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The Stratonovich Formulation of Quantum Feedback Network Rules

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We express the rules for forming quantum feedback networks using the Stratonovich form of quantum stochastic calculus rather than the Itō, or SLH form. Remarkably the feedback reduction rule implies that we obtain the Schur complement of the matrix of Stratonovich coupling operators where we short out the internal input/output coefficients.

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I. INTRODUCTION

Quantum stochastic differential equations[1,3] give the mathematical framework for modelling open quantum systems where one wish to explicitly take account of inputs and outputs, sometimes referred to as the SLH framework. There are explicit rules for forming a quantum feedback network where various outputs from component systems are fed back in as inputs elsewhere in the network[4,5]. The techniques have since been widely applied to model quantum optical systems both theoretically and experimentally[6]-[10]. The following paper gives the formulation of the rules in terms of the Stratonovich form of the quantum stochastic calculus[3,11,12]. The Stratonovich form is closer to a Hamiltonian description, and has previously shown revealed some interesting representations of open quantum systems[13,14]. The reformulation the quantum feedback connection rules for the Stratonovich representation will of course follow as an exercise in converting from the existing Itō rules, however, the rather surprising result is that the resulting rule that emerges is the direct Schur complementation procedure. This is somewhat surprising as there is no obvious simple algebraic elimination of internal inputs and outputs going on at the level of the Hamiltonian. Also, the simplest network consisting of two systems in series is not encouraging when analyzed in terms of the Stratonovich form. Nevertheless the rule that emerges is remarkable direct and succinct. It is possible that the Stratonovich form of the feedback rule may have some numerical advantages in modelling quantum feedback networks, for instance using computational packages such as QHDL[15].

A. SLH Framework

In the Markovian model of an open quantum system we consider a fixed Hilbert space $\mathfrak{h}_0$ for the system and a collection of independent quantum white noises $b_k(t)$ labeled by k belonging to some discrete set $\mathsf{k} = \{1, \cdots, n\}$, that is, we have

$$\left[b_j(t), b_k(s)^*\right] = \delta_{jk}\delta(t-s).$$

The Schrödinger equation is

$$\dot{U}(t) = -i\Upsilon(t)\, U(t) \tag{1}$$

where the stochastic Hamiltonian takes the form

$$\Upsilon(t) = E_{00} + \sum_{j\in\mathsf{k}} E_{j0} b_j(t)^* + \sum_{k\in\mathsf{k}} E_{0k}\, b_k(t) + \sum_{j,k\in\mathsf{k}} E_{jk} b_j(t)^* b_k(t). \tag{2}$$

Here we assume that the $E_{\alpha\beta}$ are operators on $\mathfrak{h}_0$ with $E^*_{\alpha\beta} = E_{\beta\alpha}$. (In the course of this paper we will take them to be bounded so as to avoid technical distractions.) We may write

$$\Upsilon(t) = \left[1, b_{\mathsf{k}}(t)^*\right] \mathbf{E} \left[\begin{array}{c} 1 \\ b_{\mathsf{k}}(t) \end{array}\right]$$

where

$$b_{\mathsf{k}}(t) = \left[\begin{array}{c} b_1(t) \\ \vdots \\ b_n(t) \end{array}\right], \quad \mathbf{E} = \left[\begin{array}{cc} E_{00} & E_{0\mathsf{k}} \\ E_{\mathsf{k}0} & E_{\mathsf{kk}} \end{array}\right]$$

with $E_{0\mathsf{k}} = [E_{01}, \cdots, E_{0n}]$, $E_{\mathsf{k}0} = E^*_{0\mathsf{k}}$ and E_{kk} the $n\times n$ matrix with entries E_{jk} with $j,k\in\mathsf{k}$. The Schrödinger equation (1) is interpreted as the Stratonovich quantum stochastic differential equation

$$\begin{aligned} dU(t) = -i\Big\{ & E_{00}\otimes dt + \sum_{j\in\mathsf{k}} E_{j0}\otimes dB_j(t)^* \\ & + \sum_{k\in\mathsf{k}} E_{0k}\otimes dB_k(t) + \sum_{j,k\in\mathsf{k}} E_{jk}\otimes d\Lambda_{jk}(t) \Big\} \circ U(t), \end{aligned} \tag{3}$$

which may be readily converted into the quantum Itō form of Hudson and Parthasarathy. (In fact the latter is accomplished by Wick ordering the noise fields $b_k(t)$ and $b^*_k(t)$ in (1).)

In the theory of quantum feedback networks, we consider interconnected open Markovian models in the limit where the connections have zero time delay. If we have several components, separately described by stochastic Hamiltonians $\Upsilon^{(i)}(t)$ for $i=1,\cdots,m$, then the natural Hamiltonian describing all components would be

$$\Upsilon(t)=\sum_{i=1}^{m}\Upsilon^{(i)}(t)=\sum_{i=1}^{m}\left[1,b_{\mathsf{k}(i)}(t)^{*}\right]\mathbf{E}^{(i)}\left[\begin{array}{c}1\\ b_{\mathsf{k}(i)}(t)\end{array}\right] \tag{4}$$

where $\mathsf{k}(i)$ is the set of input labels for the ith component, and the coupling terms are by $\mathbf{E}^{(i)}=\left[\begin{array}{cc}E_{00}^{(i)} & E_{0\mathsf{k}(i)}^{(i)}\\ E_{\mathsf{k}(i)0}^{(i)} & E_{\mathsf{k}(i)\mathsf{k}(i)}^{(i)}\end{array}\right]$. (At this stage we make no assumptions about whether the coupling operators of different components commute or not: mathematically we just take all $E_{\alpha\beta}^{(i)}$ to be defined one the same Hilbert space $\mathfrak{h}_0$ describing the all the components collectively and work at this level of generality.) The total Hamiltonian is then

$$\Upsilon^{\text{parallel}}(t)=\left[1,b_{\mathsf{k}}(t)^{*}\right]\mathbf{E}^{\text{parallel}}\left[\begin{array}{c}1\\ b_{\mathsf{k}}(t)\end{array}\right]$$

where $\mathsf{k}=\mathsf{k}(1)\cup\cdots\cup\mathsf{k}(m)$ is the collection of labels for all components, and

$$\mathbf{E}^{\text{parallel}}=\left[\begin{array}{cccc}\sum_{i=1}^{m}E_{00}^{(i)} & E_{0\mathsf{k}(1)}^{(1)} & \cdots & E_{0\mathsf{k}(m)}^{(m)}\\ E_{\mathsf{k}(1)0}^{(1)} & E_{\mathsf{k}(1)\mathsf{k}(1)}^{(1)} & & 0\\ \vdots & & \ddots & \\ E_{\mathsf{k}(m)0}^{(m)} & 0 & & E_{\mathsf{k}(m)\mathsf{k}(m)}^{(m)}\end{array}\right]. \tag{5}$$

This gives us our first network rule - how to assemble the m components together into a single model before connections are made. The second rule deals with making connections. Here we must divide our set of fields into two groups - those that are external and those that are to be fed back in as internal drives. We denote the label sets for these as e and i respectively so that the total set k is the disjoint union $\mathsf{e}\cup\mathsf{i}$. The Hamiltonian $\Upsilon^{\text{parallel}}(t)$ may then be decomposed as

$$\Upsilon^{\text{parallel}}(t)=\left[1,b_{\mathsf{e}}(t)^{*},b_{\mathsf{i}}(t)^{*}\right]\left[\begin{array}{ccc}E_{00} & E_{0\mathsf{e}} & E_{0\mathsf{i}}\\ E_{\mathsf{e}0} & E_{\mathsf{ee}} & E_{\mathsf{ei}}\\ E_{\mathsf{i}0} & E_{\mathsf{ie}} & E_{\mathsf{ii}}\end{array}\right]\left[\begin{array}{c}1\\ b_{\mathsf{e}}(t)\\ b_{\mathsf{i}}(t)\end{array}\right].$$

Applying the feedback connections we should obtain a reduced model where the internal inputs have been eliminated leaving only the set e of external fields, that is, we should obtain a stochastic Hamiltonian of the form

$$\Upsilon^{\text{fb}}(t)=\left[1,b_{\mathsf{e}}(t)^{*}\right]\mathbf{E}^{\text{fb}}\left[\begin{array}{c}1\\ b_{\mathsf{e}}(t)\end{array}\right].$$

The expression for the reduced coefficients has been derived for the Itō form, however, the remarkable result presented here is that the feedback reduction formula for the Stratonovich form is actually the Schur complement of the matrix $\mathbf{E}^{\text{parallel}}$ describing the open loop network where we short out the internal blocks. Under the assumption that the matrix E_{ii} of operators is invertible, we shall show that

$$\mathbf{E}^{\text{fb}}\equiv\left[\begin{array}{cc}E_{00}^{\text{fb}} & E_{0\mathsf{e}}^{\text{fb}}\\ E_{\mathsf{e}0}^{\text{fb}} & E_{\mathsf{ee}}^{\text{fb}}\end{array}\right]=\left[\begin{array}{cc}E_{00} & E_{0\mathsf{e}}\\ E_{\mathsf{e}0} & E_{\mathsf{ee}}\end{array}\right]-\left[\begin{array}{c}E_{0\mathsf{i}}\\ E_{\mathsf{ei}}\end{array}\right]E_{\mathsf{ii}}^{-1}\left[\begin{array}{cc}E_{\mathsf{i}0} & E_{\mathsf{ie}}\end{array}\right]. \tag{6}$$

B. Notation

Let $\mathfrak{h}$ be a fixed Hilbert space. Given a countable set j of labels we set $\mathbb{C}^{\mathsf{j}}$ which is then a Hilbert space spanned by a collection of orthonormal vectors $\{e_j : j\in\mathsf{j}\}$, we may take as canonical basis. The Hilbert space $\mathfrak{h}\otimes\mathbb{C}^{\mathsf{j}}$ may then be represented as $\oplus_{j\in\mathsf{j}}\mathfrak{h}$, that is, as the set of vectors $\Psi=[\psi_j]_{j\in\mathsf{j}}$ with each $\psi_j\in\mathfrak{h}$ and $\sum_{j\in\mathsf{j}}\|\psi_j\|^2<\infty$. An operator X on $\mathfrak{h}\otimes\mathbb{C}^{\mathsf{j}}$ may likewise we represented as the array $[X_{jk}]_{j,k\in\mathsf{j}}$ where each X_{jk} is an operator on $\mathfrak{h}$, so that $X\Psi=X[\psi_j]_{j\in\mathsf{j}}=\left[\sum_{k\in\mathsf{j}}X_{jk}\psi_k\right]_{j\in\mathsf{j}}$.

Let a be a subset of j. We shall understand that a block matrix X_{aa} is invertible (with inverse Y_{aa}) to mean the obvious property that the system of equations

$$\sum_{a'\in\mathsf{a}} X_{aa'}\phi_{a'} = \psi_a, \qquad (\forall \quad a\in\mathsf{a}),$$

has an unique solution $(\phi_a)_{a\in\mathsf{a}}$ for any given $(\psi_a)_{a\in\mathsf{a}}$ (given by $\phi_a = \sum_{a'\in\mathsf{a}} Y_{aa'}\psi_{a'}$). In particular, we shall write I_{a} for the identity on $\mathfrak{h}\otimes\mathbb{C}^{\mathsf{a}}$.

In general if a and b are subsets of the label set j we may set $X_{\mathsf{ab}} = [X_{ab}]_{a\in\mathsf{a},b\in\mathsf{b}}$ which defined as sub-block operator. If a and b are nonempty disjoint subsets such that $\mathsf{j} = \mathsf{a}\cup\mathsf{b}$ then we may reorder the operator into sub-blocks as $X \equiv \begin{bmatrix} X_{\mathsf{aa}} & X_{\mathsf{ab}} \\ X_{\mathsf{ba}} & X_{\mathsf{bb}} \end{bmatrix}$ corresponding to the direct sum decomposition $\mathbb{C}^{\mathsf{j}} \cong \mathbb{C}^{\mathsf{a}}\oplus\mathbb{C}^{\mathsf{b}}$. In such cases we define the Schur complement of X to be

$$\underset{\mathsf{b}}{\mathrm{Schur}} X \triangleq X_{\mathsf{aa}} - X_{\mathsf{ab}}X_{\mathsf{bb}}^{-1}X_{\mathsf{ba}} \tag{7}$$

where we shall always assume that X_{bb} is invertible as an operator on $\mathfrak{h}\otimes\mathbb{C}^{\mathsf{b}}$. Specifically, we say that this is the Schur complement of X obtained by shortening the set of indices b.

A key property that we shall use is that the order in which successive shortening of indices are applied is not important. In particular,

$$\underset{\mathsf{b}_1\cup\cdots\cup\mathsf{b}_n}{\mathrm{Schur}} X = \underset{\mathsf{b}_1}{\mathrm{Schur}}\cdots\underset{\mathsf{b}_n}{\mathrm{Schur}} X$$

for any disjoint sets $\mathsf{b}_1,\cdots,\mathsf{b}_n\subset\mathsf{j}$.

II. QUANTUM FEEDBACK NETWORKS

A. Quantum Stochastic Evolutions

We recall the Hudson-Parthasarathy theory of quantum stochastic evolutions[1,2] on Hilbert spaces of the form

$$\mathfrak{H} = \mathfrak{h}_0\otimes\Gamma\left(\mathfrak{K}\otimes L^2[0,\infty)\right)$$

where $\mathfrak{h}_0$ is a fixed Hilbert space, called the *initial space*, and $\mathfrak{K}$ is a fixed Hilbert space called the *internal space*. Specifically, $\mathfrak{K}$ is the multiplicity space (also known as the color space) of the noise and takes the form

$$\mathfrak{K} = \mathbb{C}^{\mathsf{k}}$$

where $\mathfrak{K} = \{1,\cdots,n\}$ are the labels of the input noise fields driving the open quantum system with Hilbert space $\mathfrak{h}_0$. As before, let $\{e_k : k\in\mathsf{k}\}$ be the canonical orthonormal basis for $\mathfrak{K}$. The annihilation processes are defined, for each $k\in\mathsf{k}$, by

$$B_k(t) \triangleq a\left(e_k\otimes 1_{[0,t]}\right)$$

where $a(\cdot)$ is the annihilation functor from $\mathfrak{K}\otimes L^2[0,\infty)$ to the Fock space $\Gamma\left(\mathfrak{K}\otimes L^2[0,\infty)\right)$ and $1_{[0,t]}$ is the indicator function for the interval $[0,t]$. Its adjoint $B_k(t)^*$ is the creation process, and scattering is described by the processes

$$\Lambda_{jk}(t) \triangleq d\Gamma\left(|e_j\rangle\langle e_k|\otimes\pi_{[0,t]}\right)$$

where $d\Gamma(\cdot)$ is the differential second quantization functor and $\pi_{[0,t]}$ is the operator of pointwise multiplication by $1_{[0,t]}$ on $L^2[0,\infty)$.

As is well known $\mathfrak{H}$ decomposes as $\mathfrak{H}_{[0,t]}\otimes\mathfrak{H}_{(t,\infty)}$ for each $t>0$ where $\mathfrak{H}_{[0,t]} = \mathfrak{h}_0\otimes\Gamma\left(\mathfrak{k}\otimes L^2[0,t)\right)$ and $\mathfrak{H}_{(t,\infty)} = \Gamma\left(\mathfrak{k}\otimes L^2(t,\infty)\right)$. We shall write $\mathfrak{A}_{t]}$ for the space of operators on $\mathfrak{H}$ that act trivially on the future component $\mathfrak{H}_{(t,\infty)}$. A quantum stochastic process $X_t = \{X_t : t\geq 0\}$ is said to be *adapted* if $X_t\in\mathfrak{A}_{t]}$ for each $t\geq 0$.

Taking $\{x_{\alpha\beta}(t) : t\geq 0\}$ to be a family of adapted quantum stochastic processes, their quantum stochastic integral is $X_t = \int_0^t x_{\alpha\beta}(s)\,dB^{\alpha\beta}(t)$ which is shorthand for

$$\int_0^t x_{00}(s)\,ds + \sum_{j\in\mathsf{j}}\int_0^t x_{j0}(s)\,dB_j(s)^* + \sum_{k\in\mathsf{j}}\int_0^t x_{0k}(s)\,dB_k(s) + \sum_{j,k\in\mathsf{j}}\int_0^t x_{jk}(s)\,d\Lambda_{jk}(s)$$

and where the differentials are understood in the Itō sense. Given a similar quantum Itō integral Y_t, with $dY_t = y_{\alpha\beta}(t)\, dB^{\alpha\beta}(t)$, we have the quantum Itō product rule

$$d(X_t.Y_t) = dX_t.Y_t + X_t.dY_t + dX_t.dY_t, \tag{8}$$

with the Itō correction given by

$$dX_t.dY_t = x_{\alpha k}(t)\, y_{k\beta}(t)\; dB^{\alpha\beta}(t). \tag{9}$$

The coefficients $\{x_{\alpha\beta}(t)\}$ may be assembled into a matrix

$$\mathbf{X}(t) = \begin{bmatrix} x_{00}(t) & x_{0\mathsf{k}}(t) \\ x_{\mathsf{k}0}(t) & x_{\mathsf{kk}}(t) \end{bmatrix} \in \mathfrak{A}_{t]}^{(1+n)\times(1+n)}, \tag{10}$$

which we term the *Itō matrix* for the process. Note that in terms of our earlier conventions, the appropriate index set is the $0 \cup \mathsf{k}$ which has $1+n$ elements.

(Here we use the convention that $x_{0\mathsf{k}}(t)$ denotes the row vector with entries $(x_{0j}(t))_{j=1}^{n}$, etc. The Itō matrix for a product $X_t Y_t$ of quantum Itō integrals will then be given by $\mathbf{X}(t)\, Y(t) + X(t)\, \mathbf{Y}(t) + \mathbf{X}(t)\, \hat{\delta} \mathbf{Y}(t)$, where $\hat{\delta} \triangleq \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{\mathsf{k}} \end{array}\right]$.

The general form of the constant operator-coefficient quantum stochastic differential equation for an adapted unitary process U is

$$dU(t) = \Big\{ -\left(\frac{1}{2} L_{\mathsf{k}}^{*} L_{\mathsf{k}} + iH\right) dt + \sum_{j\in\mathsf{k}} L_j dB_j(t)^{*} - \sum_{j,k\in\mathsf{k}} S_{jk} L_k dB_k(t) + \sum_{j,k\in\mathsf{k}} (S_{jk} - \delta_{jk}) d\Lambda_{jk}(t) \Big\} U(t) \tag{11}$$

where the S_{jk}, L_j and H are operators on the initial Hilbert space with $S_{\mathsf{kk}} = [S_{jk}]_{j,k\in\mathsf{k}}$ is unitary and H self-adjoint. (We use the convention that $L_{\mathsf{k}} = [L_k]_{k\in\mathsf{k}}$ and that $L_{\mathsf{k}}^{*} L_{\mathsf{k}} = \sum_{k\in\mathsf{k}} L_k^{*} L_k$.) The corresponding Itō matrix of coefficients is

$$\mathbf{G} = \begin{bmatrix} -\left(\frac{1}{2} L_{\mathsf{k}}^{*} L_{\mathsf{k}} + iH\right) & -S_{\mathsf{kk}} L_{\mathsf{k}}^{*} \\ L_{\mathsf{k}} & S_{\mathsf{kk}} - I_{\mathsf{k}} \end{bmatrix}, \tag{12}$$

which we have previously called the Itō generator matrix. The triple (S, L, H) are termed the Hudson-Parthasarathy parameters of the open system evolution. Explicitly, we have that[12]

$$S_{\mathsf{kk}} = \frac{I_{\mathsf{k}} - \frac{i}{2} E_{\mathsf{kk}}}{I_{\mathsf{k}} + \frac{i}{2} E_{\mathsf{kk}}}, \quad L_{\mathsf{k}} = -i\left(I_{\mathsf{k}} + \frac{i}{2} E_{\mathsf{kk}}\right)^{-1} E_{\mathsf{k}0}, \quad H = E_{00} + \frac{1}{2} \mathrm{Im}\left\{ E_{0\mathsf{k}} \left(I_{\mathsf{k}} + \frac{i}{2} E_{\mathsf{kk}}\right)^{-1} E_{\mathsf{k}0} \right\}$$

where $\mathrm{Im} X$ means $\frac{1}{2i}(X - X^{*})$. Note that we may invert to get

$$E_{\mathsf{kk}} = \frac{2}{i} \frac{I_{\mathsf{k}} - S_{\mathsf{kk}}}{I_{\mathsf{k}} + S_{\mathsf{kk}}}, \tag{13}$$

provided that $I_{\mathsf{k}} + S_{\mathsf{kk}}$ is invertible. We note that there will exist SLH models that do not possess Stratonovich representations. A simple example is an optical mirror for which $S \equiv -1$.

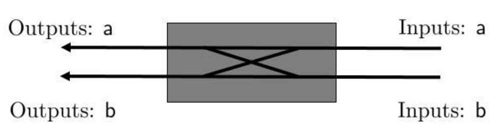

FIG. 1. A single SLH component with input labels k split into two groups a and b.

In Figure 1 we have partitions the inputs and outputs into two groups. Here we have the block partition

$$S_{\mathsf{kk}} \equiv \begin{bmatrix} S_{\mathsf{aa}} & S_{\mathsf{ab}} \\ S_{\mathsf{ba}} & S_{\mathsf{bb}} \end{bmatrix}, \quad L_{\mathsf{k}} \equiv \begin{bmatrix} L_{\mathsf{a}} \\ L_{\mathsf{b}} \end{bmatrix}.$$

Note that we did not need to group the outputs in the same way as the inputs.

1. *Belavkin-Holevo Matrix Representation*

As is well-known, the Heisenberg group gives a matrix representation of the Lie algebra of the usual canonical commutation relations, specifically in terms of upper triangular matrices. Independently, Belavkin[16] and Holevo[17] developed the analogous representation in the setting for quantum stochastic calculus. We consider the mapping from Itō matrices $\mathbf{X} \in \mathfrak{A}^{(1+n)\times(1+n)}$ to associated *Belavkin-Holevo matrices*

$$\mathbb{X} = \mathscr{H}\left(\begin{bmatrix} x_{00} & x_{0\mathsf{k}} \\ x_{\mathsf{k}0} & x_{\mathsf{kk}} \end{bmatrix} \right) = \begin{bmatrix} 0 & x_{0\mathsf{k}} & x_{00} \\ 0 & x_{\mathsf{kk}} & x_{\mathsf{k}0} \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{A}^{(1+n+1)\times(1+n+1)}, \tag{14}$$

understood for each fixed time t. The Belavkin-Holevo matrices are $1+n+1$ square dimensional and we shall write their label set as

$$\mathsf{j} = \overline{0} \cup \mathsf{k} \cup \underline{0},$$

that is, $\overline{0}$ labels the top row/column and $\underline{0}$ labels the bottom row/column. We also introduce

$$\mathbb{I} \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{\mathsf{k}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbb{J} \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{\mathsf{k}} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The *twisted involution* on the set of Belavkin-Holevo matrices is defined by

$$\mathbb{X}^\star \triangleq \mathbb{J}\mathbb{X}^\dagger\mathbb{J}$$

We have the following properties:

$$\mathscr{H}\left(\mathbf{X}^\dagger\right) = \mathscr{H}\left(\mathbf{X}\right)^\star, \text{and}\, \mathscr{H}\left(\mathbf{XPY}\right) = \mathscr{H}\left(\mathbf{X}\right)\mathscr{H}\left(\mathbf{Y}\right).$$

The main advantage of using this representation is that the Itō correction $\mathbf{XPY}$ can now be given as just the ordinary product $\mathbb{XY}$ of the Belavkin-Holevo matrices.

The Belavkin-Holevo matrix associated with the Itō generating matrix is then

$$\mathbb{G} = \mathscr{H}\left(\mathbf{G}\right) = \begin{bmatrix} 0 & -S_{\mathsf{kk}}L_{\mathsf{k}}^{*} & -\left(\frac{1}{2}L_{\mathsf{k}}^{*}L_{\mathsf{k}} + iH\right) \\ 0 & S_{\mathsf{kk}} - I_{\mathsf{k}} & L_{\mathsf{k}} \\ 0 & 0 & 0 \end{bmatrix}$$

The related matrix

$$\mathbb{V} = \mathbb{I} + \mathbb{G} = \begin{bmatrix} 1 & -S_{\mathsf{kk}}L_{\mathsf{k}}^{*} & -\left(\frac{1}{2}L_{\mathsf{k}}^{*}L_{\mathsf{k}} + iH\right) \\ 0 & S_{\mathsf{kk}} & L_{\mathsf{k}} \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

is $\star$-unitary, that is

$$\mathbb{V}\mathbb{V}^\star = \mathbb{I} = \mathbb{V}^\star\mathbb{V}. \tag{16}$$

2. *The Stratonovich Form*

One can given the Stratonovich form of the unitary dynamics. Here we define the Stratonovich integral by the algebraic relation $dX(t) \circ Y(t) \equiv dX(t)\, Y(t) + \frac{1}{2}dX(t)\, dY(t)$ however this agrees with the notion of a mid-point rule[11]. For the unitary QSDE (11) we have the Stratonovich form (3) which we may write as

$$dU(t) = -idE(t) \circ U(t)$$

end where

$$dE(t) = E_{00}dt + \sum_{j\in\mathsf{j}} E_{j0}dB_j(t)^* + \sum_{k\in\mathsf{j}} E_{0k}dB_k(t) + \sum_{j,k\in\mathsf{j}} E_{jk}d\Lambda_{jk}(t).$$

Formally the stochastic Hamiltonian $\Upsilon(t)$ is the derivative of the quantum stochastic integral process $E(t)$. As outlined above, we assemble these operators into the Stratonovich generator matrix,

$$\mathbf{E} = \begin{bmatrix} E_{00} & E_{0\mathsf{k}} \\ E_{\mathsf{k}0} & E_{\mathsf{kk}} \end{bmatrix}$$

and we have the relationship

$$\mathbf{G} = -i\mathbf{E} - \frac{i}{2}\mathbf{E}\hat{\delta}\mathbf{G}.$$

This is more clearly seen in terms of Belavkin-Holevo matrices. Set

$$\mathbb{E} = \mathscr{H}(\mathbf{E}) = \begin{bmatrix} 0 & E_{0\mathsf{k}} & E_{00} \\ 0 & E_{\mathsf{kk}} & E_{\mathsf{k}0} \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$\mathbb{V} = \frac{\mathbb{I} - \frac{i}{2}\mathbb{E}}{\mathbb{I} + \frac{i}{2}\mathbb{E}}.$$

In particular, we see that $\mathbb{V}$ is $\star$-unitary if and only if $\mathbb{E} = \mathbb{E}^\star$ which in turn implies that $E^*_{\alpha\beta} = E_{\beta\alpha}$ for all $\alpha, \beta \in 0 \cup \mathsf{k}$.

B. Quantum Feedback Networks

In[5] we introduced the series product describing the situation where one SLH drives another, see Figure 2. Here the output of the first system (S_1, L_1, H_1) is fed forward as the input to the second system (S_2, L_2, H_2) and the limit of zero time delay is assumed. (Note that the systems do not technically have to be distinct and may have the same initial space!) In the limit, the Hudson-Parthasarathy parameters of the composite system were shown to be[5],[4]

$$\begin{aligned} S_{\text{series}} &= S_2 S_1, \\ L_{\text{series}} &= L_2 + S_2 L_1, \\ H_{\text{series}} &= H_1 + H_2 + \text{Im}\left\{ L_2^\dagger S_2 L_1 \right\}. \end{aligned}$$

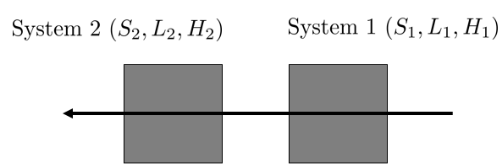

FIG. 2. Systems in series: Output of system 1 is the input of system 2.

We refer to the associative group law

$$(S_2, L_2, H_2) \lhd (S_1, L_1, H_1) \triangleq \Big(S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \text{Im}\{L_2^\dagger S_2 L_1\} \Big)$$

determined above as the series product. As remarked in[4], the series product actually arises natural in Belavkin-Holevo matrix form as

$$\mathbb{V}_{\text{series}} = \mathbb{V}_2 \mathbb{V}_1.$$

The product is clearly associative, as one would expect physically, and the general rule for several systems in series is then $\mathbb{V}_{\text{series}} = \mathbb{V}_n \cdots \mathbb{V}_2 \mathbb{V}_1$

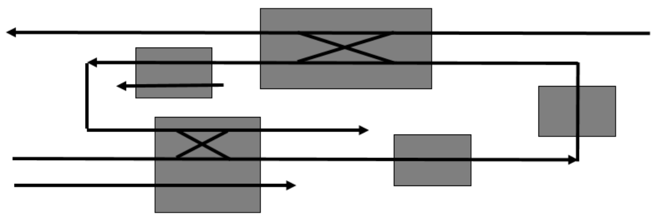

FIG. 3. A general quantum feedback network.

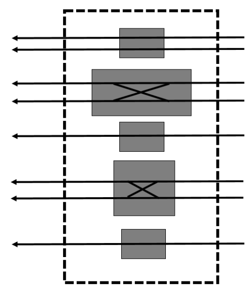

FIG. 4. The disconnected components in parallel.

More generally we gave the rules for construction an arbitrary quantum feedback network where we have several open quantum components, each described by a SLH model, and where various outputs of are fed back as driving inputs with zero time delay, see Figure 3.

In fact, it suffices to give two rules for constructing arbitrary networks of this type. The first step is to take the network description and break all the connections leaving only the individual open loop description. We can look upon this as a single SLH component, see Figure 4.

Our first network rule is that the models $(S_j, L_j, H_j)_{j=1}^n$ when concatenated in parallel, as sketched in Figure 4, correspond to the single SLH component

$$\boxplus_{j=1}^n (S_j, L_j, H_j) = \left(\begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H_1 + \cdots + H_n \right). \tag{17}$$

(Note that we have made no assumptions that the operators corresponding to different components commute!)

The second network rule tells us how to connect the various internal inputs, see Figure 5. To this end we first

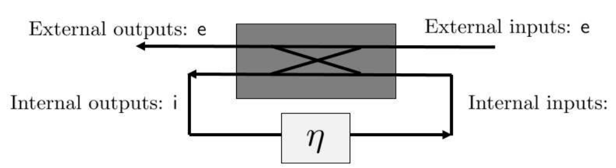

FIG. 5. Feedback

divide the inputs and outputs into external and internal groups. That is, the set k of all inputs is split into e which are the external inputs, and i which are the internal. With respect to this decomposition, we have the block partition

$$S_{\mathsf{kk}} = \begin{bmatrix} S_{\mathsf{ee}} & S_{\mathsf{ei}} \\ S_{\mathsf{ie}} & S_{\mathsf{ii}} \end{bmatrix}, \; L_{\mathsf{k}} = \begin{bmatrix} L_{\mathsf{e}} \\ L_{\mathsf{i}} \end{bmatrix},$$

We may for convenience include an adjacency matrix η which tells us which internal input label corresponds to which internal output label. The second rule then states that the reduced SLH model is

$$\begin{aligned}
S^{\mathrm{fb}} &= S_{\mathsf{ee}} + S_{\mathsf{ei}}\eta\left(I_{\mathsf{i}} - S_{\mathsf{ii}}\eta\right)^{-1} S_{\mathsf{ie}}, \\
L^{\mathrm{fb}} &= L_{\mathsf{e}} + S_{\mathsf{ei}}\eta\left(I_{\mathsf{i}} - S_{\mathsf{ii}}\eta\right)^{-1} L_{\mathsf{i}}, \\
H^{\mathrm{fb}} &= H + \sum_{i=\mathsf{i},\mathsf{e}} \mathrm{Im} L_j^{\dagger}\eta S_{j\mathsf{i}}\left(I_{\mathsf{i}} - S_{\mathsf{ii}}\eta\right)^{-1} L_{\mathsf{i}}.
\end{aligned} \tag{18}$$

We remark that the adjacency matrix is not essential and may be absorbed into the matrix S_{kk}. Indeed, we may think of the adjacency matrix as a simple device performing a scattering only and the disconnected model is $S_{\mathsf{kk}}\begin{bmatrix} I_{\mathsf{e}} & 0 \\ 0 & \eta \end{bmatrix} = \begin{bmatrix} S_{\mathsf{ee}} & S_{\mathsf{ei}}\eta \\ S_{\mathsf{ie}} & S_{\mathsf{ii}}\eta \end{bmatrix}$ by the series product. As such all we have to do is to replace $S_{\mathsf{ei}}\eta$ by S_{ei} and $S_{\mathsf{ii}}\eta$ by S_{ii}. In the following we shall always assume that this has already done, so we work with (18) with $\eta = I_{\mathsf{i}}$. We also note that we need the condition that the inverse of $I_{\mathsf{i}} - S_{\mathsf{ii}}$ exists: this is the condition needed to ensure that the network is well-posed.

We follow Smolyanov and Truman[18] in expressing the feedback reduction rule in terms of the Belavkin-Holevo matrices. This turns out to be most convenient for our purposes. Let $\overline{0} \cup \mathsf{e} \cup \mathsf{i} \cup \underline{0}$ be the usual labeling of the entries of the matrix $\mathbb{V}$, then the feedback reduced matrix has components

$$\left[\mathbb{V}^{\mathrm{fb}}\right]_{\alpha\beta} = \mathbb{V}_{\alpha\beta} + \mathbb{V}_{\alpha\mathsf{i}}\left(I_{\mathsf{i}} - \mathbb{V}_{\mathsf{ii}}\right)^{-1}\mathbb{V}_{\mathsf{i}\beta}$$

where α, β now belong to $\overline{0} \cup \mathsf{e} \cup \underline{0}$. That is, the matrix $\mathbb{V}^{\mathrm{fb}}$ is a Möbius transformation of the original $\mathbb{V}$. In fact, the feedback reduced $\mathbb{V}^{\mathrm{fb}}$ inherits the property of $\star$-unitarity from $\mathbb{V}$.

It is useful to reformulate this in terms of the associated $\mathbb{G}$ matrix. Indeed we have the following result

$$\mathbb{G}^{\mathrm{fb}} = \operatorname*{Schur}_{\mathsf{i}} \mathbb{G}. \tag{19}$$

Therefore, the feedback reduced $\mathbb{G}^{\mathrm{fb}}$ is just the original $\mathbb{G}$ shortened by the internal inputs and outputs.

III. THE STRATONOVICH VERSION OF QUANTUM FEEDBACK NETWORKS

We now reformulate network theory in terms of the Stratonovich generating matrices.

The first network rule is elementary. If we have m components each with input/output labels sets $\mathsf{k}(i)$ and Stratonovich generating matrices $\mathbf{E}^{(i)} = \begin{bmatrix} E_{00}^{(i)} & E_{0\mathsf{k}(i)}^{(i)} \\ E_{\mathsf{k}(i)0}^{(i)} & E_{\mathsf{k}(i)\mathsf{k}(i)}^{(i)} \end{bmatrix}$ for $i = 1, \cdots, m$, then the overall Stratonovich model for the components in parallel is

$$\boxplus_{i=1}^{m}\mathbf{E}^{(i)} = \mathbf{E}^{\mathrm{parallel}}$$

which of course has the set on input labels $\mathsf{k} = \cup_{i=1}^{m}\mathsf{k}(i)$ and $\mathbf{E}^{\mathrm{parallel}}$ is given by (5).

The second rule involves splitting the labels up as external and internal: $\mathsf{k} = \mathsf{e} \cup \mathsf{i}$.

Proposition 1 *Let* $\mathbf{E}$ *be the Stratonovich generator matrix labeled by* $0 \cup \mathsf{e} \cup \mathsf{i}$. *The quantum feedback network obtained by feeding the internal outputs back as internal inputs is well-posed if and only if the operator*

$$\mathscr{E}_{\mathsf{ii}} \triangleq E_{\mathsf{ii}} - \frac{i}{2}E_{\mathsf{ie}}\left(I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}}\right)^{-1} E_{\mathsf{ei}} \tag{20}$$

is strictly invertible.

Proof. We have that $\mathbb{V}_{\mathsf{ii}} = S_{\mathsf{ii}}$, and $S = \begin{bmatrix} S_{\mathsf{ee}} & S_{\mathsf{ei}} \\ S_{\mathsf{ie}} & S_{\mathsf{ii}} \end{bmatrix} = \begin{bmatrix} I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}} & \frac{i}{2}E_{\mathsf{ei}} \\ \frac{i}{2}E_{\mathsf{ie}} & I_{\mathsf{i}} + \frac{i}{2}E_{\mathsf{ii}} \end{bmatrix}^{-1} \begin{bmatrix} I_{\mathsf{e}} - \frac{i}{2}E_{\mathsf{ee}} & \frac{i}{2}E_{\mathsf{ei}} \\ \frac{i}{2}E_{\mathsf{ie}} & I_{\mathsf{i}} - \frac{i}{2}E_{\mathsf{ii}} \end{bmatrix}$ however a standard result in inverting block matrices shows that

$$\begin{bmatrix} I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}} & \frac{i}{2}E_{\mathsf{ei}} \\ \frac{i}{2}E_{\mathsf{ie}} & I_{\mathsf{i}} + \frac{i}{2}E_{\mathsf{ii}} \end{bmatrix}^{-1} \equiv \begin{bmatrix} \left(I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}}\right)^{-1} - \frac{1}{4}\left(I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}}\right)^{-1} E_{\mathsf{ei}}\Delta_{\mathsf{ii}}E_{\mathsf{ie}} & -\frac{i}{2}\left(I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}}\right)^{-1} E_{\mathsf{ei}}\Delta_{\mathsf{ii}} \\ -\frac{i}{2}\Delta_{\mathsf{ii}}E_{\mathsf{ie}}\left(I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}}\right)^{-1} & \Delta_{\mathsf{ii}} \end{bmatrix},$$

where $\Delta_{\mathsf{ii}} = \left(I_{\mathsf{i}} + \frac{i}{2}E_{\mathsf{ii}} + \frac{1}{4}E_{\mathsf{ie}}\left(I_{\mathsf{e}} + \frac{i}{2}E_{\mathsf{ee}}\right)^{-1}E_{\mathsf{ei}}\right)^{-1} \equiv \left(I_{\mathsf{i}} + \frac{i}{2}\mathscr{E}_{\mathsf{ii}}\right)^{-1}$. Substituting in yields the explicit form

$$S_{\mathsf{ii}} \equiv \left(I_{\mathsf{i}} - \frac{i}{2}\mathscr{E}_{\mathsf{ii}}\right)\left(I_{\mathsf{i}} + \frac{i}{2}\mathscr{E}_{\mathsf{ii}}\right)^{-1}.$$

The well-posed property is that $I_{\mathsf{i}} - \mathbb{V}_{\mathsf{ii}}$ is invertible, that is $I_{\mathsf{i}} - S_{\mathsf{ii}}$ is invertible. We see that this is equivalent to the requirement that $\mathscr{E}_{\mathsf{ii}}$ be invertible. ■

We have seen that the condition for an SLH model to have a Stratonovich representation is that $I_{\mathsf{k}}+S_{\mathsf{kk}}$ is invertible. We additionally have the well-posed property that $I_{\mathsf{i}} - S_{\mathsf{ii}}$ is invertible

Before stating our main theorem, we have the following lemma which will be used in the proof.

Lemma 2 *Let $\mathbb{G} = \mathscr{H}(\mathbf{G})$ be the Belavkin-Holevo matrix associated with a given Itō generating matrix $\mathbf{G}$. We double up the usual labels $\overline{0} \cup \mathsf{k} \cup \underline{0}$ to get $\left\{\overline{0} \cup \mathsf{k} \cup \underline{0}\right\} \cup \left\{\overline{0}' \cup \mathsf{k}' \cup \underline{0}'\right\}$ where the k' is a copy of k. Then*

$$\mathbb{G} = -\operatorname*{Schur}_{\overline{0}'\cup\mathsf{k}'\cup\underline{0}'}\left[\begin{array}{cc} 2\mathbb{I} & \sqrt{2}\mathbb{I} \\ \sqrt{2}\mathbb{I} & \mathbb{I} + \frac{i}{2}\mathbb{E} \end{array}\right].$$

Proof. This just says that we get $\mathbb{G}$ from shortening out the duplicate set of labels. Indeed we have

$$\operatorname*{Schur}_{\overline{0}'\cup\mathsf{k}'\cup\underline{0}'}\left[\begin{array}{cc} 2\mathbb{I} & \sqrt{2}\mathbb{I} \\ \sqrt{2}\mathbb{I} & \mathbb{I} + \frac{i}{2}\mathbb{E} \end{array}\right] = 2\mathbb{I} - 2\left(\mathbb{I} + \frac{i}{2}\mathbb{E}\right)^{-1} = \frac{i\mathbb{E}}{\mathbb{I} + \frac{i}{2}\mathbb{E}}$$

which agrees with $\mathbb{G} = \frac{\mathbb{I} - \frac{i}{2}\mathbb{E}}{\mathbb{I} + \frac{i}{2}\mathbb{E}} - \mathbb{I}$ up to the sign. ■

We are now able to state our main result which is the feedback reduction rule in terms of the Stratonovich generator matrices.

Theorem 3 *Let $\mathbf{E}$ be the Stratonovich generator matrix labeled by $0 \cup \mathsf{e} \cup \mathsf{i}$ with E_{ii} invertible. The feedback reduced Stratonovich generator (yielding (18) with $\eta = I_{\mathsf{i}}$) is*

$$\mathbf{E}^{\mathrm{fb}} = \operatorname*{Schur}_{\mathsf{i}} \mathbf{E}. \tag{21}$$

That is,

$$\left[\begin{array}{cc} E^{\mathrm{fb}}_{00} & E^{\mathrm{fb}}_{0\mathsf{e}} \\ E^{\mathrm{fb}}_{\mathsf{e}0} & E^{\mathrm{fb}}_{\mathsf{ee}} \end{array}\right] = \left[\begin{array}{cc} E_{00} & E_{0\mathsf{e}} \\ E_{\mathsf{e}0} & E_{\mathsf{ee}} \end{array}\right] - \left[\begin{array}{c} E_{0\mathsf{i}} \\ E_{\mathsf{ei}} \end{array}\right] E_{\mathsf{ii}}^{-1}\left[\begin{array}{cc} E_{\mathsf{i}0} & E_{\mathsf{ie}} \end{array}\right].$$

Proof. Combining (19) with the lemma, we see that $\mathbb{G}^{\mathrm{fb}}$ can be written as successive Schur complements:

$$\mathbb{G}^{\mathrm{fb}} = -\operatorname*{Schur}_{\mathsf{i}}\operatorname*{Schur}_{\overline{0}'\cup\mathsf{k}'\cup\underline{0}'}\left[\begin{array}{cc} 2\mathbb{I} & \sqrt{2}\mathbb{I} \\ \sqrt{2}\mathbb{I} & \mathbb{I} + \frac{i}{2}\mathbb{E} \end{array}\right].$$

We start with the doubled up set of labels $\left\{\overline{0} \cup \mathsf{e} \cup \mathsf{i} \cup \underline{0}\right\} \cup \left\{\overline{0}' \cup \mathsf{e}' \cup \mathsf{i}' \cup \underline{0}'\right\}$ and shortened by the duplicate labels $\left\{\overline{0}' \cup \mathsf{e}' \cup \mathsf{i}' \cup \underline{0}'\right\}$ and then by the labels i. However, we could alternatively shortened first by $\mathsf{i} \cup \mathsf{i}'$ and then by $\overline{0}' \cup \mathsf{e}' \cup \underline{0}'$ to get the same result. Now

$$\operatorname*{Schur}_{\mathsf{i}\cup\mathsf{i}'}\left[\begin{array}{cc} 2\mathbb{I} & \sqrt{2}\mathbb{I} \\ \sqrt{2}\mathbb{I} & \mathbb{I} + \frac{i}{2}\mathbb{E} \end{array}\right] = \operatorname*{Schur}_{\mathsf{i}\cup\mathsf{i}'}\left[\begin{array}{cccccccc} 2 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 2I_{\mathsf{e}} & 0 & 0 & 0 & \sqrt{2}I_{\mathsf{e}} & 0 & 0 \\ 0 & 0 & 2I_{\mathsf{i}} & 0 & 0 & 0 & \sqrt{2}I_{\mathsf{i}} & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 & 1 & \frac{i}{2}\mathbb{E}_{0\mathsf{e}} & \frac{i}{2}\mathbb{E}_{0\mathsf{i}} & \frac{i}{2}\mathbb{E}_{00} \\ 0 & \sqrt{2}I_{\mathsf{e}} & 0 & 0 & 0 & I_{\mathsf{e}} + \frac{i}{2}\mathbb{E}_{\mathsf{ee}} & \frac{i}{2}\mathbb{E}_{\mathsf{ei}} & \frac{i}{2}\mathbb{E}_{\mathsf{e}0} \\ 0 & 0 & \sqrt{2}I_{\mathsf{i}} & 0 & 0 & \frac{i}{2}\mathbb{E}_{\mathsf{ie}} & I_{\mathsf{i}} + \frac{i}{2}\mathbb{E}_{\mathsf{ii}} & \frac{i}{2}\mathbb{E}_{\mathsf{i}0} \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 1 \end{array}\right]$$

$$
= \begin{bmatrix} 2 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 2I_{\mathsf{e}} & 0 & 0 & \sqrt{2}I_{\mathsf{e}} & 0 \\ 0 & 0 & 2 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 1 & \frac{i}{2}\mathbb{E}_{0\mathsf{e}} & \frac{i}{2}\mathbb{E}_{00} \\ 0 & \sqrt{2}I_{\mathsf{e}} & 0 & 0 & I_{\mathsf{e}} + \frac{i}{2}\mathbb{E}_{\mathsf{ee}} & \frac{i}{2}\mathbb{E}_{\mathsf{e}0} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{i}{2}\mathbb{E}_{0\mathsf{i}} \\ 0 & \frac{i}{2}\mathbb{E}_{\mathsf{ei}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2I_{\mathsf{i}} & \sqrt{2}I_{\mathsf{i}} \\ \sqrt{2}I_{\mathsf{i}} & I_{\mathsf{i}} + \frac{i}{2}\mathbb{E}_{\mathsf{ii}} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2}\mathbb{E}_{\mathsf{ie}} & \frac{i}{2}\mathbb{E}_{\mathsf{i}0} \end{bmatrix}
$$

but

$$
\begin{bmatrix} 2I_{\mathsf{i}} & \sqrt{2}I_{\mathsf{i}} \\ \sqrt{2}I_{\mathsf{i}} & I_{\mathsf{i}} + \frac{i}{2}\mathbb{E}_{\mathsf{ii}} \end{bmatrix}^{-1} = \frac{1}{i\mathbb{E}_{\mathsf{ii}}} \begin{bmatrix} I_{\mathsf{i}} + \frac{i}{2}\mathbb{E}_{\mathsf{ii}} & -\sqrt{2}I_{\mathsf{i}} \\ -\sqrt{2}I_{\mathsf{i}} & 2I_{\mathsf{i}} \end{bmatrix}
$$

so that

$$
\underset{\mathsf{i}\cup\mathsf{i}'}{\mathrm{Schur}} \begin{bmatrix} 2\mathbb{I} & \sqrt{2}\mathbb{I} \\ \sqrt{2}\mathbb{I} & \mathbb{I} + \frac{i}{2}\mathbb{E} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 2I_{\mathsf{e}} & 0 & 0 & \sqrt{2}I_{\mathsf{e}} & 0 \\ 0 & 0 & 2 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 1 & \frac{i}{2}\mathbb{E}^{\mathrm{fb}}_{0\mathsf{e}} & \frac{i}{2}\mathbb{E}^{\mathrm{fb}}_{00} \\ 0 & \sqrt{2}I_{\mathsf{e}} & 0 & 0 & I_{\mathsf{e}} + \frac{i}{2}\mathbb{E}^{\mathrm{fb}}_{\mathsf{ee}} & \frac{i}{2}\mathbb{E}^{\mathrm{fb}}_{\mathsf{e}0} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 1 \end{bmatrix}
$$

where

$$
\mathbb{E}^{\mathrm{fb}} = \underset{\mathsf{i}}{\mathrm{Schur}}\, \mathbb{E} = \mathscr{H}\left(\underset{\mathsf{i}}{\mathrm{Schur}}\, \mathbf{E}\right) = \mathscr{H}\left(\mathbf{E}^{\mathrm{fb}}\right).
$$

Shortening over the remaining duplicate labels $\left\{\overline{0}' \cup \mathsf{e}' \cup \underline{0}'\right\}$ and multiplying by minus, we recover $\mathbb{G}^{\mathrm{fb}}$. From the expression obtained in the Lemma, we see that $\mathbf{E}^{\mathrm{fb}} = \underset{\mathsf{i}}{\mathrm{Schur}}\, \mathbf{E}$ must be the Stratonovich generating matrix. ■

It is clear from the statement of the Theorem that there are situations where the feedback network is well-posed, i.e. $\mathscr{E}_{\mathsf{ii}}$ is invertible, but the Schur complement is not invertible, i.e. E_{ii} is not invertible. The following proposition gives a simple test of when the Schur complement exists.

Proposition 4 *Let* $S_{\mathsf{kk}} = \begin{bmatrix} S_{\mathsf{ee}} & S_{\mathsf{ei}} \\ S_{\mathsf{ie}} & S_{\mathsf{ii}} \end{bmatrix}$ *be a scattering matrix decomposed with respect to a specification of internal and external input/outputs* $\mathsf{k} = \mathsf{e} \cup \mathsf{i}$, *with* $I_{\mathsf{k}} + S_{\mathsf{kk}}$ *invertible. Let us set*

$$
\mathscr{S}_{\mathsf{ii}} \triangleq S_{\mathsf{ii}} - S_{\mathsf{ie}} \left(I_{\mathsf{i}} + S_{\mathsf{ii}}\right)^{-1} S_{\mathsf{ie}}.
$$

Then the Stratonovich matrix block E_{ii} *is invertible if and only if* $I_{\mathsf{i}} - \mathscr{S}_{\mathsf{ii}}$ *is invertible.*

Proof. Starting from the identity $E_{\mathsf{kk}} = \frac{2}{i}\left(I_{\mathsf{k}} + S_{\mathsf{kk}}\right)^{-1}\left(I_{\mathsf{k}} - S_{\mathsf{kk}}\right)$ we obtain the sub-block

$$
E_{\mathsf{ii}} = \frac{2}{i}\left(I_{\mathsf{i}} + \mathscr{S}_{\mathsf{ii}}\right)^{-1}\left(I_{\mathsf{i}} - \mathscr{S}_{\mathsf{ii}}\right)
$$

using the same methods as used in the previous proposition. This formal expression shows precisely when the block E_{ii} is invertible, and we arrive at the desired conclusion. ■

A. Beam-splitter Example

As an example we consider the simple example of a beam-splitter with Stratonovich matrix

$$
E_{\mathsf{kk}} = \begin{bmatrix} E_{\mathsf{ee}} & E_{\mathsf{ei}} \\ E_{\mathsf{ie}} & E_{\mathsf{ii}} \end{bmatrix} \equiv \begin{bmatrix} \alpha & \beta \\ \beta^{*} & \gamma \end{bmatrix}
$$

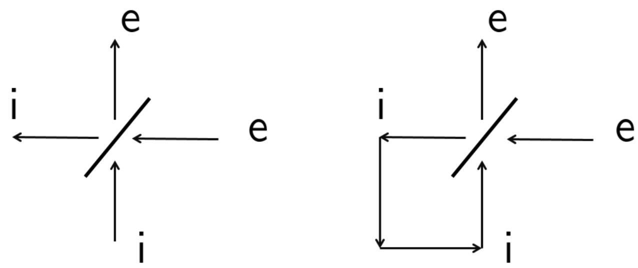

FIG. 6. (Left) a beam-splitter with matrix S_{kk}; (Right) its feedback reduction.

with α, β real and β complex. A simple algebra shows that

$$S_{\mathsf{kk}} \equiv \frac{1}{1+\frac{i}{2}(\alpha+\gamma)-\frac{1}{4}\left(\alpha\gamma-|\beta|^2\right)}\left[\begin{array}{cc} 1+\frac{i}{2}(\gamma-\alpha)+\frac{1}{4}\left(\alpha\gamma-|\beta|^2\right) & -i\beta \\ -i\beta^* & 1+\frac{i}{2}(\alpha-\gamma)+\frac{1}{4}\left(\alpha\gamma-|\beta|^2\right)\end{array}\right]$$

which gives the general form of a beam-splitter derivable from a Stratonovich form.

We see that if $\gamma=0\,(=E_{\mathsf{ii}})$ then the Schur complement is not defined so the Theorem does not apply automatically. Nevertheless we see that $\mathscr{E}_{\mathsf{ii}}=\gamma-\frac{i}{2}\frac{|\beta|^2}{1+\frac{i}{2}\alpha}$ and so the network may be well-posed even if $\gamma=0$, and it is instructive to look at this case. Here we have

$$S_{\mathsf{kk}} \equiv \frac{1}{1+\frac{i}{2}\alpha+\frac{1}{4}|\beta|^2}\left[\begin{array}{cc} 1-\frac{i}{2}\alpha-\frac{1}{4}|\beta|^2 & -i\beta \\ -i\beta^* & 1+\frac{i}{2}\alpha-\frac{1}{4}|\beta|^2\end{array}\right]$$

prior to feedback, and for all parametrizations we have that the feedback reduced scattering is

$$S_{\mathsf{ee}}^{\mathrm{fb}} \equiv -1.$$

The answer is of course not something which has a well-defined Stratonovich form: in fact we have only $\lim_{\varepsilon\to\infty}\frac{1-\frac{i}{2}\varepsilon}{1+\frac{i}{2}\varepsilon}=-1$ but that this can not be realized for finite ε.

B. The Series Product

Let us recall that the series product can be written in terms of Belavkin-Holevo matrices as $\mathbb{V}_{\text{series}}=\mathbb{V}_2\mathbb{V}_1$. If both $\mathbb{V}_2$ and $\mathbb{V}_1$ come from Stratonovich matrices $\mathbb{E}_2$ and $\mathbb{E}_1$ respectively, then the corresponding $\mathbb{E}_{\text{series}}$ should be expressible in terms of $\mathbb{E}_2$ and $\mathbb{E}_1$.

Proposition 5 *Suppose that both $\mathbb{V}_2$ and $\mathbb{V}_1$ come from Stratonovich matrices $\mathbb{E}_2$ and $\mathbb{E}_1$ respectively, then their series product $\mathbb{V}_{\text{series}}=\mathbb{V}_2\mathbb{V}_1$ has the Stratonovich matrix*

$$\mathbb{E}_{series}=\left(\mathbb{I}+\frac{i}{2}\mathbb{E}_2\right)^{-1}(\mathbb{E}_1+\mathbb{E}_2)\left(\mathbb{I}-\frac{1}{4}\mathbb{E}_2\mathbb{E}_1\right)^{-1}\left(\mathbb{I}+\frac{i}{2}\mathbb{E}_2\right).$$

Proof. Let us begin by noting that the Cayley transformation $F\mapsto V(F)=\frac{I-F}{I+F}$ is a map from matrices $\{F: I+F \text{ is invertible}\}$ to $\{V: I+V \text{ is invertible}\}$. Indeed, if $V=\frac{I-F}{I+F}$ then $F=\frac{I-V}{I+V}$, so $V^{-1}(\cdot)\equiv V(\cdot)$. We wish to solve $V(F_3)=V(F_2)V(F_1)$ for F_3 given F_1 and F_2. The identity $V(F_3)=V_2V(F_1)$ can be rearranged to give

$$F_3=\left[(I-V_2)+(I+V_2)F_1\right]\left[(I+V_2)+(I-V_2)F_1\right]^{-1}$$

and setting $F_2=V^{-1}(V_2)\equiv\frac{I-V_2}{I+V_2}$ leads to

$$F_3=(I+V_2)(F_1+F_2)(I+F_2F_1)^{-1}(I+V_2)^{-1}$$

and noting that $I+V_2=2\frac{I}{I+F_2}$ gives the result. ■

C. Adjacency Matrices

The adjacency operator corresponding to a given permutation σ on a set of n labels will be denoted by $\eta(\sigma)$. That is,

$$[\eta(\sigma)]_{jk} = \begin{cases} 1, & j = \sigma(k); \\ 0, & \text{otherwise.} \end{cases}$$

A natural question is when does there exist a E_{kk} such that $\eta(\sigma) = \frac{I_{\mathsf{k}} - \frac{i}{2}E_{\mathsf{kk}}}{I_{\mathsf{k}} + \frac{i}{2}E_{\mathsf{kk}}}$. For this to be possible we need that $I_{\mathsf{k}} - \eta(\sigma)$ is invertible, so that $E_{\mathsf{kk}} \equiv \frac{2}{i}\frac{I_n - \eta(\sigma)}{I_n + \eta(\sigma)}$.

This mapping : $\sigma \mapsto \eta(\sigma)$ from the set of permutations on n labels to the $n \times n$ matrices is a reducible representation of the permutation group: indeed each $\eta(\sigma)$ has eigenvalue unity for the eigenvector $[1, \cdots, 1]^\top$ and this is a nontrivial invariant subspace. More generally, we have that the spectrum (including explicit degeneracies) of $\eta(\sigma)$ is the multi-set $\cup_{k\geq 1}\{k\text{-th roots of unity}\}^{n_k(\sigma)}$ where $n_k(\sigma)$ counts the number of cycles of length k in the permutation. In particular, we note that -1 is not in the spectrum if and only if there are no even cycles in σ[20].

As a result, we have that $I_n + \eta(\sigma)$ does not have zero as an eigenvalue if and only if σ has no even cycles. This is the condition for $\eta(\sigma)$ to be expressed as a Cayley transorm of some 2×2 matrix E_{kk}. For instance, the swap gate

$$\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

corresponding to the cycle (12) is the simplest adjacency matrix that cannot be expressed in this way.

IV. CONCLUSIONS

The Stratonovich form of the quantum stochastic calculus has the advantage of revealing a Hamiltonian structure. This is readily seen in the rule for concatenating components to get $\Upsilon(t)$ in (4) - and so (5). The fact that the feedback reduction rule has the direct form of a Schur complement, that is equation (6), is however an unexpected feature. We might mention that the Schur complement has previously emerged as the appropriate tool in adiabatic elimination results in the SLH formalism[21–23].

In the paper, we have restricted our attention to just the proof of the mathematical form of the feedback reduced Stratonovich matrix. This is done at the very broad level of generality offered by the SLH formalism, and have deferred the application to specific models for a latter publication. It should be mentioned that situations such as completely closed feedback loops served as a motivating problem for this publication as the Schur complement form already appears in a restricted sense, and strongly suggestive of the general result presented here in Theorem 1. We have also restricted to the vacuum case for the input fields. The issue of introducing time delays into the feedback connections has also been ignored - we assume the validity of an instantaneous feedback limit - however we note that there has been some promising developments in this direction for linear quantum systems[24].

While the Stratonovich form of the feedback reduction rule is much simpler that the Itō form (18), it is not true that it is universally easier to work with. For instance, the series product formula is a corollary to the concatenation rule (17) and the feedback rule (18), and so must be derivable from the two equivalent formulations in terms of Stratonovich matrices of coefficients. However this derivation is very involved. It is therefore the case that certain operations, such as putting systems in series are better handled with the Itō, or SLH, form while other operations, such as feedback reduction may be better handled in the Stratonovich form. The situation is not unlike classical circuit theory where one chooses judicially the form of the immittances: namely impedances for components is series in a network, and admittances (their inverses) for components placed in parallel in a network. While analogy is not exact, it does suggest that a hybrid use of the Itō and Stratonovich rules may be very useful for calculating the SLH characteristics of complex quantum feedback networks.

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