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**Abstract**

We derive the filtering equation for Markovian systems undergoing homodyne measurement in the situation where the output processes being monitored are squeezed. The filtering theory applies to case where the system is driven by Fock noise (that, quantum input processes in a coherent state) and where the output is mixed with a squeezed signal. It also applies to the case of a system driven by squeezed noise, but here there is a physical restriction to emission/absorption coupling only. For the special case of a cavity mode where the dynamics is linear, we are able to derive explicitly the filtered estimate $\pi_t(a)$ for the mode annihilator $a$ based on the homodyne quadrature observations up to time $t$.

In memory of Slava Belavkin.

## 1 Introduction

The theory of quantum filtering was developed by V.P. Belavkin in the 1980's [1]-[4] as the extension of classical filtering theory [5]-[10]. It has been subsequently developed as a technique for quantum measurement and control [11]-[23].

Our aim here is to derive the filter for a system whose output is either squeezed light, or is mixed with squeezed light. For the special case of a linear dynamics, we are able to give the filter explicitly. The advantage of using squeezing as a control resource was proposed in [24] and [25]. In the experimental setup in figure 1 below we have a input-system-output device with dynamical coupling operators $(S, L, H)$, see later, whose output is sent through a beam splitter and mixed with a second input. The input into the system will be modelled as a coherent state, however we take the second input to be in a squeezed coherent state. The output processes from the beam splitter are then subject to a homodyne measurement. The scheme depicted in figure 1 has been

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proposed in [26] as a component of an entanglement with large squeezing to suppress noise.
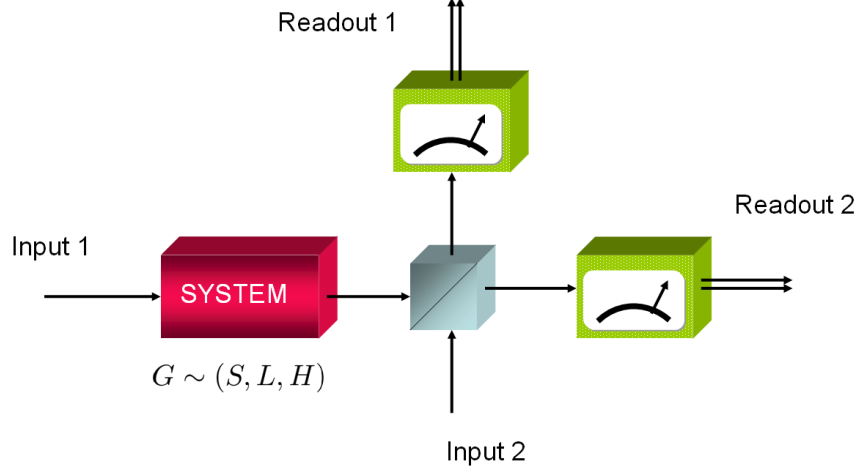


Figure 1: Experimental setup: an open quantum system is driven by a coherent state field (input 1) and the output enters a 50-50 beamsplitter along with a squeezed coherent state field (input 2). The single-line arrows stand for quantum field inputs, while the double-line arrows imply classical readout.

The input processes are described by independent quantum Wiener annihilation processes $A(\cdot)$ and $B(\cdot)$ and we have

$$[A(t), A^*(s)] = [B(t), B^*(s)] = t \wedge s,$$

where $t \wedge s = \min(t, s)$. The inputs are in Gaussian states with means

$$\mathbb{E}[dA(t)] = \alpha(t) \, dt \quad, [dB(t)] = \beta(t). \tag{1}$$

We shall take the two inputs to be independent, with input one being a coherent state (and so having the same covariances as the vacuum) while input two is squeezed. The quantum Itō table [27] will then have the form

$$
\begin{array}{c|cc}
\times & dB & dB^* \\
\hline
dB & 0 & dt \\
dB^* & 0 & 0
\end{array}
\quad , \quad
\begin{array}{c|cc}
\times & dA & dA^* \\
\hline
dA & mdt & (n+1)\,dt \\
dA^* & ndt & m^*dt
\end{array}
\quad , \tag{2}
$$

where $n > 1$ and $|m|^2 \le n(n+1)$. Note that $dAdB^* \equiv 0$, etc.

The output $B^{\text{out}}$ from the system is passed through the beam splitter along with the squeezed input $A^{\text{out}} \equiv A$ leading to the overall output

$$
\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} B^{\text{out}} \\ A^{\text{out}} \end{bmatrix}
$$

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with $C_1 = \frac{1}{\sqrt{2}}\left(B^{\mathrm{out}} + A^{\mathrm{out}}\right)$ and $C_2 = \frac{1}{\sqrt{2}i}\left(B^{\mathrm{out}} - A^{\mathrm{out}}\right)$ then being independent annihilation processes: $\left[C_j\left(t\right), C_k^*\left(s\right)\right] = \delta_{jk}\ \left(t \wedge s\right)$.

Finally we measure the quadratures

$$Y_1 = C_1 + C_1^* \equiv \frac{1}{\sqrt{2}}\left(Q_B^{\mathrm{out}} + Q_A^{\mathrm{out}}\right)$$

$$Y_2 = C_2 + C_2^* \equiv \frac{1}{\sqrt{2}}\left(P_B^{\mathrm{out}} - P_A^{\mathrm{out}}\right)$$

at the detectors. Here $Q_B^{\mathrm{out}} = B^{\mathrm{out}} + B^{\mathrm{out}*}$ and $P_B^{\mathrm{out}} = \frac{1}{i}(B^{\mathrm{out}} - B^{\mathrm{out}*})$, etc. We will see that quadratures then have increments of the form

$$dY_1\left(t\right) = \frac{1}{\sqrt{2}}\left\{j_t\left(S\right)dB\left(t\right) + j_t\left(L\right)dt + dA\left(t\right) + \mathrm{H.c.}\right\},$$

$$dY_2\left(t\right) = \frac{1}{\sqrt{2}i}\left\{j_t\left(S\right)dB\left(t\right) + j_t\left(L\right)dt - dA\left(t\right) - \mathrm{H.c.}\right\}, \qquad (3)$$

where $j_t\left(\cdot\right)$ transfers to the interaction picture, and $S$ and $L$ are prescribed operators determining how the system couples to the input field.

The outputs $Y_1$ and $Y_2$ have non-trivial correlation expressed through the following table

| $\times$ | $dY_1$ | $dY_2$ |
|---|---|---|
| $dY_1$ | $\left(1 + n + m'\right)dt$ | $m''dt$ |
| $dY_2$ | $m''dt$ | $\left(1 + n - m'\right)dt$ |

where $m = m' + im''$ is the decomposition of the squeezing parameter into real and imaginary parts. In particular, the matrix

$$K = \left[\begin{array}{cc} 1 + n + m' & m'' \\ m'' & 1 + n - m' \end{array}\right] \qquad (4)$$

has determinant $\Delta = \left(1 + n\right)^2 - |m|^2 > 1 + n$ and therefore is invertible.

Our interest will be in modelling the dynamical evolution of the system conditioned on the continuous observations of the output quadratures $Y_1$ and $Y_2$.

## 2    Quantum Filtering for General Gaussian States

In this section we derive the filter for a generic quantum Markov model with inputs in Gaussian states. Generalizing the situation in the introduction, we allow for $m_{\mathrm{Fock}}$ vacuum inputs $B_1, \cdots, B_{m_{\mathrm{Fock}}}$ and $m_{\mathrm{sq}}$ squeezed inputs $A_1, \cdots, A_{m_{\mathrm{sq}}}$. We may think of the input driving a single component (with Hilbert space $\mathfrak{h}$) which in fact is a network of several quantum inputs. The inputs are fields on a Fock space $\mathfrak{F}$. The outputs are then fed into a measurement apparatus which performs a continuous measurement on a collection $Y_1, \cdots, Y_{n_{\mathrm{obs}}}$ of commuting
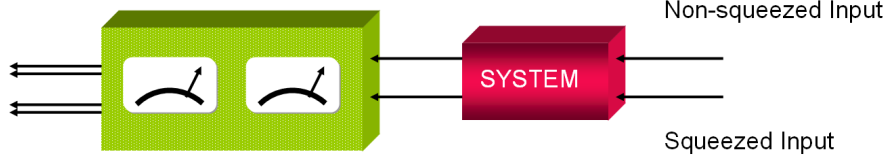
Figure 2: Equivalent block-diagram set-up of the system in figure 1.

self-adjoint processes. We shall assume that the measured processes are linear combinations of the output quadratures, and that they are linearly independent (so $n_{\mathrm{obs}} \leq m_{\mathrm{Fock}} + m_{\mathrm{sq}}$). The set-up is depicted in Figure 2.

On the joint space $\mathfrak{h} \otimes \mathfrak{F}$, we consider the quantum stochastic process $V(\cdot)$ satisfying the quantum stochastic differential equation (QSDE) [27]

$$
\begin{aligned}
dV(t) \;=\; & \{ (S_{jk} - \delta_{jk}) \otimes d\Lambda_{jk}(t) \\
& + L_j \otimes dB_j^*(t) + R_j \otimes dA_j^* \\
& - L_j^* S_{jk} \otimes dB_k(t) - R_j^* \otimes dA_j \\
& - (K_A + K_B + iH) \otimes dt \} \, V(t) \qquad (5)
\end{aligned}
$$

with $V(0) = 1$.

Here $S = [S_{jk}]$ is a unitary $m_{\mathrm{Fock}} \times m_{\mathrm{Fock}}$ matrix whose entries are bounded operators on $\mathfrak{h}$, $L_j$ $(j = 1, \cdots, m_{\mathrm{Fock}})$ and $R_j$ $(j = 1, \cdots, m_{\mathrm{sq}})$ are bounded operators and $H$ self-adjoint. This specific form of QSDE may be termed the *Hudson-Parthasarathy equation* as the algebraic conditions on the coefficients are necessary and sufficient to ensure unitarity (though the restriction of boundedness can be lifted). The process is also adapted in the sense of Hudson and Parthasarathy [27]. The operators $K_A$ and $K_B$ will be given below as (6) and a representation specific.

The processes $\Lambda_{jk}(t)$ are the scattering (or gauge) processes. Formally we may introduce $b_j(t)$ as the derivative of $B_j(t)$ in which case

$$
\Lambda_{jk}(t) \equiv \int_0^t b_j^*(\tau) b_k(\tau) \, d\tau.
$$

We adopt the summation convention that repeated Latin indices are summed from 1 to $m_{\mathrm{Fock}}$ for the non-squeezed terms (e.g. $B_j, \Lambda_{jk}, L_j, S_{jk}$), and from 1 to $m_{\mathrm{sq}}$ for the squeezed terms (e.g. $A_j, R_j$). The ranges should be clear from the context. **Note that the squeezed terms have no scattering terms!** The squeezed terms rely on the non-Fock (i.e., Araki-Woods) representation and the scattering process is not well-defined in this case.

## 2.1   Gaussian Input States

We shall aim for the most general Gaussian state for the input field. Let us introduce the Weyl displacement operator

$$
\begin{aligned}
W(g, f) \;=\; & \exp\Big[\int_0^\infty g_k(t)dB_k(t)^* + f_j(t)dA_j(t)^* \\
& - g_k(t)^* dB_k(t) - f_j(t)^* dA_j(t)\Big]
\end{aligned}
$$

with square-integrable test functions $g_k, f_k$. We require a Gaussian state where the $A$ fields are squeezed and the $B$ fields are unsqueezed (Fock): here we take

$$
\begin{aligned}
\mathbb{E}\left[W\left(g, f\right)\right] \;=\; & \exp\Big\{\int_0^\infty \left[g_k(t)\beta_k(t)^* - g_k(t)^* \beta_k(t)\right] dt - \frac{1}{2}\int g_k^*(t)g_k(t)dt\Big\} \\
& \times \exp\Big\{\int_0^\infty \left[f_k(t)\alpha_k(t)^* - f_k(t)^* \alpha_k(t)\right] dt \\
& \quad - \frac{1}{2}\int_0^\infty [\vec{f}'(t)^\top, \;\; -\vec{f}''(t)^\top]\; C \begin{bmatrix} \vec{f}'(t) \\ \vec{f}''(t) \end{bmatrix} dt\Big\}.
\end{aligned}
$$

Here the means are $\alpha_j$ are $\beta_j$ giving the multi-dimensional version of (1). The $A$ quadratures and $B$ quadratures are always independent for this state. We also have $\vec{f}'$ and $\vec{f}''$ as the vectors whose entries are the real and imaginary components of the fields $f_j$. Introducing the quadratures

$$
\begin{aligned}
Q_k \;&=\; A_k + A_k^*, \\
P_k \;&=\; \frac{1}{i}\left(A_k - A_k^*\right),
\end{aligned}
$$

and set $[X_j(t)]_1^{2m_{\text{sq}}} = \left(Q_1(t), P_1(t), \cdots, Q_{m_{\text{sq}}}(t), P_{m_{\text{sq}}}(t)\right)$. Then

$$
[X_j(t), X_k(s)] = 2iJ_{jk} \min\{t, s\}
$$

where $J = \oplus_{j=1}^{m_{\text{sq}}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The covariance matrix $C$ is then

$$
C_{jk} = \operatorname{Re} \mathbb{E}\left[\left(X_j - \bar{X}_j\right)\left(X_k - \bar{X}_k\right)\right]
$$

which is a $(2m_{\text{sq}})$-square symmetric matrix. The Heisenberg uncertainty principle implies that $C$ cannot be chosen arbitrarily, but must satisfy

$$
C + iJ \geq 0.
$$

We may write

$$
\begin{aligned}
\mathbb{E}\left[W\left(g,f\right)\right] = \exp\{ & 2\int_0^\infty \left[g_k''(t)\beta_k'(t)^* - g_k'\left(t\right)^*\beta_k''\left(t\right)\right]dt \\
& -\frac{1}{2}\int_0^\infty \left[g_j'\left(t\right)g_j'\left(t\right) + g_j''\left(t\right)g_j''\left(t\right)\right]dt \\
& +2\int_0^\infty \left[f_k''(t)\alpha_k'(t)^* - f_k'\left(t\right)^*\alpha_k''\left(t\right)\right]dt \\
& -\frac{1}{2}\int_0^\infty \left[f_j'\left(t\right)f_k'\left(t\right)C_{jk}^{PP} + f_j''\left(t\right)f_k''\left(t\right)C_{jk}^{QQ} + 2f_j'\left(t\right)f_k''\left(t\right)C_{jk}^{QP}\right]\}.
\end{aligned}
$$

### 2.1.1  The Quantum Itō Table

The quantum Itō table for the squeezed fields takes the form

$$
\begin{aligned}
dA_j dA_k^* &= \left(\delta_{jk} + N_{kj}\right)dt \\
dA_j^* dA_k &= N_{jk}\, dt \\
dA_j dA_k &= M_{jk}\, dt \\
dA_j^* dA_k^* &= M_{kj}^*\, dt
\end{aligned}
$$

where $N = N^*$ and $M = M^\top$ are complex $m_{\mathrm{sq}} \times m_{\mathrm{sq}}$ matrices. The matrices $N, M$ are fully determined from the covariance $C$ and vice versa. From the relation $C_{jk}^{QQ}dt = dQ_j dQ_k$ we see that

$$
dQ_j dQ_k = C_{jk}^{QQ}\, dt
$$

with

$$
C^{QQ} = I_{n_{\mathrm{obs}}} + N + N^\top + M + M^*.
$$

This generalizes (4) to the multidimensional case.

For the unsqueezed fields, we may additionally include the scattering process to get

$$
\begin{aligned}
dB_j dB_k^* &= \delta_{jk}\, dt \\
dB_j d\Lambda_{kl} &= \delta_{jk}\, dB_l \\
d\Lambda_{jk} dB_l^* &= \delta_{kl}\, dB_j^* \\
d\Lambda_{jk} d\Lambda_{lr} &= \delta_{kl}\, d\Lambda_{jr}.
\end{aligned}
$$

All other product of the increments $dt, dA_j, dA_j^*, dB_k, dB_k^*, d\Lambda_{jl}$ vanish.

## 2.2 The Heisenberg-Langevin Equations

The quantum stochastic process defined by (5) defines a unitary if and only if $S = [S_{jk}]$ is unitary, $H$ is self-adjoint, and

$$
\begin{aligned}
K_B &= \frac{1}{2}L_k^* L_k, \\
K_A &= \frac{1}{2}R_j^* \left( \delta_{jk} + N_{kj} \right) R_k + \frac{1}{2}R_j N_{jk} R_k^* \\
&\quad - \frac{1}{2}R_j^* M_{jk} R_k^* - \frac{1}{2}R_j M_{jk}^* R_k.
\end{aligned}
$$

(6)

For a given system operator $X$ we set

$$
j_t \left( X \right) \triangleq V^* \left( t \right) \left[ X \otimes I \right] V \left( t \right).
$$

(7)

Then from the quantum Itō calculus we get

$$
\begin{aligned}
dj_t \left( X \right) &= j_t \left( \mathcal{L}_{jk} X \right) \otimes d\Lambda_{jk}(t) \\
&\quad + j_t \left( \mathcal{L}_{j0} X \right) \otimes dB_j^* \left( t \right) + j_t \left( \mathcal{R}_{j0} X \right) \otimes dA_j^* \left( t \right) \\
&\quad + j_t \left( \mathcal{L}_{0k} X \right) \otimes dB_k \left( t \right) + j_t \left( \mathcal{R}_{0k} X \right) \otimes dA_k \left( t \right) \\
&\quad + j_t \left( \mathcal{L}_{00} X + \mathcal{R}_{00} X - i \left[ X, H \right] \right) \otimes dt
\end{aligned}
$$

(8)

where the Evans-Hudson maps $\mathcal{L}_{\mu\nu}$ are explicitly given by

$$
\begin{aligned}
\mathcal{L}_{jk} X &= S_{lj}^* X S_{lk} - \delta_{jk} X, \\
\mathcal{L}_{j0} X &= S_{lj}^* [X, L_l], \\
\mathcal{L}_{0k} X &= [L_l^*, X] S_{lk} \\
\mathcal{L}_{00} X &= \mathcal{L}_L X
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_{j0} X &= [X, R_j], \\
\mathcal{R}_{0k} X &= [R_k^*, X], \\
\mathcal{R}_{00} X &= \mathcal{L}_R X + \frac{1}{2} N_{jk} \left\{ R_k^* [X, R_j] + [R_k^*, X] R_j \right\} \\
&\quad + \frac{1}{2} N_{kj} \left\{ R_k [X, R_j^*] + [R_k, X] R_j^* \right\} \\
&\quad - \frac{1}{2} M_{jk}^* \left\{ R_k [X, R_j] + [R_k, X] R_j \right\} \\
&\quad - \frac{1}{2} M_{jk} \left\{ R_k^* [X, R_j^*] + [R_k^*, X] R_j^* \right\}
\end{aligned}
$$

where we have the standard pure Lindblad superoperator

$$
\mathcal{L}_L X \triangleq \frac{1}{2}L_l^* [X, L_l] + \frac{1}{2}[L_l^*, X] L_l.
$$

(9)

## 2.3   Output Processes

We introduce the processes

$$
\begin{aligned}
A_j^{\mathrm{out}}(t) &\triangleq V^*(t)\left[I \otimes A_j(t)\right]V(t)\\
B_j^{\mathrm{out}}(t) &\triangleq V^*(t)\left[I \otimes B_j(t)\right]V(t).
\end{aligned}
\tag{10}
$$

We note that we equivalently have $A_k^{\mathrm{out}}(t) \equiv V^*(T)\left[1 \otimes A_k(t)\right]V(T)$, for $t \leq T$, and similarly for the $B$ fields. Using the quantum Itō rules, we see that

$$
\begin{aligned}
dA_k^{\mathrm{out}} &= dA_k + j_t(R_k)dt,\\
dB_k^{\mathrm{out}} &= j_t(S_{kl})dB_l(t) + j_t(L_k)dt.
\end{aligned}
\tag{11}
$$

The readout will consist of the observations of processes $(\alpha = 1, \cdots, n_{\mathrm{obs}})$

$$
\begin{aligned}
Y_\alpha(t) &= T_{\alpha j}B_j^{\mathrm{out}}(t) + U_{\alpha k}A_k^{\mathrm{out}}(t) + \text{H.c.}\\
&\equiv V^*(t)\left[I \otimes \{T_{\alpha j}B_j(t) + U_{\alpha k}A_k(t) + \text{H.c.}\}\right]V(t),
\end{aligned}
$$

where the $T_{\alpha k}$ and $U_{\alpha j}$ are complex constants. We require that the $Y_\alpha(t)$ commute for all $\alpha$ and $t \geq 0$. This requires the following identity $Z_{\alpha\beta} = Z_{\beta\alpha}$ where we introduce

$$
Z_{\alpha\beta} \triangleq T_{\alpha k}T_{\beta k}^* + U_{\alpha j}U_{\beta k}^*.
\tag{12}
$$

We will have

$$
dY_\alpha = j_t\left(T_{\alpha j}S_{jk}\right)dB_k + U_{\alpha k}dA_k + j_t(T_{\alpha j}L_j + U_{\alpha j}R_j)dt + \text{H.c.}
\tag{13}
$$

We now consider the problem of continuously measuring processes $Y_1, \cdots, Y_{n_{\mathrm{obs}}}$. These generate the measurement algebra up to time $t$:

$$
\mathfrak{Y}_{t]} = \mathrm{vN}\left\{Y_\alpha(s) : \alpha = 1, \cdots, n_{\mathrm{obs}}, \, 0 \leq s \leq t\right\}.
\tag{14}
$$

The family $\left\{\mathfrak{Y}_{t]} : t \geq 0\right\}$ then forms an increasing family (*filtration*) of commutative von Neumann algebras.

The correlation matrix $K = [K_{\alpha\beta}]$ is defined by

$$
dY_\alpha \, dY_\beta = K_{\alpha\beta} \, dt
\tag{15}
$$

From the increment (13) we see that

$$
K_{\alpha\beta} \equiv Z_{\alpha\beta} + U_{\alpha k}N_{jk}U_{\beta j}^* + U_{\alpha k}^* N_{jk}U_{\beta j} + U_{\alpha k}M_{jk}U_{\beta j} + U_{\alpha k}^* M_{jk}^* U_{\beta j}^*.
\tag{16}
$$

## 2.4   The Filter

The filtered estimate for $j_t(X)$ given the measurement readout is then

$$
\pi_t(X) := \mathbb{E}\left[j_t(X) \mid \mathfrak{Y}_{t]}\right],
$$

where $\mathbb{E}\left[\cdot \mid \mathfrak{Y}_{t]}\right]$ is the conditional expectation onto the measurement readout algebra up to time $t$ for the given state $\mathbb{E}$ which is the product state for the

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system with the Gaussian states for the fields. The conditional expectation has the least squares property that

$$\mathbb{E}\left[\left\{\hat{X}_t - j_t\left(X\right)\right\}^2\right]$$

is a minimum over all $\hat{X}_t \in \mathfrak{Y}_{t]}$ for the choice $\hat{X}_t = \pi_t\left(X\right)$. This implies the condition that

$$\mathbb{E}\left[\left\{\pi_t\left(X\right) - j_t\left(X\right)\right\} C\left(t\right)\right] = 0$$

for every $C\left(t\right) \in \mathfrak{Y}_{t]}$.

We now state the main result which we derive in the next subsection. (We shall adopt the convention that repeated Greek indices implies a sum over the range 1 to $n_{\mathrm{obs}}$.)

**Theorem 1** *The filter satisfies the Belavkin-Kushner-Stratonovich equation*

$$d\pi_t(X) = \pi_t(\mathcal{L}X)\,dt + \mathcal{H}_t^\alpha\left(X\right) dI_\alpha\left(t\right), \qquad (17)$$

*where (suppressing the time index $t$) $\mathcal{L}X = \tilde{\mathcal{L}}_{00}X + \tilde{\mathcal{R}}_{00}X - i\left[X, H\right]$ with*

$$
\begin{aligned}
\tilde{\mathcal{L}}_{00}X &= \mathcal{L}_{00}X + \beta_j^*(t)\mathcal{L}_{j0}X + \mathcal{L}_{0k}X\beta_k(t) + \beta_j^*\mathcal{L}_{jk}X\beta_k(t), \\
\tilde{\mathcal{R}}_{00}X &= \mathcal{R}_{00}X + \alpha_j^*(t)\mathcal{R}_{j0}X + \mathcal{R}_{0k}X\alpha_k(t),
\end{aligned}
$$

*and*

$$
\begin{aligned}
\mathcal{H}_t^\alpha\left(X\right) &= \pi_t(X\tilde{L}^\alpha + \tilde{L}^{\alpha*}X) - \pi_t(X)\pi_t(\tilde{L}^\alpha + \tilde{L}^{\alpha*}) \\
&\quad + \pi_t([X, \tilde{R}^\alpha]) + \pi_t([\tilde{R}^{\alpha*}, X])
\end{aligned}
\qquad (18)
$$

*where*

$$
\begin{aligned}
\tilde{L}_\alpha &= T_{\alpha k}(L_k + S_{kl}\beta_l) + U_{\alpha j}(R_j + \alpha_j) & (19) \\
\tilde{R}_\alpha &= R_j\left[U_{\beta k}N_{jk} + U_{\beta k}^* M_{jk}^*\right]. & (20)
\end{aligned}
$$

*and with $K^{-1} = [K^{\alpha\beta}]$ the inverse matrix of $K = [K_{\alpha\beta}]$,*

$$\tilde{L}^\alpha = K^{\alpha\beta}\tilde{L}_\beta, \quad \tilde{R}^\alpha = K^{\alpha\beta}\tilde{R}_\beta$$

*and finally, the innovations processes are*

$$dI_\alpha(t) = dY_\alpha(t) - \pi_t(\tilde{L}_\alpha + \tilde{L}_\alpha^*)\,dt. \qquad (21)$$

Note that $\mathbb{E}[dI_\alpha(t)] \equiv 0$.

## 2.5 Derivation of the Filter

To derive a differential equation for the filter, we shall apply the characteristic function technique to derive the filter for this problem. The technique is to assume that the filter satisfies and equation of the form

$$d\pi_t(X) = \mathcal{F}_t(X)\,dt + \mathcal{H}_t^\alpha(X)\,dY_\alpha(t) \tag{22}$$

where we assume that the processes $\mathcal{F}_t(X)$ and $\mathcal{H}_t^j(X)$ are adapted and lie in $\mathfrak{Y}_{t]}$.

To establish the proposition, we assume the form (22) and apply a method based on introducing a process $C(t)$ satisfying the QSDE

$$dC(t) = f_\alpha(t)\,C(t)\,dY_\alpha(t), \tag{23}$$

with initial condition $C(0) = I$. Here we assume that the $f_\alpha$ are integrable, but otherwise arbitrary. These coefficients may be deduced from the identity

$$d\mathbb{E}\left[(j_t(X) - \pi_t(X))\,C(t)\right] = 0$$

which is valid since $C(t) \in \mathfrak{Y}_{t]}$. We note that the Itō product rule implies $I + II + III = 0$ where

$$
\begin{aligned}
I &= \mathbb{E}\left[\{dj_t(X) - d\pi_t(X)\}\,C(t)\right], \\
II &= \mathbb{E}\left[(j_t(X) - \pi_t(X))\,dC(t)\right], \\
III &= \mathbb{E}\left[(dj_t(X) - d\pi_t(X))\,dC(t)\right].
\end{aligned}
$$

Here we have

$$I = \mathbb{E}[\{j_t(\mathcal{L}X) - \mathcal{F}_t(X) - \mathcal{H}_t^\alpha(X)\,j_t(\tilde{L}_\alpha + \tilde{L}_\alpha^*)\}C(t)]dt \tag{24}$$

$$II = \mathbb{E}\left[(\{\pi_t(X) - j_t(X)\})\,j_t\left(\tilde{L}_\alpha + \tilde{L}_\alpha^*\right)C(t)\right]f_\alpha(t)\,dt,$$

$$
\begin{aligned}
III &= \mathbb{E}[\{j_t(\mathcal{L}_{0k}X)dB_k + j_t(\mathcal{L}_{jk}X)d\Lambda_{jk} + j_t(\mathcal{R}_{j0}X)\,dA_j^* + j_t(\mathcal{R}_{0j}X)\,dA_j \\
&\quad -\mathcal{H}_t^\beta(X)dY_\beta\}\,dY_\alpha\,C(t)]\,f_\alpha(t)
\end{aligned}
$$

To simplify term $III$ we note that

$$
\begin{aligned}
j_t(\mathcal{L}_{0k}X)dB_k dY_\alpha &= j_t(\mathcal{L}_{0k}X)\,T_{\alpha l}^*\,j_t(S_{lk}^*)\,dt = j_t\left([L_k^*, X]\right)T_{\alpha k}^*dt, \\
j_t(\mathcal{L}_{jk}X)d\Lambda_{jk}dY_\alpha &= j_t(\mathcal{L}_{jk}X)T_{\alpha l}^*\,j_t\left(S_{lk}^*\right)dB_j^* = j_t\left([S_{lj}^*, X]\right)T_{\alpha l}^*dB_j^*, \\
j_t(\mathcal{R}_{j0}X)dA_j^* dY_\alpha &= j_t(\mathcal{R}_{j0}X)\left[U_{\alpha k}N_{jk} + U_{\alpha k}^*M_{jk}^*\right]dt = j_t\left(\left[X, \tilde{R}_\alpha\right]\right)dt, \\
j_t(\mathcal{R}_{0j}X)dA_j dY_\alpha &= j_t(\mathcal{R}_{0j}X)\left[U_{\alpha k}M_{jk} + U_{\alpha k}^*(\delta_{jk} + N_{kj})\right]dt \\
&= j_t\left([R_j^*, X]\right)U_{\alpha j}^*dt + j_t\left(\left[\tilde{R}_\alpha^*, X\right]\right)dt,
\end{aligned}
$$

and so

$$
\begin{aligned}
III &= \mathbb{E}[j_t\left(\left[\tilde{L}_\alpha^*, X\right]\right) C\left(t\right)] f_\alpha(t) dt + \mathbb{E}[j_t\left(\left[X, \tilde{R}_\alpha\right]\right) C\left(t\right)] f_\alpha(t) dt \\
&\quad + \mathbb{E}[j_t\left(\left[\tilde{R}_\alpha^*, X\right]\right) C\left(t\right)] f_\alpha(t) dt - K_{\alpha\beta}\mathbb{E}[\mathcal{H}_t^\beta(X) C\left(t\right)] f_\alpha(t) dt.
\end{aligned}
$$

Now from the identity $I + II + III = 0$ we may extract separately the coefficients of $f_\alpha\left(t\right) dt$ since the $f_\alpha$ were arbitrary. In particular we have $I \equiv 0$ as this is the only term not proportional to $f_\alpha\left(t\right) dt$, and this implies that

$$
\mathcal{F}_t\left(X\right) \equiv \pi_t(\mathcal{L}X) - \mathcal{H}_t^\alpha\left(X\right)\pi_t(\tilde{L}_\alpha + \tilde{L}_\alpha^*)
$$

Similarly $II + III \equiv 0$ so

$$
\begin{aligned}
0 &= \mathbb{E}\left[\left\{\pi_t(X\tilde{L}_\alpha + X\tilde{L}_\alpha^*) - \pi_t\left(X\right)\pi_t\left(\tilde{L}_\alpha + \tilde{L}_\alpha^*\right)\right\} C\left(t\right)\right] f_\alpha\left(t\right) dt \\
&\quad + \mathbb{E}[\pi_t\left(\left[\tilde{L}_\alpha^*, X\right]\right) C\left(t\right)] f_\alpha(t) dt + \mathbb{E}[\pi_t\left(\left[X, \tilde{R}_\alpha\right]\right) C\left(t\right)] f_\alpha(t) dt \\
&\quad + \mathbb{E}[\pi_t\left(\left[\tilde{R}_\alpha^*, X\right]\right) C\left(t\right)] f_\alpha(t) dt - K_{\alpha\beta}\mathbb{E}[\mathcal{H}_t^\beta(X) C\left(t\right)] f_\alpha(t) dt.
\end{aligned}
$$

From the coefficients of $f_\alpha\left(t\right)$ we deduce

$$
\begin{aligned}
K_{\alpha\beta}\mathcal{H}_t^\beta(X) &= \pi_t(X\tilde{L}_\alpha + \tilde{L}_\alpha^* X) - \pi_t\left(X\right)\pi_t\left(\tilde{L}_\alpha + \tilde{L}_\alpha^*\right) \\
&\quad + \pi_t\left(\left[X, \tilde{R}_\alpha\right]\right) + \pi_t\left(\left[\tilde{R}_\alpha^*, X\right]\right),
\end{aligned}
$$

which is the desired form.

## 2.6 The Example

We now return to the specific model introduced in the introduction. In this case we have $m_{\text{Fock}} = 1, m_{\text{sq}} = 1$ and $n_{\text{obs}} = 2$. The system $G_{\text{sys}} \sim (S, L, H)$ is driven by the Fock input, while the squeezed input is otherwise unprocessed $(R \equiv 0)$. We therefore have

$$
\begin{aligned}
\tilde{L}_1 &= \frac{1}{\sqrt{2}}\left[L + S\beta\left(t\right)\right] + \frac{1}{\sqrt{2}}\alpha\left(t\right), \\
\tilde{L}_2 &= \frac{1}{\sqrt{2}i}\left[L + S\beta\left(t\right)\right] - \frac{1}{\sqrt{2}i}\alpha\left(t\right).
\end{aligned}
$$

The covariance matrix $K$ is now given by (4) and we find

$$
\begin{aligned}
\tilde{L}^1 &= \frac{1 + n - m^*}{\sqrt{2}\Delta}\left[L + S\beta\left(t\right)\right] + \frac{1 + n - m}{\sqrt{2}\Delta}\alpha\left(t\right), \\
\tilde{L}^2 &= \frac{1 + n + m^*}{\sqrt{2}\Delta i}\left[L + S\beta\left(t\right)\right] - \frac{1 + n + m}{\sqrt{2}\Delta i}\alpha\left(t\right),
\end{aligned}
$$

with $\Delta = \left(1 + n\right)^2 - |m|^2$, and of course $\tilde{R}^1 = \tilde{R}^2 = 0$.

# 3    Belavkin-Kalman Filters

In this section we solve the filter problem for the special case of a linear dynamical model with Gaussian state. For simplicity we take the system to be a single cavity mode $a$ and have one squeezed input and (at most) one Fock input.

Here it is possible to obtain an exact form for the filtered estimate $\pi_t(a)$ in terms of the innovations. In particular, we will encounter the following conditional covariances

$$
\begin{aligned}
\mathscr{V}(t) &\triangleq \pi_t(a^* a) - \pi_t(a^*)\pi_t(a), \\
\mathscr{W}(t) &\triangleq \pi_t(a^2) - \pi_t(a)^2.
\end{aligned}
$$

It is a feature of the linear dynamics that these covariances will be deterministic functions of time $t$. The assumption of gaussianity ensures that higher mode moments can be written as combinations of products of first and second order moments, so for instance

$$
\begin{aligned}
\pi_t(XYZ) =\ & \pi_t(X)\pi_t(YZ) + \pi_t(Y)\pi_t(XZ) + \pi_t(Z)\pi_t(XY) \\
& -2\pi_t(X)\pi_t(Y)\pi_t(Z)
\end{aligned}
$$

where $X, Y, Z$ may be $a$ or $a^*$. In particular, we will make use of the identities

$$
\begin{aligned}
\pi_t(a^* a^2) &= \pi_t(a^*)\pi_t(a^2) + 2\pi_t(a)\pi_t(a^* a) - 2\pi_t(a^*)\pi_t(a)^2 \\
&= \pi_t(a^*)\mathscr{W}(t) + 2\pi_t(a)\mathscr{V}(t) + \pi_t(a^*)\pi_t(a)^2, \qquad (25) \\
\pi_t(a^3) &= 3\pi_t(a)\pi_t(a^2) - 2\pi_t(a)^3 \\
&= 3\pi_t(a)\mathscr{W}(t) + \pi_t(a)^3. \qquad (26)
\end{aligned}
$$

## 3.1    Cavity mode driven by Fock input whose output is mixed with squeezed noise

This is the set-up described in subsection 2.6 with

$$
S = e^{i\phi}, \quad L = \sqrt{\kappa} a, \quad H = \omega a^* a.
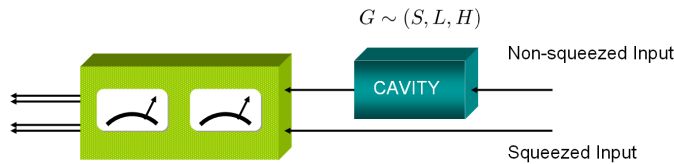$$



Figure 3: System driven by a Fock input: the output subsequently mixed with a squeezed input field.

(More generally we could take $L = \sqrt{\kappa_-}a + \sqrt{\kappa_+}a^*$ and $H = \omega a^* a + \varepsilon a^{*2} + \varepsilon^* a^2$ and still retain a linear dynamics: this is also solvable.) We now have

$$
\begin{aligned}
\tilde{L}^1 &= \frac{1 + n - m}{\sqrt{2}\Delta} \left[ \sqrt{\kappa}a + e^{i\phi}\beta(t) \right] + \frac{1 + n - m^*}{\sqrt{2}\Delta} \alpha(t), \\
\tilde{L}^2 &= \frac{1 + n + m^*}{\sqrt{2}\Delta i} \left[ \sqrt{\kappa}a + e^{i\phi}\beta(t) \right] - \frac{1 + n + m}{\sqrt{2}\Delta i} \alpha(t).
\end{aligned}
$$

In the following section we will take $m$ to be real, that is, $m = m'$ and $m'' = 0$. The filter equation then take the form

$$
\begin{aligned}
d\pi_t(X) &= \pi_t(\frac{1}{2}\kappa a^* [X, a] + \frac{1}{2}\kappa [a^*, X] a - i\omega [X, a^* a] \\
&\quad + \sqrt{\kappa} [X, a] e^{-i\phi}\beta^*(t) + \sqrt{\kappa} [a^*, X] e^{i\phi}\beta(t)) \, dt \\
&\quad + \sqrt{\frac{\kappa}{2(1 + n + m)}} \left\{ \pi_t(Xa + a^* X) - \pi_t(X)\pi_t(a + a^*) \right\} dW_1 \\
&\quad + \frac{1}{i}\sqrt{\frac{\kappa}{2(1 + n - m)}} \left\{ \pi_t(Xa - a^* X) - \pi_t(X)\pi_t(a - a^*) \right\} dW_2
\end{aligned}
$$

$$(27)$$

where we now introduce rescaled processes

$$
W_1(t) = \frac{1}{\sqrt{1 + n + m}} I_1(t), \quad W_2(t) = \frac{1}{\sqrt{1 + n - m}} I_2(t).
$$

The pair $W_1$ and $W_2$ are independent canonical Wiener processes and we have

$$
(dW_1)^2 = (dW_2)^2 = dt, \quad dW_1 dW_2 = 0.
$$

We obtain the following equation for the estimate $\pi_t(a)$ :

$$
\begin{aligned}
d\pi_t(a) &= -\left[ \left( \frac{1}{2}\kappa + i\omega \right) \pi_t(a) + \sqrt{\kappa}e^{i\phi}\beta(t) \right] dt \\
&\quad + \sqrt{\frac{\kappa}{2(1 + n + m)}} \left( \mathscr{W}(t) + \mathscr{V}(t) \right) dW_1(t) \\
&\quad + \frac{1}{i}\sqrt{\frac{\kappa}{2(1 + n - m)}} \left( \mathscr{W}(t) - \mathscr{V}(t) \right) dW_2(t)
\end{aligned}
$$

We now compute $\mathscr{V}(t)$. We have

$$
d\pi_t(a^* a) = -\left[ \kappa\pi_t(a^* a) + \sqrt{\kappa}\pi_t(a) e^{-i\phi}\beta(t)^* + \sqrt{\kappa}\pi_t(a^*) e^{i\phi}\beta(t) \right] dt + \cdots .
$$

Here, and in the following, expressions of the form $dP_t = G_t dt + \cdots$ retain drift terms, that is, the terms proportional to the increments of the innovations are indicated by the ellipsis. From the Itō product rule,

$$
d\mathscr{V}(t) = d\pi_t(a^* a) - \left[ d\pi_t(a^*)\,\pi_t(a) + \pi_t(a^*)\,d\pi_t(a) + d\pi_t(a^*)\,d\pi_t(a) \right]
$$

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Note that
$$d\pi_t\left(a^*\right)d\pi_t\left(a\right)=\Gamma\left(t\right)dt$$

where

$$
\begin{aligned}
\Gamma\left(t\right) &= \frac{\kappa}{2\left(1+n+m\right)}\left|\mathscr{W}\left(t\right)+\mathscr{V}\left(t\right)\right|^2+\frac{\kappa}{2\left(1+n-m\right)}\left|\mathscr{W}\left(t\right)-\mathscr{V}\left(t\right)\right|^2 \\
&= \frac{\kappa\left(1+n\right)}{\Delta}\left(\mathscr{V}\left(t\right)^2+\left|\mathscr{W}\left(t\right)\right|^2\right)-\frac{2\kappa m}{\Delta}\mathscr{V}\left(t\right)\operatorname{Re}\mathscr{W}\left(t\right) \quad\quad (28)
\end{aligned}
$$

and this leads to
$$d\mathscr{V}\left(t\right)=-\kappa\mathscr{V}\left(t\right)dt-\Gamma\left(t\right)dt+\cdots.$$

The non-drift term in fact vanishes. To see this we would have to look at the additional terms in $d\pi_t\left(a^*a\right)$ proportional to the increments of the innovations. To this end we need to calculate $\mathcal{H}_t^\alpha\left(a^*a\right)$ which in this case involves cubic terms $\pi_t\left(a^*a^2\right),\pi_t\left(a^{*2}a\right),\pi_t\left(a^3\right)$ and $\pi_t\left(a^{*3}\right)$. However, we may reduce this to first and second order conditional expectations using (25) and (26). By inspection, the coefficients of $dW_1$ and $dW_2$ vanish identically.

Similarly we have

$$d\mathscr{W}\left(t\right)=d\pi_t\left(a^2\right)-\left[2\pi_t\left(a\right)d\pi_t\left(a\right)+d\pi_t\left(a\right)d\pi_t\left(a\right)\right]$$

This time we have
$$d\pi_t\left(a\right)d\pi_t\left(a\right)=\Sigma\left(t\right)dt$$

where

$$
\begin{aligned}
\Sigma\left(t\right) &= \frac{\kappa}{2\left(1+n+m\right)}\left(\mathscr{W}\left(t\right)+\mathscr{V}\left(t\right)\right)^2-\frac{\kappa}{2\left(1+n-m\right)}\left(\mathscr{W}\left(t\right)-\mathscr{V}\left(t\right)\right)^2 \\
&= \frac{2\kappa\left(1+n\right)}{\Delta}\mathscr{W}\left(t\right)\mathscr{V}\left(t\right)-\frac{\kappa m}{\Delta}\left(\mathscr{W}\left(t\right)^2+\mathscr{V}\left(t\right)^2\right).
\end{aligned}
$$

This leads to the SDE

$$d\mathscr{W}\left(t\right)=-2\left(\frac{\kappa}{2}+i\omega\right)\mathscr{W}\left(t\right)dt-\Sigma\left(t\right)dt+\cdots.$$

Once again, the cubic terms appearing may be replaced using (25) and (26) with the overall result that the drift terms vanish identically. We can now give the explicit form for the filter.

**Proposition 2** *(With real squeezing parameter $m$.) The filtered estimate $\pi_t\left(a\right)$ for the cavity mode satisfies*

$$
\begin{aligned}
d\pi_t\left(a\right) = &-\left[\left(\frac{1}{2}\kappa+i\omega\right)\pi_t\left(a\right)+\sqrt{\kappa}e^{i\phi}\beta\left(t\right)\right]dt \\
&+\sqrt{\frac{\kappa}{2}\frac{1}{\left(1+n+m\right)}}\left(\mathscr{W}\left(t\right)+\mathscr{V}\left(t\right)\right)dI_1\left(t\right) \\
&+\frac{1}{i}\sqrt{\frac{\kappa}{2}\frac{1}{\left(1+n-m\right)}}\left(\mathscr{W}\left(t\right)-\mathscr{V}\left(t\right)\right)dI_2\left(t\right) \quad\quad (29)
\end{aligned}
$$

*where $\mathscr{V}(t)$ and $\mathscr{W}(t)$ are deterministic functions satisfying the ODEs*

$$
\begin{aligned}
\frac{d}{dt}\mathscr{V}(t) &= -\kappa\mathscr{V}(t) - \frac{\kappa(1+n)}{\Delta}\left(\mathscr{V}(t)^2 + |\mathscr{W}(t)|^2\right) \\
&\quad + \frac{2\kappa m}{\Delta}\mathscr{V}(t)\ Re\,\mathscr{W}(t), \tag{30} \\
\frac{d}{dt}\mathscr{W}(t) &= -2\left(\frac{\kappa}{2} + i\omega\right)\mathscr{W}(t) - 2\frac{\kappa(1+n)}{\Delta}\mathscr{W}(t)\ \mathscr{V}(t) \\
&\quad + \frac{\kappa m}{\Delta}\left(\mathscr{V}(t)^2 + \mathscr{W}(t)^2\right), \tag{31}
\end{aligned}
$$

*with initial conditions $\mathscr{V}(0) = \mathbb{E}\left[a^*a\right] - \mathbb{E}\left[a^*\right]\mathbb{E}\left[a\right]$ and $\mathscr{W}(0) = \mathbb{E}\left[a^2\right] - \mathbb{E}\left[a\right]^2$. The innovations are given by*

$$
\begin{aligned}
dI_1(t) &= dY_1(t) - \sqrt{2}\left[\frac{\sqrt{\kappa}}{2}\pi_t(a + a^*) + \mathrm{Re}\{e^{i\phi}\beta(t) + \alpha(t)\}\right]dt \\
dI_2(t) &= dY_2(t) - \sqrt{2}\left[\frac{\sqrt{\kappa}}{2}\pi_t\left(\frac{a - a^*}{i}\right) + \mathrm{Im}\{e^{i\phi}\beta(t) + \alpha(t)\}\right]dt.
\end{aligned}
$$

The generalization to complex $m$ is straightforward, but algebraically more involved due to the fact that the innovations are now correlated.

## 3.2 Quantum filtering with direct squeezed input

Our results also apply to the situation where we apply a squeezed field as input to a system. In this can there can be no scattering. This falls into general situation covered in Theorem 1 where we now ignore the Fock field $B$ (unless we wish to include further unmodelled dissipation). The output field will of course be squeezed and we measure the quadrature

$$
Y(t) = e^{-i\theta}A^{\mathrm{out}}(t) + e^{i\theta}A^{\mathrm{out}}(t)^*.
$$

We therefore set $S = I, L = 0$ with $R$, the coupling operator to the single (squeezed) input, nonzero.
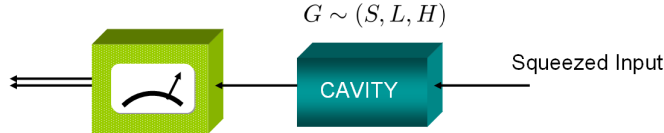


Figure 4: System driven by a squeezed input (no scattering allowed!).

The filter equation then simplifies to

$$
d\pi_t(X) = \pi_t(\mathcal{L}X)\,dt + \mathcal{H}_t(X)\,dI(t), \tag{32}
$$

where

$$\mathcal{L}X = \mathcal{R}_{00}X + \alpha^*(t)\mathcal{R}_{10}X + \mathcal{R}_{01}X\alpha(t) - i\,[X, H]$$

$$\mathcal{H}_t\,(X) = \pi_t(X\tilde{L}_1 + \tilde{L}_1^* X) - \pi_t(X)(\tilde{L}_1 + \tilde{L}_1^*) + \pi_t([X, \tilde{R}_1]) + \pi_t([\tilde{R}_1^*, X]) \quad (33)$$

where

$$\tilde{L}_1 = e^{i\theta}(R + \alpha\,(t)), \quad \tilde{R}_1 = \left(e^{i\theta}n + e^{-i\theta}m^*\right)R. \quad (34)$$

and we require

$$\tilde{L} \equiv \frac{1}{K}\tilde{L}_1, \quad \tilde{R} \equiv \frac{1}{K}\tilde{R}_1$$

with $K = 1 + 2n + 2\,(m'\cos 2\theta + m''\sin 2\theta)$. The innovations process is now

$$dI(t) = dY(t) - \pi_t(\tilde{L} + \tilde{L}^*)\,dt, \quad (35)$$

and we $(dI)^2 = K\,dt$.

Again we specialize to the case of a cavity and take

$$R = \sqrt{\gamma}a, \quad H = \omega a^* a.$$

The filter equation is

$$
\begin{aligned}
d\pi_t\,(X) \ = \ & \frac{\gamma}{2}\,(1 + n)\,\pi_t\,([a^*, X]\,a + a^*\,[X, a])\,dt \\
& + \frac{\gamma}{2}n\pi_t\,([a, X]\,a^* + a\,[X, a^*])\,dt \\
& - \frac{\gamma}{2}m^*\pi_t\,([a, X]\,a + a\,[X, a])\,dt \\
& - \frac{\gamma}{2}m\pi_t\,([a^*, X]\,a^* + a^*\,[X, a^*])\,dt \\
& - i\omega\pi_t\,([X, a^*a])\,dt \\
& + \frac{\sqrt{\gamma}}{K}\{\pi_t(e^{i\theta}Xa + e^{-i\theta}a^* X) - \pi_t(X)\pi_t(e^{i\theta}a + e^{-i\theta}a^*) \\
& + \left(e^{i\theta}n + e^{-i\theta}m^*\right)\pi_t([X, a]) + \left(e^{-i\theta}n + e^{i\theta}m\right)\pi_t([a^*, X])\}dI\,(t).
\end{aligned}
$$

For $X = a$ we obtain

$$
\begin{aligned}
d\pi_t\,(a) \ = \ & -\left(\frac{1}{2}\gamma + i\omega\right)\pi_t\,(a)\,dt \\
& + \frac{\sqrt{\gamma}}{K}\left\{e^{-i\theta}\,(\mathscr{V}\,(t) - n) + e^{i\theta}\,(\mathscr{W}\,(t) - m)\right\}dI\,(t).
\end{aligned}
$$

Observing that

$$d\pi_t\,(a^*)\,d\pi_t\,(a) = \frac{\gamma}{K}\left|e^{-i\theta}\,(\mathscr{V}\,(t) - n) + e^{i\theta}\,(\mathscr{W}\,(t) - m)\right|^2 dt,$$

and

$$
\begin{aligned}
d\pi_t\,(a^* a) \ = \ & \gamma(n - \pi_t\,(a^* a))\,dt \\
& + \frac{\sqrt{\gamma}}{K}\{e^{i\theta}\pi_t\left(a^* a^2\right) - e^{i\theta}\pi_t\left(a^* a\right)\pi_t\,(a) \\
& + e^{-i\theta}\pi_t\left(a^{*2}a\right) - e^{i\theta}\pi_t\left(a^* a\right)\pi_t\,(a^*) \\
& - \left(e^{i\theta}n + e^{-i\theta}m^*\right)\pi_t\,(a) - \left(e^{-i\theta}n + e^{i\theta}m\right)\pi_t(a^*)\}dI\,(t)
\end{aligned}
$$

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we see that

$$d\mathscr{V}(t) = d\pi_t(a^*a) - [d\pi_t(a^*)\,\pi_t(a) + \pi_t(a^*)\,d\pi_t(a) + d\pi_t(a^*)\,d\pi_t(a)]$$
$$= -\gamma\left(\mathscr{V}(t) - n\right)dt - \frac{\gamma}{K}\left|e^{-i\theta}\left(\mathscr{V}(t) - n\right) + e^{i\theta}\left(\mathscr{W}(t) - m\right)\right|^2 dt,$$

where we use the identity (25) again.

Similarly, we find

$$d\pi_t\left(a^2\right) = -2\left(\frac{1}{2}\gamma + i\omega\right)\pi_t(a)\,dt + \gamma m dt + \cdots,$$

and

$$\left(d\pi_t(a)\right)^2 = -\frac{\gamma}{K}\left\{e^{-i\theta}\left(\mathscr{V}(t) - n\right) + e^{i\theta}\left(\mathscr{W}(t) - m\right)\right\}^2 dt,$$

so that

$$d\mathscr{W}(t) = d\pi_t\left(a^2\right) - [2\pi_t(a)\,d\pi_t(a) + d\pi_t(a)\,d\pi_t(a)]$$
$$= -2\left(\frac{1}{2}\gamma + i\omega\right)\mathscr{W}(t)\,dt + \gamma m dt$$
$$-\frac{\gamma}{K}\left\{e^{-i\theta}\left(\mathscr{V}(t) - n\right) + e^{i\theta}\left(\mathscr{W}(t) - m\right)\right\}^2 dt.$$

**Proposition 3** *The filtered estimate $\pi_t(a)$ for the cavity mode satisfies*

$$d\pi_t(a) = -\left(\frac{1}{2}\gamma + i\omega\right)\pi_t(a)\,dt$$
$$+ \frac{\sqrt{\gamma}}{K}\left(e^{-i\theta}(\mathscr{V}(t) - n) + e^{i\theta}(\mathscr{W}(t) - m)\right)dI(t) \qquad (36)$$

*where $\mathscr{V}(t)$ and $\mathscr{W}(t)$ are deterministic functions satisfying the ODEs*

$$\frac{d}{dt}\mathscr{V}(t) = -\gamma(\mathscr{V}(t) - n) - \frac{\gamma}{K}\left|e^{-i\theta}(\mathscr{V}(t) - n) + e^{i\theta}(\mathscr{W}(t) - m)\right|^2, (37)$$
$$\frac{d}{dt}\mathscr{W}(t) = -2i\omega\mathscr{W} - \gamma(\mathscr{W}(t) - m)$$
$$-\frac{\kappa}{2}\left(e^{-i\theta}(\mathscr{V}(t) - n) + e^{i\theta}(\mathscr{W}(t) - m)\right)^2, \qquad (38)$$

*with initial conditions $\mathscr{V}(0) = \mathbb{E}\left[a^*a\right] - \mathbb{E}\left[a^*\right]\mathbb{E}\left[a\right]$ and $\mathscr{W}(0) = \mathbb{E}\left[a^2\right] - \mathbb{E}\left[a\right]^2$.*

# References

[1] V.P. Belavkin, Quantum filtering of Markov signals with white quantum noise. *Radiotechnika i Electronika*, **25**, 1445-1453 (1980).

[2] V.P. Belavkin, Quantum continual measurements and a posteriori collapse on CCR. *Commun. Math. Phys.*, **146**, 611-635 (1992).

[3] V. P. Belavkin, Quantum stochastic calculus and quantum nonlinear filtering. *Journal of Multivariate Analysis*, **42**, 171-201(1992) .

[4] V.P. Belavkin, Stochastic calculus of input-output processes and non-demolition filtering, Reviews of the Newest Achievements in Science and technology, Current Problems of Mathematics VINITI, Ed. A.S. Holevo, **36**, 29-67, (1989)

[5] R.L. Stratonovich, On the Theory of Optimal Nonlinear Filtering of Random Functions, *Teor. Veroyatn. Primen.*, No. 2, 239-242 (1959)

[6] R.L. Stratonovich, Conditional Markov Processes and Their Application to the Theory of Optimal Control, New York: Elsevier, (1968)

[7] M. Zakai, On the optimal filtering of diffusion processes. *Z. Wahrsch. th. verw. Geb.*, **11**, 230-243 (1969)

[8] H.J. Kushner, Jump-diffusion approximations for ordinary differential equations with wideband random right hand sides. *SIAM J. Control Optim.* **17**, 729-744, (1979)

[9] H.J. Kushner, Diffusion approximations to output processes of nonlinear systems with wideband inputs and applications. *IEEE Trans. Inf. Th.* **26**, 715-725 (1980)

[10] M.H.A. Davis and S.I. Marcus, An introduction to nonlinear filtering. In M. Hazewinkel and J. C. Willems, editors, S*tochastic Systems: The Mathematics of Filtering and Identification and Applications*, pages 53-75. D. Reidel, (1981)

[11] A. Barchielli, Direct and heterodyne detection and other applications of quantum stochastic calculus to quantum optics. *Quantum Opt.*, **2**, 423-441. (1990)

[12] A. Barchielli, V.P. Belavkin, Measurements continuous in time and posterior states in quantum mechanics, J. Phys. A, Math Gen 24 (12), 1495-1514 (1991)

[13] H.M. Wiseman and G.J. Milburn, Quantum theory of field-quadrature measurements. *Phys. Rev. A*, **47**, 642-662( 1993)

[14] M.A. Armen, J.K. Au, J.K. Stockton, A.C. Doherty, and H. Mabuchi, Adaptive homodyne measurement of optical phase. *Phys. Rev. Lett.*, 89:133602, (2002)

[15] L.M. Bouten, M. I. Guţă, and H. Maassen, Stochastic Schrödinger equations. *J. Phys. A*, **37**, 3189-3209 (2004)

[16] L.M. Bouten, S.C. Edwards, and V.P. Belavkin, Bellman equations for optimal feedback control of qubit states. *J. Phys. B, At. Mol. Opt. Phys.*, 38:151160 (2005)

[17] A.C. Doherty, S. Habib, K. Jacobs, H. Mabuchi, and S.M. Tan, Quantum feedback and classical control theory. *Phys. Rev. A*, 62:012105 (2000)

[18] J. Gough, V.P. Belavkin, and O.G. Smolyanov, Hamilton-Jacobi-Bellman equations for quantum filtering and control. *J. Opt. B: Quantum Semiclass. Opt.* **7**, S237-S244 (2005)

[19] R. Van Handel, J.K. Stockton, and H. Mabuchi, Feedback control of quantum state reduction. *IEEE Transactions on Automatic Control*, **50**, 768-780, (2005)

[20] L. Bouten, R. van Handel, M. R. James, An introduction to quantum filtering *SIAM J. Control Optim.* **46**, 2199-2241 (2007).

[21] V. P. Belavkin, P. Staszewski, Nondemolition observation of a free quantum particle, *Phys. Rev. A* 45, 1347–1356, (1992)

[22] J.E. Gough, Optimal Quantum Feedback Control for Canonical Observables in Quantum Stochastics and Information: Statistics, Filtering & Control, pp. 262-279 Eds. M. Guta and V.P. Belavkin, World Scientific (2008)

[23] J.E. Gough, M.R. James, H.I. Nurdin, J. Combes, *Quantum Filtering for Systems Driven by Fields in Single Photon States and Superposition of Coherent States*, Phys. Rev. A 86, 043819 (2012)

[24] M. Yanagisawa and H. Kimura. Transfer function approach to quantum control-part I: Dynamics of quantum feedback systems. *IEEE Trans. Automatic Control,* (48):2107-2120, 12 (2003)

[25] M. Yanagisawa and H. Kimura. Transfer function approach to quantum control-part II: Control concepts and applications. *IEEE Trans. Automatic Control*, (48):2121-2132, 12 (2003)

[26] S.G. Hofer, W. Wieczorek, M. Aspelmeyer, K. Hammerer, Quantum entanglement and teleportation in pulsed cavity-optomechanics, *Phys. Rev. A* 84, 052327 (2011)

[27] R.L. Hudson, K.R. Parthasarathy, *Commun.Math.Phys.* **93** 301-323, (1984)

[28] R.L. Hudson, J.M. Lindsay, A non-commutative martingale representation theorem for non-Fock quantum Brownian motion, *J. Funct. Anal.* **61**, 202-221 (1985)