\zeta F)\left(\tau\epsilon, \cdot, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]] + \zeta^i\right)\Bigg\|_{C^{1,\alpha}(\partial\Omega^i)} .
\end{aligned} \tag{3.74}$$

By Lemma 3.2.2 (ii) and by the Mean Value Theorem in Banach space (see, e.g., Ambrosetti and Prodi [1, Thm. 1.8]), we can estimate the $C^{1,\alpha}(\partial\Omega^i)$

norm of $T_3[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)$ (cf. (3.70) and (3.74)) as follows:

$$\begin{aligned}
&\|T_3[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad\le 2\left\|(\partial_\zeta^2 F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad\times\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]^2-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad+2\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]^2\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad\times\left\|\mathcal{N}_{\partial_\zeta^2 F}\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right.\\
&\qquad\left.-\mathcal{N}_{\partial_\zeta^2 F}\left(\tau\epsilon,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad\le 4\left\|(\partial_\zeta^2 F)\left(\tau\epsilon,\cdot,\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad\times\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad\times\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad+4C_3^2\,\|\Psi^i[\epsilon]\|^2_{C^{1,\alpha}(\partial\Omega^i)}\left\|d_v\mathcal{N}_{\partial_\zeta^2 F}(\tau\epsilon,\tilde{\psi}_3^i)\right\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))}\\
&\qquad\times\tau\epsilon\left\|\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]-\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)},
\end{aligned}\tag{3.75}$$

where

$$\tilde{\psi}_3^i=\theta_3\left(\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\psi^i]+\zeta^i\right)+(1-\theta_3)\left(\tau\epsilon\left(\frac{1}{2}I+W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]+\zeta^i\right),$$

for some $\theta_3\in]0,1[$. Let R be as in (3.61). By the same argument used to prove (3.60), one verifies the inequalities

$$\|\tilde{\psi}_1^i\|_{C^{1,\alpha}(\partial\Omega^i)}\le R,\quad \|\tilde{\psi}_2^i\|_{C^{1,\alpha}(\partial\Omega^i)}\le R,\quad \|\tilde{\psi}_3^i\|_{C^{1,\alpha}(\partial\Omega^i)}\le R.\tag{3.76}$$

By assumption (2.42), the partial derivatives $\partial_\zeta \partial^2_\epsilon F(\eta,t,\zeta)$, $\partial_\zeta\partial_\epsilon\partial_\zeta F(\eta,t,\zeta)$ and $\partial_\zeta \partial^2_\zeta F(\eta,t,\zeta)$ exist for all $(\eta,t,\zeta)\in]-\epsilon_0,\epsilon_0[\times\partial\Omega^i\times\mathbb{R}$ and by Remark 2.5.4 and Lemma 3.2.2 (ii), we obtain

$$\begin{aligned}
\|d_v\mathcal{N}_{\partial^2_\epsilon F}(\tau\eta,\tilde{\psi}^i_1)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))} &\le 2\|\mathcal{N}_{\partial_\zeta\partial^2_\epsilon F}(\tau\eta,\tilde{\psi}^i_1)\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&= 2\,\|\partial_\zeta\partial^2_\epsilon F(\tau\eta,\cdot,\tilde{\psi}^i_1(\cdot))\|_{C^{1,\alpha}(\partial\Omega^i)}\,,\\
\|d_v\mathcal{N}_{\partial_\epsilon\partial_\zeta F}(\tau\eta,\tilde{\psi}^i_2)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))} &\le 2\|\mathcal{N}_{\partial_\zeta\partial_\epsilon\partial_\zeta F}(\tau\eta,\tilde{\psi}^i_2)\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&= 2\,\|\partial_\zeta\partial_\epsilon\partial_\zeta F(\tau\eta,\cdot,\tilde{\psi}^i_2(\cdot))\|_{C^{1,\alpha}(\partial\Omega^i)}\,,\\
\|d_v\mathcal{N}_{\partial^2_\zeta F}(\tau\eta,\tilde{\psi}^i_3)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))} &\le \|\mathcal{N}_{\partial_\zeta\partial^2_\zeta F}(\tau\eta,\tilde{\psi}^i_3)\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&= 2\|\partial_\zeta\partial^2_\zeta F(\tau\eta,\cdot,\tilde{\psi}^i_3(\cdot))\|_{C^{1,\alpha}(\partial\Omega^i)}\,,
\end{aligned} \tag{3.77}$$

for all $\eta\in]0,\epsilon^*[$. By Proposition 3.2.4 (ii), there exists $C_6>0$ such that

$$\begin{aligned}
&\|\partial_\zeta\partial^2_\epsilon F(\tau\eta,\cdot,\tilde{\psi}^i_1(\cdot))\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad \le C_6\|\partial_\zeta\partial^2_\epsilon F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}\left(1+\|\tilde{\psi}^i_1\|_{C^{1,\alpha}(\partial\Omega^i)}\right)^2,\\
&\|\partial_\zeta\partial_\epsilon\partial_\zeta F(\tau\eta,\cdot,\tilde{\psi}^i_2(\cdot))\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad \le C_6\|\partial_\zeta\partial_\epsilon\partial_\zeta F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega^i\times[-R,R])}\left(1+\|\tilde{\psi}^i_2\|_{C^{1,\alpha}(\partial\Omega^i)}\right)^2,\\
&\|\partial_\zeta\partial^2_\zeta F(\tau\eta,\cdot,\tilde{\psi}^i_3(\cdot))\|_{C^{1,\alpha}(\partial\Omega^i)}\\
&\qquad \le C_6\|\partial_\zeta\partial^2_\zeta F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega^i\times[-R,R])}\left(1+\|\tilde{\psi}^i_3\|_{C^{1,\alpha}(\partial\Omega^i)}\right)^2,
\end{aligned} \tag{3.78}$$

for all $\eta\in]0,\epsilon^*[$. Now, by assumption (2.42) one deduces that the map $\tilde{\mathcal{N}}_F$ defined as in Lemma 3.2.6 (with $B=F$) is real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^i)$. Then one verifies that also the maps

$$\partial^2_\epsilon\partial_\zeta\tilde{\mathcal{N}}_F=\tilde{\mathcal{N}}_{\partial^2_\epsilon\partial_\zeta F},\quad \partial_\epsilon\partial^2_\zeta\tilde{\mathcal{N}}_F=\tilde{\mathcal{N}}_{\partial_\epsilon\partial^2_\zeta F},\quad \partial^3_\zeta\tilde{\mathcal{N}}_F=\tilde{\mathcal{N}}_{\partial^3 F},$$
$$\partial_\epsilon\partial_\zeta\tilde{\mathcal{N}}_F=\tilde{\mathcal{N}}_{\partial_\epsilon\partial_\zeta F},\quad \partial^2_\zeta\tilde{\mathcal{N}}_F=\tilde{\mathcal{N}}_{\partial^2_\zeta F}$$

are real analytic from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega^i)$. Hence, Lemma 3.2.6 (with $m=1$) implies that there exists $C_7>0$ such that

$$\begin{aligned}
&\sup_{\eta\in[-\epsilon^*,\epsilon^*]}\|\partial_\zeta\partial_\epsilon^2 F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}\le C_7,\\
&\sup_{\eta\in[-\epsilon^*,\epsilon^*]}\|\partial_\zeta\partial_\epsilon\partial_\zeta F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}\le C_7,\\
&\sup_{\eta\in[-\epsilon^*,\epsilon^*]}\|\partial_\zeta\partial_\zeta^2 F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega\times[-R,R])}\le C_7,\\
&\sup_{\eta\in[-\epsilon^*,\epsilon^*]}\|\partial_\epsilon\partial_\zeta F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega\times[\zeta^i-C_3R,\zeta^i+C_3R])}\le C_7,\\
&\sup_{\eta\in[-\epsilon^*,\epsilon^*]}\|\partial_\zeta^2 F(\tau\eta,\cdot,\cdot)\|_{C^{1,\alpha}(\partial\Omega\times[\zeta^i-C_3R,\zeta^i+C_3R])}\le C_7.
\end{aligned}\tag{3.79}$$

Thus, by (3.76), (3.77), (3.78), and (3.79), and by the membership of $\epsilon\in]0,\epsilon^*[$ and $\delta^*\in]0,1[$, we have

$$\begin{aligned}
\|d_v\mathcal{N}_{\partial_\epsilon^2 F}(\tau\epsilon,\tilde{\psi}_1^i)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))}&\le 2C_6\,C_7\,(1+R)^2\,,\\
\|d_v\mathcal{N}_{\partial_\epsilon\partial_\zeta F}(\tau\epsilon,\tilde{\psi}_2^i)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))}&\le 2C_6\,C_7\,(1+R)^2\,,\\
\|d_v\mathcal{N}_{\partial_\zeta^2 F}(\tau\epsilon,\tilde{\psi}_3^i)\|_{\mathcal{L}(C^{1,\alpha}(\partial\Omega^i),C^{1,\alpha}(\partial\Omega^i))}&\le 2C_6\,C_7\,(1+R)^2\,,
\end{aligned}\tag{3.80}$$

uniformly with respect to $\tau\in]0,1[$.

We can now bound the $C^{1,\alpha}$ norms with respect to the variable $t\in\partial\Omega^i$ of (3.68), (3.69), and (3.70) uniformly with respect to $\tau\in]0,1[$. Indeed, by (3.54), (3.71) and (3.80), we obtain

$$\|T_1[\epsilon,\psi^i,\Psi^i[\epsilon]](\tau,\cdot)\|_{C^{1,\alpha}(\partial\Omega^i)}\le 2C_6\,C_7\,(1+R)^2\delta^* \tag{3.81}$$

for all $\tau \in]0,1[$. By Proposition 3.2.4 (ii), by (3.57) and (3.79), we obtain

$$\begin{aligned}
&\left\| (\partial_\epsilon \partial_\zeta F)\left(\tau\epsilon, \cdot, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \le C_6\, C_7 \left(1 + \left\|\tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right\|_{C^{1,\alpha}(\partial\Omega^i)}\right)^2 \\
&\qquad \le C_6\, C_7 \left(1 + C_3 R_1 + |\zeta^i|\right)^2 , \\
&\left\| (\partial_\zeta^2 F)\left(\tau\epsilon, \cdot, \tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right)\right\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \le C_6\, C_7 \left(1 + \left\|\tau\epsilon\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \zeta^i\right\|_{C^{1,\alpha}(\partial\Omega^i)}\right)^2 \\
&\qquad \le C_6\, C_7 \left(1 + C_3 R_1 + |\zeta^i|\right)^2 ,
\end{aligned} \tag{3.82}$$

for all $\tau \in]0,1[$. Hence, in view of (3.57), (3.80) and (3.82) and by (3.73) and (3.75) we have

$$\begin{aligned}
&\|T_2[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot)\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \le \left\{2C_6C_7\left(1 + C_3R_1 + |\zeta^i|\right)^2 + 4C_3C_6C_7R_2(1+R)^2\right\}\delta^* , \\
&\|T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot)\|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\qquad \le \left\{4C_6C_7\left(1 + C_3R_1 + |\zeta^i|\right)^2 R + 8C_3^2C_6C_7R_2^2(1+R)^2\right\}\delta^* ,
\end{aligned} \tag{3.83}$$

for all $\tau \in]0,1[$, where to obtain the inequality for $T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot)$ we have also used that

$$\left\|\left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\psi^i] + \left(\frac{1}{2}I + W_{\partial\Omega^i}\right)[\Psi^i[\epsilon]]\right\|_{C^{1,\alpha}(\partial\Omega^i)} \le C_3(R_1 + R_2) \le R$$

(cf. (3.61)). Moreover, since the boundedness provided in (3.81) and (3.83) is uniform with respect to $\tau \in]0,1[$, one verifies that the following inequality holds:

$$\begin{aligned}
&\Big\| \int_0^1 (1-\tau) \Big\{ T_1[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) + 2\, T_2[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) \\
&+ T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) \Big\}\, d\tau \Big\|_{C^{1,\alpha}(\partial\Omega^i)} \le \int_0^1 (1-\tau) \| T_1[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) \\
&+ 2\, T_2[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) + T_3[\epsilon, \psi^i, \Psi^i[\epsilon]](\tau, \cdot) \|_{C^{1,\alpha}(\partial\Omega^i)}\, d\tau \\
&\le \Big\{ 2C_6\, C_7\, (1+R)^2 + 4C_6 C_7 \Big(1 + C_3 R_1 + |\zeta^i|\Big)^2 + 8 C_3 C_6 C_7 R_2 (1+R)^2 \\
&\quad + 4C_6 C_7 \Big(1 + C_3 R_1 + |\zeta^i|\Big)^2 R + 8 C_3^2 C_6 C_7 R_2^2 (1+R)^2 \Big\}\, \delta^* .
\end{aligned} \tag{3.84}$$

Then, by (3.67) and (3.84) we obtain

$$\begin{aligned}
&\| S_2[\epsilon, \psi^i_\epsilon] - S_2[\epsilon, \Psi^i[\epsilon]] \|_{C^{1,\alpha}(\partial\Omega^i)} \\
&\quad \le \Big\{ 2C_6\, C_7\, (1+R)^2 + 4C_6 C_7 \Big(1 + C_3 R_1 + |\zeta^i|\Big)^2 + 8 C_3 C_6 C_7 R_2 (1+R)^2 \\
&\qquad + 4C_6 C_7 \Big(1 + C_3 R_1 + |\zeta^i|\Big)^2 R + 8 C_3^2 C_6 C_7 R_2^2 (1+R)^2 \Big\}\, \delta^*
\end{aligned} \tag{3.85}$$

(also recall that $\epsilon \in]0,1[$).

- *Step 7: Conclusion for S.*

Finally, by (3.58), (3.66) and (3.85), we have

$$\| S[\epsilon, \psi^i_\epsilon] - S[\epsilon, \Psi^i[\epsilon]] \|_{C^{1,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o)} \le C_8\, \delta^*,$$

with

$$\begin{aligned}
C_8 \equiv\ & C_4\, C_5\, (1 + R^\alpha) + 2C_6\, C_7\, (1+R)^2 + 4C_6 C_7 \Big(1 + C_3 R_1 + |\zeta^i|\Big)^2 \\
&+ 8 C_3 C_6 C_7 R_2 (1+R)^2 + 4C_6 C_7 \Big(1 + C_3 R_1 + |\zeta^i|\Big)^2 R + 8 C_3^2 C_6 C_7 R_2^2 (1+R)^2 .
\end{aligned}$$

- *Step 8: Estimate for* (3.53) *and determination of* δ^*.

By (3.47) and (3.48) we conclude that the norm of the difference between $(\phi^o, \phi^i, \zeta, \psi^i)$ and

$$(\Phi^o[\epsilon], \Phi^i[\epsilon], Z[\epsilon], \Psi^i[\epsilon])$$

in the space $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times \mathbb{R} \times C^{1,\alpha}(\partial\Omega^i)$ is less than $C_1 C_8\, \delta^*$. Then, by (3.49) and by the triangle inequality we obtain

$$\|(\phi^o, \phi^i, \zeta, \psi^i) - (\phi^o_0, \phi^i_0, \zeta_0, \psi^i_0)\|_{C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0\times\mathbb{R}\times C^{1,\alpha}(\partial\Omega^i)} \leq C_1 C_8\, \delta^* + \frac{K}{2}\,.$$

Thus, in order to have $(\phi^o, \phi^i, \zeta, \psi^i) \in \mathcal{B}_{0,K}$, it suffices to take

$$\delta^* < \frac{K}{2C_1 C_8}$$

in inequality (3.50). Then, for such choice of δ^*, (3.53) holds and the theorem is proved. □

3.5 Local uniqueness for the family of solutions

As a consequence of Theorem 3.4.1, we can derive the following local uniqueness result for the family $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$.

Corollary 3.5.1. *Let assumptions* (2.13), (2.18), (2.42) *and* (2.44) *hold true. Let* $\epsilon' \in]0, \epsilon_0[$ *be as in Theorem 2.6.5. Let* $\{(u^o_\epsilon, u^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ *be as in Theorem 2.7.2. Let* $\{(v^o_\epsilon, v^i_\epsilon)\}_{\epsilon\in]0,\epsilon'[}$ *be a family of functions such that* $(v^o_\epsilon, v^i_\epsilon) \in C^{1,\alpha}(\Omega(\epsilon)) \times C^{1,\alpha}(\epsilon\Omega^i)$ *is a solution of problem* (2.2) *for all* $\epsilon \in]0, \epsilon'[$. *If*

$$\lim_{\epsilon\to 0^+} \epsilon^{-1} \left\| v^i_\epsilon(\epsilon\cdot) - u^i_\epsilon(\epsilon\cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} = 0, \qquad (3.86)$$

then there exists $\epsilon_* \in]0, \epsilon'[$ *such that*

$$(v^o_\epsilon, v^i_\epsilon) = (u^o_\epsilon, u^i_\epsilon) \quad \forall \epsilon \in]0, \epsilon_*[\,.$$

Proof. Let ϵ^* and δ^* be as in Theorem 3.4.1. By (3.86) there is $\epsilon_* \in]0, \epsilon^*[$ such that

$$\left\| v^i_\epsilon(\epsilon \cdot) - u^i_\epsilon(\epsilon \cdot) \right\|_{C^{1,\alpha}(\partial\Omega^i)} \leq \epsilon\delta^* \quad \forall \epsilon \in]0, \epsilon_*[.$$

Then the statement follows by Theorem 3.4.1. □

CHAPTER 4

Existence result for the nonlinear transmission problem (3)

This chapter is mainly devoted to prove the existence of a specific family of solutions of a boundary value problem for the Laplace equation with nonlinear non-autonomous transmission conditions on the boundary of an inclusion shaped by a parameter ϕ belonging to a suitable class of diffeomorphism. Moreover, we analyse the dependence of that specific family of solutions upon the perturbation parameter ϕ.

We recall the geometric framework of our problem already briefly described in the Introduction. We fix once for all a natural number

$$n \geq 2$$

that will be the dimension of the space $\mathbb{R}^n$ we are going to work in and a parameter

$$\alpha \in]0,1[$$

which we use to define the regularity of our sets and functions.

Then, we introduce two sets Ω^o and Ω^i that satisfy the following conditions:

$$\begin{aligned}
&\Omega^o,\ \Omega^i \text{ are bounded open connected subsets of } \mathbb{R}^n \text{ of class } C^{1,\alpha},\\
&\text{their exteriors } \mathbb{R}^n \setminus \overline{\Omega^o} \text{ and } \mathbb{R}^n \setminus \overline{\Omega^i} \text{ are connected,}\\
&\text{the origin } 0 \text{ of } \mathbb{R}^n \text{ belongs both to } \Omega^o \text{ and to } \Omega^i,\\
&\text{and } \Omega^i \subset \Omega^o.
\end{aligned} \tag{4.1}$$

Then we fix three functions

$$F_1, F_2 \in C^0(\partial\Omega^i \times \mathbb{R} \times \mathbb{R}) \text{ and } f^o \in C^{0,\alpha}(\partial\Omega^o). \tag{4.2}$$

We want to introduce a transmission problem in the pair of domains consisting of $\Omega^o \setminus \overline{\Omega^i}$ and Ω^i. The functions F_1 and F_2 determine the transmission conditions on the (inner) boundary $\partial\Omega^i$. Instead, f^o plays the role of the Neumann datum on the (outer) boundary $\partial\Omega^o$. We are now ready to introduce the following nonlinear transmission boundary value problem for a pair of functions $(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$:

$$\begin{cases}
\Delta u^o = 0 & \text{in } \Omega^o \setminus \overline{\Omega^i},\\
\Delta u^i = 0 & \text{in } \Omega^i,\\
\nu_{\Omega^o} \cdot \nabla u^o(x) = f^o(x) & \forall x \in \partial\Omega^o,\\
\nu_{\Omega^i} \cdot \nabla u^o(x) = F_1(x, u^o(x), u^i(x)) & \forall x \in \partial\Omega^i,\\
\nu_{\Omega^i} \cdot \nabla u^i(x) = F_2(x, u^o(x), u^i(x)) & \forall x \in \partial\Omega^i,
\end{cases} \tag{4.3}$$

Since problem (4.3) is nonlinear, *a priori*, it is not clear why it should admit a classical solution. We prove that under suitable conditions on F_1 and F_2, problem (4.3) has at least a solution $(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$.

Then we introduce a "perturbed" variant of problem (4.3). We fix the external domain Ω^o and we assume that the boundary of the internal domain is of the form $\phi(\partial\Omega^i)$, where ϕ is a diffeomorphism of $\partial\Omega^i$ into $\mathbb{R}^n$ and belongs to the class

$$\mathcal{A}_{\partial\Omega^i} \equiv \left\{ \phi \in C^1(\partial\Omega^i, \mathbb{R}^n) : \phi \text{ injective, } d\phi(y) \text{ injective for all } y \in \partial\Omega^i \right\}. \tag{4.4}$$

Clearly the identity function on $\partial\Omega^i$ belongs to the class $\mathcal{A}_{\partial\Omega^i}$, and, for convenience, we set

$$\phi_0 \equiv \mathrm{id}_{\partial\Omega^i}. \tag{4.5}$$

Then by the Jordan Leray Separation Theorem (cf. Deimling [31, Thm. 5.2]), $\mathbb{R}^n \setminus \phi(\partial\Omega^i)$ has exactly two open connected components for all $\phi \in \mathcal{A}_{\partial\Omega^i}$, and we define $\mathbb{I}[\phi]$ to be the unique bounded open connected component of $\mathbb{R}^n \setminus \phi(\partial\Omega^i)$. Finally we set

$$\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \equiv \left\{ \phi \in \mathcal{A}_{\partial\Omega^i} : \overline{\mathbb{I}[\phi]} \subset \Omega^o \right\}.$$

Clearly, by assumption (4.1), $\phi_0 \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$.

Now let $\phi \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$. We consider the following nonlinear non-autonomous trasmission problem in the perforated domain $\Omega^o \setminus \overline{\mathbb{I}[\phi]}$ for a pair of functions $(u^o, u^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \mathbb{I}[\phi]) \times C^{1,\alpha}(\overline{\mathbb{I}[\phi]})$:

$$\begin{cases} \Delta u^o = 0 & \text{in } \Omega^o \setminus \overline{\mathbb{I}[\phi]}, \\ \Delta u^i = 0 & \text{in } \mathbb{I}[\phi], \\ \nu_{\Omega^o} \cdot \nabla u^o(x) = f^o(x) & \forall x \in \partial\Omega^o, \\ \nu_{\mathbb{I}[\phi]} \cdot \nabla u^o(x) = F_1(\phi^{(-1)}(x), u^o(x), u^i(x)) & \forall x \in \phi(\partial\Omega^i), \\ \nu_{\mathbb{I}[\phi]} \cdot \nabla u^i(x) = F_2(\phi^{(-1)}(x), u^o(x), u^i(x)) & \forall x \in \phi(\partial\Omega^i). \end{cases} \tag{4.6}$$

4.1 Preliminary results

We start this section with the following representation result for harmonic functions in a domain with an inclusion in terms of single layer potentials plus constants.

Lemma 4.1.1. *Let Ω be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$, such that $\overline{\Omega} \subseteq \Omega^o$. Then the map $(U^o_\Omega[\cdot,\cdot,\cdot,\cdot,\cdot], U^i_\Omega[\cdot,\cdot,\cdot,\cdot,\cdot])$ from $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}^2$ to $C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega^o} \setminus \Omega) \times C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ which takes a quintuple $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ to the pair of functions*

$$(U^o_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i], U^i_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i])$$

defined by

$$\begin{aligned} U^o_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &= (v^+_{\Omega^o}[\mu^o] + v^-_\Omega[\mu^i] + \rho^o)_{|\overline{\Omega^o}\setminus\Omega} \\ U^i_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &= v^+_\Omega[\eta^i] + \rho^i \end{aligned} \tag{4.7}$$

is bijective.

Proof. The map is well defined. Indeed, by the harmonicity and regularity properties of single layer potentials (cf. Theorem 1.3.3 (i)-(ii)), we know that

$$\begin{aligned} &\Delta U^o_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] = 0 \quad \text{on } \Omega^o \setminus \overline{\Omega}, \\ &\Delta U^i_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] = 0 \quad \text{on } \Omega, \\ &(U^o_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i], U^i_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i]) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega) \times C^{1,\alpha}(\overline{\Omega}), \end{aligned}$$

for all $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}^2$. We now show it is bijective. We take a pair of functions $(h^o, h^i) \in C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega^o} \setminus \Omega) \times C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$ and we prove that there exists a unique quintuple $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in$

$C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}^2$ such that

$$(U^o_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i], U^i_\Omega[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i]) = (h^o, h^i). \tag{4.8}$$

By the uniqueness of the classical solution of the Dirichlet boundary value problem, the second equation in (4.8) is equivalent to

$$V_{\partial\Omega}[\eta^i] + \rho^i = h^i_{|\partial\Omega} \tag{4.9}$$

(notice that, by $h^i \in C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$, we have $h^i_{|\partial\Omega} \in C^{1,\alpha}(\partial\Omega) \subset C^{0,\alpha}(\partial\Omega)$). By Theorem 1.3.3 (vi), there exists a unique pair $(\eta^i, \rho^i) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that (4.9) holds. Then we are left to show that there exists a unique triple $(\mu^o, \mu^i, \rho^o) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$ such that

$$(v^+_{\Omega^o}[\mu^o] + v^-_{\Omega^i}[\mu^i] + \rho^o)_{|\overline{\Omega^o}\setminus\Omega} = h^o. \tag{4.10}$$

By the jump relation for single layer potential (cf. Theorem 1.3.3 (iv)) and by the uniqueness of the classical solution of the Neumann-Dirichlet mixed boundary value problem (cf. Evans [32, Problems 6.6 pag. 366]), equation (4.10) is equivalent to the following system of integral equations:

$$\begin{aligned} &V_{\partial\Omega^o}[\mu^o] + v^-_{\Omega}[\mu^i]_{|\partial\Omega^o} + \rho^o = h^o_{|\partial\Omega^o}, \\ &\left(\frac{1}{2}I + W^*_{\partial\Omega}\right)[\mu^i] + \nu_\Omega \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega} = \nu_\Omega \cdot \nabla h^o_{|\partial\Omega}, \end{aligned} \tag{4.11}$$

(notice that, by $h^0 \in C^{1,\alpha}_{\mathrm{harm}}(\overline{\Omega})$, we get $h^o_{|\partial\Omega^o} \in C^{1,\alpha}(\partial\Omega) \subset C^{0,\alpha}(\partial\Omega)$ and $\nu_\Omega \cdot \nabla h^o_{|\partial\Omega} \in C^{0,\alpha}(\partial\Omega)$). Then we observe that by Theorem 1.3.3 (vi), the map from $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega)$ which takes a triple (μ^o, μ^i, ρ^o) to the pair of functions

$$\left(V_{\partial\Omega^o}[\mu^o] + \rho^o, \frac{1}{2}\mu^i\right)$$

is an isomorphism. Moreover, by the properties of integral equations with real analytic kernel and no singularities (cf. Theorem A.2.2 (ii) in Appendix A.2) and by Theorem 1.3.5 (iii), the map from $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega)$ which takes a triple $(\mu^o, \mu^i.\rho^o)$ to the pair of functions $(v_\Omega^-[\mu^i]_{|\partial\Omega^o}, W^*_{\partial\Omega}[\mu^i] + \nu_\Omega \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega})$ is compact. Hence, the map from $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega)$ which takes a triple (μ^o, μ^i, ρ^o) to the pair of functions

$$\left(V_{\partial\Omega^o}[\mu^o] + v_\Omega^-[\mu^i]_{|\partial\Omega^o} + \rho^o, \left(\frac{1}{2}I + W^*_{\partial\Omega}\right)[\mu^i] + \nu_\Omega \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega}\right)$$

is a compact perturbation of an isomorphism and therefore it is a Fredholm operator of index 0. Thus, to complete the proof, it suffices to show that (4.11) with $(h^o_{|\partial\Omega^o}, \nu_\Omega \cdot \nabla h^o_{|\partial\Omega}) = (0,0)$ implies $(\mu^o, \mu^i, \rho^o) = (0,0,0)$. If

$$\left(V_{\partial\Omega^o}[\mu^o] + v_\Omega^-[\mu^i]_{|\partial\Omega^o} + \rho^o, \left(\frac{1}{2}I + W^*_{\partial\Omega}\right)[\mu^i] + \nu_\Omega \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega}\right) = (0,0), \tag{4.12}$$

then by the jump relation for the single layer potential (cf. Theorem 1.3.3 (iv)) and by the uniqueness of the classical solution of Neumann-Dirichlet mixed boundary value problem (cf. Evans [32, Problems 6.6 pag. 366]), one deduces that

$$(v^+_{\Omega^o}[\mu^o] + v^-_\Omega[\mu^i] + \rho^o)_{|\overline{\Omega^o}\setminus\Omega} = 0.$$

Moreover, by the continuity of $v_\Omega[\mu^i]$ in $\mathbb{R}^n$, we have that

$$(v^+_{\Omega^o}[\mu^o] + v^-_\Omega[\mu^i] + \rho^o)_{|\partial\Omega} = (v^+_{\Omega^o}[\mu^o] + v^+_\Omega[\mu^i] + \rho^o)_{|\partial\Omega} = 0.$$

Then by the uniqueness of the classical solution of Dirichlet boundary value problem in Ω (cf. Theorem 1.4.1 (i)) we deduce that

$$(v^+_{\Omega^o}[\mu^o] + v^+_{\Omega}[\mu^i] + \rho^o)_{|\overline{\Omega}} = 0. \qquad (4.13)$$

Then by the jump relation for single layer potential (cf. Theorem 1.3.3 (iv)), adding and subtracting the term $\nu_\Omega \cdot \nabla (v^+_{\Omega^o}[\mu^o] + \rho^o)_{|\partial\Omega}$ and taking into account (4.13), we get

$$\begin{aligned}\mu^i &= \nu_\Omega \cdot \nabla v^-_\Omega[\mu^i]_{|\partial\Omega} - \nu_\Omega \cdot \nabla v^+_\Omega[\mu^i]_{|\partial\Omega} \\ &= \nu_\Omega \cdot \nabla (v^+_{\Omega^o}[\mu^o] + v^-_\Omega[\mu^i] + \rho^o)_{|\partial\Omega} - \nu_\Omega \cdot \nabla (v^+_{\Omega^o}[\mu^o] + v^+_\Omega[\mu^i] + \rho^o)_{|\partial\Omega} = 0.\end{aligned}$$

Thus, by (4.12), we obtain $V_{\Omega^o}[\mu^o] + \rho^o = 0$ on $\partial\Omega^o$, which implies $(\mu^o, \rho^o) = (0, 0)$ (cf. Theorem 1.3.3 (vi)). Hence $(\mu^o, \mu^i, \rho^o) = (0, 0, 0)$ and the proof is complete. □

To represent the boundary condition of a linearised version of problem (4.3), we find convenient to introduce a matrix function

$$A(\cdot) = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) \\ A_{21}(\cdot) & A_{22}(\cdot) \end{pmatrix} : \partial\Omega^i \to M_2(\mathbb{R}).$$

We set

$$\tilde{A} \equiv \begin{pmatrix} A_{11} & A_{12} \\ -A_{21} & -A_{22} \end{pmatrix}.$$

We will exploit the following conditions on the matrix A:

$$
\begin{cases}
\bullet\ A \in M_2(C^{0,\alpha}(\partial\Omega^i)); \\
\bullet\ \text{For every } (\xi_1,\xi_2) \in \mathbb{R}^2, (\xi_1,\xi_2)^T \tilde{A}(\xi_1,\xi_2) \geq 0 \text{ on } \partial\Omega^i; \\
\bullet\ \text{If } (c_1,c_2) \in \mathbb{R}^2 \text{ and } A(x)(c_1,c_2) = 0 \text{ for all } x \in \partial\Omega^i, \\
\quad \text{then } (c_1,c_2) = (0,0).
\end{cases}
\tag{4.14}
$$

We remark that in literature the third condition in (4.14) is often replaced by a condition on the invertibility of the matrix A, namely

$$\bullet\ \text{There exist a point } x \in \partial\Omega^i \text{ such that } A(x) \text{ is invertible.} \tag{4.15}$$

We point out that the matrix $A(x) = \begin{pmatrix} x_1^2 & x_1 \\ -x_1 & -1 \end{pmatrix}$ with $x = (x_1, \ldots, x_n) \in \partial\Omega^i$ satisfies the third condition in (4.14) but not condition (4.15).

Then we can prove the following result on the uniqueness of the solution of a homogeneous A-linearly dependent transmission problem. We mention that we will apply this results in the Proposition 4.1.3 below and in section 4.4 in Proposition 4.4.4.

Lemma 4.1.2. *Let A be as in* (4.14)*. Then the unique solution in $C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$ of problem*

$$
\begin{cases}
\Delta u^o = 0 & \text{in } \Omega^o \setminus \overline{\Omega^i}, \\
\Delta u^i = 0 & \text{in } \Omega^i, \\
\nu_{\Omega^o} \cdot \nabla u^o(x) = 0 & \forall x \in \partial\Omega^o, \\
\nu_{\Omega^i} \cdot \nabla u^o(x) - A_{11}(x)u^0(x) - A_{12}(x)u^i(x) = 0 & \forall x \in \partial\Omega^i, \\
\nu_{\Omega^i} \cdot \nabla u^i(x) - A_{21}(x)u^0(x) - A_{22}(x)u^i(x) = 0 & \forall x \in \partial\Omega^i,
\end{cases}
\tag{4.16}
$$

is $(u^o, u^i) = (0, 0)$.

Proof. Clearly the pair of functions $(u^o, u^i) = (0, 0)$ is a solution of problem (4.16). Then we prove it is the unique one in the product space $C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$ by an energy argument. By the Divergence Theorem, we have

$$
\begin{aligned}
0 &\le \int_{\Omega^o \setminus \overline{\Omega^i}} |\nabla u^o(x)|^2 \, dx + \int_{\Omega^i} |\nabla u^i(x)|^2 \, dx \\
&\le \int_{\partial\Omega^o} (\nu_{\Omega^o} \cdot \nabla u^o(x)) u^o(x) \, d\sigma_x - \int_{\partial\Omega^i} (\nu_{\Omega^i} \cdot \nabla u^o(x)) u^o(x) \, d\sigma_x \\
&\quad + \int_{\partial\Omega^i} (\nu_{\Omega^i} \cdot \nabla u^i(x)) u^i(x) \, d\sigma_x \\
&\le - \left\{ \int_{\partial\Omega^i} (u^o(x), u^i(x))^T \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ -A_{21}(x) & -A_{22}(x) \end{pmatrix} (u^o(x), u^i(x)) \, d\sigma_x \right\} \le 0,
\end{aligned}
$$

where the last inequality holds thanks to the second assumption in (4.14). Then, we obtain

$$
\int_{\Omega^o \setminus \overline{\Omega^i}} |\nabla u^o(x)|^2 \, dx = 0 \qquad \text{and} \qquad \int_{\Omega^i} |\nabla u^i(x)|^2 \, dx = 0.
$$

Hence u^o and u^i are constant, i.e. there exists $(c_1, c_2) \in \mathbb{R}^2$ such that

$$
u^o(x) = c_1 \quad \forall x \in \overline{\Omega^o} \setminus \Omega^i \qquad \text{and} \qquad u^i(x) = c_2 \quad \forall x \in \overline{\Omega^i}.
$$

Then, by the forth and fifth equations in (4.16), we obtain $A(x)(c_1, c_2) = 0$ for all $x \in \partial\Omega^i$, which, by the third assumption in (4.14), implies $(c_1, c_2) = (0, 0)$. The proof is completed. □

In the following proposition, we investigate an auxiliary boundary operator which we will exploit in the integral formulation of our problem.

Proposition 4.1.3. *Let A be as in* (4.14). *Let J_A be the map from $L^2(\partial\Omega^o)_0 \times L^2(\partial\Omega^i) \times L^2(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $L^2(\partial\Omega^o) \times (L^2(\partial\Omega^i))^2$ which takes a quintuple*

$(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ *to the triple* $J_A[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i]$ *defined by*

$$\begin{aligned}
J_{A,1}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv \left(-\frac{1}{2}I + W^*_{\partial\Omega^o}\right)[\mu^o] + \nu_{\Omega^o} \cdot \nabla v^-_{\Omega^i}[\mu^i]_{|\partial\Omega^o} && \text{on } \partial\Omega^o, \\
J_{A,2}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv \left(\frac{1}{2}I + W^*_{\partial\Omega^i}\right)[\mu^i] + \nu_{\Omega^i} \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} \\
&\quad - (A_{11}, A_{12})^T \cdot (v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) && \text{on } \partial\Omega^i, \\
J_{A,3}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv \left(-\frac{1}{2}I + W^*_{\partial\Omega^i}\right)[\eta^i] \\
&\quad - (A_{21}, A_{22})^T \cdot (v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) && \text{on } \partial\Omega^i.
\end{aligned} \tag{4.17}$$

Then the following statements hold.

(i) J_A *is a linear isomorphism from* $L^2(\partial\Omega^o)_0 \times L^2(\partial\Omega^i) \times L^2(\partial\Omega^i)_0 \times \mathbb{R}^2$ *to* $L^2(\partial\Omega^o) \times (L^2(\partial\Omega^i))^2$.

(i) J_A *is a linear isomorphism from* $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ *to* $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$.

(ii) J_A *is a linear isomorphism from* $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ *to* $C^{0,\alpha}(\partial\Omega^o) \times (C^{0,\alpha}(\partial\Omega^i))^2$.

Proof. We first prove (i). We write J_A in the form

$$J_A = \tilde{J}^+_A \circ \tilde{J}_A \circ \tilde{J}^-_A,$$

where $\tilde{J}^-_A$ is the inclusion of $L^2(\partial\Omega^o)_0 \times L^2(\partial\Omega^i) \times L^2(\partial\Omega^i)_0 \times \mathbb{R}^2$ into $L^2(\partial\Omega^o) \times (L^2(\partial\Omega^i))^2 \times \mathbb{R}^2$, $\tilde{J}_A$ is the map from $L^2(\partial\Omega^o) \times (L^2(\partial\Omega^i))^2 \times \mathbb{R}^2$ into itself which takes a quintuple $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ to the quintuple $\tilde{J}_A[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i]$

defined by

$$
\begin{aligned}
\tilde{J}_{A,1}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i] &\equiv \left(-\frac{1}{2}I+W^*_{\partial\Omega^o}\right)[\mu^o]+\nu_{\Omega^o}\cdot\nabla v^-_{\Omega^i}[\mu^i]_{|\partial\Omega^o}\\
\tilde{J}_{A,2}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i] &\equiv \left(\frac{1}{2}I+W^*_{\partial\Omega^i}\right)[\mu^i]+\nu_{\Omega^i}\cdot\nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}\\
&\quad -(A_{11},A_{12})^T\cdot(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}+V_{\partial\Omega^i}[\mu^i],V_{\partial\Omega^i}[\eta^i])\\
\tilde{J}_{A,3}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i] &\equiv \left(-\frac{1}{2}I+W^*_{\partial\Omega^i}\right)[\eta^i]\\
&\quad -(A_{21},A_{22})^T\cdot(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}+V_{\partial\Omega^i}[\mu^i],V_{\partial\Omega^i}[\eta^i])\\
\tilde{J}_{A,4}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i] &\equiv \rho^o\\
\tilde{J}_{A,5}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i] &\equiv \rho^i
\end{aligned}
$$

and $\tilde{J}^+_A$ is the map from $L^2(\partial\Omega^o)\times(L^2(\partial\Omega^i))^2\times\mathbb{R}^2$ into $L^2(\partial\Omega^o)\times(L^2(\partial\Omega^i))^2$ which takes a quintuple (f,g_1,g_2,c_1,c_2) to the triple $\tilde{J}^+_A[f,g_1,g_2,c_1,c_2]$ defined by

$$\tilde{J}^+_A[f,g_1,g_2,c_1,c_2]=(f,g_1-(A_{11},A_{12})^T\cdot(c_1,c_2),g_2-(A_{21},A_{22})^T\cdot(c_1,c_2)).$$

Then we observe that $\tilde{J}^-_A$ is Fredholm of index -2, because

$$\mathrm{Ker}\,\tilde{J}^-_A=\{0\}\text{ and }\mathrm{Coker}\,\tilde{J}^-_A=\mathrm{Span}\,\{(1,0,0,0,0),(0,0,1,0,0)\},$$

and that $\tilde{J}^+_A$ is a Fredholm operator of index 2, because

$$\mathrm{Coker}\,\tilde{J}^+_A=\{0\}\text{ and }\mathrm{Ker}\,\tilde{J}^+_A=\mathrm{Span}\,\{(0,A_{11},A_{21},1,0),(0,A_{12},A_{22},0,1)\}.$$

Next, we observe that the map from $L^2(\partial\Omega^o)\times(L^2(\partial\Omega^i))^2\times\mathbb{R}^2$ into itself which takes a quintuple $(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)$ to the quintuple $(-\frac{1}{2}\mu^o,\frac{1}{2}\mu^i,-\frac{1}{2}\eta^i,\rho^o,\rho^i)$ is an isomorphism. Moreover, by mapping properties of integral operator with real analytic kernel and no singularity (cf. Theorem A.2.1 (ii) in the Appendix), by Theorem 1.3.5 (i), by the compactness of the operators $V_{\partial\Omega^i}$ from $L^2(\partial\Omega^i)$ into itself (see Costabel [19, Thm. 1]), and by the bilinearity and continuity of the product in $L^2(\partial\Omega^i)$, we deduce that the map from $L^2(\partial\Omega^o) \times (L^2(\partial\Omega^i))^2 \times \mathbb{R}^2$ into itself which takes quintuple $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ to the quintuple $\tilde{J}^C_A[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i]$ defined by

$$
\begin{aligned}
\tilde{J}^C_{A,1}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv W^*_{\partial\Omega^o}[\mu^o] + \nu_{\Omega^o} \cdot \nabla v^-_{\Omega^i}[\mu^i]_{|\partial\Omega^o} \\
\tilde{J}^C_{A,2}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv W^*_{\partial\Omega^i}[\mu^i] + \nu_{\Omega^i} \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} \\
&\quad - (A_{11}, A_{12})^T \cdot (v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i], V_{\partial\Omega^i}[\eta^i]) \\
\tilde{J}^C_{A,3}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv W^*_{\partial\Omega^i}[\eta^i] \\
&\quad - (A_{21}, A_{22})^T \cdot (v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i], V_{\partial\Omega^i}[\eta^i]) \\
\tilde{J}^C_{A,4}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv 0 \\
\tilde{J}^C_{A,5}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] &\equiv 0
\end{aligned}
$$

is compact. Hence, we conclude that $\tilde{J}_A$ is a compact perturbation of an isomorphism and therefore it is Fredholm of index 0. Since the index of a composition of Fredholm operators, is the sum of the indexes of the components, we deduce that J_A is a Fredholm operator of index 0. Therefore, in order to complete the proof of point (i), it suffices to prove that J_A is injective. Thus, we now assume that $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in L^2(\partial\Omega^o)_0 \times L^2(\partial\Omega^i) \times L^2(\partial\Omega^i)_0 \times \mathbb{R}^2$ and that

$$
J_A[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] = (0, 0, 0). \tag{4.18}
$$

Since the integral operators which appear on the definition of J_A (cf. (4.17)) display weak singularities, then a standard argument based on iterated kernels implies that $(\mu^o, \mu^i, \eta^i) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0$. Then mapping properties of integral operator with real analytic kernel and no singularity (cf.

Theorem A.2.1 (ii) in the Appendix) and by classical known regularity result in potential theory (cf. Miranda [57, Chap. II, §14]), we know that

$$\nu_{\Omega^o} \cdot \nabla v^-_{\Omega^i}[\mu^i]_{|\partial\Omega^o} \in C^{0,\alpha}(\partial\Omega^o),$$
$$v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i},\ \nu_{\Omega^i} \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i},\ V_{\partial\Omega^i}[\eta^i],\ V_{\partial\Omega^i}[\mu^i] \in C^{0,\alpha}(\partial\Omega^i).$$

Hence, by (4.18) and by the membership of $A \in M_2(C^{0,\alpha}(\partial\Omega^i))$ (cf. first condition in (4.14)), we obtain that

$$\left(-\frac{1}{2}I + W^*_{\partial\Omega^o}\right)[\mu^o] \in C^{0,\alpha}(\partial\Omega^o)$$

and

$$\left(\frac{1}{2}I + W^*_{\partial\Omega^i}\right)[\mu^i], \left(-\frac{1}{2}I + W^*_{\partial\Omega^i}\right)[\eta^i] \in C^{0,\alpha}(\partial\Omega^i).$$

Then Theorem 1.4.5 (ii) implies that (μ^o, μ^i, η^i) belongs to $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0$. By the jump relations (cf. Theorem 1.3.3 (iv)), by Lemma 4.1.1, and by equation (4.18), we deduce that the pair of functions

$$(U^o_{\Omega^i}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i], U^i_{\Omega^i}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i])$$

defined by (4.7) is a solution of the boundary value problem (4.16). Then by Lemma 4.1.2, we have that

$$(U^o_{\Omega^i}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i], U^i_{\Omega^i}[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i]) = (0, 0),$$

which implies that $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) = (0, 0, 0, 0, 0)$, by the uniqueness of the representation provided by Lemma 4.1.1.

In order to prove (ii), it suffices to observe that J_A is continuous from $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ and that, since the integral operators which appear on the definition of J_A (cf. (4.17)) display weak singularities, then, by a standard argument based on iterated kernels, we obtain that

$$J_A[\mu^o, \mu^i, \eta^i, \rho^o, \rho^i] \in C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$$

implies $(\mu^o, \mu^i, \eta^i) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0$ for all $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in L^2(\partial\Omega^o)_0 \times L^2(\partial\Omega^i) \times L^2(\partial\Omega^i)_0 \times \mathbb{R}^2$ (similarly to what we have done for (4.18)).

Finally (iii) can be proven in a similar way to point (ii). The proof is completed. □

4.2 Formulation of problem (3) in terms of integral equations

We are now ready to convert the nonlinear transmission problem (4.6) into a system of integral equations. We observe that, by using a suitable change of variable, the integral system is defined on a fixed set $\partial\Omega^o \times (\partial\Omega^i)^2$, which does not depend on the perturbation parameter ϕ.

Proposition 4.2.1. *Let* $(\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$. *Then the pair of functions*

$$(U^o_{\mathbb{I}[\phi]}[\mu^o, \mu^i \circ \phi^{(-1)}, \eta^i \circ \phi^{(-1)}, \rho^o, \rho^i], U^i_{\mathbb{I}[\phi]}[\mu^o, \mu^i \circ \phi^{(-1)}, \eta^i \circ \phi^{(-1)}, \rho^o, \rho^i]),$$

where $(U^o_{\mathbb{I}[\phi]}[\cdot,\cdot,\cdot,\cdot,\cdot], U^i_{\mathbb{I}[\phi]}[\cdot,\cdot,\cdot,\cdot,\cdot])$ *are defined by* (4.7), *is a solution of prob-*

lem (4.6) *if and only if*

$$\begin{aligned}
&\left(-\frac{1}{2}I + W^*_{\partial\Omega^o}\right)[\mu^o](x) + \nu_{\Omega^o}(x)\cdot\nabla v^-_{\mathbb{I}[\phi]}[\mu^i\circ\phi^{(-1)}](x) = f^o(x) \qquad \forall x\in\partial\Omega^o,\\
&\left(\frac{1}{2}I + W^*_{\partial\mathbb{I}[\phi]}\right)[\mu^i\circ\phi^{(-1)}](\phi(t)) + \nu_{\mathbb{I}[\phi]}(\phi(t))\cdot\nabla v^+_{\Omega^o}[\mu^o](\phi(t))\\
&= F_1\left(t, v^+_{\Omega^o}[\mu^o](\phi(t)) + V_{\partial\mathbb{I}[\phi]}[\mu^i\circ\phi^{(-1)}](\phi(t)) + \rho^o, V_{\partial\mathbb{I}[\phi]}[\eta^i\circ\phi^{(-1)}](\phi(t)) + \rho^i\right)\\
&\forall t\in\partial\Omega^i,\\
&\left(-\frac{1}{2}I + W^*_{\partial\mathbb{I}[\phi]}\right)[\eta^i\circ\phi^{(-1)}](\phi(t))\\
&= F_2\left(t, v^+_{\Omega^o}[\mu^o](\phi(t)) + V_{\partial\mathbb{I}[\phi]}[\mu^i\circ\phi^{(-1)}](\phi(t)) + \rho^o, V_{\partial\mathbb{I}[\phi]}[\eta^i\circ\phi^{(-1)}](\phi(t)) + \rho^i\right)\\
&\forall t\in\partial\Omega^i.
\end{aligned} \tag{4.19}$$

Proof. We first observe that, by the regularity of $\phi\in\mathcal{A}^{\Omega^o}_{\partial\Omega^i}$, if

$$(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)\in C^{0,\alpha}(\partial\Omega^o)_0\times C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}^2,$$

then

$$(\mu^o,\mu^i\circ\phi^{(-1)},\eta^i\circ\phi^{(-1)},\rho^o,\rho^i)\in C^{0,\alpha}(\partial\Omega^o)_0\times C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}^2.$$

Moreover, by the definition of $\mathcal{A}^{\Omega^o}_{\partial\Omega^i}$ and $\mathbb{I}[\phi]$ we have that $\partial\mathbb{I}[\phi]=\phi(\partial\Omega^i)$ and $\overline{\mathbb{I}[\phi]}\subseteq\Omega^o$. Hence we can apply Lemma 4.1.1 with $\Omega=\mathbb{I}[\phi]$. Then by jump relations (cf. Theorem 1.3.3 (iv)) and by a change of variable on $\phi(\Omega^i)$, we obtain that the pair of functions

$$\begin{aligned}
&U^o_{\mathbb{I}[\phi]}[\mu^o,\mu^i\circ\phi^{(-1)},\eta^i\circ\phi^{(-1)},\rho^o,\rho^i] = (v^+_{\Omega^o}[\mu^o] + v^-_{\mathbb{I}[\phi]}[\mu^i\circ\phi^{(-1)}] + \rho^o)_{|\overline{\Omega^o}\setminus\mathbb{I}[\phi]},\\
&U^i_{\mathbb{I}[\phi]}[\mu^o,\mu^i\circ\phi^{(-1)},\eta^i\circ\phi^{(-1)},\rho^o,\rho^i] = v^+_{\mathbb{I}[\phi]}[\eta^i\circ\phi^{(-1)}] + \rho^i,
\end{aligned}$$

is a solution of problem (4.6) if and only if (4.19) is satisfied. □

4.3 The limiting system and existence result for problem (2)

In this section we prove an existence theorem for the unperturbed transmission problem (4.3). In doing that we analyse the limiting system, i.e. the system of integral equation obtained by choosing $\phi = \phi_0$ in (4.19). It consists of the following three equations that for the sake of exposition we present as a three component vector field on $\partial\Omega^o \times \partial\Omega^i \times \partial\Omega^i$.

$$\begin{pmatrix} \left(-\frac{1}{2}I + W^*_{\partial\Omega^o}\right)[\mu^o] + \nu_{\Omega^o} \cdot \nabla v^-_{\Omega^i}[\mu^i]_{|\partial\Omega^o} \\ \left(\frac{1}{2}I + W^*_{\partial\Omega^i}\right)[\mu^i] + \nu_{\Omega^i} \cdot \nabla v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} \\ \left(-\frac{1}{2}I + W^*_{\partial\Omega^i}\right)[\eta^i] \end{pmatrix} = \begin{pmatrix} f^o \\ \mathcal{N}_{F_1}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \\ \mathcal{N}_{F_2}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \end{pmatrix} \tag{4.20}$$

Then, using the operator J_A introduced in Proposition 4.1.3, we can prove the following result for the transmission problem (4.3).

Proposition 4.3.1. *Let A be as in* (4.14). *Let* $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$. *Let* $(U^o_{\Omega^i}[\cdot,\cdot,\cdot,\cdot,\cdot], U^i_{\Omega^i}[\cdot,\cdot,\cdot,\cdot,\cdot])$ *be defined by* (4.7). *Let J_A be as in Proposition 4.1.3. Then*

$$(U^o_{\Omega^i}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i], U^i_{\Omega^i}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i])$$

is a solution of problem (4.3) *if and only if*

$$\begin{pmatrix}\mu^o\\ \mu^i\\ \eta^i\\ \rho^o\\ \rho^i\end{pmatrix} = J_A^{(-1)}\left[\begin{pmatrix} f^o\\ \mathcal{N}_{F_1}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}+V_{\partial\Omega^i}[\mu^i]+\rho^o, V_{\partial\Omega^i}[\eta^i]+\rho^i)\\ \mathcal{N}_{F_2}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}+V_{\partial\Omega^i}[\mu^i]+\rho^o, V_{\partial\Omega^i}[\eta^i]+\rho^i)\end{pmatrix} - \begin{pmatrix}0 & 0 & 0\\ 0 & A_{11} & A_{12}\\ 0 & A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}0\\ v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}+V_{\partial\Omega^i}[\mu^i]+\rho^o\\ V_{\partial\Omega^i}[\eta^i]+\rho^i\end{pmatrix}\right]. \tag{4.21}$$

Proof. By Lemma 4.1.1 and by the jump relation of Theorem 1.3.3 (iv), we know that if $(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)\in C^{0,\alpha}(\partial\Omega^o)_0\times C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}^2$ then $(U^o_{\Omega^i}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i], U^i_{\Omega^i}[\mu^o,\mu^i,\eta^i,\rho^o,\rho^i])$ defined by (4.7) is a solution of problem (4.3) if an only if (4.20) holds. Then, by subtracting on both sides of (4.20) the term

$$\begin{pmatrix}0 & 0 & 0\\ 0 & A_{11} & A_{12}\\ 0 & A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}0\\ v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i}+V_{\partial\Omega^i}[\mu^i]+\rho^o\\ V_{\partial\Omega^i}[\eta^i]+\rho^i\end{pmatrix}\in C^{0,\alpha}(\partial\Omega^o)\times(C^{0,\alpha}(\partial\Omega^i))^2$$

and by the invertibility of J_A from $C^{0,\alpha}(\partial\Omega^o)_0\times C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}^2$ to $C^{0,\alpha}(\partial\Omega^o)\times(C^{0,\alpha}(\partial\Omega^i))^2$ provided by Proposition 4.1.3 (iii), the validity of the statement follows. □

We now introduce an auxiliary map. If A is as in (4.14) and J_A is as in Proposition 4.1.3, we denote by T_A operator from $C^0(\partial\Omega^o)_0\times C^0(\partial\Omega^i)\times$

$C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ defined by

$$T_A(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \equiv J_A^{(-1)} \left[\begin{pmatrix} f^o \\ \mathcal{N}_{F_1}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \\ \mathcal{N}_{F_2}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \end{pmatrix} \right.$$
$$\left. - \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 \\ v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o \\ V_{\partial\Omega^i}[\eta^i] + \rho^i \end{pmatrix} \right]. \tag{4.22}$$

We study continuity and compactness properties of T_A in the following proposition.

Proposition 4.3.2. *Let A be as in* (4.14). *Let T_A be as in* (4.22). *Then T_A is a continuous operator from $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ and maps bounded sets into sets of compact closure.*

Proof. By the properties of integral operator with real analytic kernel and no singularities (cf. Theorem A.2.1 (ii) in the Appendix) and by the compactness of the embedding of $C^{0,\alpha}(\partial\Omega^i)$ into $C^0(\partial\Omega^i)$, $v^+_{\Omega^o}[\cdot]_{|\partial\Omega^i}$ is compact from $C^0(\partial\Omega^o)_0$ into $C^0(\partial\Omega^i)$. By mapping properties of single layer potential (cf. Miranda [57, Chap. II, §14, III]) and by the compactness of the embedding of $C^{0,\alpha}(\partial\Omega^i)$ into $C^0(\partial\Omega^i)$, $V_{\partial\Omega^i}$ is compact from $C^0(\partial\Omega^i)$ into itself. Hence, by the bilinear continuity of the product of continuous functions, the map from $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ which takes the quintuple $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ to the triple

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 \\ v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o \\ V_{\partial\Omega^i}[\eta^i] + \rho^i \end{pmatrix}$$

is continuous and maps bounded sets into sets with compact closure. Moreover, by assumption (4.2), one readily verifies that the operators $\mathcal{N}_{F_1}$ and $\mathcal{N}_{F_2}$ are continuous from $(C^0(\partial\Omega^i))^2$ into $C^0(\partial\Omega^i)$. Hence the map from $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ which takes the quintuple $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ to the triple

$$\begin{pmatrix} f^0 \\ \mathcal{N}_{F_1}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \\ \mathcal{N}_{F_2}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \end{pmatrix}$$

is continuous and maps bounded sets into sets of compact closure. Finally, by Proposition 4.1.3 (ii), the operator J_A is an isomorphism from $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ and, accordingly, T_A is continuous and maps bounded sets into sets of compact closure. $\square$

In the sequel we will assume the following growth condition on the pair of function F_1 and F_2 with respect to the matrix function A defined as in (4.14):

- There exist two constants $C_F \in]0, +\infty[$ and $\delta \in]0, 1[$ such that
$$\left| \begin{pmatrix} F_1(x, \zeta_1, \zeta_2) \\ F_2(x, \zeta_1, \zeta_2) \end{pmatrix} - A(x) \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \right| \leq C_F(1 + |\zeta_1| + |\zeta_2|)^\delta \qquad (4.23)$$
for all $(x, \zeta_1, \zeta_2) \in \partial\Omega^i \times \mathbb{R}^2$.

Then, by Leray Schauder Theorem (cf. Theorem A.3.1 in Appendix A.3), we can prove the following existence result in $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ for the nonlinear system (4.21). We notice that this result implies also existence of solution for the limiting system (4.20).

Proposition 4.3.3. *Let A be as in* (4.14)*. Let F_1, F_2 satisfy assumption* (4.23)*. Let J_A be as in Proposition 4.1.3. Then the nonlinear system* (4.21) *has* *at least a solution* $(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$.

Proof. We plan to apply the Leray-Schauder Theorem to the operator T_A defined by (4.22) in the Banach space $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$. By Proposition 4.3.2 we already know that T_A is a continuous operator from $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ and maps bounded sets into sets of compact closure. So in order to apply Leray-Schuder Theorem, we are left to show that if $\lambda \in [0,1]$ and if

$$(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) = \lambda T_A(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \tag{4.24}$$

with $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$, then there exists a constant $C \in]0, +\infty[$ (which does not depend on λ and $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$), such that

$$\|(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)\|_{C^0(\partial\Omega^o)\times(C^0(\partial\Omega^i))^2\times\mathbb{R}^2} \leq C. \tag{4.25}$$

By (4.24) and since $|\lambda| \leq 1$, we deduce that

$$\begin{aligned} &\|(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)\|_{C^0(\partial\Omega^o)\times(C^0(\partial\Omega^i))^2\times\mathbb{R}^2} \\ &\qquad \leq \|T_A(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)\|_{C^0(\partial\Omega^o)\times(C^0(\partial\Omega^i))^2} \end{aligned} \tag{4.26}$$

By the growth condition (4.23), one can show that

$$\left\| \begin{pmatrix} \mathcal{N}_{F_1}(h_1^i, h_2^i) \\ \mathcal{N}_{F_2}(h_1^i, h_2^i) \end{pmatrix} - A \begin{pmatrix} h_1^i \\ h_2^i \end{pmatrix} \right\|_{(C^0(\partial\Omega^i))^2} \leq C_F(1 + \|h_1^i\|_{C^0(\partial\Omega^i)} + \|h_2^i\|_{C^0(\partial\Omega^i)})^\delta \tag{4.27}$$

for all pair of functions $(h_1^i, h_2^i) \in (C^0(\partial\Omega^i))^2$. Hence, by (4.26) and by the definition of T_A in (4.22), we deduce that there exist two constants $C_1, C_2 \in]0, +\infty[$, which depend only on the operator norm of $J_A^{(-1)}$ from $C^0(\partial\Omega^o) \times (C^0(\partial\Omega^i))^2$ to $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ (cf. Theorem

4.1.3 (ii)), on $\|f^o\|_{\partial\Omega^o}$, on the constant $C_F \in]0,+\infty[$ provided by the growth condition (4.23) (cf. (4.27)), on the norm of the bounded linear operator $v^+_{\Omega^o}[\cdot]_{|\partial\Omega^i}$ from $C^0(\partial\Omega^o)$ to $C^0(\partial\Omega^i)$, and on the norm of the bounded linear operator $V_{\partial\Omega^i}$ from $C^0(\partial\Omega^i)$ into itself, such that

$$\begin{aligned} &\|(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)\|_{C^0(\partial\Omega^o)\times(C^0(\partial\Omega^i))^2\times\mathbb{R}^2} \\ &\qquad\qquad \leq C_1(C_2+\|(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)\|_{C^0(\partial\Omega^o)\times(C^0(\partial\Omega^i))^2\times\mathbb{R}^2})^\delta. \end{aligned} \tag{4.28}$$

Then by a straightforward calculation, one shows that (4.28) implies the validity of inequality (4.25) with

$$C = \max\{1, C_1(C_2+1)^{\frac{1}{1-\delta}}\}.$$

Hence, by the Leray-Schauder Theorem there exists at least one solution $(\mu^o_0,\mu^i_0,\eta^i_0,\rho^o_0,\rho^i_0) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ of the equation

$$(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i) = T_A(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i),$$

i.e. a solution in $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ of the nonlinear system (4.21). □

In the sequel we will exploit a continuity condition on the superposition operators generated by F_1 and F_2, namely

$$\begin{aligned} &\bullet \text{ The superposition operators } \mathcal{N}_{F_1} \text{ and } \mathcal{N}_{F_2} \\ &\quad \text{are continuous from } (C^{0,\alpha}(\partial\Omega^i))^2 \text{ into } C^{0,\alpha}(\partial\Omega^i). \end{aligned} \tag{4.29}$$

For conditions on F_1 and F_2 which imply the validity of assumption 4.34, we refer to Valent [76, Chap. II]. Then we can prove a regularity result for the fixed point provided by Proposition 4.3.3, and, in particular, an existence result for problem (4.3).

Proposition 4.3.4. *Let A be as in* (4.14). *Let assumptions* (4.23) *and* (4.29) *hold. Then the nonlinear system* (4.21) *is equivalent to the limiting system 4.20 and has at least a solution*

$$(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2.$$

In particular, problem (4.3) *has at least a solution* $(u_0^o, u_0^i) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \Omega^i) \times C^{1,\alpha}(\overline{\Omega^i})$ *given by*

$$(u_0^o, u_0^i) \equiv (U_{\Omega^i}^o[\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i], U_{\Omega^i}^i[\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i]) \tag{4.30}$$

where $(U_{\Omega^i}^o[\cdot,\cdot,\cdot,\cdot,\cdot], U_{\Omega^i}^i[\cdot,\cdot,\cdot,\cdot,\cdot])$ *are defined by* (4.7).

Proof. Let T_A be as in (4.22). By Proposition 4.3.3, there exists

$$(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i) \in C^{0}(\partial\Omega^o)_0 \times C^{0}(\partial\Omega^i) \times C^{0}(\partial\Omega^i)_0 \times \mathbb{R}^2$$

such that

$$(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i) = T_A(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i).$$

By the mapping properties of integral operators with real analytic kernel and no singularities (cf. Theorem A.2.1 (ii) in the Appendix)), $v_{\Omega^o}^+[\mu_0^o]_{|\partial\Omega^i}$ belongs to $C^{0,\alpha}(\partial\Omega^i)$. By classical results in potential theory (cf. Miranda [57, Chap. II, §14, III]), $V_{\partial\Omega^i}[\mu_0^i]$ and $V_{\partial\Omega^i}[\eta_0^i]$ belong to $C^{0,\alpha}(\partial\Omega^i)$. Then, by condition (4.29) and by the membership of $A \in M_2(C^{0,\alpha}(\partial\Omega^i))$, we obtain that

$$\begin{pmatrix} f^o \\ \mathcal{N}_{F_1}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \\ \mathcal{N}_{F_2}(v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o, V_{\partial\Omega^i}[\eta^i] + \rho^i) \end{pmatrix}$$
$$- \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 \\ v^+_{\Omega^o}[\mu^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i] + \rho^o \\ V_{\partial\Omega^i}[\eta^i] + \rho^i) \end{pmatrix}$$

belongs to the product space $C^{0,\alpha}(\partial\Omega^o) \times (C^{0,\alpha}(\partial\Omega^i))^2$. Finally, by the invertibility of the operator J_A from $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^{0,\alpha}(\partial\Omega^o) \times (C^{0,\alpha}(\partial\Omega^i))^2$, we obtain that $(\mu^o_0, \mu^i_0, \eta^i_0, \rho^o_0, \rho^i_0) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ and that (4.21) is equivalent to the limiting system (4.20). In particular, by Proposition 4.3.1 we deduce that the pair of functions given by (4.30) is a solution of problem (4.3) (cf. the definition of T_A in (4.22)). □

4.4 Application of the Implicit Function Theorem

In view of Proposition 4.2.1 and the equivalence of problem (4.6) with the integral equations (4.19), we introduce the auxiliary map $M = (M_1, M_2, M_3)$ from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^{0,\alpha}(\partial\Omega^o) \times (C^{0,\alpha}(\partial\Omega^i))^2$

defined by

$$
\begin{aligned}
& M_1[\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i](x) \equiv \left(-\frac{1}{2}I + W^*_{\partial\Omega^o}\right)[\mu^o](x) \\
& + \nu_{\Omega^o}(x) \cdot \nabla v^-_{\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](x) - f^o(x) \quad \forall x \in \partial\Omega^o, \\
& M_2[\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i](t) \\
& \equiv \left(\frac{1}{2}I + W^*_{\partial\mathbb{I}[\phi]}\right)[\mu^i \circ \phi^{(-1)}](\phi(t)) + \nu_{\mathbb{I}[\phi]}(\phi(t)) \cdot \nabla v^+_{\Omega^o}[\mu^o](\phi(t)) \\
& - F_1\left(t, v^+_{\Omega^o}[\mu^o](\phi(t)) + V_{\partial\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](\phi(t)) + \rho^o, V_{\partial\mathbb{I}[\phi]}[\eta^i \circ \phi^{(-1)}](\phi(t)) + \rho^i\right) \\
& \forall t \in \partial\Omega^i, \\
& M_3[\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i](t) \equiv \left(-\frac{1}{2}I + W^*_{\partial\mathbb{I}[\phi]}\right)[\eta^i \circ \phi^{(-1)}](\phi(t)) \\
& - F_2\left(t, v^+_{\Omega^o}[\mu^o](\phi(t)) + V_{\partial\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](\phi(t)) + \rho^o, V_{\partial\mathbb{I}[\phi]}[\eta^i \circ \phi^{(-1)}](\phi(t)) + \rho^i\right) \\
& \forall t \in \partial\Omega^i,
\end{aligned} \tag{4.31}
$$

for all $(\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$. Then, by the definition of M, we can deduce the following result.

Proposition 4.4.1. *Let A be as in* (4.14). *Let assumptions* (4.23) *and* (4.29) *hold. Let*

$$(\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2.$$

Then the pair of functions

$$(U^o_{\mathbb{I}[\phi]}[\mu^o, \mu^i \circ \phi^{(-1)}, \eta^i \circ \phi^{(-1)}, \rho^o, \rho^i], U^i_{\mathbb{I}[\phi]}[\mu^o, \mu^i \circ \phi^{(-1)}, \eta^i \circ \phi^{(-1)}, \rho^o, \rho^i]),$$

where $(U^o_{\mathbb{I}[\phi]}[\cdot,\cdot,\cdot,\cdot,\cdot], U^i_{\mathbb{I}[\phi]}[\cdot,\cdot,\cdot,\cdot,\cdot])$ are defined by (4.7), *is a solution of problem* (4.6) *if and only if*

$$M[\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i] = (0, 0, 0). \tag{4.32}$$

In particular,

$$M[\phi_0, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i] = (0,0,0) \tag{4.33}$$

is equivalent to the limiting system (4.20) *and has a solution* $(\mu^o_0, \mu^i_0, \eta^i_0, \rho^o_0, \rho^i_0) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$.

Proof. The statement follows by Proposition 4.2.1 and by the definition of M (cf. (4.19) and (4.31)). Finally by the definition of ϕ_0 (cf. (4.5)) , we obtain that, for all $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$, equation (4.33) is equivalent to (4.20) and the existence of $(\mu^o_0, \mu^i_0, \eta^i_0, \rho^o_0, \rho^i_0) \in C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ solution of (4.33) follows by Proposition 4.3.3. □

By Proposition 4.4.1, the study of problem (4.6) is reduced to that of equation (4.32). We now wish to apply the Implicit Function Theorem (cf. Theorem A.1.2 in Appendix A.1) to equation (4.32) around the limiting value ϕ_0. As a first step we have to analyse the regularity of the map M. In order to do that, we will need the following straightforward variant of a result proven by Lanza De Cristoforis and Rossi [53, Lemma 3.3].

Lemma 4.4.2. *Let* $\phi \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$. *Then there exists a unique element* $\tilde{\sigma}_n[\phi] \in C^{0,\alpha}(\partial\Omega^i)$ *such that*

$$\int_{\phi(\partial\Omega^i)} f(y)\, d\sigma_y = \int_{\partial\Omega^i} f(\phi(s))\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s \quad \forall f \in L^1(\phi(\partial\Omega^i))$$

Moreover the map from $A_{\partial\Omega^i}$ *to* $C^{0,\alpha}(\partial\Omega^i)$ *which takes* ϕ *to* $\tilde{\sigma}_n[\phi]$ *and the map from* $A_{\partial\Omega^i}$ *to* $C^{0,\alpha}(\partial\Omega^i)$ *which takes* ϕ *to* $(\nu_{\mathbb{I}[\phi]} \circ \phi)\, \tilde{\sigma}_n[\phi]$ *are real analytic.*

In the sequel we will exploit the following:

$$\begin{aligned}&\bullet \text{The superposition operators } \mathcal{N}_{F_1} \text{ and } \mathcal{N}_{F_2}\\&\quad \text{are real analytic from } (C^{0,\alpha}(\partial\Omega^i))^2 \text{ into } C^{0,\alpha}(\partial\Omega^i).\end{aligned} \tag{4.34}$$

For conditions on F_1 and F_2 which imply the validity of assumption (4.34), we refer to Valent [76, Chap. II]. We now show that M is real analytic.

Proposition 4.4.3. *Let assumption* (4.34) *holds. Then the map* M *is real analytic from* $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ *to* $C^{0,\alpha}(\partial\Omega^o) \times (C^{0,\alpha}(\partial\Omega^i))^2$.

Proof. We analyse separately M_1, M_2 and M_3. The map from $C^{0,\alpha}(\partial\Omega^o)$ into itself which takes μ^o to $\left(-\frac{1}{2}I + W^*_{\partial\Omega^o}\right)[\mu^o]$ is linear and continuous, so real analytic. The map from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^o)$ which takes (ϕ, μ^i) to the function of the variable x defined by

$$\begin{aligned}\nu_{\Omega^o} \cdot \nabla v^-_{\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](x) &= \nu_{\Omega^o} \cdot \nabla \left(\int_{\partial\Omega^i} S_n(x - \phi(t))\, \mu^i(t)\, \tilde{\sigma}_n[\phi](t)\, d\sigma_t \right)\\&= \left(\int_{\partial\Omega^i} (\nu_{\Omega^o} \cdot \nabla S_n(x - \phi(t)))\, \mu^i(t)\, \tilde{\sigma}_n[\phi](t)\, d\sigma_t \right)\end{aligned}$$

can be proven to be real analytic by observing that if $\phi \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$ then $x - \phi(t) \neq 0$ for all $(x,t) \in \partial\Omega^o \times \partial\Omega^i$ and by the properties of integral operators with real analytic kernels and no singularities (see Theorem A.2.1 (ii) in Appendix A.2). Finally f^o does not depend on $(\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$.

We now analyse M_2. The map from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϕ, μ^i) to the function of the variable t defined by

$$\begin{aligned}&\left(\frac{1}{2}I + W^*_{\partial\mathbb{I}[\phi]}\right)[\mu^i \circ \phi^{(-1)}](\phi(t)) = \frac{1}{2}\mu^i + W^*_{\partial\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](\phi(t))\\&= \frac{1}{2}\mu^i + \int_{\partial\Omega^i} (\nu_{\mathbb{I}[\phi]}(\phi(t))) \cdot \nabla S_n(\phi(t) - \phi(s))\, \mu^i(s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s\end{aligned}$$

is real analytic by a real analyticity result of dependence of single and double layer potentials upon perturbation of the support and of the density (see Lanza de Cristoforis and Rossi [53, Thm. 3.12] and Lanza de Cristoforis [48, Prop. 7]). The map from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϕ, μ^o) to the function of the variable t defined by

$$\nu_{\mathbb{I}[\phi]}(\phi(t)) \cdot \nabla v^+_{\Omega^o}[\mu^o](\phi(t)) = \int_{\partial\Omega^o} (\nu_{\mathbb{I}[\phi]}(\phi(t)) \cdot \nabla S_n(\phi(t) - y))\, \mu^o(y)\, d\sigma_y$$

can be proven to be real analytic by observing that if $\phi \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$ then $\phi(t) - y \neq 0$ for all $(y, t) \in \partial\Omega^o \times \partial\Omega^i$ and by the properties of integral operators with real analytic kernels and no singularities (see Theorem A.2.1 (ii) in Appendix A.2). For the third term of M_2 we proceed in this way. The map from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϕ, μ^o) to the function of the variable t defined by

$$v^+_{\Omega^o}[\mu^o](\phi(t)) = \int_{\partial\Omega^o} S_n(\phi(t) - y)\, \mu^o(y)\, d\sigma_y$$

can be proven to be real analytic by observing that if $\phi \in \mathcal{A}^{\Omega^o}_{\partial\Omega^i}$ then $\phi(t) - y \neq 0$ for all $(y, t) \in \partial\Omega^o \times \partial\Omega^i$ and by the properties of integral operators with real analytic kernels and no singularities (see Theorem A.2.1 (ii) in Appendix A.2). The map from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes (ϕ, μ^o) to the function of the variable t defined by

$$V_{\partial\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](\phi(t)) = \int_{\phi(\partial\Omega^i)} S_n(\phi(t) - y)\, \mu^i \circ \phi^{(-1)}(y)\, d\sigma_y$$

is real analytic by a result of real analytic dependence of single and double layer potentials upon perturbation of the support and of the density (see Lanza de Cristoforis and Rossi [53, Thm. 3.12] and Lanza de Cristoforis [48, Prop. 7]).

Similarly one can treat the term $V_{\partial\mathbb{I}[\phi]}[\eta^i \circ \phi^{(-1)}](\phi(\cdot))$. Hence, by the real analyticity of the composition of real analytic maps and by assumption (4.29), we conclude that the map from $\mathcal{A}^{\Omega^o}_{\partial\Omega^i} \times C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^{0,\alpha}(\partial\Omega^i)$ which takes a quintuple $(\phi, \mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ to the function

$$\mathcal{N}_{F_1}\left(v^+_{\Omega^o}[\mu^o](\phi(\cdot))_{|\partial\Omega^i} + V_{\partial\mathbb{I}[\phi]}[\mu^i \circ \phi^{(-1)}](\phi(\cdot)) + \rho^o, V_{\partial\mathbb{I}[\phi]}[\eta^i \circ \phi^{(-1)}](\phi(\cdot)) + \rho^i\right)$$

is real analytic.

Finally, we observe that M_3 can be treated in a similar way to M_2. The proof is complete. $\square$

In the sequel it will be convenient to consider F_1 and F_2 as two components of a vector field on $\partial\Omega^i \times \mathbb{R} \times \mathbb{R}$. Namely, we denote by F the function from $\partial\Omega^i \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}^2$ defined by

$$F(t, \zeta_1, \zeta_2) = (F_1(t, \zeta_1, \zeta_2), F_2(t, \zeta_1, \zeta_2)) \quad \forall (t, \zeta_1, \zeta_2) \in \partial\Omega^i \times \mathbb{R}^2. \tag{4.35}$$

Clearly, we can extend the definition of the superposition operator in a natural way, i.e. by setting

$$\mathcal{N}_F : (C^{0,\alpha}(\partial\Omega^i))^2 \to (C^{0,\alpha}(\partial\Omega^i))^2, \quad \mathcal{N}_F \equiv (\mathcal{N}_{F_1}, \mathcal{N}_{F_2}). \tag{4.36}$$

Now let $(\mu^o_0, \mu^i_0, \eta^i_0, \rho^o_0, \rho^i_0) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$ be as in Proposition 4.3.3. By standard calculus in Banach space, we have the following formula regarding the first order differential of $\mathcal{N}_F$:

$$d\mathcal{N}_F(v^+_{\Omega^o}[\mu^o_0]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu^i_0] + \rho^o_0, V_{\partial\Omega^i}[\eta^i_0] + \rho^i_0).(h_1, h_2) = A_{\mathcal{N}_F,0}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

for all $(h_1, h_2) \in (C^{0,\alpha}(\partial\Omega^i))^2$ where

$$A_{\mathcal{N}_F,0} \equiv \begin{pmatrix} \mathcal{N}_{\partial_{\zeta_1} F_1}(\alpha_0^1, \alpha_0^2) & \mathcal{N}_{\partial_{\zeta_2} F_1}(\alpha_0^1, \alpha_0^2) \\ \mathcal{N}_{\partial_{\zeta_1} F_2}(\alpha_0^1, \alpha_0^2) & \mathcal{N}_{\partial_{\zeta_2} F_2}(\alpha_0^1, \alpha_0^2) \end{pmatrix} \tag{4.37}$$

and α_0^1 and α_0^2 are functions from $\partial\Omega^i$ to $\mathbb{R}$ defined by

$$\alpha_0^1 \equiv v_{\Omega^o}^+[\mu_0^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu_0^i] + \rho_0^o,$$
$$\alpha_0^2 \equiv V_{\partial\Omega^i}[\eta_0^i] + \rho_0^i.$$

Then, we will require the following assumption:

- The matrix $A_{\mathcal{N}_F,0}$ given by (4.37) associated to the linear form $d\mathcal{N}_F(v_{\Omega^o}^+[\mu_0^o]_{|\partial\Omega^i} + V_{\partial\Omega^i}[\mu_0^i] + \rho_0^o, V_{\partial\Omega^i}[\eta_0^i] + \rho_0^i)$ satisfies assumption (4.14). (4.38)

In particular, we notice that assumption (4.34) implies the validity of the first of the three conditions of (4.14) for the matrix $A_{\mathcal{N}_F,0}$.

In order to apply the Implicit Function Theorem (cf. Theorem A.1.2 in Appendix A.1) to equation (4.32) we need to prove the invertibility of the partial differential of M. We do so, in the following proposition.

Proposition 4.4.4. *Let A be as in* (4.14)*. Let assumptions* (4.23)*,* (4.34) *and* (4.38) *hold. Then the partial differential of M with respect to $(\mu^o, \mu^i, \eta^i, \rho^o, \rho^i)$ evaluated at the point $(\phi_0, \mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i)$, which we denote by*

$$\partial_{(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)} M[\phi_0, \mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i], \tag{4.39}$$

is an isomorphism from $C^{0,\alpha}(\partial\Omega^o)_0 \times C^{0,\alpha}(\partial\Omega^i) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}^2$ to $C^{0,\alpha}(\partial\Omega^o) \times (C^{0,\alpha}(\partial\Omega^i))^2$.

Proof. By standard calculus in Banach spaces, one verifies that the partial differential (4.39) is the linear and continuous operator delivered by

$$\begin{aligned}
&\partial_{(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)}M_1[\phi_0,\mu_0^o,\mu_0^i,\eta_0^i,\rho_0^o,\rho_0^i].(\tilde{\mu}^o,\tilde{\mu}^i,\tilde{\eta}^i,\tilde{\rho}^o,\tilde{\rho}^i)(x)\\
&\qquad = \left(-\frac{1}{2}I + W^*_{\Omega^o}\right)[\tilde{\mu}^o](x) + \nu_{\Omega^o}\cdot\nabla v^-_{\Omega^i}[\tilde{\mu}^i](x) \qquad \forall x\in\partial\Omega^o,\\
&\partial_{(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)}M_2[\phi_0,\mu_0^o,\mu_0^i,\eta_0^i,\rho_0^o,\rho_0^i].(\tilde{\mu}^o,\tilde{\mu}^i,\tilde{\eta}^i,\tilde{\rho}^o,\tilde{\rho}^i)(t)\\
&\qquad = \left(\frac{1}{2}I + W^*_{\Omega^i}\right)[\tilde{\mu}^i](t) + \nu_{\Omega^i}\cdot\nabla v^+_{\Omega^o}[\mu^o](t)\\
&\qquad - \partial_{\zeta_1}F_1\left(t, v^+_{\Omega^o}[\mu_0^o](t) + V_{\Omega^i}[\mu_0^i](t) + \rho_0^o, V_{\Omega^i}[\eta_0^i](t) + \rho_0^i\right)\\
&\qquad \times\left(v^+_{\Omega^o}[\tilde{\mu}^o](t) + V_{\Omega^i}[\tilde{\mu}^i](t) + \tilde{\rho}^o\right)\\
&\qquad - \partial_{\zeta_2}F_1\left(t, v^+_{\Omega^o}[\mu_0^o](t) + V_{\Omega^i}[\mu_0^i](t) + \rho_0^o, V_{\Omega^i}[\eta_0^i](t) + \rho_0^i\right)\\
&\qquad \times\left(V_{\Omega^i}[\tilde{\eta}^i](t) + \tilde{\rho}^i\right) \qquad \forall t\in\partial\Omega^i,\\
&\partial_{(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)}M_3[\phi_0,\mu_0^o,\mu_0^i,\eta_0^i,\rho_0^o,\rho_0^i].(\tilde{\mu}^o,\tilde{\mu}^i,\tilde{\eta}^i,\tilde{\rho}^o,\tilde{\rho}^i)(t)\\
&\qquad = \left(-\frac{1}{2}I + W^*_{\Omega^i}\right)[\tilde{\eta}^i](t)\\
&\qquad - \partial_{\zeta_1}F_2\left(t, v^+_{\Omega^o}[\mu_0^o](t) + V_{\Omega^i}[\mu_0^i](t) + \rho_0^o, V_{\Omega^i}[\eta_0^i](t) + \rho_0^i\right)\\
&\qquad \times\left(v^+_{\Omega^o}[\tilde{\mu}^o](t) + V_{\Omega^i}[\tilde{\mu}^i](t) + \tilde{\rho}^o\right)\\
&\qquad - \partial_{\zeta_2}F_2\left(t, v^+_{\Omega^o}[\mu_0^o](t) + V_{\Omega^i}[\mu_0^i](t) + \rho_0^o, V_{\Omega^i}[\eta_0^i](t) + \rho_0^i\right)\\
&\qquad \times\left(V_{\Omega^i}[\tilde{\eta}^i](t) + \tilde{\rho}^i\right) \qquad \forall t\in\partial\Omega^i,
\end{aligned}$$

for all $(\tilde{\mu}^o,\tilde{\mu}^i,\tilde{\eta}^i,\tilde{\rho}^o,\tilde{\rho}^i)\in C^{0,\alpha}(\partial\Omega^o)_0\times C^{0,\alpha}(\partial\Omega^i)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}^2$. Then, by Proposition 4.1.3 (cf. (4.17)) and by assumption (4.38), we conclude that $\partial_{(\mu^o,\mu^i,\eta^i,\rho^o,\rho^i)}M[\phi_0,\mu_0^o,\mu_0^i,\eta_0^i,\rho_0^o,\rho_0^i]$ is an isomorphism of Banach spaces. □

Finally, by the Implicit Function Theorem (cf. Theorem A.1.2 in Appendix A.1), we can prove a real analytic dependence result for the densities that represent the solution of problem (4.6) upon the perturbation of the inner domain Ω^i given by the diffeomorphism ϕ.

Theorem 4.4.5. *Let A be as in* (4.14)*. Let assumptions* (4.23)*,* (4.34) *and* (4.38) *hold. Let*

$$(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i) \in C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$$

be as in Proposition 4.3.3. Then, there exist two open neighborhoods Q_0 of ϕ_0 in $\mathcal{A}^{\Omega^o}_{\partial\Omega^i}$ and U_0 of $(\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i)$ in $C^0(\partial\Omega^o)_0 \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)_0 \times \mathbb{R}^2$, and a real analytic map

$$(M^o[\cdot], M^i[\cdot], N^i[\cdot], R^o[\cdot], R^i[\cdot]) : Q_0 \to U_0$$

such that the set of zeros of M in $Q_0 \times U_0$ coincided with graph of the function $(M^o[\cdot], M^i[\cdot], N^i[\cdot], R^o[\cdot], R^i[\cdot])$. In particular we have

$$(M^o[\phi_0], M^i[\phi_0], N^i[\phi_0], R^o[\phi_0], R^i[\phi_0]) = (\mu_0^o, \mu_0^i, \eta_0^i, \rho_0^o, \rho_0^i). \tag{4.40}$$

Proof. It follows by Proposition 4.4.4, by Proposition 4.4.3 and by the Implicit Function Theorem for real analytic functions. The validity of (4.40) is a consequence of Propositions 4.3.4 and 4.4.1. □

4.5 Real analytic representation of the family of solutions

We are now ready to exhibit a family of solutions of problem (4.6) and to show its dependence upon the perturbation parameter ϕ. We begin with the following definition.

Definition 4.5.1. *Let Q_0 and $(M^o[\cdot], M^i[\cdot], N^i[\cdot], R^o[\cdot], R^i[\cdot])$ be as in Theo-*

rem 4.4.5. Then, for each $\phi \in Q_0$ *we set*

$$u^o_\phi(x) = U^o_{\mathbb{I}[\phi]}[M^o[\phi], M^i[\phi] \circ \phi^{(-1)}, N^i[\phi] \circ \phi^{(-1)}, R^o[\phi], R^i[\phi]](x) \quad \forall x \in \overline{\Omega^o} \setminus \mathbb{I}[\phi],$$
$$u^i_\phi(x) = U^i_{\mathbb{I}[\phi]}[M^o[\phi], M^i[\phi] \circ \phi^{(-1)}, N^i[\phi] \circ \phi^{(-1)}, R^o[\phi], R^i[\phi]](x) \quad \forall x \in \overline{\mathbb{I}[\phi]},$$

where $(U^o_{\mathbb{I}[\phi]}[\cdot,\cdot,\cdot,\cdot,\cdot], U^i_{\mathbb{I}[\phi]}[\cdot,\cdot,\cdot,\cdot,\cdot])$ *are defined by* (4.7).

By Proposition 4.4.1, Theorem 4.4.5 and Definition 4.5.1 we deduce that the pair (u^o_ϕ, u^i_ϕ) is a solution of problem (4.6) for all $\phi \in Q_0$ and by Proposition 4.3.4 we deduce that the pair $(u^o_{\phi_0}, u^i_{\phi_0})$ is a solution of problem (4.3) . Namely, the following holds.

Theorem 4.5.2. *Let* A *be as in* (4.14)*. Let assumptions* (4.23), (4.34) *and* (4.38) *hold. Let* Q_0 *be as in Theorem 4.4.5 and let* (u^o_ϕ, u^i_ϕ) *be as in Definition 4.5.1. Then, for all* $\phi \in Q_0$

$$(u^o_\phi, u^i_\phi) \in C^{1,\alpha}(\overline{\Omega^o} \setminus \mathbb{I}[\phi]) \times C^{1,\alpha}(\overline{\mathbb{I}[\phi]})$$

is a solution of problem (4.6)*. In particular*

$$(u^o_{\phi_0}, u^i_{\phi_0}) = (u^o_0, u^i_0)$$

is a solution of problem (4.3)*.*

We are now ready to prove our main result of this chapter, where we show that suitable restrictions of the functions u^o_ϕ and u^i_ϕ depend real analytically on the parameter ϕ which determines the domain perturbation.

Theorem 4.5.3. *Let* A *be as in* (4.14)*. Let assumptions* (4.23), (4.34) *and* (4.38) *hold. Let* Q_0 *be as in Theorem 4.4.5. The following statements hold.*

(i) Let $\Omega_{\mathtt{int}}$ *be a bounded open subset of* Ω^o*. Let* $Q_{\mathtt{int}} \subseteq Q_0$ *be an open*

neighborhood of ϕ_0 *such that*

$$\overline{\Omega_{\mathtt{int}}} \subseteq \mathbb{I}[\phi] \quad \forall \phi \in Q_{\mathtt{int}}.$$

Then there exists a real analytic map $U^i_{\mathtt{int}}$ *from* $Q_{\mathtt{int}}$ *to* $C^{1,\alpha}(\overline{\Omega_{\mathtt{int}}})$ *such that*

$$u^i_\phi(x) = U^i_{\mathtt{int}}[\phi](x) \quad \forall x \in \overline{\Omega_{\mathtt{int}}}.$$

(ii) Let $\Omega_{\mathtt{ext}}$ *be a bounded open subset of* Ω^o*. Let* $Q_{\mathtt{ext}} \subseteq Q_0$ *be an open neighborhood of* ϕ_0 *such that*

$$\overline{\Omega_{\mathtt{ext}}} \subseteq \Omega^o \setminus \overline{\mathbb{I}[\phi]} \quad \forall \phi \in Q_{\mathtt{ext}}.$$

Then there exists a real analytic map $U^o_{\mathtt{ext}}$ *from* $Q_{\mathtt{ext}}$ *to* $C^{1,\alpha}(\overline{\Omega_{\mathtt{ext}}})$ *such that*

$$u^o_\phi(x) = U^o_{\mathtt{ext}}[\phi](x) \quad \forall x \in \overline{\Omega_{\mathtt{ext}}}.$$

Proof. We first prove (i). By Definition 4.5.1 and by (4.7), we have

$$\begin{aligned} u^i_\phi(x) &= U^i_{\mathbb{I}[\phi]}[M^o[\phi], M^i[\phi] \circ \phi^{(-1)}, N^i[\phi] \circ \phi^{(-1)}, R^o[\phi], R^i[\phi]](x) \\ &= v^+_{\mathbb{I}[\phi]}[N^i[\phi] \circ \phi^{(-1)}](x) + R^i[\phi] \\ &= \int_{\phi(\partial\Omega^i)} S_n(x-y) \left(N^i[\phi] \circ \phi^{(-1)}\right)(y)\, d\sigma_y + R^i[\phi] \\ &= \int_{\partial\Omega^i} S_n(x-\phi(s))\, N^i[\phi](s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s + R^i[\phi] \qquad \forall x \in \overline{\mathbb{I}[\phi]} \end{aligned}$$

for all $\phi \in Q_0$. Then it is natural to define

$$U^i_{\mathtt{int}}[\phi](\cdot) \equiv \int_{\partial\Omega^i} S_n(\cdot - \phi(s))\, N^i[\phi](s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s + R^i[\phi] \qquad \forall \phi \in Q_{\mathtt{int}}.$$

By assumption $Q_{\mathtt{int}} \subseteq Q_0$ and Theorem 4.4.5, we know that the map from

$Q_{\mathtt{int}}$ to $\mathbb{R}$ which takes ϕ to $R^i[\phi]$ is real analytic. Moreover, by the real analyticity of $N^i[\cdot]$ (cf. Theorem 4.4.5) and by the properties of integral operators with real analytic kernels and no singularities (cf. Theorem A.2.1 (i) in Appendix A.2), one can prove that the map from $Q_{\mathtt{int}}$ to $C^{1,\alpha}(\overline{\Omega_{\mathtt{int}}})$ which takes ϕ to the function

$$\int_{\partial\Omega^i} S_n(\cdot - \phi(s))\, N^i[\phi](s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s$$

is real analytic. Hence, one deduces the validity of (i).

We now prove (ii). By Definition 4.5.1 and by (4.7), we have

$$\begin{aligned} u^o_\phi(x) &= U^o_{\mathbb{I}[\phi]}[M^o[\phi], M^i[\phi]\circ\phi^{(-1)}, N^i[\phi]\circ\phi^{(-1)}, R^o[\phi], R^i[\phi]](x) \\ &= v^+_{\Omega^o}[M^o[\phi]](x) + v^-_{\mathbb{I}[\phi]}[M^i[\phi]\circ\phi^{(-1)}](x) + R^o[\phi] \\ &= v^+_{\Omega^o}[M^o[\phi]](x) + \int_{\phi(\partial\Omega^i)} S_n(x-y)\, \left(M^i[\phi]\circ\phi^{(-1)}\right)(y)\, d\sigma_y + R^o[\phi] \\ &= v^+_{\Omega^o}[M^o[\phi]](x) + \int_{\partial\Omega^i} S_n(x-\phi(s))\, M^i[\phi](s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s + R^o[\phi] \end{aligned}$$

for all $x \in \overline{\Omega^o} \setminus \mathbb{I}[\phi]$ and for all $\phi \in Q_0$. Then it is natural to define

$$U^o_{\mathtt{ext}}[\phi](\cdot) \equiv v^+_{\Omega^o}[M^o[\phi]](\cdot) + \int_{\partial\Omega^i} S_n(\cdot - \phi(s))\, M^i[\phi](s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s + R^o[\phi]$$

for all $\phi \in Q_{\mathtt{ext}}$. Since $M^o[\cdot]$ is real analytic (cf. Theorem 4.4.5), since $v^+_{\Omega^o}[\cdot]$ is linear and continuous from $C^{0,\alpha}(\partial\Omega^o)$ to $C^{1,\alpha}(\overline{\Omega^o})$ (cf. Theorem 1.3.3 (ii)), and since the restriction operator from $C^{1,\alpha}(\overline{\Omega^o})$ to $C^{1,\alpha}(\overline{\Omega_{\mathtt{ext}}})$ is linear and continuous (cf. hypothesis $\overline{\Omega_{\mathtt{ext}}} \subseteq \Omega^o \setminus \overline{\mathbb{I}[\phi]}$ for all $\phi \in Q_{\mathtt{ext}}$), then the map from $Q_{\mathtt{ext}}$ to $C^{1,\alpha}(\overline{\Omega_{\mathtt{ext}}})$ which takes ϕ to $v^+_{\Omega^o}[M^o[\phi]]_{|\overline{\Omega_{\mathtt{ext}}}}$ is real analytic. By the assumption $Q_{\mathtt{ext}} \subseteq Q_0$ and by Theorem 4.4.5, we know that the map from $Q_{\mathtt{ext}}$ to $\mathbb{R}$ which takes ϕ to $R^i[\phi]$ is real analytic. Finally, by the real analyticity of $M^i[\cdot]$ (cf. Theorem 4.4.5) and by the properties of integral operators with real analytic kernels and no singularities (cf. Theorem A.2.1 (i) in Appendix A.2), one can prove that the map from $Q_{\mathtt{ext}}$ to $C^{1,\alpha}(\overline{\Omega_{\mathtt{ext}}})$ which takes ϕ to the function

$$\int_{\partial\Omega^i} S_n(\cdot - \phi(s))\, M^i[\phi](s)\, \tilde{\sigma}_n[\phi](s)\, d\sigma_s$$

is real analytic. Hence, one deduces the validity of (ii). □

APPENDIX A

Implicit Function Theorem, integral operators with real analytic kernels and Leray Schauder Theorem

A.1 Real Analytic maps in Banach spaces and the Implicit Function Theorem

In this section we recall the definition of real analytic map acting between Banach spaces. Moreover we present the classical Implicit Function Theorem, in the framework of real analytic maps between Banach spaces.

We will denote by X' the space of continuous linear functionals from X to $\mathbb{R}$, namely $X' = \mathcal{L}(X, \mathbb{R})$. Moreover, if $N \in \mathbb{N} \setminus \{0\}$, X_1, ..., X_N are Banach spaces, and i_1, ..., i_N are positive natural numbers, then $\mathcal{L}^{i_1,\dots,i_N}(X_1, \dots, X_N; X)$ will denote the space of continuous multilinear maps from $X_1^{i_1} \times \cdots \times X_N^{i_N}$ to X endowed with the norm

$$\|a\|_{\mathcal{L}^{i_1,\dots,i_N}(X_1,\dots,X_N;X)} = \sup_Q \|a[x_{1,1},\dots,x_{1,i_1},\dots,x_{N,1},\dots,x_{N,i_N}]\|_X$$

where

$$Q = \{(x_{1,1},\dots,x_{1,i_1},\dots,x_{N,1},\dots,x_{N,i_N}) \in X_1^{i_1} \times \cdots \times X_N^{i_N} : \\ \|x_{1,1}\|_{X_1} \le 1,\dots,\|x_{i,i_1}\|_{X_1} \le 1,\dots,\|x_{N,1}\|_{X_N} \le 1,\dots,\|x_{N,i_N}\|_{X_N} \le 1\}.$$

Finally, to shorten our notation, we set

$$[x_1^{(i_1)},\dots,x_N^{(i_N)}] = [\underbrace{x_1,\dots,x_1}_{i_1-\text{times}},\dots,\underbrace{x_N,\dots,x_N}_{i_N-\text{times}}]\,. \tag{A.1}$$

We now recall the definition of real analytic maps from a Banach space X to a Banach space Y (see, for example, Deimling [31]).

Definition A.1.1. *Let X, Y be real Banach spaces. Let U be an open subset of X. We say that a function f from U to Y is real analytic if for every $x \in U$ there are $\rho, M \in]0,+\infty[$ and multilinear maps $a_j(x) \in \mathcal{L}^j(X,Y)$, with $j \in \mathbb{N}$, such that*

$$\|a_j(x)\|_{\mathcal{L}^j(X,Y)} \le M \left(\frac{1}{\rho}\right)^j \qquad \forall j \in \mathbb{N}$$

and

$$f(y) = \sum_{j=0}^{+\infty} a_j(x)[(y-x)^j] \qquad \forall y \in B_X(x,\rho).$$

Then we have the following well known result for real analytic functions in Banach spaces: the Implicit Function Theorem. For a proof, see Deimling [31, Thm. 15.3].

Theorem A.1.2 (Implicit Function Theorem)**.** *Let X, Y, Z be real Banach spaces. Let W be an open subset of $X \times Y$ and $(x_0, y_0) \in W$. Let F from W to Z be a real analytic function and $F(x_0, y_0) = 0$. Let the differential $\partial_y F(x_0, y_0)$ of the map $F(x_0, \cdot)$ at $y = y_0$ be a homeomorphisms of Y onto Z. Then there exists an open neighborhood U of x_0 in X and an open neighborhood V of y_0 in Y such that $U \times V \subset W$ and an real analytic map ϕ of U to V such that the zeros set of F in $U \times V$ coincides with the graph of ϕ in U, namely*

$$\{(x, y) \in U \times V : F(x, y) = 0\} = \{(x, \phi(x)) : x \in U\}.$$

A.2 Integral operators with real analytic kernel and no singularities

In this section we recall some results on integral operators with real analytic kernel and no singularities. The proof of the first result, which implies all the others theorems presented in this section, can be found in Lanza de Cristoforis and Musolino [51, Prop. 4.1].

Theorem A.2.1. *Let $h_1, h_2 \in \mathbb{N} \setminus \{0\}$. Let $m \in \mathbb{N}$, $\alpha \in]0, 1]$. Let Y be a topological space. Let $\mathcal{M}$ be a σ-algebra of Y containing the Borel sets of Y. Let μ be a measure on $\mathcal{M}$. Let Z be a Banach space. Let W be a non-empty open subset of $\mathbb{R}^{h_1} \times \mathbb{R}^{h_2} \times Z$. Let G be a real analytic map from W to $\mathbb{R}$. Then the following statements holds.*

(i) Let $n \in \mathbb{N} \setminus \{0\}$. Let Ω_1 be a bounded open connected subset of $\mathbb{R}^n$. Let Ω_1 be regular in the sense of Whitney. Let

$$\tilde{F} \equiv \Big\{(\psi, \phi, z) \in C^{m,\alpha}(\overline{\Omega_1}, \mathbb{R}^{h_1}) \times C^0_b(Y, \mathbb{R}^{h_2}) \times Z : \\ \psi(\overline{\Omega_1}) \times \overline{\phi(Y)} \times \{z\} \subseteq W\Big\}.$$

Then the map $\tilde{\mathcal{H}}_G$ from $\tilde{F} \times L^1(Y)$ to $C^{m,\alpha}(\overline{\Omega_1})$ defined by

$$\tilde{\mathcal{H}}_G[\psi, \phi, z, f](t) \equiv \int_Y G(\psi(t), \phi(y), z) f(y)\, d\sigma_y \quad \forall t \in \overline{\Omega_1},$$

for all $(\psi, \phi, z, f) \in \tilde{F} \times L^1(Y)$, is real analytic.

(ii) Let $n \in \mathbb{N} \setminus \{0, 1\}$. Let Ω_1 be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{\max\{1,m\},\alpha}$. Let

$$F^\# \equiv \Big\{(\psi, \phi, z) \in C^{m,\alpha}(\partial\Omega_1, \mathbb{R}^{h_1}) \times C^0_b(Y, \mathbb{R}^{h_2}) \times Z : \\ \psi(\partial\Omega_1) \times \overline{\phi(Y)} \times \{z\} \subseteq W\Big\}.$$

Then the map $\mathcal{H}^\#_G$ from $F^\# \times L^1(Y)$ to $C^{m,\alpha}(\partial\Omega_1)$ defined by

$$\mathcal{H}^\#_G[\psi, \phi, z, f](t) \equiv \int_Y G(\psi(t), \phi(y), z) f(y)\, d\sigma_y \quad \forall t \in \partial\Omega_1,$$

for all $(\psi, \phi, z, f) \in F^\# \times L^1(Y)$, is real analytic.

Then, from the previous Theorem one can deduce the following results widely used in our work.

Theorem A.2.2. *Let $n \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let Ω, Ω_1 be bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$.*

(i) Let W be an open subset of $\mathbb{R}^n$. Let J_1, J_2 be open intervals of R such that

$$\Big\{\epsilon_1 x - \epsilon_2 y : \epsilon_1 \in J_1, \epsilon_2 \in J_2, x \in \overline{\Omega_1}, y \in \partial\Omega\Big\} \subseteq W.$$

If G is an analytic function from W to $\mathbb{R}$ and $m \in \mathbb{N}$, then the map from $J_1 \times J_2 \times L^1(\partial\Omega)$ to $C^{m,\alpha}(\overline{\Omega_1})$ which takes $(\epsilon_1, \epsilon_2, \theta)$ to the function

$$\int_{\partial\Omega} G(\epsilon_1 x - \epsilon_2 y)\theta(y)\, d\sigma_y \quad \forall x \in \overline{\Omega_1},$$

is real analytic.

(ii) *Let W be an open subset of $\mathbb{R}^n$. Let J_1, J_2 be open intervals of R such that*

$$\{\epsilon_1 x - \epsilon_2 y : \epsilon_1 \in J_1, \epsilon_2 \in J_2, x \in \partial\Omega_1, y \in \partial\Omega\} \subseteq W.$$

If G is an analytic function from W to $\mathbb{R}$, then the map from $J_1 \times J_2 \times L^1(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega_1)$ which takes $(\epsilon_1, \epsilon_2, \theta)$ to the function

$$\int_{\partial\Omega} G(\epsilon_1 x - \epsilon_2 y)\theta(y)\, d\sigma_y \quad \forall x \in \partial\Omega_1,$$

is real analytic.

(iii) *Let $h_1 \in \mathbb{N}\backslash\{0\}$. Let $m \in \{0,1\}$. Let W be an open subset of $\mathbb{R}^{h_1} \times \mathbb{R} \times \mathbb{R}$. Let G be an analytic function from W to $\mathbb{R}$. Then the map from the set*

$$\mathcal{A} \equiv \Big\{(\psi, \epsilon) \in C^{m,\alpha}(\partial\Omega_1, \mathbb{R}^{h_1}) \times \mathbb{R} : \psi(\partial\Omega_1) \times [0,1] \times \{\epsilon\} \subseteq W\Big\}$$

to $C^{m,\alpha}(\partial\Omega_1)$ which takes (ψ, ϵ) to the function

$$\int_0^1 G(\psi(x), \beta, \epsilon)\, d\beta \quad \forall x \in \partial\Omega_1,$$

is real analytic.

A.3 Leray-Schauder Theorem

In this section we recall a well known fixed point theorem which follows by the invariance of the Leray-Schauder topological degree. For a proof see Gilbarg and Trudinger [35, Thm. 11.3].

Theorem A.3.1 (Leray-Schauder Theorem)**.** *Let X be a Banach space. Let T be a continuous (nonlinear) operator from X to itself which maps bounded sets to sets with a compact closure. If there exists a constant $M \in]0, +\infty[$ such that*

$$\|x\|_X \leq M \quad \textit{for all } (x, \lambda) \in X \times [0,1] \quad \textit{satisfying } x = \lambda T(x),$$

then T has at least one fixed point $x \in X$ such that

$$\|x\|_X \leq M.$$

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Whatever I would write here, as maybe all the thesis, it will be useless. Nothing lasts forever. Once I read a quotation of Jorge Luis Borges saying:

"Nothing is built on stone, all is built on sand
but we must build as if the sand were stone".

We are what we read, our readings shape us. Personally, I deeply believe that what really matters is not what you achieve but what you overcome to get there. This holds for PhD as for life too. Hence, what really last are places, moments, and memories of people whom we share our time with.

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To my family