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Lanza de Cristoforis, Massimo; Musolino, Paolo

Published in:
Revista Matemática Complutense

DOI:
10.1007/s13163-017-0242-5

Publication date:
2018

Citation for published version (APA):
Lanza de Cristoforis, M., & Musolino, P. (2018). Two-parameter homogenization for a nonlinear periodic Robin problem for the Poisson equation: a functional analytic approach. *Revista Matemática Complutense*, *31*(1), 63-110. https://doi.org/10.1007/s13163-017-0242-5

Two-parameter homogenization for a nonlinear periodic Robin problem for the Poisson equation. A functional analytic approach*

Massimo Lanza de Cristoforis †, Paolo Musolino‡

Abstract: We consider a nonlinear Robin problem for the Poisson equation in an unbounded periodically perforated domain. The domain has a periodic structure, and the size of each cell is determined by a positive parameter δ. The relative size of each periodic perforation is instead determined by a positive parameter ϵ. We prove the existence of a family of solutions which depends on ϵ and δ and we analyze the behavior of such a family as (ϵ, δ) tends to $(0,0)$ by an approach which is alternative to that of asymptotic expansions and of classical homogenization theory.

Keywords: Nonlinear Robin problem; singularly perturbed domain; Poisson equation; periodically perforated domain; homogenization; real analytic continuation in Banach space

2010 Mathematics Subject Classification: 35J25; 31B10; 45A05; 47H30

1 Introduction

In this paper, we consider a nonlinear Robin problem for the Poisson equation in a periodically perforated domain with small holes. We fix once for all

$$n \in \mathbb{N} \setminus \{0,1\}\,, \qquad \text{and} \qquad (q_{11}, \ldots, q_{nn}) \in]0, +\infty[^n\,,$$

and we introduce a periodicity cell

$$Q \equiv \Pi_{j=1}^{n}]0, q_{jj}[\,.$$

Then we denote by q the diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 & \ldots & 0 \\ 0 & q_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & q_{nn} \end{pmatrix}$$

and by $m_n(Q)$ the n dimensional measure of the fundamental cell Q. Clearly, $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q.

Then we consider $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in]0,1[$ and a subset Ω of $\mathbb{R}^n$ satisfying the following assumption.

$$\begin{aligned} &\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha}. \\ &\text{Let } \mathbb{R}^n \setminus \mathrm{cl}\Omega \text{ be connected. Let } 0 \in \Omega\,. \end{aligned} \tag{1.1}$$

Next we fix $p \in Q$. Then there exists $\epsilon_0 \in]0, +\infty[$ such that

$$p + \epsilon \mathrm{cl}\Omega \subseteq Q \qquad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\,, \tag{1.2}$$

where cl denotes the closure. To shorten our notation, we set

$$\Omega_{p,\epsilon} \equiv p + \epsilon\Omega \qquad \forall \epsilon \in \mathbb{R}\,.$$

*The authors acknowledge the support of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). M. Lanza de Cristoforis acknowledges the support of the project BIRD168373/16 "Singular perturbation problems for the heat equation in a perforated domain" of the University of Padua and of the grant EP/M013545/1: "Mathematical Analysis of Boundary-Domain Integral Equations for Nonlinear PDEs" from the EPSRC, UK. P. Musolino acknowledges the support of an 'assegno di ricerca INdAM'. P. Musolino is a Sêr CYMRU II COFUND fellow, also supported by the 'Sêr Cymru National Research Network for Low Carbon, Energy and Environment'.

†Dipartimento di Matematica 'Tullio Levi-Civita', Università degli Studi di Padova, Via Trieste 63, 35121 Padova
‡Department of Mathematics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, UK.

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\epsilon}] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_{p,\epsilon}) , \qquad \mathbb{S}[\Omega_{p,\epsilon}]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}] ,$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. Then a function u defined either on $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ or on $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ is q-periodic if $u(x+q_{hh}e_h) = u(x)$ for all x in the domain of u and for all $h \in \{1, \dots, n\}$. Here $\{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. Next we introduce a dilation of the periodic domain $\mathbb{S}[\Omega_{p,\epsilon}]^-$ by setting

$$\mathbb{S}(\epsilon,\delta)^- \equiv \delta\mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall(\epsilon,\delta) \in]0,\epsilon_0[\times]0,+\infty[.$$

The parameter δ determines the size of the periodic cells of $\mathbb{S}(\epsilon,\delta)^-$. Next we turn to introduce the data of our problem. To do so, we fix $\rho \in]0,+\infty[$ and we consider the Roumieu function space $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ of q-periodic real analytic functions from $\mathbb{R}^n$ to $\mathbb{R}$ (see (2.2)), and we assume that

$$\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[} \text{ is a real analytic family in } C^0_{q,\omega,\rho}(\mathbb{R}^n) , \tag{1.3}$$

i.e., that the map from $]-\epsilon_0, \epsilon_0[$ to $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ which takes ϵ to f_ϵ is real analytic, and we assign a (nonlinear) continuous real valued function

$$G \in C^0(\partial\Omega \times \mathbb{R})$$

satisfying certain regularity assumptions which we specify later (cf. (3.4), (4.7).) Then we consider the following periodic nonlinear problem for the Poisson equation for each $(\epsilon,\delta) \in]0,\epsilon_0[\times]0,+\infty[$

$$\begin{cases} \Delta u(x) = f_\epsilon(\delta^{-1}x) & \forall x \in \mathbb{S}(\epsilon,\delta)^- , \\ u \text{ is } \delta q-\text{periodic in } \mathbb{S}(\epsilon,\delta)^- , & \\ \frac{\partial}{\partial\nu_{\delta\Omega_{p,\epsilon}}} u(x) + G(\delta^{-1}\epsilon^{-1}(x-\delta p), u(x)) = 0 & \forall x \in \delta\partial\Omega_{p,\epsilon} , \end{cases} \tag{1.4}$$

where $\nu_{\delta\Omega_{p,\epsilon}}$ is the outward unit normal to $\delta\Omega_{p,\epsilon}$ on $\delta\partial\Omega_{p,\epsilon}$.

Our first goal is to identify a family of solutions of problem (1.4) for ϵ and δ close to 0. Our second goal is to analyze what happens to the family of solutions when ϵ and δ tend to the degenerate value 0.

We distinguish two cases which depend on the behavior of $\int_Q f_\epsilon\, dy$ as ϵ is close to zero.

If $\int_Q f_\epsilon\, dy$ is not identically zero in $\epsilon \in]-\epsilon_0, \epsilon_0[$, our assumption (1.3) implies that there exist a unique $n_f \in \mathbb{N}$ and a unique analytic function F from $]-\epsilon_0, \epsilon_0[$ to $\mathbb{R}$ such that

$$\int_Q f_\epsilon\, dy = \epsilon^{n_f} F(\epsilon) \qquad \forall \epsilon \in]-\epsilon_0, \epsilon_0[, \qquad F(0) \neq 0 . \tag{1.5}$$

If instead $\int_Q f_\epsilon\, dy$ is identically zero, we set by definition $n_f \equiv +\infty$. Then we consider separately case $n_f \geq n-1$ and case $n_f < n-1$.

In case $n_f \geq n-1$, we look for a family of solutions $u(\epsilon,\delta,\cdot)$ such that

($\diamond$) $\lim_{(\epsilon,\delta)\to(0,0)} u(\epsilon,\delta,\delta\cdot)$ exists in the $C^{m,\alpha}$-norm on the compact subsets of $\mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$.

($\diamond\diamond$) $\lim_{(\epsilon,\delta)\to(0,0)} u(\epsilon,\delta,\delta(p+\epsilon\cdot))$ exists in the $C^{m,\alpha}$-norm on the compact subsets of $\mathbb{R}^n \setminus \Omega$.

Now by a result of [15, Prop. 4.4 (ii)], one can prove that if such a family exists, then the limit of the rescaled family of ($\diamond$) must necessarily be a constant. Thus in order to show the existence of such a family, we assume that there exists $c_\diamond \in \mathbb{R}$ such that

$$\int_{\partial\Omega} G(t, c_\diamond)\, d\sigma_t = 0 , \qquad \int_{\partial\Omega} G_u(t, c_\diamond)\, d\sigma_t \neq 0 , \qquad G_u(t, c_\diamond) \geq 0 \qquad \forall t \in \partial\Omega , \tag{1.6}$$

where G_u denotes the partial derivative of G with respect to the second argument, and we prove that for ϵ and δ small, problem (1.4) has a solution

$$u(\epsilon,\delta,\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-) ,$$

where the symbol $C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-)$ denotes the Schauder space of functions of class $C^m(\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-)$ with α-Hölder continuous derivatives of order m (see Definition 4.12.) The constant $c_\diamond$ plays the role of the limiting value of the rescaled solutions of ($\diamond$).

In case $n_f < n-1$, our problem displays a higher degree of singularity and we cannot readily identify a limiting problem as (ϵ,δ) tends to $(0,0)$, but instead we can do so if we take the limit as (ϵ,δ) tends to $(0,0)$ in a restricted way (see discussion after (3.12).) So for each $\gamma_0 \in [0,+\infty[$ and for each function $\hat{\epsilon}(\cdot)$ such that

$$\hat{\epsilon}(\cdot) \text{ is a function from }]0,+\infty[\text{ to }]0,\epsilon_0[\,, \tag{1.7}$$
$$\lim_{\delta\to 0}\hat{\epsilon}(\delta) = 0\,, \qquad \lim_{\delta\to 0}\frac{\delta}{\hat{\epsilon}(\delta)^{(n-1)-n_f}} = \gamma_0\,,$$

we look for a family of solutions $u(\delta,\cdot)$ defined on $\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-$ with $\epsilon = \hat{\epsilon}(\delta)$ such that

(∗) $\lim_{\delta\to 0} u(\delta,\delta\cdot)$ exists in the $C^{m,\alpha}$-norm on the compact subsets of $\mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$.

(∗∗) $\lim_{\delta\to 0} u(\delta,\delta(p+\hat{\epsilon}(\delta)\cdot))$ exists in the $C^{m,\alpha}$-norm on the compact subsets of $\mathbb{R}^n \setminus \Omega$.

Then again by [15, Prop. 4.4 (ii)], one can prove that if such a family exists, then the limit of the rescaled family of (∗) must necessarily be a constant. Thus in order to show the existence of such a family, we assume that there exist $c_* \in \mathbb{R}$ and $\gamma_0 \in [0,+\infty[$ such that

$$\int_{\partial\Omega} G(t,c_*)\,d\sigma_t - F(0)\gamma_0 = 0\,,\ \int_{\partial\Omega} G_u(t,c_*)\,d\sigma_t \neq 0\,,\ G_u(t,c_*) \geq 0\ \forall t\in\partial\Omega\,, \tag{1.8}$$

and we prove that for all functions $\hat{\epsilon}(\cdot)$ as in (1.7) and for δ small, problem (1.4) with $\epsilon = \hat{\epsilon}(\delta)$ has a solution

$$u(\delta,\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),\delta)^-)\,,$$

(see Definition 5.7.) The constant c_* plays the role of the limiting value of the rescaled solutions of (∗).

The goal of our paper is to investigate the behavior of $u(\epsilon,\delta,\cdot)$ and of $u(\delta,\cdot)$ as (ϵ,δ) and δ tend to $(0,0)$ and to 0, respectively. In particular, we pose the following question.

(†) What can we say on the function $(\epsilon,\delta)\mapsto u(\epsilon,\delta,\cdot)$ as (ϵ,δ) is close to $(0,0)$ in $]0,\epsilon_0[\times]0,+\infty[$, and what can we say on the function $\delta\mapsto u(\delta,\cdot)$ as δ is close to 0 in $]0,+\infty[$?

The asymptotic behavior of solutions of problems in periodically perforated domains has long been investigated in the frame of Homogenization Theory. It is perhaps difficult to provide a complete list of contributions, and here we mention, *e.g.*, Khruslov [23], Marčenko and Khruslov [38], Cioranescu and Murat [9, 10], and for nonlinear Robin problems the work of Cabarrubias and Donato [7]. We also mention Maz'ya and Movchan [39], where the assumption of periodicity of the array of inclusions has been released.

More generally, problems in singularly perturbed domains have been largely studied with the methods of asymptotic expansions. Here, we mention, *e.g.*, Ammari and Kang [1], Ammari, Kang, and Lee [2], Bonnaillie-Noël, Dambrine, and Lacave [4], Bonnaillie-Noël, Dambrine, Tordeux, and Vial [5], Dauge, Tordeux, and Vial [17], Kozlov, Maz'ya, and Movchan [25], Maz'ya, Movchan, and Nieves [40], Maz'ya, Nazarov, and Plamenewskij [41], Novotny and Sokołowski [45], Ozawa [46], Ward and Keller [49].

Here instead, we wish to represent the functions in question (†) in terms of real analytic maps of (ϵ,δ) and in terms of possibly singular at $(0,0)$, but known functions of (ϵ,δ) in case $n_f \geq (n-1)$ and in terms of real analytic maps of $(\hat{\epsilon}(\delta),\delta/\hat{\epsilon}(\delta)^{(n-1)-n_f})$ and in terms of possibly singular at $(0,\gamma_0)$, but known functions of $(\hat{\epsilon}(\delta),\delta/\hat{\epsilon}(\delta)^{(n-1)-n_f})$ in case $n_f < (n-1)$.

One of the main advantages of our approach is that it allows to justify the possibility to expand the solutions or related functionals in terms of convergent power series and of known functions of the singular perturbation parameters. Indeed, for example, if we know that a certain functional associated to $u(\epsilon,\delta,\cdot)$ can be expressed in terms of real analytic functions and known functions of (ϵ,δ), then we can expand the real analytic functions into power series and thus we can deduce representation formulas consisting of convergent power series and explicitly known maps (so as for example for the expressions of Theorem 8.5 (i), (ii), and of [34, Thm. 3, Cor. 1] for the energy integral of the solutions of a linear Dirichlet problem.)

Moreover, the coefficients of the power series expansions can be computed by solving some explicit systems of integral equations. The expression of such systems for the case of a Dirichlet problem in a bounded domain with a small hole has been obtained in [16].

This paper is a first step in the analysis of nonlinear homogenization problems by exploiting a method which has already been developed for singular perturbation problems in domains with small holes (cf. *e.g.*, [27].) Such a method has been exploited for singularly perturbed boundary value problems for the Laplace equation in [28, 29, 30], for linearized elastostatics in [13, 14] and for the Stokes equations in [11, 12]. Concerning problems in periodic domains we refer to [32, 43, 44], and in particular to [36] where the analysis of a two-parameter anisotropic homogenization problem for a Dirichlet problem for the Poisson equation is carried out.

We also observe that boundary value problems in domains with periodic inclusions can be analyzed, at least for the two dimensional case, with the method of functional equations. Here we mention, *e.g.*, Castro, Pesetskaya, and Rogosin [8], Drygas and Mityushev [19], and Kapanadze, Mishuris, and Pesetskaya [24].

2 Preliminaries and notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X}\times\mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x,y) \in \mathcal{X}\times\mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Let A be a matrix. Then A_{ij} denotes the (i,j)-entry of A. If A is invertible, A^t and A^{-1} denote the transpose and the inverse matrix of A, respectively. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. We also set

$$\mathbb{D}^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{D}\,.$$

For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{R})$, or more simply by $C^m(\Omega)$.

Let $r \in \mathbb{N}\setminus\{0\}$. Let $f \in (C^m(\Omega))^r$. The s-th component of f is denoted f_s, and Df denotes the Jacobian matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{\substack{s=1,\dots,r,\\ l=1,\dots,n}}$. Let $\eta \equiv (\eta_1,\dots,\eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1+\cdots+\eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1}\dots\partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$. The subspace of $C^m(\mathrm{cl}\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [21].) The subspace of $C^m(\mathrm{cl}\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R)))$ for all $R \in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega)$. Let $\mathbb{D} \subseteq \mathbb{R}^r$. Then $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\Omega))^r : f(\mathrm{cl}\Omega) \subseteq \mathbb{D}\}$.

We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if $\mathrm{cl}\Omega$ is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [21, §6.2].) We denote by ν_Ω the outward unit normal to $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [21] (see also [26, §2, Lem. 3.1, 4.26, Thm. 4.28], [37, §2].)

If M is a manifold imbedded in $\mathbb{R}^n$ of class $C^{m,\alpha}$, with $m \ge 1$, $\alpha \in]0,1[$, one can define the Schauder spaces also on M by exploiting the local parametrizations. In particular, one can consider the space $C^{k,\alpha}(\partial\Omega)$ on $\partial\Omega$ for $0 \le k \le m$ with Ω a bounded open set of class $C^{m,\alpha}$, and the trace operator from $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. We denote by $d\sigma$ the area element of a manifold M imbedded in $\mathbb{R}^n$. We retain the standard notation for the Lebesgue space $L^p(M)$ of p-summable functions. Also, if $\mathcal{X}$ is a vector subspace of $L^1(M)$, we find convenient to set

$$\mathcal{X}_0 \equiv \left\{ f \in \mathcal{X} : \int_M f\,d\sigma = 0 \right\}. \tag{2.1}$$

We note that throughout the paper 'analytic' means always 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [18, §15].

We set $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \ne j$ for all $i,j = 1,\dots,n$.

If Ω is an arbitrary open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, $\beta \in]0,1]$, we set

$$C^k_b(\mathrm{cl}\Omega) \equiv \{u \in C^k(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall\gamma \in \mathbb{N}^n \text{ such that } |\gamma| \le k\},$$

and we endow $C^k_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^k_b(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\le k} \sup_{x\in\mathrm{cl}\Omega} |D^\gamma u(x)| \qquad \forall u \in C^k_b(\mathrm{cl}\Omega)\,.$$

Then we set

$$C^{k,\beta}_b(\mathrm{cl}\Omega) \equiv \{u \in C^{k,\beta}(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall\gamma \in \mathbb{N}^n \text{ such that } |\gamma| \le k\},$$

and we endow $C^{k,\beta}_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^{k,\beta}_b(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\le k} \sup_{x\in\mathrm{cl}\Omega} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \mathrm{cl}\Omega|_\beta \qquad \forall u \in C^{k,\beta}_b(\mathrm{cl}\Omega)\,,$$

where $|D^\gamma u : \mathrm{cl}\Omega|_\beta$ denotes the β-Hölder constant of $D^\gamma u$.

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega) \equiv \left\{ u \in C^\infty(\mathrm{cl}\Omega) : \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} \qquad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega)\,,$$

where $|\beta| \equiv \beta_1 + \cdots + \beta_n$ for all $\beta \equiv (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. As is well known, the Roumieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\right)$ is a Banach space.

Next we turn to periodic domains. If Ω is an arbitrary subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq Q$, then we set

$$\mathbb{S}[\Omega] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega\,, \qquad \mathbb{S}[\Omega]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega]\,.$$

If $k \in \mathbb{N}$, $\beta \in]0,1]$, then we set

$$C^k_q(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{u \in C^k_b(\mathrm{cl}\mathbb{S}[\Omega]) :\ u \text{ is } q - \text{periodic}\right\}\,,$$

which we regard as a Banach subspace of $C^k_b(\mathrm{cl}\mathbb{S}[\Omega])$, and

$$C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{u \in C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega]) :\ u \text{ is } q - \text{periodic}\right\}\,,$$

which we regard as a Banach subspace of $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega])$. Then $C^k_q(\mathrm{cl}\mathbb{S}[\Omega]^-)$ and $C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega]^-)$ can be defined similarly. If $\rho \in]0,+\infty[$, then we set

$$C^0_{q,\omega,\rho}(\mathbb{R}^n) \equiv \left\{u \in C^\infty_q(\mathbb{R}^n) :\ \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!}\|D^\beta u\|_{C^0(\mathrm{cl}Q)} < +\infty\right\}\,, \tag{2.2}$$

where $C^\infty_q(\mathbb{R}^n)$ denotes the set of q-periodic functions of $C^\infty(\mathbb{R}^n)$, and

$$\|u\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!}\|D^\beta u\|_{C^0(\mathrm{cl}Q)} \qquad \forall u \in C^0_{q,\omega,\rho}(\mathbb{R}^n)\,.$$

The Roumieu class $\left(C^0_{q,\omega,\rho}(\mathbb{R}^n), \|\cdot\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)}\right)$ is a Banach space. As is well known, if f is a q-periodic real analytic function from $\mathbb{R}^n$ to $\mathbb{R}$, then there exists $\rho \in]0,+\infty[$ such that

$$f \in C^0_{q,\omega,\rho}(\mathbb{R}^n)\,.$$

As is well known, there exists a q-periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{m_n(Q)}\,,$$

where δ_{qz} denotes the Dirac measure with mass in qz (cf. *e.g.*, [31, p. 84].) The distribution $S_{q,n}$ is determined up to an additive constant, and we can take

$$S_{q,n}(x) = -\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{m_n(Q)4\pi^2|q^{-1}z|^2} e^{2\pi i(q^{-1}z)\cdot x}\,,$$

in the sense of distributions in $\mathbb{R}^n$. Moreover, $S_{q,n}$ is even, and real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and locally integrable in $\mathbb{R}^n$ (cf. *e.g.*, Ammari and Kang [1, p. 53], [31, §3].)

Let S_n be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n}\log|x| & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n = 2\,, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n > 2\,, \end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. S_n is well-known to be the fundamental solution of the Laplace operator.

Then the function $S_{q,n} - S_n$ admits an analytic extension to $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ (cf. *e.g.*, Ammari and Kang [1, Lemma 2.39, p. 54].) We find convenient to set

$$R_{q,n} \equiv S_{q,n} - S_n \qquad \text{in } (\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}\,.$$

Obviously, $R_{q,n}$ is not a q-periodic function. We note that the following elementary equality holds

$$S_{q,n}(\epsilon x) = \epsilon^{2-n} S_n(x) + \frac{1}{2\pi}(\delta_{2,n}\log\epsilon) + R_{q,n}(\epsilon x)\,, \tag{2.3}$$

for all $x \in \mathbb{R}^n \setminus \epsilon^{-1}q\mathbb{Z}^n$ and $\epsilon \in]0,+\infty[$.

If Ω is a bounded open subset of $\mathbb{R}^n$ and $f \in L^\infty(\Omega)$, then we set

$$P_n[\Omega, f](x) \equiv \int_\Omega S_n(x-y)f(y)\,dy \qquad \forall x \in \mathbb{R}^n\,.$$

If we further assume that $\Omega \subseteq Q$, then we set

$$P_{q,n}[\Omega, f](x) \equiv \int_\Omega S_{q,n}(x-y)f(y)\,dy \qquad \forall x \in \mathbb{R}^n\,.$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$. If H is any of the functions $S_{q,n}$, $R_{q,n}$ and $\mathrm{cl}\Omega \subseteq Q$ or if H equals S_n, we set

$$\begin{aligned} v[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} H(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,,\\ w[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(y)} H(x-y)\mu(y)\,d\sigma_y\\ &= -\int_{\partial\Omega} \nu_\Omega(y)\cdot DH(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,,\\ w_*[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(x)} H(x-y)\mu(y)\,d\sigma_y\\ &= \int_{\partial\Omega} \nu_\Omega(x)\cdot DH(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega\,, \end{aligned}$$

for all $\mu \in L^2(\partial\Omega)$, where DH is the Jacobian matrix of H. As is well known, if $\mu \in C^0(\partial\Omega)$, then $v[\partial\Omega, S_{q,n}, \mu]$ and $v[\partial\Omega, S_n, \mu]$ are continuous in $\mathbb{R}^n$, and we set

$$\begin{aligned} v^+[\partial\Omega, S_{q,n}, \mu] &\equiv v[\partial\Omega, S_{q,n}, \mu]_{|\mathrm{cl}\mathbb{S}[\Omega]} & v^-[\partial\Omega, S_{q,n}, \mu] &\equiv v[\partial\Omega, S_{q,n}, \mu]_{|\mathrm{cl}\mathbb{S}[\Omega]^-}\\ v^+[\partial\Omega, S_n, \mu] &\equiv v[\partial\Omega, S_n, \mu]_{|\mathrm{cl}\Omega} & v^-[\partial\Omega, S_n, \mu] &\equiv v[\partial\Omega, S_n, \mu]_{|\mathrm{cl}\Omega^-}\,. \end{aligned}$$

Also, if μ is continuous, then $w[\partial\Omega, S_{q,n}, \mu]_{|\mathbb{S}[\Omega]}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}[\Omega]$, which we denote by $w^+[\partial\Omega, S_{q,n}, \mu]$ and $w[\partial\Omega, S_{q,n}, \mu]_{|\mathbb{S}[\Omega]^-}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}[\Omega]^-$, which we denote by $w^-[\partial\Omega, S_{q,n}, \mu]$ (cf. *e.g.*, [31, §3].)

Similarly, $w[\partial\Omega, S_n, \mu]_{|\Omega}$ admits a continuous extension to $\mathrm{cl}\Omega$, which we denote by $w^+[\partial\Omega, S_n, \mu]$ and $w[\partial\Omega, S_n, \mu]_{|\Omega^-}$ admits a continuous extension to $\mathrm{cl}\Omega^-$, which we denote by $w^-[\partial\Omega, S_n, \mu]$ (cf. *e.g.*, Miranda [42], [37, Thm. 3.1].)

In the specific case in which H equals S_n, we omit S_n and we simply write $v[\partial\Omega, \mu]$, $w[\partial\Omega, \mu]$, $w_*[\partial\Omega, \mu]$ instead of $v[\partial\Omega, S_n, \mu]$, $w[\partial\Omega, S_n, \mu]$, $w_*[\partial\Omega, S_n, \mu]$, respectively. Similarly, in case H equals $S_{q,n}$, we omit $S_{q,n}$ and we write $v_q[\partial\Omega, \mu]$, $w_q[\partial\Omega, \mu]$, $w_{q,*}[\partial\Omega, \mu]$ instead of $v[\partial\Omega, S_{q,n}, \mu]$, $w[\partial\Omega, S_{q,n}, \mu]$, $w_*[\partial\Omega, S_{q,n}, \mu]$, respectively.

3 Formulation of problem (1.4) in terms of integral equations

As a first step, we transform our problem so as to remove the parameter δ from the domain of problem (1.4). We do so by exploiting the rule of change of variables.

We observe that a function $u \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-)$ satisfies problem (1.4) if and only if the function

$$u^\sharp(\cdot) = u(\delta\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1)^-)\,,$$

satisfies the following auxiliary boundary value problem

$$\begin{cases} \Delta u^\sharp(x) = \delta^2 f_\epsilon(x) & \forall x \in \mathbb{S}(\epsilon,1)^-\,,\\ u^\sharp \text{ is } q-\text{periodic in } \mathbb{S}(\epsilon,1)^-\,, & \\ \frac{\partial}{\partial\nu_{\Omega_{p,\epsilon}}} u^\sharp(x) + \delta G(\epsilon^{-1}(x-p), u^\sharp(x)) = 0 & \forall x \in \partial\Omega_{p,\epsilon}\,. \end{cases} \tag{3.1}$$

In order to convert problem (3.1) into an integral equation, we need some notation. If $G \in C^0(\partial\Omega\times\mathbb{R})$, we denote by T_G the (nonlinear nonautonomous) composition operator from $C^0(\partial\Omega)$ to itself which maps $v \in C^0(\partial\Omega)$ to the function $T_G[v]$ defined by

$$T_G[v](t) \equiv G(t, v(t)) \qquad \forall t \in \partial\Omega\,.$$

We also need the following Lemma. For a proof we refer to [32, Lem. 3.2].

Proposition 3.2. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $\mathbb{I}$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathbb{R}^n \setminus \mathrm{cl}\mathbb{I}$ is connected and such that $\mathrm{cl}\mathbb{I} \subseteq Q$. Then the map $M[\cdot,\cdot]$ from $C^{m-1,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ to $C^{m,\alpha}(\partial\mathbb{I})$ defined by*

$$M[\mu,\xi](x) \equiv v_q[\partial\mathbb{I},\mu](x) + \xi \qquad \forall x \in \partial\mathbb{I}\,,$$

for all $(\mu,\xi) \in C^{m,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ is a linear homeomorphism from $C^{m-1,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\mathbb{I})$ (see (2.1).)

We now transform problem (3.1) into a problem for integral equations by means of the following.

Theorem 3.3. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega \times \mathbb{R})$ be such that*

$$T_G \text{ maps } C^{m-1,\alpha}(\partial\Omega) \text{ to itself}\,. \tag{3.4}$$

Let $(\epsilon,\delta) \in]0,\epsilon_0[\times]0,+\infty[$. Then the map $u^\sharp[\epsilon,\delta,\cdot,\cdot]$ from the set of pairs $(\theta,c) \in C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ that solve the equation

$$\begin{aligned}
&\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) DS_n(t-s)\theta(s)\,d\sigma_s \\
&\quad + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t) DR_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s \\
&\quad + G\biggl(t, \delta\epsilon\int_{\partial\Omega} S_n(t-s)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s + c \\
&\qquad + \delta^2\left[P_{q,n}[Q,f_\epsilon](p+t\epsilon) - \int_Q f_\epsilon\,dy R_{q,n}(\epsilon t)\right] - \delta^2\epsilon^{2-n}\int_Q f_\epsilon\,dy S_n(t)\biggr) \\
&\qquad + \delta\nu_\Omega(t)\left[DP_{q,n}[Q,f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon dy DR_{q,n}(\epsilon t)\right] \\
&\qquad - \delta\epsilon^{1-n}\int_Q f_\epsilon dy \nu_\Omega(t) DS_n(t) = 0 \quad \forall t \in \partial\Omega\,,
\end{aligned} \tag{3.5}$$

to the set of $u^\sharp \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ which solve the auxiliary problem (3.1) and which takes (θ,c) to the function

$$\begin{aligned}
u^\sharp[\epsilon,\delta,\theta,c] &\equiv \omega^\sharp[\epsilon,\delta,\theta,c] \\
&\quad + \delta^2\left[\int_Q S_{q,n}(\cdot-y)f_\epsilon(y)\,dy - \int_Q f_\epsilon\,dy S_{q,n}(\cdot-p)\right]
\end{aligned} \tag{3.6}$$

where

$$\omega^\sharp[\epsilon,\delta,\theta,c] \equiv v[\partial\Omega_{p,\epsilon}, S_{q,n}, \delta\theta(\epsilon^{-1}(\cdot-p))] + c + \delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\,, \tag{3.7}$$

is a bijection.

Proof. By classical properties of volume potentials (cf. *e.g.*, [35, Prop. A1]) and by the rule of change of variables, a function $u^\sharp \in C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1)^-)$ solves problem (3.1) if and only if the function

$$\omega^\sharp \equiv u^\sharp - \delta^2\left[\int_Q S_{q,n}(\cdot-y)f_\epsilon(y)\,dy - \int_Q f_\epsilon\,dy S_{q,n}(\cdot-p)\right]\,,$$

satisfies the following boundary value problem

$$\begin{cases}
\Delta\omega^\sharp(x) = 0 & \forall x \in \mathbb{S}(\epsilon,1)^-\,,\\
\omega^\sharp \text{ is } q-\text{periodic in } \mathbb{S}(\epsilon,1)^-\,, & \\
\frac{\partial}{\partial\nu_{\Omega_{p,\epsilon}}}\omega^\sharp(x) = -\delta G\biggl(\epsilon^{-1}(x-p), \omega^\sharp(x) & \\
\quad + \delta^2\left[\int_Q S_{q,n}(x-y)f_\epsilon(y)\,dy - \int_Q f_\epsilon\,dy S_{q,n}(x-p)\right]\biggr) & \\
\quad - \delta^2\frac{\partial}{\partial\nu_{\Omega_{p,\epsilon}}(x)}\left[\int_Q S_{q,n}(x-y)f_\epsilon(y)\,dy - \int_Q f_\epsilon\,dy S_{q,n}(x-p)\right] & \forall x \in \partial\Omega_{p,\epsilon}\,.
\end{cases} \tag{3.8}$$

If $u^\sharp \in C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1)^-)$ solves problem (3.1), then Proposition 3.2 implies that there exists a unique pair $(\theta,c) \in C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$ such that

$$\int_{\partial\Omega_{p,\epsilon}} \delta\theta(\epsilon^{-1}(y-p))\,d\sigma_y = 0\,, \tag{3.9}$$

and such that

$$\omega^\sharp(x) = v_q^-[\partial\Omega_{p,\epsilon}, \delta\theta(\epsilon^{-1}(\cdot - p))](x) + c + \delta_{2,n}\delta^2 \int_Q f_\epsilon\, dy \frac{\log\epsilon}{2\pi} \qquad x \in \mathrm{cl}\mathbb{S}(\epsilon,1)^- \,.$$

By the rule of change of variables, we can rewrite (3.9) as

$$\int_{\partial\Omega} \theta\, d\sigma = 0\,. \tag{3.10}$$

Then the third equation of (3.8) and classical jump properties of single layer potentials imply that

$$\begin{aligned}
&\frac{1}{2}\delta\theta(\epsilon^{-1}(x-p)) + w_{q,*}[\partial\Omega_{p,\epsilon}, \delta\theta(\epsilon^{-1}(\cdot - p))](x) \\
&\quad = -\delta G\Biggl(\epsilon^{-1}(x-p), v_q[\partial\Omega_{p,\epsilon}, \delta\theta(\epsilon^{-1}(\cdot - p))](x) + c \\
&\qquad + \delta^2 \left[\int_Q S_{q,n}(x-y) f_\epsilon(y)\, dy - \int_Q f_\epsilon\, dy S_{q,n}(x-p) + \int_Q f_\epsilon\, dy \frac{\delta_{2,n}\log\epsilon}{2\pi}\right]\Biggr) \\
&\qquad - \delta^2 \frac{\partial}{\partial\nu_{\Omega_{p,\epsilon}}(x)} \left[\int_Q S_{q,n}(x-y) f_\epsilon(y)\, dy - \int_Q f_\epsilon\, dy S_{q,n}(x-p)\right]\,,
\end{aligned}$$

for all $x \in \partial\Omega_{p,\epsilon}$, which we rewrite as

$$\begin{aligned}
&\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) DS_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) DR_{q,n}(\epsilon(t-s))\theta(s)\, d\sigma_s \\
&\quad = -G\Biggl(t, \epsilon\delta \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \delta_{2,n}\delta \frac{(\epsilon\log\epsilon)}{2\pi} \int_{\partial\Omega} \theta\, d\sigma \\
&\qquad + \epsilon^{n-1}\delta \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta(s)\, d\sigma_s + c \\
&\qquad + \delta^2 \Biggl[P_{q,n}[Q, f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon dy \biggl(\epsilon^{2-n} S_n(t) \\
&\qquad + \frac{1}{2\pi}\delta_{2,n}(\log\epsilon) + R_{q,n}(\epsilon t)\biggr) + \int_Q f_\epsilon\, dy \frac{\delta_{2,n}\log\epsilon}{2\pi}\Biggr]\Biggr) \\
&\qquad - \delta\nu_\Omega(t) \left[DP_{q,n}[Q, f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon dy (\epsilon^{1-n} DS_n(t) + DR_{q,n}(\epsilon t))\right] \\
&\quad = -G\Biggl(t, \epsilon\delta \int_{\partial\Omega} S_n(t-s)\theta(s)\, d\sigma_s + \epsilon^{n-1}\delta \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta(s)\, d\sigma_s + c \\
&\qquad + \delta^2 \left[P_{q,n}[Q, f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon dy (\epsilon^{2-n} S_n(t) + R_{q,n}(\epsilon t))\right]\Biggr) \\
&\qquad - \delta\nu_\Omega(t) \left[DP_{q,n}[Q, f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon dy (\epsilon^{1-n} DS_n(t) + DR_{q,n}(\epsilon t))\right] \quad \forall t \in \partial\Omega\,,
\end{aligned}$$

(see (3.10).) Hence, (θ, c) belongs to $C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ and satisfies equation (3.5). Conversely, if (θ, c) belongs to $C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ and satisfies equation (3.5), then by reading backward the above computations, we see that the function $u^\sharp[\epsilon, \delta, \theta, c]$ delivered by (3.6), (3.7) satisfies problem (3.1).

On the other hand, if (θ_1, c_1), $(\theta_2, c_2) \in C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ and if

$$u^\sharp[\epsilon, \delta, \theta_1, c_1] = u^\sharp[\epsilon, \delta, \theta_2, c_2]\,,$$

then

$$\omega^\sharp[\epsilon, \delta, \theta_1, c_1] = \omega^\sharp[\epsilon, \delta, \theta_2, c_2]\,,$$

and thus the uniqueness of Proposition 3.2 implies that $\theta_1 = \theta_2$, $c_1 = c_2$, and thus the proof is complete. □ □

Next we observe that the left hand side of equation (3.5) contains only two terms which may not converge as (ϵ, δ) tends to $(0, 0)$. Namely,

$$\delta^2 \epsilon^{2-n} \int_Q f_\epsilon\, dy \qquad \text{and} \qquad \delta\epsilon^{1-n} \int_Q f_\epsilon dy\,. \tag{3.11}$$

If $n_f = +\infty$, *i.e.*, if $\int_Q f_\epsilon dy = 0$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$, then the above terms are identically equal to zero. If instead $n_f < +\infty$, the above terms can be rewritten as

$$\delta^2 \epsilon^{n_f+2-n} F(\epsilon) \qquad \text{and} \qquad \delta \epsilon^{n_f+1-n} F(\epsilon)\,, \tag{3.12}$$

(cf. (1.5)).

Now we distinguish two cases. If $n_f \geq (n-1)$, then the above terms in (3.11) have limit as (ϵ, δ) tends to $(0,0)$. Thus if $n_f \geq (n-1)$ we can take the limit as (ϵ, δ) tends to $(0,0)$ in equation (3.5) under appropriate regularity assumptions and obtain an equation which we address to as 'limiting integral equation'. Namely,

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) DS_n(t-s)\theta(s)\, d\sigma_s + G(t,c) = 0 \qquad \forall t \in \partial\Omega\,. \tag{3.13}$$

If instead $n_f < n-1$, then the second term in (3.11) (or (3.12)) cannot have a limit as (ϵ, δ) tends to $(0,0)$, and accordingly, we cannot take the limit as (ϵ, δ) tends to $(0,0)$ in equation (3.5) and we cannot identify a 'limiting integral equation'. Hence, case $n_f < n-1$ requires a different treatment. Here we observe that if we fix $\gamma_0 \in [0, +\infty[$ and if we consider the pairs (ϵ, δ) of the graph of a function $\hat{\epsilon}$ from $]0, +\infty[$ to $]0, \epsilon_0[$ such that (1.7) holds, then we can take the limit as δ tends to 0 in the terms of (3.11) (or of (3.12)) with $\epsilon = \hat{\epsilon}(\delta)$ and obtain

$$\lim_{\delta\to 0} \delta^2 \hat{\epsilon}(\delta)^{2-n} \int_Q f_{\hat{\epsilon}(\delta)}\, dy = 0 \qquad \text{and} \qquad \lim_{\delta \to 0} \delta \hat{\epsilon}(\delta)^{1-n} \int_Q f_{\hat{\epsilon}(\delta)} dy = \gamma_0 F(0)\,.$$

Hence, we can take the limit as δ tends to 0 in equation (3.5) with $\epsilon = \hat{\epsilon}(\delta)$ under appropriate regularity assumptions and obtain an equation which we address to as 'limiting integral equation associated to γ_0'. Namely,

$$\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) DS_n(t-s)\theta(s)\, d\sigma_s + G(t,c) - \gamma_0 F(0) \nu_\Omega(t) DS_n(t) = 0 \quad \forall t \in \partial\Omega\,. \tag{3.14}$$

We now turn to analyze equation (3.5) and we do so by treating separately case $n_f \geq n-1$ and case $n_f < n-1$.

4 Analysis of the integral equation (3.5) in case $n_f \geq n-1$.

We first analyze the 'limiting integral integral equation' (3.13) by means of the following.

Theorem 4.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let $G \in C^0(\partial\Omega \times \mathbb{R})$ be such that (3.4) holds. Let $c_\diamond \in \mathbb{R}$ be such that*

$$\int_{\partial\Omega} G(t, c_\diamond)\, d\sigma_t = 0\,. \tag{4.2}$$

Then the following statements hold.

(i) The limiting integral equation (3.13) with $c = c_\diamond$ has a unique solution $\theta_\diamond \in C^{m-1,\alpha}(\partial\Omega)_0$ (see (2.1).)

(ii) The 'limiting boundary value problem'

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega\,, \\ \frac{\partial u}{\partial \nu_\Omega}(x) + G(x, c_\diamond) = 0 & \forall x \in \partial\Omega\,, \\ \lim_{x\to\infty} u(x) = 0\,, \end{cases} \tag{4.3}$$

has one and only one solution $u^\sharp_\diamond \in C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ and

$$u^\sharp_\diamond = v^-[\partial\Omega, \theta_\diamond]\,. \tag{4.4}$$

Proof. By classical the classical Fredholm Theory and by Schauder regularity results, equation (3.13) has a unique solution $\theta_\diamond$ in $C^{m-1,\alpha}(\partial\Omega)$. Then assumption (4.2) and equality $w[\partial\Omega, 1]_{|\partial\Omega} = 1/2$ imply that $\int_{\partial\Omega} \theta_\diamond\, d\sigma = 0$ (cf. *e.g.*, Folland [20, Props. 3.11, 3.37] and [28, Thm. 5.1(i)].) We now consider statement (ii). By classical jump relations of the normal derivative of a single layer potential and by equation (3.13) with $c = c_\diamond$, the function $u^\sharp_\diamond = v^-[\partial\Omega, \theta_\diamond]$ satisfies the boundary condition of the limiting boundary value problem (4.3). Since $\int_{\partial\Omega} \theta_\diamond\, d\sigma = 0$, the function $u^\sharp_\diamond \equiv v^-[\partial\Omega, \theta_\diamond]$ satisfies the limiting condition of (4.3). Since $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ is connected, the uniqueness of solutions for problem (4.3) follows by classical results on the exterior Neumann problem for harmonic functions (cf. *e.g.*, Folland [20, Thm. 3.40].) □ □

We are now ready to analyze equation (3.5) around the degenerate case in which $(\epsilon,\delta)=(0,0)$ and under the assumption that $n_f \geq (n-1)$. In order treat both case $n_f<+\infty$ and case $n_f=+\infty$ at the same time, we find convenient to set

$$\tilde{n}_f \equiv n_f \qquad \text{if } n_f<+\infty\,, \qquad \tilde{n}_f \equiv n-1 \qquad \text{if } n_f=+\infty\,,$$

and to set $F(\epsilon)\equiv 0$ for all $\epsilon\in]-\epsilon_0,\epsilon_0[$ in case $n_f=+\infty$. Indeed, if so we have

$$\int_Q f_\epsilon\,dy = \epsilon^{\tilde{n}_f}F(\epsilon) \qquad \forall \epsilon\in]-\epsilon_0,\epsilon_0[\,, \tag{4.5}$$

both in case $n_f<+\infty$ and case $n_f=+\infty$, and

$$F(0)\neq 0 \qquad \text{if } n_f<+\infty\,, \qquad F(0)=0 \qquad \text{if } n_f=+\infty\,.$$

Then we are ready to introduce the following.

Theorem 4.6. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $n_f\geq n-1$. Let $G\in C^0(\partial\Omega\times\mathbb{R})$ be such that*

$$T_G \text{ is real analytic in } C^{m-1,\alpha}(\partial\Omega)\,. \tag{4.7}$$

Let $c_\diamond\in\mathbb{R}$ be such that (1.6) holds. Let $\Lambda_\diamond$ be the map from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ defined by

$$\begin{aligned}
\Lambda_\diamond[\epsilon,\delta,\theta,c](t) \equiv{}& \frac{1}{2}\theta(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\theta(s)\,d\sigma_s\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s\\
&+G\biggl(t,\delta\epsilon\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s+c\\
&\quad+\delta^2\left[P_{q,n}[Q,f_\epsilon](p+t\epsilon)-\epsilon^{\tilde{n}_f}F(\epsilon)R_{q,n}(\epsilon t)\right]-\delta^2\epsilon^{2-n}\epsilon^{\tilde{n}_f}F(\epsilon)S_n(t)\biggr)\\
&\quad+\delta\nu_\Omega(t)\biggl[DP_{q,n}[Q,f_\epsilon](p+\epsilon t)-\epsilon^{\tilde{n}_f}F(\epsilon)DR_{q,n}(\epsilon t)\biggr]\\
&\quad-\delta\epsilon^{1-n}\epsilon^{\tilde{n}_f}F(\epsilon)\nu_\Omega(t)DS_n(t) \quad \forall t\in\partial\Omega\,,
\end{aligned}$$

for all $(\epsilon,\delta,\theta,c)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$. Then the following statements hold.

(i) Equation $\Lambda_\diamond[0,0,\theta,c_\diamond]=0$ is equivalent to the limiting integral equation (3.13) with $c=c_\diamond$ and has one and only one solution $\theta_\diamond\in C^{m-1,\alpha}(\partial\Omega)_0$ (see (2.1).)

(ii) If $(\epsilon,\delta)\in]0,\epsilon_0[\times]0,+\infty[$, then equation $\Lambda_\diamond[\epsilon,\delta,\theta,c]=0$ is equivalent to equation (3.5) in the unknown $(\theta,c)\in C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$.

(iii) There exist $(\epsilon',\delta')\in]0,\epsilon_0[\times]0,+\infty[$ and an open neighborhood $\mathcal{U}$ of $(\theta_\diamond,c_\diamond)$ in $C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$, and a real analytic map $(\Theta_\diamond,C_\diamond)$ from $]-\epsilon',\epsilon'[\times]-\delta',\delta'[$ to $\mathcal{U}$ such that the set of zeros of the map $\Lambda_\diamond$ in $]-\epsilon',\epsilon'[\times]-\delta',\delta'[\times\mathcal{U}$ coincides with the graph of $(\Theta_\diamond,C_\diamond)$. In particular,

$$(\Theta_\diamond[0,0],C_\diamond[0,0])=(\theta_\diamond,c_\diamond)\,.$$

Proof. Statement (i) and (ii) are an immediate consequence of Theorem 4.1 and of the definition of $\Lambda_\diamond$. We now turn to show that $\Lambda_\diamond$ is analytic in a neighborhood of $(0,0,\theta_\diamond,c_\diamond)$. We first note that the maps from $]-\epsilon_0,\epsilon_0[\times L^1(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$, which take (ϵ,θ) to the functions

$$\begin{aligned}
&\textstyle\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s && \forall t\in\partial\Omega\,,\\
&\textstyle\int_{\partial\Omega}\partial_{x_j}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s && \forall j\in\{1,\dots,n\} \qquad \forall t\in\partial\Omega\,,
\end{aligned} \tag{4.8}$$

are real analytic (cf. [35, Lem. A.7 (i)].) Then the analyticity of $R_{q,n}$ and analyticity results on the composition operator imply that the map from $]-\epsilon_0,\epsilon_0[$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes ϵ to the function $R_{q,n}(\epsilon t)$ of $t\in\partial\Omega$

is real analytic (cf. Böhme and Tomi [3, p. 10], Henry [22, p. 29], Valent [48, Thm. 5.2, p. 44].) Moreover, by assumption (1.3) and [35, Lem. A.7 (ii)], the maps from $\mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ which take ϵ to

$$
\begin{aligned}
P_{q,n}[Q,f_\epsilon](p+\epsilon t) \qquad & \forall t\in\partial\Omega\,,\\
\partial_{x_j}P_{q,n}[Q,f_\epsilon](p+\epsilon t) \quad \forall j\in\{1,\dots,n\} \quad & \forall t\in\partial\Omega\,,
\end{aligned}
$$

are analytic. Then the analyticity of $\Lambda_\diamond$ follows by the linearity and continuity of $v[\partial\Omega,\cdot]_{|\partial\Omega}$ from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$, and by the linearity and continuity of $w_*[\partial\Omega,\cdot]_{|\partial\Omega}$ from $C^{m-1,\alpha}(\partial\Omega)$ to itself, and by the continuity of the pointwise product in Schauder spaces, and by assumptions (1.3), (4.7).

Next we turn to prove that the differential $\partial_{(\theta,c)}\Lambda_\diamond[0,0,\theta_\diamond,c_\diamond]$ of Λ at the quadruple $(0,0,\theta_\diamond,c_\diamond)$ with respect to the variable (θ,c) is a linear homeomorphism from $C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach space, the differential of $\Lambda_\diamond$ at $(0,0,\theta_\diamond,c_\diamond)$ with respect to the variable (θ,c) is delivered by the following formula

$$
\partial_{(\theta,c)}\Lambda_\diamond[0,0,\theta_\diamond,c_\diamond](\overline{\theta},\overline{c})=\frac{1}{2}\overline{\theta}(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\overline{\theta}(s)\,d\sigma_s+G_u(t,c_\diamond)\overline{c} \qquad \forall t\in\partial\Omega\,,
$$

for all $(\overline{\theta},\overline{c})\in C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ (see also [28, Prop. 6.3], which ensures the existence of G_u.) We now show that $\partial_{(\theta,c)}\Lambda_\diamond[0,0,\theta_\diamond,c_\diamond]$ is a bijection. To do so, we show that if $h\in C^{m-1,\alpha}(\partial\Omega)$, then the equation

$$
\partial_{(\theta,c)}\Lambda_\diamond[0,0,\theta_\diamond,c_\diamond](\overline{\theta},\overline{c})=h\,, \tag{4.9}
$$

has a unique solution $(\overline{\theta},\overline{c})\in C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$. If equation (4.9) holds, then $\overline{c}\in\mathbb{R}$ must necessarily satisfy the equation

$$
\int_{\partial\Omega}h(t)-G_u(t,c_\diamond)\overline{c}\,d\sigma_t=0\,. \tag{4.10}
$$

Indeed, $\int_{\partial\Omega}\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\overline{\theta}(s)\,d\sigma_s\,d\sigma_t=\int_{\partial\Omega}\frac{1}{2}\overline{\theta}(s)\,d\sigma_s=0$ (cf. *e.g.*, Folland [20, Prop. 3.19].) Thus we must have

$$
\overline{c}=\frac{\int_{\partial\Omega}h\,d\sigma}{\int_{\partial\Omega}G_u(t,c_\diamond)\,d\sigma}\,, \tag{4.11}
$$

(cf. (1.6).) If we choose $\overline{c}$ as in (4.11), then equality (4.10) and known classical results ensure that there exists a unique $\overline{\theta}\in C^{m-1,\alpha}(\partial\Omega)$ such that

$$
\frac{1}{2}\overline{\theta}(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\overline{\theta}(s)\,d\sigma_s=h(t)-G_u(t,c_\diamond)\overline{c} \qquad \forall t\in\partial\Omega\,,
$$

(cf. *e.g.*, Folland [20, Prop. 3.37] and [28, Thm. 5.1 (i)].) Then equality (4.10) and the computations following (4.10) imply that $\int_{\partial\Omega}\overline{\theta}(s)\,d\sigma_s=0$. Hence, equation (4.9) does have a unique solution for each $h\in C^{m-1,\alpha}(\partial\Omega)$ and $\partial_{(\theta,c)}\Lambda_\diamond[0,0,\theta_\diamond,c_\diamond]$ is a bijection. Then the Open Mapping Theorem implies that $\partial_{(\theta,c)}\Lambda_\diamond[0,0,\theta_\diamond,c_\diamond]$ is a homeomorphism. Since $\Lambda_\diamond$ is analytic, statement (iii) is an immediate consequence of statements (i), (ii) and of the Implicit Function Theorem in Banach spaces (cf. *e.g.*, Deimling [18, Thm. 15.3].) □ □

We are now ready to define our family of solutions of the auxiliary problem (3.1) in case $n_f\geq n-1$. We do so by means of the following.

Definition 4.12. *Let the assumptions of Theorem 4.6 hold. Then we set*

$$
\begin{aligned}
\omega^\sharp(\epsilon,\delta,x) &\equiv \omega^\sharp[\epsilon,\delta,\Theta_\diamond[\epsilon,\delta],C_\diamond[\epsilon,\delta]](x) \qquad \forall x\in \mathrm{cl}\mathbb{S}(\epsilon,1)^-\,,\\
u^\sharp(\epsilon,\delta,x) &\equiv \omega^\sharp[\epsilon,\delta,\Theta_\diamond[\epsilon,\delta],C_\diamond[\epsilon,\delta]](x)\\
&\quad +\delta^2\left[\int_Q S_{q,n}(x-y)f_\epsilon(y)\,dy-\int_Q f_\epsilon\,dy S_{q,n}(x-p)\right] \ \forall x\in \mathrm{cl}\mathbb{S}(\epsilon,1)^-\,,
\end{aligned}
$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$ (see also (3.6), (3.7).)

Then $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ is a family of solutions of the auxiliary problem (3.1) in case $n_f\geq n-1$ and our aim is to analyze the behavior of such a family as (ϵ,δ) tends to $(0,0)$.

5 Analysis of the integral equation (3.5) in case $n_f < n-1$.

We first analyze the 'limiting integral integral equation associated to γ_0' (3.14) by means of the following.

Theorem 5.1. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be as in (1.1). Let $G \in C^0(\partial\Omega \times \mathbb{R})$ be such that (3.4) holds. Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$, $F_0 \in \mathbb{R}$ be such that*

$$\int_{\partial\Omega} G(t,c_*)\, d\sigma_t - F_0\gamma_0 = 0\,. \tag{5.2}$$

Then the following statements hold.

(i) The limiting integral equation associated to γ_0 (3.14) with $c = c_*$ and with $F(0)$ replaced by F_0 has a unique solution $\theta_* \in C^{m-1,\alpha}(\partial\Omega)_0$.

(ii) The 'limiting boundary value problem'

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \mathrm{cl}\Omega\,, \\ \frac{\partial u}{\partial\nu_\Omega}(x) + G(x,c_*) - F_0\gamma_0\nu_\Omega(x)DS_n(x) = 0 & \forall x \in \partial\Omega\,, \\ \lim_{x\to\infty} u(x) = 0\,, \end{cases} \tag{5.3}$$

has one and only one solution $u^\sharp_* \in C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ and

$$u^\sharp_* = v^-[\partial\Omega,\theta_*]\,. \tag{5.4}$$

Proof. Let $a \in]0,+\infty[$ be such that $\mathrm{cl}\mathbb{B}_n(0,a) \subseteq \Omega$. Since S_n is harmonic in $\mathbb{R}^n\setminus\{0\}$, we have

$$\int_{\partial\Omega} \nu_\Omega(t)DS_n(t)\, d\sigma_t = \int_{\partial\mathbb{B}_n(0,a)} \nu_{\mathbb{B}_n(0,a)}(t)DS_n(t)\, d\sigma_t = \int_{\partial\mathbb{B}_n(0,a)} \frac{|t|^2}{s_n a^{n+1}}\, d\sigma_t = 1\,.$$

Then condition (5.2) implies that

$$\int_{\partial\Omega} G(t,c_*) - \gamma_0F_0\nu_\Omega(t)DS_n(t)\, d\sigma_t = 0\,,$$

and accordingly statement (i) follows by the classical Fredholm Theory and by Schauder regularity results (cf. *e.g.*, Folland [20, Props. 3.11, 3.37] and [28, Thm. 5.1(i)].)

We now consider statement (ii). By classical jump relations of the normal derivative of a single layer potential and by equation (3.14) with $c = c_*$ and with $F(0)$ replaced by F_0, the function $u^\sharp_* = v^-[\partial\Omega,\theta_*]$ satisfies the boundary condition of the limiting boundary value problem (5.3). Since $\int_{\partial\Omega}\theta_*\, d\sigma = 0$, the function $u^\sharp_* \equiv v^-[\partial\Omega,\theta_*]$ satisfies the limiting condition of (5.3). Since $\mathbb{R}^n\setminus\mathrm{cl}\Omega$ is connected, the uniqueness of solutions for problem (5.3) follows by classical results on the exterior Neumann problem for harmonic functions (cf. *e.g.*, Folland [20, Thm. 3.40].) □ □

Now the idea is to replace the term $\delta\epsilon^{n_f+1-n}$ which appears in (3.12) and which has no limit as (ϵ,δ) tends to $(0,0)$ by a new variable γ and to obtain a new equation which depends on ϵ and γ and which is not singular in ϵ and γ and to analyze the dependence of θ and c upon ϵ and γ. To do so, we introduce the following.

Theorem 5.5. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $n_f < n-1$. Let $G \in C^0(\partial\Omega\times\mathbb{R})$ satisfy (4.7). Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ satisfy (1.8) (cf. (1.5).) Let Λ_* be the map from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ defined by*

$$\begin{aligned}\Lambda_*[\epsilon,\gamma,\theta,c](t) \equiv{}& \frac{1}{2}\theta(t) + \int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\theta(s)\, d\sigma_s \\ &+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\theta(s)\, d\sigma_s \\ &+G\biggl(t, \gamma\epsilon^{n-n_f}\int_{\partial\Omega}S_n(t-s)\theta(s)\, d\sigma_s + \gamma\epsilon^{2(n-1)-n_f}\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta(s)\, d\sigma_s + c \\ &\quad+\gamma^2\epsilon^{2(n-1)-2n_f}\left[P_{q,n}[Q,f_\epsilon](p+t\epsilon) - \epsilon^{n_f}F(\epsilon)R_{q,n}(\epsilon t)\right] - \gamma^2\epsilon^{n-n_f}F(\epsilon)S_n(t)\biggr) \\ &+\gamma\epsilon^{n-1-n_f}\nu_\Omega(t)\Bigl[DP_{q,n}[Q,f_\epsilon](p+\epsilon t) - \epsilon^{n_f}F(\epsilon)DR_{q,n}(\epsilon t)\Bigr] \\ &-\gamma F(\epsilon)\nu_\Omega(t)DS_n(t) \quad \forall t \in \partial\Omega\,,\end{aligned}$$

for all $(\epsilon, \gamma, \theta, c) \in]-\epsilon_0, \epsilon_0[\times \mathbb{R} \times C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$. *Then the following statements hold.*

(i) Equation $\Lambda_*[0, \gamma_0, \theta, c_*] = 0$ *is equivalent to the limiting integral equation associated to* γ_0 *(3.14) with* $c = c_*$ *and has one and only one solution* $\theta_* \in C^{m-1,\alpha}(\partial\Omega)_0$ *(see (2.1).)*

(ii) Let $\hat{\epsilon}$ *be as in (1.7). Let* $\delta \in]0, +\infty[$, $\hat{\epsilon}(\delta) < \epsilon_0$. *Then equation*

$$\Lambda_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f - n+1}, \theta, c] = 0$$

is equivalent to the integral equation (3.5) with $\epsilon = \hat{\epsilon}(\delta)$ *in the unknown* $(\theta, c) \in C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$.

(iii) There exist $\epsilon' \in]0, \epsilon_0[$ *and an open neighborhood* Γ_0 *of* γ_0 *in* $\mathbb{R}$, *and an open neighborhood* $\mathcal{U}$ *of* (θ_*, c_*) *in* $C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$, *and a real analytic map* (Θ_*, C_*) *from* $]-\epsilon', \epsilon'[\times\Gamma_0$ *to* $\mathcal{U}$ *such that the set of zeros of the map* Λ_* *in* $]-\epsilon', \epsilon'[\times\Gamma_0 \times \mathcal{U}$ *coincides with the graph of* (Θ_*, C_*). *In particular,*

$$(\Theta_*[0, \gamma_0], C_*[0, \gamma_0]) = (\theta_*, c_*)\,.$$

Proof. Statement (i) is an immediate consequence of Theorem 5.1. Statement (ii) follows by Theorem 4.6 (ii) and by the definition of Λ_*. By the same arguments of the proof of Theorem 4.6 (iii), the operator Λ_* is analytic.

Next we turn to prove that the differential $\partial_{(\theta,c)}\Lambda_*[0, \gamma_0, \theta_*, c_*]$ of Λ_* at the quadruple $(0, \gamma_0, \theta_*, c_*)$ with respect to the variable (θ, c) is a linear homeomorphism from $C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$. By standard calculus in Banach space, the differential $\partial_{(\theta,c)}\Lambda_*[0, \gamma_0, \theta_*, c_*]$ of Λ_* at $(0, \gamma_0, \theta_*, c_*)$ with respect to the variable (θ, c) is delivered by the following formula

$$\partial_{(\theta,c)}\Lambda_*[0, \gamma_0, \theta_*, c_*](\overline{\theta}, \overline{c}) = \frac{1}{2}\overline{\theta}(t) + \int_{\partial\Omega} \nu_\Omega(t) DS_n(t-s)\overline{\theta}(s)\, d\sigma_s + G_u(t, c_*)\overline{c} \qquad \forall t \in \partial\Omega\,,$$

for all $(\overline{\theta}, \overline{c}) \in C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ (see also [28, Prop. 6.3], which ensures the existence of G_u), and we have already proved that the linear operator in the right hand side is a linear homeomorphism from $C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ onto $C^{m-1,\alpha}(\partial\Omega)$ (cf. Theorem 5.1 (i).) Since Λ_* is analytic, statement (iii) is an immediate consequence of statements (i), (ii) and of the Implicit Function Theorem in Banach spaces (cf. *e.g.*, Deimling [18, Thm. 15.3].) □ □

By the limiting relations in (1.7), there exists $\delta' \in]0, +\infty[$ such that

$$\hat{\epsilon}(\delta) \in]0, \epsilon'[\qquad \frac{\delta}{\hat{\epsilon}(\delta)^{(n-1)-n_f}} \in \Gamma_0 \qquad \forall \delta \in]0, \delta'[\,. \tag{5.6}$$

We are now ready to define our family of solutions of the auxiliary problem (3.1) in case $n_f < n-1$. We do so by means of the following.

Definition 5.7. *Let the assumptions of Theorem 5.5 hold. Let* $\delta' \in]0, +\infty[$ *be as in (5.6). Then we set*

$$\begin{aligned}
\omega^\sharp(\delta, x) &\equiv \omega^\sharp[\hat{\epsilon}(\delta), \delta, \Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-n+1}], C_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-n+1}]](x) \\
&\qquad\qquad \forall x \in \mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta), 1)^-\,, \\
u^\sharp(\delta, x) &\equiv \omega^\sharp[\hat{\epsilon}(\delta), \delta, \Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-n+1}], C_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-n+1}]](x) \\
&\quad + \delta^2 \left[\int_Q S_{q,n}(x-y) f_{\hat{\epsilon}(\delta)}(y)\, dy - \int_Q f_{\hat{\epsilon}(\delta)}\, dy S_{q,n}(x-p)\right] \\
&\qquad\qquad \forall x \in \mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta), 1)^-\,,
\end{aligned}$$

for all $\delta \in]0, \delta'[$ *(see also (3.6), (3.7).)*

By Theorem 5.5, $\{u^\sharp(\delta, \cdot)\}_{\delta \in]0,\delta'[}$ is a family of solutions of the auxiliary problem (3.1) in case $n_f < n-1$ and our aim is to analyze the behavior of such a family as δ tends to 0.

6 A functional analytic representation theorem for the family of solutions $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ of the auxilary problem (3.1)

We first introduce the following lemma.

Lemma 6.1. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $\epsilon^*\in]0,\epsilon_0[$. There exist $\rho'\in]0,+\infty[$ such that the map $\mathcal{P}_{\epsilon^*}$ from $]-\epsilon_0,\epsilon_0[$ to $C^0_{q,\omega,\rho'}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-)$ defined by*

$$\mathcal{P}_{\epsilon^*}[\epsilon](x)\equiv\int_Q S_{q,n}(x-y)f_\epsilon(y)\,dy-\int_Q f_\epsilon\,dy S_{q,n}(x-p)\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-\,,\tag{6.2}$$

for all $\epsilon\in]-\epsilon_0,\epsilon_0[$ is analytic.

Proof. By assumption (1.3) and [35, Prop. A.2], there exists $\rho'\in]0,\rho]$ such that the map from $]-\epsilon_0,\epsilon_0[$ to $C^0_{q,\omega,\rho'}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-)$, which takes ϵ to $P_{q,n}[Q,f_\epsilon]_{|\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-}$ is analytic. Since $S_{q,n}(\cdot-p)$ is analytic in $\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$ and $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-$ is contained in $\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$, possibly shrinking ρ', we can assume that $S_{q,n}(\cdot-p)_{|\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-}\in C^0_{q,\omega,\rho'}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*}]^-)$. By assumption (1.3), the integral $\int_Q f_\epsilon\,dy$ depends analytically on ϵ. Hence, $\mathcal{P}_{\epsilon^*}$ is analytic. □ □

We are now ready to prove a representation theorem for the family of solutions $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$.

Theorem 6.3. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $n_f\geq n-1$. Let $G\in C^0(\partial\Omega\times\mathbb{R})$ satisfy condition (4.7). Let $c_\diamond\in\mathbb{R}$ be such that (1.6) holds. Let ρ', $\mathcal{P}_{\epsilon^*}$ be as in Lemma 6.1 for all $\epsilon^*\in]0,\epsilon_0[$. Let ϵ', δ' be as in Theorem 4.6 (iii). Then the following statements hold.*

(i) Let $\tilde\Omega$ be an open subset of $\mathbb{R}^n$ with nonzero distance from $p+q\mathbb{Z}^n$. Then there exist $\epsilon^*_{\tilde\Omega}\in]0,\epsilon'[$ such that

$$\mathrm{cl}\tilde\Omega\subseteq\mathbb{S}[\Omega_{p,\epsilon}]^-\qquad\forall\epsilon\in[-\epsilon^*_{\tilde\Omega},\epsilon^*_{\tilde\Omega}]\,,$$

and $\epsilon_{\tilde\Omega}\in]0,\epsilon^*_{\tilde\Omega}[$ such that $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-\subseteq\mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon\in[-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}]$, and a real analytic map $V_{\diamond,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}$ from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$ such that

$$\begin{aligned}\omega^\sharp(\epsilon,\delta,x)&=\epsilon^{n-1}\delta V_{\diamond,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}[\epsilon,\delta](x)+C_\diamond[\epsilon,\delta]+\delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\,,\\ u^\sharp(\epsilon,\delta,x)&=\epsilon^{n-1}\delta V_{\diamond,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}[\epsilon,\delta](x)+C_\diamond[\epsilon,\delta]+\delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\\ &\qquad+\delta^2\mathcal{P}_{\epsilon^*_{\tilde\Omega}}[\epsilon](x)\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-\,,\end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon_{\tilde\Omega}[\times]0,\delta'[$. Moreover,

$$V_{\diamond,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}[0,0]=0\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-\qquad C_\diamond[0,0]=c_\diamond\,.\tag{6.4}$$

(ii) Let $\tilde\Omega$ be a bounded open subset of $\mathbb{R}^n\setminus\mathrm{cl}\Omega$. Then there exist $\epsilon_{\tilde\Omega,r}\in]0,\epsilon'[$ and a real analytic map $V^r_{\diamond,\tilde\Omega}$ from $]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times]-\delta',\delta'[$ to $C^{m,\alpha}(\mathrm{cl}\tilde\Omega)$ and a real analytic map $\mathcal{P}^r_{\tilde\Omega}$ from $]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\mathrm{cl}\tilde\Omega)$ such that

$$\begin{aligned}&p+\epsilon\mathrm{cl}\tilde\Omega\subseteq\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\qquad\forall\epsilon\in]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\,,\\ &\omega^\sharp(\epsilon,\delta,p+\epsilon t)=\epsilon\delta V^r_{\diamond,\tilde\Omega}[\epsilon,\delta](t)+C_\diamond[\epsilon,\delta]+\delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\qquad\forall t\in\mathrm{cl}\tilde\Omega\,,\\ &u^\sharp(\epsilon,\delta,p+\epsilon t)=\epsilon\delta V^r_{\diamond,\tilde\Omega}[\epsilon,\delta](t)+C_\diamond[\epsilon,\delta]+\delta^2\mathcal{P}^r_{\tilde\Omega}[\epsilon](t)\qquad\forall t\in\mathrm{cl}\tilde\Omega\,,\end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon_{\tilde\Omega,r}[\times]0,\delta'[$. Moreover,

$$V^r_{\diamond,\tilde\Omega}[0,0](t)=u^\sharp_\diamond(t)\qquad\mathcal{P}^r_{\tilde\Omega}[0](t)=\int_Q S_{q,n}(p-y)f_0(y)\,dy\qquad\forall t\in\mathrm{cl}\tilde\Omega\,.$$

(iii) There exist a real analytic map $V^r_{\diamond,\partial\Omega}$ from $]-\epsilon',\epsilon'[\times]-\delta',\delta'[$ to $C^{m,\alpha}(\partial\Omega)$ and a real analytic map $\mathcal{P}^r_{\partial\Omega}$ from $]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\partial\Omega)$ such that

$$\begin{aligned}
\omega^\sharp(\epsilon,\delta,p+\epsilon t) &= \epsilon\delta V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t) + C_\diamond[\epsilon,\delta] + \delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi} \qquad \forall t\in\partial\Omega\,,\\
u^\sharp(\epsilon,\delta,p+\epsilon t) &= \epsilon\delta V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t) + C_\diamond[\epsilon,\delta] + \delta^2\mathcal{P}^r_{\partial\Omega}[\epsilon](t) \qquad \forall t\in\partial\Omega\,,
\end{aligned} \tag{6.5}$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Moreover,

$$V^r_{\diamond,\partial\Omega}[0,0](t) = u^\sharp_\diamond(t) \qquad \mathcal{P}^r_{\partial\Omega}[0](t) = \int_Q S_{q,n}(p-y)f_0(y)\,dy \qquad \forall t\in\partial\Omega\,. \tag{6.6}$$

(iv) There exist $\epsilon_1\in]0,\epsilon'[$, $\delta_1\in]0,\delta'[$ and two analytic maps $J_1^\diamond$ from $]-\epsilon_1,\epsilon_1[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$ and $J_2^\diamond$ from $]-\epsilon_1,\epsilon_1[$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} u^\sharp(\epsilon,\delta,x)\,dx = J_1^\diamond[\epsilon,\delta] + \delta_{2,n}\delta^2\epsilon^{\tilde{n}_f}J_2^\diamond[\epsilon]\log\epsilon \quad \forall(\epsilon,\delta)\in]0,\epsilon_1[\times]0,\delta_1[\,.$$

Moreover,

$$J_1^\diamond[0,0] = c_\diamond m_n(Q) \qquad J_2^\diamond[0] = \frac{F(0)}{2\pi}m_n(Q)$$

Finally, if $\int_Q f_\epsilon\,dy = 0$ for all $\epsilon\in]-\epsilon_0,\epsilon_0[$, then we can take $J_2^\diamond$ equal to 0.

Proof. We first consider statement (i). Let $\epsilon^*_{\tilde\Omega}$, $\epsilon_{\tilde\Omega}$ be as in Lemma A.12 (i) of the Appendix. By Definition 4.12 of $\omega^\sharp(\epsilon,\delta,\cdot)$ and $u^\sharp(\epsilon,\delta,\cdot)$, and by (6.2), we have

$$\begin{aligned}
\omega^\sharp(\epsilon,\delta,x) &= \epsilon^{n-1}\int_{\partial\Omega} S_{q,n}(x-p-\epsilon s)\delta\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s\,,\\
&\qquad\qquad + C_\diamond[\epsilon,\delta] + \delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\\
u^\sharp(\epsilon,\delta,x) &= \omega^\sharp(\epsilon,\delta,x) + \delta^2\mathcal{P}_{\epsilon^*_{\tilde\Omega}}[\epsilon](x) \qquad \forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-\,,
\end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Thus we find natural to set

$$V_{\diamond,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}[\epsilon,\delta](x) \equiv \int_{\partial\Omega} S_{q,n}(x-p-\epsilon s)\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s \qquad \forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-\,,$$

for all $(\epsilon,\delta)\in]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[$. Now it suffices to show that the right hand side of the above definition defines a real analytic map from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$. Let V be an open bounded connected subset of $\mathbb{R}^n$ of class C^1 such that

$$\mathrm{cl}Q\subseteq V, \qquad \mathrm{cl}V\cap(qz+\mathrm{cl}\Omega_{p,\epsilon^*_{\tilde\Omega}}) = \emptyset \qquad \forall z\in\mathbb{Z}^n\setminus\{0\}\,,$$

Let $W\equiv V\setminus\mathrm{cl}\Omega_{p,\epsilon^*_{\tilde\Omega}}$. Since $S_{q,n}(x-p-\epsilon s)$ is analytic in (ϵ,x,s), a result on integral operators with real analytic kernels and with no singularity (cf. [33, Prop. 4.1 (i)]), and the analyticity of $\Theta_\diamond$ imply that the function from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[\times$ to $C^{m,\alpha}(\mathrm{cl}W)$ which takes (ϵ,δ) to the function

$$\int_{\partial\Omega} S_{q,n}(x-p-\epsilon s)\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s \qquad \forall x\in\mathrm{cl}W\,,$$

is real analytic. Since the restriction operator from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-$ to $\mathrm{cl}W$ induces an isomorphism from $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$ onto the subspace of $C^{m,\alpha}(\mathrm{cl}W)$ of the restrictions to $\mathrm{cl}W$ of q-periodic functions of $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-$, we conclude that the function from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$ which takes (ϵ,δ) to the above integral for $x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-$ is analytic. Also, equality (6.4) follows by the definition of $V_{\diamond,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}$ and by Theorem 4.6 (iii) and by the membership of $\theta_\diamond$ in $C^{m-1,\alpha}(\partial\Omega)_0$.

We now turn to prove statement (ii). By assumption, there exists $R>0$ such that $\mathrm{cl}\tilde\Omega\subseteq\mathbb{B}_n(0,R)$. Then we set $\Omega^*\equiv\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega$. Let $\epsilon_{\Omega^*,r}$ be as in Lemma A.12 (ii) of the Appendix with $\epsilon_1=\epsilon'$. Then we take $\epsilon_{\tilde\Omega,r}\equiv\epsilon_{\Omega^*,r}$.

It clearly suffices to show that $V^r_{\diamond,\Omega^*}$ and $\mathcal{P}^r_{\Omega^*}$ exist and are analytic and then to set $V^r_{\diamond,\tilde{\Omega}}$ and $\mathcal{P}^r_{\tilde{\Omega}}$ equal to the composition of the restriction of $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega})$ with $V^r_{\diamond,\Omega^*}$ and $\mathcal{P}^r_{\Omega^*}$, respectively. By definition of $\omega^\sharp(\epsilon,\delta,\cdot)$ and $u^\sharp(\epsilon,\delta,\cdot)$ and by equality (2.3), and by equality $\int_{\partial\Omega}\Theta_\diamond[\epsilon,\delta]\,d\sigma=0$, we have

$$
\begin{aligned}
\omega^\sharp(\epsilon,\delta,p+\epsilon t) &= \epsilon\delta\int_{\partial\Omega} S_n(t-s)\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s \\
&\quad+\int_{\partial\Omega}\epsilon^{n-1}\delta R_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s + C_\diamond[\epsilon,\delta]+\delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\,, \\
u^\sharp(\epsilon,\delta,p+\epsilon t) &= \omega^\sharp(\epsilon,\delta,p+\epsilon t) \\
&\quad+\delta^2\left[P_{q,n}[Q,f_\epsilon](p+\epsilon t)-\int_Q f_\epsilon\,dy S_{q,n}(\epsilon t)\right] \\
&= \epsilon\delta\int_{\partial\Omega} S_n(t-s)\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s+\int_{\partial\Omega}\epsilon^{n-1}\delta R_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s \\
&\quad+C_\diamond[\epsilon,\delta]+\delta_{2,n}\delta^2\epsilon^{\tilde{n}_f}F(\epsilon)\frac{\log\epsilon}{2\pi}+\delta^2\Bigg[P_{q,n}[Q,f_\epsilon](p+\epsilon t) \\
&\quad-\epsilon^{\tilde{n}_f}F(\epsilon)\left(\epsilon^{2-n}S_n(t)+\delta_{2,n}\frac{\log\epsilon}{2\pi}+R_{q,n}(\epsilon t)\right)\Bigg]\,,
\end{aligned}
\tag{6.7}
$$

for all $t\in\mathrm{cl}\Omega^*$ and for all $(\epsilon,\delta)\in]0,\epsilon_{\tilde{\Omega},r}[\times]0,\delta'[$. Thus we find natural to set

$$
\begin{aligned}
V^r_{\diamond,\Omega^*}[\epsilon,\delta] &\equiv \int_{\partial\Omega}S_n(t-s)\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s \\
&\qquad+\int_{\partial\Omega}\epsilon^{n-2}R_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s\,,
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
\mathcal{P}^r_{\Omega^*}[\epsilon](t) &\equiv P_{q,n}[Q,f_\epsilon](p+\epsilon t) \\
&\qquad-\epsilon^{\tilde{n}_f+2-n}F(\epsilon)S_n(t)-\epsilon^{\tilde{n}_f}F(\epsilon)R_{q,n}(\epsilon t)\,,
\end{aligned}
\tag{6.9}
$$

for all $t\in\mathrm{cl}\Omega^*$ and for all $(\epsilon,\delta)\in]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times]-\delta',\delta'[$. Now it suffices to prove that the right hand sides of (6.8), (6.9) define real analytic maps from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times]-\delta',\delta'[$ and from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$, respectively. Since $v[\partial\Omega,\cdot]_{|\mathrm{cl}\Omega^*}$ is linear and continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$, Theorem 4.6 (iii) implies the analyticity of the map which takes (ϵ,δ) to $v[\partial\Omega,\Theta_\diamond[\epsilon,\delta]]_{|\mathrm{cl}\Omega^*}$. Next we consider the second integral operator in the right hand side of (6.8). Since $\mathrm{cl}\Omega^*-\partial\Omega$ is compact, possibly shrinking $\epsilon_{\tilde{\Omega},r}$ we can assume that

$$
\epsilon(t-s)\in(\mathbb{R}^n\setminus q\mathbb{Z}^n)\cup\{0\}\qquad\forall(\epsilon,t,s)\in]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times\mathrm{cl}\Omega^*\times\partial\Omega\,.
\tag{6.10}
$$

Then we note that the maps from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[$ to $C^{m,\alpha}(\partial\Omega)$ and to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ which take ϵ to $\epsilon\mathrm{id}_{\partial\Omega}$ and to $\epsilon\mathrm{id}_{\mathrm{cl}\Omega^*}$ are real analytic, respectively. Then a result on integral operators with analytic kernels of [33, Prop. 4.1 (i)] implies that the map from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ which takes (ϵ,θ) to the function

$$
\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s\qquad\forall t\in\mathrm{cl}\Omega^*\,,
\tag{6.11}
$$

is analytic. Then we conclude that the map $V^r_{\diamond,\Omega^*}$ is analytic. We also note that $V^r_{\diamond,\Omega^*}[0,0]=v[\partial\Omega,\theta_\diamond]=u^\sharp_\diamond$ (cf. (4.4).) Then the analyticity of $R_{q,n}$ and analyticity results on the composition operator imply that the map from $]-\epsilon_0,\epsilon_0[$ to $C^{m-1,\alpha}(\partial\Omega)$ which takes ϵ to the function $R_{q,n}(\epsilon t)$ of $t\in\partial\Omega$ is real analytic (cf. Böhme and Tomi [3, p. 10], Henry [22, p. 29], Valent [48, Thm. 5.2, p. 44].) Then by Theorem [35, Prop. A.7 (ii)] and by the analyticity of F, it follows that $\mathcal{P}^r_{\Omega^*}$ is analytic. By setting $\epsilon=0$, we obtain the formula for $\mathcal{P}^r_{\Omega^*}[0]$.

Next we prove statement (iii). We define $V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t)$ and $\mathcal{P}^r_{\partial\Omega}[\epsilon](t)$ to be equal to the right hand side of (6.8) and (6.9) for all $t\in\partial\Omega$, respectively. Then equality (6.7) for $t\in\partial\Omega$ implies the validity of the equalities (6.5). The analyticity of $V^r_{\diamond,\partial\Omega}[\epsilon,\delta]$ and $\mathcal{P}^r_{\partial\Omega}[\epsilon]$ follows by the same arguments of the proof of statement (iii) with $\mathrm{cl}\Omega^*$ replaced by $\partial\Omega$. We also note that (6.10) with $\mathrm{cl}\Omega^*$ replaced by $\partial\Omega$ holds for all $\epsilon\in]-\epsilon_0,\epsilon_0[$. Indeed a point $p+\epsilon t$ with $t\in\partial\Omega$ can equal $p+\epsilon s+qz$ with $s\in\partial\Omega$ and $z\in\mathbb{Z}^n$ only if $z=0$.

Next we prove statement (iv). We first set

$$
\omega^\sharp_1(\epsilon,\delta,x)\equiv\epsilon^{n-1}\int_{\partial\Omega}S_{q,n}(x-p-\epsilon s)\delta\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s+C_\diamond[\epsilon,\delta]\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,
$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. By Definition 4.12 of $\omega^\sharp(\epsilon,\delta,\cdot)$, we have

$$
\omega^\sharp(\epsilon,\delta,x)=\omega^\sharp_1(\epsilon,\delta,x)+\delta_{2,n}\delta^2\int_Q f_\epsilon\,dy\frac{\log\epsilon}{2\pi}\qquad\forall x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,
$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$, and accordingly

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} u^\sharp(\epsilon,\delta,x)\,dx = \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} \omega_1^\sharp(\epsilon,\delta,x)\,dx$$
$$+\delta_{2,n}\delta^2\epsilon^{\tilde{n}_f}F(\epsilon)\frac{\log\epsilon}{2\pi}m_n(Q\setminus \mathrm{cl}\Omega_{p,\epsilon})+\delta^2\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} P_{q,n}[Q,f_\epsilon](x)$$
$$-\epsilon^{\tilde{n}_f}F(\epsilon)S_{q,n}(x-p)\,dx$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. By equality (6.5), the function $\omega_1^\sharp(\epsilon,\delta,\cdot)$ is the only solution in $C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{S}[\Omega_{p,\epsilon}]^-\,, \\ w \text{ is } q-\text{periodic in } \mathbb{S}[\Omega_{p,\epsilon}]^-\,, & \\ w(p+\epsilon t)=\epsilon\delta V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t)+C_\diamond[\epsilon,\delta] & \forall t\in\partial\Omega\,, \end{cases}$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Then Lemma A.4 of the Appendix and the limiting relation

$$\lim_{(\epsilon,\delta)\to(0,0)}\omega_1^\sharp(\epsilon,\delta,x)=C_\diamond[0,0]$$

for all $x\in\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$ and Corollary A.5 of the Appendix imply the existence of $\epsilon_1\in]0,\epsilon'[$ and $\delta_1\in]0,\delta'[$ and of an analytic map $\tilde{J}_1^\diamond$ from $]-\epsilon_1,\epsilon_1[\times]-\delta_1,\delta_1[$ to $\mathbb{R}$ such that

$$\begin{aligned} \tilde{J}_1^\diamond[\epsilon,\delta] &= \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}}\omega_1^\sharp(\epsilon,\delta,x)\,dx \qquad \forall(\epsilon,\delta)\in]0,\epsilon_1[\times]0,\delta_1[\,, \\ \tilde{J}_1^\diamond[0,0] &= C_\diamond[0,0]m_n(Q)=c_\diamond m_n(Q)\,. \end{aligned}$$

By assumption (1.3) and by [35, Prop. A.2], there exists $\rho'\in]0,\rho]$ such that the map from $]-\epsilon_0,\epsilon_0[$ to $C^0_{q,\omega,\rho'}(\mathbb{R}^n)$ which takes ϵ to $P_{q,n}[Q,f_\epsilon]$ is real analytic. Then Proposition A.8 (i) of the Appendix implies the existence of an analytic map $\tilde{J}_2^\diamond$ from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ such that

$$\begin{aligned} \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} P_{q,n}[Q,f_\epsilon]\,dx &= \tilde{J}_2^\diamond[\epsilon] \qquad \forall\epsilon\in]0,\epsilon_0[\,, \\ \tilde{J}_2^\diamond[0] &= \int_Q P_{q,n}[Q,f_0]\,dx\,. \end{aligned} \tag{6.12}$$

By Proposition A.8 (ii) of the Appendix, there exists an analytic map G_1 from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ such that

$$\begin{aligned} \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} S_{q,n}(x-p)\,dx &= G_1(\epsilon)-\delta_{2,n}\frac{\epsilon^2\log\epsilon}{2\pi}m_n(\Omega) \qquad \forall\epsilon\in]0,\epsilon_0[\,, \\ G_1(0) &= \int_Q S_{q,n}(x-p)\,dx\,. \end{aligned} \tag{6.13}$$

Hence, we conclude that

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} u^\sharp(\epsilon,\delta,x)\,dx = \tilde{J}_1^\diamond[\epsilon,\delta]+\delta^2\tilde{J}_2^\diamond[\epsilon]+\delta_{2,n}\delta^2\epsilon^{\tilde{n}_f}F(\epsilon)\frac{\log\epsilon}{2\pi}[m_n(Q)-\epsilon^n m_n(\Omega)]$$
$$-\delta^2\epsilon^{\tilde{n}_f}F(\epsilon)G_1(\epsilon)+\delta^2\epsilon^{\tilde{n}_f}F(\epsilon)\delta_{2,n}\frac{\epsilon^2\log\epsilon}{2\pi}m_n(\Omega)$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Thus if we set

$$\begin{aligned} J_1^\diamond[\epsilon,\delta] &\equiv \tilde{J}_1^\diamond[\epsilon,\delta]+\delta^2\tilde{J}_2^\diamond[\epsilon]-\delta^2\epsilon^{\tilde{n}_f}F(\epsilon)G_1(\epsilon)\,, \\ J_2^\diamond[\epsilon] &\equiv F(\epsilon)\left[\frac{\epsilon^2}{2\pi}m_n(\Omega)+(m_n(Q)-\epsilon^n m_n(\Omega))\frac{1}{2\pi}\right]\,, \end{aligned}$$

for all $(\epsilon,\delta)\in]-\epsilon_1,\epsilon_1[\times]-\delta_1,\delta_1[$, then statement (iv) holds true. □ □

Next we turn to introduce a representation theorem for the family of solutions $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ in case $n_f<n-1$.

Theorem 6.14. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon \in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $n_f < n-1$. Let $G \in C^0(\partial\Omega \times \mathbb{R})$ satisfy condition (4.7). Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ satisfy (1.8). Let $\epsilon' \in]0,\epsilon_0[$, be as in Theorem 5.5 (iii). Let Γ_0 be an open neighborhood of γ_0 in $\mathbb{R}$ as in Theorem 5.5 (iii). Let ρ', $\mathcal{P}_{\epsilon^*}$ be as in Lemma 6.1 for all $\epsilon^* \in]0,\epsilon_0[$. Let $\hat{\epsilon}$ be as in (1.7). Let $\delta' \in]0,+\infty[$ be as in (5.6). Then the following statements hold.*

(i) Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^n$ with nonzero distance from $p + q\mathbb{Z}^n$. Then there exist $\epsilon^*_{\tilde{\Omega}} \in]0,\epsilon'[$ such that

$$\mathrm{cl}\tilde{\Omega} \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall \epsilon \in [-\epsilon^*_{\tilde{\Omega}}, \epsilon^*_{\tilde{\Omega}}],$$

and $\epsilon_{\tilde{\Omega}} \in]0,\epsilon^*_{\tilde{\Omega}}[$ such that $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^- \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon \in [-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}]$, and $\delta_{\tilde{\Omega}} \in]0,\delta'[$ such that

$$\hat{\epsilon}(\delta) \in]0,\epsilon_{\tilde{\Omega}}[, \qquad \delta\hat{\epsilon}(\delta)^{n_f-(n-1)} \in \Gamma_0 \qquad \forall \delta \in]0,\delta_{\tilde{\Omega}}], \tag{6.15}$$

and a real analytic map $V_{*,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}$ from $]-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}[\times\Gamma_0$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-)$ such that

$$\begin{aligned}
\omega^\sharp(\delta,x) &= \hat{\epsilon}(\delta)^{n-1}\delta V_{*,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](x) \\
&\qquad + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2 \int_Q f_{\hat{\epsilon}(\delta)}\,dy \frac{\log\hat{\epsilon}(\delta)}{2\pi}, \\
u^\sharp(\delta,x) &= \hat{\epsilon}(\delta)^{n-1}\delta V_{*,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](x) \\
&\qquad + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2 \int_Q f_{\hat{\epsilon}(\delta)}\,dy \frac{\log\hat{\epsilon}(\delta)}{2\pi} \\
&\qquad + \delta^2\mathcal{P}_{\epsilon^*_{\tilde{\Omega}}}[\hat{\epsilon}(\delta)](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-,
\end{aligned}$$

for all $\delta \in]0,\delta_{\tilde{\Omega}}[$. Moreover,

$$V_{*,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[0,\gamma_0] = 0 \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-, \qquad C_*[0,\gamma_0] = c_*. \tag{6.16}$$

(ii) Let $\tilde{\Omega}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$. Then there exist $\epsilon_{\tilde{\Omega},r} \in]0,\epsilon'[$ and $\delta_{\tilde{\Omega},r} \in]0,\delta'[$ and a real analytic map $V^r_{*,\tilde{\Omega}}$ from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times\Gamma_0$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega})$ and a real analytic map $\mathcal{P}^r_{\tilde{\Omega}}$ from $]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega})$ such that

$$\begin{aligned}
&\hat{\epsilon}(\delta) \in]0,\epsilon_{\tilde{\Omega},r}[, \qquad \delta\hat{\epsilon}(\delta)^{n_f-(n-1)} \in \Gamma_0 \qquad \forall \delta \in]0,\delta_{\tilde{\Omega},r}], \\
&p + \hat{\epsilon}(\delta)\mathrm{cl}\tilde{\Omega} \subseteq \mathrm{cl}\mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta)}]^- \qquad \forall \delta \in]0,\delta_{\tilde{\Omega},r}[, \\
&\omega^\sharp(\delta,p+\hat{\epsilon}(\delta)t) = \hat{\epsilon}(\delta)\delta V^r_{*,\tilde{\Omega}}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](t) \\
&\qquad\qquad + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2 \int_Q f_{\hat{\epsilon}(\delta)}\,dy \frac{\log\hat{\epsilon}(\delta)}{2\pi} \qquad \forall t \in \mathrm{cl}\tilde{\Omega}, \\
&u^\sharp(\delta,p+\hat{\epsilon}(\delta)t) = \hat{\epsilon}(\delta)\delta V^r_{*,\tilde{\Omega}}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](t) \\
&\qquad\qquad + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta^2\mathcal{P}^r_{\tilde{\Omega}}[\hat{\epsilon}(\delta)](t) \qquad \forall t \in \mathrm{cl}\tilde{\Omega},
\end{aligned}$$

for all $\delta \in]0,\delta_{\tilde{\Omega},r}[$. Moreover,

$$V^r_{*,\tilde{\Omega}}[0,\gamma_0](t) = u^\sharp_*(t) \qquad \mathcal{P}^r_{\tilde{\Omega}}[0](t) = \int_Q S_{q,n}(p-y)f_0(y)\,dy \qquad \forall t \in \mathrm{cl}\tilde{\Omega}. \tag{6.17}$$

(iii) There exist a real analytic map $V^r_{*,\partial\Omega}$ from $]-\epsilon',\epsilon'[\times\Gamma_0$ to $C^{m,\alpha}(\partial\Omega)$ and a real analytic map $\mathcal{P}^r_{\partial\Omega}$ from $]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\partial\Omega)$ such that

$$\begin{aligned}
&\omega^\sharp(\delta,p+\hat{\epsilon}(\delta)t) = \hat{\epsilon}(\delta)\delta V^r_{*,\partial\Omega}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](t) \\
&\qquad\qquad + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2 \int_Q f_{\hat{\epsilon}(\delta)}\,dy \frac{\log\hat{\epsilon}(\delta)}{2\pi} \qquad \forall t \in \partial\Omega, \\
&u^\sharp(\delta,p+\hat{\epsilon}(\delta)t) = \hat{\epsilon}(\delta)\delta V^r_{*,\partial\Omega}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](t) \\
&\qquad\qquad + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta^2\mathcal{P}^r_{\partial\Omega}[\hat{\epsilon}(\delta)](t) \qquad \forall t \in \partial\Omega,
\end{aligned} \tag{6.18}$$

for all $\delta \in]0,\delta'[$. Moreover,

$$V^r_{*,\partial\Omega}[0,\gamma_0](t) = u^\sharp_*(t) \qquad \mathcal{P}^r_{\partial\Omega}[0](t) = \int_Q S_{q,n}(p-y)f_0(y)\,dy \qquad \forall t \in \partial\Omega.$$

(iv) There exist $\epsilon_1 \in]0, \epsilon'[$ and an open neighborhood Γ_1 of γ_0 contained in Γ_0 and two analytic maps J_1^*, J_2^* from $]-\epsilon_1, \epsilon_1[\times\Gamma_1$ to $\mathbb{R}$ and $\delta_1 \in]0, \delta'[$ such that

$$\hat{\epsilon}(\delta) \in]0, \epsilon_1[\,, \qquad \delta\hat{\epsilon}(\delta)^{n_f-(n-1)} \in \Gamma_1 \qquad \forall \delta \in]0, \delta_1[\,,$$

and

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)}} u^\sharp(\delta, x)\, dx = J_1^*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2\hat{\epsilon}(\delta)^{n_f} J_2^*[\hat{\epsilon}(\delta)]\log\hat{\epsilon}(\delta) \qquad \forall \delta \in]0, \delta_1[\,.$$

Moreover,

$$J_1^*[0, \gamma_0] = c_* m_n(Q) \qquad J_2^*[0] = \frac{F(0)}{2\pi} m_n(Q)$$

Finally, if $\int_Q f_\epsilon\, dy = 0$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$, then we can take J_2^* equal to 0.

Proof. Let $\epsilon^*_{\tilde{\Omega}}$, $\epsilon_{\tilde{\Omega}}$ be as in Lemma A.12 (i) of the Appendix. The existence of $\delta_{\tilde{\Omega}}$ as in (6.15) is an immediate consequence of the limiting relations in (1.7). By Definition 5.7 of $\omega^\sharp(\delta, \cdot)$, we have

$$\begin{aligned}\omega^\sharp(\delta, x) &= \hat{\epsilon}(\delta)^{n-1}\int_{\partial\Omega} S_{q,n}(x - p - \hat{\epsilon}(\delta)s)\delta\Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](s)\, d\sigma_s \\ &\quad + C_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2\int_Q f_{\hat{\epsilon}(\delta)}\, dy \frac{\log\hat{\epsilon}(\delta)}{2\pi} \\ u^\sharp(\delta, x) &= \omega^\sharp(\delta, x) + \delta^2\mathcal{P}_{\epsilon^*_{\tilde{\Omega}}}[\hat{\epsilon}(\delta)](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-\,,\end{aligned}$$

for all $\delta \in]0, \delta_{\tilde{\Omega}}[$. Thus we find natural to set

$$V_{*,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[\epsilon, \gamma](x) \equiv \int_{\partial\Omega} S_{q,n}(x - p - \epsilon s)\Theta_*[\epsilon, \gamma](s)\, d\sigma_s \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-\,,$$

for all $(\epsilon, \gamma) \in]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[\times\Gamma_0$. By the analyticity of Θ_* and by the very same argument of the proof of Theorem 6.3 (i), the map $V_{*,\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}$ is analytic. By Theorem 5.5 (iii) and by the membership of θ_* in $C^{m-1,\alpha}(\partial\Omega)_0$, we deduce the validity of (6.16).

We now turn to prove statement (ii). Then we take Ω^* and $\epsilon_{\tilde{\Omega},r} \equiv \epsilon_{\Omega^*,r}$ as in the proof of Theorem 6.3 (ii). The existence of $\delta_{\tilde{\Omega},r}$ is an immediate consequence of the limiting relations in (1.7). It clearly suffices to show that V^r_{*,Ω^*} and $\mathcal{P}^r_{\Omega^*}$ exist and are analytic and then to set $V^r_{*,\tilde{\Omega}}$ and $\mathcal{P}^r_{\tilde{\Omega}}$ equal to the composition of the restriction of $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega})$ with V^r_{*,Ω^*} and $\mathcal{P}^r_{\Omega^*}$, respectively. By definition of $\omega^\sharp(\delta, \cdot)$ and $u^\sharp(\delta, \cdot)$ and by equality (2.3), and by equality $\int_{\partial\Omega}\Theta_*[\epsilon, \delta]\, d\sigma = 0$ and by the same computations of (6.7), we have

$$\begin{aligned}\omega^\sharp(\delta, p + \hat{\epsilon}(\delta)t) &= \hat{\epsilon}(\delta)\delta\int_{\partial\Omega} S_n(t - s)\Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](s)\, d\sigma_s \\ &\quad + \int_{\partial\Omega}\hat{\epsilon}(\delta)^{n-1}\delta R_{q,n}(\hat{\epsilon}(\delta)(t - s))\Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](s)\, d\sigma_s \\ &\quad + C_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2\int_Q f_{\hat{\epsilon}(\delta)}\, dy\frac{\log\hat{\epsilon}(\delta)}{2\pi}\,, \\ u^\sharp(\delta, p + \hat{\epsilon}(\delta)t) & \\ &= \hat{\epsilon}(\delta)\delta\int_{\partial\Omega} S_n(t - s)\Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](s)\, d\sigma_s \\ &\quad + \int_{\partial\Omega}\hat{\epsilon}(\delta)^{n-1}\delta R_{q,n}(\hat{\epsilon}(\delta)(t - s))\Theta_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}](s)\, d\sigma_s \\ &\quad + C_*[\hat{\epsilon}(\delta), \delta\hat{\epsilon}(\delta)^{n_f-(n-1)}] + \delta_{2,n}\delta^2\hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))\frac{\log\hat{\epsilon}(\delta)}{2\pi} \\ &\quad + \delta^2\Bigg[P_{q,n}[Q, f_{\hat{\epsilon}(\delta)}](p + \hat{\epsilon}(\delta)t) \\ &\quad - \hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))\left(\hat{\epsilon}(\delta)^{2-n}S_n(t) + \delta_{2,n}\frac{\log\hat{\epsilon}(\delta)}{2\pi} + R_{q,n}(\hat{\epsilon}(\delta)t)\right)\Bigg]\,,\end{aligned} \tag{6.19}$$

for all $t \in \mathrm{cl}\Omega^*$ and for all $\delta \in]0, \delta_{\tilde{\Omega},r}[$. Thus we find natural to set

$$V^r_{*,\Omega^*}[\epsilon,\gamma] \equiv \int_{\partial\Omega} S_n(t-s)\Theta_*[\epsilon,\gamma](s)\,d\sigma_s \qquad (6.20)$$

$$+\int_{\partial\Omega} \epsilon^{n-2} R_{q,n}(\epsilon(t-s))\Theta_*[\epsilon,\gamma](s)\,d\sigma_s\,,$$

$$\mathcal{P}^r_{\Omega^*}[\epsilon](t) \equiv P_{q,n}[Q,f_\epsilon](p+\epsilon t) \qquad (6.21)$$

$$-\epsilon^{n_f+2-n}F(\epsilon)S_n(t) - \epsilon^{n_f}F(\epsilon)R_{q,n}(\epsilon t)\,,$$

for all $t \in \mathrm{cl}\Omega^*$ and for all $(\epsilon,\gamma) \in]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times\Gamma_0$. Now it suffices to prove that the right hand side of (6.20), (6.21) define real analytic maps from $]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times\Gamma_0$ and $]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$, respectively. We have already proved that $\mathcal{P}^r_{\Omega^*}$ is analytic and that the second equality in (6.17) holds (see proof of Theorem 6.3 (ii).) By arguing precisely as in the proof of Theorem 6.3 (ii), and by exploiting the analyticity of Θ_*, we can prove that V^r_{*,Ω^*} is analytic. We also observe that

$$V^r_{*,\Omega^*}[0,\gamma_0](t) = \int_{\partial\Omega} S_n(t-s)\Theta_*[0,\gamma_0](s)\,d\sigma_s = \int_{\partial\Omega} S_n(t-s)\theta_*(s)\,d\sigma_s = u^\sharp_*(t) \qquad \forall t \in \mathrm{cl}\Omega^*\,,$$

(cf. (5.4).) Next we prove statement (iii). We define $V^r_{*,\partial\Omega}[\epsilon,\gamma](t)$ and $\mathcal{P}^r_{\partial\Omega}[\epsilon](t)$ to be equal to the right hand side of (6.20) and (6.21) for all $t \in \partial\Omega$, respectively. Then equality (6.19) for $t \in \partial\Omega$ implies the validity of the equalities (6.18). The analyticity of $V^r_{*,\partial\Omega}[\epsilon,\gamma]$ and $\mathcal{P}^r_{\partial\Omega}[\epsilon]$ follows by the same arguments of the proof of statement (ii) with $\mathrm{cl}\Omega^*$ replaced by $\partial\Omega$. We also note that (6.11) with $\mathrm{cl}\Omega^*$ replaced by $\partial\Omega$ holds for all $\epsilon \in]-\epsilon_0,\epsilon_0[$. Indeed a point $p+\epsilon t$ with $t \in \partial\Omega$ can equal $p+\epsilon s+qz$ with $s \in \partial\Omega$ and $z \in \mathbb{Z}^n$ only if $z=0$.

Next we prove statement (iv). We first set

$$\omega^\sharp_1(\delta,x) \equiv \hat{\epsilon}(\delta)^{n-1}\int_{\partial\Omega} S_{q,n}(x-p-\hat{\epsilon}(\delta)s)\delta\Theta_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}](s)\,d\sigma_s$$

$$+C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}] \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta)}]^-\,,$$

for all $\delta \in]0,\delta'[$. By Definition 5.7 of $\omega^\sharp(\delta,\cdot)$, we have

$$\omega^\sharp(\delta,x) = \omega^\sharp_1(\delta,x) + \delta_{2,n}\delta^2\int_Q f_{\hat{\epsilon}(\delta)}\,dy\frac{\log\hat{\epsilon}(\delta)}{2\pi} \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta)}]^-\,,$$

for all $\delta \in]0,\delta'[$, and accordingly

$$\begin{aligned}\int_{Q\setminus\mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)}} u^\sharp(\delta,x)\,dx = &\int_{Q\setminus\mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)}} \omega^\sharp_1(\delta,x)\,dx\\ &+\delta_{2,n}\delta^2\hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))\frac{\log\hat{\epsilon}(\delta)}{2\pi}m_n(Q\setminus\mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)})\\ &+\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)}} P_{q,n}[Q,f_{\hat{\epsilon}(\delta)}](x)\\ &-\hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))S_{q,n}(x-p)\,dx\end{aligned}$$

for all $\delta \in]0,\delta'[$. By equality (6.18), the function $\omega^\sharp_1(\delta,\cdot)$ is the only solution in $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta)}]^-)$ of the Dirichlet problem

$$\begin{cases}\Delta w = 0 & \text{in } \mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta)}]^-\,,\\ w \text{ is } q-\text{periodic in } \mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta)}]^-\,,\\ w(p+\hat{\epsilon}(\delta)t) = \hat{\epsilon}(\delta)^{n-n_f}\delta\hat{\epsilon}(\delta)^{n_f-n+1}V^r_{*,\partial\Omega}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}](t) & \\ \qquad +C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}] & \forall t \in \partial\Omega\,,\end{cases}$$

for all $\delta \in]0,\delta'[$. By Theorems 5.5 (iii) and 6.3 (iii), the function $\epsilon^{n-n_f}\gamma V^r_{*,\partial\Omega}[\epsilon,\gamma] + C_*[\epsilon,\gamma]$ is real analytic in $(\epsilon,\gamma) \in]-\epsilon',\epsilon'[\times\Gamma_0$. Moreover, such a function equals $\hat{\epsilon}(\delta)^{n-n_f}\delta\hat{\epsilon}(\delta)^{n_f-n+1}V^r_{*,\partial\Omega}[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}] + C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}]$ if we set $\epsilon = \hat{\epsilon}(\delta)$, $\gamma = \delta\hat{\epsilon}(\delta)^{n_f-n+1}$ for $\delta \in]0,\delta'[$. Then Lemma A.4 of the Appendix and the limiting relation

$$\lim_{\delta\to 0}\omega^\sharp_1(\delta,x) = C_*[0,\gamma_0] \qquad \forall x \in \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$$

and Corollary A.5 of the Appendix imply the existence of $\epsilon_1 \in]0,\epsilon'[$ and of an open neighborhood Γ_1 of γ_0 in $\mathbb{R}$ contained in Γ_0 and of an analytic map $\tilde{J}_1^*$ from $]-\epsilon_1,\epsilon_1[\times\Gamma_1$ to $\mathbb{R}$ and of $\delta_1 \in]0,\delta'[$ such that

$$
\begin{aligned}
\hat{\epsilon}(\delta) \in]0,\epsilon_1[\,, \qquad & \delta\hat{\epsilon}(\delta)^{n_f-(n-1)} \in \Gamma_1 \qquad \forall \delta \in]0,\delta_1]\,,\\
\tilde{J}_1^*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}] &= \int_{Q\setminus \mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)}} \omega_1^\sharp(\delta,x)\,dx \qquad \forall \delta \in]0,\delta_1[\,,\\
\tilde{J}_1^*[0,\gamma_0] &= C_*[0,\gamma_0]m_n(Q) = c_* m_n(Q)\,.
\end{aligned}
$$

Next we define $\tilde{J}_2^* \equiv \tilde{J}_2^\diamond$ as in (6.12) and and G_1 as in (6.13). Hence, we conclude that

$$
\begin{aligned}
\int_{Q\setminus \mathrm{cl}\Omega_{p,\hat{\epsilon}(\delta)}} u^\sharp(\delta,x)\,dx = {}& \tilde{J}_1^*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}] + \delta^2\tilde{J}_2^*[\hat{\epsilon}(\delta)]\\
&+\delta_{2,n}\delta^2\hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))\frac{\log\hat{\epsilon}(\delta)}{2\pi}[m_n(Q)-\hat{\epsilon}(\delta)^n m_n(\Omega)]\\
&-\delta^2\hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))G_1(\hat{\epsilon}(\delta)) + \delta^2\hat{\epsilon}(\delta)^{n_f}F(\hat{\epsilon}(\delta))\delta_{2,n}\frac{\hat{\epsilon}(\delta)^2\log\hat{\epsilon}(\delta)}{2\pi}m_n(\Omega)
\end{aligned}
$$

for all $\delta \in]0,\delta_1[$. Thus if we set

$$
\begin{aligned}
J_1^*[\epsilon,\gamma] &\equiv \tilde{J}_1^*[\epsilon,\gamma] + (\gamma\epsilon^{(n-1)-n_f})^2\tilde{J}_2^*[\epsilon] - (\gamma\epsilon^{(n-1)-n_f})^2\epsilon^{n_f}F(\epsilon)G_1(\epsilon)\,,\\
J_2^*[\epsilon] &\equiv F(\epsilon)\left[\frac{\epsilon^2}{2\pi}m_n(\Omega) + (m_n(Q)-\epsilon^n m_n(\Omega))\frac{1}{2\pi}\right]\,,
\end{aligned}
$$

for all $(\epsilon,\gamma) \in]-\epsilon_1,\epsilon_1[\times\Gamma_1$, then statement (iv) holds true. □ □

7 A convergence result for the solutions of the auxiliary problem (3.1)

We now plan to analyze the behavior of the family $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and of the family $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ in a Lebesgue space as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0, respectively. The difficulty here is that the domain $\mathrm{cl}\mathbb{S}(\epsilon,1)^-$ of $u^\sharp(\epsilon,\delta,\cdot)$ depends on ϵ and that the domain $\mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),1)^-$ of $u^\sharp(\delta,\cdot)$ depends on δ. Then we extend such functions 'by zero' to the whole of $\mathbb{R}^n$, and we analyze the behavior of the extensions as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0, respectively. Thus if v is a function from $\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-$ to $\mathbb{R}$, we denote by $\mathbf{E}_{(\epsilon,\delta)}[v]$ the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$
\mathbf{E}_{(\epsilon,\delta)}[v](x) \equiv \begin{cases} v(x) & \forall x \in \mathrm{cl}\mathbb{S}(\epsilon,\delta)^-\,,\\ 0 & \forall x \in \mathbb{R}^n\setminus\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-\,. \end{cases}
$$

Then we have the following.

Proposition 7.1. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega\times\mathbb{R})$ satisfy condition (4.7). Let $r \in [1,+\infty[$. Then the following statements hold.*

(i) Let $n_f \geq n-1$. Let $c_\diamond \in \mathbb{R}$ be such that (1.6) holds. Let ϵ', δ' be as in Theorem 4.6 (ii). Let $\{(\varepsilon_j,\delta_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon'[\times]0,\delta'[$ which converges to $(0,0)$. Then

$$
\lim_{j\to\infty}\mathbf{E}_{(\varepsilon_j,1)}[u^\sharp(\varepsilon_j,\delta_j,\cdot)] = c_\diamond \qquad \text{in } L^r(V)\,,
$$

for all bounded open subsets V of $\mathbb{R}^n$.

(ii) Let $n_f < n-1$. Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ satisfy (1.8). Let $\hat{\epsilon}$ be as in (1.7). Let δ' be as in (5.6). Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\delta'[$ which converges to 0. Then

$$
\lim_{j\to\infty}\mathbf{E}_{(\hat{\epsilon}(\delta_j),1)}[u^\sharp(\delta_j,\cdot)] = c_* \qquad \text{in } L^r(V)\,,
$$

for all bounded open subsets V of $\mathbb{R}^n$.

Proof. We first show that

$$\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |u^\sharp(\varepsilon_j,\delta_j,x)| < +\infty\,. \tag{7.2}$$

By Definition 4.12, it suffices to show that

$$\sup_{j\in\mathbb{N}}\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |\omega^\sharp(\varepsilon_j,\delta_j,x)| < +\infty\,, \tag{7.3}$$

$$\sup_{j\in\mathbb{N}}\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} \delta_j^2\left|\int_Q S_{q,n}(x-y)f_{\varepsilon_j}(y)\,dy - \int_Q f_{\varepsilon_j}\,dy S_{q,n}(x-p)\right| < +\infty\,. \tag{7.4}$$

We first prove (7.3). Since $\omega^\sharp[\varepsilon_j,\delta_j,\Theta_\diamond[\varepsilon_j,\delta_j],C_\diamond[\varepsilon_j,\delta_j]]$ is harmonic and q-periodic, the Maximum Principle implies that

$$\sup_{j\in\mathbb{N}}\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |\omega^\sharp(\varepsilon_j,\delta_j,x)| \le \sup_{j\in\mathbb{N}}\sup_{x\in \partial\Omega_{p,\varepsilon_j}} |\omega^\sharp(\varepsilon_j,\delta_j,x)|\,.$$

Then equality (4.5) and Theorem 6.3 (iii) imply that

$$\begin{aligned}&\sup_{j\in\mathbb{N}}\sup_{x\in \partial\Omega_{p,\varepsilon_j}} |\omega^\sharp(\varepsilon_j,\delta_j,x)| \\ &\qquad\le \sup_{j\in\mathbb{N}}\sup_{t\in\partial\Omega}\left|\varepsilon_j\delta_j V^r_{\diamond,\partial\Omega}[\varepsilon_j,\delta_j](t) + C_\diamond[\varepsilon_j,\delta_j] + \delta_{2,n}\delta_j^2\varepsilon_j^{\tilde{n}_f}F(\varepsilon_j)\frac{\log\varepsilon_j}{2\pi}\right| < +\infty\,,\end{aligned}$$

and accordingly inequality (7.3) holds true. Indeed, $V^r_{\diamond,\partial\Omega}$ and $C_\diamond$ are continuous at $(0,0)$ and F is continuous at 0. Next we consider inequality (7.4). By assumption (1.3) and [35, Lem. A.2], there exists $\rho' \in]0,\rho]$ such that the map from $]-\epsilon_0,\epsilon_0[$ to $C^0_{q,\omega,\rho'}(\mathbb{R}^n)$, which takes ϵ to $P_{q,n}[Q,f_\epsilon]$ is analytic. Since $C^0_{q,\omega,\rho'}(\mathbb{R}^n)$ is continuously imbedded into $C^0_b(\mathbb{R}^n)$, we have

$$\sup_{j\in\mathbb{N}}\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-}\left|\delta_j^2\int_Q S_{q,n}(x-y)f_{\varepsilon_j}(y)\,dy\right| \le \sup_{j\in\mathbb{N}}\sup_{x\in\mathbb{R}^n}\left|\delta_j^2 P_{q,n}[Q,f_{\varepsilon_j}](x)\right| < +\infty\,. \tag{7.5}$$

Then equality (4.5) implies that

$$\begin{aligned}&\sup_{j\in\mathbb{N}}\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-}\left|\delta_j^2\int_Q f_{\varepsilon_j}\,dy S_{q,n}(x-p)\right| \\ &\qquad\le \sup_{j\in\mathbb{N}}\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-}\left|\delta_j^2\varepsilon_j^{\tilde{n}_f}F(\varepsilon_j)S_{q,n}(x-p)\right|\,.\end{aligned} \tag{7.6}$$

Since $-\Delta_x S_{q,n}(x-p) = \frac{1}{m_n(Q)} > 0$ for all $x\in\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$, the function $S_{q,n}(\cdot-p)$ is super-harmonic and satisfies the strong Minimum Principle in $\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$. Accordingly,

$$\begin{aligned}\min_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-}\varepsilon_j^{\tilde{n}_f}S_{q,n}(x-p) &= \min_{t\in\partial\Omega}\varepsilon_j^{\tilde{n}_f}S_{q,n}(p+t\varepsilon_j-p)\\ &= \min_{t\in\partial\Omega}\left(\varepsilon_j^{\tilde{n}_f}\varepsilon_j^{2-n}S_n(t) + \varepsilon_j^{\tilde{n}_f}\delta_{2,n}\frac{\log\varepsilon_j}{2\pi} + \varepsilon_j^{\tilde{n}_f}R_{q,n}(\varepsilon_j t)\right) \qquad \forall j\in\mathbb{N}\,.\end{aligned}$$

Since $R_{q,n}$ is continuous at 0 and $\tilde{n}_f \ge n-1$, we conclude that there exists $M_1\in\mathbb{R}$ such that

$$\min_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-}\varepsilon_j^{\tilde{n}_f}S_{q,n}(x-p) \ge M_1 \qquad \forall j\in\mathbb{N}\,. \tag{7.7}$$

On the other hand we know that $S_{q,n}(\cdot-p)$ is continuous and q-periodic in $\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$ and that

$$\lim_{x\to p}S_{q,n}(x-p) = \lim_{x\to p}\left(S_n(x-p)+R_{q,n}(x-p)\right) = -\infty\,.$$

Accordingly, $S_{q,n}(\cdot-p)$ is bounded from above and there exists $M_2\in]0,+\infty[$ such that

$$\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-}\varepsilon_j^{\tilde{n}_f}S_{q,n}(x-p) \le M_2 \qquad \forall j\in\mathbb{N}\,. \tag{7.8}$$

By combining (7.5), (7.6), (7.7) and (7.8) and the continuity of F at 0, we conclude that (7.4) holds true. Hence, (7.2) is also true. Now Theorem 6.3 (i) implies that

$$\lim_{j\to\infty}\mathbf{E}_{(\varepsilon_j,1)}[u^\sharp(\varepsilon_j,\delta_j,\cdot)] = c_\diamond \qquad \text{a.e. in } \mathbb{R}^n\,.$$

Hence, the Dominated Convergence Theorem in the set Q and the q-periodicity of $\mathbf{E}_{(\varepsilon_j,1)}[u^\sharp(\varepsilon_j,\delta_j,\cdot)]$ imply the validity of statement (i). Similarly, in order to prove statement (ii), we prove that

$$\sup_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\hat{\epsilon}(\delta_j)}]^-} |u^\sharp(\delta_j,x)| < +\infty\,, \tag{7.9}$$

and we invoke the Dominated Convergence Theorem in the set Q. As above, inequality (7.9) follows by inequalities (7.3) and (7.4) with ε_j and $\omega^\sharp(\varepsilon_j,\delta_j,x)$ replaced by $\hat{\epsilon}(\delta_j)$, $\omega^\sharp(\delta_j,x)$, respectively. We can prove such inequalities by replacing $C_\diamond[\varepsilon_j,\delta_j]$, $V^r_{\diamond,\partial\Omega}[\varepsilon_j,\delta_j]$ by $C_*[\hat{\epsilon}(\delta_j),\delta_j\hat{\epsilon}(\delta_j)^{n_f-(n-1)}]$, $V^r_{*,\partial\Omega}[\hat{\epsilon}(\delta_j),\delta_j\hat{\epsilon}(\delta_j)^{n_f-(n-1)}]$, respectively, and by exploiting the same argument of the proof of statement (i) and Theorems 5.5, 6.14 (iii) instead of Theorems 4.6, 6.3 (iii). □ □

8 A convergence result for the solutions of problem (1.4)

As we have already said at the beginning of section 3, a function $u \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-)$ satisfies problem (1.4) if and only if the function $u^\sharp(\cdot) = u(\delta\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1)^-)$, satisfies the auxiliary boundary value problem (3.1). Thus we can now introduce a family of solutions for problem (1.4) by means of the following.

Definition 8.1.

(i) Let the assumptions of Theorem 4.6 hold. Then we set

$$\begin{aligned} \omega(\epsilon,\delta,x) &\equiv \omega^\sharp(\epsilon,\delta,x/\delta) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon,\delta)^-\,,\\ u(\epsilon,\delta,x) &\equiv u^\sharp(\epsilon,\delta,x/\delta) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon,\delta)^-\,,\end{aligned}$$

for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$.

(ii) Let the assumptions of Theorem 5.5 hold. Let $\delta' \in]0,+\infty[$ be as in (5.6). Then we set

$$\begin{aligned} \omega(\delta,x) &\equiv \omega^\sharp(\delta,x/\delta) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),\delta)^-\,,\\ u(\delta,x) &\equiv u^\sharp(\delta,x/\delta) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),\delta)^-\,,\end{aligned}$$

for all $\delta \in]0,\delta'[$.

By Theorems 4.6 and 5.5, $\{u(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ is a family of solutions of problem (1.4) in case $n_f \geq n-1$, and $\{u(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ is a family of solutions of problem (1.4) in case $n_f < n-1$. Our aim is to analyze the behavior of such families as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0, respectively. To do so, we exploit the results on the families of solutions of the auxiliary problem (3.1), which we have introduced in Definitions 4.12 and 5.7.

In particular, we note that Theorem 6.3 (i), (ii) implies that the family of solutions $\{u(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ satisfies conditions $(\diamond)$, $(\diamond\diamond)$ of the introduction, and that the limit of $(\diamond)$ is precisely the constant $c_\diamond$. Similarly, Theorem 6.14 (i), (ii) implies that the family of solutions $\{u(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ satisfies conditions $(*)$, $(**)$ of the introduction, and that the limit of $(*)$ is precisely the constant c_*.

Next we show the validity of the following convergence theorem in Lebesgue spaces.

Proposition 8.2. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega\times\mathbb{R})$ satisfy condition (4.7). Let $r \in [1,+\infty[$. Then the following statements hold.*

(i) Let $n_f \geq n-1$. Let $c_\diamond \in \mathbb{R}$ be such that (1.6) holds. Let ϵ', δ' be as in Theorem 4.6 (iii). Let $\{(\varepsilon_j,\delta_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon'[\times]0,\delta'[$ which converges to $(0,0)$. Then

$$\lim_{j\to\infty} \mathbf{E}_{(\varepsilon_j,\delta_j)}[u(\varepsilon_j,\delta_j,\cdot)] = c_\diamond \qquad \text{in } L^r(V)\,,$$

for all bounded open subsets V of $\mathbb{R}^n$.

(ii) Let $n_f < n-1$. Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ satisfy (1.8). Let $\hat{\epsilon}$ be as in (1.7). Let δ', be as in (5.6). Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\delta'[$ which converges to 0. Then

$$\lim_{j\to\infty} \mathbf{E}_{(\hat{\epsilon}(\delta_j),\delta_j)}[u(\delta_j,\cdot)] = c_* \qquad \text{in } L^r(V)\,,$$

for all bounded open subsets V of $\mathbb{R}^n$.

Proof. By Lemma A.10 of the Appendix, there exists $C \in]0, +\infty[$ such that

$$\begin{aligned}\|\mathbf{E}_{(\varepsilon_j,\delta_j)}[u(\varepsilon_j,\delta_j,\cdot)] - c_\diamond\|_{L^r(V)} &= \|\mathbf{E}_{(\varepsilon_j,\delta_j)}[u^\sharp(\varepsilon_j,\delta_j,\cdot/\delta)] - c_\diamond\|_{L^r(V)} \\ &= \|\mathbf{E}_{(\varepsilon_j,1)}[u^\sharp(\varepsilon_j,\delta_j,\cdot)](\cdot/\delta) - c_\diamond\|_{L^r(V)} \\ &\leq C\|\mathbf{E}_{(\varepsilon_j,1)}[u^\sharp(\varepsilon_j,\delta_j,\cdot)] - c_\diamond\|_{L^r(Q)} \quad \forall j \in \mathbb{N}\,.\end{aligned}$$

Then the statement follows by Theorem 7.1 (i). The proof of statement (ii) follows the same lines by replacing Theorem 7.1 (i) with Theorem 7.1 (ii). □ □

The result above is akin to those obtained by variational methods, although here the methods are completely different. We now exploit our methods to describe the convergence of the families of solutions $\{u(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and $\{u(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0, respectively, in the spirit of the present paper.

We first note that if $\tilde{\Omega}$ is a nonempty open subset of $\mathbb{R}^n$, then

$$\tilde{\Omega} \cap \left(\mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}(\epsilon,\delta)^-\right) \neq \emptyset\,,$$

whenever (ϵ,δ) is sufficiently close to $(0,0)$. Hence, $u(\epsilon,\delta,\cdot)$ is not defined in the whole of $\tilde{\Omega}$ for (ϵ,δ) sufficiently close to $(0,0)$, and we cannot hope to describe the behavior of $u(\epsilon,\delta,\cdot)$ as we did for $u^\sharp(\epsilon,\delta,\cdot)$ in Theorem 6.3. Similarly, $u(\delta,\cdot)$ is not defined in the whole of $\tilde{\Omega}$ for δ sufficiently close to 0, and we cannot hope to describe the behavior of $u(\delta,\cdot)$ as we did for $u^\sharp(\delta,\cdot)$ in Theorem 6.14. Hence, we must resort to a different avenue.

We first fix $r \in [1,+\infty[$ and we identify $\mathbf{E}_{(\epsilon,\delta)}[u(\epsilon,\delta,\cdot)]$ and $\mathbf{E}_{(\hat{\epsilon}(\delta),\delta)}[u(\delta,\cdot)]$ with the corresponding functionals in the dual of the space of functions of $L^{r'}(\mathbb{R}^n)$ with compact support, where r' is the conjugate exponent to r, and we would like to describe the 'weak' behavior of $\mathbf{E}_{(\epsilon,\delta)}[u(\epsilon,\delta,\cdot)]$ as (ϵ,δ) tends to $(0,0)$ and of $\mathbf{E}_{(\hat{\epsilon}(\delta),\delta)}[u(\delta,\cdot)]$ as δ tends to 0 in terms of analytic maps. More precisely, we would like to describe the behavior of the integrals

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,\delta)}[u(\epsilon,\delta,\cdot)]\phi\,dx\,, \qquad \int_{\mathbb{R}^n} \mathbf{E}_{(\hat{\epsilon}(\delta),\delta)}[u(\delta,\cdot)]\phi\,dx \tag{8.3}$$

as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0 in terms of analytic maps, for all elements ϕ with compact support of $L^{r'}(\mathbb{R}^n)$. At the moment however, we cannot do so for all elements ϕ with compact support of $L^{r'}(\mathbb{R}^n)$, but only for all the elements ϕ which belong to a certain dense subspace $\mathcal{T}_q$ of $L^{r'}(\mathbb{R}^n)$ of functions with compact support, which we now turn to introduce by means of the following technical statement of [36, § 9]

Proposition 8.4. *Let $\mathcal{T}_q$ be the vector subspace of $L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ generated by the set of functions*

$$\{\chi_{sQ+y} : (s,y) \in (\mathbb{Q}\cap]0,+\infty[) \times \mathbb{R}^n\}\,.$$

(i) If $r \in [1,+\infty[$, then the space $\mathcal{T}_q$ is dense in $L^r(\mathbb{R}^n)$.

(ii) If $\phi \in \mathcal{T}_q$, then there exist $y_1,\ldots,y_r \in \mathbb{R}^n$, and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, and $s \in \mathbb{Q}\cap]0,+\infty[$ such that

$$\phi(x) = \sum_{l=1}^{r} \lambda_l \chi_{y_l+sQ}(x) \qquad \text{a.a. } x \in \mathbb{R}^n\,.$$

Next we turn to analyze the behavior of the integrals in (8.3) with $\phi \in \mathcal{T}_q$ as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0, respectively. In the spirit of this paper, we represent the integrals in (8.3) in terms of analytic maps evaluated at specific values of (ϵ,δ) and δ, respectively. Namely, at values of δ such that the periodic cell δQ is a certain integer fraction of the periodicity cell Q. More precisely, we require that δ equals the reciprocal of some integer $l \in \mathbb{N}\setminus\{0\}$. To do so, we first introduce the following technical statement.

Theorem 8.5. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega\times\mathbb{R})$ satisfy condition (4.7). Let $r \in [1,+\infty[$. Then the following statements hold.*

(i) Let $n_f \geq n-1$. Let $c_\diamond \in \mathbb{R}$ be such that (1.6) holds. Let ϵ', δ' be as in Theorem 4.6 (iii). Then we have the following.

(j_1) *Let* $s \in]0,+\infty[$. *Let* $\tilde{y} \in \mathbb{R}^n$. *Let* ϵ_1, δ_1, $J_1^\diamond$, $J_2^\diamond$ *be as in Theorem 6.3 (iv). Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,l^{-1}s)}[u(\epsilon,l^{-1}s,\cdot)]\chi_{\tilde{y}+sQ}\,dx$$
$$= s^n J_1^\diamond[\epsilon,l^{-1}s] + \delta_{2,n}s^n(l^{-1}s)^2\epsilon^{\tilde{n}_f}J_2^\diamond[\epsilon]\log\epsilon\,,$$

for all $l \in \mathbb{N}\setminus\{0\}$ *such that* $l > s/\delta_1$ *and for all* $\epsilon \in]0,\epsilon_1[$. *Moreover,*

$$s^n J_1^\diamond[0,0] = \int_{\mathbb{R}^n} c_\diamond \chi_{\tilde{y}+sQ}\,dx\,.$$

Finally, if $\int_Q f_\epsilon\,dy = 0$ *for all* $\epsilon \in]-\epsilon_0,\epsilon_0[$, *then we can take* $J_2^\diamond$ *equal to* 0.

(j_1j_1) *Let* $\phi \in \mathcal{T}_q$. *Let* $s \in \mathbb{Q}\cap]0,+\infty[$ *be as in Proposition 8.4 (ii). Then there exist real analytic maps* $H_{\diamond,\phi}$ *from* $]-\epsilon_1,\epsilon_1[\times]-\delta_1,\delta_1[$ *to* $\mathbb{R}$ *and* $\tilde{H}_{\diamond,\phi}$ *from* $]-\epsilon_1,\epsilon_1[$ *to* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon_j,l^{-1}s)}[u(\epsilon,l^{-1}s,\cdot)]\phi\,dx$$
$$= s^n H_{\diamond,\phi}[\epsilon,l^{-1}s] + \delta_{2,n}s^n(l^{-1}s)^2\epsilon^{\tilde{n}_f}\tilde{H}_{\diamond,\phi}[\epsilon]\log\epsilon$$

for all $l \in \mathbb{N}\setminus\{0\}$ *such that* $l > s/\delta_1$ *and for all* $\epsilon \in]0,\epsilon_1[$. *Moreover,*

$$s^n H_{\diamond,\phi}[0,0] = \int_{\mathbb{R}^n} c_\diamond \phi\,dx\,.$$

Finally, if $\int_Q f_\epsilon\,dy = 0$ *for all* $\epsilon \in]-\epsilon_0,\epsilon_0[$, *then we can take* $\tilde{H}_{\diamond,\phi}$ *equal to* 0.

(ii) *Let* $n_f < n-1$. *Let* $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ *satisfy (1.8). Let* ϵ', Γ_0 *be as in Theorem 5.5 (iii). Let* $\hat{\epsilon}$ *be as in (1.7). Let* δ' *be as in (5.6). Then we have the following.*

(j_2) *Let* $s \in]0,+\infty[$. *Let* $\tilde{y} \in \mathbb{R}^n$. *Let* δ_1, ϵ_1, Γ_1, J_1^*, J_2^* *be as in Theorem 6.14 (iv). Then*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\hat{\epsilon}(l^{-1}s),l^{-1}s)}[u(l^{-1}s,\cdot)]\chi_{\tilde{y}+sQ}\,dx$$
$$= s^n J_1^*[\hat{\epsilon}(l^{-1}s), l^{-1}s\hat{\epsilon}(l^{-1}s)^{n_f-(n-1)}]$$
$$+\delta_{2,n}s^n(l^{-1}s)^2\hat{\epsilon}(l^{-1}s)^{n_f}\log\hat{\epsilon}(l^{-1}s)J_2^*[\hat{\epsilon}(l^{-1}s)]\,,$$

for all $l \in \mathbb{N}\setminus\{0\}$ *such that* $l > s/\delta'$. *Moreover,*

$$s^n J_1^*[0,\gamma_0] = \int_{\mathbb{R}^n} c_* \chi_{\tilde{y}+sQ}\,dx\,.$$

Finally, if $\int_Q f_\epsilon\,dy = 0$ *for all* $\epsilon \in]-\epsilon_0,\epsilon_0[$, *then we can take* J_2^* *equal to* 0.

(j_2j_2) *Let* $\phi \in \mathcal{T}_q$. *Let* $s \in \mathbb{Q}\cap]0,+\infty[$ *be as in Proposition 8.4 (ii). Then there exist real analytic maps* $H_{*,\phi}$ *from* $]-\epsilon_1,\epsilon_1[\times\Gamma_1$ *to* $\mathbb{R}$ *and* $\tilde{H}_{*,\phi}$ *from* $]-\epsilon_1,\epsilon_1[$ *to* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\hat{\epsilon}(l^{-1}s),l^{-1}s)}[u(\hat{\epsilon}(l^{-1}s),l^{-1}s,\cdot)]\phi\,dx$$
$$= s^n H_{*,\phi}[\hat{\epsilon}(l^{-1}s), l^{-1}s\hat{\epsilon}(l^{-1}s)^{n_f-(n-1)}]$$
$$+\delta_{2,n}s^n(l^{-1}s)^2\hat{\epsilon}(l^{-1}s)^{n_f}\tilde{H}_{*,\phi}[\hat{\epsilon}(l^{-1}s)]\log\hat{\epsilon}(l^{-1}s)\,,$$

for all $l \in \mathbb{N}\setminus\{0\}$ *such that* $l > s/\delta'$. *Moreover,*

$$s^n H_{*,\phi}[0,\gamma_0] = \int_{\mathbb{R}^n} c_* \phi\,dx\,.$$

Finally, if $\int_Q f_\epsilon\,dy = 0$ *for all* $\epsilon \in]-\epsilon_0,\epsilon_0[$, *then we can take* $\tilde{H}_{*,\phi}$ *equal to* 0.

Proof. By Lemma A.11 of the Appendix and by Theorem 6.3 (iv), we have

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,l^{-1}s)}[u(\epsilon,l^{-1}s,\cdot)]\chi_{\tilde{y}+sQ}\,dx$$
$$= \int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,1)}[u^\sharp(\epsilon,l^{-1}s,\cdot)](x/(l^{-1}s))\chi_{\tilde{y}+sQ}(x)\,dx$$
$$= s^n\int_Q \mathbf{E}_{(\epsilon,1)}[u^\sharp(\epsilon,l^{-1}s,\cdot)]\,dx = s^n\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} u^\sharp(\epsilon,l^{-1}s,x)\,dx$$
$$= s^n J_1^\diamond[\epsilon,l^{-1}s] + s^n\delta_{2,n}l^{-2}s^2\epsilon^{\tilde{n}_f}J_2^\diamond[\epsilon]\log\epsilon$$

for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > s/\delta'$ and for all $\epsilon \in]0, \epsilon'[$. Hence, (j_1) holds true.

Similarly, we now prove (j_2). By Lemma A.11 and by Theorem 6.14 (iv), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \mathbf{E}_{(\hat{\epsilon}(l^{-1}s), l^{-1}s)}[u(l^{-1}s, \cdot)] \chi_{\tilde{y}+sQ}\, dx \\
&\qquad = s^n \int_Q \mathbf{E}_{(\hat{\epsilon}(l^{-1}s),1)}[u^\sharp(l^{-1}s, \cdot)]\, dx = s^n \int_{Q \setminus \mathrm{cl}\Omega_{p,\hat{\epsilon}(l^{-1}s)}} u^\sharp(l^{-1}s, x)\, dx \\
&\qquad = s^n J_1^*[\hat{\epsilon}(l^{-1}s), l^{-1}s\hat{\epsilon}(l^{-1}s)^{n_f-(n-1)}] \\
&\qquad\quad + s^n \delta_{2,n} l^{-2} s^2 \hat{\epsilon}(l^{-1}s)^{n_f} J_2^*[\hat{\epsilon}(l^{-1}s)] \log \hat{\epsilon}(l^{-1}s)
\end{aligned}
$$

for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > s/\delta'$. Hence, (j_2) holds true. Since ϕ is a finite linear combination of translations of functions such as $\chi_{\tilde{y}+sQ}$, statements $(j_1 j_1)$ and $(j_2 j_2)$ are an immediate consequence of statements (j_1) and (j_2), respectively. □ □

A Appendix

We first introduce the following variant of a result of Preciso [47, Prop. 1.1, p. 101].

Proposition A.1. *Let n_1, $n_2 \in \mathbb{N} \setminus \{0\}$, $\rho \in]0, +\infty[$, $m \in \mathbb{N}$, $\alpha \in]0, 1]$. Let Ω_1 be a bounded open subset of $\mathbb{R}^{n_1}$. Let Ω_2 be a bounded open connected subset of $\mathbb{R}^{n_2}$ of class C^1. Then the composition operator T from $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\Omega_2, \Omega_1)$ to $C^{m,\alpha}(\mathrm{cl}\Omega_2)$ defined by*

$$
T[u, v] \equiv u \circ v \quad \forall (u, v) \in C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\Omega_2, \Omega_1)\,,
$$

is real analytic.

Then we introduce the following statement of [43, Lem. 3.8, Prop. 3.14, Rmk. 3.15].

Theorem A.2. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\tilde{g} \in C^{m,\alpha}(\partial\Omega)$. Then there exist $\epsilon_1 \in]0, \epsilon_0[$ and an open neighborhood $\tilde{\Gamma}$ of $\tilde{g}$ in $C^{m,\alpha}(\partial\Omega)$ and a real analytic map $(\hat{\eta}[\cdot,\cdot], \hat{\xi}[\cdot,\cdot])$ from $]-\epsilon_1, \epsilon_1[\times\tilde{\Gamma}$ to $C^{m,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ such that the only solution $\varsigma[\epsilon, g] \in C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of the Dirichlet problem*

$$
\begin{cases}
\Delta u(x) = 0 & \forall x \in \mathbb{S}(\epsilon, 1)^-\,, \\
u \text{ is } q-\text{periodic in } \mathbb{S}(\epsilon, 1)^-\,, & \\
u(p + t\epsilon) = g(t) & \forall t \in \partial\Omega\,,
\end{cases}
$$

is delivered by the formula

$$
\varsigma[\epsilon, g](x) = w_q^-[\partial\Omega_{p,\epsilon}, \hat{\eta}[\epsilon, g](\epsilon^{-1}(\cdot - p))](x) + \hat{\xi}[\epsilon, g] \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon, 1)^-\,,
$$

for all $(\epsilon, g) \in]0, \epsilon_1[\times\tilde{\Gamma}$. Moreover,

$$
(\hat{\eta}[0, \tilde{g}], \hat{\xi}[0, \tilde{g}]) = (\tilde{\eta}, \tilde{\xi})\,,
$$

where $(\tilde{\eta}, \tilde{\xi}) \in C^{m,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ is the only solution of the equation

$$
-\frac{1}{2}\tilde{\eta} + w[\partial\Omega, \tilde{\eta}] + \tilde{\xi} = \tilde{g} \qquad \text{on } \partial\Omega\,.
$$

Also,

$$
\tilde{\xi} = \int_{\partial\Omega} \tilde{g}\tilde{\tau}\, d\sigma\,,
$$

where $\tilde{\tau} \in C^{m-1,\alpha}(\partial\Omega)$ is the only solution of the problem

$$
-\frac{1}{2}\tau + w_*[\partial\Omega, \tau] = 0 \qquad \text{on } \partial\Omega\,, \qquad \int_{\partial\Omega} \tau\, d\sigma = 1\,. \tag{A.3}
$$

In order to compute $\tilde{\xi}$, the following lemma is sometimes useful.

Lemma A.4. *Let the same assumptions of Theorem A.2 hold. Then*

$$
\lim_{]0,\epsilon_1[\times\tilde{\Gamma}\ni(\epsilon,g)\to(0,\tilde{g})} \varsigma[\epsilon, g](x) = \tilde{\xi} \qquad \forall x \in \mathbb{R}^n \setminus (p + q\mathbb{Z}^n)\,.
$$

Proof. Since

$$\varsigma[\epsilon,g](x) = -\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(s)DS_{q,n}(x-p-\epsilon s)\hat{\eta}[\epsilon,g](s)\,d\sigma_s + \hat{\xi}[\epsilon,g] \qquad \forall x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,$$

for all $(\epsilon,\gamma)\in]0,\epsilon_1[\times\tilde{\Gamma}$, the statement follows by the continuity of $\hat{\eta}$ and $\hat{\xi}$ at $(0,\tilde{g})$, and by the continuity of $DS_{q,n}$ in $\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$. □ □

Then we deduce the validity of the following corollary.

Corollary A.5. *Let the same assumptions of Theorem A.2 hold. Then there exist $\epsilon_1\in]0,\epsilon_0[$, and an open neighborhood $\tilde{\Gamma}$ of $\tilde{g}$ in $C^{m,\alpha}(\partial\Omega)$, and an analytic map J_1 from $]-\epsilon_1,\epsilon_1[\times\tilde{\Gamma}$ to $\mathbb{R}$ such that*

$$\int_{Q\setminus\Omega_{p,\epsilon}}\varsigma[\epsilon,g]\,dx = J_1[\epsilon,g] \qquad \forall(\epsilon,g)\in]0,\epsilon_1[\times\tilde{\Gamma}\,.$$

Moreover, $J_1[0,\tilde{g}] = m_n(Q)\int_{\partial\Omega}\tilde{g}\tilde{\tau}\,d\sigma$, where $\tilde{\tau}$ is the only solution in $C^{m-1,\alpha}(\partial\Omega)$ of problem (A.3).

Proof. We first observe that

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\varsigma[\epsilon,g]\,d\sigma = \int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} w_q^-[\partial\Omega_{p,\epsilon},\hat{\eta}[\epsilon,g](\epsilon^{-1}(\cdot-p))](x)\,dx + \hat{\xi}[\epsilon,g]m_n(Q\setminus\Omega_{p,\epsilon}) \tag{A.6}$$

for all $(\epsilon,g)\in]0,\epsilon_1[\times\tilde{\Gamma}$. Next we note that

$$\begin{aligned}
&\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} w_q^-[\partial\Omega_{p,\epsilon},\hat{\eta}[\epsilon,g](\epsilon^{-1}(\cdot-p))](x)\,dx \qquad\qquad (A.7)\\
&\qquad = -\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\int_{\partial\Omega_{p,\epsilon}}\nu_{\Omega_{p,\epsilon}}(y)DS_{q,n}(x-y)\hat{\eta}[\epsilon,g](\epsilon^{-1}(y-p))\,d\sigma_y\,dx\\
&\qquad = -\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\sum_{j=1}^n\frac{\partial}{\partial x_j}\int_{\partial\Omega_{p,\epsilon}}S_{q,n}(x-y)\hat{\eta}[\epsilon,g](\epsilon^{-1}(y-p))(\nu_{\Omega_{p,\epsilon}}(y))_j\,d\sigma_y\,dx\\
&\qquad = \int_{\partial\Omega_{p,\epsilon}}\sum_{j=1}^n(\nu_{\Omega_{p,\epsilon}}(x))_j\int_{\partial\Omega_{p,\epsilon}}S_{q,n}(x-y)\hat{\eta}[\epsilon,g](\epsilon^{-1}(y-p))(\nu_{\Omega_{p,\epsilon}}(y))_j\,d\sigma_y\,d\sigma_x\\
&\qquad = \sum_{j=1}^n\int_{\partial\Omega}(\nu_\Omega(t))_j\int_{\partial\Omega}S_{q,n}(\epsilon(t-s))\hat{\eta}[\epsilon,g](s)(\nu_\Omega(s))_j\,d\sigma_s d\sigma_t\epsilon^{2n-2}\\
&\qquad = \sum_{j=1}^n\int_{\partial\Omega}(\nu_\Omega(t))_j\int_{\partial\Omega}S_n(t-s)\hat{\eta}[\epsilon,g](s)(\nu_\Omega(s))_j\,d\sigma_s d\sigma_t\epsilon^{n}\\
&\qquad\quad +\frac{\delta_{2,n}}{2\pi}\epsilon(\epsilon\log\epsilon)\sum_{j=1}^n\int_{\partial\Omega}(\nu_\Omega(t))_j\,d\sigma_t\int_{\partial\Omega}\hat{\eta}[\epsilon,g](s)(\nu_\Omega(s))_j\,d\sigma_s\\
&\qquad\quad +\sum_{j=1}^n\int_{\partial\Omega}(\nu_\Omega(t))_j\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\hat{\eta}[\epsilon,g](s)(\nu_\Omega(s))_j\,d\sigma_s d\sigma_t\epsilon^{2n-2}\\
&\qquad = \sum_{j=1}^n\int_{\partial\Omega}(\nu_\Omega(t))_j\int_{\partial\Omega}S_n(t-s)\hat{\eta}[\epsilon,g](s)(\nu_\Omega(s))_j\,d\sigma_s d\sigma_t\epsilon^{n}\\
&\qquad\quad +\sum_{j=1}^n\int_{\partial\Omega}(\nu_\Omega(t))_j\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\hat{\eta}[\epsilon,g](s)(\nu_\Omega(s))_j\,d\sigma_s d\sigma_t\epsilon^{2n-2}\,,
\end{aligned}$$

for all $(\epsilon,g)\in]0,\epsilon_1[\times\tilde{\Gamma}$. Thus it is natural to define J_1 as the map from $]-\epsilon_1,\epsilon_1[\times\tilde{\Gamma}$ to $\mathbb{R}$ which takes (ϵ,g) to the sum of the right hand side of (A.7) and of the term $\hat{\xi}[\epsilon,g]m_n(Q\setminus\Omega_{p,\epsilon}) = \hat{\xi}[\epsilon,g](m_n(Q)-\epsilon^n m_n(\Omega))$ in the right hand side of equality (A.6). By classical potential theory, the operator $v[\partial\Omega,\cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$. Then the continuity of the pointwise product in $C^{m-1,\alpha}(\partial\Omega)$ and the analyticity of $\hat{\eta}[\cdot,\cdot]$ imply the analyticity of the first sum in the right hand side of (A.7). Then the analyticity of the map in (6.11), and the continuity of the product in $C^{m-1,\alpha}(\partial\Omega)$ and the analyticity of $\hat{\eta}[\cdot,\cdot]$ imply the analyticity of the second sum in the right hand side of (A.7) in the variable (ϵ,g). The analyticity of $\hat{\xi}[\cdot,\cdot]$ implies the analyticity of the term $\hat{\xi}[\epsilon,g](m_n(Q)-\epsilon^n m_n(\Omega))$ upon the variable (ϵ,g). Hence, $J_1[\cdot,\cdot]$ is real analytic from $]-\epsilon_1,\epsilon_1[\times\tilde{\Gamma}$ to $\mathbb{R}$. Finally,

$$J_1[0,\tilde{g}] = m_n(Q)\hat{\xi}[0,\tilde{g}] = m_n(Q)\tilde{\xi} = m_n(Q)\int_{\partial\Omega}\tilde{\tau}\tilde{g}\,d\sigma\,,$$

where $\tilde{\tau}$ is the unique solution of problem (A.3). □ □

Next we introduce the following technical statement.

Proposition A.8. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2).*

(i) Let $\rho\in]0,+\infty[$. Then there exists a real analytic map G from $]-\epsilon_0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}Q)$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\Omega_{p,\epsilon}} h\,dx = G[\epsilon,h] \qquad \forall(\epsilon,h)\in]0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}Q)$$

$$G[0,h] = \int_Q h\,dx \qquad \forall h\in C^0_{\omega,\rho}(\mathrm{cl}Q)\,.$$

(ii) There exists a real analytic function G_1 from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\Omega_{p,\epsilon}} S_{q,n}(x-p)\,dx = G_1(\epsilon)-\delta_{2,n}\frac{\epsilon^2\log\epsilon}{2\pi}m_n(\Omega) \qquad \forall\epsilon\in]0,\epsilon_0[\,.$$

Moreover,

$$G_1(0) = \int_Q S_{q,n}(x-p)\,dx\,.$$

Proof. For the existence of G, we follow the proof of Lemma 2.2 of [30] and we note that $\int_{Q\setminus\Omega_{p,\epsilon}} h\,dx = \int_Q h\,dx - \epsilon^n\int_\Omega h(p+\epsilon s)\,ds$ for all $(\epsilon,h)\in]0,\epsilon_1[\times C^0_{\omega,\rho}(\mathrm{cl}Q)$, and we define G as the map from $]-\epsilon_0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}Q)$ to $\mathbb{R}$ which takes (ϵ,h) to the right hand side of such an equality. The analyticity of G follows by Proposition A.1. The formula for $G[0,h]$ follows by the definition of G. Next we turn to prove statement (ii). By identity (2.3) and by the rule of change of variables, we have

$$\begin{aligned}\int_{Q\setminus\Omega_{p,\epsilon}} S_{q,n}(x-p)\,dx &= \int_Q S_{q,n}(x-p)\,dx \\ &\quad -\epsilon^2\int_\Omega S_n(t)\,dt - \delta_{2,n}\frac{\epsilon^2\log\epsilon}{2\pi}m_n(\Omega) - \epsilon^n\int_\Omega R_{q,n}(\epsilon t)\,dt \qquad \forall\epsilon\in]0,\epsilon_0[\,.\end{aligned}$$

Then we can set

$$G_1(\epsilon) \equiv \int_Q S_{q,n}(x-p)\,dx - \epsilon^2\int_\Omega S_n(t)\,dt - \epsilon^n\int_\Omega R_{q,n}(\epsilon t)\,dt \qquad \forall\epsilon\in]-\epsilon_0,\epsilon_0[\,.$$

By the analyticity of $R_{q,n}$ in $(\mathbb{R}^n\setminus q\mathbb{Z}^n)\cup\{0\}$ and by analyticity results on the composition operator (cf. Böhme and Tomi [3, p. 10], Henry [22, p. 29], Valent [48, Thm. 5.2, p. 44]), we deduce that the map from $]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\mathrm{cl}\Omega)$, which takes ϵ to the function $R_{q,n}(\epsilon t)$ of the variable $t\in\mathrm{cl}\Omega$ is real analytic. Then by the continuity of the linear operator from $C^{m,\alpha}(\mathrm{cl}\Omega)$ to $\mathbb{R}$ which takes a map to its integral, the function G_1 is analytic from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$. Then we obviously have $G_1(0)=\int_Q S_{q,n}(x-p)\,dx$. □ □

Next we introduce the following inequality for dilated q-periodic functions, which we prove by arguments akin to those of Braides and De Franceschi [6, ex. 27, p. 20]. We denote by u_δ the function from $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$u_\delta(x)\equiv u(x/\delta) \qquad \forall x\in\mathbb{R}^n\,, \tag{A.9}$$

for all $\delta\in]0,+\infty[$ and for all q-periodic functions $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then we have the following.

Lemma A.10. *Let $r\in[1,+\infty[$, $\delta_0\in]0,+\infty[$. Let V be a bounded open subset of $\mathbb{R}^n$. Then there exists $C\in]0,+\infty[$ such that*

$$\|u_\delta\|_{L^r(V)} \le C\|u\|_{L^r(Q)} \qquad \forall\delta\in]0,\delta_0[\,,$$

for all q-periodic $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Since V is bounded, there exists a family $\{z_l\}_{l=1}^s$ of points of $\mathbb{Z}^n$ such that

$$V \subseteq \bigcup_{l=1}^{s} (qz_l + \mathrm{cl}Q) .$$

Then the q-periodicity of u implies that

$$\begin{aligned}\int_V |u_\delta(y)|^r \, dy &\leq \sum_{l=1}^{s} \int_{qz_l + \mathrm{cl}Q} |u_\delta(y)|^r \, dy = \sum_{l=1}^{s} \int_{\delta^{-1}qz_l + \delta^{-1}\mathrm{cl}Q} |u(x)|^r \, dx\delta^n \\ &\leq \sum_{l=1}^{s} \int_{\delta^{-1}qz_l + ([\delta^{-1}]+1)\mathrm{cl}Q} |u(x)|^r \, dx\delta^n = s \int_{([\delta^{-1}]+1)\mathrm{cl}Q} |u(x)|^r \, dx\delta^n \\ &\leq C^r \int_Q |u(x)|^r \, dx \qquad \forall \delta \in]0, \delta_0[,\end{aligned}$$

for all q-periodic $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where

$$C \equiv s^{1/r} \left\{ \sup_{\delta \in]0,\delta_0[} ([\delta^{-1}] + 1)^n \delta^n \right\}^{1/r} < +\infty ,$$

and where $[\delta^{-1}]$ denotes the integer part of δ^{-1}. □ □

Next we introduce the following lemma for dilated q-periodic functions.

Lemma A.11. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a q-periodic function. Let $\tilde{y} \in \mathbb{R}^n$, $s \in]0, +\infty[$, $l \in \mathbb{N} \setminus \{0\}$. Then the following equality holds*

$$\int_{\mathbb{R}^n} u_{s/l}(x) \chi_{\tilde{y}+sQ}(x) \, dx = s^n \int_Q u \, dx ,$$

(see (A.9).)

Proof. Since $u_{s/l}$ is $l^{-1}sq$-periodic, it is also sq-periodic and accordingly,

$$\int_{\mathbb{R}^n} u_{s/l}(x) \chi_{\tilde{y}+sQ}(x) \, dx = \int_{\tilde{y}+sQ} u_{s/l}(x) \, dx = \int_{sQ} u_{s/l}(x) \, dx .$$

Next we observe that

$$\bigcup_{0 \leq z_j \leq l-1} (qz + l^{-1}Q) \subseteq Q , \qquad m_n \left(Q \setminus \bigcup_{0 \leq z_j \leq l-1} (qz + l^{-1}Q) \right) = 0 .$$

Accordingly, the $l^{-1}sq$-periodicity of $u_{s/l}(\cdot)$ implies that

$$\begin{aligned}\int_{sQ} u_{s/l}(x) \, dx &= \int_{sl^{-1}Q} u_{s/l}(x) \, dx l^n \\ &= \int_{sl^{-1}Q} u(x/(s/l)) \, dx l^n = \int_Q u(y) \, dy l^n (s/l)^n = s^n \int_Q u \, dx .\end{aligned}$$

□ □

Finally, we introduce the following elementary lemma of [32, Lem. A.5].

Lemma A.12. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $p \in Q$. Let Ω be as in (1.1). Let $\epsilon_0 \in]0, +\infty[$ be as in (1.2). Let $\epsilon_1 \in]0, \epsilon_0[$.*

(i) Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^n$ with a nonzero distance from $p + q\mathbb{Z}^n$. Then there exist $\epsilon^*_{\tilde{\Omega}} \in]0, \epsilon_1[$ such that

$$\mathrm{cl}\tilde{\Omega} \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall \epsilon \in [-\epsilon^*_{\tilde{\Omega}}, \epsilon^*_{\tilde{\Omega}}] ,$$

and $\epsilon_{\tilde{\Omega}} \in]0, \epsilon^*_{\tilde{\Omega}}[$ such that

$$\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^- \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall \epsilon \in [-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}] .$$

(ii) Let $\Omega^\sharp$ be a bounded open subset of $\mathbb{R}^n$ such that $\Omega^\sharp \subseteq \mathbb{R}^n \setminus \mathrm{cl}\Omega$. Then there exists $\epsilon_{\Omega^\sharp,r} \in]0, \epsilon_1[$ such that

$$p + \epsilon \mathrm{cl}\Omega^\sharp \subseteq Q , \qquad p + \epsilon\Omega^\sharp \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall \epsilon \in [-\epsilon_{\Omega^\sharp,r}, \epsilon_{\Omega^\sharp,r}] \setminus \{0\} .$$

References

[1] Ammari, H., Kang, H.: Polarization and moment tensors. Applied Mathematical Sciences, **162**. Springer, New York (2007).

[2] Ammari, H., Kang, H., Lee, H.: Layer potential techniques in spectral analysis. American Mathematical Society, Providence, RI (2009).

[3] Böhme, R., Tomi, F.: Zur Struktur der Lösungsmenge des Plateauproblems. Math. Z., **133**, 1–29 (1973).

[4] Bonnaillie-Noël, V., Dambrine, M., Lacave, C.: Interactions between moderately close inclusions for the 2D Dirichlet-Laplacian. Appl. Math. Res. Express. AMRX, **2016**, 1–23 (2016).

[5] Bonnaillie-Noël, V., Dambrine, M., Tordeux, S., Vial, G.: Interactions between moderately close inclusions for the Laplace equation. Math. Models Methods Appl. Sci., **19**, 1853–1882 (2009).

[6] Braides, A., Defranceschi, A.: Homogenization of multiple integrals. Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Oxford University Press, New York (1998).

[7] Cabarrubias, B., Donato, P.: Homogenization of a quasilinear elliptic problem with nonlinear Robin boundary conditions. Appl. Anal., **91**, 1111–1127 (2012).

[8] Castro, L.P., Pesetskaya, E., Rogosin, S.V.: Effective conductivity of a composite material with non-ideal contact conditions. Complex Var. Elliptic Equ., **54**, 1085–1100 (2009).

[9] Cioranescu, D., Murat, F.: Un terme étrange venu d'ailleurs, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980), volume 60 of Res. Notes in Math., pages 98–138, 389–390. Pitman, Boston, Mass. (1982).

[10] Cioranescu, D., Murat, F.: Un terme étrange venu d'ailleurs. II, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981), volume 70 of Res. Notes in Math., pages 154–178, 425–426. Pitman, Boston, Mass. (1982).

[11] Dalla Riva, M.: Energy integral of the Stokes flow in a singularly perturbed exterior domain. Opuscula Math., **32**, 647–659 (2012).

[12] Dalla Riva, M.: Stokes flow in a singularly perturbed exterior domain. Complex Var. Elliptic Equ., **58**, 231–257 (2013).

[13] Dalla Riva, M., Lanza de Cristoforis, M.: Microscopically weakly singularly perturbed loads for a nonlinear traction boundary value problem: a functional analytic approach. Complex Var. Elliptic Equ., **55** , 771–794 (2010).

[14] Dalla Riva, M., Lanza de Cristoforis, M.: A singularly perturbed nonlinear traction boundary value problem for linearized elastostatics. A functional analytic approach. Analysis (Munich), **30**, 67–92 (2010).

[15] Dalla Riva, M., Lanza de Cristoforis, M., Musolino, P.: A local uniqueness result for a quasi-linear heat transmission problem in a periodic two-phase dilute composite. In Recent Trends in Operator Theory and Partial Differential Equations - The Roland Duduchava Anniversary Volume, Operator Theory: Advances and Applications, Volume 258, 193–227, Birkhäuser Verlag, Basel (2017).

[16] Dalla Riva, M., Musolino, P., Rogosin, S.V.: Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole, Asymptot. Anal., **92**, 339–361 (2015).

[17] Dauge, M., Tordeux, S., Vial, G.: Selfsimilar perturbation near a corner: matching versus multiscale expansions for a model problem. In Around the research of Vladimir Maz'ya. II, 95–134, Int. Math. Ser. (N. Y.), 12, Springer, New York (2010).

[18] Deimling, K.: *Nonlinear functional analysis*. Springer-Verlag, Berlin (1985).

[19] Drygas, P., Mityushev, V.: Effective conductivity of unidirectional cylinders with interfacial resistance. Quart. J. Mech. Appl. Math., **62**, , 235–262 (2009).

[20] Folland, G.B.: Introduction to partial differential equations. Princeton University Press, Princeton, NJ, second edition (1995).

[21] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer Verlag, Berlin, *etc.* (1983).

[22] Henry, D.: Topics in nonlinear analysis. Trabalho de Matemática, 192, Brasilia (1982).

[23] Khruslov, E. Ja.: The method of orthogonal projections and the Dirichlet boundary value problem in domains with a “fine-grained” boundary. (Russian) Mat. Sb. (N.S.), **88** (130), 38–60 (1972).

[24] Kapanadze, D., Mishuris, G., Pesetskaya, E.: Improved algorithm for analytical solution of the heat conduction problem in doubly periodic 2D composite materials. Complex Var. Elliptic Equ., **60**, 1–23 (2015).

[25] Kozlov, V., Maz'ya, V., Movchan, A.: Asymptotic analysis of fields in multi-structures. Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York (1999).

[26] Lanza de Cristoforis, M.: Properties and pathologies of the composition and inversion operators in Schauder spaces. Acc. Naz. delle Sci. detta dei XL, **15**, 93–109 (1991).

[27] Lanza de Cristoforis, M.: Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole in Schauder spaces. Comput. Methods Funct. Theory, **2** (2002), 1–27.

[28] Lanza de Cristoforis, M.: Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach. *Complex Var. Elliptic Equ.*, **52**, 945–977 (2007).

[29] Lanza de Cristoforis, M.: Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach. Analysis (Munich), **28**, 63–93 (2008).

[30] Lanza de Cristoforis, M.: A singular domain perturbation problem for the Poisson equation. In More progresses in analysis. Proceedings of the 5th international ISAAC congress, Catania, Italy, July 25–30, 2005, 955–965, World Scientific, Hackensack, NJ (2009).

[31] Lanza de Cristoforis, M., Musolino, P.: A perturbation result for periodic layer potentials of general second order differential operators with constant coefficients. Far East J. Math. Sci. (FJMS), **52**, 75–120 (2011).

[32] Lanza de Cristoforis, M., Musolino, P.: A singularly perturbed nonlinear Robin problem in a periodically perforated domain: a functional analytic approach. Complex Var. Elliptic Equ., **58**, 511–536 (2013).

[33] Lanza de Cristoforis, M., Musolino, P.: A real analyticity result for a nonlinear integral operator. J. Integral Equations Appl., **25**, 21–46 (2013).

[34] Lanza de Cristoforis, M., Musolino, P.: A functional analytic approach to homogenization problems. In Integral Methods in Science and Engineering: Theoretical and Computational Advances, Proceedings of the 13th International Conference on Integral Methods in Science and Engineering, IMSE 2014, Karlsruhe, Germany 21-25 July 2014, 353–359, Birkhäuser Verlag, Basel (2015).

[35] Lanza de Cristoforis, M., Musolino, P.: A singularly perturbed Neumann problem for the Poisson equation in a periodically perforated domain. A functional analytic approach. Z. Angew. Math. Mech., **92**, 253–272 (2016).

[36] Lanza de Cristoforis, M., Musolino, P.: Two-parameter anisotropic homogenization for a Dirichlet problem for the Poisson equation in an unbounded periodically perforated domain. A functional analytic approach. *Submitted.*

[37] Lanza de Cristoforis, M., Rossi, L.: Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density. J. Integral Equations Appl., **16**, 137–174 (2004).

[38] Marčenko, V.A., Khruslov, E.Y.: Boundary value problems in domains with a fine-grained boundary. Izdat. “Naukova Dumka”, Kiev (1974). (in Russian)

[39] Maz'ya, V., Movchan, A.: Asymptotic treatment of perforated domains without homogenization. Math. Nachr., **283**, 104–125 (2010).

[40] Maz'ya, V., Movchan, A., Nieves, M.: Green's Kernels and Meso-scale Approximations in Perforated Domains. Lecture Notes in Mathematics, **2077**, Springer, Berlin (2013).

[41] Maz'ya, V., Nazarov, S., Plamenevskij, B.: Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vols. I, II, Operator Theory: Advances and Applications, **111**, **112**. Birkhäuser Verlag, Basel (2000).

[42] Miranda, C.: Sulle proprietà di regolarità di certe trasformazioni integrali. Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I, **7**, 303–336 (1965).

[43] Musolino, P.: A singularly perturbed Dirichlet problem for the Laplace operator in a periodically perforated domain. A functional analytic approach. Math. Methods Appl. Sci., **35**, 334–349. (2012)

[44] Musolino, P.: A singularly perturbed Dirichlet problem for the Poisson equation in a periodically perforated domain. A functional analytic approach. In Advances in Harmonic Analysis and Operator Theory, The Stefan Samko Anniversary Volume, Operator Theory: Advances and Applications, Volume 229, pages 269–289, Birkhäuser Verlag, Basel (2013).

[45] Novotny, A.A., Sokołowski, J.: Topological derivatives in shape optimization. Interaction of Mechanics and Mathematics, Springer, Heidelberg (2013).

[46] Ozawa, S.: Electrostatic capacity and eigenvalues of the Laplacian. J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30**, 53–62 (1983).

[47] Preciso, P.: Regularity of the composition and of the inversion operator and perturbation analysis of the conformal sewing problem in Romieu type spaces. Tr. Inst. Mat. Minsk, **5**, 99–104 (2000).

[48] Valent, T.: Boundary value problems of finite elasticity. Local theorems on existence, uniqueness and analytic dependence on data. Springer-Verlag, New York (1988).

[49] Ward, M.J., Keller, J.B.: Strong localized perturbations of eigenvalue problems. SIAM J. Appl. Math., **53**, 770–798 (1993).