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ON CERTAIN PROPERTIES OF CUNTZ–KRIEGER TYPE ALGEBRAS

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Abstract. The note presents a further study of the class of Cuntz–Krieger type algebras. A necessary and sufficient condition is identified that ensures that the algebra is purely infinite, the ideal structure is studied, and nuclearity is proved by presenting the algebra as a crossed product of an AF-algebra by an abelian group. The results are applied to examples of Cuntz–Krieger type algebras, such as higher rank semigraph C^*-algebras and higher rank Exel-Laca algebras.

1. Introduction

During the last two decades, Cuntz and Cuntz–Krieger algebras, in the form of graph algebras, have been studied intensively. Recent samples include [10, 9].

Based on the work of Cuntz and Krieger in [8], in [2] the first named author considered a class of so-called Cuntz–Krieger type algebras relying on a flexible generators and relations approach. This class, which is recalled in Section 2, includes (aperiodic) Cuntz–Krieger algebras [8], higher rank Exel–Laca algebras [3], (aperiodic) higher rank graph C^*-algebras [11, 12], (aperiodic) ultragraph algebras [17] and (cancelling) higher rank semigraph C^*-algebras [5].

The aim of this note is to analyse these algebras further. Pure infiniteness was introduced by J. Cuntz in [6] as a fundamental property of his Cuntz algebras. In Section 3 we show that a Cuntz–Krieger type algebra is purely infinite if and only if the projections of its core are infinite, see Theorem 3.2. Applications to higher rank semigraph C^*-algebras and higher rank Exel–Laca algebras, stated in Corollaries 3.3 and 3.4, respectively, give quite tractable conditions for checking when those algebras are purely infinite.

In Section 4 we study the ideal structure of Cuntz–Krieger type algebras. The ideal structure for Cuntz–Krieger algebras was firstly studied by J. Cuntz in [7]. There is an injection of certain ideals of the core to the ideals of the Cuntz–Krieger type algebra, see Theorem 4.6. If these certain ideals are all cancelling (Definitions 4.8 and 4.11) then this injection is even a lattice isomorphism, see Theorem 4.9, Corollary 4.10, Theorem 4.12 and Corollary 4.13. We give reformulations of such an isomorphism especially for higher rank semigraph algebras in Corollaries 4.14 and 4.15.

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In Section 5 we present the stabilised Cuntz–Krieger type algebras as crossed products of AF-algebras by abelian groups, see Theorem 5.1. This uses Takai's duality and gauge actions. Hence Cuntz–Krieger type algebras are nuclear.

2. Cuntz–Krieger type algebras

We briefly recall the basic definitions and facts of the class of Cuntz–Krieger type algebras introduced in [2] and slightly extended in [4].

Assume that we are given an alphabet $\mathcal{A}$, the free nonunital $*$-algebra $\mathbb{F}$ generated by $\mathcal{A}$, a two-sided self-adjoint ideal $\mathbb{I}$ of $\mathbb{F}$, and a closed subgroup H of $\mathbb{T}^{\mathcal{A}}$ ($\mathbb{T}$ denotes the circle). We are interested in the quotient $*$-algebra $\mathbb{F}/\mathbb{I}$ and its universal C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$. Denote the set of words of $\mathbb{F}/\mathbb{I}$ by $W = \{a_1 \dots a_n \in \mathbb{F}/\mathbb{I} |\, a_i \in \mathcal{A} \cup \mathcal{A}^*\}$. (We will always write x rather than $x + \mathbb{I}$ in the quotient $\mathbb{F}/\mathbb{I}$ for elements $x \in \mathbb{F}$ if there is no danger of confusion.) An element x of a $*$-algebra is called a partial isometry if $xx^*x = x$, and a projection if $x^2 = x^* = x$.

We are going to introduce the following properties (A), (B) and (C') for the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$.

(A) There exists a gauge action $t : H \longrightarrow \mathrm{Aut}(\mathbb{F}/\mathbb{I})$ determined by $t_\lambda(a) = \lambda_a a$ for all $a \in \mathcal{A}$ and $\lambda = (\lambda_b)_{b\in\mathcal{A}} \in H$.

Denote by $(\hat{H}, +, 0)$ the character group of $(H, \cdot, 1)$; note that we write the group operation of $\hat{H}$ additively. The gauge action t induces a so-called balance function $\mathrm{bal} : W\backslash\{0\} \longrightarrow \hat{H}$ from the nonzero words of $\mathbb{F}/\mathbb{I}$ to the character group $\hat{H}$ determined by $\mathrm{bal}(a)((\lambda_b)_{b\in\mathcal{A}}) = \lambda_a \in \mathbb{T}$, $\mathrm{bal}(xy) = \mathrm{bal}(x) + \mathrm{bal}(y)$ and $\mathrm{bal}(x^*) = -\mathrm{bal}(x)$, where $a \in \mathcal{A}$, $(\lambda_b)_{b\in\mathcal{A}} \in H \subseteq \mathbb{T}^{\mathcal{A}}$ and $x, y \in W$ (see [2, Lemma 3.1]).

Define $\mathbb{A}$ to be the linear span in $\mathbb{F}/\mathbb{I}$ of all words $x \in W\backslash\{0\}$ satisfying $\mathrm{bal}(x) = 0$. Actually, $\mathbb{A}$ is a $*$-algebra. Words x with balance $\mathrm{bal}(x) = 0$ are called zero-balanced. Write W_n for the set of words with balance $n \in \hat{H}$. Since every element of $\mathbb{F}/\mathbb{I}$ is expressable as a linear combination of words, we may write $\mathbb{F}/\mathbb{I} = \sum_{n\in\hat{H}} \mathrm{lin}(W_n)$. Note, however, that this sum might not be a direct sum.

(B) $\mathbb{A}$ is locally matricial, that is, for all $x_1, \dots, x_n \in \mathbb{A}$ there exists a finite dimensional C^*-subalgebra A of $\mathbb{A}$ such that $x_1, \dots, x_n \in A$.

(C') For every nonzero-balanced word $x \in W\backslash W_0$ and every nonzero projection $e \in \mathbb{A}$ there exists a nonzero projection $p \le e$ in $\mathbb{A}$ such that $pxp = 0$.

Definition 2.1. A system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ is called a Cuntz–Krieger type system, or $\mathbb{F}/\mathbb{I}$ is called a Cuntz–Krieger type $*$-algebra, if (A), (B) and (C') are satisfied and there exists a C^*-representation $\pi : \mathbb{F}/\mathbb{I} \longrightarrow A$ which is injective on $\mathbb{A}$.

Throughout assume that $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ is a Cuntz–Krieger type system if nothing else is said. There exists a universal enveloping C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$ for $\mathbb{F}/\mathbb{I}$, and clearly the universal representation $\zeta : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\mathbb{F}/\mathbb{I})$ is injective on $\mathbb{A}$. The enveloping C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$ is called the Cuntz–Krieger type algebra associated to $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$. A $*$-homomorphism $\mathbb{F}/\mathbb{I} \to A$ into a C^*-algebra A is called a *C^*-representation of* $\mathbb{F}/\mathbb{I}$, and $\mathbb{A}$-*faithful* if it is faithful on $\mathbb{A}$. We remark that for a system $(\mathcal{A}, H, \mathbb{F}, H)$ satisfying (A), (B) and (C'), an $\mathbb{A}$-faithful representation of $\mathbb{F}/\mathbb{I}$ into a C^*-algebra exists automatically if the word set W consists of partial isometries, see [4, Theorem 3.1].

We have the following Cuntz–Krieger uniqueness theorem.

Theorem 2.2. *If $\pi : \mathbb{F}/\mathbb{I} \longrightarrow A$ is an $\mathbb{A}$-faithful representation into a C^*-algebra A with dense image in A then A is canonically isomorphic to $C^*(\mathbb{F}/\mathbb{I})$ via $\pi(x) \mapsto \zeta(x)$, so π is essentially the universal map ζ (see* [2, Theorem 3.3] *and Theorem 2.1 and Corollary 1 of Section 3 of* [4]*).*

The next lemma states that we usually may assume without loss of generality that ζ is injective. We then usually avoid notating ζ and regard $\mathbb{F}/\mathbb{I}$ as a subset of $C^*(\mathbb{F}/\mathbb{I})$.

Lemma 2.3. *We may assume without loss of generality that the universal representation $\zeta : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\mathbb{F}/\mathbb{I})$ is injective by dividing out the kernel of ζ. The new quotient $\mathbb{F}/\mathbb{I}$ is a Cuntz–Krieger $*$-algebra again ($\mathcal{A}, \mathbb{F}$ and H remain unchanged). $\mathbb{A}$ remains unchanged under this modification.*

In a previous preprint of this note we proved the last lemma and the next lemma. However, we have reproved and published them already now in [4, Propositions 2 and 4]. The setting in [4] generalises the setting of this note by allowing the image of the balance function, here the commutative group $\hat{H}$, to be a non-commutative group. Say that a $*$-algebra X satisfies the C^*-property if for every $x \in X$, $xx^* = 0$ implies $x = 0$.

Lemma 2.4. *ζ is injective if and only if $\mathbb{F}/\mathbb{I}$ satisfies the C^*-property. The kernel of ζ is the ideal generated by $\{x \in \mathbb{F}/\mathbb{I} |\, xx^* = 0\}$.*

Lemma 2.5. *There exists a conditional expectation $F : C^*(\mathbb{F}/\mathbb{I}) \longrightarrow C^*(\mathbb{A}) \subseteq C^*(\mathbb{F}/\mathbb{I})$ determined by $F(\zeta(w)) = 1_{\{\mathrm{bal}(w)=0\}}\zeta(w)$ for words $w \in W$ (see* [4, Proposition 2]*).*

3. Pure Infiniteness

In this section we analyse the pure infiniteness of a Cuntz–Krieger type algebra $C^*(\mathbb{F}/\mathbb{I})$. We say that a C^*-algebra A is purely infinite if every nonzero hereditary sub-C^*-algebra of A contains an infinite projection. (This condition is for instance stated in [15, Proposition 4.1.1.(v)] and is also used in [14].)

Recall that a projection p in a C^*-algebra A is called infinite if it is the source projection s^*s of a partial isometry s in A with range projection ss^* being smaller than p. Recall the following simple lemma.

Lemma 3.1. *If a projection is infinite then any other projection which is bigger in Murray–von Neumann order is also infinite.*

Theorem 3.2. *A Cuntz–Krieger type algebra $C^*(\mathbb{F}/\mathbb{I})$ is purely infinite if and only if every nonzero projection of $\mathbb{A}$ is infinite in $C^*(\mathbb{F}/\mathbb{I})$.*

Proof. We assume that ζ is injective (Lemma 2.3). Define $A = C^*(\mathbb{F}/\mathbb{I})$. Assume that A is purely infinite. Then for any nonzero projection $e \in \mathbb{A}$ the hereditary

C^*-algebra eAe contains some infinite projection p. Since $p \le e$, e is infinite in A by Lemma 3.1.

To prove the other direction, assume that every nonzero projection in $\mathbb{A}$ is infinite in A. It is proved in Lemma 1 of [4] that there exists a larger Cuntz–Krieger type system $S = (\mathcal{A} \times \mathcal{P}, \mathbb{G}, \mathbb{J}, H \times \{1\})$ than $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ such that $\mathbb{G}/\mathbb{J} \cong \mathbb{F}/\mathbb{I} \otimes \mathbb{F}'/\mathbb{I}'$, where $\mathbb{F}'/\mathbb{I}'$ is a commutative unital locally matricial algebra, and the system S satisfies property (C) of [2]. This property is a sharpening of (C') and states that for every nonzero-balanced word $x \in W \backslash W_0$ and all nonzero projections $e, e_1, e_2 \in \mathbb{A}$ there exist nonzero projections $p \le e, p_1 \le e_1, p_2 \le e_2$ in $\mathbb{A}$ such that $pxp = 0$ and $p_1 x p_2 = 0$. If we can show that $C^*(\mathbb{G}/\mathbb{J}) \cong C^*(\mathbb{F}/\mathbb{I}) \otimes C^*(\mathbb{F}'/\mathbb{I}')$ is purely infinite, then it is not difficult to check that $C^*(\mathbb{F}/\mathbb{I})$ is also purely infinite. (The following fact holds in general: If $A \otimes D$ is purely infinite for two C^*-algebras A and D where D is unital and commutative, then A is purely infinite.)

That is why we may assume without loss of generality in what follows that the system $(\mathcal{A}, \mathbb{F}, \mathbb{I}, H)$ satisfies property (C) of [2]. To show that $A = C^*(\mathbb{F}/\mathbb{I})$ is purely infinite, we imitate the proof of [14, Proposition 5.11]. Let h be a nonzero positive element of A. We have to show that $\overline{hAh}$ contains an infinite projection. Let $\varepsilon > 0$, and choose $y \ge 0$ in $\mathbb{F}/\mathbb{I}$ such that $\|y - h^2\| \le \varepsilon$.

By [2, Lemma 2.6] (applied to $\pi = \zeta$) we are provided with a faithful expectation $F : A \to C^*(\mathbb{A})$ such that for every representation $y = \sum_{\gamma \in \hat{H}} y_\gamma$ (where $y_\gamma \in \text{lin}(W_\gamma)$) there exists a projection $Q \in \mathbb{A}$ satisfying $QyQ = Qy_1Q \in \mathbb{A}$ and $\|Fy\| = \|QyQ\|$.

We may assume without loss of generality that $\|Fh^2\| = 1$. We have

$$\|Fy\| \ge \|Fh^2\| - \varepsilon = 1 - \varepsilon.$$

Let $QyQ \in \mathcal{M}$ for some finite dimensional C^*-algebra $\mathcal{M} \subseteq \mathbb{A}$. We choose a system of generating matrix units for $\mathcal{M}$ such that the positive element QyQ has diagonal form in $\mathcal{M} = M_{k_1} \oplus \ldots \oplus M_{k_d}$. By projecting on the largest diagonal entry, we can choose a positive operator $R_1 \in \mathcal{M}$ such that $P = R_1 QyQ R_1$ is a projection and $\|R_1\| \le (1-\varepsilon)^{-1/2}$. By hypothesis $P \in \mathbb{A}$ is an infinite projection.

It follows that $\|R_1 Q h^2 Q R_1 - P\| \le \|R_1^2\| \|Q\|^2 \|y - h^2\| \le \varepsilon/(1-\varepsilon)$. By functional calculus one obtains $R_2 \in A_+$, so that $R_2 R_1 Q h^2 Q R_1 R_2$ is a projection and

$$\|R_2 R_1 Q h^2 Q R_1 R_2 - P\| \le 2\varepsilon/(1-\varepsilon).$$

For small ε one can then find an element R_3 in A such that

$$R_3 R_2 R_1 Q h^2 Q R_1 R_2 R_3^* = P.$$

Let $R = R_3 R_2 R_1 Q$, so that $Rh^2R^* = P$. Consequently, Rh is a partial isometry, whose initial projection hR^*Rh is a projection in hAh and whose final projection is P. Moreover, if V is a partial isometry in A such that $V^*V = P$ and $VV^* < P$, then $(hR^*)V(Rh)$ is a partial isometry in hAh with initial projection hR^*Rh and final projection strictly less than hR^*Rh. □

We shall now apply the last theorem to cancelling higher rank semigraph algebras [5], which are special Cuntz–Krieger type $*$-algebras.

Corollary 3.3. *A cancelling semigraph C^*-algebra $C^*(\mathbb{F}/\mathbb{I})$ (see* [5, Definitions 5.1 and 7.2]*) is purely infinite if and only if every standard projection (see* [5, Definition 5.14]*) is infinite in $C^*(\mathbb{F}/\mathbb{I})$.*

Proof. Cancelling semigraph algebras are algebras of amenable Cuntz–Krieger systems [4] (this follows from the discussion in [5, Section 7]), which again are Cuntz–Krieger type $*$-algebras (since the image of the balance map, $\hat{H}$, is an abelian group). So we can apply Theorem 3.2. We just need to recall that by [5, Corollary 6.4] every nonzero projection in $\mathbb{A}$ is larger or equal than a standard projection in Murray–von Neumann order, and so is infinite by Lemma 3.1 if every standard projection is infinite. □

The next corollary concerns higher rank Exel–Laca algebras [3], which are special Cuntz–Krieger type algebras.

Corollary 3.4. *Let $C^*(\mathbb{F}/\mathbb{I})$ be a higher rank Exel–Laca algebra* [3]*. Then $C^*(\mathbb{F}/\mathbb{I})$ is purely infinite if and only if every nonzero projection of the form $P_{a_1}\dots P_{a_n}$ ($a_i \in \mathcal{A}$, $P_a = aa^*$) is infinite in $C^*(\mathbb{F}/\mathbb{I})$.*

Proof. By [3, Corollary 4.14] and [3, Lemma 4.5] every projection $p \in \mathbb{A}$ allows the following estimate in Murray–von Neumann order:

$$p \precsim xx^* \precsim x^*x = Q_{a_1}\dots Q_{a_n} \geq P_{b_1}\dots P_{b_n} \neq 0$$

for some word x in the letters of the alphabet $\mathcal{A}$, and some letters $a_i, b_i \in \mathcal{A}$. Hence, the claim follows from Lemma 3.1 and Theorem 3.2. □

4. Ideal structure

In this section we investigate the ideal structure of a Cuntz–Krieger type algebra $C^*(\mathbb{F}/\mathbb{I})$. We assume that ζ is injective (Lemma 2.3).

Write Σ for the set of two-sided self-adjoint ideals in $\mathbb{F}/\mathbb{I}$. Denote by $\mathcal{I}$ the set of closed two-sided ideals in $C^*(\mathbb{F}/\mathbb{I})$. Suppose that $\mathbb{B}$ is a $*$-subalgebra of $\mathbb{A}$. Write $\Sigma^{\mathbb{B}}$ for the set of self-adjoint two-sided ideals in $\mathbb{B}$. Define

$$\Sigma_{\mathbb{B}} = \{\, J \cap \mathbb{B} \in \Sigma^{\mathbb{B}} \mid J \in \Sigma \,\}.$$

For a subset X of $\mathbb{F}/\mathbb{I}$, define $\Sigma(X) \in \Sigma$ to be the two-sided self-adjoint ideal in $\mathbb{F}/\mathbb{I}$ generated by X, and $\mathcal{I}(X) \in \mathcal{I}$ the closed two-sided ideal in $C^*(\mathbb{F}/\mathbb{I})$ generated by X. Denote by $q_X : \mathbb{F}/\mathbb{I} \longrightarrow (\mathbb{F}/\mathbb{I})/\Sigma(X)$ the quotient map.

Lemma 4.1. *For all $J \in \Sigma$ one has $J \cap \mathbb{B} = (\Sigma(J \cap \mathbb{B})) \cap \mathbb{B}$.*

Proof. $J \cap \mathbb{B} \subseteq J \cap \mathbb{B} \cap \mathbb{B} \subseteq (\Sigma(J \cap \mathbb{B})) \cap \mathbb{B} \subseteq \Sigma(J) \cap \mathbb{B} = J \cap \mathbb{B}$. □

Lemma 4.2. *We have $\Sigma_{\mathbb{B}} = \{\, J \cap \mathbb{B} \in \Sigma^{\mathbb{B}} \mid J \in \Sigma,\, J = \Sigma(J \cap \mathbb{B}) \,\}$.*

Proof. Given $J \in \Sigma$, consider $I = \Sigma(J \cap \mathbb{B})$. By Lemma 4.2 we have $I = \Sigma(I \cap \mathbb{B})$ and $J \cap \mathbb{B} = I \cap \mathbb{B}$, which proves the claim. □

Lemma 4.3. *We have $\Sigma_{\mathbb{B}} = \{\, I \in \Sigma^{\mathbb{B}} \mid \Sigma(I) \cap \mathbb{B} = I \,\}$.*

Proof. Given $I \in \Sigma_{\mathbb{B}}$, we have $I = J \cap \mathbb{B}$ for some ideal $J \in \Sigma$. By Lemma 4.1 we obtain $\Sigma(I) \cap \mathbb{B} = I$. The reverse implication is obvious. □

Lemma 4.4. *We have*

$$\Sigma_{\mathbb{A}} = \{ I \in \Sigma^{\mathbb{A}} \mid \forall x, y \in W : \mathrm{bal}(x) + \mathrm{bal}(y) = 0 \implies xIy \subseteq I \}. \tag{4.1}$$

Hence $\Sigma_{\mathbb{A}}$ is closed under the lattice operation $I + J$.

Proof. Write $\mathcal{J}$ for the righthanded set of (4.1). Consider $I \in \Sigma_{\mathbb{A}}$ and write it as $I = J \cap \mathbb{A}$ for some $J \in \Sigma$. If $i \in I$ and $x, y \in W$ with $\mathrm{bal}(x) + \mathrm{bal}(y) = 0$ then $xiy \in \mathbb{A} \cap J$. This shows that $\Sigma_{\mathbb{A}} \subseteq \mathcal{J}$.

To prove $\mathcal{J} \subseteq \Sigma_{\mathbb{A}}$, consider $I \in \mathcal{J}$. Since $I \subseteq \mathbb{A}$, $I \subseteq \Sigma(I) \cap \mathbb{A}$. For the reverse inclusion consider $z \in \Sigma(I) \cap \mathbb{A}$. We may write $z = \sum \alpha_k x_k i_k y_k$ for some scalars $\alpha_k \in \mathbb{C}$, some $i_k \in I$, and some (possibly empty) words $x_k, y_k \in W$. We have $F(z) = z$ for the conditional expectation F of Lemma 2.5 as $z \in \mathbb{A}$. Hence $z = \sum \beta_k x_k i_k y_k$ for some scalars $\beta_k \in \mathbb{C}$ such that $\beta_k = 0$ if $\mathrm{bal}(x_k) + \mathrm{bal}(y_k) \neq 0$. This shows that $z \in I$ as $I \in \mathcal{J}$. We have proved that $I = \Sigma(I) \cap \mathbb{A}$, which is in $\Sigma_{\mathbb{A}}$. □

In the next lemma we state a result of Bratteli [1], now for not necessarily separable AF-algebras. We skip the proof which just consists of a slight adaption of Bratteli's proof.

Lemma 4.5. *Let A be a locally matricial algebra and $\overline{A}$ its C^*-algebraic norm closure. There is a bijection γ between the family of self-adjoint two-sided ideals in A and the family of closed two-sided ideals in $\overline{A}$ through $\gamma(I) = \overline{I}$ and $\gamma^{-1}(I) = I \cap A$.*

Theorem 4.6. *Every $*$-subalgebra $\mathbb{B}$ of $\mathbb{A}$ induces an injective map $\Phi_{\mathbb{B}} : \Sigma_{\mathbb{B}} \longrightarrow \mathcal{I}$ given by $\Phi_{\mathbb{B}}(I) = \mathcal{I}(I)$ for $I \in \Sigma_{\mathbb{B}}$. The inverse map is determined by $\Phi_{\mathbb{B}}^{-1}(D) = D \cap \mathbb{B}$ for $D \in \mathcal{I}$. For all $I, J \in \Sigma_{\mathbb{B}}$ we have*

$$\begin{aligned} \Phi_{\mathbb{B}}(I + J) &= \Phi_{\mathbb{B}}(I) + \Phi_{\mathbb{B}}(J) && \text{if } I + J \in \Sigma_{\mathbb{B}}, \\ \Phi_{\mathbb{B}}(I \cap J) &= \Phi_{\mathbb{B}}(I) \cap \Phi_{\mathbb{B}}(J) && \text{if } \Phi_{\mathbb{B}}(I) \cap \Phi_{\mathbb{B}}(J) \in \Phi_{\mathbb{B}}(\Sigma_{\mathbb{B}}). \end{aligned}$$

Proof. Step 1. At first we are going to check injectivity of $\Phi_{\mathbb{A}}$. Let $I \in \Sigma_{\mathbb{A}}$, and put $D = \mathcal{I}(I)$. Then $\overline{I} \subseteq \overline{D \cap \mathbb{A}}$ (norm-closures in $C^*(\mathbb{F}/\mathbb{I})$). To prove the reverse inclusion $\overline{D \cap \mathbb{A}} \subseteq \overline{I}$, suppose that $x \in D \cap \mathbb{A}$. Let $\varepsilon > 0$. Since $D = \overline{\Sigma(I)}$, there is some $y \in \Sigma(I)$ such that $\|x - y\| \leq \varepsilon$. Let F be the conditional expectation of Lemma 2.5. Since $Fx = x$, we have

$$\|x - Fy\| = \|Fx - Fy\| \leq \|x - y\| \leq \varepsilon.$$

Choose for y a representation $y = \sum \alpha_i a_i x_i b_i$ for some scalars $\alpha_i \in \mathbb{C}$, some (possibly empty) words $a_i, b_i \in W$, and some elements $x_i \in J$. Since $\mathrm{bal}(x_i) = 0$, either $F(a_i x_i b_i) = a_i x_i b_i$ or $F(a_i x_i b_i) = 0$. Hence $Fy = \sum \beta_i a_i x_i b_i \in \mathbb{A}$ for some scalars $\beta_i \in \mathbb{C}$, and consequently $Fy \in \Sigma(I) \cap \mathbb{A} = I$ by Lemma 4.3. Since $\varepsilon > 0$ was arbitrary, $x \in \overline{I}$.

We have proved that $\overline{I} = \overline{D \cap \mathbb{A}}$, and so $I = D \cap \mathbb{A}$ by Lemma 4.5. Hence $\Phi_{\mathbb{A}}^{-1} \Phi_{\mathbb{A}}(I) = I$ if we set $\Phi_{\mathbb{A}}^{-1}(D) = D \cap \mathbb{A}$. Hence $\Phi_{\mathbb{A}}$ is injective.

Step 2. In this step we will show that $\Phi_{\mathbb{B}}$ injective. Define $\mu : \Sigma_{\mathbb{B}} \to \Sigma_{\mathbb{A}}$ by $\mu(I) = \Sigma(I) \cap \mathbb{A}$. The map μ is injective as $\mu^{-1}(J) = J \cap \mathbb{B}$ is an inverse for μ by

Lemma 4.3. The identity

$$\Phi_{\mathbb{A}}(\mu(I)) = \Phi_{\mathbb{A}}(\Sigma(I) \cap \mathbb{A}) = \overline{\Sigma(\Sigma(I) \cap \mathbb{A})} = \overline{\Sigma(I)} = \Phi_{\mathbb{B}}(I)$$

shows that $\Phi_{\mathbb{B}} = \Phi_{\mathbb{A}}\mu$, and so $\Phi_{\mathbb{B}}$ is injective by the proved injectivity of $\Phi_{\mathbb{A}}$. To prove the formula for $\Phi_{\mathbb{B}}^{-1}$ we note that

$$\Phi_{\mathbb{B}}^{-1}(D) = \mu^{-1}\Phi_{\mathbb{A}}^{-1}(D) = (D \cap \mathbb{A}) \cap \mathbb{B} = D \cap \mathbb{B}.$$

Step 3. To prove the lattice rules for $\Phi_{\mathbb{B}}$ we consider $I_1, I_2 \in \Sigma_{\mathbb{B}}$ and set $D_1 = \Phi_{\mathbb{B}}(I_1), D_2 = \Phi_{\mathbb{B}}(I_2)$. If $D_1 \cap D_2 \in \Phi_{\mathbb{B}}(\Sigma_{\mathbb{B}})$ then

$$\Phi_{\mathbb{B}}^{-1}(D_1 \cap D_2) = \Phi_{\mathbb{B}}^{-1}(D_1) \cap \Phi_{\mathbb{B}}^{-1}(D_2) = I_1 \cap I_2,$$

which shows $D_1 \cap D_2 = \Phi_{\mathbb{B}}(I_1 \cap I_2)$. If $I_1 + I_2 \in \Sigma_{\mathbb{B}}$ then

$$\Phi_{\mathbb{B}}(I_1 + I_2) = \overline{\Sigma(I_1 + I_2)} = \overline{\Sigma(D_1 + D_2)} = D_1 + D_2.$$

□

We need a lemma which is often used in the theory of Cuntz–Krieger type algebras.

Lemma 4.7. *Let J be a subset of $\mathbb{A}$. Then the gauge actions exist on $(\mathbb{F}/\mathbb{I})/\Sigma(J)$, so (A) is satisfied for the same H. One has $\mathrm{bal}(q_J(x)) = \mathrm{bal}(x)$ for all words $x \in W$ with $q_J(x) \neq 0$. If π is a representation of $\mathbb{F}/\mathbb{I}$, X a linear subspace of $\mathbb{A}$ and $J := \ker(\pi|_X)$ then the representation $\tilde{\pi}$ induced by π by dividing out J is injective on $q_J(X)$ ($\pi = \tilde{\pi} q_J$).*

Proof. It is well known that $\mathbb{A}$ is the fixed point algebra of the gauge action t. Hence, $t_\lambda(j) = j$ for $j \in J$ and $\lambda \in H$ since $J \subseteq \mathbb{A} = \mathrm{lin}(W_0)$. Since an $x \in \Sigma(J)$ allows a representation $x = \sum_i \alpha_i a_i j_i b_i$ for scalars $\alpha_i \in \mathbb{C}$, (possibly empty) words $a_i, b_i \in W$, and elements $j_i \in J$, this shows that $t_\lambda(\Sigma(J)) \subseteq \Sigma(J)$ ($\lambda \in H$). Hence the gauge actions exist on $(\mathbb{F}/\mathbb{I})/\Sigma(J)$. For the last claim, if $\tilde{\pi}(q_J(x)) = 0$ for $x \in X$, then $\pi(x) = 0$, then $x \in \ker(\pi|_X)$, then $x \in J$, then $q_J(x) = 0$, showing that $\tilde{\pi}$ is injective on $q_J(X)$. □

Definition 4.8. An ideal $I \in \Sigma_{\mathbb{A}}$ is called cancelling if $\mathbb{F}/\mathbb{I}$ divided by I satisfies property (C').

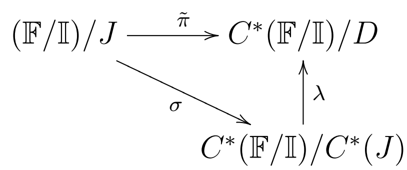
The proof of the next theorem will reveal that I is cancelling if and only if $\mathbb{F}/\mathbb{I}$ divided by I is a Cuntz–Krieger type $*$-algebra. Write $\Omega_{\mathbb{A}} \subseteq \Sigma_{\mathbb{A}}$ for the family of all cancelling ideals.

Theorem 4.9. *We have $\Phi_{\mathbb{A}}(\Omega_{\mathbb{A}}) = \{ D \in \mathcal{I} \mid D \cap \mathbb{A} \in \Omega_{\mathbb{A}} \}$.*

Proof. Define $\mathcal{J} = \{ D \in \mathcal{I} \mid D \cap \mathbb{A} \in \Omega_{\mathbb{A}} \}$. To prove $\Phi_A(\Omega_{\mathbb{A}}) \subseteq \mathcal{J}$, consider an element $I \in \Omega_{\mathbb{A}}$, and note that $\Phi_{\mathbb{A}}^{-1}(\Phi_{\mathbb{A}}(I)) = I = \Phi_{\mathbb{A}}(I) \cap \mathbb{A} \in \Omega_{\mathbb{A}}$ by Theorem 4.6. Hence $\Phi_{\mathbb{A}}(I) \in \mathcal{J}$.

To prove $\mathcal{J} \subseteq \Phi_A(\Omega_{\mathbb{A}})$ consider an element $D \in \mathcal{J}$. Define $J = \Sigma(D \cap \mathbb{A})$. Write $\pi : \mathbb{F}/\mathbb{I} \longrightarrow C^*(\mathbb{F}/\mathbb{I})/D$ for the canonical quotient map. Write $C^*(J)$ for the norm closure of J in $C^*(\mathbb{F}/\mathbb{I})$. As J is a two-sided self-adjoint ideal in $\mathbb{F}/\mathbb{I}$ by definition, $C^*(J)$ is a two-sided closed ideal in the norm closure $C^*(\mathbb{F}/\mathbb{I})$ of $\mathbb{F}/\mathbb{I}$. Since $C^*(J) \subseteq D$, π induces a homomorphism $\tilde{\pi} : (\mathbb{F}/\mathbb{I})/J \longrightarrow C^*(\mathbb{F}/\mathbb{I})/D$.

There is also a canonical homomorphism $\sigma : (\mathbb{F}/\mathbb{I})/J \longrightarrow C^*(\mathbb{F}/\mathbb{I})/C^*(J)$. Hence, by introducing a further quotient map λ, we obtain a commutative diagram



Since $D \cap \mathbb{A} = \ker(\pi|_{\mathbb{A}})$, by Lemma 4.7 the algebra $(\mathbb{F}/\mathbb{I})/J$ is invariant under the gauge actions and $\tilde{\pi}$ is injective on $q_J(\mathbb{A})$, which is the new core "$\mathbb{A}$" for the algebra $(\mathbb{F}/\mathbb{I})/J$ since $\mathrm{bal}(q_J(x)) = \mathrm{bal}(x)$. So $(\mathbb{F}/\mathbb{I})/J$ is an algebra which satisfies (A) and (B), and there exists an $\mathbb{A}$-faithful C^*-representation $\tilde{\pi}$. Since J is generated by the cancelling ideal $D \cap \mathbb{A} \in \Omega_{\mathbb{A}}$, by Definition 4.8 $(\mathbb{F}/\mathbb{I})/J$ satisfies also (C') and so is a Cuntz–Krieger $*$-algebra.

Hence, by Theorem 2.2 the images of $\tilde{\pi}$ and σ are canonically isomorphic, and so λ is proved to be an isomorphism. By the definition of λ this implies $C^*(J) = D$. Since $D \in \mathcal{J}$, $D \cap \mathbb{A} \in \Omega_{\mathbb{A}}$, and so $D = C^*(J) = \Phi_{\mathbb{A}}(D \cap \mathbb{A}) \in \Phi_{\mathbb{A}}(\Omega_{\mathbb{A}})$ as we wanted to show. □

Corollary 4.10. *If all ideals in $\Sigma_{\mathbb{A}}$ are cancelling then $\Phi_{\mathbb{A}}$ is a lattice isomorphism.*

Proof. Since all ideals in $\Sigma_{\mathbb{A}}$ are cancelling, $\Omega_{\mathbb{A}} = \Sigma_{\mathbb{A}}$. By Theorem 4.9, $\Phi_{\mathbb{A}}$ is surjective. By Theorem 4.6 and Lemma 4.4, $\Phi_{\mathbb{A}}$ is an injective lattice homomorphism. □

We aim to generalise the last theorem by allowing $\mathbb{A}$ to be a smaller algebra $\mathbb{B}$. The sense of the next definition will become clear in Corollary 4.13 or in the proof of Corollary 4.14.

Definition 4.11. An ideal $I \in \Sigma_{\mathbb{B}}$ is called $\mathbb{B}$-cancelling if $X := (\mathbb{F}/\mathbb{I})/\Sigma(I)$ satisfies property (C'), and every arbitrarily given C^*-representation of X is injective on $q_I(\mathbb{A})$ if and only if it is injective on $q_I(\mathbb{B})$.

Note that cancelling is the same as $\mathbb{A}$-cancelling. Write $\Omega_{\mathbb{B}} \subseteq \Sigma_{\mathbb{B}}$ for the family of $\mathbb{B}$-cancelling ideals. The next theorem and corollary generalise the last ones.

Theorem 4.12. *We have $\Phi_{\mathbb{B}}(\Omega_{\mathbb{B}}) = \{ D \in \mathcal{I} \mid D \cap \mathbb{B} \in \Omega_{\mathbb{B}} \}$.*

Proof. This is proved exactly like Theorem 4.9. One just replaces $\mathbb{A}$ by $\mathbb{B}$ and $\Omega_{\mathbb{A}}$ by $\Omega_{\mathbb{B}}$ everywhere. □

Corollary 4.13. *If all ideals in $\Sigma_{\mathbb{B}}$ are $\mathbb{B}$-cancelling then $\Phi_{\mathbb{B}}$ is a bijection.*

Proof. Since all ideals in $\Sigma_{\mathbb{B}}$ are $\mathbb{B}$-cancelling, $\Omega_{\mathbb{B}} = \Sigma_{\mathbb{B}}$. By Theorem 4.12 $\Phi_{\mathbb{B}}$ is surjective and by Theorem 4.6 $\Phi_{\mathbb{B}}$ is injective. □

We shall now apply the last corollary to cancelling higher rank semigraph algebras [5].

Corollary 4.14. *Let $\mathbb{F}/\mathbb{I}$ be a cancelling semigraph algebra (see* [5, Definitions 5.1 and 7.2]*), and $\mathbb{B}$ the $*$-subalgebra of $\mathbb{A}$ generated by the standard projections* *(see* [5, Definition 5.14]*). Then every quotient of* $\mathbb{F}/\mathbb{I}$ *by an ideal in* $\Sigma_{\mathbb{B}}$ *is a semigraph algebra by* [5, Lemma 8.1]*. Now if every such quotient is cancelling (as a semigraph algebra), then* $\Phi_{\mathbb{B}}$ *is a bijection.*

Proof. A C^*-representation of a cancelling semigraph algebra is injective on $\mathbb{A}$ if and only it is injective on $\mathbb{B}$ by [5, Corollary 6.4]. If I is an ideal in $\Sigma_{\mathbb{B}}$, then the image of q_I is a semigraph algebra by [5, Lemma 8.1]. The set of standard projections (see [5, Definition 5.14]) in the semigraph algebra $q_I(\mathbb{F}/\mathbb{I})$ are the image of the standard projections in $\mathbb{F}/\mathbb{I}$; so $q_I(\mathbb{B})$ is the $*$-algebra generated by the standard projections in $q_I(\mathbb{F}/\mathbb{I})$. Note also that $q_I(\mathbb{A})$ is the core, or the "$\mathbb{A}$", of $q_I(\mathbb{F}/\mathbb{I})$. Hence by [5, Corollary 6.4], a C^*-representation of $q_I(\mathbb{F}/\mathbb{I})$ is injective on $q_I(\mathbb{A})$ if and only if it is injective on $q_I(\mathbb{B})$. So if we assume that $q_I(\mathbb{F}/\mathbb{I})$ is cancelling (as a semigraph algebra), then it is a Cuntz–Krieger type $*$-algebra, and so satisfies (C'), and by Definition 4.11 I is $\mathbb{B}$-cancelling.

So if we assume that $q_I(\mathbb{F}/\mathbb{I})$ is cancelling for every $I \in \Sigma_{\mathbb{B}}$, then $\Sigma_{\mathbb{B}}$ conists of $\mathbb{B}$-cancelling ideals only, and so $\Sigma_{\mathbb{B}} = \Omega_{\mathbb{B}}$. The claim follows thus by Corollary 4.13. □

Corollary 4.15. *If every quotient of a cancelling semigraph algebra* $\mathbb{F}/\mathbb{I}$ *by an ideal in* $\Sigma_{\mathbb{A}}$ *is cancelling (as a semigraph algebra), then* $\Phi_{\mathbb{A}}$ *is a lattice isomorphism.*

Proof. One repeats the last three sentences of the proof of Corollary 4.14 and replaces $\mathbb{B}$ by $\mathbb{A}$ everwhere. □

5. Crossed Product Representation and Nuclearity

By using the Cuntz–Krieger uniqueness theorem, Theorem 2.2, we can extend each gauge action $t_\lambda \in \mathrm{Aut}(\mathbb{F}/\mathbb{I})$ to a gauge actions $\theta_\lambda \in \mathrm{Aut}(C^*(\mathbb{F}/\mathbb{I}))$ $(\lambda \in H)$. We may thus apply Takai's duality theorem [16] and obtain the following result.

Theorem 5.1. *By Takai's duality theorem we have*

$$C^*(\mathbb{F}/\mathbb{I}) \otimes \mathcal{K}(L^2(\mathcal{H})) \cong C^*(\mathbb{F}/\mathbb{I}) \rtimes_\theta H \rtimes_{\widehat{\theta}} \widehat{H}.$$

Moreover, $C^*(\mathbb{F}/\mathbb{I}) \rtimes_\theta H$ *is the norm closure of a locally matricial algebra. Hence* $C^*(\mathbb{F}/\mathbb{I})$ *is nuclear.*

Proof. The nuclearity is concluded from the observation that $C^*(\mathbb{F}/\mathbb{I})$ is then evidently the corner of a crossed product of a (possibly non-separable) AF-algebra by an abelian group.

We assume that ζ is injective (Lemma 2.3). *Step 1.* In the first step we follow the idea in [13, Lemma 3.1]. We denote the crossed product $C^*(\mathbb{F}/\mathbb{I}) \rtimes_\theta H$ by A. Let $\mathcal{M}(A)$ be the multiplier algebra of A. Let $(U_\lambda)_{\lambda \in H} \subseteq \mathcal{M}(A)$ be the unitaries inducing the actions $(\theta_\lambda)_{\lambda \in H}$. Let

$$\chi(F) := \int_H F(\lambda) U_\lambda d\lambda \qquad \forall F \in \widehat{H},$$

where we integrate in $\mathcal{M}(A)$, and where $d\lambda$ denotes the normalized Haar measure on H. It is easy to see that $(\chi(F))_{F \in \widehat{H}}$ forms a family of mutually orthogonal projections in $\mathcal{M}(A)$.

Recall that $\text{bal}(a)_\lambda a = \lambda_a a = \theta_\lambda(a)$ for $a \in \mathcal{A}$ and $\lambda \in H$, and we write the group operation of $\hat{H}$ additively. Notice that

$$\chi(F)a = a\chi(F + \text{bal}(a)) \qquad \forall a \in \mathcal{A} \, \forall F \in \widehat{H}. \tag{5.1}$$

Notice that $a\chi(F) \in A$ for all $a \in \mathcal{A}$ and $F \in \widehat{H}$. By an application of the Stone-Weierstrass theorem the linear span of $\widehat{H}$ is dense in $L^1(H)$. Hence A is the norm closure of

$$B := \text{lin}\{\, \chi(F)x \mid x \in W,\, F \in \widehat{H}\,\}.$$

Step 2. It remains to show that B is locally matricial. Consider a finite subset

$$\Gamma = \{\chi(F_1)x_1, \chi(F_2)x_2, \ldots, \chi(F_n)x_n\}$$

for some fixed nonzero $x_1, \ldots, x_n \in W$ and $F_1, \ldots, F_n \in \widehat{H}$. By enlarging Γ, if necessary, we can assume that Γ is self-adjoint (possible by identity (5.1)).

Let ω be the set of nonzero words in the alphabet Γ. By identity (5.1) each $y \in \omega$ has a representation

$$y = \chi(F_{j_1})x_{j_1}\chi(F_{j_2})x_{j_2} \ldots \chi(F_{j_m})x_{j_m} = \chi(F_{j_1})x_{j_1}x_{j_2} \ldots x_{j_m}$$

for some $1 \le j_1, \ldots, j_m \le n$. Since $y \ne 0$, we necessarily have

$$F_{j_{k+1}} = F_{j_k} + \text{bal}(x_{j_k}) \qquad \forall k = 1, \ldots, m-1.$$

Let

$$\begin{aligned} K \;\; = \;\; \{\, & x_{j_1}x_{j_2} \ldots x_{j_m} \in \mathbb{F}/\mathbb{I} \,|\, m \ge 1,\, 1 \le j_1, \ldots, j_{m+1} \le n, \\ & F_{j_{k+1}} = F_{j_k} + \text{bal}(x_{j_k}) \;\; \forall k = 1, \ldots, m\,\}. \end{aligned}$$

Notice that

$$\omega \subseteq \Gamma \cup \{\chi(F_1), \ldots, \chi(F_n)\}K\Gamma$$

(products in A). Thus, if we can show that K lies in some finite dimensional space $\mathcal{M}_n$ then $\text{lin}(\omega) = \text{Alg}^*(\Gamma)$ is a subspace of the finite dimensional space

$$\text{lin}(\Gamma \cup \{\chi(F_1), \ldots, \chi(F_n)\}\mathcal{M}_n\Gamma),$$

and we are done.

We shall construct $\mathcal{M}_n$ by induction. Let $\gamma \subseteq \{1, \ldots, n\}$ and

$$L_\gamma := \{\, x_{j_1}x_{j_2} \ldots x_{j_m} \in K \mid \{F_{j_1}, F_{j_2}, \ldots, F_{j_{m+1}}\} \subseteq \{\, F_i \mid i \in \gamma\,\}\,\}.$$

If $|\gamma| = 1$ then all x_{j_k} of $x_{j_1}x_{j_2} \ldots x_{j_m} \in L_\gamma$ are zero-balanced. Let $\mathcal{M}_1 \subseteq \mathbb{A}$ be a finite dimensional $*$-algebra containing $\{\, x_i \in \mathbb{A} \mid 1 \le i \le n,\, \text{bal}(x_i) = 0\,\}$. Then it is clear that $L_\gamma \subseteq \mathcal{M}_1$.

By induction hypothesis on $N = 1, \ldots, n-1$ we assume that there exists a finite dimensional vector space $\mathcal{M}_N$, such that $L_\gamma \subseteq \mathcal{M}_N$ for all $\gamma \subseteq \{1, \ldots, n\}$ with $|\gamma| = N$.

Let $\delta \subseteq \{1, \ldots, n\}$ with $|\delta| = N + 1$. Let $x = x_{j_1}x_{j_2} \ldots x_{j_m} \in L_\delta$. Let

$$\{\, 1 \le i \le m+1 \mid F_{j_i} = F_{j_1}\,\} =: \{1 = i_1 \le \ldots \le i_M \le m+1\}.$$

For $k = 1, \ldots, M-1$ let

$$y_k = \prod_{t=i_k}^{i_{k+1}-1} x_{j_t}.$$

Since y_k is a partial word of the word $x = x_{j_1} x_{j_2} \dots x_{j_m}$ which lives in K, we get

$$\mathrm{bal}(y_k) = \sum_{t=i_k}^{i_{k+1}-1} \mathrm{bal}(x_{j_t}) = \sum_{t=i_k}^{i_{k+1}-1} F_{j_{t+1}} - F_{j_t} = F_{j_{i_{k+1}}} - F_{j_{i_k}} = F_{j_1} - F_{j_1} = 0.$$

Hence y_k is zero-balanced and lives in $\mathbb{A}$. We have

$$x = y_1 y_2 \dots y_{M-1} x_{j_{i_M}} x_{j_{i_M+1}} \dots x_{j_m}.$$

Notice that for all $k = 1, \dots, M$, both the 'middle term' of y_k, i.e.

$$x_{j_{i_k+1}} x_{j_{i_k+2}} \dots x_{j_{i_{k+1}-2}},$$

and the 'end term' of x, i.e. $x_{j_{i_M+1}} \dots x_{j_m}$, lie in $L_{\delta \setminus \{j_1\}} \subseteq \mathcal{M}_N$ (the inclusion is by induction hypothesis). Thus $y_1, \dots, y_{M-1}$ lie in the finite dimensional vector space

$$Y = \Big(\sum_{s=1}^{n} \mathbb{C} x_s + \sum_{s,t=1}^{n} \mathbb{C} x_s x_t + \sum_{s,t=1}^{n} x_s \mathcal{M}_N x_t \Big) \cap \mathbb{A}.$$

Hence $Z = \mathrm{Alg}^*(Y)$ is a finite dimensional vector space since $Y \subseteq \mathbb{A}$. Thus $y_1 \dots y_{M-1} \in Z$, and x lies in the finite dimensional vector space

$$\mathcal{M}_{N+1} = Z + \sum_{s=1}^{n} Z x_s + \sum_{s=1}^{n} Z x_s \mathcal{M}_N.$$

Notice that the choice of $\mathcal{M}_{N+1}$ is independent of δ and $x \in L_\delta$. This completes the induction. If $N+1 = n$ then the proof is complete since then $K = L_{\{1,\dots,n\}} \subseteq \mathcal{M}_n$. □

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References

1. O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.
2. B. Burgstaller, *The uniqueness of Cuntz–Krieger type algebras*, J. reine angew. Math. **594** (2006), 207–236.
3. B. Burgstaller, *A class of higher rank Exel–Laca algebras*, Acta Sci. Math. (Szeged) **73** (2007), 209–235.
4. B. Burgstaller, *Representations of crossed products by cancelling actions and applications*, Houston J. Math. **38** (2012), no. 3, 761–774.
5. B. Burgstaller, *A Cuntz–Krieger uniqueness theorem for semigraph C^*-algebras*, Banach J. Math. Anal. **6** (2012), no. 2, 38–57.
6. J. Cuntz, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173–185.
7. J. Cuntz, *A class of C^*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C^*-algebras*, Invent. Math. **63** (1981), 25–40.
8. J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

9. S. Eilers, T. Katsura, M. Tomforde and J. West, *The ranges of K-theoretic invariants for nonsimple graph algebras*, Trans. Amer. Math. Soc. **368** (2016), no. 6, 3811–3847.
10. E. Gillaspy, *K-theory and homotopies of 2-cocycles on higher-rank graphs*, Pacific J. Math. **278** (2015), no. 2, 407–426.
11. A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.
12. I. Raeburn, A. Sims and T. Yeend, *The C^*-algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. **213** (2004), 206–240.
13. I. Raeburn and W. Szymanski, *Cuntz–Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), 39–59.
14. G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras*, J. reine angew. Math. **513** (1999) 115–144.
15. M. Rørdam and E. Størmer, *Classification of nuclear C^*-algebras. Entropy in operator algebras*, Encyclopaedia of Mathematical Sciences 126. Operator Algebras and Non-Commutative Geometry 7. Berlin: Springer. 198 p. (2002).
16. H. Takai, *On a duality for crossed products of C^*-algebras*, J. Funct. Anal. **19** (1975), 25–39.
17. M. Tomforde, *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs*, J. Operator Theory **50** (2003), no. 2, 345–368.

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