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The Gisin-Percival stochastic Schrödinger equation from standard quantum filtering theory

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We show that the quantum state diffusion equation of Gisin and Percival, driven by complex Wiener noise, is equivalent up to a global stochastic phase to quantum trajectory models. With an appropriate feedback scheme, we set up an analog continuous measurement model which exactly simulates the Gisin-Percival quantum state diffusion. *Published by AIP Publishing.* https://doi.org/10.1063/1.5007917

I. INTRODUCTION

Standard quantum mechanics tells us that closed systems evolve according to the Schrödinger equation, which provides a deterministic and reversible rule for states. The idea that quantum systems may evolve as stochastic processes has emerged in several ways: as an extension of standard formalism to allow for continuous measurements, as a numerical technique for simulating open systems, and as a new law of nature. These approaches have very similar requirements and this leads to a convergence of structure that is pleasing from a mathematical point of view.[1] However, this commonality implies that distinguishing between different physical situations is difficult.

The Gisin-Percival model[2] is exceptional in that it is postulated on the grounds of covariance (up to a global phase) under transformations of the coupling operators that leave the GKS-Lindblad generator, named after Gorini-Kossakowski-Sudarshan and Lindblad, invariant (a symmetry that is typically broken in quantum filtering/trajectory models), and it achieves this by using specific complex Wiener possesses as driving noise instead of the more typical real Brownian motions.

This leads to the question of whether it is possible to realize the Gisin-Percival equation as a special instance of a quantum trajectory model or whether it is truly distinguished from this class. The problem was first raised and addressed by Wiseman and Milburn:[3] they consider a heterodyne detection scheme with a local oscillator with a finite detuning Ω from the system and show that the Gisin equation follows as the infinite detuning limit of the associated stochastic Schrödinger equation; see also Ref. 4—the particular structure of the Gisin-Percival equation arising in effect from the vanishing of the rapidly oscillating terms.[5]

We treat this problem directly in Sec. II without recourse to limits. In Proposition 3 we show that an explicit homodyne quantum trajectory problem can reproduce the Gisin-Percival solution up to a global stochastic phase that depends causally on the noise. Though, in Proposition 1, we show that the stochastic process must have nontrivial quadratic variation. In Sec. IV, we show that it is actually possible, with the additional use of feedback, to obtain a standard homodyne detection scheme that exactly reproduces the Gisin-Percival evolution.

A. Stochastic Schrödinger equations

We fix a Hilbert space $\mathfrak{h}$ as the system state space, and all operators will be assumed to act on this space. Let $\mathbf{L} = [L_1, L_2, \ldots, L_n]^\top$ be a collection of operators, and H be a self-adjoint operator. The $(\mathbf{L}, H)$ *Belavkin-Schrödinger equation* takes the form[1]

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$$d|\psi_t\rangle = \left(-iH - \frac{1}{2}\sum_k \left(L_k^* L_k - \lambda_k L_k + \frac{1}{4}\lambda_k^2\right)\right)|\psi_t\rangle\, dt + \sum_k \left(L_k - \frac{1}{2}\lambda_k\right)|\psi_t\rangle\, dI_k(t), \qquad (1)$$

where

$$\lambda_k(t) = \langle\psi_t|\left(L_k + L_k^*\right)\psi_t\rangle \qquad (2)$$

and $\{I_k\}$ are a family of independent standard Wiener processes,

$$dI_j(t)\, dI_k(t) = \delta_{jk}dt. \qquad (3)$$

By contrast, let $\mathbf{R} = [R_1, R_2, \ldots, R_m]^\top$ be a collection of operators, then the $(\mathbf{R}, H)$ *Gisin-Percival-Schrödinger equation* takes the form[1]

$$d|\tilde\psi_t\rangle = \left(-iH - \frac{1}{2}\sum_k \left(R_k^* R_k - 2c_k^* R_k + |c_k|^2\right)\right)|\tilde\psi_t\rangle\, dt + \sum_k (R_k - c_k)|\tilde\psi_t\rangle\, d\xi_k(t)^*, \qquad (4)$$

where now

$$c_k(t) = \langle\psi_t|L_k\psi_t\rangle \qquad (5)$$

and $\{\xi_k\}$ are a family of independent *complex* Wiener processes,

$$d\xi_j(t)^*\, d\xi_k(t) = \delta_{jk}dt,$$
$$d\xi_j(t)\, d\xi_k(t) = 0. \qquad (6)$$

Equations (1) and (4) are both stochastic analogs of the Schrödinger equation, driven by classical noise. Note that every complex Wiener process ξ can be written as a sum, $\frac{1}{\sqrt{2}}(B_1(t) + iB_2(t))$, of two independent standard Wiener processes. Both equations are nonlinear—on account of $\lambda(t)$'s in (1) and the $c(t)$'s in (4).

The equations however have very different origins. The Belavkin-Schrödinger equation comes from conditioning the state of the system on the results of continuous indirect measurement[6] and is best understood in terms of a quantum filtering theory,[7–9] based on quantum stochastic calculus.[10–12] It has also arisen independently as a quantum Monte Carlo technique.[13,14] The Gisin-Percival-Schrödinger equation however is postulated as a new law of physics: it is proposed as the equation describing a quantum state diffusion leading to a collapse of the wavefunction even for *closed* systems.[15,17] It should be mentioned, however, that (1) also occurs as a proposal for spontaneous collapse and localization,[16] again as a new law of physics.

The question we ask in this article is whether or not one can realize a particular Gisin-Percival-Schrödinger equation as some Belavkin-Schrödinger equation. What we show is that for any fixed $(\mathbf{R}, H)$, leading to a solution $\tilde\psi_t$ of the corresponding Gisin-Percival-Schrödinger equation with initial state ψ_0, there will exist an $(\mathbf{L}, H)$ such that (with appropriate identification of the noises)

$$|\psi_t\rangle = e^{i\Theta(t)}|\tilde\psi_t\rangle, \qquad (7)$$

where ψ_t is to be the solution of some $(\mathbf{L}, H)$ Belavkin-Schrödinger equation with the same initial state ψ_0.

In particular, the wavefunctions are equivalent up to some c-number phase, $\Theta(t)$. However, this phase will have to be time-dependent. In fact, $\Theta(t)$ will have to be random, with a causal dependence on the noise up to time t, and in particular it must be a diffusion process with unbounded variation, that is,

$$d\Theta(t)\, d\Theta(t) \neq 0. \qquad (8)$$

In fact, we shall show in Proposition 1 that if $\Theta(t)$ is of bounded variation, then (7) cannot hold.

In Sec. II A, we shall give explicit constructions to establish the representation (7). After a discussion of the underlying network theory in Sec. III, we show in Sec. IV that the Gisin-Percival

Schrödinger equation can in fact be modeled exactly as a Belavkin-Schrödinger equation if we allow feedback.

B. Relation to other work

It is well known that the Belavkin-Schrödinger equation has the linear form

$$d|\chi_t\rangle = -\left(\frac{1}{2} \sum_k L_k^* L_k + iH\right)|\chi_t\rangle \, dt + \sum_k L_k |\chi_t\rangle \, dY_k(t), \tag{9}$$

where $Y_k(t)$ are the measured quadrature processes (usually from a).[6–9] The Wiener processes $I_k(t)$ are then the innovation processes given by

$$dI_k(t) = dY_k(t) - \lambda_k(t) \, dt. \tag{10}$$

Underlying this, of course, is a quantum stochastic description for a definite input-system-output model. Effectively, one is homodyning the quantum output fields, with the kth channel interpreted as a continuous measurement of the observable $L_k + L_k^*$. The linearity is no great surprise and is the well known classically as the Zakai equation in filtering theory.

The fact that the Y_k are *not* Wiener processes (otherwise we would be conditioning on white noise) is often missed in the physics literature. One can consider the mathematical equation (9) with the Y_k replaced by independent Wiener processes Z_k, but this is physically meaningless: it becomes physically correct only when one replaces the Z_k with the processes having the same statistical distribution as the Y_k and mathematically this can be done using a re-weighting of the path probabilities known as a Cameron-Martin-Girsanov transformation; see, for instance, Ref. 18.

1. Wiseman Milburn limit

The derivation by Wiseman and Milburn[3] is based on an unequivocally clear physical model. We consider a system with a single input field with a coupling operator

$$L = \sqrt{\gamma} e^{-i\varphi} e^{i\Omega t} a, \tag{11}$$

where a is a fixed system operator (say the annihilator for a cavity mode), φ is a fixed phase, and Ω is the detuning frequency from the local oscillator. Substituting into (9) we have

$$d|\chi_t\rangle = -\left(\frac{1}{2}\gamma a^* a + iH\right)|\chi_t\rangle \, dt + \sqrt{\gamma} e^{-i\varphi} e^{i\Omega t} a|\chi_t\rangle \left(dI(t) + \sqrt{\gamma}\langle e^{-i\varphi} e^{i\Omega t} a + e^{i\varphi} e^{-i\Omega t} a^*\rangle dt\right).$$

They now take the limit $\Omega \to \infty$ and drop the rapidly oscillating terms to obtain

$$d|\chi_t\rangle \approx -\left(\frac{1}{2}\gamma a^* a + iH\right)|\chi_t\rangle \, dt + \sqrt{\gamma} a|\chi_t\rangle \left(d\xi_\Omega(t)^* + \sqrt{\gamma}\langle a^*\rangle dt\right), \tag{12}$$

where they introduce the process $\xi_\Omega(t)$ defined by

$$d\xi_\Omega(t)^* = e^{-i\varphi} e^{i\Omega t} \, dI(t). \tag{13}$$

One may argue that $d\xi_\Omega(t)^* d\xi_\Omega(t) = (dI(t))^2 = dt$ while $d\xi_\Omega(t)^2 = e^{2i\varphi} e^{-2i\Omega t} \, dt \approx 0$ so that in (12) we may replace $\xi_\Omega(t)$ with a limit complex Wiener process $\xi(t)$. The resulting equation then leads to the $(R = \sqrt{\gamma} a, H)$ Gisin-Percival equation for $|\chi_t\rangle / \|\chi_t\|$. A precise derivation would presumably involve a Fourier analysis of the input (pre-measurement) process $Z(t) = e^{-i\varphi} e^{i\Omega t} B(t) e^{i\varphi} e^{-i\Omega t} B(t)^*$ considered in Ref. 8 and the use of the Riemann-Lebesgue Lemma[19] to justify the omission of rapidly oscillating terms, but this lies outside the scope of this paper.[21–25]

2. Girsanov transformation

We mention that a recent result by Parthasarathy and Usha Devi[20] shows how to derive the Gisin-Percival equation from a quantum stochastic evolution using an appropriate Girsanov transformation.

In fact, their scheme uses the same construction as that appearing in our simplest representation of the Gisin-Percival equations in terms of the Belavkin equation, see Subsection II C.

C. Covariance

Equations (1) and (4) give rise to quantum dynamical semigroups: for instance, taking the average over the ensemble of Wiener paths $\{I_k\}$, we obtain

$$\mathbb{E}[\langle \psi_t | X \, \psi_t \rangle] \equiv \Phi_t(X),$$

where Φ_t is the completely positive semigroup on the operators of $\mathfrak{h}$ with the GKS-Lindbladian associated with $(\mathbf{L}, H)$ to be

$$\mathcal{L}_{(\mathbf{L},H)}(X) = \frac{1}{2} \sum_k \left(\left[L_k^*, \cdot \right] L_k + L_k^* \left[\cdot, L_k \right] \right) - i \left[\cdot, H \right]. \tag{14}$$

The corresponding expression involving $\tilde{\psi}_t$ averaged over the complex Wiener noise leads to the quantum dynamical semigroup with GKS-Lindbladian $(\mathbf{R}, H)$.

Nevertheless, there are several differences. The GKS-Lindbladian $(\mathbf{L}, H)$ is invariant under the transformations[1]

$$L_k \rightarrow L_k' = \sum_j u_{kj} L_k, \tag{15}$$

with $U = [u_{jk}]$ unitary, and the transformations

$$
\begin{aligned}
L_k \rightarrow L_k' &= L_k + \beta_k, \\
H \rightarrow H' &= H + \sum_k \operatorname{Im} \left\{ \beta_k^* L_k \right\} + \epsilon,
\end{aligned}
\tag{16}
$$

with ϵ real.

The Gisin-Percival-Schrödinger equation transforms covariantly under these transformations. For the unitary rotation transformations (15), we need only rotate the complex Wiener processes, while under the translation transformations (16) we have

$$d|\tilde{\psi}_t'\rangle = \left(-iH - \frac{1}{2} \sum_k \left(R_k^* R_k - 2 c_k^* R_k + |c_k|^2 \right) \right) |\tilde{\psi}_t'\rangle \, dt - i\epsilon(t) |\tilde{\psi}_t'\rangle \, dt + \sum_k \left(R_k - c_k \right) |\tilde{\psi}_t'\rangle \, d\xi_k(t)^*, \tag{17}$$

where the additional term $\epsilon(t) = \sum_k \operatorname{Im} \left\{ \beta_k^* c_k \right\}$ is a time-dependent phase.

The Belavkin-Schrödinger equation, however, does not typically possess such a covariance.

D. Notation

We will write the Belavkin-Schrödinger equation (1) in the form

$$d|\psi_t\rangle = dF_{(\mathbf{L},H)}(t) \, |\psi_t\rangle, \tag{18}$$

with

$$dF_{(\mathbf{L},H)}(t) = \sum_k \left(L_k - \frac{1}{2} \lambda_k \right) dI_k(t) + \left(-iH - \frac{1}{2} \sum_k \left(L_k^* L_k - \lambda_k L_k + \frac{1}{4} \lambda_k^2 \right) \right) dt. \tag{19}$$

Similarly, we will write the Gisin-Percival-Schrödinger equation (4) in the form

$$d|\tilde{\psi}_t\rangle = dM_{(\mathbf{L},H)}(t) \, |\tilde{\psi}_t\rangle, \tag{20}$$

with

$$dM_{(\mathbf{R},H)}(t) = \sum_k \left(R_k - c_k \right) d\xi_k(t)^* + \left(-iH - \frac{1}{2} \sum_k \left(R_k^* R_k - 2 c_k^* R_k + |c_k|^2 \right) \right) dt. \tag{21}$$

II. REPRESENTATION

For clarity we take the Gisin-Percival-Schrödinger equation with just a single collapse operator R,

$$d|\tilde{\psi}_t\rangle = \left(-iH - \frac{1}{2}\left(R^*R - 2c^*R + |c|^2\right)\right)|\tilde{\psi}_t\rangle\, dt + (R - c)\,|\tilde{\psi}_t\rangle\, d\xi\,(t)^*. \tag{22}$$

Our question is whether there exists a choice of $(\mathbf{L}, H')$ for which the associated Belavkin-Schrödinger equation reproduces the (R, H) Gisin-Percival-Schrödinger equation.

Proposition 1. There is no direct Belavkin-Schrödinger equation which will reproduce the (R, H) Gisin-Percival-Schrödinger equation, in the sense that

$$|\psi(t)\rangle \equiv e^{i\theta(t)}|\tilde{\psi}(t)\rangle,$$

for some real valued differentiable process θ.

Proof. Let us suppose that there is a $(\mathbf{L}, H')$ Belavkin-Schrödinger equation (1) reproducing (22), up to some ignorable phase term. That is,

$$dF_{(\mathbf{L},H')}(t) = dM_{(R,H)}(t) + i\dot{\theta}(t)\,dt.$$

Looking at the skew-adjoint time-independent terms in the dt coefficients, we see that we need to have the same Hamiltonian, $H' = H$ and $\theta = 0$. Moreover, we then have

$$\sum_k \mathrm{Re}\left(L_k^*L_k - \lambda_k\,(t)\,L_k + \frac{1}{4}\lambda_k\,(t)^2\right) \equiv \mathrm{Re}\left(R^*R - 2c\,(t)^*\,R + |c\,(t)\,|^2\right),$$

and comparing the time-independent parts we see that

$$\sum_k L_k^*L_k = R^*R. \tag{23}$$

In addition, we have

$$\sum_k \left(L_k + L_k^*\right)\lambda_k\,(t) - \frac{1}{2}\sum_k \lambda_k\,(t)^2 = 2c\,(t)^*\,R + 2c\,(t)\,R^* - 2\,|c\,(t)|^2$$

and averaging gives

$$\frac{1}{4}\sum_k \lambda_k\,(t)^2 = |c\,(t)|^2. \tag{24}$$

Subtracting this last part off, we obtain

$$\sum_k \left(L_k + L_k^*\right)\lambda_k\,(t) = 2c\,(t)^*\,R + 2c\,(t)\,R^*. \tag{25}$$

Now, equating the noise terms leads to

$$\sum_k \left(L_k - \frac{1}{2}\lambda_k\,(t)\right)dI_k\,(t) \equiv (R - c\,(t))\,d\xi\,(t)^*, \tag{26}$$

and from the requirement that the I_k are independent canonical Wiener processes and that $(d\xi^*)^2 = 0$ we get that

$$0 = \sum_k \left(L_k - \frac{1}{2}\lambda_k\,(t)\right)^2 = \sum_k (L_k^2 - \lambda_k\,(t)\,L_k + \frac{1}{4}\lambda_k\,(t)^2).$$

As this should be true for time-dependent $\lambda_k\,(t)$, we require that

$$\sum_k L_k^2 = 0 \tag{27}$$

and therefore

$$\sum_k \lambda_k(t) L_k = \frac{1}{4} \sum_k \lambda_k(t)^2 .$$

We must then have $\sum_k \lambda_k(t) \left(L_k + L_k^*\right) = \frac{1}{2} \sum_k \lambda_k(t)^2$. The average of this last equation for vector state ψ_t yields $\sum_k \lambda_k(t)^2 = \frac{1}{2} \sum_k \lambda_k(t)^2$ which is a contradiction. ∎

A. Representation up to a stochastic phase

We may however try and find a relation such as

$$|\psi(t)\rangle \equiv e^{i\Theta(t)} |\tilde{\psi}(t)\rangle, \tag{28}$$

where $\Theta(t)^* = \Theta(t)$ is a self-adjoint stochastic process of unbounded variation. This now leads to

$$dF_{(\mathbf{L},H')}(t) = dM_{(R,H)}(t) + id\Theta(t) + id\Theta(t)\, dM_{(R,H)}(t). \tag{29}$$

We may now relax condition (26) and replace it by

$$\sum_k \left(L_k - \frac{1}{2}\lambda_k(t)\right) dI_k(t) \equiv (R - c(t))\, d\xi(t)^* + id\Theta(t). \tag{30}$$

This then implies

$$\sum_k \left(L_k + L_k^* - \lambda_k(t)\right) dI_k(t) \equiv (R - c(t))\, d\xi(t)^* + (R^* - c(t)^*)\, d\xi(t). \tag{31}$$

We will now show that for an elementary class, which we called the canonical class, it is possible to find a stochastic phase Θ so that this is achieved.

For further discussion about the gauge invariance of stochastic master equations when the wavefunction is multiplied by a stochastic phase, see the work of Wiseman and Milburn,[4] p. 170.

B. The canonical class

Let us suppose that each collapse operator L_k is proportional to R,

$$L_k \equiv z_k R, \tag{32}$$

then (23) is satisfied if

$$\sum_k |z_k|^2 = 1. \tag{33}$$

We now assume the relaxed condition (30), and from (31) we get

$$\sum_k \left(z_k R + z_k^* R^* - \lambda_k(t)\right) dI_k(t) \equiv (R - c(t))\, d\xi(t)^* + (R^* - c(t)^*)\, d\xi(t),$$

where we now have $\lambda_k(t) = z_k c(t) + z_k^* c(t)^*$. The time-independent coefficients may be equated and this leads to the identity $\sum_k \left(z_k R + z_k^* R^*\right) dI_k(t) \equiv R\, d\xi(t)^* + R^*\, d\xi(t)$ and assuming that $R \neq R^*$ one sees that one should take

$$\xi(t)^* \equiv \sum_k z_k I_k(t). \tag{34}$$

One easily checks that the rest of (31) then follows: that is, $\sum_k \lambda_k(t)\, dI_k(t) \equiv c(t)\, d\xi(t)^* + c(t)^*\, d\xi(t)$. We also see that (33) implies that $d\xi \cdot d\xi^* = dt$, and for the identification (34) we see that the other condition $(d\xi^*)^2 = 0$ requires

$$\sum_k z_k^2 = 0. \tag{35}$$

The skew adjoint part of (30) is

$$\sum_k \left(z_k R - z_k^* R^*\right) dI_k(t) \equiv (R - c(t)) d\xi(t)^* - (R^* - c(t)^*) d\xi(t) + 2id\Theta(t)$$

and, eliminating terms using (34), we find

$$d\Theta(t) = \frac{1}{2i} (c(t) d\xi(t)^* - c(t)^* d\xi(t)) \equiv \sum_k \mathrm{Im}\{z_k c(t)\} dI_k(t).$$

We note that for the canonical class the phase Θ is proportional to the identity operator on $\mathfrak{h}$, so we may think of it as a (stochastic) phase function.

This stochastic "phase" Θ has the nontrivial Ito table

$$d\Theta \, d\xi^* = d\xi^* \, d\Theta = -\frac{1}{2i} c(t) \, dt,$$

$$d\Theta \, d\xi = d\xi \, d\Theta = \frac{1}{2i} c(t)^* \, dt,$$

$$d\Theta \, d\Theta = \frac{1}{2} |c(t)|^2 \, dt.$$

Now let us check that this gives the correct answer, specifically, that we obtain the correct dt terms. Starting from (19) for the canonical class, we note (from the conditions $\sum_k |z_k|^2 = 1$, $\sum_k z_k^2 = 0$) that

$$\sum_k L_k^* L_k = R^* R,$$

$$\sum_k L_k^* \lambda_k = \sum_k (z_k c + z_k^* c^*) z_k R = c^* R,$$

$$\sum_k \lambda_k^2 = \sum_k (z_k c + z_k^* c^*)^2 = 2|c|^2,$$

$$\sum_k (L_k - \frac{1}{2} \lambda_k) I_k = \sum_k (z_k R - \frac{1}{2} c z_k - \frac{1}{2} + z_k^* c^*) I_k = (R - \frac{1}{2} c) d\xi^* - \frac{1}{2} c^* s\xi.$$

So for the canonical class we have

$$dF_{(\mathbf{L},H)} = \left(-iH - \frac{1}{2} (R^* R \underline{-c^* R + \frac{1}{2} |c|^2}) \right) dt + (R - \frac{1}{2} c) d\xi^* - \frac{1}{2} c^* \, d\xi.$$

The underlined terms are half of what they should be in the dt part of $dM_{(R,H)}$—this was exactly the problem we ran into in Proposition 1. However, let us look at the effect of the stochastic phase Θ. The right-hand side of (29) is

$$dM_{(R,H)}(t) + id\Theta(t) + id\Theta(t) \, dM_{(R,H)}(t) = \left(-iH - \frac{1}{2} (R^* R - 2c^* R + |c|^2) \right) dt$$

$$+ (R - c) d\xi^* + \frac{1}{2} (c d\xi^* - c^* d\xi) + \frac{1}{2} (c d\xi^* - c^* d\xi)(R - c) d\xi^*,$$

and the Ito correction $id\Theta(t) dM_{(R,H)}(t) \equiv -\frac{1}{2} c^* (R - c) \, dt$ gives precisely the missing dt term contribution. By inspection, we see that we have recovered (29).

Remark 2. For $n = 2$, the equations $\sum_k |z_k|^2 = 1$ and $\sum_k z_k^2 = 0$ have solutions

$$z_1 = \frac{1}{\sqrt{2}} e^{i\phi}, \quad z_1 = \pm i \frac{1}{\sqrt{2}} e^{i\phi}, \tag{36}$$

for some phase ϕ. For each integer $n \geq 3$, we have a larger set of solutions, but a subclass is given by $\{z_1, \ldots, z_n\}$, where $z_k = \frac{1}{\sqrt{n}} e^{i\pi(k-1)/n}$, for $k = 1, \ldots, n$.

C. Simplest example

From the remark, we see that the simplest realization (up to a phase) of the canonical class is given by taking $n = 2$ collapse operators

$$L_1 = \frac{1}{\sqrt{2}}R, \quad L_2 = \frac{i}{\sqrt{2}}R. \tag{37}$$

Here we will have

$$\lambda_1(t) = \frac{1}{\sqrt{2}}\langle\psi_t|(R + R^*)\psi_t\rangle,$$

$$\lambda_2(t) = \frac{i}{\sqrt{2}}\langle\psi_t|(R - R^*)\psi_t\rangle$$

so that

$$\langle\psi_t|R\psi_t\rangle = \frac{1}{\sqrt{2}}(\lambda_1(t) - i\lambda_2(t)),$$

and the complex Wiener process is

$$\xi(t)^* = \frac{1}{\sqrt{2}}I_1(t) + \frac{i}{\sqrt{2}}I_2(t). \tag{38}$$

D. The filters

Let X be an arbitrary system operator, then its filtered expectation at time t from the Belavkin theory is

$$\pi_t(X) = \langle\psi_t|X\psi_t\rangle \tag{39}$$

and from (1) we get

$$d\pi_t(X) = \pi_t(\mathcal{L}_{(\mathbf{L},H)}X)\,dt + \sum_k \{\pi_t(XL_k + L_k^*X) - \lambda_k(t)\,\pi_t(X)\}dI_k(t), \tag{40}$$

and we recall that $\lambda_k(t)$ now equals $\pi_t(L_k + L_k^*)$. Here the Lindbladian is the one determined by collapse operators $\mathbf{L} = \{L_k\}$ and Hamiltonian H.

Although not interpreted as a filter, we may consider the equivalent in Gisin-Percival's theory

$$\tilde{\pi}_t(X) = \langle\tilde{\psi}_t|X\tilde{\psi}_t\rangle. \tag{41}$$

This time, using (4), we find

$$d\tilde{\pi}_t(X) = \tilde{\pi}_t(\mathcal{L}_{(\mathbf{R},H)}X)\,dt + \sum_k \{\tilde{\pi}_t(XR_k) - c_k(t)\,\tilde{\pi}_t(X)\}\,d\xi_k(t)^*$$

$$+ \sum_k \left\{\tilde{\pi}_t(R_k^*X) - c_k(t)^*\,\tilde{\pi}_t(X)\right\}\,d\xi_k(t). \tag{42}$$

As the Belavkin-Schrödinger wavefunction $|\psi_t\rangle$ in the canonical class is equal to the Gisin-Percival-Schrödinger wavefunction up to a phase, the following result should not be surprising.

Proposition 3. The Belavkin filter $\pi_t(X)$ corresponding to the canonical class $\left(L_1 = \frac{1}{\sqrt{2}}R,\right.$ $\left.L_2 = \frac{i}{\sqrt{2}}R, H\right)$ Belavkin-Schrödinger model is identical to the Gisin-Percival "filter" $\tilde{\pi}_t(X)$ for the (R, H) Gisin-Percival-Schrödinger equation.

The proof is routine and simply amounts to substituting (37) into (40) and reassembling the components using the definition (38) for the complex Wiener process. The result is just the single collapse operator R version of (4). If both are initialized on the same state ψ_0, then we must have $\pi_t \equiv \tilde{\pi}_t$. The extension to the multi-dimensional situation is obvious.

III. SYSTEM THEORY APPROACH

We recall the "SLH" formulation of open quantum Markov systems, so called because of the operator coefficients in the central equation (43) below. Our system, with Hilbert space $\mathfrak{h}$, is coupled to an environment which is a Bose reservoir with Fock space $\mathfrak{F}$. For an n input model, we take $\mathfrak{F}$ to be the Bose Fock space with one particle space $\mathbb{C}^n \otimes L^2[0, \infty)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$, then $e_k \otimes f$ gives a one particle state corresponding to an input quantum in the kth channel with wavefunction $f = f(t), t \geq 0$. (We may think of the reservoir quanta traveling through the system, and the parameter t labels the part of the input field passing through the system at time t.) We take $B_k(t)$ to be the annihilation operator for the one-particle vector $e_k \otimes 1_{[0,t]}$—that is, annihilating a reservoir quantum of type k sometime over the interval $[0, t]$. Along with the creators, we have the canonical commutation relations

$$\left[B_j(t), B_k(s)^* \right] = \delta_{jk} \max\{t, s\}.$$

Formally, we may introduce densities $b_k(t)$ interpreted as annihilators of a reservoir quantum of type k at time t, with $\left[b_j(t), b_k(s)^* \right] = \delta_{jk} \delta(t-s)$. Then $B_j(t) \equiv \int_0^t b_j(s)\, ds$, etc. We may furthermore introduce the operators

$$\Lambda_{jk}(t) \equiv \int_0^t b_j(s)^* b_k(s)\, ds$$

describing the instantaneous scattering from channel k to channel j of the reservoir quanta at some time $s \in [0, t]$.

It is well known that the most general form of a unitary adapted process on $\mathfrak{h} \otimes \mathfrak{F}$ from a quantum stochastic differential equation with constant coefficients is $U(t)$ given by

$$dU(t) = dG(t)\, U(t), \quad U(0) = I,$$

where

$$dG(t) = -\left(iH + \frac{1}{2} \sum_k L_k^* L_k \right) \otimes dt + \sum_k L_k \otimes dB_k(t)^* - \sum_{jk} L_j^* S_{jk} \otimes dB_k(t) + \sum_{jk} \left(S_{jk} - \delta_{jk} \right) \otimes d\Lambda_{jk}(t),$$

(43)

where the coefficients S_{jk}, L_j, and H are operators on $\mathfrak{h}$ with $S = \left[S_{jk} \right]$ unitary and $H = H^*$. We write $G \sim (S, \mathbf{L}, H)$ for the coefficients and refer to them as the Hudson-Parthasarathy parameters.

We set $j_t(X) = U(t)^* [X \otimes I]\, U(t)$ for a system operator X, and we have

$$dj_t(X) = j_t(\mathcal{L}X) \otimes dt + \sum_{jk} j_t \left(\left[L_j^*, X \right] S_{jk} \right) \otimes dB_k(t)$$

$$+ \sum_{jk} j_t \left(S_{jk}^* [X, L_k] \right) \otimes dB_j(t)^* + \sum_{jk} \left(\sum_l S_{lj}^* X S_{lk} - \delta_{jk} X \right) \otimes d\Lambda_{jk}(t),$$

where $\mathcal{L}X = \frac{1}{2} \sum_k \left[L_k^*, X \right] L_k + \frac{1}{2} \sum_k L_k^* [X, L_k] - i [X, H]$ is a GKS-Lindblad generator.

The output fields are

$$B_{\text{out},k}(t) = U(t)^* [I \otimes B_k(t)]\, U(t)$$

and so $dB_{\text{out},j}(t) = \sum_k j_j \left(S_{jk} \right) \otimes dB_k(t) + j_t(L_k) \otimes dt$.

A. Filtering

Suppose we wish to monitor the output quadratures

$$Y_k(t) = B_{\text{out},k}(t) + B_{\text{out},k}(t)^*.$$

They form a commuting set of observables. We have that $Y_k(t) \equiv U(t)^* [I \otimes Z_k(t)]\, U(t)$, where $Z_k(t) = B_k(t) + B_k(t)^*$: they are a commuting set of observables on the Fock space, having the distribution of independent Wiener processes for the Fock vacuum state.

The aim of filtering is to compute

$$\pi_t(X) = \mathbb{E}\left[j_t(X)\,|\mathfrak{Y}_t\right]$$

which is the conditional expectation of $j_t(X)$ onto the (commutative) algebra $\mathfrak{Y}_t$ generated by the measured observables $\{Y_k(s) : k, 0 \geq s \leq s\}$.

We restrict our attention to the non-scattering case $S = I_n$, that is, $G \sim (I_n, \mathbf{L}, H)$ and

$$dG(t) = -\left(iH + \frac{1}{2}\sum_k L_k^* L_k\right) \otimes dt - \sum_k L_k^* \otimes dB_k(t) + \sum_k L_k \otimes dB_k(t)^*$$

so that

$$dY_k(t) = I \otimes dZ_k(t) + j_t\left(L_k + L_k^*\right) \otimes dt.$$

In this case, the filter is then given by

$$\pi_t(X) = \langle \psi_t | X\,\psi_t \rangle,$$

where ψ_t is the solution to the Belavkin-Schrödinger equation (1). The processes $I_k(t)$ are the innovations

$$dI_k(t) = dY_k - \pi_t\left(L_k + L_k^*\right)dt = I \otimes dZ_k(t) + \left[j_t\left(L_k + L_k^*\right) - \pi_t\left(L_k + L_k^*\right)\right] \otimes dt.$$

It is easy to see that the innovations form a multidimensional Wiener process.

B. The series product

We now consider the situation where the output of one system, $G_1 \sim (S_1, \mathbf{L}_1, H_1)$, is fed in as input to another, $G_2 \sim (S_2, \mathbf{L}_2, H_2)$. In the limit of instantaneous feedforward, we find the combined model $G_2 \triangleleft G_1$, where the series product is defined by

$$(S_2, \mathbf{L}_2, H_2) \triangleleft (S_1, \mathbf{L}_1, H_1) = \left(S_2 S_1, \mathbf{L}_2 + S_2 \mathbf{L}_1, H_1 + H_2 + \text{Im}\{\mathbf{L}_2^* S_2 \mathbf{L}_1\}\right).$$

The covariance of the Lindblad generator, discussed in Subsection I C, can be described in the following terms. A triple (U, β, ε) is said to belong to the central extension of the Euclidean group over the Hilbert space, denoted by $Eu(\mathfrak{h})$ if U is an $n \times n$ matrix with complex scalar entries, $\beta = [\beta_1, \ldots, \beta_n]^\top$ is column vector of complex scalars, and ε is real. We then have the covariance

$$\mathcal{L}_{E \triangleleft G} = \mathcal{L}_G,$$

for all $E \in Eu(\mathfrak{h})$. Note that (15) and (16) correspond to $(I_n, \mathbf{L}, H) \mapsto (U, 0, 0) \triangleleft (I_n, \mathbf{L}, H)$ and $(I_n, \mathbf{L}, H) \mapsto (I_n, \beta, \epsilon) \triangleleft (I_n, \mathbf{L}, H)$, respectively.

C. Weyl displacement

Let β be a collection of square-integrable complex-valued functions $\beta_k = \beta_k(t)$. We can consider a "Weyl Box" to be a component which displaces the vacuum inputs by the amplitudes $\beta(t)$. That is,

$$\text{Weyl}_\beta(t) \sim (I_n, \beta(\mathbf{t}), 0). \tag{44}$$

The output annihilator process of the Weyl Box will then be $B_k(t) + \int_0^t \beta_k(s)ds$.

Placing a Weyl Box after a system $G \sim (I_n, \mathbf{L}, H)$ as a post-filter results in the combined model

$$\text{Weyl}_\beta(t) \triangleleft G \sim \left(I_n, \mathbf{L} + \beta(\mathbf{t}), H + \text{Im}\sum_k \beta_k^*(t)L_k\right).$$

Note that the Hudson-Parthasarathy coefficients are now time-dependent.

IV. SIMULATING THE GISIN-PERCIVAL EQUATION

We now show that it is possible to use our system theory approach to build feedback systems such that the conditioned state corresponds to the Gisin-Percival Schrödinger wavefunction. The setup is described in Fig. 1. We take the system G to be the simple model $\left(L_1 = \frac{1}{\sqrt{2}}R, L_2 = \frac{i}{\sqrt{2}}R, H\right)$, that is,

$$G \sim \left(I_2, \begin{bmatrix} \frac{1}{\sqrt{2}}R \\ \frac{i}{\sqrt{2}}R \end{bmatrix}, H\right). \tag{45}$$

Both inputs are displaced by amplitudes $\beta_k(t)$ using Weyl Box post-filters that lead to the composite system $G \lhd \mathrm{Weyl}_\beta(t)$ given by

$$G \sim \left(I_2, \begin{bmatrix} \frac{1}{\sqrt{2}}R + \beta_1(t) \\ \frac{i}{\sqrt{2}}R + \beta_2(t) \end{bmatrix}, H + \frac{1}{\sqrt{2}}\mathrm{Im}\left\{[\beta_1(t) + i\beta_2(t)]^*R\right\}\right). \tag{46}$$

We perform homodyne measurements on the quadratures, recording the essentially classical processes $Y_1(t)$ and $Y_2(t)$. Using this, we may compute the conditional wavefunction for the Belavkin-Schrödinger equation, but now with the parameters $(I_2, \mathbf{L}, H)$ replaced by the modulated ones in (46).

At this stage, we compute a pair of processes $\alpha_k(t)$ using the conditional state ψ_t. For the moment we make only one assumption:

(A) *The functions $\alpha_k(t)$ are purely imaginary functions, depending causally on the measurement records $\{Y_k(s): k = 1, 2; 0 \leq s \leq t\}$.*

The input displacements are then taken to be $\beta_k(t)$ which are Fock space based processes defined by the *pull-back transformation*

$$I \otimes \beta_k(t) = U(t)\alpha_k(t)U(t)^*. \tag{47}$$

Note that the $\beta_k(t)$ defined in this way are functions of the measurement records $\{Z_k(s): k = 1, 2; 0 \leq s \leq t\}$.

We now try and introduce feedback following the approach in Ref. 26. In terms of consistency, we are now taking the displacements $\beta_k(t)$ to be adapted processes on the Fock space and describable in terms of the processes Z_1 and Z_2. By this manner we make the feedback loops. While the model is more involved than the constant coefficient or even time-dependent coefficient case, the dynamics are still well defined, and arguably Markovian.

Let us look at the resultant filter in detail. First we have $dY_k(t) = dZ_k(t) + j_t(L_k + \beta_k(t) + \mathrm{H.c.})\,dt$. But as the β_k are imaginary, this reduces to
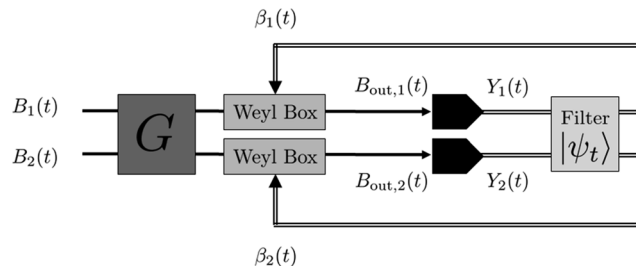

FIG. 1. The setup is as follows. The overall system is driven by a pair of vacuum quantum input fields $B_1(t)$ and $B_2(t)$. These drive the system and the output is subsequently modulated by Weyl Boxes with displacements $\beta_1(t)$ and $\beta_2(t)$, respectively. We perform a pair of homodyne measurements on the (modulated) quadratures $Y_k(t) = B_{\mathrm{out},k}(t) + B_{\mathrm{out},k}(t)^*$ and use this information to compute the conditioned state $|\psi_t\rangle$ using the Belavkin filter. Using the filter we compute a pair of processes $\alpha_k(t)$ which are then fed back in as the displacements ($\beta_k(t)$) for the respective Weyl Boxes.

$$dY_1(t) = dZ_1(t) + \frac{1}{2\sqrt{2}} j_t(R + R^*) \, dt,$$

$$dY_2(t) = dZ_2(t) + \frac{i}{2\sqrt{2}} j_t(R - R^*) \, dt, \tag{48}$$

and in both cases $dI_k(t) \equiv dY_k(t) - \pi_t(L_k + L_k^*) \, dt$.

The Belavkin-Schrödinger equation is now the one with the term (18) given by

$$dF(t) = \sum_k \left(L_k + \alpha_k(t) - \frac{1}{2}\lambda_k \right) dI_k(t) + \left(-iH - i \sum_k \mathrm{Im}\{\alpha_k^*(t)L_k\} - \frac{1}{2} \sum_k \left((L_k + \alpha_k(t))^*(L_k + \alpha_k(t)) \right. \right.$$

$$\left. \left. + \frac{1}{2} \sum_k \lambda_k(L_k + \alpha_k(t)) - \frac{1}{8} \sum_k \lambda_k^2 \right) \right) dt. \tag{49}$$

Some comments are in order. First of all (49) is obtained from (19) with the change $L_k \mapsto L_k + \alpha_k(t)$ and $H \mapsto H + \sum_k \mathrm{Im}\{L_k^* \alpha_k(t)\}$. We note that these equations involve the $\alpha_k(t)$, which are functions of the measurements $\{Y_k\}$, rather than the $\beta_k(t)$, and the reason for this is that we have to push forward to the output picture; see Ref. 8. The $\lambda_k(t)$ should in principle be shifted to $\lambda_k(t) + \alpha_k(t) + \alpha_k(t)^*$. But this shift is by the real part of the $\beta_k(t)$ and we recall that these vanish by assumption (A).

We now seek to arrange things so that (49) gives us the Gisin-Percival term $dM_{(R,H)}(t)$ and for the noise term, with the previous identification (38) for the noise, we see that we must have

$$L_1 + \alpha_1(t) - \frac{1}{2}\lambda_1(t) \equiv \frac{1}{\sqrt{2}}(R - c(t)),$$

$$L_2 + \alpha_2(t) - \frac{1}{2}\lambda_2(t) \equiv \frac{i}{\sqrt{2}}(R - c(t)), \tag{50}$$

and this implies the following choice:

$$\alpha_1(t) \equiv -\frac{1}{2\sqrt{2}} \langle \psi_t | (R - R^*)\psi_t \rangle,$$

$$\alpha_2(t) \equiv -\frac{i}{2\sqrt{2}} \langle \psi_t | (R + R^*)\psi_t \rangle. \tag{51}$$

We remark that both of these choices determine purely imaginary expressions, and so the ansatz (A) is in place. In fact, if we decompose $c(t) = \langle \psi_t | R \, \psi_t \rangle$ into real and imaginary parts $c'(t) + ic''(t)$, then

$$\lambda_1(t) = \sqrt{2}c'(t), \quad \lambda_2(t) = -\sqrt{2}c''(t)$$

$$\alpha_1(t) = -\frac{i}{\sqrt{2}}c''(t), \quad \alpha_2(t) = -\frac{i}{\sqrt{2}}c'(t). \tag{52}$$

With this choice of the $\alpha_k(t)$'s, we find that the dt term in (49) is

$$-iH - \frac{1}{2} \sum_k L_k^* L_k - \sum_k \left(\alpha_k^*(t) - \frac{1}{2}\lambda_k(t) \right) L_k - \frac{1}{2} \sum_k \left(\alpha_k(t) \right)^* \alpha_k(t) - \lambda_k \alpha_k(t) + \frac{1}{4}\lambda_k^2 \right), \tag{53}$$

and it is straightforward to check that

$$\sum_k \left(\alpha_k^*(t) - \frac{1}{2}\lambda_k(t) \right) L_k \equiv c(t)^* R,$$

$$\frac{1}{2} \sum_k \alpha_k(t))^* \alpha_k(t) = \frac{1}{4} \sum_k \lambda_k^2 \equiv \frac{1}{2}|c(t)|^2,$$

$$\sum_k \lambda_k \alpha_k(t) = 0. \tag{54}$$

Substituting these explicit expressions in to (49) leads to $dF(t) = dM_{(R,H)}(t)$, which is now exactly as in (21). In other words, the setup with feedback leads to a conditioned state dynamics which is the same as the Gisin-Percival-Schrödinger equation.

V. CONCLUSION

The Gisin-Percival equation appears distinct from the usual stochastic Schrödinger equations obtained through quantum filtering models. However, as shown by Wiseman and Milburn,[3] it may be obtained as a high detuning limit of a homodyne measurement. Here we have shown that the Gisin-Percival equation is equivalent up to an overall phase to a specific quantum trajectories problem (in fact, we obtain a general class and work with the simplest such representative). The phase term will however be a stochastic process with non-trivial quadratic variation. The quantum measurement scheme needs to ensure that the covariance symmetry of the Gisin-Percival equation holds, and in the simplest case this is done by indirect measurement where two input-output channels couple to the same system operator R and we make homodyne measurements of orthogonal quadratures of the output fields.

In fact, we show that an appropriate feedback of the measured signals allows us to include this phase term, leading to an exact analog simulation of the Gisin-Percival state diffusion in terms of quantum trajectories.

The quantum state diffusion equation of Gisin and Percival is therefore not distinguishable from standard quantum trajectory models.

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[1] J. Kupsch, "Open quantum systems," in *Decoherence and the Appearance of a Classical World in Quantum Theory*, edited by E. Joos, H. D. Zeh, C. Kiefer, D. J. W. Giulini, J. Kupsch, and I.-O. Stamatescu (Springer-Verlag, Berlin, Heidelberg, 2003), Chap. 7.

[2] N. Gisin and I. C. Percival, J. Phys. A: Math. Gen. **25**, 5677–5691 (1992).

[3] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 1652–1666 (1993).

[4] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, 2009).

[5] V. P. Belavkin, Lect. Notes Control Inf. Sci. **121**, 245–265 (1989).

[6] V. P. Belavkin, Radiotechnika Electronika **25**, 1445–1453 (1980); "Stochastic calculus of input-output processes and non-demolition filtering," in *Reviews of the Newest Achievements in Science and Technology*, Volume 36 of Current Problems of Mathematics, edited by A. S. Holevo (VINITI, 1989), pp. 29–67; J. Multivar. Anal. **42**, 171–201 (1992).

[7] L. M. Bouten, M. I. Guţă, and H. Maassen, J. Phys. A **37**, 3189–3209 (2004).

[8] L. Bouten and R. van Handel, "Quantum filtering: A reference probability approach," e-print arXiv:math-ph/0508006 (unpublished).

[9] L. Bouten, R. van Handel, and M. R. James, SIAM J. Control Optim. **46**, 2199–2241 (2007).

[10] R. L. Hudson and K. R. Parthasarathy, Commun. Math. Phys. **93**, 301 (1984).

[11] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).

[12] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus* (Birkhauser, 1992).

[13] H. J. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics (Springer, 1993), Vol. 18.

[14] J. Dalibard, Y. Castin, and K. Molmer, Phys. Rev. Lett. **68**, 580–583 (1992).

[15] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, 2000).

[16] G. C. Ghirardi, P. Pearle, and A. Rimini, Phys. Rev. A **42**, 78–89 (1990).

[17] P. Goetsch and R. Graham, Phys. Rev. A **50**, 5242–5255 (1994).

[18] D. Gatarek and N. Gisin, J. Math. Phys. **32**(8), 2152–2157 (1991).

[19] S. Bochner and K. Chandrasekharan, *Fourier Transforms* (Princeton University Press, 1949).

[20] K. R. Parthasarathy and A. R. Usha Devi, "From quantum stochastic differential equations to Gisin-Percival state diffusion," preprint arXiv:1705.00520v1.

[21] C. M. Mora and R. Rebolledo, Ann. Appl. Probab. **18**(2), 591–619 (2008).

[22] F. Fagnola and C. Mora, Indian J. Pure Appl. Math. **46**(4), 399–414 (2015).

[23] F. Fagnola and C. M. Mora, ALEA, Lat. Am. J. Probab. Math. Stat. **10**(1), 191–223 (2013).

[24] J. Gough and M. R. James, Commun. Math. Phys. **287**, 1109 (2009).

[25] J. Gough and M. R. James, IEEE Trans. Autom. Control **54**, 2530 (2009).

[26] J. E. Gough, "Non-Markovian quantum feedback networks II: Controlled flows," J. Math. Phys. **58**, 063517 (2017).