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Zeno Dynamics for Open Quantum Systems

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Dedicated to the memory of Slava Belavkin, the father of quantum cybernetics.

Abstract

In this paper we formulate limit Zeno dynamics of general open systems as the adiabatic elimination of fast components. We are able to exploit previous work on adiabatic elimination of quantum stochastic models to give explicitly the conditions under which open Zeno dynamics will exist. The open systems formulation is further developed as a framework for Zeno master equations, and Zeno filtering (that is, quantum trajectories based on a limit Zeno dynamical model). We discuss several models from the point of view of quantum control. For the case of linear quantum stochastic systems we present a condition for stability of the asymptotic Zeno dynamics.

1 Introduction

The quantum Zeno effect is the basic principle that repeated or continual external interventions on a quantum dynamical system may lead to a constraining of the system to a sub-dynamics. The original formulation by Misra and Sudarshan [1] dealt with a system initially in an unstable system $|\psi_0\rangle$ which is repeatedly measured to see if it is in this state (a von Neumann projective measurement with orthogonal projection $P_{\mathbf{z}} = |\psi_0\rangle\langle\psi_0|$). At each stage the survival probability $p(t) = |\langle\psi_0|e^{-itH}\psi_0\rangle|^2$ is $1 + O(t^2)$ for small t, so for N measurements at time separations t/N the probability of measuring the same state on all N attempts is $p(t/N)^N$ which converges to 1 as $N \uparrow \infty$. The state is in principle "frozen" as $|\psi_0\rangle$, up to a phase.

More generally the projection $P_{\mathbf{z}}$ may be of rank greater than one, in which case we find that the state is constrained to remain the Zeno subspace

$$\mathfrak{h}_{\text{Zeno}} = P_{\mathbf{z}}\, \mathfrak{h}$$

where $\mathfrak{h}$ is the total Hilbert space of states. Typically, we still have a reduced evolution on the Zeno subspace governed by the Zeno Hamiltonian $H_{\text{Zeno}} =$

$P_{\mathsf{z}} H P_{\mathsf{z}}$. (The unitary $e^{-itP_{\mathsf{z}}HP_{\mathsf{z}}}$ on $\mathfrak{h}_{\mathrm{Zeno}}$ sometimes being referred to as a "non-abelian phase"). One can obtain results of the form

$$\lim_{N\to\infty}\left[P_{\mathsf{z}}e^{-iHt/N}\right]^N = P_{\mathsf{z}}e^{-itP_{\mathsf{z}}HP_{\mathsf{z}}}$$

The intervention does not necessarily have to take the form of a measurement, see the authoritative review by Facchi and Pascazio [2]. An alternative is to apply stroboscopic unitary kicks and take the limit as the time between kicks vanishes. Another possibility is to apply a strong perturbation kV to obtain a result of the form

$$\lim_{k\to\infty} e^{+ikVt}e^{-i(H+kV)t}P_{\mathsf{z}} = P_{\mathsf{z}}e^{-itP_{\mathsf{z}}HP_{\mathsf{z}}}$$

where P_{z} is projection onto the kernel of V. This is a case of the adiabatic theorem.

Despite the fact that quantum Zeno effect is typically described in terms of open systems concepts such as quantum measurement, external interaction or environmental coupling, it is rarely if ever formulated mathematically in these terms. In practice, on never makes direct measurements on quantum mechanical systems, but instead couples them to external quantum fields which are then monitored continually. A complete model would include the probing field [3]-[5]. Similarly, the coupling of the environment requires an open systems model.

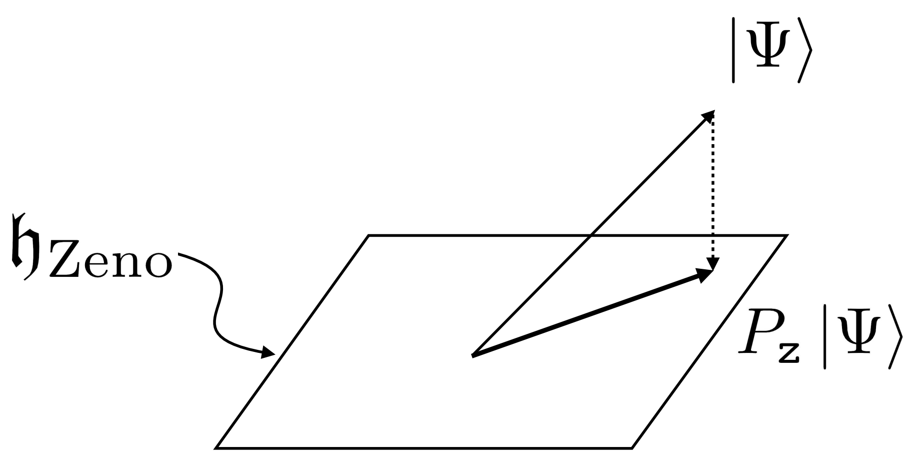

Figure 1: In a Zeno limit we find that a quantum state $|\Psi\rangle$ will have component perpendicular to a Zeno subspace $\mathfrak{h}_{\mathrm{Zeno}}$ which so strongly coupled to the environment that it effectively disappears in real time, leaving the system constrained to the Zeno subspace. The component may be adiabatically eliminated.

In this paper we start in the general framework of an open quantum Markov model for the system undergoing a quantum stochastic evolution in conjunction with a Boson field environment [6, 7, 8]. We shall equate the limit procedure leading to a Zeno dynamics as adiabatic elimination. This choice is a matter of convenience as we at once have access to powerful results due to Luc Bouten and Andrew Silberfarb [9], and later with Ramon van Handel [10], on the adiabatic elimination procedure in the framework of quantum stochastic models. This is a significantly richer theory than for closed systems. In addition we get clear conditions on when a candidate subspace will be a Zeno subspace for a sequence of models. We can the readily develop the theory of how we actually probe the Zeno sub-dynamics by for instance quantum trajectories methods for the constrained dynamics [11] - [14].

1.1 Open Quantum Systems: The "S, L, H" Formalism

In figure 2 below, we sketch an open quantum Markov system as a black box with a Bosonic input driving field and a Bosonic output. The quantum mechanical system (plant) will have underlying Hilbert space $\mathfrak{h}$ while the input will be a continuous quantum field with Fock space $\mathfrak{F}$. The coupled model will have joint Hilbert space $\mathfrak{h} \otimes \mathfrak{F}$, which is also the space on which the output observables act.

We shall outline below the theory of quantum stochastic evolutions due to Hudson-Parthasarathy. This shows that the unitary dynamics of the combined system and its Bosonic environment are specified by a triple of coefficients

$$\mathbf{G} \sim (S, L, H). \tag{1}$$

We shall in turns refer to $\mathbf{G}$ as the Hudson-Parthasarathy (HP) parameters, the generating coefficients, or just the SLH for the model.

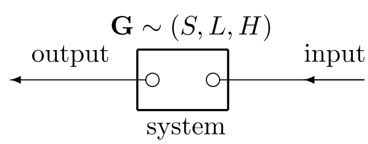

Figure 2: input-system-output component

The input-plant-output model can be summarized as

$$\begin{array}{rcl} \textbf{plant dynamics} & : & j_t(X) = U(t)^* (X \otimes I) U(t); \\ \textbf{output process} & : & B_{\mathrm{out},i}(t) = U(t)^* (I \otimes B_i(t)) U(t). \end{array}$$

where X is an arbitrary plant observable and $B_i(t)$ is the ith input field's annihilator process.

In the following we shall specify to the category of model where $U(\cdot)$ is a unitary family of operators on $\mathfrak{h} \otimes \mathfrak{F}$, satisfying a differential equation of the form [6, 7]

$$dU(t) = \Bigg\{ \sum_{ij} (S_{ij} - \delta_{ij}) \otimes d\Lambda_{ij}(t) + \sum_i L_i \otimes dB_i^*(t)$$

$$- \sum_{ij} L_i^* S_{ij} \otimes dB_j(t) + K \otimes dt \Bigg\} U(t), \quad U(0) = I, \tag{2}$$

Formally, we can introduce input process $b_i(t)$ for $i = 1, \cdots, n$ satisfying singular commutation relations $[b_i(t), b_j(t')^*] = \delta_{ij}\delta(t - t')$, so that the processes appearing in are

$$\begin{aligned} \Lambda_{ij}(t) &\triangleq \int_0^t b_i(t')^* b_j(t') dt', \\ B_i(t)^* &\triangleq \int_0^t b_i(t')^* dt', \quad B_j(t) \triangleq \int_0^t b_j(t') dt'. \end{aligned}$$

More exactly, the are rigorously defined as creation and annihilation field operators on the Boson Fock space $\mathfrak{F}$ over $L^2_{\mathbb{C}^n}(\mathbb{R})$. The increments in are understood to be future pointing in the Ito sense. We have the following table of non-vanishing products

$$\begin{aligned} d\Lambda_{ij} d\Lambda_{kl} &= \delta_{jk} d\Lambda_{il}, \qquad d\Lambda_{ij} dB_k^* = \delta_{jk} dB_i^* \\ dB_i d\Lambda_{kl} &= \delta_{ik} dB_l, \qquad dB_i dB_k^* = \delta_{ij} dt. \end{aligned}$$

Necessary and sufficient conditions for unitarity [6, 7] are that we can collect the coefficients of (2) to form a triple (S, L, H), which we call the *Hudson-Parthasarathy (HP) or SLH parameters*, consisting of a unitary matrix S, a column vector L, and a self-adjoint operator H,

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix},$$

with S_{ij}, L_i, H are all operators on $\mathfrak{h}$, and where

$$K \equiv -\frac{1}{2} \sum_i L_i^* L_i - iH.$$

We shall refer to $U(t)$ as the unitary determined by the parameters (S, L, H). In differential form, the input-plant-output model then becomes [6, 7]

plant dynamical (Heisenberg) equation:

$$\begin{aligned} dj_t(X) &= j_t(\mathscr{L}X)\, dt + \sum_i j_t(\mathscr{M}_i X)\, dB_i^*(t) \\ &\quad + \sum_i j_t(\mathscr{N}_i X)\, dB_i(t) + \sum_{j,k} j_t(\mathscr{S}_{jk} X)\, d\Lambda_{jk}(t); \end{aligned}$$

input-output relations:

$$dB_{\mathrm{out},i}\left(t\right)=j_{t}\left(S_{ik}\right)dB_{k}\left(t\right)+j_{t}\left(L_{i}\right)dt.$$

Here

$$\mathscr{L}X=\frac{1}{2}\sum_{i}L_{i}^{\ast}\left[X,L_{i}\right]+\frac{1}{2}\sum_{i}\left[L_{i}^{\ast},X\right]L_{i}-i\left[X,H\right]$$

is the Lindbladian, $\mathscr{M}_{i}X=S_{ji}^{\ast}[X,L_{j}]$, $\mathscr{N}_{i}X=[L_{k}^{\ast},X]S_{ki}$ and $\mathscr{S}_{ik}X=S_{ji}^{\ast}XS_{jk}$ $-\delta_{ik}X$.

2 Zenofiability

The essential idea is that there exists a subspace $\mathfrak{h}_{\mathrm{Zeno}}$ of $\mathfrak{h}$ to which the system may end up restricted (though entangled with the environment) due to the continuous open system dynamics. We decompose $\mathfrak{h}$ as

$$\mathfrak{h}=\mathfrak{h}_{\mathrm{Zeno}}\oplus\mathfrak{h}_{\mathrm{Fast}} \tag{3}$$

where the subspace $\mathfrak{h}_{\mathrm{Fast}}=\mathfrak{h}\ominus\mathfrak{h}_{\mathrm{Zeno}}$ contains the vector states that are most strongly coupled to the environment so that any drift out of $\mathfrak{h}_{\mathrm{Zeno}}$ will decay to zero almost instantaneously.

To make these concepts mathematically rigorous, we introduce a strength parameter $k>0$ for the coupling and make the generating operators k-dependent according to $\mathbf{G}(k)$ with SLH operators of the form

$$\begin{aligned}S(k) &= S, \quad \text{(independent of } k\text{);}\\ L\left(k\right) &= kL^{(1)}+L^{(0)};\\ H\left(k\right) &= k^{2}H^{(2)}+kH^{(1)}+H^{(0)},\end{aligned} \tag{4}$$

leading to the open unitary dynamics $U(t,k)$.

The singular limit $k\uparrow\infty$ should then separate $\mathfrak{h}_{\mathrm{Fast}}$ out as the fast subspace which may be eliminated to yield a dynamics on the open system over $\mathfrak{h}_{\mathrm{Zeno}}$. At this stage, the Zeno limit can be reformulated explicitly as an adiabatic elimination problem with the Zeno subspace $\mathfrak{h}_{\mathrm{Zeno}}$ being the slow space. At this stage we can invoke the results of Bouten, van Handel and Silberfarb [23],[9],[10] on adiabatic elimination for quantum stochastic systems.

For a given operator X on $\mathfrak{h}$, we write

$$X=\left[\begin{array}{cc}X_{\mathtt{zz}} & X_{\mathtt{zf}}\\ X_{\mathtt{fz}} & X_{\mathtt{ff}}\end{array}\right].$$

More generally we use this notation when X is an array of operators on $\mathfrak{h}$. The projections onto $\mathfrak{h}_{\mathrm{Zeno}}$ and $\mathfrak{h}_{\mathrm{Fast}}$ are denoted respectively by

$$P_{\mathtt{z}}\equiv\left[\begin{array}{cc}1 & 0\\ 0 & 0\end{array}\right],\quad P_{\mathtt{f}}\equiv\left[\begin{array}{cc}0 & 0\\ 0 & 1\end{array}\right].$$

2.1 Conditions

Condition 1 (Scaling) *We assume that the SLH operators in (4) satisfy*

$$L^{(1)}P_{\mathtt{z}}=0,\quad H^{(1)}P_{\mathtt{z}}=0,\quad P_{\mathtt{z}}H^{(1)}=0,\quad P_{\mathtt{z}}H^{(2)}P_{\mathtt{z}}=0. \tag{5}$$

Equivalently stated we require the following forms with respect to the decomposition:

$$\begin{aligned} L^{(1)} &\equiv \left[\begin{array}{cc} 0 & L_{\mathtt{zf}}^{(1)} \\ 0 & L_{\mathtt{ff}}^{(1)} \end{array}\right], \\ H &\equiv \left[\begin{array}{cc} H_{\mathtt{zz}}^{(0)} & H_{\mathtt{zf}}^{(0)}+kH_{\mathtt{zf}}^{(1)} \\ H_{\mathtt{fz}}^{(0)}+kH_{\mathtt{fz}}^{(1)} & H_{\mathtt{ff}}^{(0)}+kH_{\mathtt{ff}}^{(1)}+k^2H_{\mathtt{ff}}^{(2)} \end{array}\right]. \end{aligned}$$

In what follows, it is instructive to write

$$\begin{aligned} S &= \left[\begin{array}{cc} S_{\mathtt{zz}} & S_{\mathtt{zf}} \\ S_{\mathtt{fz}} & S_{\mathtt{ff}} \end{array}\right], \\ L(k) &= k\left[\begin{array}{cc} 0 & L_{\mathtt{zf}}^{(1)} \\ 0 & L_{\mathtt{ff}}^{(1)} \end{array}\right]+\left[\begin{array}{cc} L_{\mathtt{zz}}^{(0)} & * \\ L_{\mathtt{fz}}^{(0)} & * \end{array}\right], \\ H(k) &= k^2\left[\begin{array}{cc} 0 & 0 \\ 0 & H_{\mathtt{ff}}^{(2)} \end{array}\right]+k\left[\begin{array}{cc} 0 & H_{\mathtt{zf}}^{(1)} \\ H_{\mathtt{fz}}^{(1)} & * \end{array}\right]+\left[\begin{array}{cc} H_{\mathtt{zz}}^{(0)} & * \\ * & * \end{array}\right] \end{aligned}$$

where the terms denote as $*$ will not contribute to the $k\uparrow\infty$ limit. We also note that

$$K\left(k\right)=-\frac{1}{2}L(k)^{\ast}L(k)-iH(k)\equiv k^2A+kM+R,$$

where we will have

$$A=\left[\begin{array}{cc} 0 & 0 \\ 0 & A_{\mathtt{ff}} \end{array}\right], M=\left[\begin{array}{cc} 0 & M_{\mathtt{zf}} \\ M_{\mathtt{fz}} & * \end{array}\right], R=\left[\begin{array}{cc} R_{\mathtt{zz}} & * \\ * & * \end{array}\right]$$

with

$$\begin{aligned} A_{\mathtt{ff}} &= -\frac{1}{2}L_{\mathtt{af}}^{(1)\ast}L_{\mathtt{af}}^{(1)}-iH_{\mathtt{ff}}^{(2)} \\ M_{\mathtt{zf}} &= -\frac{1}{2}L_{\mathtt{cz}}^{(0)\ast}L_{\mathtt{cf}}^{(1)}-iH_{\mathtt{zf}}^{(1)} \\ M_{\mathtt{fz}} &= -\frac{1}{2}L_{\mathtt{cf}}^{(1)\ast}L_{\mathtt{cz}}^{(0)}-iH_{\mathtt{fz}}^{(1)} \\ R_{\mathtt{zz}} &= -\frac{1}{2}L_{\mathtt{cf}}^{(0)\ast}L_{\mathtt{cz}}^{(0)}-iH_{\mathtt{zz}}^{(0)} \end{aligned}$$

and we introduce a summation convention where repeated indices imply a summation over the values `z` and `f`.

Condition 2 (Kernel) *We require $\mathfrak{h}_{\mathtt{z}}$ to be the kernel space of A.*

Equivalently, we require that the operator $A_{\mathtt{ff}}$ is invertible on $\mathfrak{h}_{\mathtt{z}}$. Making this assumption we then introduce the operators

$$\hat{S}_{\mathtt{ab}} \triangleq \left(\delta_{\mathtt{ac}} + L^{(1)}_{\mathtt{af}} \frac{1}{A_{\mathtt{ff}}} L^{(1)*}_{\mathtt{cf}}\right) S_{\mathtt{cb}}, \tag{6}$$

$$\hat{L}_{\mathtt{a}} \triangleq L^{(0)}_{\mathtt{az}} - L^{(1)}_{\mathtt{af}} \frac{1}{A_{\mathtt{ff}}} M_{\mathtt{fz}}, \tag{7}$$

$$\hat{H} \triangleq H^{(0)}_{\mathtt{zz}} + \mathrm{Im}\left\{ M_{\mathtt{zf}} \frac{1}{A_{\mathtt{ff}}} M_{\mathtt{fz}} \right\}. \tag{8}$$

With these definitions we can state the final condition.

Condition 3 (Decoupling) *The operators $\hat{S}_{\mathtt{zf}}, \hat{S}_{\mathtt{fz}}$ and $\hat{L}_{\mathtt{f}}$ vanish.*

2.2 Asymptotic Open Zeno Dynamics

We now recall the adiabatic elimination result of Bouten and Silberfarb which establishes convergence to a limit unitary $\hat{U}(t)$ with reduced system space $\mathfrak{h}_{\mathrm{Zeno}}$ and SLH operators

$$\mathbf{G}_{\mathrm{Zeno}} \sim \left(\hat{S}_{\mathtt{zz}}, \hat{L}_{\mathtt{z}}, \hat{H}\right). \tag{9}$$

Theorem 4 (Bouten and Silberfarb 2008 [9]) *Suppose we are given a sequence of bounded operator parameters $(S, L(k), H(k))$ satisfying the scaling, kernel and decoupling conditions. Then the quantum stochastic process $U_k(t) P_{\mathtt{z}}$ converges strongly to $\hat{U}(t) P_{\mathtt{z}}$, that is*

$$\lim_{k \to \infty} \| U_k(t) \psi - U(t) \psi \| = 0$$

for all $\psi \in \mathfrak{h} \otimes \mathfrak{F}$ with $P_{\mathtt{f}} \otimes I_{\mathfrak{F}} \psi = 0$. The convergence is moreover uniform in the time coordinate t for compact time sets.

The restriction to bounded operators was lifted in a subsequent publication [10].

Definition 5 *Let $(S(k), L(k), H(k))$ be a sequence of SLH operators based on a given space $\mathfrak{h}$ having the dependence on a scaling parameter $k > 0$ as in equation (4). A subspace $\mathfrak{h}_{\mathrm{Zeno}}$ is said to be* ***zenofiable*** *if, with respect to the orthogonal decomposition $\mathfrak{h} = \mathfrak{h}_{\mathrm{Zeno}} \oplus \mathfrak{h}_{\mathrm{Fast}}$, the scaling, kernel and decoupling conditions are satisfied. The subspace $\mathfrak{h}_{\mathrm{Zeno}}$ is called the* ***Zeno subspace*** *for the $k \uparrow \infty$ asymptotic dynamics. The operators $\left(\hat{S}_{\mathtt{zz}}, \hat{L}_{\mathtt{z}}, \hat{H}\right)$ are called the* ***Zeno scattering matrix****, the* ***Zeno coupling (or collapse) operators*** *and the* ***Zeno Hamiltonian*** *operators respectively.*

3 Controllability and Observability Issues

The issue of controllability and the Zeno effect has been recently addressed by Burgarth *et al.* [18] who have shown that for a system with controlled Hamiltonian of the form $\sum_{k=1}^{n} H_k \, u_k(t)$, with deterministic control policies u_k and self-adjoint operators H_k, there are some unexpected benefits of constraints. In particular, for the case dim $\mathfrak{h} < \infty$, they show that the degree of controllability of the original system may be smaller than the degree of controllability of the Zeno limit. That is, the dimension of the Lie algebra $\mathfrak{L}_{\mathtt{z}}$ generated by $\{-iP_{\mathtt{z}}H_1P_{\mathtt{z}}, \cdots, -iP_{\mathtt{z}}H_nP_{\mathtt{z}}\}$ may be strictly greater than that of $\mathfrak{L}$ generated by $\{-iH_1, \cdots, -iH_n\}$, despite the fact that the Zeno dynamics is constrained to the smaller subspace.

We should also remark that an experiment to obtain a limiting Zeno dynamical behaviour for a cavity mode through interaction with and repeated measurements of Rydberg atom has been proposed by the Ecole Normale Supérieure group [19].

3.1 Zeno Master Equation

Let us prepare the system in an initial state $\eta \in \mathfrak{h}_{\mathrm{Zeno}}$, then for $t \geq 0$ we have the expectation of an observable X on the Zeno subspace given by

$$\mathbb{E}_t(X) = \langle \eta \otimes \Omega | j_t(X) \, \eta \otimes \Omega \rangle \equiv tr_{\mathtt{z}} \left\{ \rho_{\mathtt{z}} X \right\},$$

which introduces the Zeno density matrix $\rho_{\mathtt{z}}$. (The trace is over the Zeno subspace $\mathfrak{h}_{\mathrm{Zeno}}$.) We obtain the Zeno-Ehrenfest equation

$$\frac{d}{dt}\mathbb{E}_t(X) = \mathbb{E}_t^{\mathrm{vac}}(\mathscr{L}_{\mathrm{Zeno}} X),$$

where the Zeno Lindbladian is

$$\mathscr{L}_{\mathrm{Zeno}} X = \hat{L}_{\mathtt{z}}^* X \hat{L}_{\mathtt{z}} - \frac{1}{2} X \hat{L}_{\mathtt{z}}^* \hat{L}_{\mathtt{z}} - \frac{1}{2} \hat{L}_{\mathtt{z}}^* \hat{L}_{\mathtt{z}} X - i \left[X, \hat{H} \right].$$

The equivalent master equation is

$$\frac{d}{dt}\rho_{\mathtt{z}} = \mathscr{D}_{\mathrm{Zeno}} \rho_{\mathtt{z}}, \tag{10}$$

where $\mathscr{D}_{\mathrm{Zeno}} \rho_{\mathtt{z}} = \hat{L}_{\mathtt{z}} \rho_{\mathtt{z}} \hat{L}_{\mathtt{z}}^* - \frac{1}{2} \rho_{\mathtt{z}} \hat{L}_{\mathtt{z}}^* \hat{L}_{\mathtt{z}} - \frac{1}{2} \hat{L}_{\mathtt{z}}^* \hat{L}_{\mathtt{z}} \rho_{\mathtt{z}} + i \left[\rho_{\mathtt{z}}, \hat{H} \right]$.

3.2 Zeno Quantum Trajectories for Vacuum Input

We now consider possible continual measurements of the output field. We set

1. Homodyne $Z(t) = B_{\mathrm{in}}(t) + B_{\mathrm{in}}^*(t)$;
2. Number counting $Z(t) = \Lambda_{\mathrm{in}}(t)$.

In both cases, the family $\{Z(t) : t \geq 0\}$ is self-commuting, as is the set of observables

$$Y(t) = V(t)^* (I_{\text{sys}} \otimes Z(t)) V(t) \tag{11}$$

which constitute the actual measured process. We note the non-demolition property $[j_t(X), Y_s] = 0$ for all $t \leq s$.

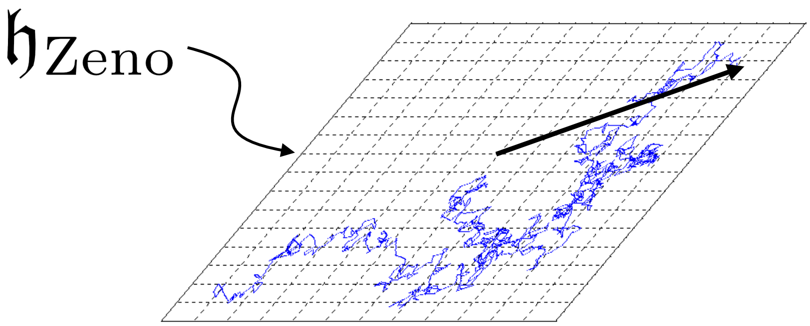

Figure 3: We may still measure the environment and obtain a conditioned state for the system within the Zeno subspace $\mathfrak{h}_{\text{Zeno}}$.

The aim of filtering theory is to obtain a tractable expression for the least-squares estimate of $j_t(X)$ given the output observations $Y(\cdot)$ up to time t. Mathematically, this is the conditional expectation

$$\pi_t(X) = \mathbb{E}_{\eta\Omega} \left[j_t(X) | \mathcal{Y}_t \right]$$

onto the measurement algebra $\mathcal{Y}_t$ generated by $Y(s)$ for $s \leq t$, with the fixed state $\eta \otimes \Omega$.

The Zeno filters are given respectively by

$$d\hat{\pi}_t(X) = \hat{\pi}_t(\mathcal{L}_{\text{Zeno}} X) dt + \hat{\mathcal{H}}_t \left(X \right) d\hat{I}(t) \tag{12}$$

where we have

	Homodyne	Number Counting
$\hat{\mathcal{H}}_t \left(X \right)$	$\hat{\pi}_t(X\hat{L} + \hat{L}^* X) - \hat{\pi}_t(X)\hat{\pi}_t(\hat{L} + \hat{L}^*)$	$dY(t) - \hat{\pi}_t(\hat{L} + \hat{L}^*)dt$
$d\hat{I}(t)$	$\hat{\pi}_t(\hat{L}^* X \hat{L})/\hat{\pi}_t(\hat{L}^* \hat{L}) - \hat{\pi}_t(X)$	$dY(t) - \hat{\pi}_t(\hat{L}^* \hat{L})dt$

.

The stochastic processes $\hat{I}$ are the innovations and correspond to the difference between observations dY and the expectations given the present conditioned state.

Alternatively we may introduce the conditioned Zeno state $\hat{\varrho}(t)$ defined by $\hat{\pi}_t(X) \equiv tr_{\mathsf{z}} \{ \hat{\varrho}_{\mathsf{z}} X \}$ for any bounded operator X on the Zeno subspace. From the filter equation we deduce the Zeno stochastic master equations

$$\hat{\varrho}(t) = \mathscr{D}_{\text{Zeno}} \hat{\varrho}(t)\, dt + \hat{\mathcal{G}}(\hat{\varrho}(t))\, d\hat{I}(t), \tag{13}$$

where

	Homodyne	Number Counting
$\hat{\mathcal{G}}_t(\varrho)$	$\hat{L}\varrho + \varrho\hat{L}^* - tr_{\mathtt{z}}\left\{\varrho(\hat{L}+\hat{L}^*)\right\}\varrho$	$dY(t) - tr_{\mathtt{z}}\left\{\varrho(\hat{L}+\hat{L}^*))\right\}dt$
$d\hat{I}(t)$	$\hat{L}\varrho\hat{L}^*/tr_{\mathtt{z}}\left\{\hat{\varrho}\hat{L}^*\hat{L}\right\} - \varrho$	$dY(t) - tr_{\mathtt{z}}\left\{\varrho\hat{L}^*\hat{L}\right\}dt$

.

3.3 Zeno Dynamics Within Networks

Let $\mathbf{G}_\alpha(k)$ be a collection of zenofiable models with state spaces $\mathfrak{h}_\alpha$, Zeno spaces $\mathfrak{h}_{\text{Zeno},\alpha}$, and limit Zeno generators $\mathbf{G}_{\text{Zeno},\alpha}$.

It is possible to form a quantum feedback network [15], [16] by connecting the output fields of some of these components as the inputs to others. In the limit of instantaneous feedback/forward we obtain a single effective SLH model for the remaining inputs. The resulting model will again be zenofiable with network Zeno subspace

$$\mathfrak{h}_{\text{Zeno}} = \bigotimes_\alpha \mathfrak{h}_{\text{Zeno},\alpha}.$$

The consistency of the construction follows from earlier work which establishes that the feedback reduction procedure for determining network models commutes with the procedure for adiabatic elimination [21], [22]. We obtain the same Zeno dynamics by first reducing the components systems to their Zeno dynamics and then performing the network interconnections.

4 Examples

In this section we discuss some well-known examples from the perspective of Zeno dynamics and control theory.

4.1 No scattering, and trivial damping

Let us set $S = I$, $L^{(1)} = 0$, $L^{(0)}_{\mathtt{fz}} = 0$ and $H^{(1)}_{\mathtt{zf}} = H^{(1)*}_{\mathtt{fz}} = 0$. In this case the only damping of significance is that of the Zeno component. Then we have $A_{\mathtt{ff}} = -iH^{(2)}_{\mathtt{ff}}$ and we require that $H^{(2)}_{\mathtt{ff}}$ is invertible on $\mathfrak{h}_{\mathtt{z}}$. It is easy to see that the decoupling conditions now apply and we obtain the open Zeno dynamics with $(\hat{S} = I, \hat{L} = L^{(0)}_{\mathtt{zz}}, \hat{H})$ where the Zeno Hamiltonian is

$$\hat{H} = H^{(0)}_{\mathtt{zz}},$$

and we note that

$$\begin{bmatrix} \hat{H} & 0 \\ 0 & 0 \end{bmatrix} \equiv P_{\mathtt{z}} H(k) P_{\mathtt{z}}.$$

The conditions are however still met if we take the off-diagonal terms $H^{(1)}_{\mathtt{zf}} = H^{(1)*}_{\mathtt{fz}}$ to be non-zero, and here we have the Zeno Hamiltonian

$$\hat{H} = H^{(0)}_{\mathtt{zz}} - H^{(1)}_{\mathtt{zf}} \frac{1}{H^{(2)}_{\mathtt{ff}}} H^{(1)}_{\mathtt{fz}}$$

Now $\hat{H}$ is the shorted version (Schur complement) of $H(1) = \begin{bmatrix} H^{(0)}_{\mathtt{zz}} & H^{(1)}_{\mathtt{zf}} \\ H^{(1)}_{\mathtt{fz}} & H^{(2)}_{\mathtt{ff}} \end{bmatrix}$. Equivalently, $\hat{H}$ is the the limit $k \uparrow \infty$ of shorted version of $H(k)$.

4.1.1 Qubit Limit

Let us consider a cavity consisting of a single photon mode with annihilator a, so that $[a, a^*] = I$. The number states $|n\rangle$, $(n = 0, 1, \cdots)$, span the infinite dimensional Hilbert space. The following model due to Mabuchi [17] shows how a large Kerr non-linearity leads to a Zeno dynamics where we are restricted to the ground and first excited state of the mode, and so have an effective qubit dynamics. We consider the $n = 2$ input model with

$$\begin{aligned} [S(k)]_{jk} &= \delta_{jk} I, \\ [L(k)]_j &= \sqrt{\kappa_j} e^{i\omega t} a, \quad (j = 1, 2) \\ H(k) &= k^2 \chi_0 a^{*2} a^2 + \Delta a^* a - i\sqrt{\kappa_1} \left(\alpha(t) a^* - \alpha^*(t) a \right). \end{aligned}$$

In the model we are in a rotating frame with frequency ω and the cavity is detuned from this frequency by an amount Δ. There is a Kerr non-linearity of strength $\chi(k) = \chi_0 k^2$ which will be the large parameter. We have two input fields with damping rate κ_j $(j = 1, 2)$, and the first input introduces a coherent driving field $\alpha(t)$.

We have $A \equiv \chi_0 a^{*2} a^2 = \chi_0 N (N - 1)$ where $N = a^* a$ is the number operator. The kernel space of A is therefore

$$\mathfrak{h}_{\mathtt{z}} = \text{span } \{|0\rangle, |1\rangle\}.$$

For this situation we have $P_{\mathtt{z}} = |0\rangle\langle 0| + |1\rangle\langle 1|$, and we find $L^{(0)}_{\mathtt{fz}} = 0$ since $P_{\mathtt{f}} a P_{\mathtt{z}} \equiv 0$. The zenofiability conditions are then satisfied and we have

$$H^{(0)}_{\mathtt{zz}} = P_{\mathtt{z}} \Delta a^* a P_{\mathtt{z}} \equiv \Delta \sigma^* \sigma$$

where $\sigma \triangleq P_{\mathtt{z}} a P_{\mathtt{z}} \equiv |0\rangle\langle 1|$. We then have that

$$\begin{aligned} \left[\hat{S}_{\mathtt{zz}}\right]_{jk} &= \delta_{jk} I_{\mathtt{z}}, \\ \left[\hat{L}_{\mathtt{z}}\right]_j &= \sqrt{\kappa_j} e^{i\omega t} \sigma, \\ \hat{H} &= \Delta \sigma^* \sigma - i\sqrt{\kappa_1} \left(\alpha(t) \sigma^* - \alpha^*(t) \sigma \right). \end{aligned}$$

The system is then completely controllable through the policy α, and observable through quadrature measurement (homodyning with $B_{\text{out},1}(t) - B_{\text{out},1}(t)^*$, and $-iB_{\text{out},1}(t) + iB_{\text{out},1}(t)^*$) and by photon counting.

4.2 No scattering, but non-trivial damping

We consider the case where $S = I$, $L^{(0)}_{\mathtt{fz}} = 0$ and $L^{(1)}_{\mathtt{ff}} = 0$, but $L^{(1)}_{\mathtt{zf}} \neq 0$. The decoupling conditions are automatically satisfied, so this set-up is zenofiable with Zeno provided that $A_{\mathtt{ff}}$, which is now given by

$$A_{\mathtt{ff}} \equiv -\frac{1}{2} L^{(1)*}_{\mathtt{zf}} L^{(1)}_{\mathtt{zf}} - iH^{(2)}_{\mathtt{ff}},$$

is invertible. If so the Zeno SLH takes the simplified form

$$\begin{aligned}
\hat{S}_{\mathtt{zz}} &\equiv I_{\mathtt{z}} + L^{(1)}_{\mathtt{zf}} \frac{1}{A_{\mathtt{ff}}} L^{(1)*}_{\mathtt{zf}}, \\
\hat{L}_{\mathtt{z}} &\equiv L^{(0)}_{\mathtt{zz}} - L^{(1)}_{\mathtt{zf}} \frac{1}{A_{\mathtt{ff}}} M_{\mathtt{fz}}, \\
\hat{H} &\equiv H^{(0)}_{\mathtt{zz}} + \mathrm{Im} \left\{ M_{\mathtt{zf}} \frac{1}{A_{\mathtt{ff}}} M_{\mathtt{fz}} \right\},
\end{aligned}$$

where now

$$\begin{aligned}
M_{\mathtt{zf}} &\equiv -\frac{1}{2} L^{(0)*}_{\mathtt{zz}} L^{(1)}_{\mathtt{zf}} - iH^{(1)}_{\mathtt{zf}}, \\
M_{\mathtt{fz}} &\equiv -\frac{1}{2} L^{(1)*}_{\mathtt{zf}} L^{(0)}_{\mathtt{zz}} - iH^{(1)}_{\mathtt{fz}}.
\end{aligned}$$

4.2.1 Alkali Atom

Consider a model for an atomic electron which has two energy states, a ground state $|g\rangle$ and an excited state $|e\rangle$ spanning $\mathfrak{h}_{\mathrm{level}} = \mathbb{C}^2$, and an intrinsic spin one-half with states $|+\rangle$ and $|-\rangle$ spanning $\mathfrak{h}_{\mathrm{spin}} = \mathbb{C}^2$. The total Hilbert space is then the 4-dimensional $\mathfrak{h} = \mathfrak{h}_{\mathrm{level}} \otimes \mathfrak{h}_{\mathrm{spin}}$. Taking $n = 3$ inputs, one for each spatial coordinate, the scaled model is [14]

$$\begin{aligned}
[L(k)]_j &= k\sqrt{\gamma} |g\rangle\langle e| \otimes \sigma_j, \quad j = 1, 2, 3, \\
H(k) &= k^2 \Delta |e\rangle\langle e| \otimes I + \sum_{j=1,2,3} I \otimes \mathcal{B}_j \sigma_j
\end{aligned}$$

where $\Delta > 0$ has the interpretation of a detuning frequency for the excited state, and $\mathcal{B}_j$ as the j component of an external magnetic field. We find that

$$A = -\left(\frac{3}{2}\gamma + i\Delta \right) |e\rangle\langle e| \otimes I$$

so if we set $P = |g\rangle\langle g| \otimes I$, we have a zenofiable model with 2-dimensional Zeno space

$$\mathfrak{h}_{\mathrm{Zeno}} = |g\rangle \otimes \mathfrak{h}_{\mathrm{spin}} \equiv \mathrm{span}\left\{ |g\rangle \otimes |+\rangle, |g\rangle \otimes |-\rangle \right\}.$$

The Zeno SLH is

$$\begin{aligned}
\left[\hat{S}\right]_{jk} &= |g\rangle\langle g| \otimes \left\{ \delta_{jk} - \frac{\gamma}{\frac{3}{2}\gamma + i\Delta} \sigma_j \sigma_k \right\}, \\
\left[\hat{L}\right]_{j} &= 0, \\
\hat{H} &= \sum_{j=1,2,3} |g\rangle\langle g| \otimes \mathcal{B}_j \sigma_j .
\end{aligned}$$

In this case the Zeno dynamics is undamped ($\hat{L} = 0$) so filtering is not possible for vacuum input. However the Zeno system (effectively the spin) is controllable through the magnetic field.

4.2.2 Λ-systems

Consider a three level atom with ground states $|g1\rangle$,$|g2\rangle$ and an excited state $|e\rangle$ with Hilbert space $\mathfrak{h}_{\text{level}} = \mathbb{C}^3$. The atom is contained in a cavity with quantum mode a with Hilbert space $\mathfrak{h}_{\text{mode}}$ where $[a, a^*] = 1$ and a annihilates a photon of the cavity mode. The combined system and cavity has Hilbert space $\mathfrak{h} = \mathfrak{h}_{\text{level}} \otimes \mathfrak{h}_{\text{mode}}$, and consider the following [20], [14],

$$\begin{aligned}
L(k) &= k\sqrt{\gamma} I \otimes a, \\
H(k) &= ik^2 \mathsf{g} \left\{ |e\rangle\langle g1| \otimes a - |g1\rangle\langle e| \otimes a^* \right\} + ik \left\{ |e\rangle\langle g2| \otimes \alpha - |g2\rangle\langle e| \otimes \alpha^* \right\}.
\end{aligned}$$

Here the cavity is lossy and leaks photons with decay rate γ, we also have a transition from $|e\rangle$ to $|g1\rangle$ with the emission of a photon into the cavity, and a scalar field α driving the transition from $|e\rangle$ to $|g2\rangle$. We see that

$$A \equiv -\frac{1}{2}\gamma I \otimes a^* a + \mathsf{g} \left\{ |e\rangle\langle g1| \otimes a - |g1\rangle\langle e| \otimes a^* \right\}.$$

and that A has a 2-dimensional kernel space spanned by the pair of states

$$|\Psi_1\rangle = |g1\rangle \otimes |0\rangle, \quad |\Psi_2\rangle = |g2\rangle \otimes |0\rangle.$$

The Zeno subspace is then the span of $|\Psi_1\rangle$ and $|\Psi_2\rangle$, and the resulting SLH operators are

$$\begin{aligned}
\hat{S} &= |\Psi_1\rangle\langle\Psi_1| - |\Psi_2\rangle\langle\Psi_2| \equiv I - 2\sigma^*\sigma, \\
\hat{L} &= -\frac{\gamma\alpha}{\mathsf{g}} |\Psi_1\rangle\langle\Psi_2| \equiv -\frac{\gamma\alpha}{\mathsf{g}}\sigma, \\
\hat{H} &= 0,
\end{aligned}$$

where $\sigma = |\Psi_1\rangle\langle\Psi_2|$. Here the Zeno dynamics has a vanishing Zeno Hamiltonian, but is partially observable through filtering as $\hat{L} \neq 0$.

5 Oscillator Models

The original treatment of adiabatic elimination problem for quantum stochastic problems dealt with systems coupled to oscillator components which decayed rapidly to their ground states [23]. Decomposing the system Hilbert space as $\mathfrak{h} = \mathfrak{h}_{\text{slow}} \otimes \mathfrak{h}_{\text{osc}}$, one can establish the conditions for adiabatic elimination of the oscillators lead to the Zeno subspace

$$\mathfrak{h}_{\text{Zeno}} = \mathfrak{h}_{\text{slow}} \otimes |0\rangle, \tag{14}$$

where $|0\rangle$ is the ground state of the oscillatory system. The effective Zeno subspace obviously being the slow space.

We consider an open model described by the generators $\mathbf{G}(k)$ with

$$\begin{aligned} S(k) &= \hat{S} \otimes I, \\ L(k) &= k \sum_i \hat{C}_i \otimes a_i + \hat{G} \otimes I, \\ K(k) &= k^2 \sum_{ij} \hat{A}_{ij} \otimes a_i^* a_j + k \sum_i \hat{Z}_i \otimes a_i^* + k \sum_i \hat{X}_i \otimes a_i + \hat{R} \otimes I, \end{aligned} \tag{15}$$

where k is a positive scaling parameter and $\hat{S}, \hat{C}_i, \hat{G}, \hat{A}_{ij}, \hat{X}_i, \hat{Z}_i, \hat{R}$ are bounded operators on $\mathfrak{h}_{\text{slow}}$ with $\hat{A} = \left[\hat{A}_{ij} \right]$ invertible with bounded inverse. Here a_i is the annihilator corresponding to the ith local oscillator, say with $i = 1, \cdots, m$.

As $k \to \infty$ the oscillators become increasingly strongly coupled to the driving noise field and in this limit we would like to consider them as being permanently relaxed to their joint ground state. The oscillators are then the fast degrees of freedom of the system, with the auxiliary space $\mathfrak{h}_{\text{slow}}$ describing the slow degrees.

We say that $\hat{A}$ is *strictly Hurwitz stable* if

$$\text{Re}\langle \psi | \hat{A} \psi \rangle < 0, \ \text{ for all } \psi \neq 0.$$

It is shown in [21] that $A = \sum_{ij} \hat{A}_{ij} \otimes a_i^* a_j$ will have kernel space given by (14) whenever $\hat{A}$ is strictly Hurwitz. Assuming that $\hat{A}$ is Hurwitz, we then obtain a Zeno limit with $\mathbf{G}_{\text{Zeno}}$

$$\begin{aligned} \hat{S} &= (I + \hat{C} \hat{A}^{-1} \hat{C}^*) \hat{S}, \\ \hat{L} &= \hat{G} - \hat{C} \hat{A}^{-1} \hat{Z}, \\ \hat{K} &= \hat{R} - \hat{X} \hat{A}^{-1} \hat{Z}, \end{aligned} \tag{16}$$

where we drop the "$\otimes |0\rangle\langle 0|_{\text{osc}}$" for convenience.

5.1 Quantum Linear Models

In principle, the slow degrees of freedom may also correspond to an assembly of oscillators too, and we may take the overall model (fast and slow oscillators) to be linear. The limit dynamics will then be linear, with the model coefficients determined by the limit theorems. These will in fact be similar in structure to singular perturbation results for linear control systems [24]. In this case it is known that the strict Hurwitz property of $\hat{A}$ (now a matrix with scalar entries) guarantees that there exist a finite constant k_0 such the total linear model is stable for $k > k_0$. This justifies the neglect of parasitic modes in engineering modelling [25]. For quantum linear passive systems, this will also be the case, and the non-Zeno part of the dynamics may be safely ignored.

Specifically, we may take the slow system to be comprised of m quantum oscillators with annihilators $b_1, \cdots, b_r$ and we obtain a linear model with fast oscillators a_i and slow (Zeno!) oscillators b_i by taking

$$\hat{G} = \sum_i (G_i^+ b_i^* + G_i^- b_i),$$
$$\hat{Z}_j = \sum_i (Z_{ij}^+ b_i^* + Z_{ij}^- b_i),$$
$$\hat{X}_j = \sum_i (X_{ij}^+ b_i^* + X_{ij}^- b_i),$$
$$\hat{R}_j = \sum_i (R_{ij}^{++} b_i^* b_j^* + R_{ij}^{--} b_i b_j + R_{ij}^{+-} b_i^* b_j),$$

and the $\hat{A}_{ij}$ scalars.

From the quantum stochastic calculus [6] the oscillators satisfy linear Heisenberg equations that take the form

$$db(t) = \Gamma_1 b \, dt + \Gamma_2 z + \Phi \, dB_{\text{in}}(t),$$
$$\frac{1}{k^2} dz(t) = \Gamma_3 b \, dt + \Gamma_4 z + \Psi \, dB_{\text{in}}(t),$$

where we set $z \equiv ka$. The coefficient matrices Γ_j ($j = 1, 2, 3, 4$), Φ, Ψ are independent of the scaling parameter k. In particular $\Gamma_4 \equiv \hat{A}$.

We are interested primarily in the stability of these equations and it is sufficient to look at the averages of the operators where the environment state is vacuum, for which the input processes are martingales, and here we have

$$\frac{d}{dt}\bar{b}(t) = \Gamma_1 \bar{b}(t) + \Gamma_2 \bar{z}(t), \tag{17}$$
$$\frac{1}{k^2}\frac{d}{dt}\bar{z}(t) = \Gamma_3 \bar{b}(t) + \Gamma_4 \bar{z}(t),$$

Proposition 6 *If $\hat{A}$ is invertible, then the necessary and sufficient conditions for the existence of a $k_0 < \infty$ such that the system (18) is asymptotically stable for $k > k_0$ are that both $\hat{A}$ and Γ_0 are strictly Hurwitz.*

This is a consequence of the following result [26].

Lemma 7 *If $\Gamma_4 \equiv \hat{A}$ is invertible, then for k sufficiently large the eigenvalues for the system of linear dynamical equations (18) are $\sigma(\Gamma_0)(1 + O(\frac{1}{k^2}))$, and $k\sigma(\Gamma_4)(1 + O(\frac{1}{k^2}))$, where $\Gamma_0 = \Gamma_1 - \Gamma_2 \Gamma_4^{-1} \Gamma_3$.*

(Here $\sigma(X)$ denotes the spectrum of a matrix X.) The strict Hurwitz property for $\hat{A}$ is essential for the existence of the Zeno dynamical limit for the system of oscillators b_i. The additional requirement that Γ_0 also be Hurwitz is required for stability of the limit system, compare with (16).

From a modelling perspective, the Zeno dynamics is a model reduction of the total assembly of fast and slow oscillators. In control applications, one would want to have the performance under the total dynamics approximated arbitrarily closely by the Zeno model by taking k sufficiently large. In this way any further feedback construction [15] would be robust against the modelling errors introduced by reducing to the Zeno description. To this end we remark again that feedback in SLH models is compatible with model reduction through adiabatic elimination [21, 22].

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