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Communicated by: S. Nicaise

We investigate the behavior of the solutions of a mixed problem for the Laplace equation in a domain Ω. On a part of the boundary $\partial\Omega$, we consider a Neumann condition, whereas in another part, we consider a nonlinear Robin condition, which depends on a positive parameter δ in such a way that for $\delta = 0$ it degenerates into a Neumann condition. For δ small and positive, we prove that the boundary value problem has a solution $u(\delta, \cdot)$. We describe what happens to $u(\delta, \cdot)$ as $\delta \to 0$ by means of representation formulas in terms of real analytic maps. Then, we confine ourselves to the linear case, and we compute explicitly the power series expansion of the solution.

KEYWORDS
boundary value problems for second-order elliptic equations, integral equations methods, Laplace operator, Neumann problem, Robin problem, singularly perturbed problem

1 | INTRODUCTION

In this paper, we study the asymptotic behavior of the solutions of a boundary value problem for the Laplace equation with a (nonlinear) Robin boundary condition, which degenerates into a Neumann condition.

Boundary value problems with perturbed Robin or mixed conditions have been investigated by several authors. For example, Wendland et al[1] considered a family of Poincaré problems approximating a mixed boundary value problem for the Laplace equation in the plane. Kirsch[2] studied the convergence of the solution of the Helmholtz equation with boundary condition of the type $-\epsilon\frac{\partial u}{\partial\nu}+u=g$ to the solution with Dirichlet condition $u = g$ as $\epsilon \to 0$. Costabel and Dauge[3] studied a mixed Neumann-Robin problem for the Laplace operator, where the Robin condition contains a parameter ϵ so that it tends to a Dirichlet condition as $\epsilon \to 0$. An extension to nonlinear equation has been considered, for example, in Berestycki and Wei.[4] Degenerating nonlinear Robin conditions in the frame of homogenization problems have been studied by Gómez et al.[5] Singularly perturbed boundary conditions for the Maxwell equations have been analyzed, for example, in Ammari and Nédélec.[6] Moreover, Schmidt and Hiptmair[7] have exploited integral equation methods for singularly perturbed boundary conditions in the frame of transmission problems. Furthermore, an approach based on potential theory to prove the solvability of a small nonlinear perturbation of a homogeneous linear transmission problem can be found

in Dalla Riva and Mishuris.[8] Concerning existence and uniqueness results for boundary value problems with nonlinear Robin conditions, we also mention, eg, Donato et al.[9]

We note that the transmission problem for a composite domain with imperfect (nonnatural) conditions along the joint boundary is, in fact, a generalization of the classical Robin problem. Such transmission conditions frequently appear in practical applications for various nonlinear multiphysics problems (eg, Mishuris et al[10,11] and Mishuris[12]). Moreover, the imperfect transmission conditions allow one to perform numerical analysis of practical problems with thin interphases at low cost with sufficient accuracy (see Mishuris and Öchsner,[13] Mishuris et al,[14] and Sonato et al[15]).

In this paper, instead, we are interested in the case where the Robin condition degenerates into a Neumann condition. To introduce the problem, we first define the geometric setting. We fix once for all a natural number

$$n \in \mathbb{N}\setminus\{0, 1\}.$$

Then, we consider $\alpha \in]0, 1[$ and two subsets Ω^i, Ω^o of $\mathbb{R}^n$ satisfying the following assumption:

$$\Omega^i \text{ and } \Omega^o \text{ are bounded open connected subsets of } \mathbb{R}^n \text{ of class } C^{1,\alpha}$$

$$\text{such that } \overline{\Omega^i} \subseteq \Omega^o \text{ and that } \mathbb{R}^n\setminus\overline{\Omega^i} \text{ and } \mathbb{R}^n\setminus\overline{\Omega^o} \text{ are connected.}$$

For the definition of sets and functions of the Schauder class $C^{k,\alpha}$ ($k \in \mathbb{N}$), we refer, eg, to Gilbarg and Trudinger.[16] The letter "i" stands for "inner" and the letter "o" stands for "outer." The symbol "‾" denotes the closure. Then, we introduce the domain Ω by setting

$$\Omega \equiv \Omega^o\setminus\overline{\Omega^i}.$$

We note that the boundary $\partial\Omega$ of Ω consists of the two connected components $\partial\Omega^o$ and $\partial\Omega^i$. Therefore, we can identify, for example, $C^{0,\alpha}(\partial\Omega)$ with the product $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$. To define the boundary data, we fix two functions

$$g^o \in C^{0,\alpha}(\partial\Omega^o), \qquad g^i \in C^{0,\alpha}(\partial\Omega^i).$$

Then, we take $\delta_0 > 0$ and a family $\{F_\delta\}_{\delta\in]0,\delta_0[}$ of functions from $\mathbb{R}$ to $\mathbb{R}$. Next, for each $\delta \in]0, \delta_0[$, we want to consider a nonlinear boundary value problem for the Laplace operator. Namely, we consider a Neumann condition on $\partial\Omega^o$ and a nonlinear Robin condition on $\partial\Omega^i$. Thus, for each $\delta \in]0, \delta_0[$, we consider the following boundary value problem:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega, \\ \frac{\partial}{\partial\nu_{\Omega^o}}u(x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{\partial}{\partial\nu_{\Omega^i}}u(x) = \delta F_\delta(u(x)) + g^i(x) & \forall x \in \partial\Omega^i, \end{cases} \tag{1}$$

where ν_{Ω^o} and ν_{Ω^i} denote the outward unit normal to $\partial\Omega^o$ and to $\partial\Omega^i$, respectively.

As a first step, under suitable assumptions, in this paper, we show that for each δ positive and small enough, problem (1) has a solution, which we denote by $u(\delta, \cdot)$. Then, we are interested in studying the behavior of $u(\delta, \cdot)$ as $\delta \to 0$, and thus, we pose the following questions.

(1) Let x be a fixed point in Ω. What can be said of the map $\delta \mapsto u(\delta, x)$ when δ is close to 0 and positive?
(2) What can be said of the map $\delta \mapsto \int_\Omega |\nabla u(\delta, x)|^2 dx$ when δ is close to 0 and positive?

We also note that if in correspondence of the limiting value $\delta = 0$, we omit the term

$$\delta F_\delta(u(x))$$

in (1), then we obtain the Neumann problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega, \\ \frac{\partial}{\partial\nu_{\Omega^o}}u(x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{\partial}{\partial\nu_{\Omega^i}}u(x) = g^i(x) & \forall x \in \partial\Omega^i. \end{cases} \tag{2}$$

On the other hand, by the divergence theorem and classical existence results for the Neumann problem, problem (2) has (at least) a solution if and only if

$$\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma = 0. \tag{3}$$

This means, in particular, that if (3) does not hold, then $u(\delta, \cdot)$ cannot converge to a solution of problem (2) as $\delta \to 0$.

In contrast with asymptotic expansion methods, in this paper, we answer the questions in (1), (2) by representing the maps of (1), (2) in terms of real analytic maps in Banach spaces and in terms of known functions of δ (for the definition and properties of real analytic maps, we refer to Deimling[17, p. 150]). We observe that if, for example, we know that the function in (1) equals for $\delta > 0$ a real analytic function defined in a whole neighborhood of $\delta = 0$, then we know that such a map can be expanded in power series for δ small.

Such an approach has been proposed by Lanza de Cristoforis[18] for the analysis of singularly perturbed problems in perforated domain as an alternative to asymptotic expansion methods (cf, eg, Maz'ya et al[19] and Maz'ya et al[20,21]). In particular, it has been exploited to analyze singularly perturbed (linear and nonlinear) Robin and mixed problems in domains with small holes (cf, eg, Lanza de Cristoforis[22] and Dalla Riva and Musolino[23] for the Laplace equation and Dalla Riva and Lanza de Cristoforis[24,25] for the Lamé equations).

The paper is organized as follows. In Section 2, we consider some model problems in an annular domain where we can explicitly construct the solutions and discuss the behavior as δ tends to 0. In Section 3, we formulate our problem in terms of integral equations. In Section 4, we prove our main result, which answers our questions (1), (2) above, and in Section 5 we discuss a local uniqueness property of the family of solutions. Finally, in Section 6, we make some comments on the linear case and compute the power series expansion of the solution.

2 | MODEL PROBLEMS

To illustrate some aspects of the problem under investigation, in this section, we consider the set

$$\Omega \equiv \mathbb{B}_n(0,1) \backslash \overline{\mathbb{B}_n(0,1/2)},$$

ie, we take $\Omega^o \equiv \mathbb{B}_n(0,1)$ and $\Omega^i \equiv \mathbb{B}_n(0,1/2)$, where, for $r > 0$, the symbol $\mathbb{B}_n(0,r)$ denotes the open ball in $\mathbb{R}^n$ of center 0 and radius r.

2.1 | A linear problem

We begin with a linear problem and to do so, we take $a, b \in \mathbb{R}$. Then, for each $\delta \in]0, +\infty[$, we consider the problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_n(0,1) \backslash \overline{\mathbb{B}_n(0,1/2)}, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1)}} u(x) = a & \forall x \in \partial \mathbb{B}_n(0,1), \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1/2)}} u(x) = \delta u(x) + b & \forall x \in \partial \mathbb{B}_n(0,1/2). \end{cases} \tag{4}$$

As is well known, for each $\delta \in]0, +\infty[$, problem (4) has a unique solution in $C^{1,\alpha}(\bar{\Omega})$, and we denote it by u_δ. On the other hand, if instead we put $\delta = 0$ in (4) we obtain

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_n(0,1) \backslash \overline{\mathbb{B}_n(0,1/2)}, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1)}} u(x) = a & \forall x \in \partial \mathbb{B}_n(0,1), \\ \frac{\partial}{\partial \nu_{\mathbb{B}_n(0,1/2)}} u(x) = b & \forall x \in \partial \mathbb{B}_n(0,1/2). \end{cases} \tag{5}$$

The solvability of problem (5) is subject to a compatibility condition on the Neumann data on $\partial \mathbb{B}_n(0,1)$ and on $\partial \mathbb{B}_n(0,1/2)$. More precisely, problem (5) has a solution if and only if

$$a \int_{\partial \mathbb{B}_n(0,1)} d\sigma - b \int_{\partial \mathbb{B}_n(0,1/2)} d\sigma = 0,$$

ie, if and only if

$$a s_n - b \frac{1}{2^{n-1}} s_n = 0, \tag{6}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial \mathbb{B}_n(0,1)$. Condition (6) can be rewritten as follows:

$$a = \frac{b}{2^{n-1}}. \tag{7}$$

In particular, if $a = \frac{b}{2^{n-1}}$ then, the Neumann problem (5) has a 1-dimensional space of solutions; if instead $a \neq \frac{b}{2^{n-1}}$, problem (5) does not have any solution.

This implies that in general the solution u_δ of problem (4) cannot converge to a solution of (5) as $\delta \to 0$, if the compatibility condition (7) does not hold. Therefore, we wish to understand the behavior of u_δ as $\delta \to 0$, and we do so by constructing explicitly u_δ.

To construct the solution u_δ, we consider separately case $n = 2$ and $n \geq 3$.

If $n = 2$, we look for the function u_δ in the form

$$u_\delta(x) \equiv A_\delta \log |x| + B_\delta \qquad \forall x \in \bar{\Omega},$$

with A_δ and B_δ to be set so that the boundary conditions of problem (4) are satisfied.

We first note that

$$\nabla u_\delta(x) = A_\delta \frac{x}{|x|^2},$$

and that accordingly

$$\frac{\partial u_\delta(x)}{\partial \nu_{\mathbb{B}_2(0,1)}} = \frac{x}{|x|} \cdot A_\delta \frac{x}{|x|^2} = A_\delta \qquad \forall x \in \partial \mathbb{B}_2(0, 1),$$

which implies that we must have

$$A_\delta = a$$

in order to fulfill the Neumann condition on $\partial \mathbb{B}_n(0, 1)$. On the other hand, as far as the Robin condition on $\partial \mathbb{B}_2(0, 1/2)$ is concerned, we must find B_δ such that

$$\frac{x}{|x|} \cdot a \frac{x}{|x|^2} = \delta(a \log |x| + B_\delta) + b \qquad \forall x \in \partial \mathbb{B}_2(0, 1/2).$$

Then, a straightforward computation implies that we must have

$$B_\delta = \frac{1}{\delta}(2a - b) + a \log 2.$$

As a consequence, if $n = 2$, we have

$$u_\delta(x) \equiv a \log |x| + a \log 2 + \frac{1}{\delta}(2a - b) \qquad \forall x \in \bar{\Omega}. \tag{8}$$

Then, we turn to consider the case of dimension $n \geq 3$, and we look for a solution of problem (4) in the form

$$u_\delta(x) \equiv A_\delta \frac{1}{(2 - n)|x|^{n-2}} + B_\delta \qquad \forall x \in \bar{\Omega},$$

with A_δ and B_δ to be set so that the boundary conditions of problem (4) are satisfied. By arguing as above, one deduces that

$$A_\delta = a, \qquad B_\delta = \frac{1}{\delta}(2^{n-1}a - b) + a \frac{2^{n-2}}{n - 2},$$

and thus,

$$u_\delta(x) \equiv a \frac{1}{(2 - n)|x|^{n-2}} + a \frac{2^{n-2}}{n - 2} + \frac{1}{\delta}(2^{n-1}a - b) \qquad \forall x \in \bar{\Omega}. \tag{9}$$

Thus, by looking at (8) and (9), we note that if condition (7) does not hold, then,

$$\lim_{\delta \to 0} \|u_\delta\|_\infty = +\infty.$$

Comparing (8) and (9) one can write the solutions in a uniform manner:

$$u_\delta(x) = u^{(0)}(x) + \frac{1}{\delta}u^{(1)}(x), \tag{10}$$

where

$$u^{(0)}(x) \equiv \begin{cases} a\log|x| + a\log 2 & \text{if } n = 2, \\ a\frac{1}{(2-n)|x|^{n-2}} + a\frac{2^{n-2}}{n-2} & \text{if } n \geq 3, \end{cases} \quad u^{(1)}(x) \equiv 2^{n-1}a - b, \qquad \forall x \in \bar{\Omega},$$

and both functions $u^{(0)}, u^{(1)} \in L_\infty(\Omega)$. In particular, we note that $u^{(0)}$ is the unique solution of (5) such that

$$\int_{\partial\mathbb{B}_n(0,1/2)} u^{(0)} \, d\sigma = 0.$$

On the other hand, if (7) holds, we have $u^{(1)} \equiv 0$ and

$$u_\delta(x) \equiv u^{(0)}(x),$$

for all $\delta \in]0, +\infty[$, and u_δ is also a solution to problem (5).

2.2 │ A nonlinear problem

In this section, we analyze a nonlinear problem, and for the sake of simplicity, we confine to the case of dimension $n = 2$.

For each $\delta \in]0, +\infty[$, we consider the problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1)\backslash\overline{\mathbb{B}_2(0,1/2)}, \\ \frac{\partial}{\partial\nu_{\mathbb{B}_2(0,1)}}u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1), \\ \frac{\partial}{\partial\nu_{\mathbb{B}_2(0,1/2)}}u(x) = \delta^2 u(x) + \delta^2 u^2(x) + b & \forall x \in \partial\mathbb{B}_2(0,1/2). \end{cases} \tag{11}$$

Now, we note that we can collect δ in the right hand side of the third equation in (11), and thus, we can write the Robin condition as follows:

$$\frac{\partial}{\partial\nu_{\mathbb{B}_2(0,1/2)}}u(x) = \delta\left(\delta u(x) + \delta u^2(x)\right) + b \qquad \forall x \in \partial\mathbb{B}_2(0,1/2).$$

If for each $\delta \in]0, +\infty[$, we introduce the function

$$F_\delta(\tau) = \delta\tau + \delta\tau^2 \qquad \forall \tau \in \mathbb{R},$$

we can rewrite problem (11) as follows:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1)\backslash\overline{\mathbb{B}_2(0,1/2)}, \\ \frac{\partial}{\partial\nu_{\mathbb{B}_2(0,1)}}u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1), \\ \frac{\partial}{\partial\nu_{\mathbb{B}_2(0,1/2)}}u(x) = \delta F_\delta(u(x)) + b & \forall x \in \partial\mathbb{B}_2(0,1/2). \end{cases} \tag{12}$$

Then again, we look for a solution u_δ in the form

$$u_\delta(x) \equiv A_\delta \log|x| + B_\delta \qquad \forall x \in \bar{\Omega},$$

with A_δ and B_δ to be set so that the boundary conditions of problem (12) are satisfied.

As we have seen, to ensure the validity of the Neumann condition on $\partial\mathbb{B}_2(0,1)$, we must have

$$A_\delta = a.$$

On the other hand, in order to satisfy the Robin condition, we have to find B_δ such that

$$2a = \delta F_\delta(-a\log 2 + B_\delta) + b.$$

Motivated by the linear case, we find it convenient to replace B_δ by $\tilde{B}_\delta/\delta + a \log 2$. In other words, we look for a solution u_δ in the form

$$u_\delta(x) \equiv A \log |x| + \frac{\tilde{B}_\delta}{\delta} + a \log 2 \qquad \forall x \in \bar{\Omega},$$

with $\tilde{B}_\delta$ such that

$$2a = \delta F_\delta \left(\frac{1}{\delta}\tilde{B}_\delta\right) + b. \qquad (13)$$

Then, we note that if we set

$$\tilde{F}(\xi, \delta) = \delta\xi + \xi^2 \qquad \forall(\xi, \delta) \in \mathbb{R}^2,$$

we have

$$\delta F_\delta \left(\frac{1}{\delta}\xi\right) = \tilde{F}(\xi, \delta) \qquad \forall \xi \in \mathbb{R}. \qquad (14)$$

As a consequence, we can rewrite Equation 13 as follows:

$$2a = \tilde{F}\left(\tilde{B}_\delta, \delta\right) + b. \qquad (15)$$

For general $\tilde{F}$, under suitable assumptions, one can try to resolve Equation 15 by means of the implicit function theorem. On the other hand, for our specific case, for each $\delta \in\,]0, +\infty[$, one has that the solutions ζ in $\mathbb{C}$ of equation

$$2a = \tilde{F}(\zeta, \delta) + b$$

are delivered by

$$\frac{-\delta + z_0}{2}, \qquad \frac{-\delta - z_0}{2},$$

where

$$z_0^2 = \delta^2 - 4(b - 2a).$$

Thus, if we look for solutions $\tilde{B}_\delta \in \mathbb{R}$ of Equation 15 for δ positive and close to 0, we may have 1, 2, or no solutions to (15) depending on the sign of

$$-4(b - 2a).$$

Therefore, for δ small and positive, we may have 1, 2, or no solutions to the nonlinear problem (11). In particular, a crucial role for the solvability of problem (11) is played by the function $\tilde{F}$, which ensures the validity of Equation 14.

2.2.1 | A family of nonlinear problems

To play with the structure of the nonlinear boundary condition, for each $\delta \in\,]0, +\infty[$, we consider the family of problems

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0, 1)\backslash\overline{\mathbb{B}_2(0, 1/2)}, \\ \dfrac{\partial}{\partial \nu_{\mathbb{B}_2(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_2(0, 1), \\ \dfrac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta u(x)(1 + c\delta^{\gamma_1} u^{\gamma_2}(x)) + b & \forall x \in \partial\mathbb{B}_2(0, 1/2), \end{cases} \qquad (16)$$

where $c \in \mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathbb{N}$. Note that such type of boundary conditions is crucially important for practical applications. For example, in metallurgy and metal forming processes, the typical boundary condition involves $\gamma_2 = 4$ where the respective term corresponds to the heat exchange due to the radiation at high temperature (see Golitsyna,[26] Letavin and Mishuris,[27] and Letavin and Shestakov[28]).

Now, we note that we can rewrite the Robin condition as follows:

$$\frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta\left(u(x) + c\delta^{\gamma_1} u^{\gamma_2+1}(x)\right) + b \qquad \forall x \in \partial\mathbb{B}_2(0, 1/2).$$

As above, for each $\delta \in]0, +\infty[$, we introduce the function

$$F_\delta(\tau) = \tau + c\delta^{\gamma_1}\tau^{\gamma_2+1} \qquad \forall \tau \in \mathbb{R}.$$

Then, we can rewrite problem (16) as follows:

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \mathbb{B}_2(0,1)\backslash\overline{\mathbb{B}_2(0,1/2)}, \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1)}} u(x) = a & \forall x \in \partial\mathbb{B}_2(0,1), \\ \frac{\partial}{\partial \nu_{\mathbb{B}_2(0,1/2)}} u(x) = \delta F_\delta(u(x)) + b & \forall x \in \partial\mathbb{B}_2(0,1/2). \end{cases} \qquad (17)$$

Again, we look for a solution u_δ in the form

$$u_\delta(x) \equiv A_\delta \log |x| + a \log 2 + \frac{1}{\delta}\tilde{B}_\delta \qquad \forall x \in \bar{\Omega},$$

with A_δ and $\tilde{B}_\delta$ to be set so that the boundary conditions of problem (17) are satisfied.

As we have seen, we must have

$$A_\delta = a,$$

and, in order to satisfy the Robin condition, we have to find $\tilde{B}_\delta$ such that

$$2a = \delta F_\delta\left(\frac{1}{\delta}\tilde{B}_\delta\right) + b. \qquad (18)$$

Then, we note that if we set

$$\tilde{F}(\xi, \omega) = \xi + c\omega\xi^{\gamma_2+1} \qquad \forall (\xi, \omega) \in \mathbb{R}^2,$$

we have

$$\delta F_\delta\left(\frac{1}{\delta}\xi\right) = \tilde{F}(\xi, \delta^{\gamma_1-\gamma_2}) \qquad \forall \xi \in \mathbb{R}, \delta \in]0, +\infty[.$$

Since we want to pass to the limit in $\tilde{F}(\xi, \delta^{\gamma_1-\gamma_2})$ as $\delta \to 0$, we find it convenient to assume that

$$\gamma_1 \geq \gamma_2.$$

As a consequence, we rewrite Equation 18 as follows:

$$2a = \tilde{F}\left(\tilde{B}_\delta, \delta^{\gamma_1-\gamma_2}\right) + b. \qquad (19)$$

We try to resolve Equation 19 around $\delta = 0$ by means of the implicit function theorem. We treat separately the case $\gamma_1 = \gamma_2$ and the case $\gamma_1 > \gamma_2$. If $\gamma_1 > \gamma_2$, then, there exists a unique $\tilde{B}_0$ such that

$$2a - b = \tilde{F}\left(\tilde{B}_0, 0\right),$$

ie,

$$\tilde{B}_0 = 2a - b.$$

Then, by applying the implicit function theorem around the pair $(2a - b, 0)$, one can prove that there exist a small neighborhood $]2a - b - \epsilon_1, 2a - b + \epsilon_1[\times] - \epsilon_2, \epsilon_2[$ of $(2a - b, 0)$ and a function

$$] - \epsilon_2, \epsilon_2[\ni \omega \mapsto \tilde{B}(\omega) \in]2a - b - \epsilon_1, 2a - b + \epsilon_1[$$

such that

$$\tilde{B}(0) = 2a - b, \qquad 2a - b = \tilde{F}\left(\tilde{B}(\omega), \omega\right) \qquad \forall \omega \in] - \epsilon_2, \epsilon_2[.$$

Therefore, there exists $\delta_1 \in]0, +\infty[$ small enough, such that

$$2a - b = \tilde{F}\left(\tilde{B}(\delta^{\gamma_1-\gamma_2}), \delta^{\gamma_1-\gamma_2}\right) \qquad \forall \delta \in]0, \delta_1[,$$

and thus, we can take

$$\tilde{B}_\delta = \tilde{B}(\delta^{\gamma_1 - \gamma_2}) \qquad \forall \delta \in]0, \delta_1[.$$

Accordingly,

$$u_\delta(x) = a \log |x| + a \log 2 + \frac{\tilde{B}(\delta^{\gamma_1 - \gamma_2})}{\delta} \qquad \forall x \in \bar{\Omega}, \qquad \forall \delta \in]0, \delta_1[.$$

Now, we turn to consider the case $\gamma_1 = \gamma_2$, and we note that

$$\tilde{F}(\xi, \delta^{\gamma_1 - \gamma_1}) = \tilde{F}(\xi, 1) = \xi + c\xi^{\gamma_1 + 1} \qquad \forall (\xi, \delta) \in \mathbb{R}^2.$$

As a consequence, there are $\gamma_1 + 1$ complex solutions ζ to the equation

$$2a - b = \zeta + c\zeta^{\gamma_1 + 1}. \tag{20}$$

Then, if we denote by $\{\tilde{B}_j\}_{j=1}^k$ the set of (distinct) real solutions to Equation 20 for each of them, we can construct the corresponding function, and thus, we can define a family of solutions $\{u_{j,\delta}\}_{\delta \in]0, +\infty[}$ to problem (16), by setting

$$u_{j,\delta}(x) \equiv a \log |x| + a \log 2 + \frac{1}{\delta} \tilde{B}_j \qquad \forall x \in \bar{\Omega}, \qquad \forall \delta \in]0, +\infty[,$$

for each $j \in \{1, \ldots, k\}$. Note that this can be presented in the form:

$$u_{j,\delta}(x) = u^{(0)}(x) + \frac{1}{\delta} u^{(j)}(x), \tag{21}$$

thus, the nonuniqueness is related to the second term of this representation only. Moreover, it makes sense also to underline that the first term in the solutions for the linear (10) and nonlinear (21) problems coincides.

3 | AN INTEGRAL EQUATION FORMULATION OF THE BOUNDARY VALUE PROBLEM

To analyze problem (1) for δ close to 0, we exploit classical potential theory, which allows to obtain an integral equation formulation of (1). To do so, we need to introduce some notation.

Let S_n be the function from $\mathbb{R}^n \backslash \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log |x| & \forall x \in \mathbb{R}^n \backslash \{0\}, \quad \text{if} \quad n = 2, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \backslash \{0\}, \quad \text{if} \quad n > 2. \end{cases}$$

S_n is well known to be a fundamental solution of the Laplace operator.

We now introduce the single layer potential. If $\mu \in C^0(\partial\Omega)$, we set

$$v[\partial\Omega, \mu](x) \equiv \int_{\partial\Omega} S_n(x - y)\mu(y) \, d\sigma_y \qquad \forall x \in \mathbb{R}^n,$$

where $d\sigma$ denotes the area element of a manifold imbedded in $\mathbb{R}^n$. As is well known, if $\mu \in C^0(\partial\Omega)$, then $v[\partial\Omega, \mu]$ is continuous in $\mathbb{R}^n$. Moreover, if $\mu \in C^{0,\alpha}(\partial\Omega)$, then the function $v^+[\partial\Omega, \mu] \equiv v[\partial\Omega, \mu]_{|\bar{\Omega}}$ belongs to $C^{1,\alpha}(\bar{\Omega})$, and the function $v^-[\partial\Omega, \mu] \equiv v[\partial\Omega, \mu]_{|\mathbb{R}^n \backslash \Omega}$ belongs to $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \backslash \Omega)$. Then, we set

$$w_*[\partial\Omega, \mu](x) \equiv \int_{\partial\Omega} v_\Omega(x) \cdot \nabla S_n(x - y)\mu(y) \, d\sigma_y \qquad \forall x \in \partial\Omega,$$

where v_Ω denotes the outward unit normal to $\partial\Omega$. If $\mu \in C^{0,\alpha}(\partial\Omega)$, the function $w_*[\partial\Omega, \mu]$ belongs to $C^{0,\alpha}(\partial\Omega)$, and we have

$$\frac{\partial}{\partial v_\Omega} v^\pm[\partial\Omega, \mu] = \mp\frac{1}{2}\mu + w_*[\partial\Omega, \mu] \qquad \text{on } \partial\Omega.$$

Then, we have the technical Lemma 1 below on the representation of harmonic functions as the sum of a single layer potential with a density with zero integral mean and a constant. Therefore, we find it convenient to set

$$
C^{0,\alpha}(\partial\Omega)_0 \equiv \left\{ f \in C^{0,\alpha}(\partial\Omega) : \int_{\partial\Omega} f \, d\sigma = 0 \right\}.
$$

The proof of Lemma 1 can be deduced by classical potential theory (cf Folland[29, ch. 3]).

Lemma 1. *Let $u \in C^{1,\alpha}(\bar{\Omega})$ be such that $\Delta u = 0$ in Ω. Then, there exists a unique pair $(\mu, c) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that*

$$
u = v^+[\partial\Omega, \mu] + c \qquad in \, \bar{\Omega}.
$$

By exploiting Lemma 1, we can establish a correspondence between the solutions of problem (1) and those of a (nonlinear) system of integral equations.

Proposition 1. *Let $\delta \in \,]0, \delta_0[$. Then, the map from the set of pairs (μ, ξ) of $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that*

$$
\begin{cases} -\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) = \delta F_\delta \left(v[\partial\Omega, \mu](x) + \frac{\xi}{\delta} \right) + g^i(x) & \forall x \in \partial\Omega^i, \end{cases} \tag{22}
$$

to the set of those functions $u \in C^{1,\alpha}(\bar{\Omega})$ that solve problem (1), which takes a pair (μ, ξ) to the function

$$
v^+[\partial\Omega, \mu] + \frac{\xi}{\delta} \tag{23}
$$

is a bijection.

Proof. If $(\mu, \xi) \in C^{0,\alpha}(\partial\Omega) \times \mathbb{R}$, then we know that $v^+[\partial\Omega, \mu] + \frac{\xi}{\delta}$ belongs to $C^{1,\alpha}(\bar{\Omega})$ and is harmonic in Ω. Moreover, if (μ, ξ) satisfies system (22), then the jump formulas for the normal derivative of the single layer potential imply the validity of the boundary condition in problem (1). Hence, the function in (23) solves problem (1).

Conversely, if $u \in C^{1,\alpha}(\bar{\Omega})$ satisfies problem (1), then the representation Lemma 1 for harmonic functions in terms of single layer potentials plus constants ensures that there exists a unique pair $(\mu, \xi) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that $u = v^+[\partial\Omega, \mu] + \frac{\xi}{\delta}$. Then, the jump formulas for the normal derivative of a single layer potential and the boundary condition in (1) imply that the system of integral equations of (22) is satisfied. Hence, the map of the statement is a bijection. □

Now that the correspondence between the solutions of boundary value problem (1) and those of the system of integral equations (22) is established, we wish to study the behavior of the solutions to system (22) as $\delta \to 0$. Then, we note that we can write

$$
\delta F_\delta \left(v[\partial\Omega, \mu](x) + \frac{\xi}{\delta} \right) = \delta F_\delta \left(\frac{1}{\delta} (\delta v[\partial\Omega, \mu](x) + \xi) \right).
$$

Therefore, to analyze the second equation in (22) for δ small, we need to make some other assumptions on the structure of the family of functions

$$
\mathbb{R} \ni \tau \mapsto \delta F_\delta \left(\frac{1}{\delta} \tau \right) \qquad \text{as } \delta \to 0.
$$

So we assume that

$$
\begin{aligned}
&\text{there exist } \delta_1 \in \,]0, \delta_0[, \, m \in \mathbb{N}, \text{ a real analytic function } \tilde{F} \text{ from } \mathbb{R}^{m+1} \text{ to } \mathbb{R}, \\
&\text{a function } \omega(\cdot) \text{ from }]0, \delta_1[\text{ to } \mathbb{R}^m \text{ such that } \omega_0 \equiv \lim_{\delta \to 0} \omega(\delta) \in \mathbb{R}^m \text{ and that} \\
&\delta F_\delta \left(\frac{1}{\delta} \tau \right) = \tilde{F}(\tau, \omega(\delta)) \text{ for all } (\tau, \delta) \in \mathbb{R} \times]0, \delta_1[.
\end{aligned} \tag{24}
$$

Thus, under the additional assumption (24), if we let δ tend to 0 in (22), we obtain the following limiting system of integral equations:

$$\begin{cases} -\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) = \tilde{F}(\xi, \omega_0) + g^i(x) \; \forall x \in \partial\Omega^i. \end{cases} \quad (25)$$

Then, as a preliminary step in the analysis of the system of integral equations (22) for δ close to 0, in the following lemma, we study the limiting system (25).

Lemma 2. *Let assumption (24) hold. Assume that $\tilde{\xi} \in \mathbb{R}$ is such that*

$$\tilde{F}(\tilde{\xi}, \omega_0) = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma \right), \quad (26)$$

where $|\partial\Omega^i|_{n-1}$ denotes the $(n-1)$-dimensional measure of $\partial\Omega^i$. Then, there exists a unique $\tilde{\mu} \in C^{0,\alpha}(\partial\Omega)_0$ such that

$$\begin{cases} -\frac{1}{2}\tilde{\mu}(x) + w_*[\partial\Omega, \tilde{\mu}](x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{1}{2}\tilde{\mu}(x) - w_*[\partial\Omega, \tilde{\mu}](x) = \tilde{F}(\tilde{\xi}, \omega_0) + g^i(x) \; \forall x \in \partial\Omega^i. \end{cases}$$

Proof. We note that if $\tilde{\xi}$ is as in (26), then, the function $\tilde{g}$ defined as follows:

$$\tilde{g}(x) \equiv \begin{cases} g^o(x) & \forall x \in \partial\Omega^o, \\ -\left(\tilde{F}(\tilde{\xi}, \omega_0) + g^i(x)\right) & \forall x \in \partial\Omega^i, \end{cases}$$

belongs to $C^{0,\alpha}(\partial\Omega)_0$. Then, by classical potential theory (cf Folland[29, ch. 3]), there exists a unique $\tilde{\mu} \in C^{0,\alpha}(\partial\Omega)_0$ such that

$$-\frac{1}{2}\tilde{\mu}(x) + w_*[\partial\Omega, \tilde{\mu}](x) = \tilde{g}(x) \qquad \forall x \in \partial\Omega,$$

and the validity of the statement follows. □

In view of Proposition 1 and under assumption (24), in order to study the solutions of (22), we find it convenient to introduce the map $\Lambda \equiv (\Lambda^o, \Lambda^i)$ from $\mathbb{R}^{m+1} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega)$ defined by setting

$$\begin{aligned} \Lambda^o[\delta, \omega, \mu, \xi] &\equiv -\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) - g^o(x) & \forall x \in \partial\Omega^o, \\ \Lambda^i[\delta, \omega, \mu, \xi] &\equiv \frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) - \tilde{F}\left(\delta v[\partial\Omega, \mu](x) + \xi, \omega\right) - g^i(x) & \forall x \in \partial\Omega^i, \end{aligned}$$

for all $(\delta, \omega, \mu, \xi) \in \mathbb{R}^{m+1} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$.

In the following proposition, we investigate the solutions of the system of integral equations (22), by applying the implicit function theorem to Λ, under suitable assumptions on the partial derivative $\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)$ of the function $(\tau, \omega) \mapsto \tilde{F}(\tau, \omega)$ with respect to the variable τ computed at the point $(\tilde{\xi}, \omega_0)$.

Proposition 2. *Let assumption (24) hold. Let $(\tilde{\mu}, \tilde{\xi})$ be as in Lemma 2. Assume that*

$$\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0) \neq 0. \quad (27)$$

Then, there exist $\delta_2 \in]0, \delta_1[$, an open neighborhood $\mathcal{U}$ of ω_0 in $\mathbb{R}^m$, an open neighborhood $\mathcal{V}$ of $(\tilde{\mu}, \tilde{\xi})$ in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$, and a real analytic map (M, Ξ) from $] - \delta_2, \delta_2[\times\mathcal{U}$ to $\mathcal{V}$ such that

$$\omega(\delta) \in \mathcal{U} \qquad \forall \delta \in]0, \delta_2[,$$

and such that the set of zeros of Λ in $] - \delta_2, \delta_2[\times\mathcal{U} \times \mathcal{V}$ coincides with the graph of (M, Ξ). In particular,

$$(M[0, \omega_0], \Xi[0, \omega_0]) = (\tilde{\mu}, \tilde{\xi}).$$

Proof. We first note that by classical potential theory (cf Miranda[30] and Lanza de Cristoforis and Rossi[31, thm. 3.1]), by assumption (24), and by analyticity results for the composition operator (cf Böhme and Tomi,[32, p. 10] Henry,[33, p. 29] and

Valent[34, Thm. 5.2, p. 44]), we conclude that Λ is real analytic. Then, we note that the partial differential $\partial_{(\mu,\xi)}\Lambda[0, \omega_0, \tilde{\mu}, \tilde{\xi}]$ of Λ at $(0, \omega_0, \tilde{\mu}, \tilde{\xi})$ with respect to the variable (μ, ξ) is delivered by

$$\partial_{(\mu,\xi)}\Lambda^o[0, \omega_0, \tilde{\mu}, \tilde{\xi}](\bar{\mu}, \bar{\xi})(x) \equiv -\frac{1}{2}\bar{\mu}(x) + w_*[\partial\Omega, \bar{\mu}](x) \qquad \forall x \in \partial\Omega^o,$$

$$\partial_{(\mu,\xi)}\Lambda^i[0, \omega_0, \tilde{\mu}, \tilde{\xi}](\bar{\mu}, \bar{\xi})(x) \equiv \frac{1}{2}\bar{\mu}(x) - w_*[\partial\Omega, \bar{\mu}](x) - \partial_\tau\tilde{F}(\tilde{\xi}, \omega_0)\bar{\xi} \qquad \forall x \in \partial\Omega^i,$$

for all $(\bar{\mu}, \bar{\xi}) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. Then, by assumption (27) and by classical potential theory (cf Folland[29, ch. 3]), we deduce that $\partial_{(\mu,\xi)}\Lambda[0, \omega_0, \tilde{\mu}, \tilde{\xi}]$ is a homeomorphism from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ onto $C^{0,\alpha}(\partial\Omega)$. Then, by the implicit function theorem for real analytic maps in Banach spaces (cf, eg, Deimling[17, theorem 15.3]), we deduce the validity of the statement. $\square$

Now that we have converted problem (1) into an equivalent system of integral equations for which we have exhibited a real analytic family of solutions, we are ready to introduce the family of solutions to (1).

Definition 1. Let the assumptions of Proposition 2 hold. Then, we set

$$u(\delta, x) \equiv v^+[\Omega, M[\delta, \omega(\delta)]](x) + \frac{\Xi[\delta, \omega(\delta)]}{\delta} \qquad \forall x \in \bar{\Omega},$$

for all $\delta \in]0, \delta_2[$.

By Propositions 1 and 2 and by Definition 1, we deduce that for each $\delta \in]0, \delta_2[$ the function $u(\delta, \cdot) \in C^{1,\alpha}(\bar{\Omega})$ is a solution to problem (1).

4 | A FUNCTIONAL ANALYTIC REPRESENTATION THEOREM FOR THE FAMILY OF SOLUTIONS

In the following theorem, we exploit the analyticity result of Proposition 2 concerning the solutions of the system of integral equations (22) in order to prove representation formulas for $u(\delta, \cdot)$ and its energy integral in terms of real analytic maps and thus to answer to questions (1), (2) of the Introduction.

Theorem 1. *Let the assumptions of Proposition 2 hold. Then, the following statements hold.*

(1) *There exists a real analytic map U from $]-\delta_2, \delta_2[\times\mathcal{V}$ to $C^{1,\alpha}(\bar{\Omega})$ such that*

$$u(\delta, x) = U[\delta, \omega(\delta)](x) + \frac{\Xi[\delta, \omega(\delta)]}{\delta} \qquad \forall x \in \bar{\Omega}, \tag{28}$$

for all $\delta \in]0, \delta_2[$. Moreover, $U[0, \omega_0]$ is a solution of the Neumann problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega, \\ \frac{\partial}{\partial\nu_{\Omega^o}}u(x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{\partial}{\partial\nu_{\Omega^i}}u(x) = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma\right) + g^i(x) & \forall x \in \partial\Omega^i, \end{cases} \tag{29}$$

and

$$\Xi[0, \omega_0] = \tilde{\xi}.$$

(2) *There exists a real analytic map E from $]-\delta_2, \delta_2[\times\mathcal{V}$ to $\mathbb{R}$ such that*

$$\int_\Omega |\nabla u(\delta, x)|^2\, dx = E[\delta, \omega(\delta)], \tag{30}$$

for all $\delta \in]0, \delta_2[$. Moreover,

$$E[0, \omega_0] = \int_\Omega |\nabla\tilde{u}(x)|^2\, dx,$$

where $\tilde{u}$ is any solution of the Neumann problem (29).

Proof. We first prove statement (1). We set

$$U[\delta, \omega](x) \equiv v^+[\partial\Omega, M[\delta, \omega]](x) \qquad \forall x \in \bar{\Omega},$$

for all $(\delta, \omega) \in]-\delta_2, \delta_2[\times \mathcal{U}$. Then, by Proposition 2 and by classical mapping properties of layer potentials (cf Miranda[30] and Lanza de Cristoforis and Rossi[31, thm. 3.1]), we conclude that U is real analytic. Since

$$\tilde{F}(\tilde{\xi}, \omega_0) = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma \right),$$

and $M[0, \omega_0] = \tilde{\mu}$, then

$$\begin{cases} \frac{\partial}{\partial\nu_{\Omega^o}} v^+[\partial\Omega, \tilde{\mu}](x) = -\frac{1}{2}\tilde{\mu}(x) + w_*[\partial\Omega, \tilde{\mu}](x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{\partial}{\partial\nu_{\Omega^i}} v^+[\partial\Omega, \tilde{\mu}](x) = \frac{1}{2}\tilde{\mu}(x) - w_*[\partial\Omega, \tilde{\mu}](x) = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma \right) + g^i(x) & \forall x \in \partial\Omega^i. \end{cases}$$

As a consequence, $v^+[\partial\Omega, \tilde{\mu}]$ solves problem (29). Then, we deduce the validity of statement (1) (see also Proposition 2).

We now consider statement (2). By the divergence theorem and standard properties of harmonic functions and their normal derivatives, we have

$$\int_\Omega |\nabla u(\delta, x)|^2 \, dx = \int_{\partial\Omega} \left(v^+[\partial\Omega, M[\delta, \omega(\delta)]] + \frac{\Xi[\delta, \omega(\delta)]}{\delta} \right) \frac{\partial v^+[\partial\Omega, M[\delta, \omega(\delta)]]}{\partial\nu_\Omega} \, d\sigma,$$

$$= \int_{\partial\Omega} v^+[\partial\Omega, M[\delta, \omega(\delta)]] \frac{\partial v^+[\partial\Omega, M[\delta, \omega(\delta)]]}{\partial\nu_\Omega} \, d\sigma,$$

for all $\delta \in]0, \delta_2[$. Thus, we find natural to set

$$E[\delta, \omega] = \int_{\partial\Omega} v^+[\partial\Omega, M[\delta, \omega]] \frac{\partial v^+[\partial\Omega, M[\delta, \omega]]}{\partial\nu_\Omega} \, d\sigma \qquad \forall(\delta, \omega) \in]-\delta_2, \delta_2[\times \mathcal{U}.$$

By Proposition 2, by mapping properties of layer potentials, and by standard calculus in Schauder spaces, we deduce the real analyticity of E from $]-\delta_2, \delta_2[\times \mathcal{U}$ to $\mathbb{R}$. Since

$$E[0, \omega_0] = \int_{\partial\Omega} v^+[\partial\Omega, \tilde{\mu}] \frac{\partial v^+[\partial\Omega, \tilde{\mu}]}{\partial\nu_\Omega} \, d\sigma$$

$$= \int_\Omega |\nabla v^+[\partial\Omega, \tilde{\mu}]|^2 \, dx,$$

and $v^+[\partial\Omega, \tilde{\mu}]$ is a solution of problem (29), we deduce the validity of statement (2). □

Remark 1. We observe that Theorem 1 implies that the quantities in the left-hand sides of (28) and of (30) can be represented as convergent power series of $(\delta, \omega(\delta) - \omega_0)$.

5 | LOCAL UNIQUENESS OF THE FAMILY OF SOLUTIONS

We now show by means of the following theorem that the family $\{u(\delta, \cdot)\}_{\delta \in]0, \delta_2[}$ is locally essentially unique (cf Lanza de Cristoforis[22, thm. 4.1 (iii)]).

Theorem 2. *Let the assumptions of Proposition 2 hold. If $\{d_j\}_{j \in \mathbb{N}}$ is a sequence of $]0, \delta_0[$ converging to 0 and if $\{u_j\}_{j \in \mathbb{N}}$ is a sequence of functions such that*

$$
\begin{aligned}
&u_j \in C^{1,\alpha}(\bar{\Omega}), \\
&u_j \quad \text{solves (1) for } \delta = d_j, \\
&\lim_{j\to\infty} d_j u_{j|\partial\Omega^i} = \tilde{\xi} \qquad \text{in} \quad C^{0,\alpha}(\partial\Omega^i),
\end{aligned} \tag{31}
$$

then, there exists $j_0 \in \mathbb{N}$ such that $u_j(\cdot) = u(d_j, \cdot)$ for all $j \geq j_0$.

Proof. Since u_j solves problem (1), Proposition 1 ensures that for each $j \in \mathbb{N}$ there exists a pair $(\mu_j, \xi_j) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that

$$
u_j = v^+[\partial\Omega, \mu_j] + \frac{\xi_j}{d_j} \qquad \text{in} \quad \bar{\Omega}. \tag{32}
$$

We now rewrite equation

$$
\Lambda[\delta, \omega, \mu, \xi] = 0
$$

in the following form

$$
\begin{cases}
-\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) = g^o(x) \quad \forall x \in \partial\Omega^o, \\
\frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) - \partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\left(\delta v[\partial\Omega, \mu](x) + \xi\right) \\
\quad = -\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\left(\delta v[\partial\Omega, \mu](x) + \xi\right) + \tilde{F}\left(\delta v[\partial\Omega, \mu](x) + \xi, \omega\right) + g^i(x) \quad \forall x \in \partial\Omega^i,
\end{cases} \tag{33}
$$

for all $(\delta, \omega, \mu, \xi) \in \mathbb{R}^{m+1} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. Next, we denote by $N[\delta, \omega, \mu, \xi] = (N^o[\delta, \omega, \mu, \xi], N^i[\delta, \omega, \mu, \xi])$ and by $B[\delta, \omega, \mu, \xi] = (B^o[\delta, \omega, \mu, \xi], B^i[\delta, \omega, \mu, \xi])$ the left and right hand side of such an equality, respectively. By standard properties of single layer potentials, we conclude that N is real analytic (cf Miranda[30] and Lanza de Cristoforis and Rossi[31, thm. 3.1]).

Next, we note that $N[\delta, \omega, \cdot, \cdot]$ is linear for all fixed $(\delta, \omega) \in \mathbb{R}^{m+1}$. Accordingly, the map from $\mathbb{R}^{m+1}$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}, C^{0,\alpha}(\partial\Omega))$ that takes (δ, ω) to $N[\delta, \omega, \cdot, \cdot]$ is real analytic. Here, $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}, C^{0,\alpha}(\partial\Omega))$ denotes the Banach space of linear and continuous operators from $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega)$. We also note that

$$
N[0, \omega_0, \cdot, \cdot] = \partial_{(\mu, \xi)}\Lambda[0, \omega_0, \tilde{\mu}, \tilde{\xi}](\cdot, \cdot),
$$

and that accordingly, $N[0, \omega_0, \cdot, \cdot]$ is a linear homeomorphism (see the proof of Proposition 2). Since the set of linear homeomorphisms is open in the set of linear and continuous operators, and since the map which takes a linear invertible operator to its inverse is real analytic (cf, eg, Hille and Phillips[35, thms. 4.3.2 and 4.3.4]), there exists an open neighborhood $\mathcal{W}$ of $(0, \omega_0)$ in $]-\delta_2, \delta_2[\times \mathcal{U}$ such that the map, which takes (δ, ω) to $N[\delta, \omega, \cdot, \cdot]^{(-1)}$ is real analytic from $\mathcal{W}$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R})$. Clearly, there exists $j_1 \in \mathbb{N}$ such that

$$
(d_j, \omega(d_j)) \in \mathcal{W} \qquad \forall j \geq j_1.
$$

Since $\Lambda[d_j, \omega(d_j), \mu_j, \xi_j] = 0$, the invertibility of $N[\delta, \omega, \cdot, \cdot]$ and equality (33) guarantee that

$$
(\mu_j, \xi_j) = N[d_j, \omega(d_j), \cdot, \cdot]^{(-1)}[B[d_j, \omega(d_j), \mu_j, \xi_j]] \qquad \forall j \geq j_1.
$$

By (32), we have

$$
d_j u_j = d_j v^+[\partial\Omega, \mu_j] + \xi_j,
$$

for all $j \geq j_1$. Accordingly,

$$
\begin{aligned}
&-\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\left(d_j v[\partial\Omega, \mu_j](x) + \xi_j\right) + \tilde{F}\left(d_j v[\partial\Omega, \mu_j](x) + \xi_j, \omega(d_j)\right) + g^i(x) \\
&\quad = -\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\left(d_j u_j(x)\right) + \tilde{F}\left(d_j u_j(x), \omega(d_j)\right) + g^i(x) \qquad \forall x \in \partial\Omega^i,
\end{aligned}
$$

for all $j \geq j_1$. Then, by assumptions (24) and (31) and by analyticity results for the composition operator (cf Böhme and Tomi,[32, p. 10] Henry,[33, p. 29] and Valent[34, thm. 5.2, p. 44]), we have

$$
\lim_{j\to\infty} -\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\left(d_j u_{j|\partial\Omega^i}\right) + \tilde{F}\left(d_j u_{j|\partial\Omega^i}, \omega(d_j)\right) + g^i = -\partial_\tau \tilde{F}(\tilde{\xi}, \omega_0)\tilde{\xi} + \tilde{F}\left(\tilde{\xi}, \omega_0\right) + g^i, \tag{34}
$$

in $C^{0,\alpha}(\partial\Omega^i)$. The analyticity of the map that takes (δ,ω) to $N[\delta,\omega,\cdot,\cdot]^{(-1)}$ implies that

$$\lim_{j\to\infty} N[d_j,\omega(d_j),\cdot,\cdot]^{(-1)} = N[0,\omega_0,\cdot,\cdot]^{(-1)}$$

$$\text{in} \quad \mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}). \tag{35}$$

Since the evaluation map from $\mathcal{L}(C^{0,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R})\times C^{0,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$, which takes a pair (A,v) to $A[v]$ is bilinear and continuous, the limiting relations (34) and (35) imply that

$$\lim_{j\to\infty}(\mu_j,\xi_j) = N[0,\omega_0,\cdot,\cdot]^{(-1)}[g^o, -\partial_\tau\tilde{F}(\tilde{\xi},\omega_0)\tilde{\xi} + \tilde{F}(\tilde{\xi},\omega_0) + g^i], \tag{36}$$

in $C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}$. Since $\Lambda[0,\omega_0,\tilde{\mu},\tilde{\xi}]=0$, the right hand side of (36) equals $(\tilde{\mu},\tilde{\xi})$. Hence,

$$\lim_{j\to\infty}(d_j,\omega(d_j),\mu_j,\xi_j) = (0,\omega_0,\tilde{\mu},\tilde{\xi}) \quad \text{in} \quad \mathbb{R}^{m+1}\times C^{0,\alpha}(\partial\Omega)_0\times\mathbb{R}.$$

Then, Proposition 2 implies that there exists $j_0\in\mathbb{N}$ such that

$$\xi_j = \Xi[d_j,\omega(d_j)], \qquad \mu_j = M[d_j,\omega(d_j)] \qquad \forall j\geq j_0.$$

Accordingly, $u_j = u(d_j,\cdot)$ for $j\geq j_0$ (see Definition 1). □

6 | REMARKS ON THE LINEAR CASE

In this section, we wish to make further considerations on the linear case. In particular, we plan to compute asymptotic expansions of the solutions as the parameter δ tends to 0.

We first note that the results of Section 4 apply to the linear case. In particular, in case

$$F_\delta(\tau) = \tau \qquad \forall(\tau,\delta)\in\mathbb{R}\times]0,+\infty[,$$

problem (1) reduces to the following linear problem:

$$\begin{cases} \Delta u(x) = 0 & \forall x\in\Omega, \\ \frac{\partial}{\partial\nu_{\Omega^o}}u(x) = g^o(x) & \forall x\in\partial\Omega^o, \\ \frac{\partial}{\partial\nu_{\Omega^i}}u(x) = \delta u(x) + g^i(x) & \forall x\in\partial\Omega^i. \end{cases} \tag{37}$$

For each $\delta\in]0,+\infty[$, we know that problem (37) has a unique solution in $C^{1,\alpha}(\overline{\Omega})$, and we denote it by $u[\delta]$.

Clearly,

$$\delta F_\delta\left(\frac{1}{\delta}\tau\right) = \tau \qquad \forall(\tau,\delta)\in\mathbb{R}\times]0,+\infty[,$$

and thus, we can take, for example,

$$\omega(\delta) = 0 \qquad \forall\delta\in]0,+\infty[,$$

and

$$\tilde{F}(\tau,\omega) = \tau \qquad \forall(\tau,\omega)\in\mathbb{R}^2.$$

In particular,

$$\omega_0 = 0, \qquad \tilde{\xi} = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o}g^o\,d\sigma - \int_{\partial\Omega^i}g^i\,d\sigma\right),$$

and

$$\partial_\tau\tilde{F}(\tilde{\xi},\omega_0) = 1 \neq 0.$$

Therefore, the results of Sections 3 and 4 apply to the present case. More precisely, by simplifying the arguments of Propositions 1 and 2, we deduce the validity of the following proposition.

Proposition 3. *Let $\Lambda_{\#} \equiv (\Lambda_{\#}^o, \Lambda_{\#}^i)$ be the map from $\mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega)$ defined by setting*

$$\Lambda_{\#}^o[\delta, \mu, \xi] \equiv -\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) - g^o(x) \qquad\qquad \forall x \in \partial\Omega^o,$$
$$\Lambda_{\#}^i[\delta, \mu, \xi] \equiv \frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) - \delta v[\partial\Omega, \mu](x) - \xi - g^i(x) \quad \forall x \in \partial\Omega^i,$$

for all $(\delta, \mu, \xi) \in \mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. Let $(\tilde{\mu}_{\#}, \tilde{\xi}_{\#})$ be the unique solution in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of

$$\begin{cases} -\frac{1}{2}\mu(x) + w_*[\partial\Omega, \mu](x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{1}{2}\mu(x) - w_*[\partial\Omega, \mu](x) = \xi + g^i(x) & \forall x \in \partial\Omega^i. \end{cases}$$

Then, there exist $\delta_0 \in]0, +\infty[$, an open neighborhood $\mathcal{V}$ of $(\tilde{\mu}_{\#}, \tilde{\xi}_{\#})$ in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$, and a real analytic map $(M_{\#}, \Xi_{\#})$ from $]-\delta_0, \delta_0[$ to $\mathcal{V}$ such that the set of zeros of $\Lambda_{\#}$ in $]-\delta_0, \delta_0[\times\mathcal{V}$ coincides with the graph of $(M_{\#}, \Xi_{\#})$. In particular,

$$(M_{\#}[0], \Xi_{\#}[0]) = (\tilde{\mu}_{\#}, \tilde{\xi}_{\#}). \tag{38}$$

Moreover,

$$u[\delta](x) \equiv v^+[\Omega, M_{\#}[\delta]](x) + \frac{\Xi_{\#}[\delta]}{\delta} \qquad \forall x \in \bar{\Omega},$$

for all $\delta \in]0, \delta_0[$.

Then, we can follow the lines of the proof of Theorem 1 and obtain the following real analytic representation result for $u[\delta]$ and its energy integral.

Theorem 3. *Let $\Xi_{\#}$ be as in Proposition 3. Then, the following statements hold.*

(1) There exists a real analytic map $U_{\#}$ from $]-\delta_0, \delta_0[$ to the space $C^{1,\alpha}(\bar{\Omega})$ such that

$$u[\delta](x) = U_{\#}[\delta](x) + \frac{\Xi_{\#}[\delta]}{\delta} \qquad \forall x \in \bar{\Omega},$$

for all $\delta \in]0, \delta_0[$. Moreover, $U_{\#}[0]$ is a solution of the Neumann problem

$$\begin{cases} \Delta u(x) = 0 & \forall x \in \Omega, \\ \frac{\partial}{\partial\nu_{\Omega^o}}u(x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{\partial}{\partial\nu_{\Omega^i}}u(x) = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma\right) + g^i(x) & \forall x \in \partial\Omega^i, \end{cases} \tag{39}$$

and

$$\Xi_{\#}[0] = \frac{1}{|\partial\Omega^i|_{n-1}}\left(\int_{\partial\Omega^o} g^o\, d\sigma - \int_{\partial\Omega^i} g^i\, d\sigma\right).$$

(2) There exists a real analytic map $E_{\#}$ from $]-\delta_0, \delta_0[$ to $\mathbb{R}$ such that

$$\int_{\Omega} |\nabla u(\delta, x)|^2\, dx = E_{\#}[\delta],$$

for all $\delta \in]0, \delta_0[$. Moreover,

$$E_{\#}[0] = \int_{\Omega} |\nabla\tilde{u}(x)|^2\, dx,$$

where $\tilde{u}$ is any solution of the Neumann problem (39).

6.1 │ Asymptotic expansion of $u[\delta]$

By Theorem 3 (1), we know that there exist a sequence of functions $\{u_{\#,k}\}_{k\in\mathbb{N}} \subseteq C^{1,\alpha}(\bar{\Omega})$ and a sequence of real numbers $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ such that

$$u[\delta](x) = \sum_{k=0}^{+\infty} u_{\#,k}(x)\delta^k + \sum_{k=0}^{+\infty} \xi_{\#,k}\delta^{k-1} \qquad \forall x \in \bar{\Omega}, \tag{40}$$

where the series are uniformly convergent for δ in a neighborhood of 0. As for the model problem (10), we note that we can rewrite equation (40) in the form

$$u[\delta](x) = u^{(0)}(x) = \frac{1}{\delta} u_\delta^{(1)}(x) \qquad \forall x \in \bar{\Omega},$$

where in this case in general $u_\delta^{(1)}$ depends on δ.

To construct the sequences $\{u_{\#,k}\}_{k\in\mathbb{N}} \subseteq C^{1,\alpha}(\bar{\Omega})$ and $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$, we wish to exploit the integral equation formulation of problem (37) and the approach of Dalla Riva et al.[36]

Now, we observe that the real analyticity result of Proposition 3 implies that there exists $\delta_1 \in]0, \delta_0[$ small enough such that we can expand $M_\#[\delta]$ and $\Xi_\#[\delta]$ into power series of δ, ie,

$$M_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\mu_{\#,k}}{k!}\delta^k, \qquad \Xi_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\xi_{\#,k}}{k!}\delta^k, \tag{41}$$

for some $\{\mu_{\#,k}\}_{k\in\mathbb{N}}$, $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ and for all $\delta \in]-\delta_1, \delta_1[$. Moreover,

$$\mu_{\#,k} = \left(\partial_\delta^k M_\#[\delta]\right)_{|\delta=0}, \qquad \xi_{\#,k} = \left(\partial_\delta^k \Xi_\#[\delta]\right)_{|\delta=0},$$

for all $k \in \mathbb{N}$. Therefore, in order to obtain a power series expansion for $u[\delta]$ for δ close to 0, we want to exploit the expansion of $(M_\#[\delta], \Xi_\#[\delta])$. Since the coefficients of the expansions in (41) are given by the derivatives with respect to δ of $M_\#[\delta]$ and $\Xi_\#[\delta]$, we would like to obtain some equations identifying $\left(\partial_\delta^k M_\#[\delta]\right)_{|\delta=0}$ and $\left(\partial_\delta^k \Xi_\#[\delta]\right)_{|\delta=0}$. The plan is to obtain such equations by deriving with respect to δ equality (38), which then leads to

$$\partial_\delta^k \left(\Lambda_\#[\delta, M_\#[\delta], \Xi_\#[\delta]]\right) = 0 \qquad \forall \delta \in]-\delta_1, \delta_1[, \qquad \forall k \in \mathbb{N}. \tag{42}$$

Then, as Proposition 4 below shows, by taking $\delta = 0$ in (42), we will obtain integral equations identifying $\left(\partial_\delta^k M_\#[\delta]\right)_{|\delta=0}$ and $\left(\partial_\delta^k \Xi_\#[\delta]\right)_{|\delta=0}$.

Proposition 4. *Let $\delta_0, M_\#[\cdot]$, and $\Xi_\#[\cdot]$ be as in Proposition 3. Then, there exist $\delta_1 \in]0, \delta_0[$ and a sequence of functions $\{\mu_{\#,k}\}_{k\in\mathbb{N}} \subseteq C^{0,\alpha}(\partial\Omega)_0$ and a sequence of real numbers $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ such that*

$$M_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\mu_{\#,k}}{k!}\delta^k \qquad and \qquad \Xi_\#[\delta] = \sum_{k=0}^{+\infty} \frac{\xi_{\#,k}}{k!}\delta^k \qquad \forall \delta \in]-\delta_1, \delta_1[, \tag{43}$$

where the two series converge uniformly for $\delta \in]-\delta_1, \delta_1[$. Moreover, the following statements hold.

(1) The pair $(\mu_{\#,0}, \xi_{\#,0})$ is the unique solution in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of the following system of integral equations:

$$\begin{cases} -\frac{1}{2}\mu_{\#,0}(x) + w_*[\partial\Omega, \mu_{\#,0}](x) = g^o(x) & \forall x \in \partial\Omega^o, \\ \frac{1}{2}\mu_{\#,0}(x) - w_*[\partial\Omega, \mu_{\#,0}](x) = \xi_{\#,0} + g^i(x) & \forall x \in \partial\Omega^i. \end{cases}$$

Moreover,

$$\xi_{\#,0} = \frac{1}{|\partial\Omega^i|_{n-1}} \left(\int_{\partial\Omega^o} g^o \, d\sigma - \int_{\partial\Omega^i} g^i \, d\sigma \right).$$

(2) For all $k \in \mathbb{N}\backslash\{0\}$, the pair $(\mu_{\#,k}, \xi_{\#,k})$ is the unique solution in $C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of the following system of integral equations:

$$\begin{cases} -\frac{1}{2}\mu_{\#,k}(x) + w_*[\partial\Omega, \mu_{\#,k}](x) = 0 & \forall x \in \partial\Omega^o, \\ \frac{1}{2}\mu_{\#,k}(x) - w_*[\partial\Omega, \mu_{\#,k}](x) = kv[\partial\Omega, \mu_{\#,k-1}](x) + \xi_{\#,k} & \forall x \in \partial\Omega^i. \end{cases} \tag{44}$$

Moreover,

$$\xi_{\#,k} = \frac{1}{|\partial\Omega^i|_{n-1}} \left(-\int_{\partial\Omega^i} k v[\partial\Omega, \mu_{\#,k-1}] \, d\sigma \right). \tag{45}$$

Proof. We first note that Proposition 3 implies the existence of δ_1 and of sequences $\{\mu_{\#,k}\}_{k\in\mathbb{N}}$ and $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ such that (43) holds. Moreover, Proposition 3 immediately implies the validity of statement (1). Then, observe that $\Lambda_\#[\delta, M_\#[\delta], \Xi_\#[\delta]] = 0$ for all $\delta \in]-\delta_0, \delta_0[$. Accordingly, the map that takes δ to $\Lambda_\#[\delta, M_\#[\delta], \Xi_\#[\delta]]$ has derivatives which are equal to zero, ie, $\partial_\delta^k(\Lambda_\#[\delta, M_\#[\delta], \Xi_\#[\delta]]) = 0$ for all $\delta \in]-\delta_0, \delta_0[$ and all $k \in \mathbb{N}\setminus\{0\}$. Then, a straightforward calculation shows that

$$\partial_\delta^k \left(\Lambda_\#^o[\delta, M_\#[\delta], \Xi_\#[\delta]] \right)(x) = -\frac{1}{2} \partial_\delta^k M_\#[\delta](x) + w_* \left[\partial\Omega, \partial_\delta^k M_\#[\delta] \right](x) = 0 \qquad \forall x \in \partial\Omega^o, \tag{46}$$

$$\partial_\delta^k \left(\Lambda_\#^i[\delta, M_\#[\delta], \Xi_\#[\delta]] \right)(x) = \frac{1}{2} \partial_\delta^k M_\#[\delta](x) - w_* \left[\partial\Omega, \partial_\delta^k M_\#[\delta] \right](x)$$

$$- \sum_{j=0}^{k} \binom{k}{j} \left(\partial_\delta^{k-j} \delta \right) v \left[\partial\Omega, \partial_\delta^j M_\#[\delta] \right](x) - \partial_\delta^k \Xi_\#[\delta] = 0 \qquad \forall x \in \partial\Omega^i, \tag{47}$$

for all $\delta \in]-\delta_0, \delta_0[$ and all $k \in \mathbb{N}\setminus\{0\}$. Then, one verifies that system (46), (47) with $\delta = 0$ can be rewritten as system (44) for all $k \in \mathbb{N}\setminus\{0\}$. Hence, classical potential ensures that the solution $(\mu_{\#,k}, \xi_{\#,k}) \in C^{0,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ of system (44) exists and is unique. Then, by integrating, one deduces the validity of (45). The proof is now complete. $\qquad\square$

Finally, by Propositions 3 and 4, Theorem 4 and standard calculus in Banach spaces, one deduces the validity of the following.

Theorem 4. *Let δ_1, $\{u_{\#,k}\}_{k\in\mathbb{N}}$, and $\{\xi_{\#,k}\}_{k\in\mathbb{N}}$ be as in Proposition 4. Let*

$$u_{\#,k}(x) \equiv v^+[\partial\Omega, \mu_{\#,k}](x) \qquad \forall x \in \bar\Omega, \quad \forall k \in \mathbb{N}.$$

Then, there exists $\delta_2 \in]0, \delta_1[$ such that

$$u[\delta](x) = \sum_{k=0}^{+\infty} u_{\#,k}(x)\delta^k + \sum_{k=0}^{+\infty} \xi_{\#,k}\delta^{k-1} \qquad \forall x \in \bar\Omega, \quad \forall \delta \in]0, \delta_2[,$$

where the series are uniformly convergent for δ in $]-\delta_2, \delta_2[$.

Then, by Proposition 4 and Theorem 4, we can deduce a representation formula similar to the one of the solution of the model problem (10) (where $u^{(1)}$ is replaced by the δ-dependent function $u_\delta^{(1)}$).

Corollary 1. *Let the assumptions of Theorem 4 hold. Let*

$$u^{(0)}(x) \equiv u_{\#,0}(x) - \xi_{\#,1} \qquad \forall x \in \bar\Omega,$$

and

$$u_\delta^{(1)}(x) \equiv \xi_{\#,0} + \sum_{k=2}^{+\infty} \xi_{\#,k}\delta^k + \sum_{k=1}^{+\infty} u_{\#,k}(x)\delta^{k+1} \qquad \forall x \in \bar\Omega, \quad \forall \delta \in]0, \delta_2[.$$

Then,

$$u[\delta](x) = u^{(0)}(x) + \frac{1}{\delta} u_\delta^{(1)}(x) \qquad \forall x \in \bar\Omega, \quad \forall \delta \in]0, \delta_2[,$$

and $u^{(0)}$ is the unique solution in $C^{1,\alpha}(\bar\Omega)$ of problem (39) such that

$$\int_{\partial\Omega^i} u^{(0)} \, d\sigma = 0.$$

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REFERENCES

1. Wendland WL, Stephan E, Hsiao GC. On the integral equation method for the plane mixed boundary value problem of the Laplacian. *Math Methods Appl Sci*. 1979;1(3):265-321.

2. Kirsch A. The Robin problem for the Helmholtz equation as a singular perturbation problem. *Numer Funct Anal Optim*. 1985;8(1-2):1-20.

3. Costabel M, Dauge M. A singularly perturbed mixed boundary value problem.*Commun Part Diff Eq*. 1996;21(11-12):1919-1949.

4. Berestycki H, Wei J. On singular perturbation problems with Robin boundary condition. *Ann Sc Norm Super Pisa Cl Sci (5)*. 2003;2(1):199-230.

5. Gómez D, Lobo M, Pérez E, Sanchez-Palencia E. Homogenization in perforated domains: a Stokes grill and an adsorption process. *Appl Anal*. to appear. https://doi.org/10.1080/00036811.2017.1395863.

6. Ammari H, Nédélec J-C. Generalized impedance boundary conditions for the Maxwell equations as singular perturbations problems. *Commun Part Diff Eq*. 1999;24(5-6):821-849.

7. Schmidt K, Hiptmair R. Asymptotic expansion techniques for singularly perturbed boundary integral equations. *Numer Math*. 2017;137(2):397-415.

8. Dalla Riva M, Mishuris G. Existence results for a nonlinear transmission problem. *J Math Anal Appl*. 2015;430(2):718-741.

9. Donato P, Monsurrò S, Raimondi F. Existence and uniqueness results for a class of singular elliptic problems in perforated domains. *Ric Mat*. 2017;66(2):333-360.

10. Mishuris G, Miszuris W, Öchsner A. Evaluation of transmission conditions for thin reactive heat-conducting interphases. *Defect Diffus Forum*. 2008;273:394-399. Trans Tech Publications.

11. Mishuris G, Miszuris W, Öchsner A. Transmission conditions for thin reactive heat-conducting interphases: general case; 2009:521-526. Trans Tech Publications.

12. Mishuris G. Imperfect transmission conditions for a thin weakly compressible interface. 2D problems. *Arch Mech (Arch Mech Stos)*. 2004;56(2):103-115.

13. Mishuris G, Öchsner A. 2D modelling of a thin elasto-plastic interphase between two different materials: plane strain case. *Compos Struct*. 2007;80(3):361-372.

14. Mishuris G, Miszuris W, Öchsner A. Finite element verification of transmission conditions for 2D heat conduction problems. *Materials Science Forum*. 2007;553:93-99. Trans Tech Publications.

15. Sonato M, Piccolroaz A, Miszuris W. General transmission conditions for thin elasto-plastic pressure-dependent interphase between dissimilar materials. *Int J Solids Struct*. 2015;64-65:9-21.

16. Gilbarg D, Trudinger NS. *Elliptic Partial Differential Equations of Second Order*. second ed, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224. Berlin: Springer-Verlag; 1983.

17. Deimling K. *Nonlinear Functional Analysis*. Berlin: Springer-Verlag; 1985.

18. Lanza de Cristoforis M. Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole in Schauder spaces. *Comput Methods Funct Theory*. 2002;2(1):1-27.

19. Maz'ya V, Movchan A, Nieves M.. *Green's Kernels and Meso-scale Approximations in Perforated Domains*, Lecture Notes in Mathematics, vol. 2077. Heidelberg: Springer; 2013.

20. Maz'ya V, Nazarov S, Plamenevskij B. *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Vol. I*, Operator Theory: Advances and Applications, vol. 111. Basel: Birkhäuser Verlag; 2000. Translated from the German by Georg Heinig and Christian Posthoff.

21. Maz'ya V, Nazarov S, Plamenevskij B. *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Vol. II*, Operator Theory: Advances and Applications, vol. 112. Basel: Birkhäuser Verlag; 2000. Translated from the German by Plamenevskij.

22. Lanza de Cristoforis M. Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach. *Complex Var Elliptic Equ*. 2007;52(10-11):945-977.

23. Dalla Riva M, Musolino P. A mixed problem for the Laplace operator in a domain with moderately close holes. *Commun Part Diff Eq*. 2016;41(5):812-837.

24. Dalla Riva M, Lanza de Cristoforis M. Microscopically weakly singularly perturbed loads for a nonlinear traction boundary value problem: a functional analytic approach. *Complex Var Elliptic Equ*. 2010;55(8-10):771-794.

25. Dalla Riva M, Lanza de Cristoforis M. Weakly singular and microscopically hypersingular load perturbation for a nonlinear traction boundary value problem: a functional analytic approach. *Complex Anal Oper Theory*. 2011;5(3):811-833.

26. Golitsyna EV. Mathematical simulation of temperature field in a hollow rotating cylinder with nonlinear boundary conditions. *High Temperature*. 2008;46(6):835-840.

27. Letavin MI, Mishuris GS. A nonstationary temperature boundary layer in a cross section of a rotating cylinder. *Differentsial'nye Uravneniya*. 1991;27(9):1596-1603 (in Russian). Translation to English in Differential Equations, 1991, 27(9):1134–1140.

28. Letavin MI, Shestakov NI. Solution of equations of thermoelasticity in a cross section of a rotating cylinder by the method of singular perturbations. *Prikl Mat Mekh*. 1993;57(2):124-132 (in Russian). Translation to English: Singular perturbations used to solve the equations of thermoelasticity in a cross-section of a rotating cylinder, in: J. Appl. Math. Mech., 1993;57(2):345–353.

29. Folland GB. *Introduction to Partial Differential Equations*. second ed. Princeton NJ: Princeton University Press; 1995.

30. Miranda C. Sulle proprietà di regolarità di certe trasformazioni integrali. *Atti Accad Naz Lincei Mem Cl Sci Fis Mat Natur Sez I (8)*. 1965;7:303-336.

31. Lanza de Cristoforis M, Rossi L. Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density. *J Integral Equations Appl*. 2004;16(2):137-174.

32. Böhme R, Tomi F. Zur Struktur der Lösungsmenge des Plateauproblems. *Math Z*. 1973;133:1-29.

33. Henry D. *Topics in Nonlinear Analysis*. Brasilia:Trabalho de Matemática; 1982.

34. Valent T. *Boundary Value Problems of Finite Elasticity*, Springer Tracts in Natural Philosophy, vol. 31. New York: Springer-Verlag; 1988. Local theorems on existence, uniqueness, and analytic dependence on data.

35. Hille E, Phillips RS. *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, vol. 31. Providence, R. I.: American Mathematical Society; 1957. rev. ed.

36. Dalla Riva M, Musolino P, Rogosin SV. Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole. *Asymptot Anal*. 2015;92(3-4):339-361.