

## Aberystwyth University

### *Feedback Network Models for Quantum Transport*

Gough, John

*Published in:*  
Physical Review E

*DOI:*  
[10.1103/PhysRevE.90.062109](https://doi.org/10.1103/PhysRevE.90.062109)

*Publication date:*  
2014

*Citation for published version (APA):*  
Gough, J. (2014). Feedback Network Models for Quantum Transport. *Physical Review E*, 90, [062109].  
<https://doi.org/10.1103/PhysRevE.90.062109>

#### **General rights**

Copyright and moral rights for the publications made accessible in the Aberystwyth Research Portal (the Institutional Repository) are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Aberystwyth Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Aberystwyth Research Portal

#### **Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

tel: +44 1970 62 2400  
email: [is@aber.ac.uk](mailto:is@aber.ac.uk)

## Feedback network models for quantum transport

John Gough\*

*Aberystwyth University, Aberystwyth SY23 3BZ, United Kingdom*

(Received 29 August 2014; published 3 December 2014)

Quantum feedback networks have been introduced in quantum optics as a framework for constructing arbitrary networks of quantum mechanical systems connected by unidirectional quantum optical fields, and has allowed for a system theoretic approach to open quantum optics systems. Our aim here is to establish a network theory for quantum transport systems where typically the mediating fields between systems are bidirectional. Mathematically, this leads us to study quantum feedback networks where fields arrive at ports in input-output pairs, making it a special case of the unidirectional theory where inputs and outputs are paired. However, it is conceptually important to develop this theory in the context of quantum transport theory—the resulting theory extends traditional approaches which tend to view the components in quantum transport as scatterers for the various fields, in the process allowing us to consider emission and absorption of field quanta by these components. The quantum feedback network theory is applicable to both Bose and Fermi fields, moreover, it applies to nonlinear dynamics for the component systems. We advance the general theory, but study the case of linear passive quantum components in some detail.

DOI: [10.1103/PhysRevE.90.062109](https://doi.org/10.1103/PhysRevE.90.062109)

PACS number(s): 03.65.–w, 05.60.Gg, 02.30.Yy, 42.50.Lc

### I. INTRODUCTION

The aim of this paper is to extend the formalism of quantum feedback networks [1,2] from their current applications in quantum optical and, more recently, optomechanical systems, into the rapidly developing field of quantum transport networks. In quantum optics applications, one usually treats the noise fields interacting with the system as unidirectional. In the input-output approach of Gardiner and Collett (see Ref. [3]), this arises naturally and may be understood as a specific case of the Lehmann-Symanzik-Zimmermann (LSZ) formalism of quantum field theory, however, physically this is also justified by the fact that bidirectional quantum optical fields may always be made unidirectional by using an optical isolator.

The quantum feedback network theory is built on the general theory of open quantum stochastic evolutions developed by Hudson and Parthasarathy [4], which goes beyond Gardiner’s theory by allowing the system to scatter noise quanta as well as emit and absorb them—now generally referred to as the SLH formalism, which we recall in the next section.

There has been an increasing motivation to develop control theory for quantum transport models. An example is the control of solid state cavity quantum electrodynamics (QED) devices (see, e.g., Refs. [5,6] for superconducting qubit examples), which replaces traditional photonic systems as hardware. This has spurred the application of control theoretic techniques, originally devised to control quantum optical devices, to different settings. Coupling a QED cavity to a quantum dot has been shown to allow control of the cavity reflectivity [7], as well as the possibility to generate nonclassical states of light [8] (see Refs. [9,10] for an overview of recent applications to photonics and quantum dots in photonic-crystal technologies). Quantum dots have also been used to stabilize mesoscopic electric currents by means of feedback [11], with proposals for delayed feedback [12] and stabilization of pure qubit states [13]. As with quantum optical devices, there has been a move away from table top experimental setups towards on-chip

devices, and strong photon-photon interactions have been shown to be implementable on integrated photonic chips were quantum dots embedded in photonic-crystal nanocavities [14].

A first step in extending quantum feedback networks to quantum transport problems has been made in Ref. [15]: Here, control methodologies were introduced for purely scattering models. However, we now wish to extend the theory to general linear systems which allow for more general models of dissipation. This leads to the framework in which to apply the standard techniques of measurement-based and coherent quantum feedback techniques. We expect that the theory presented here should be readily implementable with existing toolboxes for simulating quantum feedback networks [16,17].

Although the theory is applicable to general coupling of the fields to the components, we will develop the linear theory in some detail. Here the chain-scattering representation proves to be the essential concept. We point out that there exists a well-developed theory of control based on this approach due to Kimura [18], and which we exploit here. The results on lossless systems are particularly relevant to the linear passive models which we consider here. We also wish to acknowledge the prior work of Yanagisawa and Kimura [19,20] on quantum linear models, which as far as we know was the first to apply chain-scattering techniques to linear quantum networks.

For transparency we restrict to passive systems [21], however, it is clear that many of the results presented here should carry over to quantum transport networks having active components [22].

### II. SLH FORMALISM

For open Markov systems driven by  $n$  vacuum noise inputs, the model is specified by a triple

$$\mathbf{G} \sim (S, L, H),$$

referred to as the set of Hudson-Parthasarathy coefficients, or more prosaically as the “SLH.” Their roles are to describe respectively the input-to-output scattering  $S = [S_{jk}]$  of the external noise fields  $b_k(t)$ , the coupling  $L = [L_j]$  of the noise to the system, and the internal Hamiltonian  $H$  of the system.

\*jug@aber.ac.uk

The SLH formalism for quantum Markov models deals with the category of models

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, \quad H,$$

where the  $S_{jk}, L_k, H$  are operators on the component system Hilbert space.

These may be assimilated into the *model matrix*

$$\begin{aligned} V &= \begin{bmatrix} -\frac{1}{2}L^*L - iH & -L^*S \\ L & S \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sum_j L_j^*L_j - iH & -\sum_j L_j^*S_{j1} & \cdots & -\sum_j L_j^*S_{jm} \\ L_1 & S_{11} & \cdots & S_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ L_n & S_{n1} & \cdots & S_{nn} \end{bmatrix} \\ &= \begin{bmatrix} V_{00} & V_{01} & \cdots & V_{0m} \\ V_{10} & V_{11} & \cdots & V_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n0} & V_{n1} & \cdots & V_{nm} \end{bmatrix}. \end{aligned}$$

We recall briefly the class of Markov models for open quantum systems. The system with Hilbert space  $\mathfrak{h}$  driven by  $n$  independent Bose quantum processes with Fock space  $\mathfrak{F}$  will have a unitary evolution  $V_G(t)$  on the space  $\mathfrak{h} \otimes \mathfrak{F}$ , where  $V_G(t)$  is the solution to the quantum stochastic differential equation [4]

$$dV_G(t) = \left\{ (S_{jk} - \delta_{jk}) \otimes d\Lambda_{jk}(t) + L_j \otimes dB_j^*(t) - L_j^*S_{jk} \otimes dB_k(t) - \left(\frac{1}{2}L_k^*L_k + iH\right) \otimes dt \right\} V_G(t),$$

with initial condition  $V_G(0) = I$ . (We adopt the convention that repeated latin indices imply a summation over the range  $1, \dots, n$ .) Formally, the Bose noise can be thought of as arising from quantum white noise processes  $b_k(t)$  satisfying the singular of commutation relations

$$[b_j(t), b_k(s)^*] = \delta_{jk}\delta(t-s),$$

with

$$B_j(t) = \int_0^t b_j(s)ds, \quad B_j^*(t) = \int_0^t b_j(s)^*ds,$$

$$\Lambda_{jk}(t) = \int_0^t b_j(s)^*b_k(s)ds.$$

The conditions guaranteeing unitarity are that  $S = [S_{jk}]$  is unitary,  $L = [L_j]$  is bounded, and  $H$  self-adjoint. In the autonomous case we may assume that the operator coefficients  $S_{jk}, L_j, H$  are fixed system operators, however, there is little difficulty in allowing them to be time dependent, or, more generally, be adapted processes, that is,  $S_{jk}(t), L_j(t), H(t)$  depend on the noise up to time  $t$ . The process  $V_G(t)$  will inherit this adaptedness property. (See Fig. 1.)

For a fixed system operator  $X$  we set

$$j_t^G(X) \triangleq V_G(t)^*[X \otimes I]V_G(t). \quad (1)$$

Then from the quantum Itô calculus [4] we get the *Heisenberg-Langevin equations*,

$$\begin{aligned} dj_t^G(X) &= j_t^G(\mathcal{L}_{jk}X) \otimes d\Lambda_{jk}(t) + j_t^G(\mathcal{L}_{j0}X) \otimes dB_j^*(t) \\ &+ j_t^G(\mathcal{L}_{0k}X) \otimes dB_k(t) + j_t^G(\mathcal{L}_{00}X) \otimes dt, \quad (2) \end{aligned}$$

where the Evans-Hudson superoperators  $\mathcal{L}_{\mu\nu}$  are explicitly given by

$$\begin{aligned} \mathcal{L}_{jk}X &= S_{lj}^*XS_{lk} - \delta_{jk}X, \\ \mathcal{L}_{j0}X &= S_{lj}^*[X, L_l], \quad \mathcal{L}_{0k}X = [L_l^*, X]S_{lk} \\ \mathcal{L}_{00}X &= \frac{1}{2}L_l^*[X, L_l] + \frac{1}{2}[L_l^*, X]L_l + i[X, H]. \end{aligned}$$

In particular,  $\mathcal{L}_{00}$  takes the generic form of a Lindblad generator.

The *output processes* are then defined to be

$$B_j^{\text{out}}(t) \triangleq V_G(t)^*[I \otimes B_j(t)]V_G(t).$$

Again using the quantum Itô rules, we see that

$$dB_k^{\text{out}} = j_t^G(S_{kl})dB_l(t) + j_t^G(L_k)dt.$$

The input-output relations for the column vector  $B^{\text{out}} = [B_j^{\text{out}}]$  can be written as a *Galilean transformation*,

$$\begin{bmatrix} dt^{\text{out}} \\ dB^{\text{out}}(t) \end{bmatrix} = j_t^G(M) \begin{bmatrix} dt \\ dB(t) \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ L & S \end{bmatrix}.$$

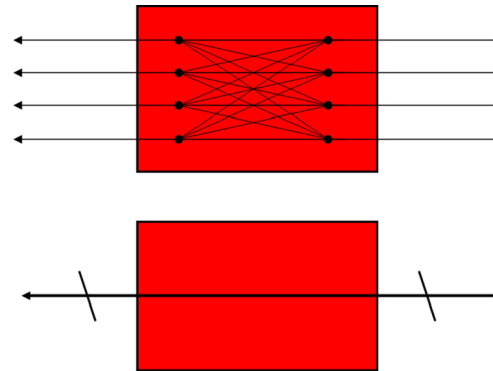


FIG. 1. (Color online) A component representing a quantum mechanical system driven by several input fields. There will be the same number of output fields. It is often convenient to think of grouped inputs with multiplicity greater than one, as in the lower figure.

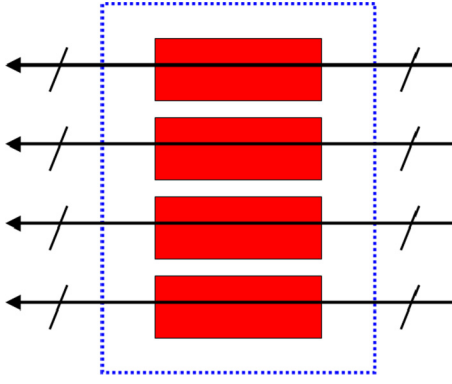


FIG. 2. (Color online) Several SLH models run in parallel: They may be collected into one single SLH model.

### A. Networks

The rules for construction arbitrary network architectures were derived in Ref. [1].

#### 1. Parallel sum rule

If we have several quantum Markov models with independent inputs, then they may be assembled into a single SLH model (see Fig. 2),

$$\boxplus_{j=1}^n (S_j, L_j, H_j) = \left( \begin{bmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, H_1 + \dots + H_n \right).$$

Note that the components need not be distinct—that is, observables associated with one component are not assumed to commute with those of others. In this case the definition is not quite so trivial as it may first appear.

#### 2. Feedback reduction rule

If we wish to feedback an output back in as an input, we obtain a reduced model, as depicted in Fig. 3.

The feedback reduction yields the model matrix [1]

$$[\mathcal{F}_{(r,s)}(\mathbf{V}, T)]_{\alpha\beta} = V_{\alpha\beta} + V_{\alpha r} T (1 - V_{rs} T)^{-1} V_{s\beta} \quad (3)$$

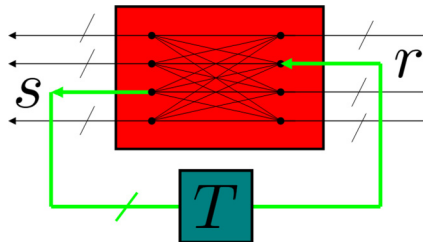


FIG. 3. (Color online) We feed selected outputs back in as inputs to get a reduced model.

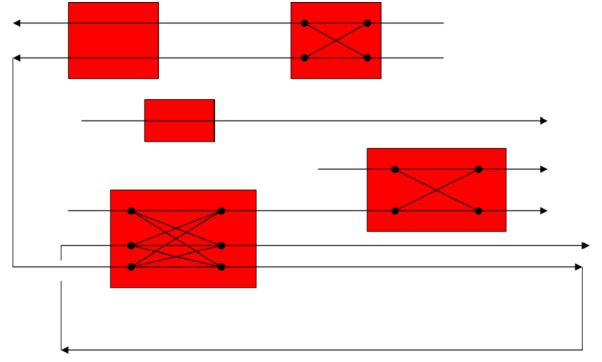


FIG. 4. (Color online) An arbitrary quantum feedback network.

for  $\alpha \neq r$  and  $\beta \neq s$ . We remain in the category of SLH models provided that  $T$  is unitary and the network is well posed, that is,  $1 - V_{rs} T$  is invertible.

### 3. Construction

If, for instance, we wished to determine the effective SLH model for the network shown in Fig. 4, then we would proceed as follows: First of all, we disconnect all the internal lines, and this leaves us with an “open-loop” description where all the components are have independent inputs and outputs. At this stage we use the parallel sum to collect all these components into a single open-loop quantum Markov component. The next step is to make the connections and this involves feeding selected outputs back in as inputs from the open-loop description, and to this end we use the feedback reduction formula. This process has recently been automated using a workflow capture software QHDL [16,17].

### B. Systems in series

The simplest model consists of two systems cascaded together, as shown in Fig. 5, and is equivalent to the single component (see Ref. [2]),

$$\begin{aligned} (S_2, L_2, H_2) \triangleleft (S_1, L_1, H_1) \\ = (S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \text{Im}\{L_2^\dagger S_2 L_1\}). \end{aligned}$$

We refer to  $G = G_2 \triangleleft G_1$  above as the series product of the  $G_1$  and  $G_2$ . It is an associative, but clearly noncommutative, product on the class of suitably composable *SLH* models.

### C. Fermion fields

In the above, we have set out the theory for bosonic field inputs, however, in many applications to quantum transport it would be natural to also consider fermionic fields. We are in the fortunate situation that the quantum stochastic calculus has a fermionic version where we may consider anticommuting fields  $b_k^{\text{in}}(t), b_k^{\text{int}}(t)$ . The theory turns out to be structurally

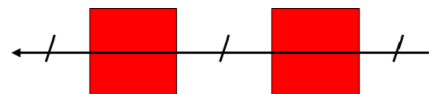


FIG. 5. (Color online) Systems in series.

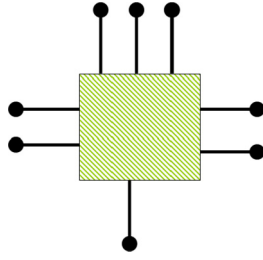


FIG. 6. (Color online) Single component with multiple lead contacts.

identical to the Bose theory provided the  $S$  and  $H$  operators are even parity (commuting with the fields) and the  $L$  operators are odd (anticommuting with the fields). The theory of fermion quantum stochastic calculus is presented in Ref. [23].

### III. QUANTUM TRANSPORT NETWORKS

In quantum transport models we encounter devices which may have several contact points (or leads) that accept quantum field signals. For definiteness, let us label the leads as  $1, 2, \dots, m$  and let  $n_k$  denote the multiplicity of the  $k$ th lead. Our aim is to describe these devices as open quantum Markov models using the SLH formalism, and to develop network rules to describe interconnected quantum transport components. (See Fig. 6.)

The main difference between the quantum transport models and quantum feedback networks is that in the former the fields are bidirectional while in the latter they are unidirectional. This means that we may use the SLH models to describe quantum transport components, but typically have to have both an input and an output field to model each field terminating at a given lead (see Fig. 7). As such, the quantum transport models can be thought of as a special form of the SLH model, and their networks as a restricted class of quantum feedback networks.

A two-lead system is sketched in Fig. 8 (for simplicity we may assume that each lead has multiplicity one, but this readily extends to multiple fields) and we formally identify this as a two-input, two-output port SLH system with

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad H.$$

The usual convention of displaying an SLH model, with all inputs on one side and all outputs on the other, needs to be modified so that we end up with both input 1 and output 1 on one side, and both input 2 and output 2 on the other. The transmission and reflection coefficients are listed in Fig. 9 and we identify the matrix  $S$  with the usual quantum transport

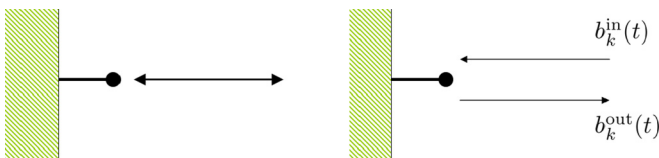


FIG. 7. (Color online) A bidirectional contact may be considered as an equivalent unidirectional input/output pair.

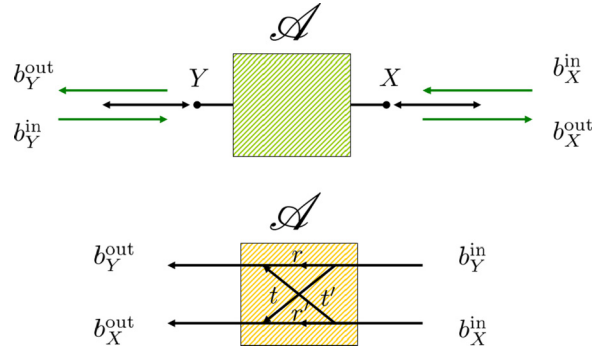


FIG. 8. (Color online) A two-lead quantum transport device is naturally modeled as a two-input, two-output SLH component.

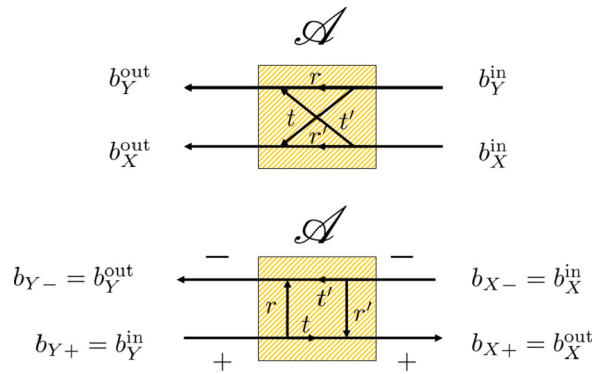


FIG. 9. (Color online) The usual input/output description is modified to have inputs and outputs corresponding to a given lead all appearing grouped on one side.

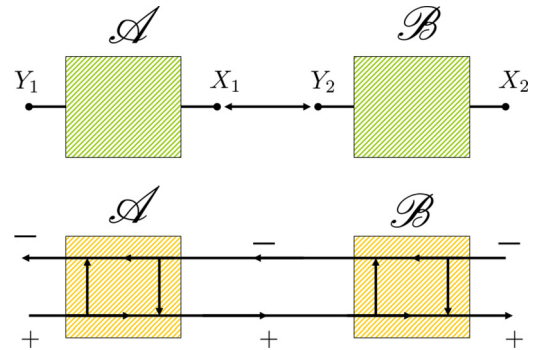


FIG. 10. (Color online) A pair of cascaded quantum transport systems is reinterpreted as a quantum feedback network.

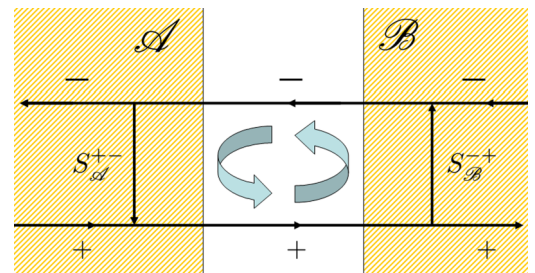


FIG. 11. (Color online) The algebraic loop appearing in the cascaded quantum transport setup in Fig. 10.

scattering matrix as

$$S = \begin{bmatrix} S_{YY} & S_{YX} \\ S_{XY} & S_{XX} \end{bmatrix} \equiv \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}. \quad (4)$$

### Quantum transport components in series

Our first step to build a network is to place two components,  $\mathcal{A}$  and  $\mathcal{B}$ , in series, as shown in Fig. 10. Here we connect the quantum transmission line between two contact leads, as indicated in the upper part. In the SLH framework, we connect up the inputs and outputs, as shown in the lower part of Fig. 10.

Let us take the SLH description of device  $\mathcal{A}$  to be

$$G_{\mathcal{A}} \sim \left( S_{\mathcal{A}} = \begin{bmatrix} S_{\mathcal{A}}^{--+} & S_{\mathcal{A}}^{--} \\ S_{\mathcal{A}}^{++} & S_{\mathcal{A}}^{+-} \end{bmatrix}, L_{\mathcal{A}} = \begin{bmatrix} L_{\mathcal{A}}^- \\ L_{\mathcal{A}}^+ \end{bmatrix}, H_{\mathcal{A}} \right),$$

with a similar convention for  $\mathcal{B}$ . Here the indices  $+$  and  $-$  indicate right and left propagating noise fields.

The situation of two quantum transport systems in series differs dramatically from the series product for unidirectional networks as now we have the presence of an algebraic feedback loop (see Fig. 11).

In particular, we need the feedback reduction formula (3) of Ref. [1] to compute the resulting SLH. The construction is a *Redheffer star product*, and is given by (see Ref. [1])

$$G_{\mathcal{A}\star\mathcal{B}} \sim (S_{\mathcal{A}\star\mathcal{B}}, L_{\mathcal{A}\star\mathcal{B}}, H_{\mathcal{A}\star\mathcal{B}}),$$

where

$$\begin{aligned} S_{\mathcal{A}\star\mathcal{B}} &= \begin{bmatrix} S_{\mathcal{A}}^{--+} + S_{\mathcal{A}}^{--} W_{\mathcal{B}\mathcal{A}\mathcal{B}}^{-+} S_{\mathcal{B}}^{++} & S_{\mathcal{A}}^{--} Z_{\mathcal{B}\mathcal{A}}^{-} S_{\mathcal{B}}^{--} \\ S_{\mathcal{B}}^{++} Z_{\mathcal{A}\mathcal{B}}^{+} S_{\mathcal{A}}^{++} & S_{\mathcal{B}}^{+-} + S_{\mathcal{B}}^{++} W_{\mathcal{A}\mathcal{B}\mathcal{A}}^{+-} S_{\mathcal{A}}^{--} \end{bmatrix}, \\ L_{\mathcal{A}\star\mathcal{B}} &= \begin{bmatrix} L_{\mathcal{A}}^- + S_{\mathcal{A}}^{--} Z_{\mathcal{B}\mathcal{A}}^{-} (L_{\mathcal{B}}^- + S_{\mathcal{B}}^{+-} L_{\mathcal{B}}^+), \\ L_{\mathcal{B}}^+ + S_{\mathcal{B}}^{++} Z_{\mathcal{A}\mathcal{B}}^{+} (L_{\mathcal{A}}^+ + S_{\mathcal{A}}^{+-} L_{\mathcal{A}}^-) \end{bmatrix}, \\ H_{\mathcal{A}\star\mathcal{B}} &= H_{\mathcal{A}} + H_{\mathcal{B}} + \text{Im} \left\{ [L_{\mathcal{A}}^{+\dagger} + L_{\mathcal{B}}^{+\dagger} S_{\mathcal{B}}^{++}, L_{\mathcal{B}}^{-\dagger} + L_{\mathcal{A}}^{-\dagger} S_{\mathcal{A}}^{--}] \begin{bmatrix} Z_{\mathcal{A}\mathcal{B}}^{+} & W_{\mathcal{A}\mathcal{B}\mathcal{A}}^{+-} \\ W_{\mathcal{B}\mathcal{A}\mathcal{B}}^{-+} & Z_{\mathcal{B}\mathcal{A}}^{-} \end{bmatrix} \begin{bmatrix} L_{\mathcal{A}}^+ \\ L_{\mathcal{B}}^- \end{bmatrix} \right\}, \end{aligned} \quad (5)$$

with the following operators arising from the algebraic loop:

$$\begin{aligned} Z_{\mathcal{A}\mathcal{B}}^{+} &= (1 - S_{\mathcal{A}}^{+-} S_{\mathcal{B}}^{-+})^{-1}, \\ Z_{\mathcal{B}\mathcal{A}}^{-} &= (1 - S_{\mathcal{B}}^{-+} S_{\mathcal{A}}^{+-})^{-1}, \\ W_{\mathcal{A}\mathcal{B}\mathcal{A}}^{+-} &= S_{\mathcal{A}}^{+-} Z_{\mathcal{B}\mathcal{A}}^{-} = Z_{\mathcal{A}\mathcal{B}}^{+} S_{\mathcal{A}}^{+-}, \\ W_{\mathcal{B}\mathcal{A}\mathcal{B}}^{-+} &= S_{\mathcal{B}}^{-+} Z_{\mathcal{A}\mathcal{B}}^{+} = Z_{\mathcal{B}\mathcal{A}}^{-} S_{\mathcal{B}}^{-+}. \end{aligned}$$

## IV. QUANTUM LINEAR PASSIVE MARKOV MODELS

It is convenient to assemble the inputs into the following column vectors of length  $n$ ,

$$\mathbf{b}^{\text{in}}(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}.$$

The input-output relations may then be written more compactly as  $\mathbf{b}^{\text{out}}(t) = j_t(S) \mathbf{b}^{\text{in}}(t) + j_t(L)$ .

We now specialize to a linear model of a quantum mechanical system consisting of a family of harmonic oscillators  $\{a_j : j = 1, \dots, m\}$  with canonical commutation relations  $[a_j, a_k] = 0 = [a_j^{\dagger}, a_k^{\dagger}]$  and  $[a_j, a_k^{\dagger}] = \delta_{jk}$ . We collect into column vectors:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}. \quad (6)$$

Our interest is in the general linear open dynamical system and this corresponds to the following situation: (1) The  $S_{jk}$  are scalars. (2) The  $L_j$ 's are linear, i.e., there exist constants  $c_{jk}$

such that  $L_j \equiv \sum_k c_{jk} a_k$ . (3)  $H$  is quadratic, i.e., there exist constants  $\omega_{jk}$  such that  $H = \sum_{jk} a_j^{\dagger} \omega_{jk} a_k$ .

The complex damping is  $\frac{1}{2} L^{\dagger} L + iH = -\mathbf{a}^{\dagger} \mathbf{A} \mathbf{a}$ , where

$$\mathbf{A} = -\frac{1}{2} C^{\dagger} C - i\Omega, \quad (7)$$

with  $C = (c_{jk})$  and  $\Omega = (\omega_{jk})$ . Note that  $\Omega = \Omega^{\dagger}$  because  $H$  is self-adjoint, hence the real part of  $\mathbf{A}$  is  $-\frac{1}{2} C^{\dagger} C \leq 0$ .

The Heisenberg-Langevin equations for  $\mathbf{a}(t) = j_t(\mathbf{a})$  and the input-output relations then become

$$\dot{\mathbf{a}}(t) = \mathbf{A} \mathbf{a}(t) - C^{\dagger} \mathbf{S} \mathbf{b}(t), \quad \mathbf{b}^{\text{out}}(t) = \mathbf{S} \mathbf{b}(t) + \mathbf{C} \mathbf{a}(t).$$

These linear equations are amenable to Laplace transform techniques [19,20]. We define for  $\text{Re } s > 0$ ,

$$X[s] = \int_0^{\infty} e^{-st} X(t) dt,$$

where  $X$  is now any of our stochastic processes. Note that  $\dot{\mathbf{a}}[s] = s\mathbf{a}[s] - \mathbf{a}$ . We find that

$$\begin{aligned} \mathbf{a}[s] &= -(sI_m - \mathbf{A})^{-1} C^{\dagger} \mathbf{S} \mathbf{b}^{\text{in}}[s] + (sI_m - \mathbf{A})^{-1} \mathbf{a}, \\ \mathbf{b}^{\text{out}}[s] &= \mathbf{S} \mathbf{b}^{\text{in}}[s] + \mathbf{C} \mathbf{a}[s]. \end{aligned}$$

The operator  $\mathbf{a}[s]$  may be eliminated to give

$$\mathbf{b}^{\text{out}}[s] = \Xi(s) \mathbf{b}^{\text{in}}[s] + \xi(s) \mathbf{a}, \quad (8)$$

where the *transfer matrix function* is

$$\Xi(s) \triangleq S - C(sI_m - \mathbf{A})^{-1} C^{\dagger} S, \quad (9)$$

and  $\xi[s] = C(sI_m - \mathbf{A})^{-1}$ .

If we average over the vacuum state of the environment, then we would find that  $\frac{d}{dt}\langle \mathbf{a}(t) \rangle_{\text{vac}} = A\langle \mathbf{a}(t) \rangle_{\text{vac}}$ . The system is said to be internally stable if  $\langle \mathbf{a}(t) \rangle_{\text{vac}} \rightarrow 0$  as  $t \rightarrow \infty$ . This occurs if and only if  $A$  is *Hurwitz*, that is, all its eigenvalues have a negative real part.

As an example, consider a single mode cavity coupling to the input field via  $L = \sqrt{\gamma}a$ , and with Hamiltonian  $H = \omega a^\dagger a$ . This implies  $A = -(\frac{\gamma}{2} + i\omega)$  and  $C = \sqrt{\gamma}$ . If the output picks up an additional phase  $S = e^{i\phi}$ , the corresponding transfer function is then computed to be

$$\Xi_{\text{cavity}}(s) = e^{i\phi} \frac{s + i\omega - \frac{\gamma}{2}}{s + i\omega + \frac{\gamma}{2}}. \quad (10)$$

For a single mode  $a$  with two inputs  $b_1^{\text{in}}$  and  $b_2^{\text{in}}$ , the choice

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \end{bmatrix}, \quad \Omega = \omega_0$$

describes the damped harmonic oscillator with unperturbed Hamiltonian  $H = \omega_0 a^\dagger a$  and coupling operators  $L_1 = \sqrt{\gamma_1}a$  and  $L_2 = \sqrt{\gamma_2}a$  to the respective inputs. The transfer function is then

$$\Xi(s) = \frac{1}{s + \frac{1}{2}(\gamma_1 + \gamma_2) + i\omega_0} \times \begin{bmatrix} s - \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + i\omega_0 & \sqrt{\gamma_1\gamma_2} \\ \sqrt{\gamma_1\gamma_2} & s + \frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_2 + i\omega_0 \end{bmatrix}. \quad (11)$$

The models are therefore determined completely by the matrices  $(S, C, \Omega)$  with  $S \in \mathbb{C}^{n \times n}$ ,  $C \in \mathbb{C}^{n \times m}$ , and  $\Omega \in \mathbb{C}^{m \times m}$ , which of course give the SLH coefficients. We shall use the convention  $[\frac{A}{C} | \frac{B}{D}](s) = D + C(s - A)^{-1}B$  for matrices  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{m \times n}$ ,  $C \in \mathbb{C}^{n \times m}$ , and  $D \in \mathbb{C}^{n \times n}$ , and write the transfer matrix function as

$$\Xi(s) = \left[ \frac{A}{C} \middle| \frac{-C^\dagger S}{S} \right](s), \quad (12)$$

where  $A = -\frac{1}{2}C^\dagger C - i\Omega$ . We note the decomposition

$$\Xi = [I_n - C(sI_m - A)^{-1}C^\dagger]S \equiv \left[ \frac{A}{C} \middle| \frac{-C^\dagger}{I_n} \right] S.$$

*Lemma 1.* (All-pass representation of  $\Xi$ .) We may write the transfer function  $\Xi$  for a passive linear quantum system as

$$\Xi(s) = \frac{1 - \frac{1}{2}\Sigma(s)}{1 + \frac{1}{2}\Sigma(s)} S, \quad (13)$$

where

$$\Sigma(s) = C \frac{1}{s + i\Omega} C^\dagger. \quad (14)$$

*Proof.* From the Woodbury matrix identity we find

$$\begin{aligned} & \frac{1}{s + \frac{1}{2}C^\dagger C + i\Omega} \\ &= \frac{1}{s + i\Omega} - \frac{1}{2} \frac{1}{s + i\Omega} C^\dagger \frac{1}{1 + \frac{1}{2}C \frac{1}{s + i\Omega} C^\dagger} C \frac{1}{s + i\Omega}, \end{aligned}$$

which we substitute into (9) to get the result.  $\blacksquare$

*Theorem 1.* The transfer function of a passive system is inner, that is,  $\Xi(i\omega)$  is unitary for all real  $\omega$  not an eigenvalue of  $\Omega$ .

*Proof.* For  $\omega$  not an eigenvalue of  $\Omega$ ,  $\Sigma(i\omega) = -iC \frac{1}{\omega + i\Omega} C^\dagger$  is well defined and we have  $\Sigma(i\omega)^\dagger = -\Sigma(i\omega)$ , so that unitarity of  $\Xi(i\omega)$  follows from (13).  $\blacksquare$

Transfer functions that are inner are otherwise referred to as *all-pass* transfer functions as classically this means that harmonic signals of arbitrary frequency pass through without attenuation. In the current context it relates the fact that the output processes are again canonical field processes.

### A. Chain-scattering representation

We now consider a linear transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = K \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} K_{12} & K_{11} \\ K_{22} & K_{21} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (15)$$

where  $u_1, u_2, z_1, z_2$  are all column vectors of equal length. Our aim is to rewrite this in the form

$$\begin{bmatrix} z_1 \\ u_1 \end{bmatrix} = \text{CHAIN}(K) \begin{bmatrix} u_2 \\ z_2 \end{bmatrix}, \quad (16)$$

which is possible if  $K_{21}$  is invertible, in which case we have

$$\text{CHAIN}(K) \triangleq \begin{bmatrix} K_{12} - K_{11}K_{21}^{-1}K_{22} & K_{12}K_{21}^{-1} \\ -K_{21}^{-1}K_{22} & K_{21}^{-1} \end{bmatrix}, \quad (17)$$

with inverse transformation

$$\text{CHAIN}^{-1}(M) \triangleq \begin{bmatrix} M_{12}M_{22}^{-1} & M_{11} - M_{12}M_{22}^{-1}M_{21} \\ M_{22}^{-1} & M_{22}^{-1}M_{21} \end{bmatrix}.$$

The linear system (15) is an input-output representation, while the system (16) is called the chain-scattering representation. In the former we have a state-based model where the system is driven by the inputs  $u_1, u_2$  and produces the outputs  $z_1, z_2$ , while in the latter the system is a wave scatterer from the wave  $u_2, z_2$  at port 2 to the wave  $z_1, u_1$  at port 1.

In the case where  $K$  is unitary, we have that

$$|u_1|^2 + |u_2|^2 = |z_1|^2 + |z_2|^2,$$

and rearranging gives

$$|u_1|^2 - |z_1|^2 = |z_2|^2 - |u_2|^2.$$

This suggests that if  $K$  implements a unitary transformation for field inputs, then  $\text{CHAIN}(K)$  implements a Bogoliubov transformation. We shall establish this fact next.

### Invariance symmetries of chain pairs

*Definition.* The  $\flat$ -conjugation is defined on matrices of dimension  $2n$  by

$$X^\flat = J_n X^\dagger J_n,$$

where

$$J_n = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}.$$

We say that  $X$  is a  $\flat$ -isometry,  $\flat$ -coisometry, if we have  $X^\flat X = I_{2n}$ ,  $XX^\flat = I_{2n}$ , respectively. If  $X$  is both a  $\flat$ -isometry and a  $\flat$ -coisometry, then we say that it is a  $\flat$ -unitary.

We now state the main structural properties of the chain-scattering transformation.

*Theorem 2.* A matrix  $K$  is an isometry, coisometry, unitary if and only if  $M = \text{CHAIN}(K)$  is a  $\mathfrak{b}$ -isometry,  $\mathfrak{b}$ -coisometry,  $\mathfrak{b}$ -unitary, respectively

The proof is somewhat cumbersome and not very enlightening, so we relegate it to the Appendix.

### B. Wave scattering in quantum transport

It is convenient to relabel the input and output fields (and their Laplace transforms) appearing in the upper picture in Fig. 9 as

$$b_{X+} = b_X^{\text{out}}, \quad b_{X-} = b_X^{\text{in}}, \quad b_{Y+} = b_Y^{\text{in}}, \quad b_{Y-} = b_Y^{\text{out}},$$

where the subscripts  $+$  and  $-$  now indicate right and left propagating fields, respectively. The relation between these fields is then

$$\begin{bmatrix} b_{Y-} \\ b_{X+} \end{bmatrix} = \Xi \begin{bmatrix} b_{Y+} \\ b_{X-} \end{bmatrix} \equiv \begin{bmatrix} \Xi_{YY}^{+-} & \Xi_{YX}^{--} \\ \Xi_{XY}^{++} & \Xi_{XX}^{+-} \end{bmatrix} \begin{bmatrix} b_{Y+} \\ b_{X-} \end{bmatrix},$$

where we break down the transfer matrix into block form. We now perform the swap described in the lower picture in Fig. 9.

The equations may be rearranged as

$$\begin{bmatrix} b_{Y-} \\ b_{Y+} \end{bmatrix} = \begin{bmatrix} \Gamma_{YX}^{--} & \Gamma_{YX}^{+-} \\ \Gamma_{YX}^{+-} & \Gamma_{YX}^{++} \end{bmatrix} \begin{bmatrix} b_{X-} \\ b_{X+} \end{bmatrix}, \quad (18)$$

or equivalently,

$$\overleftrightarrow{b}_Y = \Gamma_{YX} \overleftrightarrow{b}_X,$$

where we introduce the following shorthand notation for the inputs and outputs at a contact lead  $X$ :

$$\overleftrightarrow{b}_X \triangleq \begin{bmatrix} b_{X-} \\ b_{X+} \end{bmatrix}. \quad (19)$$

It immediately follows that

$$\Gamma_{YX}(s) \equiv \text{CHAIN}(\Xi(s)), \quad (20)$$

that is,

$$\Gamma_{XY} = \begin{bmatrix} \Xi_{YX}^{--} - \Xi_{YY}^{+-}(\Xi_{XY}^{++})^{-1}\Xi_{XX}^{+-} & \Xi_{YY}^{+-}(\Xi_{XY}^{++})^{-1} \\ -(\Xi_{XY}^{++})^{-1}\Xi_{XX}^{+-} & (\Xi_{XY}^{++})^{-1} \end{bmatrix}.$$

Inversely, we have

$$\Xi = \begin{bmatrix} \Gamma_{YX}^{+-}(\Gamma_{YX}^{++})^{-1} & \Gamma_{YX}^{--} - \Gamma_{YX}^{+-}(\Gamma_{YX}^{++})^{-1}\Gamma_{YX}^{+-} \\ (\Gamma_{YX}^{++})^{-1} & -(\Gamma_{YX}^{++})^{-1}\Gamma_{YX}^{+-} \end{bmatrix}.$$

*Theorem 3.* The function  $\Gamma_{YX}(s) = \text{CHAIN}(\Xi(s))$  is a  $\mathfrak{b}$ -unitary for  $s$  on the imaginary axis, except for eigenvalues of  $-i\Omega$ .

This is an immediate corollary to Theorems 1 and 2.

Note that the transfer function  $\Gamma_{YX}$  connects the inputs and outputs at contact lead  $X$  to those at  $Y$ . As before,  $\Gamma_{YX}^{--}$  is a Schur complement of the transfer function  $\Xi$  in block matrix form. We shall always suppose that the chain-scattering representation is valid, that is,  $\Xi_{XY}^{++}$  is invertible so that  $\Gamma_{YX}$  is well defined.

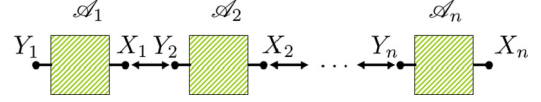


FIG. 12. (Color online) Several quantum transport components in a chain series.

#### 1. Chain scattering for quantum transport devices in series

Let us return to the devices in series shown in the upper diagram in Fig. 10. We have

$$\overleftrightarrow{b}_{Y_1} = \Gamma_{Y_1 X_1} \overleftrightarrow{b}_{X_1}, \quad \overleftrightarrow{b}_{Y_2} = \Gamma_{Y_2 X_2} \overleftrightarrow{b}_{X_2},$$

however, the identification  $b_{Y_2,+} \equiv b_{X_1,+}$  and  $b_{Y_2,-} \equiv b_{X_1,-}$  (i.e.,  $\overleftrightarrow{b}_{Y_2} \equiv \overleftrightarrow{b}_{X_1}$ ) now implies that

$$\overleftrightarrow{b}_{Y_1} = \Gamma_{Y_1 X_1} \Gamma_{Y_2 X_2} \overleftrightarrow{b}_{Y_2}. \quad (21)$$

The general rule is easy to state at this stage. For the chain of components shown in Fig. 12 we have

$$\Gamma_{Y_1 X_n} = \Gamma_{Y_1 X_1} \Gamma_{Y_2 X_2} \cdots \Gamma_{Y_n X_n}. \quad (22)$$

#### 2. Coprime factorizations

We say that transfer function  $\Gamma_{YX}$  has a factorization if we may write it as

$$\Gamma_{YX} = \Upsilon_Y^{-1} \Upsilon_X,$$

and in this way we may write the lead-to-lead equations in a more symmetric form as

$$\Upsilon_X \overleftrightarrow{b}_X = \Upsilon_Y \overleftrightarrow{b}_Y.$$

So far we have not done anything particularly useful, however, we could ask for more properties of the factorization.

Let  $H_\infty$  denote the set of Hardy functions, that is, the class of complex-matrix valued functions  $M(s)$  that are analytic in the closed right hand complex plane ( $\text{Re } s \geq 0$ ) with the property that the limit values  $M(i\omega + 0^+)$  exist for almost all  $\omega \in \mathbb{R}$  and there is a finite upper bound on the largest singular value of  $M(s)$  over  $\text{Re } s \geq 0$ .

A factorization  $\Gamma_{YX} = \Upsilon_Y^{-1} \Upsilon_X$  will be useful for control and design purposes if both  $\Upsilon_Y$  and  $\Upsilon_X$  are rational functions in the Hardy class with the property that they have no common zeros on the closed right hand plane, including  $s = \infty$ . The appropriate definition from control theory is given below (see, for instance, Ref. [24]).

*Definition.* A pair of matrix valued functions  $\Upsilon_X$  and  $\Upsilon_Y$  are left coprime if there exists a pair of rational matrix functions  $Q, P$  in the Hardy class such that

$$\Upsilon_Y P - Q \Upsilon_X = I.$$

A left coprime factorization of rational proper function  $\Gamma_{YX}$  is a factorization  $\Gamma_{YX} = \Upsilon_Y^{-1} \Upsilon_X$ , where  $\Upsilon_X$  and  $\Upsilon_Y$  are left coprime, with  $\Upsilon_Y^{-1}$  proper.

An important property of the chain-scattering representation is that

$$\Gamma_{YX} = \begin{bmatrix} I & -\Xi_{YY}^{+-} \\ 0 & \Xi_{XY}^{++} \end{bmatrix}^{-1} \begin{bmatrix} \Xi_{YX}^{--} & 0 \\ -\Xi_{XX}^{+-} & I \end{bmatrix}, \quad (23)$$



and if the original transfer function  $\Xi$  is stable, this corresponds to a left coprime factorization of  $\Gamma_{YX}$  (see Kimura [18], Sec. 4.1). A right coprime factorization is given by

$$\Gamma_{YX} = \begin{bmatrix} \Xi_{YX}^- & \Xi_{YY}^+ \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \Xi_{XX}^+ & \Xi_{XY}^+ \end{bmatrix}^{-1}.$$

### 3. Worked example

We study a simple device corresponding to a single mode  $a$  with a pair of contact leads  $X$  and  $Y$ , both of which have one input and one output field. Here we assume that the device scatters the input fields as a beam splitter, but also is damped by these inputs, as well as undergoing its own harmonic frequency  $\omega_0$ . In the Heisenberg-Langevin picture we consider the dynamical equations

$$\frac{d}{dt}a(t) = -(\gamma + i\omega_0)a(t) - \sqrt{\gamma}b_X^{\text{in}}(t) - \sqrt{\gamma}b_Y^{\text{in}}(t),$$

$$b_X^{\text{out}}(t) = \frac{1}{\sqrt{2}}b_X^{\text{in}}(t) - \frac{1}{\sqrt{2}}b_Y^{\text{in}}(t) + \sqrt{\gamma}a(t),$$

$$b_Y^{\text{out}}(t) = \frac{1}{\sqrt{2}}b_X^{\text{in}}(t) + \frac{1}{\sqrt{2}}b_Y^{\text{in}}(t) + \sqrt{\gamma}a(t).$$

which correspond to the choice

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} \sqrt{\gamma}a \\ \sqrt{\gamma}a \end{bmatrix}, \quad H = \omega_0 a^\dagger a,$$

that is,  $C = \sqrt{\gamma} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\Omega = \omega_0$ . The transfer function is then

$$\Xi[s] = \frac{1}{\sqrt{2}} \begin{bmatrix} \Theta(s) & -1 \\ \Theta(s) & 1 \end{bmatrix},$$

with  $\Theta(s) = \frac{s-\gamma+i\omega_0}{s+\gamma+i\omega_0}$ . Using the chain transformation (20) we find that

$$\Gamma_{YX} = \begin{bmatrix} -\sqrt{2} & 1 \\ -1/\Theta(s) & \sqrt{2}/\Theta(s) \end{bmatrix},$$

which admits the coprime factorization  $\Gamma_{YX} = \Upsilon_Y^{-1} \Upsilon_X$  with

$$\Upsilon_Y = \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}}\Theta(s) \\ 0 & \frac{1}{\sqrt{2}}\Theta(s) \end{bmatrix}, \quad \Upsilon_X = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

### C. Stability and the lossless property

We have seen from Theorem 1 that the transfer function  $\Xi$  of a linear passive quantum system is inner (unitary almost everywhere on the imaginary axis). In addition, if the system is stable, that is, the matrix  $A$  appearing in the state-based model is Hurwitz, then following control theoretic terminology we say that the system  $\Xi$  is lossless. For lossless systems, we have that

$$\Xi^\dagger(s)\Xi(s) \leq I$$

in the closed right hand complex plane.

Similarly, we say that a chain-scattering transfer function  $\Gamma$  is  $b$ -lossless if

$$\Gamma^b(s)\Gamma(s) \leq I,$$

for all  $\text{Re } s \geq 0$ .

Generally speaking, connecting an assembly of stable components into a network may result in marginal instability. For instance, some marginal stability may arise, which in quantum devices corresponds to a decoherence free subspace, which may be of importance in designing quantum memory storage. It is imperative to know when a given system is lossless. Fortunately, the two notions of losslessness above coincide.

*Theorem 4.*  $\Xi$  is  $b$ -lossless if and only if it takes the form  $\Gamma = \text{CHAIN}(\Xi)$ , where  $\Xi$  is lossless.

This is proved as Lemma 4.4 of Kimura's book [18].

### State space realizations

If we have the triple  $(S, C, \Omega)$  of the form

$$S = \begin{bmatrix} S_{YY} & S_{YX} \\ S_{XY} & S_{XX} \end{bmatrix}, \quad C = \begin{bmatrix} C_Y \\ C_X \end{bmatrix},$$

leading to the transfer function

$$\Xi = \left[ \begin{array}{c|cc} A & B_Y & B_X \\ \hline C_Y & S_{YY} & S_{YX} \\ C_X & S_{XY} & S_{XX} \end{array} \right],$$

where  $A = -\frac{1}{2}C_X^\dagger C_X - \frac{1}{2}C_Y^\dagger C_Y - i\Omega$  and  $B_k = -\sum_j C_j^\dagger S_{jk}$ . It follows that

$$\Gamma_{YX} \equiv \left[ \begin{array}{c|cc} A - B_Y S_{XY}^{-1} C_X & B_X - B_Y S_{XY}^{-1} S_{XX} & B_Y S_{XY}^{-1} \\ \hline C_Y - S_{YY} S_{XY}^{-1} C_X & S_{YX} - S_{YY} S_{XY}^{-1} S_{XX} & S_{YY} S_{XY}^{-1} \\ -S_{XY}^{-1} C_X & -S_{XY}^{-1} S_{XX} & S_{XY}^{-1} \end{array} \right]$$

(see Kimura [18], Sec. 4.2).

### D. Feedback and termination

We wish to consider now the effect of terminating a scattering sequence with a terminal component  $\mathcal{T}$  (see Fig. 13). The chain-scattering picture has a corresponding input-output representation where we see that the component  $\Delta$  is in fact

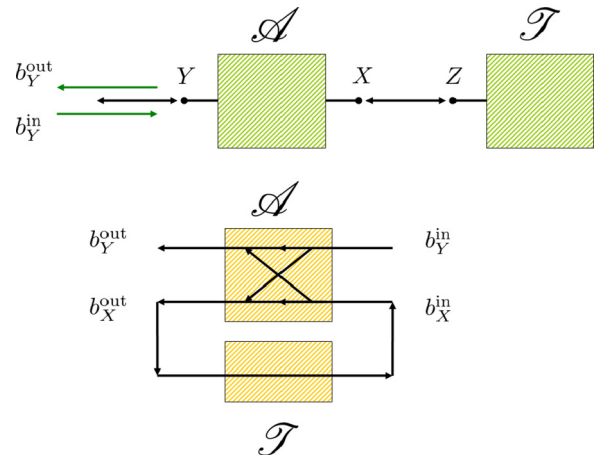


FIG. 13. (Color online) A standard procedure in circuit theory is to terminate a cascade of devices with a terminal load  $\mathcal{T}$ . In the input-output representation, this amounts to a feedback arrangement as shown.

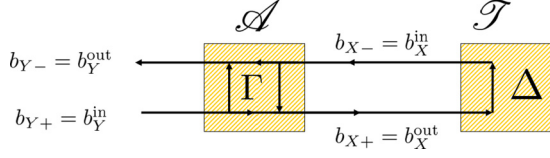


FIG. 14. (Color online) The chain-scattering representation of the terminal load.

in loop. With the identifications

$$b_Y^{\text{out}} = b_{Y-}, \quad b_Y^{\text{in}} = b_{Y+}, \quad (24)$$

and from the relation  $b_{X-}[s] = \Delta(s)b_{X+}[s]$ , we may derive the input-output relation

$$b_{Y-}[s] = \Phi(s)b_{Y+}[s],$$

where  $\Phi$  is a fractional linear transformation (see Ref. [21]),

$$\Phi = \Xi_{YY}^{-+} + \Xi_{YX}^{-} \Delta [I - \Xi_{XX}^{+-} \Delta]^{-1} \Xi_{XY}^{++}. \quad (25)$$

We stress that the setup in Fig. 13 is the special linear dynamical situation of the more general situation appearing in the feedback reduction rule in Fig. 3, where we have the operator theoretic fractional linear transformation (3).

We may obtain a similar expression in terms of the chain-scattering representation  $\Gamma$  (see Fig. 14). Indeed, we have

$$\begin{aligned} b_{Y-} &= (\Gamma_{YX}^{-} \Delta + \Gamma_{YX}^{+-}) b_{X+}, \\ b_{Y+} &= (\Gamma_{YX}^{+-} \Delta + \Gamma_{YX}^{++}) b_{X+}, \end{aligned}$$

and so we deduce that

$$\Phi = (\Gamma_{YX}^{-} \Delta + \Gamma_{YX}^{+-})(\Gamma_{YX}^{+-} \Delta + \Gamma_{YX}^{++})^{-1}. \quad (26)$$

This is equivalent to the homographic transformation from classical circuit theory, and we write  $\Phi \equiv \text{HM}(\Gamma, \Delta)$ , following Kimura [18].

*Theorem 5.* Let  $\Xi$  be a quantum passive transfer functions determined by  $(\begin{bmatrix} S_{YY} & S_{YX} \\ S_{XY} & S_{XX} \end{bmatrix}, [C_X], \Omega)$  and let  $\Delta = \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix}$ , where both representations are minimal [25]. A minimal realization of  $\Phi = \text{HM}(\text{CHAIN}(\Xi), \Delta)$  is given by

$$\Phi = \left[ \frac{A_\Phi}{C_\Phi} \middle| \frac{B_\Phi}{D_\Phi} \right],$$

where

$$\begin{aligned} A_\Phi &= \begin{bmatrix} -\frac{1}{2}C^\dagger C - i\Omega & -C_Y^\dagger C_\Phi \\ 0 & A_\Delta \end{bmatrix} \\ &\quad + E_\Phi (S_{XY} D_\Delta + S_{XX})^{-1} [C_X, S_{XY} C_\Delta], \\ B_\Phi &= E_\Phi (S_{XY} D_\Delta + S_{XX})^{-1}, \\ C_\Phi &= [C_Y - D_\Phi C_X, (S_{YY} - D_\Phi S_{XY}) C_\Delta], \\ D_\Phi &= (S_{YY} D_\Delta + S_{YX})(S_{XY} D_\Delta + S_{XX})^{-1}, \end{aligned}$$

where  $E_\Phi$  is

$$\left[ \begin{array}{c} (C_Y^\dagger S_{YY} + C_X^\dagger S_{XY}) D_\Delta + (C_Y^\dagger S_{YX} + C_X^\dagger S_{XX}) \\ B S_\Delta \end{array} \right],$$

provided that the network is well posed (that is, the operator  $S_{XY} S_\Delta + S_{XX}$  is invertible). Given  $\Xi$  fixed, there will exist

a  $\Delta$  such that the network is internally stable, that is,  $A_\Phi$  is Hurwitz, if and only if  $\Xi$  is lossless.

*Proof.* The state-based representation derives from Eqs. (4.84)–(4.87) of Kimura with the explicit form of a quantum transfer function employed for  $\Xi$ . The internal stability result follows from Theorem 4.15 of Kimura, which states that for  $\mathfrak{b}$ -unitary  $\Gamma$ , there will exist a  $\Delta$  that make  $\Phi = \text{HM}(\Gamma, \Delta)$  internally stable if and only if  $\Gamma$  is  $\mathfrak{b}$ -lossless, along with the observation that  $\Gamma = \text{CHAIN}(\Xi)$  is automatically  $\mathfrak{b}$ -lossless whenever  $\Xi$  is lossless (see Theorem 4). ■

Note that the theorem makes no claim that the stabilizing  $\Delta$  belongs to the class of transfer function corresponding to a (active or passive) quantum transfer function, only that it takes on a state-based model form. In favorable situations, this may be synthesized as another quantum device, however, it may entail using classical components.

### E. Time delays

We now consider the network with delays in the transmission line. In the case of linear circuits, this is easily modeled by the transfer function

$$\theta(s) = e^{-s\tau},$$

where  $\tau$  is the time delay in the transmission line. A simple network of two connected systems with delay is depicted in Fig. 15.

In this case the formula (21) for the cascade of quantum transport models takes the form

$$\overleftrightarrow{b}_{Y_1} = \Gamma_{Y_1 X_1} \Theta \Gamma_{Y_2 X_2} \overleftrightarrow{b}_{X_2},$$

where

$$\Theta = \begin{bmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{bmatrix}.$$

#### Trapped mode

In the special case where we have only scattering, the transfer function takes the form

$$\begin{aligned} \Xi^{-+} &= r_{\mathcal{A}} + \theta^2 t'_{\mathcal{A}} t_{\mathcal{A} \mathcal{B}} \frac{1}{1 - \theta^2 r_{\mathcal{A}} r'_{\mathcal{A}}}, \\ \Xi^{--} &= \theta t'_{\mathcal{A}} t'_{\mathcal{B}} \frac{1}{1 - \theta^2 r_{\mathcal{A}} r'_{\mathcal{A}}}, \\ \Xi^{++} &= \theta t_{\mathcal{A}} t_{\mathcal{B}} \frac{1}{1 - \theta^2 r_{\mathcal{A}} r'_{\mathcal{A}}}, \\ \Xi^{+-} &= r'_{\mathcal{B}} + \theta t_{\mathcal{B}} t'_{\mathcal{B}} r'_{\mathcal{A}} \frac{1}{1 - \theta^2 r_{\mathcal{A}} r'_{\mathcal{A}}}. \end{aligned}$$

If we suppose that the transmittivity is weak with both  $|t_{\mathcal{A}}|^2$  and  $|t_{\mathcal{B}}|^2$  of order  $\tau$ , then we may obtain a well-defined limit

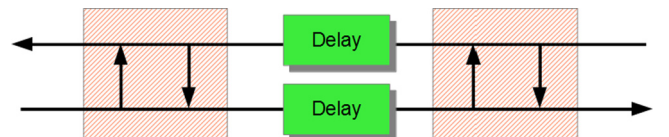


FIG. 15. (Color online) Cascaded system with delay.

for small delay  $\tau$ . In particular, we set

$$S_{\mathcal{A}} = \begin{bmatrix} \sqrt{1-2\gamma_{\mathcal{A}}\tau} & -\sqrt{2\gamma_{\mathcal{A}}\tau} \\ \sqrt{2\gamma_{\mathcal{A}}\tau} & \sqrt{1-2\gamma_{\mathcal{A}}\tau} \end{bmatrix},$$

$$S_{\mathcal{B}} = \begin{bmatrix} \sqrt{1-2\gamma_{\mathcal{B}}\tau} & -\sqrt{2\gamma_{\mathcal{B}}\tau} \\ \sqrt{2\gamma_{\mathcal{B}}\tau} & \sqrt{1-2\gamma_{\mathcal{B}}\tau} \end{bmatrix},$$

where  $\gamma_{\mathcal{A}}$  and  $\gamma_{\mathcal{B}}$  are positive constants. The limit  $\tau \rightarrow 0$  leads to

$$\lim_{\tau \rightarrow 0} \Xi(s, \tau) = \frac{1}{s + \frac{1}{2}(\gamma_{\mathcal{A}} + \gamma_{\mathcal{B}})}$$

$$\times \begin{bmatrix} s - \frac{1}{2}\gamma_{\mathcal{A}} + \frac{1}{2}\gamma_{\mathcal{B}} & \sqrt{\gamma_{\mathcal{A}}\gamma_{\mathcal{B}}} \\ \sqrt{\gamma_{\mathcal{A}}\gamma_{\mathcal{B}}} & s + \frac{1}{2}\gamma_{\mathcal{A}} - \frac{1}{2}\gamma_{\mathcal{B}} \end{bmatrix},$$

which is the transfer function of a single mode with  $\omega_0 = 0$  and two inputs damping with rates  $\gamma_{\mathcal{A}}$  and  $\gamma_{\mathcal{B}}$  [compare with (11)]. The limit corresponds to an effective trapped mode associated with the algebraic loop (see, for instance, Refs. [22,26]).

## V. NONLINEAR ELEMENTS

In our final example, we consider a nonlinear element in a quantum transport network. Our example will consist of a quantum dot, modeled as a qubit system, acting as the terminal load of a network, as shown in Fig. 16.

$$\mathbf{V} = \begin{bmatrix} K_0 & -(\sqrt{\gamma_+ r} + \sqrt{\gamma_- t})a^\dagger & -(\sqrt{\gamma_+ t'} + \sqrt{\gamma_- r'})a^\dagger & -e^{i\phi}\sqrt{\kappa}\sigma^\dagger \\ \sqrt{\gamma_+}a & r & t' & 0 \\ \sqrt{\gamma_-}a & t & r' & 0 \\ \sqrt{\kappa}\sigma & 0 & 0 & e^{i\phi} \end{bmatrix},$$

with  $K_0 = -\frac{1}{2}(\gamma_+ + \gamma_-)a^\dagger a - \frac{1}{2}\kappa\sigma^\dagger\sigma - i\omega_0 a^\dagger a - i\omega'\sigma^\dagger\sigma$ .

We now need to specify the connections we need to make: These are the pairs  $(s, r)$  consisting of an output source  $s$  and an input range  $r$  label, and to wire up the open-loop system (Fig. 17) these are (1,3) and (3,2). The corresponding adjacency matrix is

$$\eta = \begin{bmatrix} \eta_{12} & \eta_{13} \\ \eta_{32} & \eta_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where  $\eta_{sr} = 1$  if  $(s, r)$  is a connection, and  $=0$  otherwise. (The adjacency matrix ranges over the indices  $s, r$  labeling the outputs and inputs, respectively, that are to be connected to form the feedback network.)

The feedback reduction formula (3) now gives the closed-loop network as

$$\mathcal{F}(\mathbf{V}, \eta^{-1}) = \begin{bmatrix} K_0 & -(\sqrt{\gamma_+ r} + \sqrt{\gamma_- t})a^\dagger \\ \sqrt{\kappa}\sigma & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -(\sqrt{\gamma_+ t'} + \sqrt{\gamma_- r'})a^\dagger & -e^{i\phi}\sqrt{\kappa}\sigma^\dagger \\ 0 & e^{i\phi} \end{bmatrix}$$

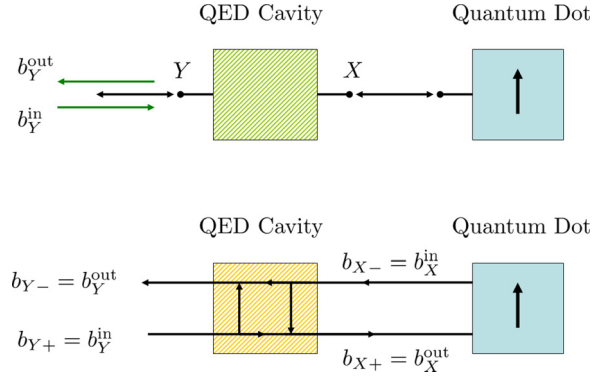


FIG. 16. (Color online) A cavity QED component is connected to a quantum dot (qubit system) so that the qubit is a nonlinear terminal load element. The chain-scattering representation is sketched underneath.

We take the SLH coefficients for the two-lead component device (a cavity QED mode  $a$ ) and the quantum dot to be respectively

$$G_{\text{QED}} \sim \left( \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}, \begin{bmatrix} \sqrt{\gamma_+}a \\ \sqrt{\gamma_-}a \end{bmatrix}, \omega_0 a \right)$$

and

$$G_{\text{QD}} \sim (e^{i\phi}, \sqrt{\kappa}\sigma, \omega'\sigma_z),$$

where  $\sigma$  is the lowering operator for the quantum dot qubit.

Following the network rules, we first form the parallel sum  $G_{\text{QED}} \boxplus G_{\text{QD}}$  which has the model matrix

$$\times \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} t' & 0 \\ r' & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \sqrt{\gamma_+}a & r \\ \sqrt{\gamma_-}a & t \end{bmatrix}$$

$$\equiv \begin{bmatrix} K & -L^\dagger S \\ L & S \end{bmatrix},$$

with  $K = -\frac{1}{2}L^\dagger L - iH$ . After a little algebra we find the equivalent SLH model to be

$$S = e^{i\phi} \left( r + \frac{tt'}{1-r'} \right), \quad (27)$$

$$L = e^{i\phi} \left( \sqrt{\gamma_+} + \frac{t'}{1-r'}\sqrt{\gamma_-} \right) a + \sqrt{\kappa}\sigma,$$

$$K = -\left( \frac{\gamma_+ + \gamma_-}{2} + \sqrt{\gamma_-} \frac{(\sqrt{\gamma_+ t'} + \sqrt{\gamma_- r'})}{1-r'} + i\omega_0 \right) a^\dagger a$$

$$- \left( \frac{1}{2}\kappa + i\omega' \right) \sigma^\dagger \sigma - e^{i\phi} \sqrt{\kappa} \left( \sqrt{\gamma_+} + \frac{t'}{1-r'}\sqrt{\gamma_-} \right) \sigma^\dagger a, \quad (28)$$

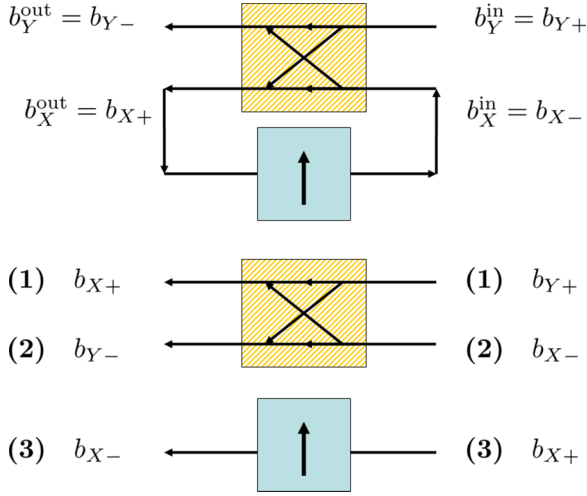


FIG. 17. (Color online) The equivalent quantum feedback network for the cavity-qubit system. Below it is the open-loop setup before the feedback connections are made—note that the outputs and input are labeled so that output 1 goes to input 3, output 3 goes to input 2, while output 2 and input 1 are the external fields that remain after the feedback reduction.

with the Hamiltonian  $H$  determined as the solution to

$$-\frac{1}{2}L^\dagger L - iH = K_0 - \sqrt{\gamma^-}(\sqrt{\gamma^+}t' + \sqrt{\gamma^-}r')\frac{1}{1-r'}a^\dagger a - e^{i\phi}\sqrt{\kappa}\left(\sqrt{\gamma^+} + \frac{t'}{1-r'}\sqrt{\gamma^-}\right)\sigma^\dagger a.$$

The input-output relation is then (with  $b_{Y+}^{\text{in}} = b_{Y+}$  and  $b_{Y-}^{\text{out}} = b_{Y-}$ )

$$b_{Y-}^{\text{out}}(t) = e^{i\phi}\left(r + \frac{tt'}{1-r'}\right)b_{Y+}^{\text{in}}(t) + e^{i\phi}\left(\sqrt{\gamma^+} + \frac{t'}{1-r'}\sqrt{\gamma^-}\right)j_t(a) + \sqrt{\kappa}j_t(\sigma),$$

where  $j_t(a)$  and  $j_t(\sigma)$  are the Heisenberg picture values of the operators  $a$  and  $\sigma$ . The master equation for joint density states

$\rho$  of the QED cavity and qubit quantum dot are therefore

$$\frac{d}{dt}\rho = \frac{1}{2}L\rho L^\dagger - \rho K^\dagger - K\rho,$$

with  $L$  and  $K$  given by (27) and (28), respectively.

## VI. CONCLUSION

We have started the program of developing a systematic network theory underlying interconnections of quantum transport components in the direction that has proved successful so far for quantum photonic networks. The existing quantum feedback network is shown to be capable of describing a large class of nonlinear quantum transport components assembled into a network and is applicable to modeling control design, especially as there is a growing interest in on-chip networks for solid state quantum networks, and hybrid quantum transport-photonic circuits. We did not consider applications to control in this paper *per se*, but it is clear that many of the techniques currently used in quantum feedback control for photonic networks are immediately applicable to this domain.

## ACKNOWLEDGMENTS

The author wishes to thank EPSRC for funding under the grant EP/L006111/1 Quantum Stochastic Analysis For Nanophotonic Circuit Design. He has also the pleasant duty to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the program Quantum Control Engineering where work on this paper was completed. He also acknowledges fruitful discussion with Clive Emery on feedback control of quantum transport systems.

## APPENDIX: PROOF OF THEOREM 2

The isometry condition  $K^\dagger K = I_{2n}$  implies the identities

$$K_{1i}^\dagger K_{1j} + K_{2i}^\dagger K_{2j} = \delta_{ij} I_n.$$

Now

$$M^b = \begin{bmatrix} M_{12}^\dagger & -M_{22}^\dagger \\ -M_{11}^\dagger & M_{21}^\dagger \end{bmatrix}.$$

We establish the  $b$ -isometry property of  $M$  (we collect together in square brackets the various terms where we use the isometric property of  $K$ ):

$$\begin{aligned} [M^b M]_{11} &= M_{12}^\dagger M_{11} - M_{22}^\dagger M_{21} = (K_{12}^\dagger - K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger)(K_{11} K_{21}^{-1}) - (-K_{22}^\dagger K_{21}^{\dagger-1})(K_{21}^{-1}) \\ &= [K_{12}^\dagger K_{11}] K_{21}^{-1} - K_{22}^\dagger K_{21}^{\dagger-1} [K_{11}^\dagger K_{11}] K_{21}^{-1} + K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} \\ &= (-K_{22}^\dagger K_{21}) K_{21}^{-1} - K_{22}^\dagger K_{21}^{\dagger-1} (I_{n_2} - K_{21}^\dagger K_{21}) K_{21}^{-1} + K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} = 0, \\ [M^b M]_{12} &= M_{12}^\dagger M_{12} - M_{22}^\dagger M_{22} = (K_{12}^\dagger - K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger)(K_{12} - K_{11} K_{21}^{-1} K_{22}) - (K_{22}^\dagger K_{21}^{\dagger-1}) K_{21}^{-1} K_{22} \\ &= K_{12}^\dagger K_{12} - [K_{12}^\dagger K_{11}] K_{21}^{-1} K_{22} - K_{22}^\dagger K_{21}^{\dagger-1} [K_{11}^\dagger K_{12}] + K_{22}^\dagger K_{21}^{\dagger-1} [K_{11}^\dagger K_{11}] K_{21}^{-1} K_{22} - K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} K_{22} \\ &= K_{12}^\dagger K_{12} + (K_{22}^\dagger K_{21}) K_{21}^{-1} K_{22} + K_{22}^\dagger K_{21}^{\dagger-1} (K_{21}^\dagger K_{22}) + K_{22}^\dagger K_{21}^{\dagger-1} (I_{n_2} - K_{21}^\dagger K_{21}) K_{21}^{-1} K_{22} - K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} K_{22} \\ &= K_{12}^\dagger K_{12} + K_{22}^\dagger K_{22} = I_{n_1}, \end{aligned}$$

$$\begin{aligned}
[M^b M]_{22} &= -M_{11}^\dagger M_{12} + M_{21}^\dagger M_{22} = -(K_{21}^{\dagger-1} K_{11}^\dagger)(K_{12} - K_{11} K_{21}^{-1} K_{22}) + K_{21}^{\dagger-1} (-K_{21}^{-1} K_{22}) \\
&= -K_{21}^{\dagger-1} [K_{11}^\dagger K_{12}] + K_{21}^{\dagger-1} [K_{11}^\dagger K_{11}] K_{21}^{-1} K_{22} - K_{21}^{\dagger-1} K_{21}^{-1} K_{22} \\
&= K_{21}^{\dagger-1} K_{21}^\dagger K_{22} + K_{21}^{\dagger-1} [I_{n_2} - K_{21}^\dagger K_{21}] K_{21}^{-1} K_{22} - K_{21}^{\dagger-1} K_{21}^{-1} K_{22} = 0,
\end{aligned}$$

and

$$\begin{aligned}
[M^b M]_{21} &= -M_{11}^\dagger M_{11} + M_{21}^\dagger M_{21} \\
&= -(K_{21}^{\dagger-1} K_{11}^\dagger)(K_{11} K_{21}^{-1}) + K_{21}^{\dagger-1} K_{21}^{-1} \\
&= -K_{21}^{\dagger-1} [K_{11}^\dagger K_{11}] K_{21}^{-1} + K_{21}^{\dagger-1} K_{21}^{-1} \\
&= -K_{21}^{\dagger-1} (I_{n_2} - K_{21}^\dagger K_{21}) K_{21}^{-1} + K_{21}^{\dagger-1} K_{21}^{-1} \\
&= I_{n_2}.
\end{aligned}$$

Therefore  $M^b M = I_n$  as required. The demonstration that the coisometry of  $K$  implies the  $b$ -coisometry of  $M$  is similar.

We now establish the “only if” part of the theorem. We note that the  $b$ -isometry implies that

$$\begin{aligned}
M_{12}^\dagger M_{12} - M_{22}^\dagger M_{22} &= I_{n_1}, \quad M_{21}^\dagger M_{21} - M_{11}^\dagger M_{11} = I_{n_2}, \\
M_{12}^\dagger M_{11} - M_{22}^\dagger M_{21} &= 0, \quad M_{21}^\dagger M_{22} - M_{11}^\dagger M_{12} = 0,
\end{aligned}$$

and these imply respectively the following identities:

$$\begin{aligned}
I_{n_1} &= K_{12}^\dagger K_{12} - K_{12}^\dagger K_{11} K_{21}^{-1} K_{22} - K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{12} \\
&\quad + K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} K_{22} - K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} K_{22}, \tag{A1}
\end{aligned}$$

$$I_{n_2} = K_{21}^{\dagger-1} K_{21}^{-1} - K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1}, \tag{A2}$$

$$0 = K_{12}^\dagger K_{11} K_{21}^{-1} - K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} + K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1}, \tag{A3}$$

$$\begin{aligned}
0 &= -K_{21}^{\dagger-1} K_{21}^{-1} K_{22} - K_{21}^{\dagger-1} K_{11} K_{12} \\
&\quad + K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} K_{22}. \tag{A4}
\end{aligned}$$

To show the isometry property of  $K$  we note that

$$\begin{aligned}
&K_{12}^\dagger K_{11} K_{21}^{-1} K_{22} \\
&\stackrel{A3}{=} [K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} - K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1}] K_{22} \\
&= K_{22}^\dagger [K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} - K_{21}^{\dagger-1} K_{21}^{-1}] K_{22} \\
&\stackrel{A2}{=} -K_{22}^\dagger K_{22}. \tag{A5}
\end{aligned}$$

Now we can compute the matrix elements:

$$\begin{aligned}
&K_{12}^\dagger K_{12} + K_{22}^\dagger K_{22} \\
&\stackrel{A1}{=} I_{n_1} + K_{12}^\dagger K_{11} K_{21}^{-1} K_{22} + K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{12} + K_{22}^\dagger K_{22} \\
&\quad - K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} M_{22} + K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} K_{22} \\
&= I_{n_1} + K_{22}^\dagger [-K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} + K_{21}^{\dagger-1} K_{21}^{-1}] K_{22} \\
&\quad + K_{12}^\dagger K_{11} K_{21}^{-1} K_{22} + (K_{12}^\dagger K_{11} K_{21}^{-1} K_{22})^\dagger + K_{22}^\dagger K_{22} \\
&\stackrel{A2}{=} I_{n_1} + 2K_{22}^\dagger K_{22} + K_{12}^\dagger K_{11} K_{21}^{-1} K_{22} + (K_{12}^\dagger K_{11} K_{21}^{-1} K_{22})^\dagger \\
&\stackrel{A5}{=} I_{n_1},
\end{aligned}$$

$$\begin{aligned}
&K_{12}^\dagger K_{11} + K_{22}^\dagger K_{21} \\
&= (K_{12}^\dagger K_{11} K_{21}^{-1} + K_{22}^\dagger) K_{21} \\
&\stackrel{A3}{=} (K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} - K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1} + K_{22}^\dagger) K_{21} \\
&= K_{22}^\dagger (K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} - K_{21}^{\dagger-1} K_{21}^{-1} + I_{n_2}) K_{21} \\
&\stackrel{A2}{=} 0, \\
&K_{11}^\dagger K_{12} + K_{21}^\dagger K_{22} \\
&= K_{21}^\dagger ((K_{12}^\dagger K_{11} K_{21}^{-1})^\dagger + K_{22}) \\
&\stackrel{A5}{=} K_{21}^\dagger (K_{22}^\dagger K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} - K_{22}^\dagger K_{21}^{\dagger-1} K_{21}^{-1})^\dagger + K_{21}^\dagger K_{22} \\
&= K_{21}^\dagger (K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1} - K_{21}^{\dagger-1} K_{21}^{-1} + I_{n_2}) K_{22} \\
&\stackrel{A2}{=} 0,
\end{aligned}$$

and

$$\begin{aligned}
&K_{11}^\dagger K_{11} + K_{21}^\dagger K_{21} \\
&\stackrel{A2}{=} K_{11}^\dagger K_{11} + K_{21}^\dagger (K_{21}^{\dagger-1} K_{21}^{-1} - K_{21}^{\dagger-1} K_{11}^\dagger K_{11} K_{21}^{-1}) K_{21} \\
&= I_{n_2},
\end{aligned}$$

from where  $K^\dagger K = I_n$ , as required.

- [1] J. E. Gough and M. R. James, *Commun. Math. Phys.* **287**, 1109 (2009).  
[2] J. E. Gough and M. R. James, *IEEE Trans. Autom. Control* **54**, 2530 (2009).  
[3] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, 2000).  
[4] R. L. Hudson and K. R. Parthasarathy, *Commun. Math. Phys.* **93**, 301 (1984).  
[5] S. Datta, *Electronic Transport in Mesoscopic Systems*, Cambridge Studies in Semiconductor Physics and Microelectronic Engineering No. 3 (Cambridge University Press, Cambridge, UK, 1997).

- [6] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, *Nature (London)* **431**, 162 (2004).  
[7] D. Englund, A. Faraon, I. Fushman, N. Stoltz, P. Petroff, and J. Vučković, *Nature (London)* **450**, 857 (2007).  
[8] A. Faraon, I. Fushman, D. Englund, N. Stoltz, P. Petroff, and J. Vučković, *Nat. Phys.* **4**, 859 (2008).  
[9] J. L. O’Brien, A. Furusawa, and J. Vučković, *Nat. Photonics* **3**, 687 (2009).  
[10] J. Vučković, [arXiv:1402.2541](https://arxiv.org/abs/1402.2541).  
[11] T. Brandes, *Phys. Rev. Lett.* **105**, 060602 (2010).  
[12] C. Emary, *Philos. Trans. R. Soc., A* **371**, 1999 (2013).

- [13] C. Pörtl, C. Emary, and T. Brandes, *Phys. Rev. B* **84**, 085302 (2011).
- [14] A. Reinhard, T. Volz, M. Winger, A. Badolato, K. J. Hennessy, E. L. Hu, and A. Imamoglu, *Nat. Photonics* **6**, 93 (2012).
- [15] C. Emary and J. E. Gough, *Phys. Rev. B* **90**, 205436 (2014).
- [16] N. Tezak, A. Niederberger, D. S. Pavlichin, G. Sarma, and H. Mabuchi, *Philos. Trans. R. Soc., A* **320**, 5270 (2012).
- [17] G. Sarma, R. Hamerly, N. Tezak, D. S. Pavlichin, and H. Mabuchi, *IEEE Photonics* **5**, 7500111 (2013).
- [18] H. Kimura, *Chain-Scattering Approach to  $H^\infty$ -Control*, System & Control: Foundations & Applications Series (Birkhäuser, Boston, 1997).
- [19] M. Yanagisawa and H. Kimura, *IEEE Trans. Autom. Control* **48**, 2107 (2003).
- [20] M. Yanagisawa and H. Kimura, *IEEE Trans. Autom. Control* **48**, 2121 (2003).
- [21] J. E. Gough, R. Gohm, and M. Yanagisawa, *Phys. Rev. A* **78**, 062104 (2008).
- [22] J. E. Gough, M. R. James, and H. I. Nurdin, *Phys. Rev. A* **81**, 023804 (2010).
- [23] R. L. Hudson and K. R. Parthasarathy, *Commun. Math. Phys.* **104**, 457 (1986).
- [24] G. E. Dullerud and F. Paganini, *A Course in Robust Control theory: A Convex Approach*, Texts in Applied Mathematics No. 36 (Springer, Berlin, 1999).
- [25] The requirement of minimality means that there are no uncontrollable or unobservable modes of the system: This is a standard concept in linear control theory which may be established in the Kalman state-based theory (see Refs. [18, 24]).
- [26] H. A. Bachor and T. C. Ralph, *A Guide to Experiments in Quantum Optics*, 2nd ed. (Wiley-VCH, Weinheim, 2004).