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ASYMPTOTIC BEHAVIOR OF THE LONGITUDINAL PERMEABILITY OF A PERIODIC ARRAY OF THIN CYLINDERS

PAOLO MUSOLINO, VLADIMIR MITYUSHEV

Abstract. We consider a Newtonian fluid flowing at low Reynolds numbers along a spatially periodic array of cylinders of diameter proportional to a small nonzero parameter ϵ. Then for $\epsilon \neq 0$ and close to 0 we denote by $K_{II}[\epsilon]$ the longitudinal permeability. We are interested in studying the asymptotic behavior of $K_{II}[\epsilon]$ as ϵ tends to 0. We analyze $K_{II}[\epsilon]$ for ϵ close to 0 by an approach based on functional analysis and potential theory, which is alternative to that of asymptotic analysis. We prove that $K_{II}[\epsilon]$ can be written as the sum of a logarithmic term and a power series in ϵ^2. Then, for small ϵ, we provide an asymptotic expansion of the longitudinal permeability in terms of the sum of a logarithmic function of the square of the capacity of the cross section of the cylinders and a term which does not depend of the shape of the unit inclusion (plus a small remainder).

1. Introduction

In this article, we study the asymptotic behavior of the longitudinal permeability of a periodic array of cylinders, when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders. We assume that the diameter of the cylinders is proportional to a small nonzero parameter ϵ and that the driving pressure gradient is parallel to the cylinders. Then the velocity field has only one non-zero component which, by the Stokes equations, satisfies a Poisson equation (cf. problem (1.2)). By means of the longitudinal component of the velocity field, one can define for each ϵ close to 0 the longitudinal permeability $K_{II}[\epsilon]$. Here, we are interested in studying the behavior of $K_{II}[\epsilon]$ as ϵ approaches the degenerate value $\epsilon = 0$, in correspondence of which the diameter of the cylinders collapses.

Several authors have studied the longitudinal permeability of arrays of cylinders by exploiting different techniques. Emersleben [17] considered a square array of contours of a constant value of a special Epstein zeta function. Happel [20] provided an analysis for the dilute case on the basis that two concentric cylinders can serve as the model for fluid moving through an assemblage of cylinders. In the seminal paper [21], Hasimoto investigated the viscous flow past a cubic array of spheres, in particular for the case when the radius of the spheres is small compared to the

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mutual distance. He also applied his results to the two-dimensional flow past a square array of circular cylinders. His techniques are based on the construction of a spatially periodic fundamental solution for the Stokes' system and apply to specific shapes (circular/spherical obstacles and square/cubic arrays). Schmid [42] investigated the longitudinal laminar flow in an infinite square array of circular cylinders.

Sangani and Yao [41] extended a method described in [40] in order treat the problem of determining the permeability of random arrays of infinitely long cylinders. In [41] the transverse and longitudinal permeabilities averaged over several configurations of random arrays of circular cylinders are expressed as a function of the area fraction of the cylinders. Adler and the second author in [34, 35] considered the longitudinal permeability of periodic rectangular arrays of circular cylinders. By methods of complex variable, they transformed the boundary value problem defining the permeability into a functional equation. Then they expressed the solution to such a functional equation in terms of a series of the radius of the cylinders, which can then be exploited to derive a formula for the longitudinal permeability as a logarithmic term and a power series in the radius of the cylinder. More precisely, first in [34] they considered the case of a single circular cylinder in the unit cell and then in [35] they turned to study the case of an arbitrary (finite) distribution of circular cylinders inside the periodicity cell.

In this article instead we investigate the asymptotic behavior of the longitudinal permeability of a rectangular array of thin cylinders, corresponding to the so called dilute case. Our method allows to treat general shapes of the cross section of the cylinders without restrictions to circular cylinders. Our aim is twofold. From one side, we extend the results of [34, 35] to very general shapes and we prove rigorously that the longitudinal permeability can be represented as the sum of a logarithmic term and a power series of the square of the 'size parameter' of the cylinders. This is deduced by an analyticity result for a singularly perturbed boundary value problem. Such a result is based on an operator theoretical reformulation of the problem and on the Implicit Function Theorem in Banach spaces. From the other side, we provide and justify an asymptotic expansion of the longitudinal permeability in terms of the square of the logarithmic capacity of the cross section of the cylinders. Once more, since there is no particular restriction on the shape of the cylinders, this expression allows comparisons between different geometric assumptions.

In contrast, for example, with the functional equation method of [34, 35], here we exploit a potential theoretical method to investigate to behavior of the longitudinal permeability in the dilute case. Integral equation techniques have indeed revealed to be a powerful tool to analyze the asymptotic behavior of a wide class of quantities of physical interest. For example, in Vogelius and Volkov [44] the time-harmonic Maxwell equation is considered and the leading order boundary perturbations are derived. Ammari, Kang and Touibi [5] derived asymptotic expansions of the effective electrical conductivity of periodic dilute composites in terms of the volume fraction occupied by the inclusions. In Ammari, Kang, and Lee [4] asymptotic formulas for the effective parameters of a two-phase elastic medium consisting of materials of different elastic properties are shown. Analogous results for the effective viscosity properties of dilute suspensions of arbitrarily shaped particles are obtained in Ammari, Garapon, Kang, and Lee [2]. For an extensive list of applications of this method, we refer, e.g., to the monograph by Ammari and Kang [3].

Furthermore, the potential theoretical method proposed by Lanza de Cristoforis for real analytic continuation properties of solutions of singularly perturbed linear or nonlinear problems has been used to analyzed different problems of concrete nature. As an example, in [12] the effective conductivity of periodic two-phase dilute composites with nonideal contact condition was investigated, whereas in [27] a quasi-linear heat transmission problem in a dilute composite was studied.

To introduce the mathematical problem, we fix

$$l \in]0,+\infty[\,, \quad Q \equiv]0,l[\times]0,1/l[\,, \quad q \equiv \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}.$$

Hence, the area $|Q|_2$ of the cell Q holds unity. We denote by q^{-1} the inverse matrix of q. Clearly, $q\mathbb{Z}^2 \equiv \{qz : z \in \mathbb{Z}^2\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the fundamental periodicity cell Q. We also take

$$\begin{aligned}&\alpha \in]0,1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^2 \text{ of class}\\ &C^{1,\alpha} \text{ such that } \mathbb{R}^2 \setminus \mathrm{cl}\Omega \text{ is connected and that } 0 \in \Omega.\end{aligned} \tag{1.1}$$

The symbol cl denotes the closure. Moreover, we fix

$$p \in Q \text{ and } \epsilon_0 \in]0,+\infty[\text{ such that } p + \epsilon \mathrm{cl}\Omega \subseteq Q \text{ for all } \epsilon \in]-\epsilon_0, \epsilon_0[\,.$$

To shorten our notation, we set

$$\Omega_{p,\epsilon} \equiv p + \epsilon\Omega \quad \forall \epsilon \in \mathbb{R}\,.$$

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\epsilon}] \equiv \cup_{z\in\mathbb{Z}^2} (qz + \Omega_{p,\epsilon})\,, \quad \mathbb{S}[\Omega_{p,\epsilon}]^- \equiv \mathbb{R}^2 \setminus \cup_{z\in\mathbb{Z}^2} (qz + \mathrm{cl}\,\Omega_{p,\epsilon}) \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\,.$$

If $\epsilon \in]-\epsilon_0, \epsilon_0[$, the set $\mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}] \times \mathbb{R}$ represents an infinite array of parallel cylinders. Instead, the set $\mathbb{S}[\Omega_{p,\epsilon}]^- \times \mathbb{R}$ is the region where a Newtonian fluid of viscosity μ is flowing at low Reynolds number. Then we assume that the driving pressure gradient is constant and parallel to the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (cf., e.g., Adler [1, Ch. 4], Sangani and Yao [41], and [34, 35]), one reduces the Stokes system to a Poisson equation for the non-zero component of the velocity field. Since we are working with dimensionless quantities, we may assume that the viscosity of the fluid and the pressure gradient are both set equal to 1. For a more complete discussion on spatially periodic structures, we refer to Adler [1, Ch. 4]. Accordingly, if $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, we consider the following Dirichlet problem for the Poisson equation:

$$\begin{gathered}\Delta u = 1 \quad \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-\,,\\ u(x+qz) = u(x) \quad \forall x \in \mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-\,, \forall z \in \mathbb{Z}^2\,,\\ u(x) = 0 \quad \forall x \in \partial\Omega_{p,\epsilon}\,.\end{gathered} \tag{1.2}$$

If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, then the solution of problem (1.2) in the space $C^{1,\alpha}_q(\mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of q-periodic functions in $\mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-$ of class $C^{1,\alpha}$ (cf. Subsection 2) is unique and we denote it by $u[\epsilon]$. From the physical point of view, the function $u[\epsilon]$ represents the non-zero component of the velocity field (cf. [34, §2]). By means of the function $u[\epsilon]$, we can introduce the effective permeability $K_{II}[\epsilon]$ which we define as the integral of the opposite of the flow velocity over the unit cell (cf. Adler [1], [34, §3]), i.e.,

$$K_{II}[\epsilon] \equiv -\int_{Q\setminus\Omega_{p,\epsilon}} u[\epsilon](x)\,dx \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}\,,$$

and we pose the following question:

$$\text{What can be said on the map } \epsilon \mapsto K_{II}[\epsilon] \text{ when } \epsilon \text{ is close to 0?} \tag{1.3}$$

Questions of this type have long been investigated with the methods of asymptotic analysis, with the aim of providing asymptotic expansions for the quantities of interest. Those techniques have been exploited to analyze a large class of singular perturbation problems by several authors. For example, in Maz'ya, Nazarov, and Plamenevskij [30] one can find the method of compound asymptotic expansions which allows the analysis of general Douglis-Nirenberg elliptic boundary value problems in domains with holes and corners. Maz'ya, Movchan, and Nieves [29] studied the asymptotic treatment of the Green's kernel in domains with small holes. Bonnaillie-Noël, Dambrine, Tordeux, and Vial [7] exploited the method of multi-scale asymptotic expansions to study the behavior of the solution of the Poisson equation in a domain with moderately close holes. Bonnaillie-Noël, Lacave, and Masmoudi [8] investigated the effect of small inclusions of size ϵ on the behavior of an ideal fluid governed by the 2D Euler system. Problems in perforated domains have been considered also in the frame of shape optimization. For example, in Novotny and Sokołowsky [39], the authors apply the topological derivative (i.e., the first term of the asymptotic expansion of a given shape functional with respect to a small parameter that measures the size of singular domain perturbations) to problems in elasticity and heat diffusion. Furthermore, boundary value problems in periodic domains have been analyzed with the method of functional equations (cf., *e.g.*, Castro, Kapanadze, and Pesetskaya [10], Castro, Pesetskaya, and Rogosin [9], Kapanadze, Mishuris, and Pesetskaya [22], and the works of the second-named author and his collaborators Adler, Drygaś, and Rylko [16, 33, 34, 35, 36]). Such a method is based on complex variable techniques and has revealed to be a useful tool for specific shapes of the inclusions/obstacles.

Here, instead, we answer the question in (1.3) by showing that

$$K_{II}[\epsilon] = -\frac{\log|\epsilon|}{2\pi} + a_0 + a_1\epsilon^2 + \epsilon^4 J[\epsilon] \tag{1.4}$$

for $\epsilon \neq 0$ close to 0, where a_0, a_1 are suitable real numbers (that we compute in terms of the solution of explicit boundary value problems and integral equations) and J is a real analytic function defined in a neighborhood of 0. We observe that our approach does have its advantages. Indeed, equality (1.4) and the parity of J (cf. Proposition 3.10) imply that

$$K_{II}[\epsilon] = -\frac{\log|\epsilon|}{2\pi} + a_0 + a_1\epsilon^2 + \epsilon^4 \sum_{j=0}^{+\infty} b_j \epsilon^{2j}, \tag{1.5}$$

for a suitable sequence of coefficients $\{b_j\}_{j=0}^{+\infty}$, where the series converges absolutely on a whole neighborhood of the degenerate value $\epsilon = 0$. Such an approach has been previously exploited by Lanza de Cristoforis, Dalla Riva and the first-named author to analyze several singularly perturbed boundary value problems in periodically perforated domains. This method has shown to be a flexible tool to analyze problems of different nature. Indeed, linear and nonlinear boundary value problems for the Laplace equation have been studied for example in [25, 37]. In [13], a nonlinear traction problem in linearized elastostatics is studied. In [12] the functional analytic approach is exploited to investigate the asymptotic behavior of the effective conductivity of a dilute composite in presence of a thermal resistance at the interface. This approach has also allowed to study quasi-linear transmission problems. Indeed, by this method, the asymptotic behavior of the temperature distribution of a two-phase composite has been analyzed under the assumption that the thermal conductivities of the materials depend nonlinearly upon the temperature (see [27]).

Then we show that by equation (1.5) we can deduce that the right-hand side of (1.5) is equal to

$$-\frac{\log\rho}{4\pi}+c+O(\rho)\quad \rho\equiv(\mathrm{cap}(\epsilon\Omega))^2\,.$$

The quantity $\rho\equiv(\mathrm{cap}(\epsilon\Omega))^2$ is the square of the logarithmic capacity of the inclusion (cf., e.g., Goluzin [19, Ch. VII,§3] and Kirsch [23, §2]), i.e., of the cross section of the cylinder. Since the area of the cell is normalized to unity, $\rho\equiv(\mathrm{cap}(\epsilon\Omega))^2$ is a measure of the concentration of inclusions in the host medium. Moreover, if we set $\rho\equiv(\mathrm{cap}(\epsilon\Omega))^2$ then the function $K_{II}[\epsilon]$ equals the sum of $-\log\rho/(4\pi)$ and of a real analytic function of ρ at zero. In the present paper, it is established that the coefficient c does not depend on the shape of Ω. In particular,

$$c=-R_{q,2}(0)+\int_Q S_{q,2}(x)\,dx\,,$$

where $S_{q,2}$ and $R_{q,2}$ are as in (2.1) and (2.2), respectively. The dimension of permeability is equal to the surface, i.e., length^2 (cf. Adler [1]).

This article is organized as follows. Section 2 is a section of preliminaries and notation. In Section 3 we prove our main result on the asymptotic behavior of $K_{II}[\epsilon]$. In Section 4 we exploit conformal maps to further investigate $K_{II}[\epsilon]$. Finally, in Section 5 we present some remarks and conclusions.

2. Preliminaries

Notation. Let $\mathcal{O}\subseteq\mathbb{R}^2$. Then $\mathrm{cl}\,\mathcal{O}$ denotes the closure of $\mathcal{O}$ and $\partial\mathcal{O}$ denotes the boundary of $\mathcal{O}$. For all $R>0$, $x\in\mathbb{R}^2$, x_j denotes the j-th coordinate of x, x^T the transpose vector, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^2$, and $\mathbb{B}_2(x,R)\equiv\{y\in\mathbb{R}^2:|x-y|<R\}$. Let Ω be an open subset of $\mathbb{R}^2$. If $f\in C^m(\Omega)$, then Df denotes the gradient $\big(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\big)$ and D^2f denotes the Hessian matrix. For a multi-index $\eta\equiv(\eta_1,\eta_2)\in\mathbb{N}^2$ we set $|\eta|\equiv\eta_1+\eta_2$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|}f}{\partial x_1^{\eta_1}\partial x_2^{\eta_2}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta|\le m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th order derivatives which are uniformly Hölder continuous with exponent $\alpha\in]0,1[$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_2(0,R))}\in C^{m,\alpha}(\mathrm{cl}(\Omega\cap\mathbb{B}_2(0,R)))$ for all $R\in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega)$. Now let Ω be a bounded open subset of $\mathbb{R}^2$. Then $C^m(\mathrm{cl}\,\Omega)$ and $C^{m,\alpha}(\mathrm{cl}\Omega)$ are endowed with their usual norm and are well known to be Banach spaces. Similarly, we define the space $C^m(\mathrm{cl}\,\Omega,\mathbb{R}^2)$ and $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^2)$ of functions from $\mathrm{cl}\,\Omega$ to $\mathbb{R}^2$ of class C^m and $C^{m,\alpha}$, respectively. Then we denote by $C^{m,\alpha}(\mathrm{cl}\Omega,\mathcal{O})$ the subset of those functions $f\in C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^2)$ such $f(\mathrm{cl}\,\Omega)\subseteq\mathcal{O}$. We say that a bounded open subset Ω of $\mathbb{R}^2$ is of class C^m or of

class $C^{m,\alpha}$, if $\mathrm{cl}\Omega$ is a manifold with boundary imbedded in $\mathbb{R}^2$ of class C^m or $C^{m,\alpha}$, respectively. We define the spaces $C^{k,\alpha}(\partial\Omega)$ for $k \in \{0,\dots,m\}$ by exploiting the local parametrizations (cf., e.g., Gilbarg and Trudinger [18, §6.2]). For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [18]. We denote by ν_Ω the outward unit normal to $\partial\Omega$, by $d\sigma$ the arc length element on $\partial\Omega$, and by $|\Omega|_2$ the two-dimensional measure of Ω. Finally, for the definition and properties of real analytic operators, we refer, e.g., to Deimling [15, p. 150].

Spaces of bounded and periodic functions. If Ω is an arbitrary open subset of $\mathbb{R}^2$, $k \in \mathbb{N}$, $\beta \in]0,1]$, we set

$$C^k_b(\mathrm{cl}\Omega) \equiv \{u \in C^k(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^2 \text{ such that } |\gamma| \le k\}\,,$$

and we endow $C^k_b(\mathrm{cl}\Omega)$ with its usual norm $\|u\|_{C^k_b(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\le k} \sup_{x\in \mathrm{cl}\Omega} |D^\gamma u(x)|$ for all $u \in C^k_b(\mathrm{cl}\Omega)$. Then we set

$$C^{k,\beta}_b(\mathrm{cl}\Omega) \equiv \{u \in C^{k,\beta}(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^2 \text{ such that } |\gamma| \le k\}\,.$$

We endow $C^{k,\beta}_b(\mathrm{cl}\Omega)$ with its usual norm $\|u\|_{C^{k,\beta}_b(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\le k} \sup_{x\in \mathrm{cl}\Omega} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \mathrm{cl}\Omega|_\beta$ for all $u \in C^{k,\beta}_b(\mathrm{cl}\Omega)$, where $|D^\gamma u : \mathrm{cl}\Omega|_\beta$ denotes the β-Hölder constant of $D^\gamma u$. Next we turn to periodic domains. If Ω_Q is an arbitrary subset of $\mathbb{R}^2$ such that $\mathrm{cl}\Omega_Q \subseteq Q$, we set

$$\mathbb{S}[\Omega_Q] \equiv \cup_{z\in\mathbb{Z}^2}(qz + \Omega_Q) = q\mathbb{Z}^2 + \Omega_Q\,, \quad \mathbb{S}[\Omega_Q]^- \equiv \mathbb{R}^2 \setminus \mathrm{cl}\mathbb{S}[\Omega_Q]\,.$$

Then a function u from $\mathrm{cl}\mathbb{S}[\Omega_Q]$ or from $\mathrm{cl}\mathbb{S}[\Omega_Q]^-$ to $\mathbb{R}$ is q-periodic if $u(x+qz) = u(x)$ for all x in the domain of definition of u and for all $z \in \mathbb{Z}^2$. If Ω_Q is an open subset of $\mathbb{R}^2$ such that $\mathrm{cl}\Omega_Q \subseteq Q$ and if $k \in \mathbb{N}$ and $\beta \in]0,1[$, then we denote by $C^k_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C^k_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and $C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ the subsets of the q-periodic functions belonging to $C^k_b(\mathrm{cl}\mathbb{S}[\Omega_Q])$, to $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega_Q])$, to $C^k_b(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and to $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, respectively. We regard the sets $C^k_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C^k_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and $C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$ as Banach subspaces of the space $C^k_b(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega_Q])$, $C^k_b(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, respectively.

Preliminaries of potential theory. To investigate problem (1.2) by a potential theoretical approach, we need to introduce a periodic analogue of the fundamental solution of the Laplace operator. As is well known there exists a q-periodic tempered distribution $S_{q,2}$ such that

$$\Delta S_{q,2} = \sum_{z\in\mathbb{Z}^2} \delta_{qz} - 1\,,$$

where δ_{qz} denotes the Dirac distribution with mass in qz. The distribution $S_{q,2}$ is determined up to an additive constant, and we can take

$$S_{q,2}(x) \equiv -\sum_{z\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{4\pi^2|q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x}\,, \tag{2.1}$$

where the series converges in the sense of distributions on $\mathbb{R}^2$ (cf., e.g., Hasimoto [21], [37, Thm. 2.1]). Then, $S_{q,2}$ is real analytic in $\mathbb{R}^2 \setminus q\mathbb{Z}^2$ and is locally integrable in $\mathbb{R}^2$. Now, let S_2 be the function from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_2(x) \equiv \frac{1}{2\pi} \log|x| \quad \forall x \in \mathbb{R}^2 \setminus \{0\}\,.$$

S_2 is well known to be the fundamental solution of the Laplace operator on the plane. Then $S_{q,2} - S_2$ is analytic in $(\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\}$ and we find convenient to set

$$R_{q,2} \equiv S_{q,2} - S_2 \quad \text{in } (\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\} \tag{2.2}$$

(see also [6]). Moreover, one has

$$R_{q,2}(x) = R_{q,2}(-x) \quad \forall x \in (\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\}\,. \tag{2.3}$$

As a consequence,

$$D^\beta R_{q,2}(0) = 0 \quad \text{for all } \beta \in \mathbb{N}^2 \text{ such that } |\beta| \text{ is odd}\,.$$

As observed in [34, p. 336], we note that in the case $Q \equiv]0,1[^2$ (or equivalently $l \equiv 1$), we have

$$S_{q,2}(x_1, x_2) = \frac{1}{2\pi} \log |\sigma(x_1 + ix_2)| - \frac{1}{4\pi} \left(cx_1^2 + (2\pi - c)x_2^2\right)$$

for all $x \equiv (x_1, x_2) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, where i is the imaginary unit, and the constant c and the Weierstrass function $\sigma(\cdot)$ are derived in [34, Appendix A].

Let $\alpha \in]0,1[$ and let $\tilde{\Omega}$ be a bounded open subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. If $\mu \in C^{1,\alpha}(\partial\tilde{\Omega})$, we define the classical double layer potential by setting

$$w[\partial\tilde{\Omega}, \mu](x) \equiv -\int_{\partial\tilde{\Omega}} DS_2(x - y) \cdot \nu_{\tilde{\Omega}}(y)\mu(y)\, d\sigma_y \quad \forall x \in \mathbb{R}^2\,.$$

As is well known, $w[\partial\tilde{\Omega}, \mu]_{|\tilde{\Omega}}$ admits a continuous extension to $\mathrm{cl}\tilde{\Omega}$, which we denote by $w^+[\partial\tilde{\Omega}, \mu]$ and $w[\partial\tilde{\Omega}, \mu]_{|\mathbb{R}^2\setminus \mathrm{cl}\,\tilde{\Omega}}$ admits a continuous extension to $\mathbb{R}^2 \setminus \tilde{\Omega}$, which we denote by $w^-[\partial\tilde{\Omega}, \mu]$. Moreover, $w^+[\partial\tilde{\Omega}, \mu] \in C^{1,\alpha}(\mathrm{cl}\tilde{\Omega})$ and $w^-[\partial\tilde{\Omega}, \mu] \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \tilde{\Omega})$ (cf., e.g., Miranda [32], Lanza de Cristoforis and Rossi [28, Thm. 3.1]).

Analogously, we introduce the periodic simple layer potential. Let $\alpha \in]0,1[$ and let Ω_Q be a bounded open subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ such that $\mathrm{cl}\Omega_Q \subseteq Q$. If $\mu \in C^{0,\alpha}(\partial\Omega_Q)$, we set

$$v_q[\partial\Omega_Q, \mu](x) \equiv \int_{\partial\Omega_Q} S_{q,2}(x - y)\mu(y)\, d\sigma_y \quad \forall x \in \mathbb{R}^2\,.$$

The function $v_q^+[\partial\Omega_Q, \mu] \equiv v_q[\partial\Omega_Q, \mu]_{|\mathrm{cl}\mathbb{S}[\Omega_Q]}$ belongs to $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, and the function $v_q^-[\partial\Omega_Q, \mu] \equiv v_q[\partial\Omega_Q, \mu]_{|\mathrm{cl}\mathbb{S}[\Omega_Q]^-}$ belongs to $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$. Similarly, we introduce the periodic double layer potential. If $\mu \in C^{1,\alpha}(\partial\Omega_Q)$, we set

$$w_q[\partial\Omega_Q, \mu](x) \equiv -\int_{\partial\Omega_Q} DS_{q,2}(x - y) \cdot \nu_{\Omega_Q}(y)\mu(y)\, d\sigma_y \quad \forall x \in \mathbb{R}^2\,.$$

The restriction $w_q[\partial\Omega_Q, \mu]_{|\mathbb{S}[\Omega_Q]}$ can be extended uniquely to an element $w_q^+[\partial\Omega_Q, \mu]$ of $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q])$, and the restriction $w_q[\partial\Omega_Q, \mu]_{|\mathbb{S}[\Omega_Q]^-}$ can be extended uniquely to an element $w_q^-[\partial\Omega_Q, \mu]$ of $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_Q]^-)$, and we have $w_q^\pm[\partial\Omega_Q, \mu] = \pm\frac{1}{2}\mu + w_q[\partial\Omega_Q, \mu]$ on $\partial\Omega_Q$. Moreover,

$$w_q[\partial\Omega_Q, \mu](x) = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \int_{\partial\Omega_Q} S_{q,2}(x - y)(\nu_{\Omega_Q}(y))_j \mu(y)\, d\sigma_y\,, \tag{2.4}$$

for all $x \in \mathbb{R}^2 \setminus \partial\mathbb{S}[\Omega_Q]$. Here $(\nu_{\Omega_Q}(\cdot))_j$ denotes the j-th component of $\nu_{\Omega_Q}(\cdot)$.

3. Asymptotic behavior of $K_{II}[\epsilon]$

As a first thing, we face the issue of transforming the boundary value problem (1.2) for the Poisson equation into a problem for the Laplace equation. To do so, we need to introduce a periodic function B_ϵ such that $\Delta B_\epsilon = 1$. We introduce such a function in the following lemma, whose validity follows immediately by [37, Thm. 2.1].

Lemma 3.1. *Let $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$. Let B_ϵ be the function from $\mathbb{R}^2 \setminus (p + q\mathbb{Z}^2)$ to $\mathbb{R}$ defined by*

$$B_\epsilon(x) \equiv -S_{q,2}(x-p) + \frac{\log|\epsilon|}{2\pi} \quad \forall x \in \mathbb{R}^2 \setminus (p + q\mathbb{Z}^2)\,.$$

Then

$$B_{\epsilon|\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-} \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)\,, \quad \Delta B_\epsilon = 1 \quad \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-\,,$$
$$B_\epsilon(p+\epsilon t) = -S_2(t) - R_{q,2}(\epsilon t) \quad \forall t \in \partial\Omega\,.$$

By means of the function B_ϵ introduced in Lemma 3.1, we are in the position to convert the homogeneous Dirichlet problem (1.2) for the Poisson equation into a non-homogeneous Dirichlet problem for the Laplace equation. If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, we denote by $u_\#[\epsilon]$ the unique solution in $C^{1,\alpha}_q(\mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of the auxiliary boundary value problem

$$\begin{aligned} \Delta u = 0 \quad &\text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-\,,\\ u(x+qz) = u(x) \quad &\forall x \in \mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-\,, \forall z \in \mathbb{Z}^2\,,\\ u(x) = -B_\epsilon(x) \quad &\forall x \in \partial\Omega_{p,\epsilon}\,. \end{aligned} \tag{3.1}$$

Clearly,

$$u[\epsilon] = u_\#[\epsilon] + B_\epsilon \quad \text{in } \mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-\,,$$

and accordingly

$$K_{II}[\epsilon] = -\int_{Q\setminus\Omega_{p,\epsilon}} u_\#[\epsilon]\,dx - \int_{Q\setminus\Omega_{p,\epsilon}} B_\epsilon\,dx\,. \tag{3.2}$$

Now for each $\epsilon \in]-\epsilon_0, \epsilon_0[$, we define the function $\Gamma[\epsilon] \in C^{1,\alpha}(\partial\Omega)$, by setting

$$\Gamma[\epsilon](t) \equiv S_2(t) + R_{q,2}(\epsilon t) \quad \forall t \in \partial\Omega\,. \tag{3.3}$$

As a consequence, if $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, the Dirichlet condition in problem (3.1) can be rewritten as

$$u(x) = \Gamma[\epsilon]\big((x-p)/\epsilon\big) \quad \forall x \in \partial\Omega_{p,\epsilon}\,.$$

Now we would like to exploit the results of [38] to study the behavior of the integral of $u_\#[\epsilon]$ in the perforated cell $Q \setminus \Omega_{p,\epsilon}$. Indeed, in [38] real analytic continuation properties for the solution of a Dirichlet problem in $\mathbb{S}[\Omega_{p,\epsilon}]^-$ upon ϵ and the Dirichlet datum have been shown. Hence, in order to exploit those results, in the following lemma we verify the analytic dependence of $\Gamma[\epsilon]$ upon ϵ.

Lemma 3.2. *The map Γ from $]-\epsilon_0, \epsilon_0[$ to $C^{1,\alpha}(\partial\Omega)$, which takes ϵ to the function of the variable $t \in \partial\Omega$ defined by (3.3), is real analytic.*

Proof. We first note that if $\epsilon \in]-\epsilon_0, \epsilon_0[$, then $\epsilon t \in (\mathbb{R}^2 \setminus q\mathbb{Z}) \cup \{0\}$ for all $t \in \mathrm{cl}\,\Omega$. Clearly, the map from $]-\epsilon_0, \epsilon_0[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega, (\mathbb{R}^2 \setminus q\mathbb{Z}) \cup \{0\})$ which takes ϵ to the function ϵt of the variable $t \in \mathrm{cl}\,\Omega$ is real analytic. Then by the analyticity of $R_{q,2}$ in $(\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\}$ and by analyticity results for the composition operator (cf., e.g., Valent [43, Thm. 5.2, p. 44]), we deduce that the map from $]-\epsilon_0,\epsilon_0[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega)$ which takes ϵ to the function $R_{q,2}(\epsilon t)$ of the variable $t\in\mathrm{cl}\,\Omega$ is real analytic. Finally, by the continuity of the trace operator from $C^{1,\alpha}(\mathrm{cl}\,\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ we immediately deduce the validity of the lemma. □

In the following proposition, we provide an integral formulation of the auxiliary problem (3.1). In order to do so, we need to introduce the space $C^{1,\alpha}(\partial\Omega)_0\equiv\{\theta\in C^{1,\alpha}(\partial\Omega)\colon \int_{\partial\Omega}\theta\,d\sigma=0\}$.

Proposition 3.3. *Let Λ be the map from $]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\Lambda[\epsilon,\theta,\xi]\equiv{}&-\frac{1}{2}\theta(t)-\int_{\partial\Omega}DS_2(t-s)\cdot\nu_\Omega(s)\theta(s)\,d\sigma_s\\&-\epsilon\int_{\partial\Omega}DR_{q,2}(\epsilon(t-s))\cdot\nu_\Omega(s)\theta(s)\,d\sigma_s+\xi-\Gamma[\epsilon](t)\quad\forall t\in\partial\Omega\,,\end{aligned}$$

for all $(\epsilon,\theta,\xi)\in]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega)_0\times\mathbb{R}$. Then the following three statements hold:

(i) If $\epsilon\in]-\epsilon_0,\epsilon_0[$, then there exists a unique pair (θ,ξ) in $C^{1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ such that $\Lambda[\epsilon,\theta,\xi]=0$, and we denote such a pair by $(\theta_\epsilon,\xi_\epsilon)$. In particular, θ_0 is the unique function in $C^{1,\alpha}(\partial\Omega)_0$ such that

$$-\frac{1}{2}\theta_0(t)-\int_{\partial\Omega}DS_2(t-s)\cdot\nu_\Omega(s)\theta_0(s)\,d\sigma_s=S_2(t)-\lim_{s\to\infty}H_0(s)\quad\forall t\in\partial\Omega\,,\tag{3.4}$$

$$\xi_0=\lim_{t\to\infty}H_0(t)+R_{q,2}(0)\,,\tag{3.5}$$

where H_0 is the unique function in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2\setminus\Omega)$ such that

$$\begin{gathered}\Delta H_0(t)=0\quad\forall t\in\mathbb{R}^2\setminus\mathrm{cl}\,\Omega\,,\\H_0(t)=S_2(t)\quad\forall t\in\partial\Omega\,,\\\sup_{t\in\mathbb{R}^2\setminus\Omega}|H_0(t)|<+\infty\,.\end{gathered}$$

Moreover,

$$w^-[\partial\Omega,\theta_0]=H_0-\lim_{t\to\infty}H_0(t)\quad\text{in }\mathbb{R}^2\setminus\Omega\,.\tag{3.6}$$

(ii) There exist $\epsilon_1\in]0,\epsilon_0[$ and a real analytic map (Θ,Ξ) from $]-\epsilon_1,\epsilon_1[$ to $C^{1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ such that $(\Theta[\epsilon],\Xi[\epsilon])=(\theta_\epsilon,\xi_\epsilon)$ for all $\epsilon\in]-\epsilon_1,\epsilon_1[$. Moreover,

$$(\Theta[\epsilon],\Xi[\epsilon])=(\Theta[-\epsilon],\Xi[-\epsilon])\quad\forall\epsilon\in]-\epsilon_1,\epsilon_1[\,.\tag{3.7}$$

(iii) If $\epsilon\in]-\epsilon_1,\epsilon_1[\setminus\{0\}$, then

$$u_\#[\epsilon](x)=w_q^-[\partial\Omega_{p,\epsilon},\Theta[\epsilon]((\cdot-p)/\epsilon)](x)+\Xi[\epsilon]\quad\forall x\in\mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-\,.$$

As a consequence,

$$u_\#[\epsilon](x)=-\epsilon\int_{\partial\Omega}DS_{q,2}(x-p-\epsilon s)\cdot\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s+\Xi[\epsilon]\quad\forall x\in\mathbb{S}[\Omega_\epsilon]^-\,.$$

Proof. We first consider statement (i). If $\epsilon\in]-\epsilon_0,\epsilon_0[$, the existence and uniqueness of a pair (θ,ξ) in $C^{1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ such that $\Lambda[\epsilon,\theta,\xi]=0$ follows by [37, Lem. 3.4, Prop. A.5]. By [38, Lem. 3.2], we deduce that

$$\xi_0=\int_{\partial\Omega}\Gamma[0]\tau_0\,d\sigma\,,$$

where τ_0 is the unique solution in $C^{0,\alpha}(\partial\Omega)$ of the problem

$$\begin{aligned} -\frac{1}{2}\tau(t) + \int_{\partial\Omega} DS_2(t-s)\cdot\nu_\Omega(t)\tau(s)\,d\sigma_s = 0 \quad \forall t\in\partial\Omega\,, \\ \int_{\partial\Omega}\tau\,d\sigma = 1\,. \end{aligned} \tag{3.8}$$

Next we note that

$$\int_{\partial\Omega}\Gamma[0]\tau_0\,d\sigma = \int_{\partial\Omega} S_2(t)\tau_0(t)\,d\sigma_t + R_{q,2}(0)\,.$$

Then by classical potential theory (cf. [14, §7]), one shows that

$$\int_{\partial\Omega} S_2\tau_0\,d\sigma = \lim_{t\to\infty} H_0(t)\,,$$

and, as a consequence, the validity of (3.5) follows. Then by equality (3.5) one verifies that θ_0 is the unique solution of (3.4). By classical potential theory, we deduce that $w^-[\partial\Omega,\theta_0]$ and $H_0 - \lim_{t\to\infty} H_0(t)$ are harmonic in $\mathbb{R}^2\setminus \mathrm{cl}\,\Omega$, coincide on $\partial\Omega$, and vanish at infinity. Accordingly, the validity of (3.6) follows.

Statement (ii) follows by [37, Prop. 3.14] and Lemma 3.2. In particular, equality (3.7) follows by (2.3) and (3.3). Finally, statement (iii) is a consequence of the theorem of change of variables in integrals and of standard results for the periodic double layer potential (cf. [37, Thm. 2.3, Lem. 3.4, and Rem. 3.15]). □

By Proposition 3.3, we can provide an asymptotic expansion of $\Xi[\epsilon]$ for ϵ close to 0.

Lemma 3.4. *There exist $\epsilon_2\in]0,\epsilon_1[$ and a real analytic function $\tilde{\Xi}$ from $]-\epsilon_2,\epsilon_2[$ to $\mathbb{R}$ such that*

$$\begin{aligned} \Xi[\epsilon] = \lim_{t\to\infty} H_0(t) + R_{q,2}(0) + \epsilon^2\Big(\int_{\partial\Omega}\int_{\partial\Omega}(t-s)^T D^2R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s\tau_0(t)\,d\sigma_t \\ + \sum_{\beta\in\mathbb{N}^2,|\beta|=2}\frac{D^\beta R_{q,2}(0)}{\beta!}\int_{\partial\Omega} t^\beta\tau_0(t)\,d\sigma_t\Big) + \epsilon^4\tilde{\Xi}[\epsilon] \quad \forall\epsilon\in]-\epsilon_2,\epsilon_2[\,, \end{aligned}$$

where τ_0 is the unique solution in $C^{0,\alpha}(\partial\Omega)$ of (3.8). Here, if $\beta=(\beta_1,\beta_2)\in\mathbb{N}^2$ and $t=(t_1,t_2)\in\mathbb{R}^2$, then $t^\beta\equiv t_1^{\beta_1}t_2^{\beta_2}$.

Proof. By Proposition 3.3, we deduce the existence of $\epsilon_2\in]0,\epsilon_1[$, $\xi_*\in\mathbb{R}$, and of a real analytic function $\tilde{\Xi}$ from $]-\epsilon_2,\epsilon_2[$ to $\mathbb{R}$ such that $\Xi[\epsilon]=\xi_0+\epsilon^2\xi_*+\epsilon^4\tilde{\Xi}[\epsilon]$ for all $\epsilon\in]-\epsilon_2,\epsilon_2[$. As a consequence,

$$\begin{aligned} &-\frac{1}{2}\Theta[\epsilon](t) - \int_{\partial\Omega} DS_2(t-s)\cdot\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s \\ &-\epsilon\int_{\partial\Omega} DR_{q,2}(\epsilon(t-s))\cdot\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s + \lim_{t\to\infty} H_0(t) + R_{q,2}(0) + \epsilon^2\xi_* + \epsilon^4\tilde{\Xi}[\epsilon] \\ &= S_2(t) + R_{q,2}(\epsilon t) \quad \forall t\in\partial\Omega\,,\forall\epsilon\in]-\epsilon_2,\epsilon_2[\,. \end{aligned}$$

By the analyticity of $R_{q,2}$ in $(\mathbb{R}^2\setminus q\mathbb{Z}^2)\cup\{0\}$ and by equality (2.3), we deduce the existence of $\tilde{\epsilon}\in]0,\epsilon_2[$ and of a bounded function $\tilde{R}_1$ from $]-\tilde{\epsilon},\tilde{\epsilon}[\times\partial\Omega$ to $\mathbb{R}$ such that

$$R_{q,2}(\epsilon t) = R_{q,2}(0) + \epsilon^2\sum_{\beta\in\mathbb{N}^2,|\beta|=2}\frac{D^\beta R_{q,2}(0)}{\beta!}t^\beta + \epsilon^4\tilde{R}_1(\epsilon,t) \quad \forall t\in\partial\Omega\,,\forall\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[\,.$$

Similarly, by possibly taking a smaller $\tilde{\epsilon}$, we can assume that there exists a bounded function $\tilde{R}_2$ from $]-\tilde{\epsilon},\tilde{\epsilon}[\times\partial\Omega\times\partial\Omega$ to $\mathbb{R}$ such that

$$DR_{q,2}(\epsilon(t-s))\cdot\nu_\Omega(s)=\epsilon(t-s)^TD^2R_{q,2}(\epsilon(t-s))\nu_\Omega(s)+\epsilon^2\tilde{R}_2(\epsilon,t,s)$$

for all $t,s\in\partial\Omega$ and all $\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[$. As a consequence

$$\begin{aligned}
&-\frac{1}{2}\Theta[\epsilon](t)-\int_{\partial\Omega}DS_2(t-s)\cdot\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s\\
&-\epsilon^2\int_{\partial\Omega}(t-s)^TD^2R_{q,2}(\epsilon(t-s))\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s\\
&-\epsilon^3\int_{\partial\Omega}\tilde{R}_2(\epsilon,t,s)\Theta[\epsilon](s)\,d\sigma_s+\lim_{t\to\infty}H_0(t)+R_{q,2}(0)+\epsilon^2\xi_*+\epsilon^4\tilde{\Xi}[\epsilon]\\
&=S_2(t)+R_{q,2}(0)+\epsilon^2\sum_{\beta\in\mathbb{N}^2,|\beta|=2}\frac{D^\beta R_{q,2}(0)}{\beta!}t^\beta+\epsilon^4\tilde{R}_1(\epsilon,t)
\end{aligned}\tag{3.9}$$

for all $t\in\partial\Omega$ and all $\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[$. Now let τ_0 be as in the proof of Proposition 3.3. By the classical Fredholm Theory, we have

$$\int_{\partial\Omega}\Big(-\frac{1}{2}\Theta[\epsilon](t)-\int_{\partial\Omega}DS_2(t-s)\cdot\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s\Big)\tau_0(t)\,d\sigma_t=0\,,$$

for all $\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[$. Moreover,

$$\lim_{t\to\infty}H_0(t)+R_{q,2}(0)=\int_{\partial\Omega}S_2(t)\tau_0(t)\,d\sigma_t+R_{q,2}(0)\,.$$

Hence, by multiplying (3.9) by $\epsilon^{-2}\tau_0(t)$ and integrating on $\partial\Omega$ with respect to the variable t, we obtain

$$\begin{aligned}
&-\int_{\partial\Omega}\int_{\partial\Omega}(t-s)^TD^2R_{q,2}(\epsilon(t-s))\nu_\Omega(s)\Theta[\epsilon](s)\,d\sigma_s\tau_0(t)\,d\sigma_t\\
&-\epsilon\int_{\partial\Omega}\int_{\partial\Omega}\tilde{R}_2(\epsilon,t,s)\Theta[\epsilon](s)\,d\sigma_s\tau_0(t)\,d\sigma_t+\xi_*+\epsilon^2\tilde{\Xi}[\epsilon]\\
&=\sum_{\beta\in\mathbb{N}^2,|\beta|=2}\frac{D^\beta R_{q,2}(0)}{\beta!}\int_{\partial\Omega}t^\beta\tau_0(t)\,d\sigma_t\\
&\quad+\epsilon^2\int_{\partial\Omega}\tilde{R}_1(\epsilon,t)\tau_0(t)\,d\sigma_t\quad\forall t\in\partial\Omega\,,\forall\epsilon\in]-\tilde{\epsilon},\tilde{\epsilon}[\,.
\end{aligned}$$

Thus, by letting ϵ tend to 0, we deduce

$$\begin{aligned}
\xi_*=&\int_{\partial\Omega}\int_{\partial\Omega}(t-s)^TD^2R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s\tau_0(t)\,d\sigma_t\\
&+\sum_{\beta\in\mathbb{N}^2,|\beta|=2}\frac{D^\beta R_{q,2}(0)}{\beta!}\int_{\partial\Omega}t^\beta\tau_0(t)\,d\sigma_t\,.
\end{aligned}$$

Hence, the validity of the statement follows. □

Remark 3.5. Let τ_0 be the unique solution in $C^{0,\alpha}(\partial\Omega)$ of (3.8). By classical potential theory (cf. [14, §7]), one verifies that

$$\int_{\partial\Omega}t^\beta\tau_0(t)\,d\sigma_t=\lim_{t\to\infty}H_{2,\beta}(t)\quad\forall\beta\in\mathbb{N}^2\,,|\beta|=2\,,$$

where if $\beta \in \mathbb{N}^2$, $|\beta| = 2$ the function $H_{2,\beta}$ is the unique element of $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega)$ such that

$$\begin{gathered}
\Delta H_{2,\beta}(t) = 0 \quad \forall t \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega\,,\\
H_{2,\beta}(t) = t^\beta \quad \forall t \in \partial\Omega\,,\\
\sup_{t\in\mathbb{R}^2\setminus\Omega} |H_{2,\beta}(t)| < +\infty\,.
\end{gathered}$$

In the following proposition, we study the behavior of $\int_{Q\setminus\Omega_{p,\epsilon}} u_\#[\epsilon]\,dx$.

Proposition 3.6. *There exist $\epsilon_3 \in]0, \epsilon_2[$ and a real analytic function $J_{\#,1}$ from $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$, such that*

$$\begin{aligned}
&\int_{Q\setminus\Omega_{p,\epsilon}} u_\#[\epsilon](x)\,dx\\
&= \lim_{t\to\infty} H_0(t) + R_{q,2}(0) + \epsilon^2\Big(- |\Omega|_2\big(\lim_{t\to\infty} H_0(t) + R_{q,2}(0)\big)\\
&\quad + \int_{\partial\Omega}\int_{\partial\Omega} (t-s)^T D^2 R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s \tau_0(t)\,d\sigma_t\\
&\quad + \sum_{\beta\in\mathbb{N}^2, |\beta|=2} \frac{D^\beta R_{q,2}(0)}{\beta!} \int_{\partial\Omega} t^\beta \tau_0(t)\,d\sigma_t\\
&\quad + \sum_{j=1}^2 \int_{\partial\Omega}\Big(\int_{\partial\Omega} S_2(t-s)\theta_0(s)(\nu_\Omega(s))_j\,d\sigma_s\Big)(\nu_\Omega(t))_j\,d\sigma_t\Big) + \epsilon^4 J_{\#,1}[\epsilon]
\end{aligned} \tag{3.10}$$

for all $\epsilon \in]-\epsilon_3, \epsilon_3[\setminus\{0\}$, where τ_0 is the unique solution in $C^{0,\alpha}(\partial\Omega)$ of (3.8). Here $(\nu_\Omega(\cdot))_j$ denotes the j-th component of $\nu_\Omega(\cdot)$.

Proof. We proceed as in the proof of [38, Thm. 4.4]. Let $\epsilon \in]-\epsilon_2, \epsilon_2[\setminus\{0\}$. Clearly,

$$\begin{aligned}
&\int_{Q\setminus\mathrm{cl}\,\Omega_{p,\epsilon}} u_\#[\epsilon](x)\,dx\\
&= \int_{Q\setminus\mathrm{cl}\,\Omega_{p,\epsilon}} w_q^-\big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot-p)/\epsilon)\big](x)\,dx\\
&+ \big(\lim_{t\to\infty} H_0(t) + R_{q,2}(0)\big)\big(1-\epsilon^2|\Omega|_2\big)\\
&+ \epsilon^2\Big(\int_{\partial\Omega}\int_{\partial\Omega} (t-s)^T D^2 R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s \tau_0(t)\,d\sigma_t\\
&+ \sum_{\beta\in\mathbb{N}^2, |\beta|=2} \frac{D^\beta R_{q,2}(0)}{\beta!} \int_{\partial\Omega} t^\beta \tau_0(t)\,d\sigma_t\Big)\big(1-\epsilon^2|\Omega|_2\big) + \epsilon^4\tilde{\Xi}[\epsilon]\big(1-\epsilon^2|\Omega|_2\big)\,.
\end{aligned}$$

By (2.4), we have

$$\begin{aligned}
&w_q^-\big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot-p)/\epsilon)\big](x)\\
&= -\sum_{j=1}^2 \frac{\partial}{\partial x_j} v_q^-\big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot-p)/\epsilon)(\nu_{\Omega_{p,\epsilon}}(\cdot))_j\big](x) \quad \forall x \in \mathrm{cl}\,Q\setminus\mathrm{cl}\,\Omega_{p,\epsilon}\,.
\end{aligned}$$

Now let $j \in \{1,2\}$. By the Divergence Theorem and the periodicity of the periodic simple layer potential, we have

$$\int_{Q\setminus\mathrm{cl}\,\Omega_{p,\epsilon}} \frac{\partial}{\partial x_j} v_q^-\big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot-p)/\epsilon)(\nu_{\Omega_{p,\epsilon}}(\cdot))_j\big](x)\,dx$$

$$
\begin{aligned}
&= \int_{\partial Q} v_q^- \big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot - p)/\epsilon)(\nu_{\Omega_{p,\epsilon}}(\cdot))_j\big](x)(\nu_Q(x))_j \, d\sigma_x \\
&\quad - \int_{\partial\Omega_{p,\epsilon}} v_q^- \big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot - p)/\epsilon)(\nu_{\Omega_{p,\epsilon}}(\cdot))_j\big](x)(\nu_{\Omega_{p,\epsilon}}(x))_j \, d\sigma_x \\
&= -\epsilon \int_{\partial\Omega} v_q^- \big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot - p)/\epsilon)(\nu_{\Omega_{p,\epsilon}}(\cdot))_j\big](p+\epsilon t)(\nu_\Omega(t))_j \, d\sigma_t \,.
\end{aligned}
$$

Then we note that

$$
\begin{aligned}
&v_q^- \big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot - p)/\epsilon)(\nu_{\Omega_{p,\epsilon}}(\cdot))_j\big](p+\epsilon t) \\
&= \epsilon \int_{\partial\Omega} S_2(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \\
&\quad + \epsilon \int_{\partial\Omega} R_{q,2}(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s \quad \forall t \in \partial\Omega \,.
\end{aligned}
$$

We now observe that if $\epsilon \neq 0$ and $x \in \mathbb{R}^2 \setminus \{0\}$ then we have

$$
S_2(\epsilon x) = S_2(x) + \frac{1}{2\pi} \log|\epsilon| \,. \tag{3.11}
$$

Moreover, by the Divergence Theorem, we have

$$
\begin{aligned}
&\int_{\partial\Omega} \Big(\int_{\partial\Omega} \Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s\Big)(\nu_\Omega(t))_j \, d\sigma_t \\
&= \Big(\int_{\partial\Omega} \Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s\Big)\Big(\int_{\partial\Omega} (\nu_\Omega(t))_j \, d\sigma_t\Big) = 0 \,.
\end{aligned} \tag{3.12}
$$

Hence, by equalities (3.11) and (3.12), if $\epsilon \in]-\epsilon_2, \epsilon_2[\setminus\{0\}$, we have

$$
\begin{aligned}
&\int_{Q \setminus \mathrm{cl}\,\Omega_{p,\epsilon}} w_q^- \big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot - p)/\epsilon)\big](x) \, dx \\
&= \sum_{j=1}^{2} \epsilon^2 \Big[\int_{\partial\Omega} \Big(\int_{\partial\Omega} S_2(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s\Big)(\nu_\Omega(t))_j \, d\sigma_t \\
&\quad + \int_{\partial\Omega} \Big(\int_{\partial\Omega} R_{q,2}(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s\Big)(\nu_\Omega(t))_j \, d\sigma_t\Big] \,.
\end{aligned}
$$

Thus we set

$$
\begin{aligned}
\tilde{J}_{\#,1}[\epsilon] \equiv \sum_{j=1}^{2} &\Big[\int_{\partial\Omega} \Big(\int_{\partial\Omega} S_2(t-s)\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s\Big)(\nu_\Omega(t))_j \, d\sigma_t \\
&+ \int_{\partial\Omega} \Big(\int_{\partial\Omega} R_{q,2}(\epsilon(t-s))\Theta[\epsilon](s)(\nu_\Omega(s))_j \, d\sigma_s\Big)(\nu_\Omega(t))_j \, d\sigma_t\Big] \,,
\end{aligned}
$$

for all $\epsilon \in]-\epsilon_2, \epsilon_2[$. Then the analyticity of Θ, the continuity of the linear map from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ which takes f to the function $\int_{\partial\Omega} S_2(t-s)f(s)\,d\sigma_s$ of the variable $t \in \partial\Omega$ (cf., e.g., Miranda [32], Lanza de Cristoforis and Rossi [28, Thm. 3.1]), the continuity of the pointwise product in Schauder spaces, standard properties of integral operators with real analytic kernels and with no singularity (cf., e.g., [26, §4]), and standard calculus in Banach spaces imply that $\tilde{J}_{\#,1}$ is a real analytic function from $]-\epsilon_2, \epsilon_2[$ to $\mathbb{R}$. Moreover, since $\int_{\partial\Omega} (\nu_\Omega(t))_j \, d\sigma_t = 0$, we have

$$
\tilde{J}_{\#,1}[0] = \sum_{j=1}^{2} \int_{\partial\Omega} \Big(\int_{\partial\Omega} S_2(t-s)\theta_0(s)(\nu_\Omega(s))_j \, d\sigma_s\Big)(\nu_\Omega(t))_j \, d\sigma_t
$$

Furthermore, by (2.3) and by (3.7), $\tilde{J}_{\#,1}[\epsilon] = \tilde{J}_{\#,1}[-\epsilon]$ for all $\epsilon \in]-\epsilon_2, \epsilon_2[$. As a consequence, there exist $\epsilon_3 \in]0, \epsilon_2[$ and a real analytic function $\tilde{J}_{\#}$ from $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$ such that

$$\begin{aligned}
&\int_{Q\setminus \mathrm{cl}\,\Omega_{p,\epsilon}} w_q^- \big[\partial\Omega_{p,\epsilon}, \Theta[\epsilon]((\cdot - p)/\epsilon)\big](x)\,dx \\
&= \epsilon^2 \sum_{j=1}^{2} \int_{\partial\Omega} \Big(\int_{\partial\Omega} S_2(t-s)\theta_0(s)(\nu_\Omega(s))_j\,d\sigma_s \Big)(\nu_\Omega(t))_j\,d\sigma_t + \epsilon^4 \tilde{J}_{\#}[\epsilon]
\end{aligned}$$

for all $\epsilon \in]-\epsilon_3, \epsilon_3[\setminus\{0\}$. Hence, if we set

$$\begin{aligned}
J_{\#,1}[\epsilon] = \tilde{J}_{\#}[\epsilon] &+ \tilde{\Xi}[\epsilon]\big(1 - \epsilon^2|\Omega|_2\big) \\
&- |\Omega|_2 \Big(\int_{\partial\Omega}\int_{\partial\Omega} (t-s)^T D^2 R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s \tau_0(t)\,d\sigma_t \\
&+ \sum_{\beta\in\mathbb{N}^2,\,|\beta|=2} \frac{D^\beta R_{q,2}(0)}{\beta!} \int_{\partial\Omega} t^\beta \tau_0(t)\,d\sigma_t \Big),
\end{aligned}$$

for all $\epsilon \in]-\epsilon_3, \epsilon_3[$, we immediately deduce that $J_{\#,1}$ is a real analytic function from $]-\epsilon_3, \epsilon_3[$ to $\mathbb{R}$ such that equality (3.10) holds, and thus the proof is complete. □

Then we turn to analyze the behavior of $\int_{Q\setminus\Omega_{p,\epsilon}} B_\epsilon(x)\,dx$ and we prove the following.

Proposition 3.7. *There exist $\epsilon_4 \in]0, \epsilon_0[$ and a real analytic function $J_{\#,2}$ from $]-\epsilon_4, \epsilon_4[$ to $\mathbb{R}$, such that*

$$\begin{aligned}
\int_{Q\setminus\Omega_{p,\epsilon}} B_\epsilon(x)\,dx = &-\int_Q S_{q,2}(x)\,dx + \frac{\log|\epsilon|}{2\pi} \\
&+ \epsilon^2\Big(R_{q,2}(0)|\Omega|_2 + \int_\Omega S_2(t)\,dt \Big) + \epsilon^4 J_{\#,2}[\epsilon],
\end{aligned} \tag{3.13}$$

for all $\epsilon \in]-\epsilon_4, \epsilon_4[\setminus\{0\}$.

Proof. We first note that

$$\begin{aligned}
\int_{Q\setminus\Omega_{p,\epsilon}} B_\epsilon(x)\,dx &= \int_Q B_\epsilon(x)\,dx - \int_{\Omega_{p,\epsilon}} B_\epsilon(x)\,dx \\
&= (1-\epsilon^2|\Omega|_2)\frac{\log|\epsilon|}{2\pi} - \int_Q S_{q,2}(x-p)\,dx + \epsilon^2 \int_\Omega S_{q,2}(\epsilon t)\,dt,
\end{aligned}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$. Then we note that by the periodicity of $S_{q,2}$ we have $\int_Q S_{q,2}(x-p)\,dx = \int_Q S_{q,2}(x)\,dx$ (cf., e.g., Cioranescu and Donato [11, Lem. 2.3, p. 27]). Moreover,

$$\int_\Omega S_{q,2}(\epsilon t)\,dt = \int_\Omega S_2(t)\,dt + \frac{|\Omega|_2 \log|\epsilon|}{2\pi} + \int_\Omega R_{q,2}(\epsilon t)\,dt,$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$. By the analyticity of $R_{q,2}$ in $(\mathbb{R}^2 \setminus q\mathbb{Z}^2) \cup \{0\}$ and by analyticity results for the composition operator (cf., e.g., Valent [43, Thm. 5.2, p. 44]), the map from $]-\epsilon_0, \epsilon_0[$ to $C^0(\mathrm{cl}\,\Omega)$ which takes ϵ to the function $R_{q,2}(\epsilon t)$

of the variable $t \in \mathrm{cl}\,\Omega$ is real analytic. Hence, if we denote by $\tilde{J}$ the function from $]-\epsilon_0, \epsilon_0[$ to $\mathbb{R}$ which takes ϵ to

$$\tilde{J}[\epsilon] \equiv \int_\Omega R_{q,2}(\epsilon t)\,dt\,,$$

by standard calculus in Banach spaces, we deduce that $\tilde{J}$ is real analytic. Moreover, by (2.3), we have $\tilde{J}[\epsilon] = \tilde{J}[-\epsilon]$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$ and

$$\tilde{J}[0] = R_{q,2}(0)|\Omega|_2\,.$$

As a consequence, there exist $\epsilon_4 \in]0, \epsilon_0[$ and a real analytic function $J_{\#,2}$ from $]-\epsilon_4, \epsilon_4[$ to $\mathbb{R}$ such that

$$\tilde{J}[\epsilon] = R_{q,2}(0)|\Omega|_2 + \epsilon^2 J_{\#,2}[\epsilon] \quad \forall \epsilon \in]-\epsilon_4, \epsilon_4[\,.$$

Hence, equality (3.13) holds and the proof is complete. □

By equality (3.2) and Propositions 3.6 and 3.7, we can immediately deduce the validity of our main result which concerns the behavior of $K_{II}[\epsilon]$.

Theorem 3.8. *Let ϵ_3, H_0 be as in Proposition 3.6. Let ϵ_4 be as in Theorem 3.7. Let $\epsilon_\# \equiv \min\{\epsilon_3, \epsilon_4\}$. Let τ_0 be the unique solution in $C^{0,\alpha}(\partial\Omega)$ of (3.8). Then there exists a real analytic function J from $]-\epsilon_\#, \epsilon_\#[$ to $\mathbb{R}$ such that*

$$\begin{aligned} K_{II}[\epsilon] &= -\frac{\log|\epsilon|}{2\pi} - \lim_{t\to\infty} H_0(t) - R_{q,2}(0) + \int_Q S_{q,2}(x)\,dx \\ &\quad + \epsilon^2\Big(|\Omega|_2 \lim_{t\to\infty} H_0(t) - \int_\Omega S_2(t)\,dt \\ &\quad - \int_{\partial\Omega}\int_{\partial\Omega} (t-s)^T D^2 R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s \tau_0(t)\,d\sigma_t \\ &\quad - \sum_{\beta\in\mathbb{N}^2\,,|\beta|=2} \frac{D^\beta R_{q,2}(0)}{\beta!} \int_{\partial\Omega} t^\beta \tau_0(t)\,d\sigma_t \\ &\quad - \sum_{j=1}^{2} \int_{\partial\Omega}\Big(\int_{\partial\Omega} S_2(t-s)\theta_0(s)(\nu_\Omega(s))_j\,d\sigma_s\Big)(\nu_\Omega(t))_j\,d\sigma_t\Big) + \epsilon^4 J[\epsilon] \end{aligned} \tag{3.14}$$

for all $\epsilon \in]-\epsilon_\#, \epsilon_\#[\setminus\{0\}$.

Now we want to investigate parity properties of the function J of Theorem 3.8. In order to do so, we need the following lemma on K_{II}.

Lemma 3.9. *We have*

$$K_{II}[\epsilon] = K_{II}[-\epsilon] \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}\,.$$

Proof. If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$ a simple computation shows that $u[\epsilon](x) = u[-\epsilon](-x+2p)$ for all $x \in \mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-$. Then, by the Theorem of change of variable in integrals, we have

$$\begin{aligned} -K_{II}[\epsilon] = \int_{Q\setminus\Omega_{p,\epsilon}} u[\epsilon](x)\,dx &= \int_{Q\setminus\Omega_{p,\epsilon}} u[-\epsilon](-x+2p)\,dx \\ &= \int_{-(Q\setminus\Omega_{p,\epsilon})+2p} u[-\epsilon](x)\,dx \\ &= \int_{(-Q+2p)\setminus\Omega_{p,-\epsilon}} u[-\epsilon](x)\,dx\,. \end{aligned}$$

Moreover, by the periodicity of $u[-\epsilon]$ we have

$$\int_{(-Q+2p)\setminus\Omega_{p,-\epsilon}} u[-\epsilon](x)\,dx = \int_{Q\setminus\Omega_{p,-\epsilon}} u[-\epsilon](x)\,dx = -K_{II}[-\epsilon]$$

(cf., e.g., Cioranescu and Donato [11, Lem. 2.3, p. 27]). As a consequence, $K_{II}[\epsilon] = K_{II}[-\epsilon]$. □

By Theorem 3.8 and Lemma 3.9, we deduce the validity of the following statement.

Proposition 3.10. *Let ϵ_5, J be as in Theorem 3.8. Then*

$$J[\epsilon] = J[-\epsilon] \quad \forall \epsilon \in]-\epsilon_5, \epsilon_5[\,.$$

4. Lower order coefficients in $K_{II}[\epsilon]$

In this section, we compute via conformal mappings the quantity $\lim_{t\to\infty} H_0(t)$, which is the only Ω-dependent term in the zero order expansion of the quantity $K_{II}[\epsilon] + (\log|\epsilon|)/(2\pi)$ for ϵ close to 0. To do so, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Therefore, we suppose that $\Omega^- \equiv \mathbb{C} \setminus \operatorname{cl}\Omega$ is the exterior domain which is outside the Jordan curve $\Gamma \equiv \partial\Omega$. Then, there exists a unique conformal map f_{Ω^-} of Ω^- onto the set $\mathbb{D}^- \equiv \{w \in \mathbb{C} \colon |w| > 1\}$ normalized by the conditions

$$f_{\Omega^-}(\infty) = \infty\,, \quad \lim_{z\to\infty} \frac{f_{\Omega^-}(z)}{z} \in]0, +\infty[\,.$$

The positive real number

$$\operatorname{cap}(\Omega) \equiv \Big(\lim_{z\to\infty} \frac{f_{\Omega^-}(z)}{z}\Big)^{-1}$$

is called the *logarithmic capacity* of Ω and is equal to the *transfinite diameter* of Ω (cf., e.g., Goluzin [19, Ch. VII,§3] and Kirsch [23, §2]). Then one verifies that the function which takes a point $z \in \operatorname{cl}\Omega^-$ to

$$\frac{1}{2\pi} \log \frac{|z|}{|f_{\Omega^-}(z)|}$$

is harmonic in Ω^-, bounded at infinity, and that

$$\frac{1}{2\pi} \log \frac{|z|}{|f_{\Omega^-}(z)|} = \frac{1}{2\pi} \log|z| \quad \forall z \in \partial\Omega\,.$$

Thus

$$H_0(z) = \frac{1}{2\pi} \log \frac{|z|}{|f_{\Omega^-}(z)|} \quad \forall z \in \operatorname{cl}\Omega^-\,.$$

In particular,

$$\lim_{z\to\infty} H_0(z) = \frac{\log \operatorname{cap}(\Omega)}{2\pi}\,. \tag{4.1}$$

Application of (4.1) to formula (3.14) allows us to deduce that

$$K_{II}[\epsilon] = -\frac{\log|\epsilon|}{2\pi} - \frac{\log \operatorname{cap}(\Omega)}{2\pi} - R_{q,2}(0) + \int_Q S_{q,2}(x)\,dx + O(\epsilon^2)\,.$$

which implies

$$K_{II}[\epsilon] = -\frac{\log|\epsilon| \operatorname{cap}(\Omega)}{2\pi} - R_{q,2}(0) + \int_Q S_{q,2}(x)\,dx + O(\epsilon^2)\,. \tag{4.2}$$

Thus, if we set $\rho \equiv (\mathrm{cap}(\epsilon\Omega))^2$, then equation (4.2) can be rewritten as

$$K_{II}[\epsilon] = -\frac{\log\rho}{4\pi} - R_{q,2}(0) + \int_Q S_{q,2}(x)\,dx + O(\rho) \quad \rho \equiv (\mathrm{cap}(\epsilon\Omega))^2\,. \tag{4.3}$$

5. Conclusions

In the present paper, we prove that the longitudinal permeability of regular arrays of cylinders can be expanded as in (1.5). Moreover, it can be rewritten as a power series of the square of the logarithmic capacity of the inclusion $\rho \equiv (\mathrm{cap}(\epsilon\Omega))^2$ plus a logarithmic term. Such quantity is a measure of the size of the section of the cylinder inside the unit cell. It is justified that the zero-th order term does not depend on the shape of cylinders.

The results can be extended to the case when the domain Ω is not connected. Arbitrary sets of n disks were considered in [35]. In this case, the number n enters in all the terms of the permeability including the logarithm. The dependence of the permeability on locations of circular holes can be analyzed following [35, 36].

We also note that by Theorem 3.8 and Proposition 3.10, we can deduce the existence of $\epsilon'_\# \in]0, \epsilon_\#[$ and of a sequence $\{b_j\}_{j=0}^{+\infty}$ of real numbers, such that

$$\begin{aligned} K_{II}[\epsilon] = &-\frac{\log|\epsilon|}{2\pi} - \lim_{t\to\infty} H_0(t) - R_{q,2}(0) + \int_Q S_{q,2}(x)\,dx \\ &+ \epsilon^2\Big(|\Omega|_2 \lim_{t\to\infty} H_0(t) - \int_\Omega S_2(t)\,dt \\ &- \int_{\partial\Omega}\int_{\partial\Omega} (t-s)^T D^2 R_{q,2}(0)\nu_\Omega(s)\theta_0(s)\,d\sigma_s \tau_0(t)\,d\sigma_t \\ &- \sum_{\beta\in\mathbb{N}^2,|\beta|=2} \frac{D^\beta R_{q,2}(0)}{\beta!} \int_{\partial\Omega} t^\beta \tau_0(t)\,d\sigma_t \\ &- \sum_{j=1}^2 \int_{\partial\Omega}\Big(\int_{\partial\Omega} S_2(t-s)\theta_0(s)(\nu_\Omega(s))_j\,d\sigma_s\Big)(\nu_\Omega(t))_j\,d\sigma_t\Big) \\ &+ \epsilon^4 \sum_{j=0}^\infty b_j \epsilon^{2j} \quad \forall \epsilon \in]-\epsilon'_\#, \epsilon'_\#[\setminus\{0\}\,, \end{aligned} \tag{5.1}$$

where the series in the right hand side converges absolutely on $]-\epsilon'_\#, \epsilon'_\#[$. Formula (5.1) provides a constructive asymptotic expansion valid for Ω as in (1.1). To prove rigorously a result of as in (5.1), it is necessary to obtain an analyticity result for $K_{II}[\epsilon] + \frac{\log|\epsilon|}{2\pi}$, which, to the best of our knowledge, is new. We emphasize that such an analysis holds for any shape of the cylinders with the only assumption (1.1). Therefore we are not confined to specific shapes as, e.g., circles or ellipses.

Once an analyticity result of this type is shown, it is of interest to compute the coefficients $\{b_j\}_{j=0}^{+\infty}$. In [14], a completely constructive method has been shown to compute the coefficients for the solution of a Dirichlet problem for the Laplace equation in a planar domain with a small hole. The computation is based on the solutions of systems of integral equations. An approach of this type can be exploited also in the case of the present paper and can allow to provide explicit expression for *all* the coefficients b_j. Once again, this would hold for all the domains which satisfy assumption (1.1). As in [14], in the case of circular cylinders, one expects to have simplified expressions that recover certainly known formulas. Thus the present paper provides the theoretical background for this aim. However, for the sake of brevity, we decided not to perform this analysis here and this may be the object of future investigations by the authors. Moreover, we plan to investigate the dependence of the longitudinal permeability upon perturbations of the shape of the cross-section of the cylinders. The integral approach of this paper may be used to derive the shape differential and the topological derivative of the longitudinal permeability. These quantities can then be used to find optimal shapes.

Furthermore, one can see that the constant term in expansion (4.3) does not depend on the shape of Ω. Therefore, if $Q \equiv]0,1[^2$, we can take the coefficient for the circular inclusions calculated by formula [33, (41)] and (4.3) becomes

$$K_{II}[\epsilon] = -\frac{\log \rho}{4\pi} - \frac{1}{4\pi}(\log \pi + 1.47644) + O(\rho), \tag{5.2}$$

where $\rho \equiv (\mathrm{cap}(\epsilon\Omega))^2$ is a measure of the concentration of the inclusions.

Formula (5.2) presents the asymptotic dependence of the macroscopic permeability K_{II} on the shape of $\Omega_{p,\epsilon} \equiv p + \epsilon\Omega$ via the square of the logarithmic capacity, i.e., $\rho \equiv (\mathrm{cap}(\epsilon\Omega))^2 = \epsilon^2(\mathrm{cap}(\Omega))^2$. Previous constructive expansions for K_{II} and for general macroscopic constants of porous media and composites (conductivity and elastic constants) were made by concentration or by a contrast parameter (cf. Milton [31], and [34, 35, 36]). Shapes parameters were introduced (see, for instance Landau, Lifshitz, and Pitaevskii [24]) to describe the dependence of the macroscopic constants on the inclusion. Here we propose to use the logarithmic capacity of the cross section of the cylinder to describe the effective permeability. In this way, one obtains an approximate expression for the longitudinal permeability as a sum of a logarithmic function of the capacity and a term which does not depend on the shape of the unit inclusion (plus a small remainder).

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