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Characteristic operator functions for quantum input-plant-output models and coherent control

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We introduce the characteristic operator as the generalization of the usual concept of a transfer function of linear input-plant-output systems to arbitrary quantum nonlinear Markovian input-output models. This is intended as a tool in the characterization of quantum feedback control systems that fits in with the general theory of networks. The definition exploits the linearity of noise differentials in both the plant Heisenberg equations of motion and the differential form of the input-output relations. Mathematically, the characteristic operator is a matrix of dimension equal to the number of outputs times the number of inputs (which must coincide), but with entries that are operators of the plant system. In this sense, the characteristic operator retains details of the effective plant dynamical structure and is an essentially quantum object. We illustrate the relevance to model reduction and simplification definition by showing that the convergence of the characteristic operator in adiabatic elimination limit models requires the same conditions and assumptions appearing in the work on limit quantum stochastic differential theorems of Bouten and Silberfarb [Commun. Math. Phys. **283**, 491-505 (2008)]. This approach also shows in a natural way that the limit coefficients of the quantum stochastic differential equations in adiabatic elimination problems arise algebraically as Schur complements and amounts to a model reduction where the fast degrees of freedom are decoupled from the slow ones and eliminated. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4906136]

I. INTRODUCTION

There has been much interest lately in the behavior and control of quantum linear systems, particularly as these are amenable to transfer matrix function techniques. In this note, we wish to exploit the structural features of quantum Markovian models to construct an analogue of the transfer matrix function for non-linear systems. Coming from the classical direction, there has been fruitful application of operator techniques to control systems in recent years[1–4] employing for instance characteristic functions techniques, multi-analytic operators, and commutant lifting methods. Here, we introduce a natural characteristic operator function associated with a quantum Markov (or SLH) model.

As in standard quantum mechanics, the model is formulated by representing physical quantities (observables) as self-adjoint operators on a Hilbert space. The quantum mechanical system (plant) will have underlying Hilbert space $\mathfrak{h}$ while the input will be a continuous quantum field with Hilbert space $\mathfrak{F}$. The coupled model will have joint Hilbert space $\mathfrak{h} \otimes \mathfrak{F}$, which is also the space on which the output observables act.

The input-plant-output model can be summarized as

$$\textbf{plant dynamics} : j_t(X) = U(t)^* (X \otimes I) U(t),$$
$$\textbf{output process} : B_{\text{out},i}(t) = U(t)^* (I \otimes B_i(t)) U(t)$$

where X is an arbitrary plant observable, $B_i(t)$ is a component of the input field, and $U(t)$ is the unitary entangling the plant with the portion of the bath that has interacted with it over the time period $[0,t]$.

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A. The "SLH" formalism

In the following, we shall specify to a category of model where $U(\cdot)$ is a unitary family of operators on $\mathfrak{h} \otimes \mathfrak{F}$, satisfying a differential equation of the form[5–8]

$$dU(t) = \left\{ \sum_{ij} (S_{ij} - \delta_{ij}) \otimes d\Lambda_{ij}(t) + \sum_i L_i \otimes dB_i^*(t) - \sum_{ij} L_i^* S_{ij} \otimes dB_j(t) + K \otimes dt \right\} U(t),$$

$$U(0) = I. \tag{1}$$

Formally, we can introduce input process $b_{\text{in},i}(t)$ for $i = 1,\ldots,n$ satisfying singular commutation relations of the form $[b_i(t), b_j(t')^*] = \delta_{ij}\delta(t - t')$, so that the processes appearing in (1) are

$$\Lambda_{ij}(t) \triangleq \int_0^t b_i(t')^* b_j(t') dt', B_i(t)^* \triangleq \int_0^t b_i(t')^* dt', \quad B_j(t) \triangleq \int_0^t b_j(t') dt'.$$

More exactly, they are rigorously defined as creation and annihilation field operators on the Boson Fock space $\mathfrak{F}$ over $L^2_{\mathbb{C}^n}(\mathbb{R})$. The increments in (1) are understood to be future pointing in the Ito sense. We have the following table of non-vanishing products:

$$d\Lambda_{ij} d\Lambda_{kl} = \delta_{jk} d\Lambda_{il}, \qquad d\Lambda_{ij} dB_k^* = \delta_{jk} dB_i^*,$$
$$dB_i d\Lambda_{kl} = \delta_{ik} dB_l, \qquad dB_i dB_k^* = \delta_{ij} dt.$$

Necessary and sufficient conditions for unitarity[5,6] are that we can collect the coefficients of (1) to form a triple (S, L, H), which we call the *Hudson-Parthasarathy (HP) parameters*, consisting of a unitary matrix S, a column vector L, and a self-adjoint operator H,

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, \tag{2}$$

with S_{ij}, L_i, H are all operators on $\mathfrak{h}$, and where

$$K \equiv -\frac{1}{2} \sum_i L_i^* L_i - iH. \tag{3}$$

It has become fashionable to refer to this, plus the related feedback network models,[13,20] as the "SLH" formalism.

We shall refer to $U(t)$ as the unitary determined by the coupling parameters (S, L, H). In differential form, the input-plant-output model then becomes[5,6]

plant dynamical (Heisenberg) equation

$$dj_t(X) = j_t(\mathscr{L}X) dt + \sum_i j_t(\mathscr{M}_i X) dB_i^*(t) + \sum_i j_t(\mathscr{N}_i X) dB_i(t) + \sum_{j,k} j_t(\mathscr{S}_{jk} X) d\Lambda_{jk}(t), \tag{4}$$

here,

$$\mathscr{L}X = \frac{1}{2} \sum_i L_i^*[X, L_i] + \frac{1}{2} \sum_i [L_i^*, X] L_i - i[X, H], \quad \text{(the Lindbladian!)}, \tag{5}$$

$$\mathscr{M}_i X = S_{ji}^*[X, L_j], \tag{6}$$

$$\mathscr{N}_i X = [L_k^*, X] S_{ki}, \tag{7}$$

$$\mathscr{S}_{ik} X = S_{ji}^* X S_{jk} - \delta_{ik} X, \tag{8}$$

input-output relations

$$dB_{\text{out},i}(t) = j_t(S_{ik}) dB_k(t) + j_t(L_i) dt. \tag{9}$$

B. Linear quantum models

If we specify to a system of quantum mechanical oscillators with modes $a_1, \ldots, a_m$ satisfying canonical commutation relations,

$$\left[a_\alpha, a_\beta^* \right] = \delta_{\alpha\beta}, \tag{10}$$

then we obtain a linear dynamical model with the prescription

$$S_{ij} = D_{ij}, \; L_i = \sum_{\alpha=1}^m C_{i\alpha} a_\alpha, \; H = \sum_{\alpha,\beta} a_\alpha^* \omega_{\alpha\beta} a_\beta. \tag{11}$$

Specifically, the plant dynamics and input-output relations are affine linear in the mode variables a_i

$$da_\alpha(t) = \sum_\beta A_{\alpha\beta} a_\beta(t) \, dt + \sum_i B_{\alpha i} dB_i(t),$$

$$dB_{\text{out},i}(t) = \sum_\beta C_{i\beta} a_\beta(t) \, dt + \sum_k D_{ik} dB_k(t).$$

where setting $D = \left[D_{ij} \right] \in \mathbb{C}^{n \times n}$, $C = [C_{i\alpha}] \in \mathbb{C}^{n \times m}$, and $\Omega = \left[\omega_{\alpha\beta} \right] \in \mathbb{C}^{m \times m}$, we have

$$A = -\frac{1}{2} C^* C - i\Omega, \quad B = -C^* S. \tag{12}$$

In turn, a model having this specific structure is said to be physically realizable. The transfer matrix associated with the linear dynamics is then defined to be[9–11]

$$T(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] (s) \triangleq D + C(sI - A)^{-1} B, \tag{13}$$

and there exists a well-established literature developing control theory from analysis of these functions.

The definition here leads to transfer functions that are positive real functions of the complex variable s, and they model passive systems. The generalization to active linear models, which we do not need here, is given in Ref. 12.

C. Characteristic operators

In the mathematical formulation of open quantum Markov systems, a natural role is played by the model matrix, introduced in Ref. 13,

$$\mathbf{V} = \begin{bmatrix} -\dfrac{1}{2} L^* L - iH & -L^* S \\ L & S \end{bmatrix} \tag{14}$$

We now use it as the basis for the definition of an operator-valued generalization of the characteristic function.

Definition 1 (The characteristic operator). For given (S, L, H), we define the corresponding characteristic operator by

$$\mathscr{T}(s) \triangleq \left[\begin{array}{c||c} -\dfrac{1}{2} L^* L - iH & -L^* S \\ \hline L & S \end{array} \right] (s) = S - L\left(sI + \frac{1}{2} L^* L + iH\right)^{-1} L^* S. \tag{15}$$

We shall often write $\mathscr{T}_{(S,L,H)}$ for emphasis.

Lemma 2. The characteristic operator $\mathscr{T}(s)$ is a bounded operator for Re, $s > 0$. For all $\omega \in \mathbb{R}$, such that $i\omega$ lies in the resolvent set of $K = -\frac{1}{2} L^* L - iH$ (that is, whenever $i\omega - K$ is invertible), we have $\mathscr{T}(i\omega)$ well-defined and unitary

$$\mathscr{T}(i\omega)^* \mathscr{T}(i\omega) = \mathscr{T}(i\omega)\mathscr{T}(i\omega)^* = I. \tag{16}$$

The proof follows mutatis mutandis of the proof of an analogous result in Ref. 11.

D. Examples

1. Lossless system

Suppose that we have no coupling $L = 0$, then the characteristic operator is $\mathcal{T}(s) \equiv S$, constant. This is true even if H is non-zero. Without coupling, we cannot infer anything about the system Hamiltonian.

2. Quantum linear passive system

For the model considered in Subsection I B, we have

$$\mathcal{T}(s) = S - Ca\frac{1}{s - a^*Aa}a^*C^*S, \tag{17}$$

where $a^* = [a_1^*, \ldots, a_m^*]$. For the $m = n = 1$ case, we have explicitly

$$\mathcal{T}(s) = S - C\frac{1}{s - A(N + 1)}C^*S, \tag{18}$$

where $N = a^*a$ is the number operator for the single mode. In fact, we see that

$$\langle \mathcal{T}(s) \rangle_{\text{vac}} = T(s), \tag{19}$$

where $T(s)$ is transfer function (13). The same vacuum expectation is obtained for the cases n, m greater than one.

3. Qubit example

A simple example is a qubit system with master equation

$$\frac{d}{dt}\varrho = \mathcal{D}_L\varrho + i\left[\varrho, H\right], \tag{20}$$

where $\mathcal{D}_L\varrho = L\varrho L^* - \frac{1}{2}\{\varrho L^*L + L^*L\varrho\}$ and we set $L = \sqrt{\gamma(n + 1)}\sigma_- + \sqrt{\gamma n}\sigma_+$, $H = \omega\sigma_z$. This models a qubit in a thermal bath with $0 \leq n \leq 1$ being the equilibrium occupancy of the state $|\uparrow\rangle$ in the presence of the oscillation $\omega\sigma_z$. We shall take the scattering to be by a polarization-dependent phase

$$S = e^{i\varphi_+}|\uparrow\rangle\langle\uparrow| + e^{i\varphi_-}|\downarrow\rangle\langle\downarrow|. \tag{21}$$

In the σ_z-basis $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $K = \begin{bmatrix} -\frac{1}{2}\gamma(n+1) - i\omega & 0 \\ 0 & -\frac{1}{2}\gamma n + i\omega \end{bmatrix}$, and the characteristic operator explicitly is

$$\mathcal{T}(s) = \begin{bmatrix} \dfrac{s - \frac{1}{2}\gamma n - i\omega}{s + \frac{1}{2}\gamma n + i\omega}e^{i\varphi_+} & 0 \\ 0 & \dfrac{s - \frac{1}{2}\gamma(n + 1) + i\omega}{s + \frac{1}{2}\gamma(n + 1) - i\omega}e^{i\varphi_-} \end{bmatrix}. \tag{22}$$

The characteristic operator is diagonal in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, but this would no longer be true if $[S, \sigma_z] \neq 0$.

4. Opto-mechanical example

We consider a model of a cavity mode a between a fixed leaky mirror and a perfect mirror with quantum mechanical position $X = b + b^*$, see Fig. 1. The SLH model takes the form

$$S = 1, \quad L = \sqrt{\gamma}a, \quad H = \Delta a^*a + \omega_0 b^*b + gXa^*a, \tag{23}$$
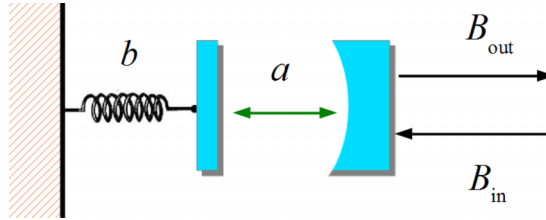
FIG. 1. A mechanical mode (moveable mirror) coupled to an open cavity.

where γ is the damping to the input field at the leaky mirror, Δ is the cavity detuning, ω_0 is the harmonic frequency of the mirror, and g is the coupling strength associated with mirror-mode inter-action. Note that the interaction $g_0 X a^* a$ couples the position of the mirror to the cavity mode photon number in accordance with the notion of radiation pressure. This is a standard opto-mechanical model and we obtain the Langevin equations

$$dj_t(a) = -\left(\frac{1}{2}\gamma + i\Delta + ig_0 X\right) j_t(a)\, dt - \sqrt{\gamma}dB(t),$$

$$dj_t(b) = -i\omega_0 j_t(b)\, dt - g_0 j_t(a^*a)\, dt.$$

A simplifying assumption is that the mechanical processes are much slower than the optical ones, in which case, we set $\omega_0 \equiv 0$. The characteristic operator in this case is

$$\mathscr{T}_{\text{optomech}}(s) = \mathscr{T}_{(I, \sqrt{\gamma}a, (\Delta + gX)a^*a)}(s).$$

This is recognizable as the characteristic operator of a quantum linear passive system as in (17), but with the operator A taking the form $A = -\left(\frac{1}{2}\gamma + i\Delta + igX\right)$. That is, A is no longer scalar valued, but depends explicitly on the position observable X of the mirror. Note that A is still strictly Hurwitz since X is self-adjoint. We remark that position dependent transfer functions have been proposed for single photon input-output models for this type of model with one-particle fields related by[14]

$$\xi_{\text{out}}(s) = \frac{s - \frac{1}{2}\gamma - i(\Delta + gX)}{s + \frac{1}{2}\gamma - i(\Delta + gX)}\xi_{\text{in}}(s),$$

and here, the transfer function corresponds to the partial trace of $\mathscr{T}_{\text{optomech}}(s)$ over the vacuum state of the cavity.

E. Properties of the characteristic operator

Lemma 3 (All-pass representation). The characteristic operator admits the following "all-pass" representation:

$$\mathscr{T}(s) = \frac{1 - \frac{1}{2}\Sigma(s)}{1 + \frac{1}{2}\Sigma(s)}S, \tag{24}$$

where $\Sigma(s) = L(s + iH)^{-1}L^*$.

This is proved in Ref. 15, and we recall briefly the proof.

Proof. An application of the Woodbury matrix identity[16] $(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$ shows that

$$\frac{1}{sI + iH + \frac{1}{2}L^*L} = \frac{1}{sI + iH} - \frac{1}{2}\frac{1}{sI + iH}\frac{1}{1 + \frac{1}{2}L(sI + iH)^{-1}L^*}\frac{1}{sI + iH}. \tag{25}$$

Substituting into (15) then gives the above relation after some straightforward algebra.   □

Note that $\Sigma(i\omega)^* = -\Sigma(i\omega)$ for real ω, so that we could alternatively have deduced unitarity by a Cayley transformation argument.

Corollary 4. Suppose that the model parameters satisfy the condition $[L, H] \equiv 0$, *then the characteristic operator takes the form*

$$\mathscr{T}(s) = \frac{s - \frac{1}{2}LL^* + iH}{s + \frac{1}{2}LL^* + iH}S. \tag{26}$$

The condition $[L, H] \equiv 0$ arises as the QND condition for measurement disturbance in the sense of Braginsky.[17]

Remark 5 (Equivalence to passive systems). For a finite-dimensional system, say with Hilbert space $\mathfrak{h} = \mathbb{C}^m$, *we may fix an orthonormal basis of* m *vectors for* $\mathfrak{h}$. *In this representation, we may describe* H *as an* $m \times m$ *matrix which we denote as* $\Omega \in \mathbb{C}^{m \times m}$. *The coupling operator* L *is then a column vector of* n *operators, each represented as an* $m \times m$ *matrix, so that* L *may be represented as an* $nm \times m$ *matrix which we denote as* $C \in \mathbb{C}^{nm \times m}$. *In this manner,* S *becomes a complex valued matrix* $D \in \mathbb{C}^{nm \times nm}$. *We then have the equivalence*

$$\mathscr{T}(s) = \left[\begin{array}{c|c} -\frac{1}{2}L^*L - iH & -L^*S \\ \hline L & S \end{array} \right](s) \equiv \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right](s), \tag{27}$$

where $A = -\frac{1}{2}C^*C - i\Omega$ *and* $B = -C^*D$. *In this, we realise the characteristic function as the transfer operator of a linear passive system* A, B, C, D, *structurally similar to those considered in Sec. I B, with a state space of* m *dimensions and* nm *inputs.*

F. Stratonovich form of the characteristic operator

We show now that the characteristic operator function can be described in terms of the coefficient operators in the Stratonovich QSDE.

The Stratonovich differential is defined using the midpoint rule convention which leads to the algebraic rule[18,19]

$$dX_t \circ Y_t \triangleq dX_t Y_t + \frac{1}{2}dX_t dY_t.$$

It can then be shown that the Stratonovich form of QSDE (1) takes the form

$$dU(t) = \left\{ \sum_{ij} E_{ij} \otimes d\Lambda_{ij}(t) + \sum_i E_{i0} \otimes dB_i^*(t) + \sum_j E_{0j} \otimes dB_j(t) + E_{00} \otimes dt \right\} \circ U(t),$$

$$U(0) = I,$$

with $E_{ij}^* = E_{ji}$, $E_{i0}^* = E_{0i}$ and $E_{00}^* = E_{00}$. It is convenient to collect all the coefficients into a (Hermitian) matrix

$$\mathbf{E} = \begin{bmatrix} E_{00} & E_{0\ell} \\ E_{\ell 0} & E_{\ell\ell} \end{bmatrix}. \tag{28}$$

The components of $\mathbf{E}$ are related to the (S, L, H) by the transformation:[18,19] $S = [S_{ij}]_{1 \leq i, j \leq n}$ is the Cayley transform $E_{\ell\ell} = [E_{ij}]_{1 \leq i, j \leq n}$,

$$S = \frac{1 - \frac{i}{2}E_{\ell\ell}}{1 + \frac{i}{2}E_{\ell\ell}}, \tag{29}$$

and therefore, S is unitary, while

$$L = i\frac{1}{1 + \frac{i}{2}E_{\ell\ell}}E_{\ell 0}, \quad H = E_{00} + \frac{1}{2}\text{Im}\left\{ E_{0\ell}\frac{1}{1 + \frac{i}{2}E_{\ell\ell}}E_{\ell 0} \right\} \tag{30}$$

with H self-adjoint. Note that the operator K is then given by

$$K \equiv -iE_{00} - \frac{1}{2}E_{0\ell}\frac{1}{1+\frac{i}{2}E_{\ell\ell}}E_{\ell 0}.$$

Lemma 6 (Stratonovich form of the Characteristic Operator). We may write the characteristic operator in terms of the coefficients making up Stratonovich matrix **E** *(28) as*

$$\mathscr{T}(s) \equiv \frac{I - \frac{i}{2}E_{\ell\ell} - \frac{1}{2}E_{\ell 0}\frac{1}{s+iE_{00}}E_{0\ell}}{I + \frac{i}{2}E_{\ell\ell} + \frac{1}{2}E_{\ell 0}\frac{1}{s+iE_{00}}E_{0\ell}}. \tag{31}$$

Proof. We have explicitly that

$$\mathscr{T}(s) = \frac{I - \frac{i}{2}E_{\ell\ell}}{I + \frac{i}{2}E_{\ell\ell}} - \frac{1}{I + \frac{i}{2}E_{\ell\ell}}E_{\ell 0}\frac{1}{s + iE_{00} + \frac{1}{2}E_{0\ell}\frac{1}{1+\frac{i}{2}E_{\ell\ell}}E_{0\ell}}E_{0\ell}\frac{1}{I + \frac{i}{2}E_{\ell\ell}}.$$

The Woodbury matrix identity[16] with $A = I + \frac{i}{2}E_{\ell\ell}, U = \frac{1}{\sqrt{2}}E_{\ell 0}, V = \frac{1}{\sqrt{2}}E_{0\ell}, C = (s + iE_{00})^{-1}$ shows that

$$\frac{1}{I + \frac{i}{2}E_{\ell\ell} + \frac{1}{2}E_{\ell 0}\frac{1}{s+iE_{00}}E_{0\ell}} = \frac{1}{2}(\mathscr{T}(s) + I).$$

Rearranging for then gives the desired result.   □

Note that we have the correct limit $\lim_{|s|\to\infty} = \frac{1-\frac{i}{2}E_{\ell\ell}}{1+\frac{i}{2}E_{\ell\ell}} = S$.

Suppose that we have $E_{00} = kF_{00}, E_{\ell 0} = kF_{\ell 0}$, and $E_{\ell\ell} = F_{\ell\ell}$ independent of k, then the associated transfer operator $\mathscr{T}_k(s)$ has the well-defined limit

$$\lim_{k\to\infty}\mathscr{T}_k(s) = \widehat{S},$$

provided that F_{00} is invertible. Here, $\widehat{S} = \frac{1-\frac{i}{2}\widehat{E}_{\ell\ell}}{1+\frac{i}{2}\widehat{E}_{\ell\ell}}$ with $\widehat{E}_{\ell\ell} = F_{\ell\ell} - F_{\ell 0}(F_{00})^{-1}F_{0\ell}$. This limit, which corresponds physically to high-energy and strong damping, leads to a purely scattering model but with a shifted scattering matrix $\widehat{S}$. We shall study more general examples of this type of scaling leading to SLH models with nontrivial couplings $\widehat{L}$ and Hamiltonians $\widehat{H}$.

II. MODEL SIMPLIFICATION AND REDUCTION

As we have seen, the characteristic operator for a system with underlying Hilbert space $\mathfrak{h}$ with n inputs is a function taking values in $\mathfrak{B}(\mathfrak{h}) \otimes \mathbb{C}^{n\times n}$, the set of $n \times n$ matrices with entries in $\mathfrak{B}(\mathfrak{h})$, the bounded operators on $\mathfrak{h}$.

Let A and B be models with the same input dimension n and having coefficient parameters (S_A, L_A, H_A) and (S_B, L_B, H_B), respectively. We may *cascade* the systems by feeding the output of A and input to B and in the instantaneous feedforward limit, we get the model $B \triangleleft A$ on $\mathfrak{h} = \mathfrak{h}_B \otimes \mathfrak{h}_A$ with parameters given by the series product, see Refs. 13 and 20 $(S_B \otimes S_A, L_B \otimes I_A + S_B \otimes L_A, H_B \otimes I_A + I_B \otimes H_A + \text{Im}\{L_B^* S_B \otimes L_A\})$. In this case, we typically have

$$\mathscr{T}_{B\triangleleft A}(z) \neq \mathscr{T}(z)_B \otimes \mathscr{T}_A(z). \tag{32}$$

(Here, we employ the shorthand $S_B \otimes S_A$ for the matrix with j,k-entries $\sum_{l=1}^{n}[S_B]_{jl} \otimes [S_A]_{lk}$, etc.)

Thus, characteristic function for cascaded systems is not naturally the product of their characteristic operators. For cascaded classical systems, the state spaces take the forms $\mathcal{X}_A$ and $\mathcal{X}_B$ so that the combined state space is the direct sum $\mathcal{X}_B \oplus \mathcal{X}_A$. The rule in quantum theory is that the combined Hilbert state space for the cascaded systems is the tensor product and not the direct sum. (Note that for quantum linear systems, the Hilbert space is the Fock space $\mathfrak{h} = \Gamma(\mathcal{X})$ over $\mathcal{X}$, and for combined linear systems, we have $\Gamma(\mathcal{X}_A) \otimes \Gamma(\mathcal{X}_B) \cong \Gamma(\mathcal{X}_A \oplus \mathcal{X}_B)$, which is the usual rule for Fock spaces.[6] In this way, the usually cascade rule re-emerges for the corresponding transfer functions.[11])

With this observation, we see that model reduction techniques based around the characteristic operator should involve direct sum decompositions, say

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \tag{33}$$

into orthogonal subspaces. Each of the coefficients $X = S_{jk}, L_j, H$, etc., can be represented as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \tag{34}$$

where X_{ab} maps from $\mathfrak{h}_{\mathsf{b}}$ to $\mathfrak{h}_{\mathsf{a}}$. The characteristic operator may similarly be decomposed as

$$\mathscr{T}(s) = \begin{bmatrix} \mathscr{T}_{11}(s) & \mathscr{T}_{12}(s) \\ \mathscr{T}_{21}(s) & \mathscr{T}_{22}(s) \end{bmatrix} \tag{35}$$

with

$$\mathscr{T}_{\mathsf{ab}}(s) \equiv \left\{ \delta_{\mathsf{ac}} - L_{\mathsf{ad}}\left[\frac{1}{s-K}\right]_{\mathsf{de}} L^*_{\mathsf{ce}} \right\} S_{\mathsf{cb}}. \tag{36}$$

(Here, we have the convention that repeated sans serif indices are summed over the ranges 1 and 2. We also adopt the notation that S_{jk} is the $\mathfrak{B}(\mathfrak{h})$-valued output j, input k entry of S, while S_{ab} is the component of S mapping from $\mathfrak{h}_{\mathsf{b}}$ to $\mathfrak{h}_{\mathsf{a}}$, which is an $n \times n$ matrix of maps from $\mathfrak{h}_{\mathsf{b}}$ to $\mathfrak{h}_{\mathsf{a}}$. Similarly, L_{ad} is the n-column vector of maps from $\mathfrak{h}_{\mathsf{b}}$ to $\mathfrak{h}_{\mathsf{a}}$.)

Using the Schur-Feshbach identity, we may write the resolvent $\frac{1}{s-K}$ as

$$\begin{bmatrix} s - K_{11} & -K_{12} \\ -K_{21} & s - K_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{\Delta}_{11}(s) & \widehat{\Delta}_{12}(s) \\ \widehat{\Delta}_{21}(s) & \widehat{\Delta}_{22}(s) \end{bmatrix}, \tag{37}$$

where, introducing

$$\widehat{K}_{11}(s) = K_{11} + K_{12}\frac{1}{s-K_{22}}K_{21} \tag{38}$$

and $\Delta_{22}(s) = \frac{1}{s-K_{22}}$, we have

$$\widehat{\Delta}_{11}(s) = \frac{1}{s - \widehat{K}_{11}(s)},$$
$$\widehat{\Delta}_{12}(s) = \widehat{\Delta}_{11}(s) K_{12}\Delta_{22}(s),$$
$$\widehat{\Delta}_{21}(s) = \Delta_{22}(s) K_{21}\widehat{\Delta}_{11}(s),$$
$$\widehat{\Delta}_{22}(s) = \Delta_{22}(s) + \Delta_{22}(s) K_{21}\widehat{\Delta}_{11}(s) K_{12}\Delta_{22}(s).$$

The blocks of the characteristic operator partitioned with respect to the direct sum $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ are then

$$\mathscr{T}_{\mathsf{ab}}(s) \equiv \left\{ \delta_{\mathsf{ac}} - L_{\mathsf{ad}}\widehat{\Delta}_{\mathsf{de}}(s)L^*_{\mathsf{ce}} \right\} S_{\mathsf{cb}}. \tag{39}$$

Definition 7. Given the direct sum $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, we say that orthogonal subspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are decoupled if the characteristic operator takes the block diagonal form

$$\mathscr{T}(s) = \begin{bmatrix} \mathscr{T}_{11}(s) & 0 \\ 0 & \mathscr{T}_{22}(s) \end{bmatrix}, \tag{40}$$

that is, $\mathscr{T}_{21}(s) = 0$ and $\mathscr{T}_{12}(s) = 0$.

We note that if V is a unitary on the system space, then the basic unitary rotation behaviour for characteristic operators is

$$\mathscr{T}_{(V^*SV, V^*LV, V^*HV)} \equiv V^*\mathscr{T}_{(S,L,H)}V. \tag{41}$$

However, we note following result, which is easily derived.

Proposition 8. For any unitary V on the plant Hilbert space, the HP parameters (S, LV, V^*HV) generate the same characteristic operator as (S, L, H). More generally, we have the following invariance property of the characteristic function:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{c|c} V^*AV & V^*B \\ \hline CV & D \end{array} \right]. \tag{42}$$

Therefore, while the characteristic operator is a quantum object—for n inputs, it is an $n \times n$ matrix with entries that are operators on the plant space—its dependence on the plant operators is only up to a unitary equivalence as outlined in the proposition.

Definition 9. Let (S, L, H) be given HP parameters for a fixed plant Hilbert space $\mathfrak{h}$. If (S', L', H') are HP parameters for a proper subspace $\mathfrak{h}'$ of the plant space, then we say that (S', L', H') is a reduced model of (S, L, H) if we have

$$\mathscr{T}_{(S,L,H)} = \begin{bmatrix} \mathscr{T}_{(S',L',H')} & 0 \\ 0 & I \end{bmatrix}, \tag{43}$$

with respect to the decomposition $\mathfrak{h} = \mathfrak{h}' \oplus (\mathfrak{h}')^\perp$. A reduced model is minimal if it allows no further model reduction.

A. Examples

1. Detuned two-level atom

As a simple toy model, let us consider a two-level atom with ground and excited states $|g\rangle$ and $|e\rangle$. We fix the open system as being a single input model with $S = I$, $L = \sqrt{\gamma}\sigma_z + \sqrt{\kappa}\sigma_-$, and Hamiltonian

$$H(k) = k^2 \Delta \sigma_+\sigma_- + k\beta\sigma_+ + k\beta^*\sigma_- + \omega_0,$$

where $\sigma_- = |g\rangle\langle e|$, etc. Here, $\Delta > 0$ is interpreted as a detuning parameter and β as the amplitude of a drive. Both the detuning and amplitude are assume to be large, which correspond to the limit $k \to \infty$.

The characteristic operator for the two-level system is then given by

$$\mathscr{T}_k(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \sqrt{\gamma} & 0 \\ \sqrt{\kappa} & -\sqrt{\gamma} \end{bmatrix} \begin{bmatrix} s + \frac{1}{2}(\gamma+\kappa) + ik^2\Delta + i\omega_0 & -\frac{1}{2}\sqrt{\kappa\gamma} + ik\beta \\ -\frac{1}{2}\sqrt{\kappa\gamma} + ik\beta^* & s + \frac{1}{2}\gamma + i\omega_0 \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\gamma} & \sqrt{\kappa} \\ 0 & -\sqrt{\gamma} \end{bmatrix},$$

which can be calculated explicitly as a 2×2 matrix whose entries are rational polynomials in s of degree 2. What is of interest here is that for large k, the characteristic operator takes the limit form

$$\lim_{k\to\infty} \mathscr{T}_k(s) = \begin{bmatrix} 1 & 0 \\ 0 & \mathscr{T}_g(s) \end{bmatrix},$$

where $\mathscr{T}_g(s) = \frac{s - \frac{1}{2}\gamma + i\omega_0'}{s + \frac{1}{2}\gamma + i\omega_0'}$, where we have the shifted frequency $\omega_0' = \omega_0 - \frac{|\beta|^2}{\Delta}$. The limit model corresponds to the transfer function of a linear system with a single degree of freedom having the damping γ and frequency ω_0'.

What is happening in this limit is that the excited state plays an increasingly negligible role in the model as its decay rate starts to increase: The limit is a reduced model, however, with a shift of the frequency.

2. Qubit

As a next example, we consider a qubit driven by three input fields, with

$$S = I_3, L = \begin{bmatrix} \sqrt{\kappa_1}\sigma \\ \sqrt{\kappa_2}\sigma \\ \sqrt{\kappa_3}\sigma \end{bmatrix}, H = \Delta\sigma^*\sigma - i\sqrt{\kappa_1}\left(\alpha\sigma^* - \alpha^*\sigma\right),$$

where σ, σ^* are the lowering and raising operators, Δ is a fixed detuning, and α the amplitude of a drive field. The characteristic operator now takes the form $\mathcal{T}(s) = \left[\mathcal{T}_{jk}(s)\right]$, where we have the components

$$\mathcal{T}_{jk}(s) = \delta_{jk}I_2 - \frac{\sqrt{\kappa_j\kappa_k}s}{s^2 + (\frac{1}{2}\kappa + i\Delta)s + \kappa_1|\alpha|^2}\sigma\sigma^*,$$

for $j, k \in \{1, 2, 3\}$ and where $\kappa = \kappa_1 + \kappa_2 + \kappa_3$. In the special case where $\alpha = 0$, there is a zero-pole cancellation.

III. ASYMPTOTIC MODEL REDUCTION VIA ADIABATIC ELIMINATION

We begin by considering the description of perturbations to open system models in terms of their characteristic operators. We discuss regular perturbations first for completeness: suppose, we have a model (S, L, H) which is a perturbation of solvable model (S, L, H_0) with

$$H = H_0 + \lambda V, \tag{44}$$

so that $K = -\frac{1}{2}L^*L - iH \equiv K_0 - i\lambda V$. The resolvents $R(z) = (z - K)^{-1}$ and $R_0(z) = (z - K_0)^{-1}$ are then related by $R(z) = R_0(z) - i\lambda R(z)V R_0(z)$. For bounded perturbation V, we have the Neumann series $R(z) = \sum_{n=0}^{\infty} R_0(z)(-i\lambda V R_0(z))^n$ so that the characteristic operators are related by

$$\mathcal{T}(z) = \mathcal{T}_0(z) - \sum_{n=1}^{\infty}(-i\lambda)^n L R_0(z)(V R_0(z))^n L^* S. \tag{45}$$

This formula will be valid for suitably small constants λ. In principle, this formula may be useful for perturbative approaches to system modelling.

Our main focus, however, will be singular perturbations corresponding to adiabatic elimination.

A. Fast and slow subspace decomposition

There exists a large body of results under the name of *adiabatic elimination* applicable to open quantum models. A universal mathematical approach has been developed by Bouten, Silberfarb, and van Handel.[21,22] We formulate their presentation in a slightly different language. Essentially, the common element in adiabatic elimination problems is that the system space can be decomposed into a fast space, which is viewed as increasingly strongly coupled to the bath, and a slow space. Specifically, we assume a decomposition of the system space as

$$\mathfrak{h} = \mathfrak{h}_{\text{slow}} \oplus \mathfrak{h}_{\text{fast}}. \tag{46}$$

A recent example of this is the approximate qubit regime for nonlinear optical cavities.[23] The coupling parameters are then taken as $(S, L(k), H(k))$, where k is a strength parameter which we eventually take to be large. For a given operator X on $\mathfrak{h}$, we write

$$X = \begin{bmatrix} X_{\text{ss}} & X_{\text{sf}} \\ X_{\text{fs}} & X_{\text{ff}} \end{bmatrix}. \tag{47}$$

More generally, we use this notation when X is an array of operators on $\mathfrak{h}$. The projections onto $\mathfrak{h}_{\text{slow}}$ and $\mathfrak{h}_{\text{fast}}$ are denoted by $P_{\text{s}} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $P_{\text{f}} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, respectively.

B. Assumptions: Characteristic operator limit

1. The coupling operator takes the form

$$L(k) = kL^{(1)} + L^{(0)},\tag{48}$$

where $L^{(1)}P_s = 0$, that is,

$$L^{(1)} \equiv \begin{bmatrix} 0 & L^{(1)}_{sf} \\ 0 & L^{(1)}_{ff} \end{bmatrix}.\tag{49}$$

2. The Hamiltonian takes the form $H(k) = H^{(0)} + kH^{(1)} + k^2 H^{(2)}$, where $H^{(1)}P_s = P_s H^{(1)} = 0$ and $P_s H^{(2)} P_s = 0$, that is,

$$H \equiv \begin{bmatrix} H^{(0)}_{ss} & H^{(0)}_{sf} + kH^{(1)}_{sf} \\ H^{(0)}_{fs} + kH^{(1)}_{fs} & H^{(0)}_{ff} + kH^{(1)}_{ff} + k^2 H^{(2)}_{ff} \end{bmatrix}.\tag{50}$$

3. In the expansion,

$$K(k) = -\frac{1}{2}L(k)^*L(k) - iH(k) \equiv k^2 A + kZ + R,\tag{51}$$

we require that the operator

$$A_{ff} = -\frac{1}{2}\sum_{a=s,f} L^{(1)*}_{af} L^{(1)}_{af} - iH^{(2)}_{ff}\tag{52}$$

be invertible on $\mathfrak{h}_f$.

Employing a repeated index summation convention over the index range $\{s,f\}$ from now on, we find that the operator R has components $R_{ab} = -\frac{1}{2}L^{(0)*}_{ca}L^{(0)}_{cb} - iH^{(0)}_{ab}$ with respect to the slow-fast block decomposition. Likewise,

$$A \equiv \begin{bmatrix} 0 & 0 \\ 0 & A_{ff} \end{bmatrix},$$

$$Z \equiv \begin{bmatrix} 0 & -\frac{1}{2}L^{(0)*}_{cs}L^{(1)}_{cf} - iH^{(1)}_{sf} \\ -\frac{1}{2}L^{(1)*}_{cf}L^{(0)}_{cs} - iH^{(1)}_{fs} & -\frac{1}{2}L^{(0)*}_{cf}L^{(1)}_{cf} - \frac{1}{2}L^{(1)*}_{cf}L^{(0)}_{cf} - iH^{(1)}_{ff} \end{bmatrix}.$$

In particular, we note the identities

$$R_{ss} + R^*_{ss} = -L^{(0)*}_{cs}L^{(0)}_{cs},\tag{53}$$

$$Z_{sf} + Z^*_{fs} = -L^{(0)*}_{cs}L^{(1)}_{cf},\tag{54}$$

$$A_{ff} + A^*_{ff} = -L^{(1)*}_{cf}L^{(1)}_{cf}.\tag{55}$$

C. The characteristic operator limit

In an adiabatic elimination problem, the coupling parameters $(S, L(k), H(k))$ lead to the associated characteristic operator

$$\mathcal{T}_k(s) = S - L(k)[s - K(k)]^{-1}L(k)^*S.\tag{56}$$

Lemma 10. Let $M(k)$ be a matrix parametrized by scalar k of the form

$$M(k) = \begin{bmatrix} M_{11} & kM_{12} + o(k) \\ kM_{21} + o(k) & k^2 M_{22} + o(k) \end{bmatrix}$$

with M_{22} invertible. Then, we have the limit

$$\lim_{k \to \infty} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} [s + M(k)]^{-1} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s + \widehat{M}_{11}} & -\dfrac{1}{s + \widehat{M}_{11}} M_{12} \dfrac{1}{M_{22}} \\[3mm] -\dfrac{1}{M_{22}} M_{21} \dfrac{1}{s + \widehat{M}_{11}} & \dfrac{1}{M_{22}} + \dfrac{1}{M_{22}} M_{21} \dfrac{1}{s + \widehat{M}_{11}} M_{12} \dfrac{1}{M_{22}} \end{bmatrix}.$$

Proof. See Appendix A.   □

Proposition 11. In the situation where the $L(k)$ and $H(k)$ are bounded operators for each k fixed, the characteristic operator has the strong limit

$$\widehat{\mathcal{T}}(s) = \lim_{k \to \infty} \mathcal{T}_k(s) \tag{57}$$

for Re $s > 0$, where we have

$$\widehat{\mathcal{T}}_{\mathsf{ab}}(s) = \left\{ \delta_{\mathsf{ab}} + L^{(1)}_{\mathsf{af}} \frac{1}{A_{\mathsf{ff}}} L^{(1)*}_{\mathsf{cf}} - \left[L^{(0)}_{\mathsf{as}} - L^{(1)}_{\mathsf{af}} \frac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}} \right] \frac{1}{s - \widehat{K}_{\mathsf{ss}}} \left[L^{(0)*}_{\mathsf{cs}} - Z_{\mathsf{sf}} \frac{1}{A_{\mathsf{ff}}} L^{(1)*}_{\mathsf{cf}} \right] \right\} S_{\mathsf{cb}},$$

where

$$\widehat{K}_{\mathsf{ss}} = R_{\mathsf{ss}} - Z_{\mathsf{sf}} \frac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}}. \tag{58}$$

Proof. This is a corollary to Lemma 10. In this case, we have the limit

$$\lim_{k \to \infty} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} [s - K(k)]^{-1} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s - \widehat{K}_{\mathsf{ss}}}, & -\dfrac{1}{s - \widehat{K}_{\mathsf{ss}}} Z_{\mathsf{sf}} \dfrac{1}{A_{\mathsf{ff}}} \\[3mm] -\dfrac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}} \dfrac{1}{s - \widehat{K}_{\mathsf{ss}}}, & -\dfrac{1}{A_{\mathsf{ff}}} + \dfrac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}} \dfrac{1}{s - \widehat{K}_{\mathsf{ss}}} Z_{\mathsf{sf}} \dfrac{1}{A_{\mathsf{ff}}} \end{bmatrix}.$$

□

Proposition 12. The limit characteristic operator is given by

$$\widehat{\mathcal{T}} = \left[\begin{array}{c|c} -\dfrac{1}{2} \widehat{L}^* \widehat{L} - i\widehat{H} & -\widehat{L}^* \widehat{S} \\ \hline \widehat{L} & \widehat{S} \end{array} \right] = \widehat{S} - \widehat{L} \left(s + \frac{1}{2} \widehat{L}^* \widehat{L} + i\widehat{H} \right)^{-1} \widehat{L}^* \widehat{S},$$

where the parameters $(\widehat{S}, \widehat{L}, \widehat{H})$ are defined by

$$\widehat{S} = \begin{bmatrix} \widehat{S}_{\mathsf{ss}} & \widehat{S}_{\mathsf{sf}} \\ \widehat{S}_{\mathsf{fs}} & \widehat{S}_{\mathsf{ff}} \end{bmatrix}, \ \widehat{L} = \begin{bmatrix} \widehat{L}_{\mathsf{s}} & 0 \\ \widehat{L}_{\mathsf{f}} & 0 \end{bmatrix}, \ \widehat{H} = \begin{bmatrix} \widehat{H}_{\mathsf{ss}} & 0 \\ 0 & 0 \end{bmatrix}, \tag{59}$$

with

$$\widehat{S}_{\mathsf{ab}} \triangleq \left(\delta_{\mathsf{ac}} + L^{(1)}_{\mathsf{af}} \frac{1}{A_{\mathsf{ff}}} L^{(1)*}_{\mathsf{cf}} \right) S_{\mathsf{cb}}, \tag{60}$$

$$\widehat{L}_{\mathsf{a}} \triangleq L^{(0)}_{\mathsf{as}} - L^{(1)}_{\mathsf{af}} \frac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}}, \tag{61}$$

$$\widehat{H}_{\mathsf{ss}} \triangleq H^{(0)}_{\mathsf{ss}} + \mathrm{Im} \left\{ Z_{\mathsf{sf}} \frac{1}{A_{\mathsf{ff}}} Z_{\mathsf{fs}} \right\}. \tag{62}$$

Proof. See Appendix B.   □

D. Further assumptions

We may impose additional constraints

$$\widehat{L}_{\mathsf{f}} = \widehat{S}_{\mathsf{sf}} = \widehat{S}_{\mathsf{fs}} = 0 \tag{63}$$

to ensure that limit dynamics excludes the possibility of transitions that terminate in any of the fast states. In this case, $\widehat{S}_{ss}$ is unitary.

Proposition 13. If additionally (63) holds, then the slow and fast subspaces are decoupled

$$\widehat{\mathcal{T}}(s) = \begin{bmatrix} \widehat{\mathcal{T}}_{ss}(s) & 0 \\ 0 & \widehat{S}_{ff} \end{bmatrix}, \tag{64}$$

where

$$\widehat{\mathcal{T}}_{ss}(s) = \left[\begin{array}{c|c} -\dfrac{1}{2}\widehat{L}_s^*\widehat{L}_s - i\widehat{H}_{ss} & -\widehat{L}_s^*\widehat{S}_{ss} \\ \hline \widehat{L}_s & \widehat{S}_{ss} \end{array} \right]. \tag{65}$$

Proof. This follows directly from

$$\widehat{\mathcal{T}}(s) = \left[\begin{array}{c|c} \begin{bmatrix} -\dfrac{1}{2}\widehat{L}_s^*\widehat{L}_s - i\widehat{H}_{ss} & 0 \\ 0 & 0 \end{bmatrix} & -\begin{bmatrix} \widehat{L}_s^*\widehat{S}_{ss} & 0 \\ 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} \widehat{L}_s & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \widehat{S}_{ss} & 0 \\ 0 & \widehat{S}_{ff} \end{bmatrix} \end{array} \right]. \tag{66}$$

$\square$

E. Adiabatic elimination for quantum stochastic models

The convergence of the characteristic operator is not sufficient to guarantee the convergence of the corresponding unitary processes. In paper[21] extra condition (63) is required.

Theorem 14 (Bouten and Silberfarb[21]). *Suppose, we are given a sequence of bounded operator parameters $(S, L(k), H(k))$ satisfying the assumptions in Eqn. (63). Then, $U_k(t) P_s$ converges strongly to $U(t) P_s$, that is,*

$$\lim_{k \to \infty} \|U_k(t)\psi - U(t)\psi\| = 0 \tag{67}$$

for all $\psi \in \mathfrak{h} \otimes \mathfrak{F}$ with $P_f \otimes I\psi = 0$.

The restriction to bounded coefficients was lifted in a subsequent publication.[22]

F. Related limits

It is possible to consider more specific limits which may exist in favourable circumstances. For instance, the all-pass form will lead to the scaled Σ-function

$$\Sigma_k(s) = L(k) \frac{1}{s + iH(k)} L(k)^*$$

which will converge provided $H_{ss}^{(2)}$ is invertible on the slow space. In this case, it happens the limit is well-defined and given by

$$\lim_{k \to \infty} \Sigma_k(s) = \begin{bmatrix} 0 & L_{sf}^{(1)} \\ 0 & L_{ff}^{(1)} \end{bmatrix} \begin{bmatrix} \dfrac{1}{s + i\tilde{H}_{ss}} & -\dfrac{1}{s + i\tilde{H}_{ss}} H_{sf}^{(1)} \dfrac{1}{H_{ff}^{(2)}} \\ -\dfrac{1}{H_{ff}^{(2)}} H_{fs}^{(1)} \dfrac{1}{s + i\tilde{H}_{ss}} & -i\dfrac{1}{H_{ff}^{(2)}} + \dfrac{1}{H_{ff}^{(2)}} H_{fs}^{(1)} \dfrac{1}{s + i\tilde{H}_{ss}} H_{sf}^{(1)} \dfrac{1}{H_{ff}^{(2)}} \end{bmatrix} \begin{bmatrix} 0 & L_{sf}^{(1)} \\ 0 & L_{ff}^{(1)} \end{bmatrix}^*$$

$$= \begin{bmatrix} L_{sf}^{(1)} \\ L_{ff}^{(1)} \end{bmatrix} \left(-i\dfrac{1}{H_{ff}^{(2)}} + \dfrac{1}{H_{ff}^{(2)}} H_{fs}^{(1)} \dfrac{1}{s + i\tilde{H}_{ss}} H_{sf}^{(1)} \dfrac{1}{H_{ff}^{(2)}} \right) \begin{bmatrix} L_{sf}^{(1)*} & L_{ff}^{(1)*} \end{bmatrix},$$

with $\tilde{H}_{\text{ss}} = H_{\text{ss}}^{(1)} - H_{\text{sf}}^{(1)} \frac{1}{H_{\text{ff}}^{(2)}} H_{\text{fs}}^{(1)}$. We shall refer to this the existence of a limit in all pass. As we have seen, however, the general limit may exists even when the Hamiltonian is zero.

More robust, however, is the limit formulated in terms of the Stratonovich form, where we have suitably-scaled coefficients $\mathbf{E}(k)$ and we use Stratonovich form (31) along with Lemma 10. We note the inverse relations

$$E_{\ell\ell} = 2i\frac{S-1}{S+1},$$

$$E_{\ell 0} = \frac{2i}{S+1}L,$$

$$E_{00} = H + \frac{1}{4}L^*E_{\ell\ell}L.$$

As S is required to be k-independent, the same must be true for $E_{\ell\ell}$. For convenience, we will fix the decompositions as $\mathfrak{h} = \mathfrak{h}_\text{s} \oplus \mathfrak{h}_\text{f}$ and assume that $E_{\ell\ell}$ is block diagonal

$$E_{\ell\ell} \equiv \begin{bmatrix} E_{\ell\ell}^{(\text{s})} & 0 \\ 0 & E_{\ell\ell}^{(\text{f})} \end{bmatrix}.$$

Taking form (49) for $L(k)$, it follows that

$$E_{\ell 0}(k) \equiv \begin{bmatrix} 0 & E_{\ell 0}^{(\text{sf})} \\ 0 & E_{\ell 0}^{(\text{ff})} \end{bmatrix},$$

with $E_{\ell 0}^{(\text{af})} = i\left(1 + \frac{i}{2}E_{\ell\ell}^{(\text{a})}\right)L_{\text{af}}^{(1)}$ (no summation!), for $\text{a} = \text{s}$ or f. It follows that in this case,

$$E_{00}(k) \equiv H(k) + \frac{k^2}{4}\begin{bmatrix} 0 & 0 \\ 0 & E_{\ell 0}^{(\text{sf})\dagger}E_{\ell\ell}^{(\text{s})}E_{\ell 0}^{(\text{sf})} + E_{\ell 0}^{(\text{ff})\dagger}E_{\ell\ell}^{(\text{f})}E_{\ell 0}^{(\text{ff})} \end{bmatrix} \equiv \begin{bmatrix} E_{00}^{(\text{ss})} & kE_{00}^{(\text{sf})} \\ kE_{00}^{(\text{fs})} & k^2E_{00}^{(\text{ff})} \end{bmatrix}$$

which is again of the same form of the general matrix appearing in Lemma 10. Provided that the self-adjoint term $E_{00}^{(\text{ff})}$ is invertible on $\mathfrak{h}_\text{f}$, the limit for the Stratonovich expression exists and will agree with the previous limits. We omit the more general situation where $E_{\ell\ell}$ is not block diagonal as it is more complicated and not very enlightening.

IV. HAMILTONIAN FORMULATION OF THE QUANTUM MODEL

In this section, we describe how the unitary process $U(t)$ can alternatively be viewed as Dirac picture unitaries relating a (singularly) perturbed Hamiltonian dynamics to a free Hamiltonian dynamics.

A. Dynamical perturbations

Let $V_0(t)$ and $V(t)$ be strongly continuous one-parameter groups, that is, $V_0(t+s) = V_0(t)V_0(s)$ and $V(t+s) = V(t)V(s)$, then we may view V as a perturbed dynamics with respect to the free dynamics of V_0 by transforming to the interaction picture via the wave operator

$$U(t) = V_0(t)^*V(t). \tag{68}$$

Physically, $U(t)$ transforms to the Dirac picture.[28] It inherits unitarity and strong continuity, but does not form a group. Instead, we have the so-called *cocycle property*

$$U(t+s) = \Theta_t(U(s))U(t), \tag{69}$$

where $\Theta_t(x) = V_0(t)^*XV_0(t)$. By Stone's theorem, both V_0 and V possess self-adjoint (Hamiltonian) infinitesimal generators $\widehat{H}_0$ and $\widehat{H}$, respectively: $i\dot{V}_0(t) = \widehat{H}_0V_0(t)$ and $i\dot{V}(t) = \widehat{H}V(t)$. We say that $\widehat{H}$ is a regular perturbation of $\widehat{H}_0$ if $\Upsilon = \widehat{H} - \widehat{H}_0$ defines an operator with dense domain. In this case, $U(t)$ will be strongly differentiable and

$$i\dot{U}(t) = \Upsilon(t)U(t), \tag{70}$$

where the time-dependent Hamiltonian is $\Upsilon(t) = \Theta_t(\Upsilon)$. In situations where Υ is not densely defined, we will have a singular perturbation and $U(t)$ will not generally be strongly differentiable.

B. Quantum stochastic evolutions

The quantum input processes $b_i(t)$ may view these processes as singular operators acting formally on the Hilbert space with the Fock space $\mathfrak{F}$ over $\mathbb{C}^n \otimes L^2(\mathbb{R})$. For $\Psi \in \mathfrak{F}$, we have a well-defined amplitude $\langle \tau_1, i_1; \cdots; \tau_m, i_m | \Psi \rangle$ which is completely symmetric under interchange of the m pairs of labels $(\tau_1, i_1), \cdots, (\tau_m, i_m)$, and this represent the amplitude to have m quanta with a particle of type i_1 at τ_1, particle of type i_2 at τ_2, etc. We have the following resolution of identity on $\mathfrak{F}$:

$$\sum_{m=0}^{\infty} (\int d\tau_1 \cdots d\tau_m)(\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n}) \times |\tau_1, i_1; \cdots; \tau_m, i_m\rangle\langle\tau_1, i_1; \cdots; \tau_m, i_m| = I. \tag{71}$$

The annihilator input process $b_i(t)$ is then defined almost everywhere as

$$\langle \tau_1, i_1; \cdots; \tau_m, i_m | b_i(t)\Psi \rangle = \sqrt{m+1} \, \langle t, i; \tau_1, i_1; \cdots; \tau_m, i_m | \Psi \rangle. \tag{72}$$

The annihilation operators, together with their formal adjoints the creator operators $b_i(t)^*$ satisfy the singular canonical commutation relations $[b_i(t), b_j^*(s)] = \delta_{ij}\delta(t-s)$.

1. The time shift

Let us introduce the following operator on the Fock space:

$$\widehat{H}_0 = \sum_{j=1}^{n} \int_{-\infty}^{\infty} dt \, b^*(t)_j \, i \frac{\partial}{\partial t} \, b(t)_j \tag{73}$$

which is the second quantization of the one-particle operator $i\frac{\partial}{\partial t}$. This is clearly a self-adjoint operator and the unitary group $V_0(t) = e^{-it\widehat{H}_0}$ it generates is just the time shift:

$$\langle \tau_1, i_1; \cdots; \tau_m, i_m | V_0(t)\Psi \rangle = \langle \tau_1 + t, i_1; \cdots; \tau_m + t, i_m | \Psi \rangle. \tag{74}$$

The free evolution $\Theta_t(\cdot) = V_0(t)^*(\cdot)V_0(t)$ will translate the input processes in time: $\Theta_\tau(b_i(t)) = b_i(t + \tau)$, $\Theta_\tau(b_i^*(t)) = b_i^*(t + \tau)$.

C. Unitary QSDEs as singular perturbations

The stochastic process $U(t)$ is strongly continuous, but due to the presence of the noise fields dB_i^*, dB_j, and $d\Lambda_{ij}$ is not typically strongly differentiable. Here, we see that the local interaction Υ is a singular perturbation of the generator of time-shift (73). We remark that nevertheless $U(t)$ is a Θ-cocycle and that if we now define $V(t)$ by

$$V(t) = \begin{cases} V_0(t) U(t), & t \geq 0, \\ U(-t)^* V_0(t), & t < 0 \end{cases} \tag{75}$$

then $V(t)$ is a strongly continuous unitary group and therefore admits an infinitesimal generator $\widehat{H}$. Surprising as it may seem, the quantum stochastic process $U(t)$ may be considered as the wave-operator for a quantum dynamics with Hamiltonian $\widehat{H}$ with respect to the free dynamics of the time shift generated by $\widehat{H}_0$. The relation

$$\widehat{H} = \widehat{H}_0 + \Upsilon \tag{76}$$

however has only a formal meaning as the Υ is singular with respect to $\widehat{H}_0$.

D. Global hamiltonian as singular perturbation of the time shift generator

It has been a long standing problem to characterize the associated Hamiltonian $\widehat{H}$ for SLH models.[29] The major breakthrough came in 1997 when Chebotarev solved this problem for the class of quantum stochastic evolutions satisfying Hudson-Parthasarathy differential equations with bounded commuting system coefficients.[30] His insight was based on scattering theory of a one-dimensional system with a Dirac potential, say, with formal Hamiltonian

$$k = i\partial + E\delta \tag{77}$$

describing a one-dimensional particle propagating along the negative x-axis with a delta potential of strength E at the origin. (In Chebotarev's analysis, the δ-function is approximated by a sequence of regular functions and a strong resolvent limit is performed.) The mathematical techniques used in this approach were subsequently generalized by Gregoratti[31] to relax the commutativity condition. More recently, the analysis has been further extended to treat unbounded coefficients.[32]

Independently, several authors have been engaged in the program of describing the Hamiltonian nature of quantum stochastic evolutions by interpreting the time-dependent function $\Upsilon(t)$ as being an expression involving quantum white noises satisfying a singular CCR.[33–36] This would naturally suggest that Υ should be interpreted as a sesquilinear expression in these noises at time $t = 0$.

The generator of the free dynamics $k_0 = i\partial$ is not semi-bounded and the δ-perturbation is viewed as a singular rank-one perturbation. Here, methods introduced by Albeverio and Kurasov[37–39] may be employed to construct self-adjoint extensions of such models, which we show in Sec. IV E for a wave on a 1-D wire.

E. The global Hamiltonian

The form of the Hamiltonian $\widehat{H}$ is known to be[31]

$$-i\widehat{H}\Psi = -i\tilde{H}_0\Psi - (\frac{1}{2}L_i^*L_i + iH)\Psi - L_i^*S_{ij}b_j(0^+)\Psi, \tag{78}$$

on the domain of suitable functions satisfying the boundary condition

$$b_i(0^-)\Psi = L_i\Psi + S_{ij}\,b_j(0^+)\Psi \tag{79}$$

here, the suitable functions in question are those on the joint system and Fock space that are in the domain of the free translation along the edges (excluding the vertex at the origin) and in the domain of the one-sided annihilators $b_i(0^\pm)$. This agrees with the expression found in Refs. 30 and 31. The global Hamiltonian form is essential for building up arbitrary quantum feedback networks.[13]

F. Formal linear system behind the SLH model

We now specify to the case where the plant has finite dimensional Hilbert space, say $\dim \mathfrak{h} = m < \infty$. In this case, the operators (S, L, H) are naturally represented as complex-valued matrices with dimensions

$$\mathsf{S} \in \mathbb{C}^{nm \times nm}, \quad \mathsf{L} \in \mathbb{C}^{nm \times m}, \quad \mathsf{H} \in \mathbb{C}^{m \times m}. \tag{80}$$

That is, we have the matrix representations $\mathsf{S}_{ij}, \mathsf{L}_j, \mathsf{H} \in \mathbb{C}^{m \times m}$ for a fixed orthonormal basis of $h \cong \mathbb{C}^m$. In terms of the (A, B, C, D), we then have

$$A = \mathsf{K} = -\frac{1}{2}\sum_{j=1}^{n}\mathsf{L}_j^*\mathsf{L}_j - i\mathsf{H} \in \mathbb{C}^{m \times m},$$

$$B = -\mathsf{L}^*\mathsf{S} = -[\sum_{j=1}^{n}\mathsf{L}_j^*\mathsf{S}_{j1}, \cdots, \sum_{j=1}^{n}\mathsf{L}_j^*\mathsf{S}_{jn}] \in \mathbb{C}^{m \times nm},$$

$$C = \mathsf{L} = \begin{bmatrix} \mathsf{L}_1 \\ \vdots \\ \mathsf{L}_n \end{bmatrix} \in \mathbb{C}^{nm \times m},$$

$$D = \mathsf{S} = \begin{bmatrix} \mathsf{S}_{11} & \cdots & \mathsf{S}_{1n} \\ \vdots & \ddots & \vdots \\ \mathsf{S}_{n1} & \cdots & \mathsf{S}_{nn} \end{bmatrix} \in \mathbb{C}^{nm \times nm}.$$

This is essentially the equivalent linear passive model considered in Remark 5. Explicitly, the input-state-output equations behind this will be

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where x is a $\mathbb{C}^m$-values state variable and u and y should be $\mathbb{C}^{nm}$-valued functions. Let Ψ be a solution to global Hamiltonian problem (78) and satisfying correct boundary conditions (79). This system may be rewritten as

$$\dot{\Psi} + i\tilde{H}_0\Psi = \mathsf{K}\Psi - \mathsf{L}^*\mathsf{S}u, \tag{81}$$

$$y = \mathsf{L}\Psi + \mathsf{S}u. \tag{82}$$

where now the input and output functions are

$$u_j = b_j(0^+)\Psi, \quad y_j = b_j(0^-)\Psi. \tag{83}$$

Absorbing the relatively unimportant free dynamics due to $\tilde{H}_0$, we see that (81) and (82) are linear systems with "input signal" u and "output signal" y.

The functions u and y are boundary terms related by (83) and not to be interpreted literally as control functions which we can assign.

V. EXAMPLES

We now discuss some well-known examples from the perspective of control theory.

A. No scattering and trivial damping

Let us set $S = I$, $L^{(1)} = 0$, and $L_{\mathsf{fs}}^{(0)} = 0$. In this case, the only damping of significance is that of the slow component. Then, we have $A_{\mathsf{ff}} = -iH_{\mathsf{ff}}^{(2)}$ and we require that $H_{\mathsf{ff}}^{(2)}$ is invertible on $\mathfrak{h}_{\mathsf{s}}$. It is easy to see that the decoupling conditions now apply and we obtain the open dynamics with $(\widehat{S} = I, \widehat{L} = L_{\mathsf{ss}}^{(0)}, \widehat{H})$, where the reduced Hamiltonian is

$$\widehat{H} = H_{\mathsf{ss}}^{(0)} - H_{\mathsf{sf}}^{(1)} \frac{1}{H_{\mathsf{ff}}^{(2)}} H_{\mathsf{fs}}^{(1)}.$$

Now, $\widehat{H}$ is the shorted version (Schur complement) of $H(1) = \begin{bmatrix} H_{\mathsf{ss}}^{(0)} & H_{\mathsf{sf}}^{(1)} \\ H_{\mathsf{fs}}^{(1)} & H_{\mathsf{ff}}^{(2)} \end{bmatrix}$. Equivalently, $\widehat{H}$ is the limit $k \uparrow \infty$ of shorted version of $H(k)$.

The detuned two level atom model considered in Subsection II A 1 is a special case.

B. Qubit limit

Let us consider a cavity consisting of a single photon mode with annihilator a, so that $[a, a^*] = I$. The number states $|n\rangle$, $(n = 0, 1, \ldots)$, span an infinite dimensional Hilbert space. Mabuchi[23] shows how a large Kerr non-linearity leads to a reduced dynamics where we are restricted to the ground and first excited state of the mode, and so have an effective qubit dynamics. We consider the

$n = 2$ input model with

$$[S(k)]_{jk} = \delta_{jk}I,$$
$$[L(k)]_j = \sqrt{\kappa_j}e^{i\omega t}a, \quad (j = 1,2),$$
$$H(k) = k^2\chi_0 a^{*2}a^2 + \Delta a^*a - i\sqrt{\kappa_1}(\alpha(t)a^* - \alpha^*(t)a).$$

In the model, we are in a rotating frame with frequency ω and the cavity is detuned from this frequency by a fixed amount Δ. There is a Kerr non-linearity of strength $\chi(k) = \chi_0 k^2$ which will be the large parameter. We have two input fields with damping rate $\kappa_j (j = 1,2)$, and the first input introduces a coherent driving field $\alpha(t)$.

We now have $A \equiv \chi_0 a^{*2}a^2 = \chi_0 N(N - 1)$, where $N = a^*a$ is the number operator. The kernel space of A is therefore

$$\mathfrak{h}_s = \text{span}\{|0\rangle, |1\rangle\}.$$

For this situation, we have $P_s = |0\rangle\langle 0| + |1\rangle\langle 1|$, and we find $L_{fs}^{(0)} = 0$ since $P_f a P_s \equiv 0$. The Bouten-Silberfarb conditions are then satisfied and we have

$$H_{ss}^{(0)} = P_s\Delta a^*a P_s \equiv \Delta\sigma^*\sigma,$$

where $\sigma \triangleq P_s a P_s \equiv |0\rangle\langle 1|$. We then have that

$$\left[\widehat{S}_{ss}\right]_{jk} = \delta_{jk}\, I_s,$$
$$\left[\widehat{L}_s\right]_j = \sqrt{\kappa_j}e^{i\omega t}\sigma,$$
$$\widehat{H} = \Delta\sigma^*\sigma - i\sqrt{\kappa_1}(\alpha(t)\sigma^* - \alpha^*(t)\sigma).$$

The system is then completely controllable through the control policy α and observable through quadrature measurement (homodyning with $B_{\text{out},1}(t) - B_{\text{out},1}(t)^*$ and $-iB_{\text{out},1}(t) + iB_{\text{out},1}(t)^*$) and by photon counting. The characteristic operator is as computed in Subsection II A 2. The limit characteristic operator is then ($\kappa = \kappa_1 + \kappa_2$),

$$\mathscr{T}_{\text{qubit}}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{s^2 - \frac{1}{2}\kappa s + i\Delta s + \kappa_1|\alpha|^2}{s^2 + \frac{1}{2}\kappa s + i\Delta s + \kappa_1|\alpha|^2} \end{bmatrix}.$$

C. No scattering, but non-trivial damping

We consider the case where $S = I$, $L_{fs}^{(0)} = 0$ and $L_{ff}^{(1)} = 0$, but $L_{sf}^{(1)} \neq 0$. The decoupling conditions are automatically satisfied, so all that is further required is that A_{ff}, which is now given by

$$A_{ff} \equiv -\frac{1}{2}L_{sf}^{(1)*}L_{sf}^{(1)} - iH_{ff}^{(2)}$$

is invertible. If so, the reduced SLH takes the simplified form

$$\widehat{S}_{ss} \equiv I_s + L_{sf}^{(1)}\frac{1}{A_{ff}}L_{sf}^{(1)*},$$

$$\widehat{L}_s \equiv L_{ss}^{(0)} - L_{sf}^{(1)}\frac{1}{A_{ff}}M_{fs},$$

$$\widehat{H} \equiv H_{ss}^{(0)} + \text{Im}\left\{M_{sf}\frac{1}{A_{ff}}M_{fs}\right\},$$

where now,

$$M_{sf} \equiv -\frac{1}{2}L_{ss}^{(0)*}L_{sf}^{(1)} - iH_{sf}^{(1)}, \quad M_{fs} \equiv -\frac{1}{2}L_{sf}^{(1)*}L_{ss}^{(0)} - iH_{fs}^{(1)}.$$

D. Λ-systems

Consider a three level atom with ground states $|g_1\rangle, |g_2\rangle$ and an excited state $|e\rangle$ with Hilbert space $\mathfrak{h}_{level} = \mathbb{C}^3$. The atom is contained in a cavity with quantum mode a with Hilbert space $\mathfrak{h}_{mode}$, where $[a, a^*] = 1$ and a annihilates a photon of the cavity mode. The combined system and cavity has Hilbert space $\mathfrak{h} = \mathfrak{h}_{level} \otimes \mathfrak{h}_{mode}$, and consider the following,[22,24,25]

$$L(k) = k\sqrt{\gamma} I \otimes a,$$
$$H(k) = ik^2 \mathsf{g} \{|e\rangle\langle g_1| \otimes a - \text{h.c.}\} + ik \{|e\rangle\langle g_2| \otimes \alpha - \text{h.c.}\}.$$

Here, the cavity is lossy and leaks photons with decay rate γ, we also have a transition from $|e\rangle$ to $|g_1\rangle$ with the emission of a photon into the cavity and a scalar field α driving the transition from $|e\rangle$ to $|g_2\rangle$. We see that

$$A \equiv -\frac{1}{2}\gamma I \otimes a^* a + \mathsf{g}\{|e\rangle\langle g_1| \otimes a - |g_1\rangle\langle e| \otimes a^*\}$$

and that A has a 2-dimensional kernel space spanned by the pair of states

$$|\Psi_1\rangle = |g_1\rangle \otimes |0\rangle, \quad |\Psi_2\rangle = |g_2\rangle \otimes |0\rangle.$$

The reduced subspace is then the span of $|\Psi_1\rangle$ and $|\Psi_2\rangle$, and the resulting SLH operators are

$$\widehat{S} = |\Psi_1\rangle\langle\Psi_1| - |\Psi_2\rangle\langle\Psi_2| \equiv I - 2\sigma^*\sigma,$$
$$\widehat{L} = -\frac{\gamma\alpha}{\mathsf{g}}|\Psi_1\rangle\langle\Psi_2| \equiv -\frac{\gamma\alpha}{\mathsf{g}}\sigma,$$
$$\widehat{H} = 0,$$

where $\sigma = |\Psi_1\rangle\langle\Psi_2|$. Here, the dynamics has a vanishing Hamiltonian, but is partially observable through filtering as $\widehat{L} \neq 0$. The limit characteristic operator is then

$$\mathscr{T}_\Lambda(s) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{s - \frac{\gamma^2}{2\mathsf{g}^2}|\alpha|^2}{s + \frac{\gamma^2}{2\mathsf{g}^2}|\alpha|^2} \end{bmatrix}.$$

Further examples of adiabatic elimination, particularly, where the fast degrees of freedom are oscillators, can be found in Refs. 26, 27, and 40.

VI. CONCLUSIONS

The characteristic operator is introduced here as a mathematical object containing information about quantum input-output relations when processed by a quantum mechanical system. The concept allows us to characterise quantum systems, and many of the features associated with classical transfer functions carry over. We have shown that it picks out the particular scaling introduced by Bouten and Silberfarb for adiabatic elimination for quantum open systems as being the one which leads to the convergence of characteristic operators using Schur-Feshbach type resolvent expansions. It is useful to note that strong coupling that restricts the degrees of freedom adiabatically may also be interpreted as a projection onto a Zeno subspace, though generally of an open systems character.[40]

We expect that the concept will play an important role in studying features of quantum control systems such as model reduction, controllability, and observability.

ACKNOWLEDGMENTS

APPENDIX A: PROOF OF LEMMA 10

Again, by the Schur-Feshbach identity, we may write the resolvent $\frac{1}{s + M(k)}$ as

$$\begin{bmatrix} s + M_{11}(k) & M_{12}(k) \\ M_{21}(k) & s + M_{22}(k) \end{bmatrix}^{-1} = \begin{bmatrix} \Delta_{11}(s,k) & \Delta_{12}(s,k) \\ \Delta_{21}(s,k) & \Delta_{22}(s,k) \end{bmatrix},$$

where setting

$$\widehat{M}_{11}(s,k) \triangleq M_{11}(k) - M_{12}(k)\frac{1}{s + M_{22}(k)}M_{21}(k), \tag{A1}$$

we have

$$\Delta_{11}(s,k) = \frac{1}{s + \widehat{M}_{11}(s,k)},$$

$$\Delta_{12}(s,k) = \widehat{M}_{11}(s,k)M_{12}(k)\frac{1}{s + M_{22}(k)},$$

$$\Delta_{21}(s,k) = \frac{1}{s + M_{22}(k)}M_{21}(k)\widehat{M}_{11}(s,k),$$

$$\Delta_{22}(s,k) = \frac{1}{s + M_{22}(k)} + \frac{1}{s + M_{22}(k)}M_{21}(k)\frac{1}{s + \widehat{M}_{11}(s,k)}M_{12}(k)\frac{1}{s + M_{22}(k)}.$$

Using the fact that $M_{12}(k) = kM_{12} + o(k)$, $M_{21}(k) = kM_{21} + o(k)$, and $M_{22}(k) = k^2 M_{22} + o(k)$, we note

$$\widehat{M}_{11}(s,k) \equiv M_{11} - k^2 M_{12}\frac{1}{s + k^2 M_{22} + o(k)}M_{21},$$

and the following scaled limit

$$\widehat{M}_{11} \triangleq \lim_{k \to \infty} \widehat{M}_{11}(s,k) = M_{11} - M_{12}\frac{1}{M_{22}}M_{21}, \tag{A2}$$

so that $\widehat{M}_{11}$ is a Schur complement of $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. Similarly, it follows that

$$\lim_{k \to \infty} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}[s + M(k)]^{-1}\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s + \widehat{M}_{11}} & -\dfrac{1}{s + \widehat{M}_{11}}M_{12}\dfrac{1}{M_{22}} \\ -\dfrac{1}{M_{22}}M_{21}\dfrac{1}{s + \widehat{M}_{11}} & \dfrac{1}{M_{22}} + \dfrac{1}{M_{22}}M_{21}\dfrac{1}{s + \widehat{M}_{11}}M_{12}\dfrac{1}{M_{22}} \end{bmatrix}.$$

APPENDIX B: PROOF OF PROPOSITION 12

Let us first note that we may define $\widehat{K}$ by $\widehat{K} = -\frac{1}{2}\widehat{L}^*\widehat{L} - i\widehat{H}$ in which case

$$\widehat{K} = \begin{bmatrix} -\dfrac{1}{2}\widehat{L}_{\mathsf{a}}^*\widehat{L}_{\mathsf{a}} - i\widehat{H}_{\mathsf{ss}} & 0 \\ 0 & 0 \end{bmatrix}. \tag{B1}$$

We note that $-\frac{1}{2}\widehat{L}_a^*\widehat{L}_a - i\widehat{H}_{ss}$ can be written as

$$-\frac{1}{2}\left(L_{as}^{(0)*} - Z_{fs}^*\frac{1}{A_{ff}^*}L_{af}^{(1)*}\right)\left(L_{as}^{(0)} - L_{af}^{(1)}\frac{1}{A_{ff}}Z_{fs}\right) - i\widehat{H}_{ss} = R_{ss} - \frac{1}{2}Z_{sf}\frac{1}{A_{ff}}Z_{fs} + \frac{1}{2}Z_{fs}^*\frac{1}{A_{ff}^*}Z_{sf}^*$$

$$-\frac{1}{2}\left(Z_{sf} + Z_{fs}^*\right)\frac{1}{A_{ff}}Z_{fs} - \frac{1}{2}Z_{fs}^*\frac{1}{A_{ff}^*}\left(Z_{sf} + Z_{fs}^*\right)$$

$$+\frac{1}{2}Z_{fs}^*\frac{1}{A_{ff}^*}\left(A_{ff} + A_{ff}^*\right)\frac{1}{A_{ff}}Z_{fs}$$

$$= R_{ss} - Z_{sf}\frac{1}{A_{ff}}Z_{fs},$$

where we use (55).

Therefore, with $\widehat{K}_{ss}$ is as defined in (58), we have

$$\widehat{K} = \begin{bmatrix} \widehat{K}_{ss} & 0 \\ 0 & 0 \end{bmatrix}. \tag{B2}$$

Moreover, we see that $\widehat{S}$ is unitary. To see this, set $\widehat{T} = \widehat{S}S^{-1}$ then,

$$\widehat{T}_{ca}^*\widehat{T}_{cb} = [\delta_{ca} + L_{af}^{(1)}\frac{1}{A_{ff}^*}L_{cf}^{(1)*}][\delta_{cb} + L_{cf}^{(1)}\frac{1}{A_{ff}}L_{bf}^{(1)*}] = \delta_{ab} + L_{af}^{(1)}\frac{1}{A_{ff}^*}\left\{A_{ff} + A_{ff}^* + L_{cf}^{(1)*}L_{cf}^{(1)}\right\}\frac{1}{A_{ff}}L_{bf}^{(1)*},$$

however, the expression in braces vanishes identically leaving $\widehat{T}^*\widehat{T} = I$. The proof of the co-isometric property of $\widehat{T}\widehat{T}^* = I$ is similar.

We note that

$$\widehat{S} \equiv \lim_{|s|\to\infty} \widehat{\mathscr{T}}(s). \tag{B3}$$

It remains to show that the limit characteristic function $\widehat{\mathscr{T}}$ has the stated form. Substituting in form (57), we have

$$\widehat{\mathscr{T}}_{ab}(s) - \left[\widehat{S}_{ab} - \widehat{L}_a\left(s - \widehat{K}_{ss}\right)^{-1}\widehat{L}_c\widehat{S}_{cb}\right] = \widehat{L}_a\frac{1}{s - \widehat{K}_{ss}}\left\{-(L_{ds}^{(0)} - Z_{sf}\frac{1}{A_{ff}}L_{df}^{(1)*}) + \widehat{L}_c^*(\delta_{cd} + L_{cf}^{(1)}\frac{1}{A_{ff}}L_{df}^{(1)*})\right\}S_{db}$$

and the term in braces equals

$$\left[Z_{sf}\frac{1}{A_{ff}} - Z_{fs}^*\frac{1}{A_{ff}} + L_{cs}^{(0)*}L_{cf}^{(1)}\frac{1}{A_{ff}} - L_{fs}^{(1)*}\frac{1}{A_{ff}}L_{cf}^{(1)*}L_{cf}^{(1)}\frac{1}{A_{ff}}\right]L_{df}^{(1)*} \tag{B4}$$

and using (55) again, we see that the term in square brackets is

$$Z_{sf}\frac{1}{A_{ff}} - Z_{fs}^*\frac{1}{A_{ff}} - \left(Z_{sf} + Z_{fs}^*\right)\frac{1}{A_{ff}} - Z_{fs}^*\frac{1}{A_{ff}}\left(A_{ff} + A_{ff}\right)\frac{1}{A_{ff}} \tag{B5}$$

which vanishes identically.

We note that we have the alternative form

$$\widehat{H}_{ss} = H_{ss}^{(0)} - Z_{fs}^*\frac{1}{A_{ff}}H_{fs}^{(1)} - H_{sf}^{(1)}\frac{1}{A_{ff}}Z_{fs} + Z_{fs}^*\frac{1}{A_{ff}}H_{ff}^{(2)}\frac{1}{A_{ff}}Z_{fs}. \tag{B6}$$

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