Aberystwyth University

Generalized Adaptive Fuzzy Rule Interpolation

Yang, Longzhi; Chao, Fei; Shen, Qiang

Published in:
IEEE Transactions on Fuzzy Systems

DOI:
10.1109/TFUZZ.2016.2582526

Publication date:
2017

Citation for published version (APA):
Yang, L., Chao, F., & Shen, Q. (2017). Generalized Adaptive Fuzzy Rule Interpolation. *IEEE Transactions on Fuzzy Systems*, *25*(4), 839-853. https://doi.org/10.1109/TFUZZ.2016.2582526

Generalized Adaptive Fuzzy Rule Interpolation

Longzhi Yang, *Member, IEEE*, Fei Chao, *Member, IEEE*, and Qiang Shen

***Abstract*—As a substantial extension to fuzzy rule interpolation that works based on two neighboring rules flanking an observation, adaptive fuzzy rule interpolation is able to restore system consistency when contradictory results are reached during interpolation. The approach first identifies the exhaustive sets of candidates, with each candidate consisting of a set of interpolation procedures which may jointly be responsible for the system inconsistency. Then, individual candidates are modified such that all contradictions are removed, and thus, interpolation consistency is restored. It has been developed on the assumption that contradictions may only be resulted from the underlying interpolation mechanism, and that all the identified candidates are not distinguishable in terms of their likelihood to be the real culprit. However, this assumption may not hold for real-world situations. This paper, therefore, further develops the adaptive method by taking into account observations, rules, and interpolation procedures, all as diagnosable and modifiable system components. In addition, given the common practice in fuzzy systems that observations and rules are often associated with certainty degrees, the identified candidates are ranked by examining the certainty degrees of its components and their derivatives. From this, the candidate modification is carried out based on such ranking. This study significantly improves the efficacy of the existing adaptive system by exploiting more information during both the diagnosis and modification processes.**

***Index Terms*—Adaptive fuzzy rule interpolation (AFRI), assumption-based truth maintenance system (ATMS), fuzzy inference, general diagnostic engine (GDE).**

I. Introduction

FUZZY inference systems have been successfully applied to many real-world applications, but the systems may suffer from either too sparse or too complex rule bases. Fuzzy rule interpolation (FRI) alleviates this by supporting inference with incomplete sparse rule bases, or by simplifying complex fuzzy systems that involve very dense rule bases through approximating certain parts of the model with their neighboring rules [1], [2]. Many important FRI methods and their analysis or variations have been presented in the literature, including [1]–[22]. What is common to most of these techniques is that multiple values may be derived for a single variable. This implies that inconsistencies have been generated in the interpolated results.

Adaptive fuzzy rule interpolation (AFRI) was proposed in an effort to address this problem [23], [24]. It was developed upon FRI approaches by which two neighboring rules that flank an observation are utilized for interpolation. The approach efficiently detects inconsistencies, directly locates possible sets of fault components (namely, candidates), and effectively modifies the candidates in order to restore consistency, by removing detected inconsistencies. The approach artificially treats an FRI system as a component-based mechanism, where system components are defined as interpolation procedures. An assumption-based truth maintenance system (ATMS) [25]–[27] is employed to record the depending relationships between interpolated results and their dependent system components (i.e., its proceeding interpolation procedures). Then, the classical general diagnostic engine (GDE) [28] is utilized to hypothesize a set of candidates that each may have led to all the system contradictions. Finally, the system consistency is restored by modifying an identified single candidate.

The adaptive approach outlined above assumes that all the contradictory interpolated results are caused by the underpinning interpolation procedures. This assumption restricts the applications of AFRI to problems with defective fuzzy interpolation procedures only, but observations and rules in a fuzzy inference system may also be ill-specified (to a certain extent). Thankfully, this limitation is not a fundamental restriction of the idea underlying the adaptive approach. Supported by the initial preliminary investigations of [29], this paper further develops the work of [24], to allow the diagnosis and modification of observations and rules. This significantly enhances the robustness of the original method as one consistent inference result may still be derived when the original fails, often with intuitively more reasonable interpolated results.

Due to the introduction of more complex and uncertain information to the underlying information and knowledge representation scheme, the number of generated candidates may increase dramatically. However, these candidates can be discriminated as: 1) two different values derived for a given variable that have led to a contradiction may not be equally reliable (besides, one may be correct and the other wrong); and 2) all the elements which jointly support one of the two contradictory values may not be equally reliable. A candidate prioritization mechanism is, therefore, introduced here to reinforce the present work, starting from the initial report of [30], such that only the most important candidates are considered during the modification stage. First, the classical ATMS is extended to record dependences and also to log the extent to which such dependences are deemed reliable. The candidates are then prioritized using a modified GDE

Manuscript received February 4, 2016; accepted April 20, 2016. Date of publication July 7, 2016; date of current version August 2, 2017. This work was supported in part by the National Natural Science Foundation of China under Grant 61203336.

L. Yang is with the Department of Computer Science and Digital Technologies, Faculty of Engineering and Environment, Northumbria University, Newcastle upon Tyne NE1 8ST, U.K. (e-mail: longzhi.yang@northumbria.ac.uk).

F. Chao is with the Department of Cognitive Science, School of Information Science and Engineering, Xiamen University, Xiamen 361005, China (e-mail: fchao@xmu.edu.cn).

Q. Shen is with the Department of Computer Science, Institute of Mathematics, Physics and Computer Science, Aberystwyth University, Aberystwyth SY23 3DB, U.K. (e-mail: qqs@aber.ac.uk).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TFUZZ.2016.2582526

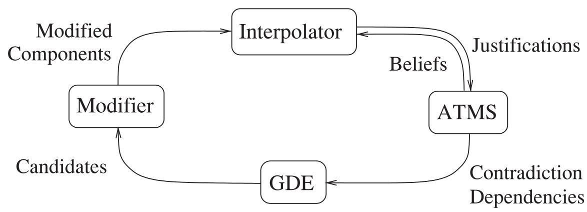

Fig. 1. Adaptive fuzzy interpolation.

by taking the reliability information into consideration. Thanks to the prioritization of candidates, a consistent solution can be rapidly derived with saved computational cost.

The remainder of this paper is organized as follows. A brief review of the theoretical underpinnings of AFRI is presented in Section II. An extension of the candidate generation procedure is reported in Section III, by which a candidate element can be an observation, rule, or fuzzy interpolation procedure. A generalization of the candidate modification procedure is discussed in Section IV, which allows the modification of all types of diagnosable candidate components. To facilitate comparison, the application problem considered in [24] is reinvestigated in Section V, where the proposed approach is employed. This paper is concluded in Section VI, with important future directions of improvements pointed out.

II. Adaptive Fuzzy Rule Interpolation

AFRI ensures that interpolated results remain consistent to a certain degree throughout the entire interpolation process [24]. In this paper, given two fuzzy sets A_i and A_j with respect to the same variable x within the domain D_x, the degree of consistency between them is represented as the degree of matching as follows:

$$M(A_i, A_j) = \sup_{x \in D_x} [\min(\mu_{A_i}(x), \mu_{A_j}(x))]. \quad (1)$$

Based on this, the degree β of a contradiction regarding two propositions P (x is A_i) and P'(x is A_j) is defined as

$$\beta = 1 - M(A_i, A_j). \quad (2)$$

This study adopts a predefined threshold β_0 $(0 \leq \beta_0 \leq 1)$ to examine whether a pair of values associated with a common variable is unacceptably contradictory. A β_0-contradiction appears if the corresponding contradictory degree between the two concerned propositions is greater than β_0.

As with [24], each pair of neighboring rules, which may be utilized together for interpolation, is termed as a fuzzy interpolation component (FIC). The input of such a component is a vector of observations and/or previous inferred results, which is hereafter referred to an interpolation input for simplicity. The output is the consequence of the interpolated rule, which takes such an input as its antecedent. The working process of AFRI is illustrated in Fig. 1. Given a fuzzy inference problem with a sparse rule base, the interpolator performs inference through FRI, and the ATMS records the dependences of contradictions upon the preceding FICs. Then, the GDE diagnoses the cause of the contradictions and generates candidates for modification, and finally, the modifier revises the candidates to remove contradictions and restore system consistency.

A. Rule Interpolation by the Interpolator

Suppose that the interpolation input is

$$O: \ x_1 = A^*_{1x} \text{ and ... and } x_m = A^*_{mx} \quad (3)$$

and that rules

$$\begin{aligned} R_i &: \text{ IF } x_1 = A_{1i} \text{ and ... and } x_m = A_{mi}, \text{ THEN } y = B_i \\ R_j &: \text{ IF } x_1 = A_{1j} \text{ and ... and } x_m = A_{mj}, \text{ THEN } y = B_j \end{aligned} \quad (4)$$

are the neighboring ones used for interpolation regarding the input O. The scale and move transformation-based FRI, upon which AFRI has been introduced, is outlined in Fig. 2. Further details of this approach can be found in [12] and [13], but this is out of the scope of this paper.

In this figure, there are m repeated subcomponents, each of which takes A^*_{kx}, A_{ki}, and A_{kj} $(1 \leq k \leq m)$ as inputs and produces a relative placement factor λ_k, an intermediate fuzzy set $A^*_{kx}{}'$, and a number of similarity measurements between A_{kx} and A^*_{kx}. Each subcomponent first uses the so-called representative values a_{ki}, a_{kj}, and a^*_{kx} to express the overall positions of A_{ki}, A_{kj}, and A^*_{kx}, respectively, computed using the function f_1. The relation regarding the relative locations between the interpolation input term A^*_{kx} and the corresponding antecedents terms (A_{ki} and A_{kj}) of a pair of neighboring rules is computed next, resulting in the required λ_k, which is computed by the real function f_2. From this, an antecedent term of the intermediate rule $A^*_{kx}{}'$ is calculated by applying real function f_3 with a parameter λ_k to A_{ki} and A_{kj}. Next, a set of similarity degrees between A^*_{kx} and $A^*_{kx}{}'$, expressed as the scale rate s_k, scale ratio $\mathbb{S}_k$, and move rate $\mathbb{M}_k$, is obtained by applying the function f_4 (which stands for a predefined similarity metric). Function f_6 is then introduced to combine all the resultant λ_k $(k \in \{1, 2, ..., m\})$ to an overall single scale λ, as is f_7 to combine all the similarity rates (s_k, $\mathbb{S}_k$, $\mathbb{M}_k$) to (s, $\mathbb{S}$, $\mathbb{M}$). The conclusion B^* can finally be approximated by transforming the consequent $B^{*\prime}$ of the intermediate rule. This is implemented by applying the combined single-scale similarity rates, through the transformation function f_5:

$$T(B^*, B^{*\prime}) = T((A_{1x}, ..., A_{mx}), (A^*_{1x}{}', ..., A^*_{mx}{}')). \quad (5)$$

B. Truth Maintenance by the Assumption-Based Truth Maintenance System

In implementing AFRI, ATMS is utilized to record the dependence of interpolated results and that of contradictions, upon the FICs from which they are inferred. Using ATMS' terminology, observations, interpolated results, contradictions, and FICs can all be represented as ATMS nodes, each of which is formed by a name (standing for its logical or physical meaning), a set of justifications, and a label.

A justification expresses a logical implication through which a node may be derived from other relevant nodes. An inferred proposition represented as an ATMS node is of the following justification:

$$M_1, M_2, ..., M_n, R_i R_j \Rightarrow C \quad (6)$$

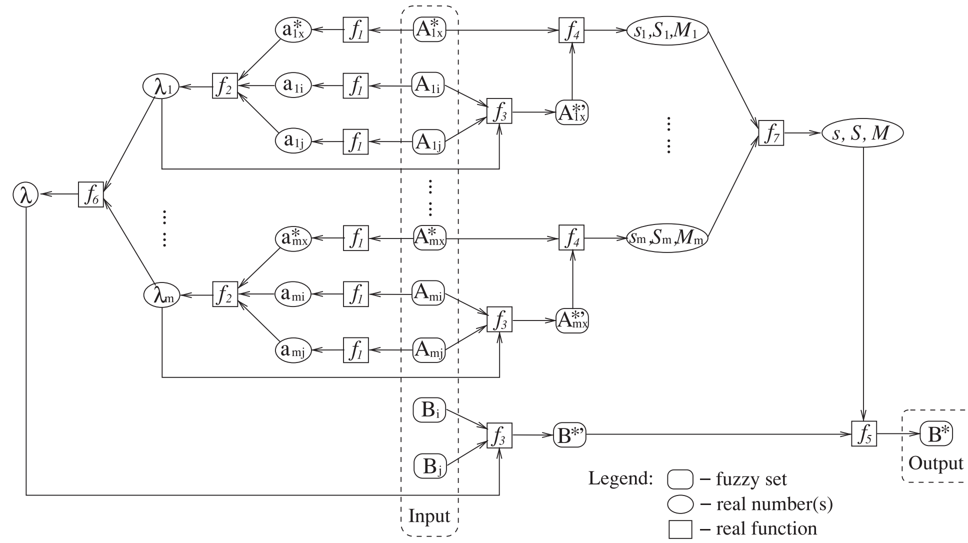

Fig. 2. Outline of transformation-based FRI.

where R_iR_j denotes the FIC formed by the two neighboring rules R_i and R_j $(i \neq j)$ which infers the interpolated result C from n other nodes $M_1, M_2, ..., M_n$ (that are observations and/or interpolated results). Based on the definition of contradiction, a β_0-contradiction is reached if the contradiction degree β between any two propositions P $(x\ is\ A_i)$ and $P'(x\ is\ A_j)$ is greater than a predefined threshold β_0, which is expressed in the format of proposition as

$$P, P' \Rightarrow_{\beta_0} \perp. \tag{7}$$

A label is a set of environments, each of which is a minimal set of FICs that jointly entail the supported node. An environment is said to be β_0-inconsistent if β_0-contradiction is logically derivable by the environment and a given justification; otherwise, the environment is $(1 - \beta_0)$-consistent. The ATMS label updating algorithm ensures that the label of each node is $(1 - \beta_0)$-consistent, sound, minimal, and complete, except that the label of the special "false" node is β_0-inconsistent rather than $(1 - \beta_0)$-consistent. Whenever a β_0-contradiction is detected, each environment in its label is added into the label of the "false" node, and all such environments and their supersets are removed from the label of every other node. In addition, any such environment which is a superset of another is removed from the label of the "false" node. Therefore, the label of the special "false" node collectively holds the minimal complete set of environments, each of which leads to a β_0-contradiction.

C. Candidate Generation by the General Diagnostic Engine

A set of minimal candidates for modification can be generated by GDE [28] from the label of the "false" node. A candidate is a set of FICs that may have led to all detected contradictions. Since a β_0-inconsistent environment contained in the label of the "false" node indicates that at least one of its elements is inconsistent (or faulty), a candidate must have a nonempty intersection with each β_0-inconsistent environment. Based on this observation, each candidate is constructed by taking just one FIC from each environment that supports the "false" node. The candidates are guaranteed to be minimal by removing all the supersets of others. As a result of this, the successful correction of any single candidate will remove all contradictions.

D. Candidate Modification by the Modifier

AFRI always modifies the candidate with the smallest cardinality first. With respect to a given queue of candidates $\mathbb{Q}$, the overall modification procedure is outlined in Algorithm 1. The main subprocedure MODIFY(C) takes a single candidate (C) as input and returns a Boolean value to indicate whether the modification succeeds or not.

Algorithm 1: The CONSISTENCYRESTORING procedure.

CONSISTENCYRESTORING($\mathbb{Q}$)
Input: $\mathbb{Q}$, a queue of candidates, each of which is a set of FICs.
Output: **True**, if succeeds; **False**, otherwise.

```
modified ←False
do
    C← Dequeue(Q)
    modified ← MODIFY(C)
while ((modified ==False) && (Q! = ∅))
return modified
```

To illustrate the basic ideas embedded in this subprocedure, suppose that the defective FIC is formed by the pair of neighboring rules as given in (4), which flanks the interpolation inputs $O_x(x \in \{1, 2, ..., n\})$ in the form of (3). The implementation of the modification procedure for a candidate consisting of a single FRI can then be summarized in the following steps:

Step 1: Find the interpolated rule "IF $x_1 = A^*_{1k}$ and $\cdots$ and $x_m = A^*_{mk}$, THEN $y = B^*_k$" whose antecedent is located in the middle most of the neighborhood of the antecedents of the two rules used for interpolation in terms of their representative values that are calculated using a particular integration formula [24]. Suppose that the relative placement factor of its consequence λ_k is modified to $\widehat{\lambda}_k$. The correction rate pair can then be calculated as

$$\begin{cases} c^- = \frac{\widehat{\lambda}_k}{\lambda_k} \\ c^+ = \frac{1-\widehat{\lambda}_k}{1-\lambda_k}. \end{cases} \tag{8}$$

Step 2: Obtain the modified relative placement factors of the consequences of all other interpolated rules, which have been created with respect to the same defective FIC in the same way as that used to compute the correction rate pair above, where $p \in \{1, 2, ..., k-1\}$ and $q \in \{k+1, k+2, ..., n\}$:

$$\begin{cases} \widehat{\lambda}_p = \lambda_p \cdot c^- \\ 1 - \widehat{\lambda}_q = (1 - \lambda_q) \cdot c^+. \end{cases} \tag{9}$$

Step 3: Compute the modified consequences of the intermediate rules corresponding to all interpolated rules that have been generated from the same defective FIC in accordance with their modified relative placement factors. Suppose that the intermediate rule corresponding to defective rule "IF $x_1 = A^*_{1x}$ and $\cdots$ and $x_m = A^*_{mx}$, THEN $y = B^*_x$" is "IF $x_1 = A'_{1x}$ and $\cdots$ and $x_m = A'_{mx}$, THEN $y = B'_x$." From this, the modified consequence of the intermediate rule $\widehat{B}'_x$ is

$$\widehat{B}'_x = (1 - \widehat{\lambda}_x) B_i + \widehat{\lambda}_x B_j \tag{10}$$

where $x \in \{1, 2, ..., n\}$. That is, the modified intermediate rule becomes "IF $x_1 = A'_{1x}$ and $\cdots$ and $x_m = A'_{mx}$, THEN $y = \widehat{B}'_x$."

Step 4: Compute the modified consequences of all interpolated rules from the consequences of the modified intermediate rules through scale and move transformations:

$$T((A_{1x}', ..., A_{mx}'), (A^*_{1x}, ..., A^*_{mx})) = T(\widehat{B}'_x, \widehat{B}^*_x) \tag{11}$$

where $x \in \{1, 2, ..., n\}$, and $T(\cdot, \cdot)$ represents the transformations based on the scale and move measures [12], [13].

Step 5: Impose restriction over the modified consequence such that it becomes consistent with the interpolation context. Suppose that m object values $B_1, B_2, ..., B_m$ are obtained for the variable y. If they are $(1 - \beta_0)$-consistent, they must satisfy

$$\bigcap_{j=1}^{m} (B_j)_{\beta_0} \neq \varnothing \tag{12}$$

where $(B_j)_{\beta_0}$ denotes the β_0-cut of fuzzy set B_j.

Step 6: Constrain the propagations of all modified consequences so that they are consistent with the rest. Propagate the modified result through the entire reasoning network. For a given variable z, suppose that m object values of the variable z have been modified via the propagation, resulting in modified values $\widehat{C}_i$, $i \in \{1, 2, ..., m\}$, and that n object values C_j, $j \in \{1, 2, ..., n\}$, of z are not affected by the propagation. These modified consequences must satisfy the following such that they are all $(1 - \beta_0)$-consistent:

$$\left(\bigcap_{i=1}^{m} (\widehat{C}_i)_{\beta_0} \right) \bigcap \left(\bigcap_{l=1}^{n} (C_j)_{\beta_0} \right) \neq \varnothing. \tag{13}$$

Step 7: Solve the set of simultaneous equalities and inequalities as posed above. The solutions imply successfully modified results which guarantee the system reasoning consistency.

III. Generalizing Candidate Generation

Only FICs are regarded as diagnosable and modifiable candidate elements in the original AFRI approach outlined above. However, observations and rules may also be faulty to a certain extent. This section extends the existing AFRI such that observations and rules can also be diagnosed and modified. To facilitate this, the certainty degrees of observations, rules, and FICs are discussed first.

A. Certainty Degrees of Observations and Rules

There are generally four categories of inexact information [31]: 1) vagueness, 2) uncertainty, 3) both vagueness and uncertainty with the latter represented as real numbers, and 4) both vagueness and uncertainty with the latter also defined as fuzzy sets. The existing FRI [24] only considers type 1 information, which is extended in this study by introducing type 2 information into the system, thereby resulting in the exploitation of type 3 information overall.

With the extra information, an observation is represented as

$$O: x_i = A^*_{ij}\ (c_O) \tag{14}$$

where $0 \leq c_O \leq 1$ expresses the certainty degree of the observation O. Conceptually, the vagueness of an object value can be modeled as a fuzzy set due to the lack of a precise boundary between given bits of information. Here, the certainty degree of an observation is represented as a crisp number, which is either assigned subjectively [32] or estimated from other mechanisms such as statistical data analysis. It indicates the confident level at which the current description of the object value may be regarded as of confidence or being reliable.

Denote the certainty degree of an observation O as c_O. Then, the uncertainty degree of the same piece of information is naturally expressed as $1 - c_O$. Thus, the modifiable range of the object value O is intuitively bounded to the proportion of $1 - c_O$ in reference to the entire variable domain. This means that the factual object value of O can be obtained by shifting the fuzzy set representation of the defective observation toward either side of the variable domain to a maximal distance of $\frac{1-c_O}{2}(\max_i - \min_i)$, where the domain of the variable x_i is $D_{x_i} = [\min_i, \max_i]$. Given that the shifting of a vague term is restricted from changing the shape and area of the underlying fuzzy set, the shifting process is equivalent to adding a real number to the original fuzzy set [33]. Formally, the factual value of A^*_{ij}, denoted as $\widehat{A}^*_{ij}$, of the observation O as given in (14) must satisfy

$$\begin{cases} \widehat{A}^*_{ij} \geq A_{ij} - \frac{1-c_O}{2}(\max_i - \min_i) \\ \widehat{A}^*_{ij} \leq A_{ij} + \frac{1-c_O}{2}(\max_i - \min_i). \end{cases} \tag{15}$$

It is possible that the shifting may be out of the variable domain due to the inaccuracy of the uncertainty information. Therefore, to ensure that the final shifting result is within the value range of the variable, the following must be satisfied:

$$\begin{cases} \min(\text{supp}(\widehat{A}^*_{ij})) \geq \min_i \\ \max(\text{supp}(\widehat{A}^*_{ij})) \leq \max_i \end{cases} \tag{16}$$

where $\text{supp}(\widehat{A}^*_{ij})$ represents the support of $\widehat{A}^*_{ij}$.

Similarly, with the uncertainty information, rules given in (4) are then extended to be of the following form:

$$\begin{aligned} &R_i : \text{ IF } x_1 = A_{1i} \text{ and } \cdots \text{ and } x_m = A_{mi}, \\ &\quad \text{THEN } y = B_i \ (c_{R_i}); \\ &R_j : \text{ IF } x_1 = A_{1j} \text{ and } \cdots \text{ and } x_m = A_{mj}, \\ &\quad \text{THEN } y = B_j \ (c_{R_j}). \end{aligned} \tag{17}$$

This means that rules R_i and R_j are certain to the degree of c_{R_i} and c_{R_j}, respectively. As with the certainty degrees associated with observations, certainty degrees attached to the rules are either subjectively provided or objectively learned.

B. Certainty Degrees of Fuzzy Interpolation Components

An FIC consisted of two neighboring rules is utilized in this study to represent the fuzzy interpolation mechanism. Essentially, this mechanism is an extension of classical linear interpolation on fuzzy rules. Thus, intuitively, if an FIC is defined on a pair of neighboring rules that are more certain to derive correct interpolated results, such an artificially created component is deemed to be more reliable, under the linearity assumption. Suppose that the FIC R_iR_j consists of the following two single-antecedent rules:

$$\begin{aligned} &R_i : \text{ IF } x = A_i, \quad \text{THEN } y = B_i \ (c_{R_i}) \\ &R_j : \text{ IF } x = A_j, \quad \text{THEN } y = B_j \ (c_{R_j}). \end{aligned} \tag{18}$$

Then, reflecting this intuition, the certainty degree $c_{R_iR_j}$ of the component R_iR_j can be defined by

$$c_{R_iR_j} = 1 - \left| \frac{d(A_i, A_j)}{\max_x - \min_x} - \frac{d(B_i, B_j)}{\max_y - \min_y} \right| \tag{19}$$

where $d(A, A')$ is the distance between A and A' (given a certain distance metric), and $\max_z$ and $\min_z$ are the maximum and minimum of the domain values of the variable $z (z = x, y)$, respectively. Note that $c_{R_iR_j} \in [0, 1]$.

For the more general cases where the FIC R_iR_j is composed by two multiantecedent rules as given in (17), the calculation of the certainty degree can be readily extended. The result is given as follows:

$$c_{R_iR_j} = 1 - \left| \frac{\sum_{k=1}^{m} \frac{d(A_{ki}, A_{kj})}{\max_{x_k} - \min_{x_k}}}{m} - \frac{d(B_i, B_j)}{\max_y - \min_y} \right|. \tag{20}$$

In this equation, the distance between the two sets of antecedents of two multiantecedent fuzzy rules is defined as the average of the distances between all pairs of corresponding antecedent terms regarding each corresponding variable. This is again to reflect the underlying linearity assumption.

C. Certainty Degrees of Interpolated Results

Given an interpolation input $M_1, M_2, ..., M_n$, two neighboring rules R_i and R_j that flank the given interpolation input, and the corresponding FIC R_iR_j, a logical consequence C can be generated by applying FRI. Then, the certainty degree c_C of the conclusion C can be derived from the certainty degrees of the input terms, the certainty degree of the neighboring rules, and the certainty degree of the corresponding FIC, which is calculated by

$$c_C = c_{M_1} \otimes c_{M_2} \otimes \cdots \otimes c_{M_n} \otimes c_{R_i} \otimes c_{R_j} \otimes c_{R_iR_j} \tag{21}$$

where the composition operator $\otimes$ is a t-norm operator, such as minimum and algebraic product. Note that multiple applications of different interpolation procedures may lead to the same interpolated result C. However, they may be associated with different certainty degrees, say $c_{C_1}, c_{C_2}, ..., c_{C_n}$. Then, the overall certainty degree c of the interpolated result C is revised as

$$c = c_{C_1} \oplus c_{C_2} \oplus \cdots \oplus c_{C_n} \tag{22}$$

where $\oplus$ is an s-norm operator, such as maximum.

D. Dependence Recording With Extended Assumption-Based Truth Maintenance System

In the previous work of [24], ATMS records the dependences of the contradictions (or interpolated results) upon FICs. However, in general, such contradictions may also depend upon the observations and rules used to perform FRI. Therefore, observations, interpolated results, contradictions, FICs, and rules are all represented as ATMS nodes in this study, which are originally assumed to be true and which may be established to be false (of a certain degree) subsequently. Recall that a justification describes how a node is derivable from other nodes. In general, any ATMS node with an interpolated result C from an interpolation input $M_1, M_2, ..., M_n$ based on neighboring rules R_i and R_j may now be verified by the following ATMS justification:

$$M_1, M_2, ..., M_n, R_i, R_j, R_iR_j \Rightarrow C. \tag{23}$$

Equation (23) degenerates to (6) when rules R_i and R_j $(i \neq j)$ are fixed and true and hence not needed to be kept in the dependence records.

The above justification not only explicitly describes how the consequence C is logically derived from other nodes, but also implicitly expresses to what extent C can be derived from the nodes M_1, M_2, ..., M_n, R_i, R_j, and R_iR_j, with the support of their certainty values. This implicit information is explicitly held in extended ATMS nodes. The certainty degrees of primitive ATMS nodes, including observations, rules, and FICs have been discussed in the previous sections, which can be directly used here to extend the corresponding ATMS nodes. The certainty degree of an interpolated result can be derived from its entire set of label environments, based on (22), while the extent to which each individual environment entails the concerned interpolated result can be computed on the basis of (21). The process of calculating and updating of the certainty degrees of interpolated results is effectively managed by an extended ATMS label-updating mechanism. As a result, an extended ATMS node not only expresses how it is entailed by its label environments, but also indicates to what extent the node is derivable from the label environments.

E. Candidate Generation With Extended General Diagnostic Engine

A β_0-contradiction occurs if two object values are observed and/or derived for a common variable that differ to the extent of at least β_0, and therefore, one or both of the two values are faulty. Due to lack of differentiating information, both contradictory values are supposed to be equally faulty in [24]. With the support of additional information of certainty degrees as recorded in the extended ATMS, two values for a common variable can be distinguished in response to the extent to which each of them is derivable. In addition, for any one of the two ATMS nodes representing the two observations/interpolated results, the elements in its label environments are also distinguishable as some of the elements are of higher certainty degrees than others. Within the label environment of either of the two contradictory values, those elements with the smallest certainty degree are intuitively regarded as the most likely to be the real culprit. Based on these observations, the candidates generated by GED can be prioritized. In order to do so, all the elements in the label environments of the "false" node are ranked first.

Suppose that $E_\perp$ is one of the label environments of the "false" node which is deduced by two contradictory proposition P and P'. Then, there must exist environments $E = \{e_1, e_2, ..., e_m\}$ and $E' = \{e'_1, e'_2, ..., e'_n\}$, which entail the corresponding propositions such that $E \cup E' = E_\perp$. Suppose that the certainty degrees associated with the propositions P and P' are c and c', respectively. The procedure of prioritizing the label elements of $E_\perp$, by assigning a ranking value to each element, is shown in Algorithm 2. Assuming that $c \leq c'$, this algorithm guarantees that $e_i \leq e'_j, i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$, and vice versa.

Algorithm 2: The ELEMENTRANKING procedure.

ELEMENTRANKING(E,E',c,c')
1) $E_\perp = E \cup E'$
2) **foreach** $e \in E_\perp$
3)   **if** $(c \leq c'\ \&\&\ e \in E') || (c' \leq c\ \&\&\ e \in E)$
4) $r_e = c_e + 1$
5)   **else**
6) $r_e = c_e$

Recall that each label environment of the "false" node entails a contradiction. Thus, by taking one element from each environment of the "false" node, a candidate is constructed. Repeating this will generate all possible candidates. If all the duplications are deliberately kept, all the originally generated candidates will have the same cardinality, equaling to the number of label environments in the "false" node. From this, all candidates can be prioritized according to the ranking values of their members. Algorithm 3 shows a two-step sorting method for this. After the ranking, duplications of candidate elements are removed, and all those candidates which are a superset of one other candidate are also removed to guarantee that the candidate set is minimal. Obviously, such removals do not alter the ranking order of the remaining candidates.

Algorithm 3: The CANDIDATESORTING procedure.

CANDIDATESORTING($\mathbb{S}$)
Input: $\mathbb{S}$, a set of candidates with the same cardinality.
1) **foreach** $C \in \mathbb{S}$
2)   SORT (C) // Sort all the members of C in ascending order by their ranking values
3) **foreach** $i = |C| : 1$
4)   STABLESORT($\mathbb{S}, i$) // Sort all the candidates in ascending order by the ranking values of their i^{th} members

Note that a number of extensions to the classic ATMS and GDE have been proposed in the literature. A possibilistic ATMS was proposed in [34], where all the assumptions and justifications are associated with possibility values and handled in the framework of possibility theory [35]. A credibilistic ATMS was proposed in [36], which is developed on the basis of credibility theory [37]. The approach of [38] and [39] generalized the classical ATMS to work with reasoning systems using multivalued logic. The present work differs from these extensions as reliability values are used to reflect certainty degrees. Note too that classical GDE has also been extended from other perspectives, such as for reducing search spaces [40] and for modeling in situations where connections may also be faulty [40]. All these extensions to ATMS and GDE are interesting in further generalizing the present study, but are beyond the scope of this paper.

F. Illustrative Example—Part 1

The running example in the original work on AFRI [24] is reconsidered herein, but all the rules and observations are now associated with the information of certainty degrees. For completeness, the rule base is provided below:

R_1: IF $x_1 = A_1$, THEN $x_2 = B_1$ (0.80)
R_2: IF $x_1 = A_2$, THEN $x_2 = B_2$ (0.90)
R_3: IF $x_2 = B_3$, THEN $x_3 = C_3$ (0.60)
R_4: IF $x_2 = B_4$, THEN $x_3 = C_4$ (0.70)
R_5: IF $x_3 = C_5$, THEN $x_6 = F_5$ (0.70)
R_6: IF $x_3 = C_6$, THEN $x_6 = F_6$ (0.80)
R_7: IF $x_3 = C_7$ and $x_4 = D_7$, THEN $x_5 = E_7$ (0.90)
R_8: IF $x_3 = C_8$ and $x_4 = D_8$, THEN $x_5 = E_8$ (0.60)
R_9: IF $x_6 = F_9$, THEN $x_7 = G_9$ (0.90)
R_{10}: IF $x_6 = F_{10}$, THEN $x_7 = G_{10}$ (0.80)
R_{11}: IF $x_5 = E_{11}$, THEN $x_7 = G_{11}$ (0.70)
R_{12}: IF $x_5 = E_{12}$, THEN $x_7 = G_{12}$ (0.90).

The parameter set and representation schemes used in [24] are also utilized in this study, and thus, the details are omitted. With the support of extra information, suppose that the four observations are now: $O_1 : x_1 = A^* = (9.0, 9.5, 10.0, 10.5)$ (0.70), $O_2 : x_2 = B^* = (7.0, 7.5, 8.0, 8.5)$ (0.60), $O_3 : x_4 = D^* = (5.5, 6.0, 6.5, 7.0)$ (0.90), and $O_4 : x_6 = F^* = (11.0, 11.5, 12.0, 12.5)$ (0.80). By applying the classical scale and move transformation-based FRI, multiple pairs of

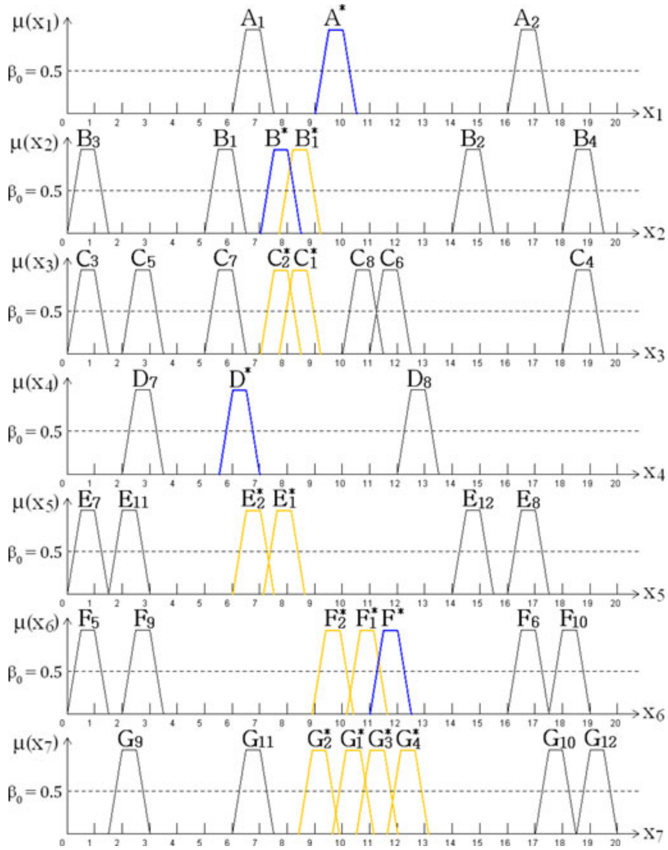

Fig. 3. Fuzzy sets and contradictions involved in the example.

contradictions result (e.g., F^* and F_2^*), which are summarized in Fig. 3.

The interpolation procedures are outlined as a component-based diagram, as illustrated in Fig. 4. In this figure, all the ATMS nodes and contradictions are shown as circles. Take node P_5 as an example. This node is inferred from the nodes P_3 and O_3 by the FIC F_4, which uses the rules R_7 and R_8, whose justification is, therefore, $P_3, O_3, R_7, R_8, F_4 \Rightarrow P_5$, where O_3 is an observation, and P_3 is a previously interpolated result. By running the label-updating algorithm of the extended ATMS, the label of the node $P_5(\{\{O_2, O_3, R_3, R_4, F_2, R_7, R_8, F_4\}\})$ can be derived from the labels of: the observation $O_3(\{\{O_3\}\})$, the interpolated result $P_3(\{\{O_2, R_3, R_4, F_2\}\})$, the rules $R_7(\{\{F_7\}\})$ and $R_8(\{\{R_8\}\})$, and the FIC $F_4(\{\{F_4\}\})$.

The certainty degrees of all FICs can be obtained by applying the approach introduced in Section III-B. For instance, the certainty degree of the FIC F_1 is calculated as follows:

$$\begin{aligned}
c_{F_1} &= 1 - \left|\frac{d(A_1, A_2)}{\max_{x_1} - \min_{x_1}} - \frac{d(B_1, B_2)}{\max_{x_2} - \min_{x_2}}\right| \\
&= 1 - \left|\frac{\mathrm{Rep}(A_2) - \mathrm{Rep}(A_1)}{\max_{x_1} - \min_{x_1}} - \frac{\mathrm{Rep}(B_2) - \mathrm{Rep}(B_1)}{\max_{x_2} - \min_{x_2}}\right| \\
&= 1 - \left|\frac{16.75 - 6.75}{20 - 0} - \frac{14.75 - 5.75}{20 - 0}\right| \\
&= 0.05
\end{aligned}$$

where Rep(A) denotes the representative value of the fuzzy set A [12]. The certainty degrees of derived nodes can be computed by following (22). As an example, the certainty degree of the derived node P_{10} is computed as follows:

$$\begin{aligned}
c_{P_{10}} &= (c_{O_2} \otimes c_{O_3} \otimes c_{R_3} \otimes c_{R_4} \otimes c_{F_2} \otimes c_{R_7} \otimes c_{R_8} \otimes c_{F_4} \\
&\quad \otimes c_{R_{11}} \otimes c_{R_{12}} \otimes c_{F_6}) \oplus (c_{O_4} \otimes c_{R_9} \otimes c_{R_{10}} \otimes c_{F_5}) \\
&= \max(0.60 * 0.90 * 0.60 * 0.70 * 1.00 * 0.90 * 0.60 * \\
&\quad 0.75 * 0.70 * 0.90 * 1.00, 0.80 * 0.90 * 0.80 * 1.00) \\
&= 0.58.
\end{aligned}$$

The certainty degrees of all other derived nodes can be calculated in the same manner. All the ATMS nodes (i.e., observations, rules, and FICs) and contradictions are summarized below:

$R_1 : \langle x_1 = A_1 \Rightarrow x_2 = B_1, 0.80, \{\{R_1\}\}\rangle$
$R_2 : \langle x_1 = A_2 \Rightarrow x_2 = B_2, 0.90, \{\{R_2\}\}\rangle$
$R_3 : \langle x_2 = B_3 \Rightarrow x_3 = C_3, 0.60, \{\{R_3\}\}\rangle$
$R_4 : \langle x_2 = B_4 \Rightarrow x_3 = C_4, 0.70, \{\{R_4\}\}\rangle$
$R_5 : \langle x_3 = C_5 \Rightarrow x_6 = F_5, 0.70, \{\{R_5\}\}\rangle$
$R_6 : \langle x_3 = C_6 \Rightarrow x_6 = F_6, 0.80, \{\{R_6\}\}\rangle$
$R_7 : \langle x_3 = C_7, x_4 = D_7 \Rightarrow x_5 = E_7, 0.90, \{\{R_7\}\}\rangle$
$R_8 : \langle x_3 = C_8, x_4 = D_8 \Rightarrow x_5 = E_8, 0.60, \{\{R_8\}\}\rangle$
$R_9 : \langle x_6 = F_9 \Rightarrow x_7 = G_9, 0.90, \{\{R_9\}\}\rangle$
$R_{10} : \langle x_6 = F_{10} \Rightarrow x_7 = G10, 0.80, \{\{R_{10}\}\}\rangle$
$R_{11} : \langle x_5 = E_{11} \Rightarrow x_7 = G11, 0.70, \{\{R_{11}\}\}\rangle$
$R_{12} : \langle x_5 = E_{12} \Rightarrow x_7 = G12, 0.90, \{\{R_{12}\}\}\rangle$
$F_1 : \langle R_1 R_2, 0.95, \{\{F_1\}\}\rangle$
$F_2 : \langle R_3 R_4, 1.00, \{\{F_2\}\}\rangle$
$F_3 : \langle R_5 R_6, 0.65, \{\{F_3\}\}\rangle$
$F_4 : \langle R_7 R_8, 0.75, \{\{F_4\}\}\rangle$
$F_5 : \langle R_9 R_{10}, 1.00, \{\{F_5\}\}\rangle$
$F_6 : \langle R_{11} R_{12}, 1.00, \{\{F_6\}\}\rangle$
$O_1 : \langle x_1 = A^*, 0.70, \{\{O_1\}\}\rangle$
$O_2 : \langle x_1 = B^*, 0.60, \{\{O_2\}\}\rangle$
$O_3 : \langle x_4 = D^*, 0.90, \{\{O_3\}\}\rangle$
$O_4 : \langle x_6 = F^*, 0.80, \{\{O_4\}\}\rangle$
$P_1 : \langle x_2 = B_1^*, 0.48, \{\{O_1, R_1, R_2, F_1\}\}\rangle$
$P_2 : \langle x_3 = C_1^*, 0.20, \{\{O_1, R_1, R_2, F_1, R_3, R_4, F_2\}\}\rangle$
$P_3 : \langle x_3 = C_2^*, 0.25, \{\{O_2, R_3, R_4, F_2\}\}\rangle$
$P_4 : \langle x_5 = E_1^*, 0.07, \{\{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4\}\}\rangle$
$P_5 : \langle x_5 = E_2^*, 0.09, \{\{O_2, O_3, R_3, R_4, F_2, R_7, R_8, F_4\}\}\rangle$
$P_6 : \langle x_6 = F_2^*, 0.09, \{\{O_2, R_3, R_4, F_2, R_5, R_6, F_3\}\}\rangle$
$P_7 : \langle x_6 = F_1^*, 0.07, \{\{O_1, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3\}\}\rangle$
$P_8 : \langle x_7 = G_2^*, 0.06, \{\{O_2, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\}\rangle$
$P_9 : \langle x_7 = G_1^*, 0.05, \{\{O_1, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\}\rangle$
$P_{10} : \langle x_7 = G_3^*, 0.58, \{\{O_2, O_3, R_3, R_4, F_2, R_7, R_8, F_4, R_{11}, R_{12}, F_6\}, \{O_4, R_9, R_{10}, F_5\}\}\rangle$
$P_{11} : \langle x_7 = G_4^*, 0.05, \{\{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_{11}, R_{12}, F_6\}\}\rangle$
$\perp_1 : \langle \perp, \{\{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4\}\}\rangle$
$\perp_2 : \langle \perp, \{\{O_2, O_4, R_3, R_4, F_2, R_5, R_6, F_3\}\}\rangle$

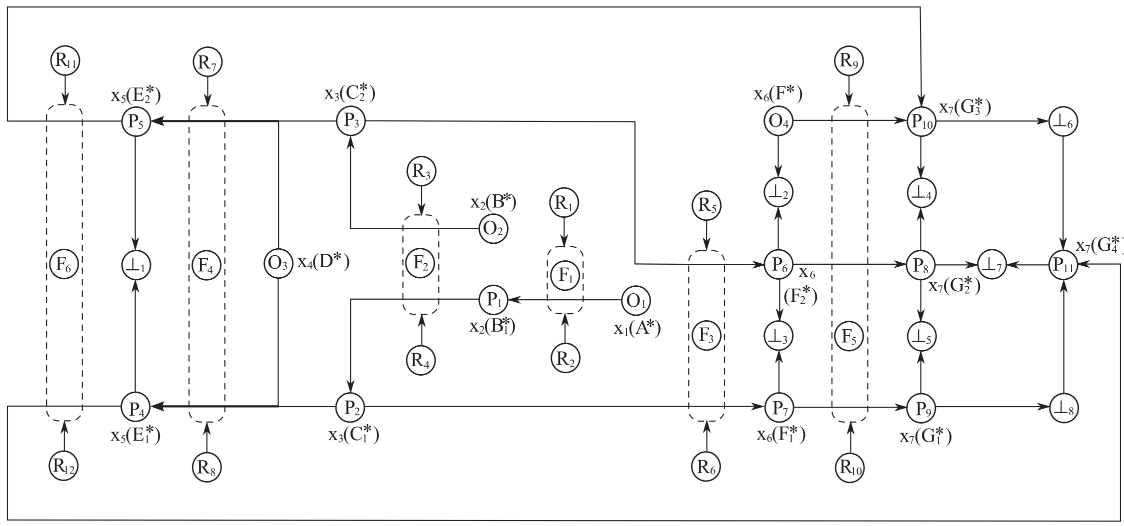

Fig. 4. Discrepancy records in ATMS.

$\perp_3 : \langle \perp, \{\{O_1, O_2, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3\}\}\rangle$
$\perp_4 : \langle \perp, \{\{O_2, O_3, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_2, O_4, R_3, R_4, F_2, R_5, R_6, F_3, R_9, F_5\}\}\rangle$
$\perp_5 : \langle \perp, \{\{O_1, O_2, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_9, R_{10}, F_5\}\}\rangle$
$\perp_6 : \langle \perp, \{\{O_1, O_3, O_4, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_{11}, R_{12}, F_6\}\}\rangle$
$\perp_7 : \langle \perp, \{\{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}\}\rangle$
$\perp_8 : \langle \perp, \{\{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}\}\rangle$.

The "false" node, denoted by $P_\perp$, collectively represents all the contradictions $\perp_1, \perp_2, ..., \perp_8$ by only containing a minimal set of label environments, which is given as follows:

$P_\perp : \langle \perp, \{\{O_1, O_2, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4\}, \{O_2, O_4, R_3, R_4, F_2, R_5, R_6, F_3\}, \{O_1, O_2, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3\}, \{O_2, O_3, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_1, O_3, O_4, R_1, R_2, F_1, R_3, R_4, F_2, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}, \{O_1, O_3, R_1, R_2, F_1, R_3, R_4, F_2, R_5, R_6, F_3, R_7, R_8, F_4, R_9, R_{10}, F_5, R_{11}, R_{12}, F_6\}\}\rangle$.

Applying the extended GDE as introduced in Section III-D, a ranked list of minimal candidates (including 85 candidates) is generated as follows:

$C_1 = [R3, 0.6]$, $C_2 = [O2, 0.6; R8, 0.6]$
$C_3 = [R8, 0.6; F3, 0.65]$, $C_4 = [O2, 0.6; F3, 0.65; O4, 0.8]$
$C_5 = [O2, 0.6; O1, 0.7]$, $C_6 = [R8, 0.6; R5, 0.7]$
$C_7 = [O2, 0.6; R11, 0.7]$, $C_8 = [O2, 0.6; R5, 0.7; O4, 0.8]$
$C_9 = [R8, 0.6; O1, 0.7; O4, 0.8]$, $C_{10} = [O2, 0.6; F4, 0.75]$
$C_{11} = [O2, 0.6; R1, 0.8]$, $C_{12} = [O2, 0.6; O4, 0.8; R6, 0.8]$
$C_{13} = [R8, 0.6; O4, 0.8; R1, 0.8]$
$C_{14} = [R8, 0.6; R6, 0.8]$, $C_{15} = [O2, 0.6; R10, 0.8]$
$C_{16} = [R8, 0.6; O4, 0.8; R2, 0.9]$
$C_{17} = [R8, 0.6; O4, 0.8; F1, 0.95]$
$C_{18} = [O2, 0.6; O3, 0.9]$, $C_{19} = [O2, 0.6; R2, 0.9]$
$C_{20} = [O2, 0.6; R7, 0.9]$, $C_{21} = [O2, 0.6; R9, 0.9]$
$C_{22} = [O2, 0.6; R12, 0.9]$, $C_{23} = [O2, 0.6; F1, 0.95]$
$C_{24} = [O2, 0.6; F5, 1.0]$, $C_{25} = [O2, 0.6; F6, 1.0]$
$C_{26} = [F3, 0.65; O1, 0.7]$, $C_{27} = [F3, 0.65; F4, 0.75]$
$C_{28} = [F3, 0.65; R1, 0.8]$, $C_{29} = [F3, 0.65; O3, 0.9]$
$C_{30} = [F3, 0.65; R2, 0.9]$, $C_{31} = [F3, 0.65; R7, 0.9]$
$C_{32} = [F3, 0.65; F1, 0.95]$
$C_{33} = [O1, 0.7; R5, 0.7; O1, 0.7]$, $C_{34} = [R4, 0.7]$
$C_{35} = [O1, 0.7; R11, 0.7; O4, 0.8]$
$C_{36} = [R5, 0.7; F4, 0.75]$
$C_{37} = [O1, 0.7; F4, 0.75; O4, 0.8]$
$C_{38} = [O1, 0.7; R6, 0.8]$
$C_{39} = [O1, 0.7; O4, 0.8; R10, 0.8]$
$C_{40} = [R5, 0.7; R1, 0.8]$, $C_{41} = [O1, 0.7; O4, 0.8; O3, 0.9]$
$C_{42} = [O1, 0.7; O4, 0.8; R7, 0.9]$
$C_{43} = [O1, 0.7; O4, 0.8; R9, 0.9]$
$C_{44} = [O1, 0.7; O4, 0.8; R12, 0.9]$
$C_{45} = [O1, 0.7; O4, 0.8; F5, 1.0]$
$C_{46} = [O1, 0.7; O4, 0.8; F6, 1.0]$
$C_{47} = [R5, 0.7; O3, 0.9]$, $C_{48} = [R5, 0.7; R2, 0.9]$
$C_{49} = [R5, 0.7; R7, 0.9]$, $C_{50} = [R5, 0.7; F1, 0.95]$
$C_{51} = [R11, 0.7; R1, 0.8; O4, 0.8; R1, 0.8]$
$C_{52} = [R11, 0.7; R1, 0.8; R6, 0.8]$
$C_{53} = [R11, 0.7; O4, 0.8; R2, 0.9]$
$C_{54} = [R11, 0.7; O4, 0.8; F1, 0.95]$
$C_{55} = [F4, 0.75; O4, 0.8; R1, 0.8]$
$C_{56} = [F4, 0.75; R6, 0.8]$
$C_{57} = [F4, 0.75; O4, 0.8; R2, 0.9]$
$C_{58} = [F4, 0.75; O4, 0.8; F1, 0.95]$
$C_{59} = [R1, 0.8; R6, 0.8; R1, 0.8]$
$C_{60} = [R1, 0.8; O4, 0.8; R1, 0.8; R10, 0.8]$
$C_{61} = [R1, 0.8; O4, 0.8; R1, 0.8; R9, 0.9]$
$C_{62} = [R1, 0.8; O4, 0.8; R1, 0.8; R12, 0.9]$
$C_{63} = [R1, 0.8; O4, 0.8; R1, 0.8; F5, 1.0]$
$C_{64} = [R1, 0.8; O4, 0.8; R1, 0.8; F6, 1.0]$
$C_{65} = [O4, 0.8; R1, 0.8; O3, 0.9]$
$C_{66} = [O4, 0.8; R1, 0.8; R7, 0.9]$, $C_{67} = [R6, 0.8; O3, 0.9]$

$C_{68} = [R6, 0.8; R2, 0.9]$, $C_{69} = [R6, 0.8; R7, 0.9]$
$C_{70} = [O4, 0.8; R10, 0.8; R2, 0.9]$
$C_{71} = [R6, 0.8; F1, 0.95]$
$C_{72} = [O4, 0.8; R10, 0.8; F1, 0.95]$
$C_{73} = [O4, 0.8; R2, 0.9; O3, 0.9]$
$C_{74} = [O4, 0.8; R2, 0.9; R7, 0.9]$
$C_{75} = [O4, 0.8; R2, 0.9; R9, 0.9]$
$C_{76} = [O4, 0.8; R2, 0.9; R12, 0.9]$
$C_{77} = [O4, 0.8; O3, 0.9; F1, 0.95]$
$C_{78} = [O4, 0.8; R7, 0.9; F1, 0.95]$
$C_{79} = [O4, 0.8; R2, 0.9; F5, 1.0]$
$C_{80} = [O4, 0.8; R2, 0.9; F6, 1.0]$
$C_{81} = [O4, 0.8; R9, 0.9; F1, 0.95]$
$C_{82} = [O4, 0.8; R12, 0.9; F1, 0.95]$
$C_{83} = [O4, 0.8; F1, 0.95; F5, 1.0]$
$C_{84} = [O4, 0.8; F1, 0.95; F6, 1.0]$, $C_{85} = [F2, 1.0]$.

From this, the reasoning consistency can be restored by successfully modifying one of the above candidates, which is detailed in Section IV.

G. Discussion on Generated Candidates

In order to effectively modify a candidate, it is necessary to examine if multiple related diagnosable ATMS nodes regarding a single interpolation step can be included in one candidate. If this is the case, the modifications of the related components must be considered jointly; otherwise, the modification of the candidate can be decomposed into that of its individual members.

Given a step of interpolation $M_1, M_2, \ldots, M_n, R_i, R_j, R_iR_j \Rightarrow C$, for notational simplicity, let N_{M_1}, N_{M_2}, ..., N_{M_n}, N_{R_i}, N_{R_j}, $N_{R_iR_j}$, and N_C denote the following nodes: M_1, M_2, $\ldots, M_n$, R_i, R_j, R_iR_j and the consequence C, respectively. Recall that the environment of each primitive ATMS node, which may be an observation, a rule or an FIC, contains only one node which represents itself [25]–[27]. Based on the label updating algorithm, every combination of one label environment from each node $N_{M_1}, i \in \{1, 2, ..., n\}$, and those label environments of nodes $\{N_{R_i}, N_{R_j}, N_{R_iR_j}\}$ jointly form a label environment of the node N_C. Assume that N_C contributes to a certain contradiction. Then, if any of its label environments contains $N_{R_iR_j}$, it must also contain N_{R_i} and N_{R_j}, and vice versa. Since a candidate is generated by taking one element from every label environment of each contradiction and any candidate which is a superset of another is removed, it is impossible that $\{N_{R_i}, N_{R_iR_j}\}$ or $\{N_{R_j}, N_{R_iR_j}\}$ is contained within a minimal candidate. Similarly, suppose that the node N is any element in the label environments of the nodes $N_{M_1}, N_{M_2}, \ldots$, and N_{M_n}, then $\{N, N_{R_i}\}$, $\{N, N_{R_j}\}$, or $\{N, N_{R_iR_j}\}$ cannot jointly appear in any single minimal candidate.

Note that N_{R_i} may also be used in conjunction with another rule rather than N_{R_j} to perform interpolation, and vice versa. Thus, it is possible that one label environment of the "false" node only contains N_{R_i} but not N_{R_j} while another only contains N_{R_j} but not N_{R_i}. Therefore, a minimal candidate may contain both N_{R_i} and N_{R_j}. In this situation, the modification of related candidate elements N_{R_i} and N_{R_j} needs to be considered jointly.

IV. Generalizing Candidate Modification

Having generated and prioritized all the candidates, one (and only one) of them needs to be modified in order to restore system consistency. This process naturally starts from the highest prioritized candidate. The principle underlying the consistency-restoring algorithm as given in Algorithm 1 is extended here by treating all observations, rules, and FICs as modifiable candidate elements. Recall that a candidate in general consists of a number of elements. Given a candidate, the modification of each of its elements will lead to a set of constraints in the format of equalities and inequalities. A satisfied solution of all joint equalities and inequalities imposed by the modifications of all the elements within a candidate will guarantee the modified result to be β_0-contradiction-free. The modification of FICs has been briefed in Section II-D and thus omitted here. The modification processes regarding observations, individual rules, and pairs of rules corresponding to a single interpolation step, are discussed below.

A. Observation Modification

It has an intuitive appeal to amend an observation based on the uncertainty value without changing the vagueness level associated with the relevant piece of information, which is reflected by the shape and area of the underlying fuzzy set. Such amendment may help maintain the interpretability of the fuzzy sets while offering an opportunity of removing inconsistencies in interpolation during the process of inference. Thus, the modification of a defective observation associated with a certainty degree of c is to shift the fuzzy set within its value range while keeping its shape and area unchanged. The shifting is required to satisfy the following.

1) The range of the shifting is bounded by (15) and (16), regarding the given c.
2) The shifted result should not cause disruption regarding the definitions of the other object values of the same variable, maintaining consistency in the specification of that variable's value domain. This is a similar constraint as that imposed in Step 5 for the modification of an FIC, as described in Section II-D.
3) The propagation of the shifted result should maintain mutual consistency with that of any other object value of the same variable. This is a similar constraint as that imposed in Step 6 for the modification of an FIC, again as described in Section II-D.

All three constraints listed above can be satisfied by constructing and then solving a set of simultaneous equalities and inequalities. The modification of observations can then be readily propagated by applying the modified results as interpolation inputs within the process of FRI. Note that as indicated above, constraints 2 and 3 are enforced in a way similar to those required over the case of modifying an FIC, while the computation implementing such modification has been generally presented in detail in [24]. Therefore, such common subprocedures of modification are omitted here; they are also omitted from the description of the modifications of interpolation rules that is to be described next.

B. Single-Rule Modification

The problem considered here is for situations where only one of a given pair of neighboring rules is identified as defective. Following the scale and move transformation-based FRI (which AFRI is developed upon), the interpolated result in response to a given input (that may be an observation or a previously inferred value) is derived from the consequent of an artificially created intermediate rule through the process outlined in Section II-A. This process involves the use of a pair of neighboring rules regarding the given input. While the antecedent of the intermediate rule and the input share the same overall location, the interpolated value is achieved by transferring the consequence of the intermediate rule with the same proportion of the area and shape differences between them. Therefore, in order to maintain interpretability, the single defective rule should be modified while keeping the shape and area of its consequence unchanged. This study follows on this intuition.

Similar to the process of modifying an observation, the modification of a defective rule is to shift the consequence of the rule within its value range by satisfying the three constraints listed in the last subsection. However, all the interpolated results that have been generated by applying this defective rule also need to be modified accordingly, as the defective rule has been utilized for their interpolation.

Although AFRI is applicable to fuzzy inference problems with multiple-antecedent rules, for illustrative simplicity, rules with two antecedents are taken in this study as an example to show the underlying approach. The method can be extended to rules with more than two antecedent variables in a straightforward manner. Given an input (A_k^*, B_k^*), suppose that the (closest) neighboring rules $A_i, B_i \Rightarrow C_i$ and $A_j, B_j \Rightarrow C_j$ flank this input. Without losing generality, assume that the second rule is defective and is included in the candidate to be modified, and that (A_i, B_i) is less than (A_j, B_j) in accordance with the integration of their representative values (for a given integration method). Based on the location of the antecedent of this defective rule, in reference to the other rule that was jointly fired with it to derive the detected contradictory interpolated result, two mirrored cases need to be addressed.

First, consider the case where the location relation between the input (A_k^*, B_k^*) and its corresponding interpolated consequence C_k^* is mapped by the line P_1P_3 within the assumed 3-D space, as shown in Fig. 5. This line is determined by the locations of the two neighboring rules used for interpolation. Suppose that the defective rule consequence is modified from C_j to $\widehat{C}_j$; then, the original mapping line P_1P_3 is accordingly shifted to the line P_1P_5. To quantitatively measure the extent of such shifting, the following correction rate c^- is introduced:

$$c^- = \frac{d(C_i, \widehat{C}_j)}{d(C_i, C_j)} \tag{24}$$

where $d(C, C')$ stands for the distance between the fuzzy sets C and C', computed as the distance between the representative values of these two fuzzy sets. Suppose that the modified result of C_k^* is denoted as $\widehat{C}_k^*$. Then, by applying the correction rate c^- to the distance between C_i and C_k^*, the distance from C_i to $\widehat{C}_k^*$

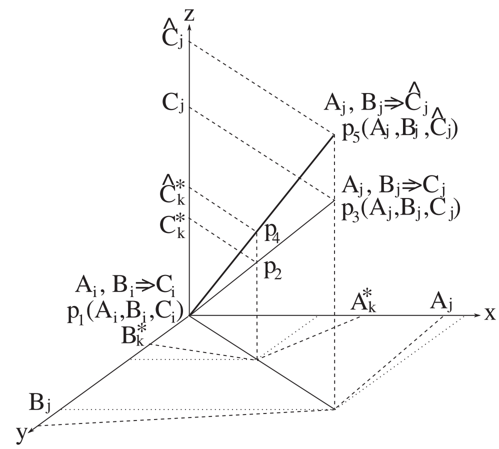

Fig. 5. Propagation of rule modification.

can be determined. Having known the locations of C_i and C_k^*, the location of $\widehat{C}_k^*$ can be computed, resulting in the modified interpolated value.

The case discussed above covers the case where an input which has invoked the defective rule for interpolation is less than the integrated antecedent of the rule. For the case where an input is greater than the antecedent, a mirrored procedure is followed to perform the modification, with a different correction rate c^+. Assume that the input (A_k^*, B_k^*) is flanked by the defective rule $A_i, B_i \Rightarrow C_i$ and the other neighboring rule, $A_j, B_j \Rightarrow C_j$; then, c^+ is defined as

$$c^+ = \frac{d(\widehat{C}_i, C_j)}{d(C_i, C_j)}. \tag{25}$$

The modified result of (A_k^*, B_k^*) can then be calculated using this correction rate, in a way similar to that utilized in the first case.

C. Modification of Both Neighboring Rules

Having addressed the situations where only one of the two neighboring rules appears in a candidate for modification, this subsection discusses the modification of both neighboring rules which are defective (i.e., both are included in a given candidate).

Suppose that the two defective neighboring rules are $A_i, B_i \Rightarrow C_i$ and $A_j, B_j \Rightarrow C_j$, and denote the (to be) modified consequences of them as $\widehat{C}_i$ and $\widehat{C}_j$, respectively. For easy reference, call the defective rule whose integrated antecedent is less than the input the left rule and the other the right. If the left rule is modified first as illustrated in Fig. 6(a), then the right defective rule will be modified using the result of modifying the left rule, as shown in Fig. 6(b). Then, the final modification can be represented by shifting the original defective location mapping line P_1P_3 to the line P_6P_5 as also illustrated in Fig. 6(b). If, however, the modification begins with the right defective rule, the modification will be performed as illustrated in Fig. 7, which also results in the final result that is the same as the one represented by the line P_6P_5 in Fig. 6(b). From this, due to the generality in the expression of the two rules, it can be concluded

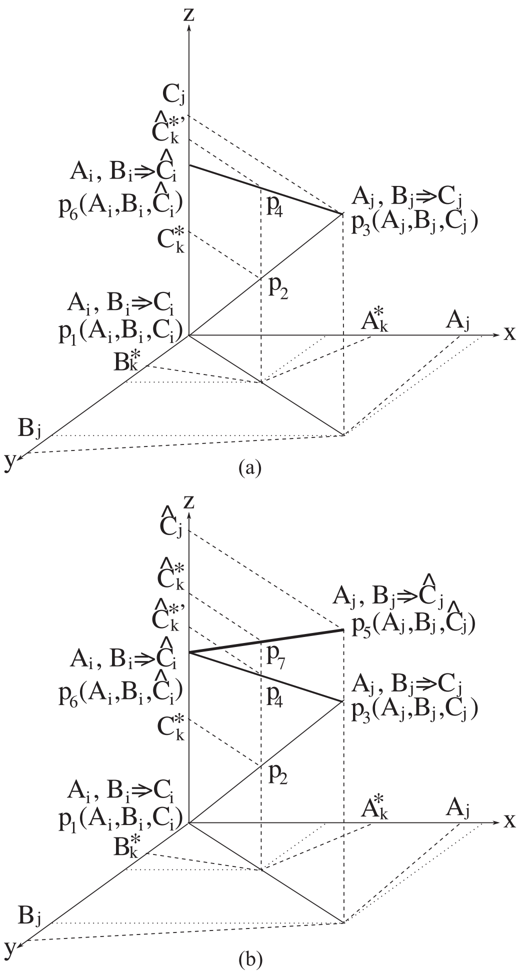

Fig. 6. Rule modification starting from left defective rule. (a) Left rule modification first. (b) Right rule modification second.

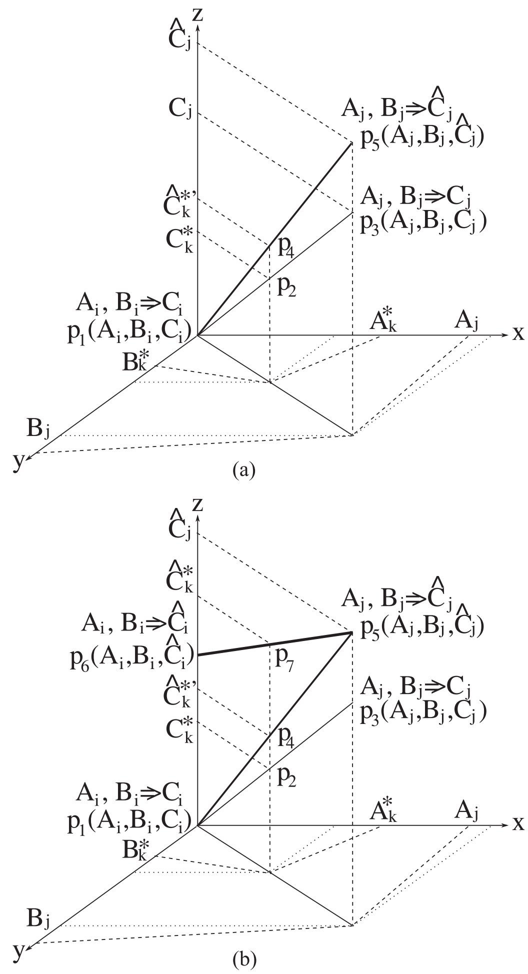

Fig. 7. Rule modification starting from right defective rule. (a) Right rule modification first. (b) Left rule modification second.

that the revised result is independent of the order of modifications. Therefore, the modification of both neighboring rules in a single candidate can be done by revising the two individual defective rules separately in either order.

D. Illustrative Example—Part 2

Continue the example given in Section III-F, the candidate C_1, which is of the highest priority, will be modified first. As only one modifiable element R_3 (If $x_2 = B_3$, THEN $x_3 = C_3$) is contained in this candidate, the modification procedure given in Section IV-B is applied. With respect to (15) and (16), the modification of the defective rule, R_3 needs to satisfy

$$\begin{cases} \widehat{C}_3 \geq C_3 - \frac{1-0.6}{2}(20-0) \\ \widehat{C}_3 \leq C_3 + \frac{1-0.6}{2}(20-0) \\ \min(\mathrm{supp}(\widehat{C}_3)) \geq 0 \\ \max(\mathrm{supp}(\widehat{C}_3)) \leq 20. \end{cases}$$

Running interpolation with the two neighboring rules consisting of the rule R_4 and the defective one R_3 leads to the following two interpolated rules:

$$IR_1: \text{ IF } x_2 \text{ is } B^*, \text{ THEN } x_3 \text{ is } C_2^*$$
$$IR_2: \text{ IF } x_2 \text{ is } B_1^*, \text{ THEN } x_3 \text{ is } C_1^*.$$

Since both antecedents of IR_1 and IR_2 are greater than the antecedent of the defective rule, C^+ is applied

$$c^+ = \frac{d(\widehat{C}_3, C_4)}{d(C_3, C_4)}.$$

From this, the overall location of the modified results will then satisfy

$$\begin{cases} d(\widehat{C}_1^*, C_4) = d(C_1^*, C_4) \cdot c^+ \\ d(\widehat{C}_2^*, C_4) = d(C_2^*, C_4) \cdot c^+. \end{cases}$$

These results are then utilized to further constrain the modified interpolated values such that

$$\begin{cases} \widehat{C}_1^* = C_1^* + (d(\widehat{C}_1^*, C_4) - d(C_1^*, C_4)) \\ \widehat{C}_2^* = C_2^* + (d(\widehat{C}_2^*, C_4) - d(C_2^*, C_4)). \end{cases}$$

The remaining process of the modification is to ensure that the modified results and their propagations are consistent with the rest. This subprocess is again the same as that of the modification of an FIC, as previously reported [24]. However, by solving all the simultaneous equalities and inequalities as listed above, including those imposed by the consistency-ensuring subprocess, there is no solution found. Therefore, the candidate with the second highest priority, that is C_2 in this example, is modified next.

The candidate C_2 includes two elements, the observation O_2 and the rule R_8, both of which need to be modified simultaneously in order to remove inconsistency. The modifications of O_2 and R_8 are carried out based on the procedures given in Sections IV-A and IV-B, respectively. In particular, according to constraint number 1 of the observation modification process, the modified value of O_2 must satisfy

$$\begin{cases} \widehat{B}^* \geq B^* - \frac{1-0.6}{2}(20-0) \\ \widehat{B}^* \leq B^* + \frac{1-0.6}{2}(20-0) \\ \min(\text{supp}(\widehat{B}^*)) \geq 0 \\ \max(\text{supp}(\widehat{B}^*)) \leq 20. \end{cases}$$

Similar constraints are also applied to the modified result of the consequence of R_8. As the modification procedure of R_8 is the same as that of R_3, as described above, the computational details are omitted here. By solving the equalities and inequalities, including those posed for consistency-ensuring, one solution is obtained as illustrated in Fig. 8. With the consistency restored, this concludes this illustrative example.

E. Computational Complexity

As the generalization of AFRI, it may be expected that the generalized AFRI will involve more computation than its original. In particular, as compared with the computational complexity of AFRI, that of the generalized version can be considered from the following two viewpoints: 1) impact of adding rules and observations as diagnosable candidate elements during candidate generation; and 2) impact of the constraints led by these extra candidate elements during candidate modification.

The computational complexity of candidate generation mainly depends on the complexity of the ATMS. It is well known that the standard ATMS has a computational complexity of exponential order in the worst case [41], but the average-case complexity can be greatly improved during practice use [42], [43]. The introduction of observations and rules as diagnosable candidate elements certainly increases the processing time because of a more sophisticated problem being addressed. However, this does not affect the general time complexity of the underlying ATMS. The complexity of the candidate modification stage is mainly determined by the constraint satisfaction mechanism which, for the problem of FRI in general, can be resolved in polynomial time complexity [24]. Although the introduction of additional constraints may increase the absolute computing time, the general time complexity will not be affected as the constraints introduced by the extra modifiable candidate elements are of the same type with those used in AFRI. Putting both aspects together, at the system level, the overall computational complexity of the generalized version does not deteriorate from that of the original AFRI approach.

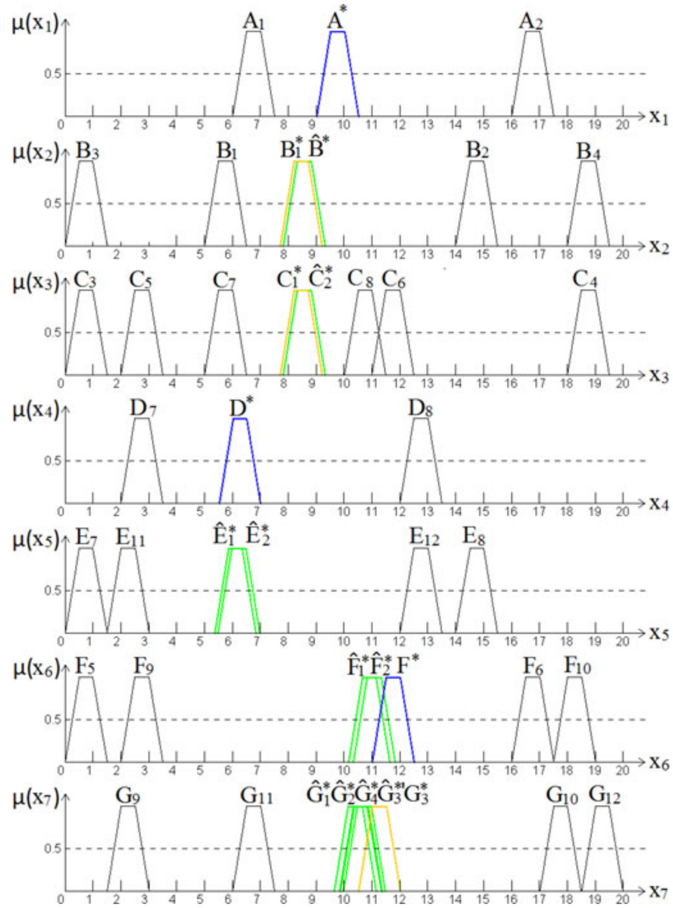

Fig. 8. One solution of the running example.

V. Application and Discussion

Disease burden may result from environmental changes [44]–[46]. An example study of this concerns how a previously roadless area in northern coastal Ecuador may be affected by the construction of a new road or railway in term of epidemiology of infectious diseases [47]. The causal relationship between the key factors driven by road construction has been established in the work of [47], which has been further quantitatively investigated using AFRI in [24]. As the theoretical development reported in this paper carries a substantial extension of [24], the application problem is reconsidered in this paper to facilitate direct comparison. For completeness, the sparse rule base is given below, and the fuzzy values included in the rules are listed in Table I.

R_1: IF $x_1 = A_1$ and $x_2 = B_1$, THEN $x_3 = C_1$ (0.9)
R_2: IF $x_1 = A_2$ and $x_2 = B_2$, THEN $x_3 = C_2$ (0.9)
R_3: IF $x_3 = C_3$ and $x_4 = D_3$, THEN $x_5 = E_3$ (0.7)
R_4: IF $x_3 = C_4$ and $x_4 = D_4$, THEN $x_5 = E_4$ (0.8)
R_5: IF $x_5 = E_5$, THEN $x_6 = F_5$ (0.8)
R_6: IF $x_5 = E_6$, THEN $x_6 = F_6$ (0.6)
R_7: IF $x_6 = F_7$, THEN $x_7 = G_7$ (0.7)
R_8: IF $x_6 = F_8$, THEN $x_7 = G_8$ (0.7)

TABLE I
FUZZY VARIABLES AND THEIR NORMALIZED OBJECT VALUES

Var	Meaning	Object value
x_1	Railway station proximity	$A_1 = \{0.02, 0.04, 0.06, 0.08\}$; $A_2 = \{0.28, 0.30, 0.32, 0.34\}$
x_2	Road proximity	$B_1 = \{0.18, 0.20, 0.22, 0.24\}$; $B_2 = \{0.39, 0.41, 0.43, 0.45\}$
x_3	Connectivity to transportation systems	$C_1 = \{0.46, 0.48, 0.50, 0.52\}$; $C_2 = \{0.62, 0.64, 0.66, 0.68\}$ $C_3 = \{0.52, 0.54, 0.56, 0.58\}$; $C_4 = \{0.85, 0.87, 0.89, 0.91\}$
x_4	Distance to the closest town	$D_3 = \{0.52, 0.54, 0.56, 0.58\}$; $D_4 = \{0.82, 0.84, 0.86, 0.88\}$
x_5	Remoteness	$E_3 = \{0.41, 0.43, 0.45, 0.47\}$; $E_4 = \{0.72, 0.74, 0.76, 0.78\}$ $E_5 = \{0.27, 0.29, 0.31, 0.33\}$; $E_6 = \{0.58, 0.60, 0.62, 0.64\}$ $E_9 = \{0.39, 0.41, 0.43, 0.45\}$; $E_{10} = \{0.62, 0.64, 0.66, 0.68\}$
x_6	Contact outside of the community	$F_5 = \{0.62, 0.64, 0.66, 0.68\}$; $F_6 = \{0.30, 0.32, 0.34, 0.36\}$ $F_7 = \{0.38, 0.40, 0.42, 0.44\}$; $F_8 = \{0.70, 0.72, 0.74, 0.76\}$
x_7	Reintroduction of pathogenic strains	$G_7 = \{0.46, 0.48, 0.50, 0.52\}$; $G_8 = \{0.65, 0.67, 0.69, 0.71\}$ $G_{15} = \{0.30, 0.32, 0.34, 0.36\}$; $G_{16} = \{0.60, 0.62, 0.64, 0.66\}$
x_8	Demographic changes	$H_9 = \{0.60, 0.62, 0.64, 0.66\}$; $H_{10} = \{0.30, 0.32, 0.34, 0.36\}$ $H_{11} = \{0.46, 0.48, 0.50, 0.52\}$; $H_{12} = \{0.68, 0.70, 0.72, 0.74\}$
x_9	Social connectedness	$I_{11} = \{0.52, 0.54, 0.56, 0.58\}$; $I_{12} = \{0.20, 0.22, 0.24, 0.26\}$ $I_{13} = \{0.28, 0.30, 0.32, 0.34\}$; $I_{14} = \{0.55, 0.57, 0.59, 0.61\}$
x_{10}	Hygiene and sanitation infrastructure	$J_{13} = \{0.26, 0.28, 0.30, 0.32\}$; $J_{14} = \{0.61, 0.63, 0.65, 0.67\}$ $J_{15} = \{0.36, 0.38, 0.40, 0.42\}$; $J_{16} = \{0.58, 0.60, 0.62, 0.64\}$
x_{11}	Infectious disease rates	$K_{15} = \{0.18, 0.20, 0.22, 0.24\}$; $K_{16} = \{0.68, 0.70, 0.72, 0.74\}$

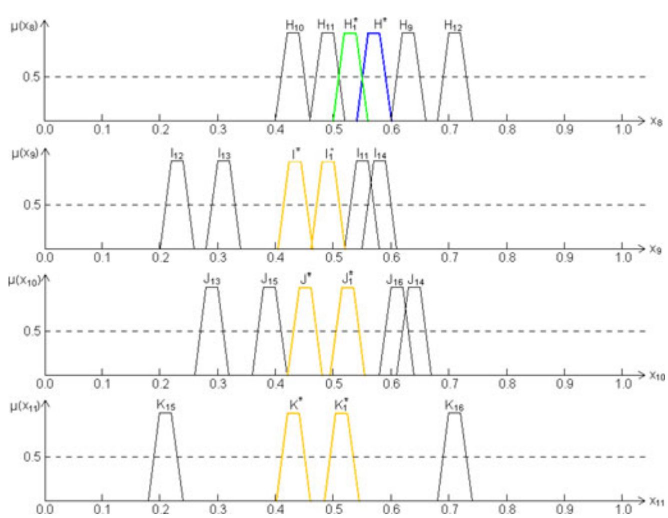

Fig. 9. Interpolated result by the HS method.

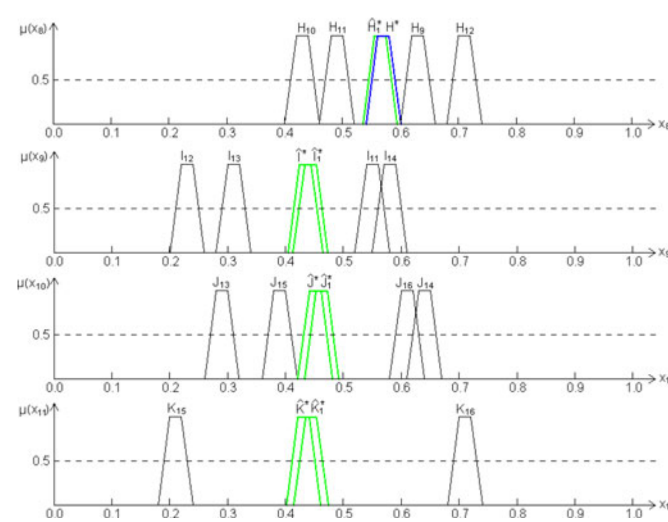

Fig. 10. Interpolated result by the adaptive approach (based on the HS method).

R_9: IF $x_5 = E_9$, THEN $x_8 = H_9$ (0.8)
R_{10}: IF $x_5 = E_{10}$, THEN $x_8 = H_{10}$ (0.6)
R_{11}: IF $x_8 = H_{11}$, THEN $x_9 = I_{11}$ (0.7)
R_{12}: IF $x_8 = H_{12}$, THEN $x_9 = I_{12}$ (0.9)
R_{13}: IF $x_9 = I_{13}$, THEN $x_{10} = J_{13}$ (0.7)
R_{14}: IF $x_9 = I_{14}$, THEN $x_{10} = J_{14}$ (0.8)
R_{15}: IF $x_7 = G_{15}$ and $x_{10} = J_{15}$, THEN $x_{11} = K_{15}$ (0.6)
R_{16}: IF $x_7 = G_{16}$ and $x_{10} = J_{16}$, THEN $x_{11} = K_{16}$ (0.8).

Suppose that four pieces of uncertain information are observed: $O_1 : x_1 = A^* = (0.16, 0.18, 0.20, 0.22)(0.7)$, $O_2 : x_2 = B^* = (0.34, 0.36, 0.38, 0.40)(0.9)$, $O_4 : x_4 = D^* = (0.65, 0.67, 0.69, 0.71)(0.6)$, and $O_8 : x_8 = H^* = (0.54, 0.56, 0.58, 0.60)(0.7)$. These observations do not invoke any rule in the rule base (with only B^* overlapping with the second antecedent attribute B_2 of the rule R_2). Thus, traditional fuzzy system techniques that are based on the use of compositional rule of inference cannot be employed to address the problem. However, FRI may help.

Assume that the set-theory-based similarity measure is utilized to compute the degree of contradiction, and let $\beta_0 = 0.5$.

β_0-contradictions will result from most of the existing interpolation methods [24]. In particular, the interpolated result using the scale and move transformation-based FRI, which the proposed work is built upon, leads to multiple (intermediate) β_0-inconsistencies, as shown in Fig. 9.

To obtain a consistent solution, the proposed adaptive fuzzy interpolation approach is applied. From the modifiable components (i.e., observations, rules, and FICs) upon which the detected contradictions depend, GDE generates 16 minimal candidates: $C_1 = [R_{10}, 0.6]$, $C_2 = [O_1, 0.7]$, $C_3 = [R_3, 0.7]$, $C_4 = [R_{11}, 0.7]$, $C_5 = [O_3, 0.8]$, $C_6 = [R_4, 0.8]$, $C_7 = [R_9, 0.8]$, $C_8 = [O_2, 0.9]$, $C_9 = [R_1, 0.9]$, $C_{10} = [R_2, 0.9]$, $C_{11} = [R_{12}, 0.9]$, $C_{12} = [F_6, 0.92]$, $C_{13} = [F_5, 0.93]$, $C_{14} = [F_1, 0.94]$, $C_{15} = [F_2, 0.99]$, and $C_{16} = [O_4, 1.6]$. One solution resulted from the modification of the first prioritized candidate C_1 is shown in Fig. 10.

From this figure, it can be seen that there is no more β_0-contradiction, and thus, consistency has been successfully restored. That is, the original inconsistent interpolated result has been successfully removed, demonstrating the effectiveness of this study. Interestingly, different from the problem-solving process of the previous work reported in [24], this solution has resulted from the modification of the very first candidate C_1. This is due to the employment of the proposed candidate prioritization method. By this method, the priority of each candidate is calculated from their reliability rather than from the informal intuition as used previously.

VI. Conclusion

This paper has presented a generalized framework for AFRI. The generalization allows the identification and modification of observations and rules, in addition to that of interpolation procedures that were addressed in the previous work. This is supported by introducing extra information of certainty degrees associated such basic elements of FRI. The work also allows for all candidates for modification to be prioritized, based on the extent to which a candidate is likely to lead to all detected contradictions, by extending the classic ATMS and GDE. The working of the extended approach is illustrated with a running example throughout Sections III and IV, and also demonstrated by a realistic application in Section V.

This research can be further improved in several directions. At the present, it works with interpolation involving just two multiple-antecedent rules. It is worthwhile to investigate how this study may be generalized to perform interpolation and extrapolation with multiple multiantecedent rules. Note that the FRI approach proposed in [48] also deals with inconsistency problems, but in a different way by considering the relevant degrees of rules relevant to a given observation. In particular, the relevant degree of a certain rule is determined by the reciprocal distance from the observation to the rule. An interesting piece of further work is, therefore, to compare these two approaches. In addition, the proposed adaptive approach is developed on the HS method only. It is desirable to apply the adaptive approach to other FRI methods, such as those implemented in MATLAB FRI toolbox [49], and to compare the generated results. Finally, it is of great interest to study how the classical ATMS and GDE can be utilized to support traditional fuzzy inference systems and to develop an integrated inconsistency detection and fault-correction platform that supports both standard fuzzy inference and FRI.

Acknowledgment

The authors are very grateful to the reviewers and the Associate Editor for their very positive and encouraging comments.

References

[1] L. T. Kóczy and K. Hirota, “Approximate reasoning by linear rule interpolation and general approximation,” *Int. J. Approx. Reason.*, vol. 9, no. 3, pp. 197–225, 1993.

[2] L. T. Kóczy and K. Hirota, “Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases,” *Inf. Sci.*, vol. 71, no. 1–2, pp. 169–201, 1993.

[3] P. Baranyi, L. T. Kóczy, and T. D. Gedeon, “A generalized concept for fuzzy rule interpolation,” *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 6, pp. 820–837, Dec. 2004.

[4] Y.-C. Chang, S.-M. Chen, and C.-J. Liau, “Fuzzy interpolative reasoning for sparse fuzzy-rule-based systems based on the areas of fuzzy sets,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 5, pp. 1285–1301, Oct. 2008.

[5] S. Chen, Y. Ko, Y. Chang, and J. Pan, “Weighted fuzzy interpolative reasoning based on weighted increment transformation and weighted ratio transformation techniques,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 6, pp. 1412–1427, Dec. 2009.

[6] S.-M. Chen and Y.-C. Chang, “Weighted fuzzy rule interpolation based on GA-based weight-learning techniques,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 4, pp. 729–744, Aug. 2011.

[7] S.-M. Chen and W.-C. Hsin, “Weighted fuzzy interpolative reasoning based on the slopes of fuzzy sets and particle swarm optimization techniques,” *IEEE Trans. Cybern.*, vol. 45, no. 7, pp. 1250–1261, Jul. 2015.

[8] S.-M. Chen and Y.-K. Ko, “Fuzzy interpolative reasoning for sparse fuzzy rule-based systems based on α-cuts and transformations techniques,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 6, pp. 1626–1648, Dec. 2008.

[9] D. Dubois and H. Prade, “On fuzzy interpolation,” *Int. J. Gen. Syst.*, vol. 28, no. 2, pp. 103–114, 1999.

[10] S. Garcia-Jimenez, H. Bustince, E. Hullermeier, R. Mesiar, N. Pal, and A. Pradera, “Overlap indices: Construction of and application to interpolative fuzzy systems,” *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 4, pp. 1259–1273, Aug. 2015.

[11] W.-H. Hsiao, S.-M. Chen, and C.-H. Lee, “A new interpolative reasoning method in sparse rule-based systems,” *Fuzzy Sets Syst.*, vol. 93, no. 1, pp. 17–22, 1998.

[12] Z. Huang and Q. Shen, “Fuzzy interpolative reasoning via scale and move transformations,” *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 2, pp. 340–359, Apr. 2006.

[13] Z. Huang and Q. Shen, “Fuzzy interpolation and extrapolation: A practical approach,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 1, pp. 13–28, Feb. 2008.

[14] S. Jin, R. Diao, C. Quek, and Q. Shen, “Backward fuzzy rule interpolation,” *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 6, pp. 1682–1698, Dec. 2014.

[15] L. T. Kóczy and K. Hirota, “Size reduction by interpolation in fuzzy rule bases,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 27, no. 1, pp. 14–25, Feb. 1997.

[16] S. Kovás, “Extending the fuzzy rule interpolation “FIVE” by fuzzy observation,” in *Computational Intelligence, Theory and Applications*, B. Reusch, Ed. Berlin, Germany: Springer, 2006, pp. 485–497.

[17] Q. Shen and L. Yang, “Generalisation of scale and move transformation-based fuzzy interpolation,” *J. Adv. Comput. Intell. Intell. Informat.*, vol. 15, no. 3, pp. 288–298, 2011.

[18] D. Tikk and P. Baranyi, “Comprehensive analysis of a new fuzzy rule interpolation method,” *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 3, pp. 281–296, Jun. 2000.

[19] K. W. Wong, D. Tikk, T. Gedeon, and L. Kóczy, “Fuzzy rule interpolation for multidimensional input spaces with applications: A case study,” *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 6, pp. 809–819, Dec. 2005.

[20] Y. Yam and L. Kóczy, "Representing membership functions as points in high-dimensional spaces for fuzzy interpolation and extrapolation," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 761–772, Dec. 2000.

[21] Y. Yam, M. L. Wong, and P. Baranyi, "Interpolation with function space representation of membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 398–411, Jun. 2006.

[22] L. Yang and Q. Shen, "Closed form fuzzy interpolation," *Fuzzy Sets Syst.*, vol. 225, pp. 1–22, 2013.

[23] S.-H. Cheng, S.-M. Chen, and C.-L. Chen, "Adaptive fuzzy interpolation based on ranking values of polygonal fuzzy sets and similarity measures between polygonal fuzzy sets," *Inf. Sci.*, vol. 342, pp. 176–190, 2016.

[24] L. Yang and Q. Shen, "Adaptive fuzzy interpolation," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 6, pp. 1107–1126, Dec. 2011.

[25] J. de Kleer, "An assumption-based TMS," *Artif. Intell.*, vol. 28, no. 2, pp. 127–162, 1986.

[26] J. de Kleer, "Extending the ATMS," *Artif. Intell.*, vol. 28, no. 2, pp. 163–196, 1986.

[27] J. de Kleer, "Problem solving with the ATMS," *Artif. Intell.*, vol. 28, no. 2, pp. 197–224, 1986.

[28] J. de Kleer and B. C. Williams, "Diagnosing multiple faults," *Artif. Intell.*, vol. 32, no. 1, pp. 97–130, 1987.

[29] L. Yang and Q. Shen, "Adaptive fuzzy interpolation with uncertain observations and rule base," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, 2011, pp. 471–478.

[30] L. Yang and Q. Shen, "Adaptive fuzzy interpolation with prioritized component candidates," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, 2011, pp. 428–435.

[31] X. Fu and Q. Shen, "Fuzzy compositional modeling," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 4, pp. 823–840, Aug. 2010.

[32] E. H. Shortliffe and B. G. Buchanan, "A model of inexact reasoning in medicine," *Math. Biosci.*, vol. 23, no. 3, pp. 351–379, 1975.

[33] Q. Shen and R. Leitch, "Fuzzy qualitative simulation," *IEEE Trans. Syst., Man Cybern.*, vol. 23, no. 4, pp. 1038–1061, Jul./Aug. 1993.

[34] D. Dubois, J. Lang, and H. Prade, "A possibilistic assumption-based truth maintenance system with uncertain justifications, and its application to belief revision," in *Proc. Workshop Truth Maintenance Syst.*, 1991, pp. 87–106.

[35] D. Dubois, H. M. Prade, H. Farreny, R. Martin-Clouaire, and C. Testemale, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, vol. 2. New York, NY, USA: Plenum, 1988.

[36] Q. Shen and R. Zhao, "A credibilistic approach to assumption-based truth maintenance," *IEEE Trans. Syst., Man Cybern. A, Syst. Humans*, vol. 41, no. 1, pp. 85–96, Jan. 2011.

[37] P. Li and B. Liu, "Entropy of credibility distributions for fuzzy variables," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 1, pp. 123–129, Feb. 2008.

[38] J. L. Castro and J. M. Zurita, "A generic ATMS," *Int. J. Approx. Reason.*, vol. 14, no. 4, pp. 259–280, 1996.

[39] J. L. Castro and J. M. Zurita, "A multivalued logic ATMS," *Int. J. Intell. Syst.*, vol. 11, no. 4, pp. 185–195, 1996.

[40] J. De Kleer, "Modeling when connections are the problem," U.S. Patent 7 962 321, 2011.

[41] G. M. Provan, "The computational complexity of truth maintenance systems," Univ. British Columbia, Vancouver, BC, Canada, *Tech. Rep.*, Rep. No. TR-88-11, 1988.

[42] A. Urbański, "Reducing computational complexity of truth maintenance systems," in *Proc. 6th Int. Conf. Artif. Intell. Inf. Control Syst. Robots*, River Edge, NJ, USA, 1994, pp. 397–404.

[43] G. Kelleher and L. van der Gaag, "The lazyRMS: Avoiding work in the ATMS," *Comput. Intell.*, vol. 9, no. 3, pp. 239–253, 1993.

[44] R. R. Colwell "Climate change and human health," *Science*, vol. 279, no. 5353, pp. 968–969, 1998.

[45] S. S. Morse, "Factors in the emergence of infectious diseases," *Emerging Infectious Diseases*, vol. 1, no. 1, pp. 7–15, 1995.

[46] WorldBank, *Health Aspects of Environmental Impact Assessment. Environmental Assessment Sourcebook Update 18*. Washington, DC, USA: World Bank, 1997.

[47] J. N. S. Eisenberg, "Environmental change and infectious disease: How new roads affect the transmission of diarrheal pathogens in rural Ecuador," *Proc. Nat. Acad. Sci. USA*, vol. 103, no. 51, pp. 19460-1–19460-5, 2006.

[48] D. Tikk, I. Joó, L. Kóczy, P. Várlaki, B. Moser, and T. D. Gedeon, "Stability of interpolative fuzzy KH controllers," *Fuzzy Sets Syst.*, vol. 125, no. 1, pp. 105–119, 2002.

[49] Z. C. Johanyak, D. Tikk, S. Kovacs, and K. W. Wong, "Fuzzy rule interpolation MATLAB toolbox—FRI toolbox," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, 2006, pp. 351–357.

Longzhi Yang (M'12) received the B.Sc. degree from the Nanjing University of Science and Technology, Nanjing, China, the M.Sc. degree from Coventry University, Coventry, U.K., and the Ph.D. degree from Aberystwyth University, Aberystwyth, U.K., all in computer science in 2003, 2006, and 2011, respectively.

He is currently a Senior Lecturer with Northumbria University, Newcastle, U.K. His research interests include fuzzy inference systems, machine learning, optimisation algorithms, intelligent control systems, and the application of such techniques in real-world uncertain environments.

Dr. Yang received the Best Student Paper Award at the 2010 IEEE International Conference on Fuzzy Systems.

Fei Chao (M'11) received the B.Sc. degree in mechanical engineering from Fuzhou University, Fuzhou, China, and the M.Sc. degree with distinction in computer science from the University of Wales, Cardiff, U.K., in 2004 and 2005, respectively, and the Ph.D. degree in robotics from Aberystwyth University, Aberystwyth, U.K., in 2009.

He was a Research Associate under the supervision of Prof. M. H. Lee with Aberystwyth University from 2009 to 2010. He is currently an Associate Professor with the Cognitive Science Department, Xiamen University, Xiamen, China. He has published 30 peer-reviewed journal and conference papers. His research interests include developmental robotics, machine learning, and optimization algorithms.

Dr. Chao is the Vice-Chair of the IEEE Computer Intelligence Society Xiamen Chapter. He is also a member of CCF.

Qiang Shen received the Ph.D. degree in computing and electrical engineering and the D.Sc. degree in computational intelligence.

He holds the established chair in computer science and is the Director of the Institute of Mathematics, Physics and Computer Science, Aberystwyth University, Aberystwyth, U.K. His research interests include computational intelligence, reasoning under uncertainty, pattern recognition, data mining, and real-world applications of such techniques for intelligent decision support (e.g., crime detection, consumer profiling, systems monitoring, and medical diagnosis). He has authored two research monographs and more than 350 peer-reviewed papers in these areas.

Dr. Shen has received an Outstanding Transactions Paper Award from the IEEE.