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Quantum trajectories for a class of continuous matrix product input states

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Abstract

We introduce a new class of continuous matrix product (CMP) states and establish the stochastic master equations (quantum filters) for an arbitrary quantum system probed by a bosonic input field in this class of states. We show that this class of CMP states arise naturally as outputs of a Markovian model, and that input fields in these states lead to master and filtering (quantum trajectory) equations which are matrix-valued. Furthermore, it is shown that this class of CMP states include the (continuous-mode) single photon and time-ordered multi-photon states.

Keywords: CMP states, quantum filter, quantum trajectories

1. Introduction

Continuous matrix product (CMP) states were introduced by Verstraete and Cirac [1–3] as the generalization of finitely correlated states to continuous-variable quantum input processes [4]. Here we introduce a new class of CMP states and derive the quantum filtering (quantum trajectory) equations [5] for state inputs in this class. We show that both the master equation and filter equations become matrix-valued. In particular, we show that this class includes

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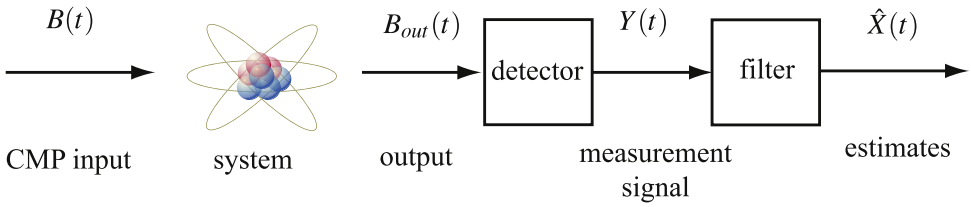

Figure 1. A schematic representation of a continuous measurement process, where the measurement signal produced by a detector is filtered to produce estimates $\hat{X}(t) = \pi_t(X) = \mathrm{tr}[\rho(t)X]$ of system operators X at time t. The system is driven by a field in a CMP state defined in this paper.

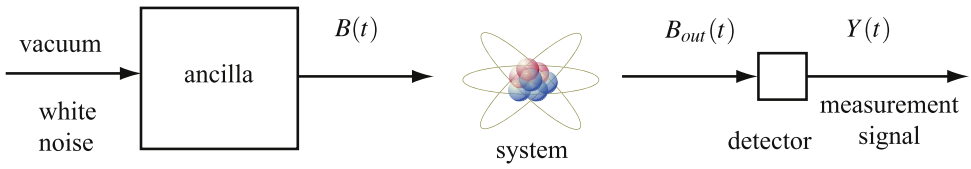

Figure 2. An ancilla system is used to model the effect of the CMP field state for the input field $B(t)$ on the system.

(continuous-mode) single photon and multi-photon states of a boson field. Whereas discrete matrix product states have been proposed to model approximations for efficient simulation of certain continuous stochastic master equation [8], our work here differs in so far as we wish to deal with continuous variable models for an open quantum system with a quantum input field in a state general enough to enable efficient derivation of quantum trajectory equations (that is construct the quantum filter for determining estimates of system operators) for important classes of non-classical field states, figure 1.

As mentioned, a particularly important motivating class of inputs are multi-photon states. The production and verification of single-photon states [9] has become routine, achievable through a variety of experimental architectures such as cavity quantum electrodynamics (QED), quantum dots in semiconductors, and circuit QED. Single photon and multi-photon states are important because they are of interest in various applications; see, e.g., [10–13] for single photon states and [19, 20] for multi-photon states. We shall first outline the solution in the standard case of Markovian systems driven by a vacuum input field state in section 2, then give the generalization to our class of CMP state inputs. The filtering equations make use of an ancilla cascaded with the system, as shown in figure 2.

Furthermore, it is shown that the class of CMP states defined in this paper include the (continuous-mode) single photon and time-ordered multi-photon states, and we derive explicit Markovian generators for these multi-photon states that allow the filtering equations for systems driven by fields in these states to be obtained from the general formulas of this paper.

The structure of this paper is as follows. Section 2 provides a brief overview of quantum Markov input-output models, quantum stochastic differential equations and the (S, L, H) formalism, and the well-known quantum master and filtering equations for open Markov models driven by a vacuum state input field. In section 3 we introduce a new class of CMP states that is defined in terms of a Hudson–Parthasarathy quantum stochastic differential equation and which can be viewed as the output states of an (S, L, H) model. Then in section 4

we derive the quantum master equation and quantum filtering (quantum trajectory) equations for an open Markov model driven by a field in the newly defined CMP states, in the form of matrix-valued equations with operator entries. This is followed in section 5 by some explicit examples of the novel CMP states: continuous-mode single photon states, and time-ordered continuous-mode multi-photon states. Section 6 then provides a summary of the contributions of this paper. The main text is also supplemented by two appendices. Appendix A details a Markovian generator model for time-ordered two-photon states that is then generalized to time-ordered multi-photon states in appendix B.

2. Quantum input-output models

We review quantum Markov input-output models. For simplicity, we shall consider single-input single-output (SISO) models only, however the ideas are readily extended to multiple inputs. To describe external inputs, we fix a Fock space $\mathfrak{F}$ for the quantum field inputs. Formally, we have quantum input processes $b_{\rm in}(t)$ satisfying a singular canonical commutation relations (CCR) $\left[b_{\rm in}(t),\, b_{\rm in}(s)^*\right] = \delta(t-s)$, and define regular processes of annihilation, creation and number: $B_{\rm in}(t) = \int_0^t b_{\rm in}(s)\,{\rm d}s$, $B_{\rm in}(t)^* = \int_0^t b_{\rm in}(s)^*{\rm d}s$ and $\Lambda_{\rm in}(t) = \int_0^t b_{\rm in}(s)^* b_{\rm in}(s)\,{\rm d}s$. We shall use the framework of the Hudson–Parthasarathy calculus of quantum stochastic integration with respect to these processes [14], which contains the quantum input–output theory of Gardiner and Collett [4] as a special case. The Fock vacuum state will be denoted by $|\,\Omega\rangle$.

We shall be interested in a new class *CMP states* which are of the form

$$|\,\Psi(T)\,\rangle \equiv \vec{T}e^{\int_0^T R(t)\otimes {\rm d}B_{\rm in}(t)^* + Q(t)\otimes {\rm d}t}\,|\,\phi\rangle \otimes |\,\Omega\rangle \tag{1}$$

with $T > 0$, $R(\,\cdot\,)$ and $Q(\,\cdot\,)$ $\mathbb{C}^{D\times D}$-valued functions, and ϕ a fixed unit vector in $\mathbb{C}^D$. We may think of $\mathbb{C}^D$ as being an auxiliary finite-dimensional Hilbert space. Without loss of generality one takes $Q(t) \equiv -\frac{1}{2}R(t)^*R(t) - iH_{\rm aux}(t)$, where $H_{\rm aux}$ is Hermitean valued.

Note that this class of CMP states is distinct from the one introduced by Verstraete–Cirac [1] which defines an unnormalized CMP state as a state of the form

$$|\,\Psi(T)\,\rangle_{\rm VC} \equiv {\rm tr}_{\rm aux}\left(B\vec{T}e^{\int_0^T R(t)\otimes {\rm d}B_{\rm in}(t)^* + Q(t)\otimes {\rm d}t}\right)|\,\Omega\rangle,$$

where B is a fixed operator on the ancilla space called the boundary operator. Typical choices studied are $B = I$ and $B = \left|e_k\right\rangle\left\langle e_j\right|$, where $\left\{\left|e_k\right\rangle\right\}$ is an orthonormal basis for the ancilla space $\mathbb{C}^D$. Whereas $|\,\Psi(T)\,\rangle_{\rm VC}$ is a pure state on the quantum field Hilbert space, our new CMP state (1) is a pure state on the composite auxiliary and quantum field Hilbert space. Nonetheless, our CMP states share the continuous product property of the Verstraete–Cirac CMP states.

To connect the classes, we introduce Fock space vectors $\left|\Psi_{jk}(t)\right\rangle$ defined by

$$\left\langle \Phi\middle|\Psi_{jk}(T)\right\rangle := \left\langle e_j \otimes \Phi\middle|\vec{T}e^{\int_0^T R(t)\otimes {\rm d}B_{\rm in}(t)^* + Q(t)\otimes {\rm d}t}\middle|e_k \otimes \Omega\right\rangle,$$

for arbitrary Fock space vector $|\,\Phi\rangle$. It follows that we work with the $\mathbb{C}^D$-valued Fock space vector

$$|\Psi(T)\rangle = \left[\sum_k \left|\Psi_{jk}(T)\right\rangle \langle e_k|\ \phi\rangle\right]_{j=1}^{D},$$

while Verstraete–Cirac work with Fock space vectors

$$|\Psi(T)\rangle_{VC} = \sum_{jk} B_{kj} \left|\Psi_{jk}(T)\right\rangle.$$

We remark that the treatment of Cirac and Verstraete employs a spatial observable x rather than a time variable. In our interpretation we think of a travelling input field, which we may take as a quantum optical field propagating at the speed of light c, so that initial the element of the field that will interaction in a Markovian manner with the system at time t will have to travel the distance $x = ct$ to the system. In fact, Haegeman, Cirac and Osborne have noted that the spatial interpretation for their CMP states is equivalent to this type of Markov model, see the discussion at the end of section E of [3] on physical interpretations. Indeed, they note that $\left|\Psi_{jk}(T)\right\rangle$, corresponding to their choice $B = \left|e_k\right\rangle\left\langle e_j\right|$, corresponds to an initialization of the ancilla in state $\left|e_j\right\rangle$ followed by a post-selection of state $\left|e_k\right\rangle$ by measurement at time T.

In either case, the natural mathematical setting for the theory is the Hudson–Parthasarathy quantum stochastic calculus [14]. Let us fix a Hilbert space $\mathfrak{h}_0$ which we call the initial space. On the joint Hilbert space $\mathfrak{h}_0 \otimes \mathfrak{F}$ we may consider the quantum stochastic differential equation (QSDE)

$$\begin{aligned} V(t, s) = I + \int_s^t \Big\{ &\left(S - I_{\text{sys}}\right) \otimes \mathrm{d}\Lambda_{\text{in}}(\tau) + L \otimes \mathrm{d}B_{\text{in}}(\tau)^* \\ &- L^* S \otimes \mathrm{d}B_{\text{in}}(\tau) - \left(\frac{1}{2}L^*L + iH\right) \otimes \mathrm{d}\tau \Big\} V(\tau, s) \end{aligned} \tag{2}$$

for $t > s$. We then have the propagation law $V(t, s) = V(t, r)\, V(r, s)$ whenever $t > r > s$. The differentials are all understood to be future-pointing (Itō convention) and by the CCR we have the following non-zero products

$$\begin{aligned} &\mathrm{d}B_{\text{in}}(t)\,\mathrm{d}B_{\text{in}}(t)^* = \mathrm{d}t, \qquad \mathrm{d}\Lambda_{\text{in}}(t)\,\mathrm{d}B_{\text{in}}(t)^* = \mathrm{d}B_{\text{in}}(t)^* \\ &\mathrm{d}B_{\text{in}}(t)\,\mathrm{d}\Lambda_{\text{in}}(t) = \mathrm{d}B_{\text{in}}(t) \quad \mathrm{d}\Lambda_{\text{in}}(t)\,\mathrm{d}\Lambda_{\text{in}}(t) = \mathrm{d}\Lambda_{\text{in}}(t). \end{aligned}$$

Provided the operator S is unitary, and H is self-adjoint, there will exist a unique solution to the QSDE which is unitary. We shall refer to the triple (S, L, H) as the (possibly time-dependent) *Hudson–Parthasarathy* (HP) parameters. In particular, we shall be interested in a fixed origin of time and will set $V(t) = V(t, 0)$.

We now fix $\mathfrak{h}_0$ as the Hilbert space $\mathfrak{h}_{\text{sys}}$ of a quantum system of interest. For a given system observable X, we set $j_t(X) = V(t)^*\left(X \otimes I_{\mathfrak{F}}\right)V(t)$ in which case we obtain the QSDE

$$\begin{aligned} dj_t(X) = {} & j_t\left(\mathcal{L}_{00}^{\text{sys}}X\right)\mathrm{d}t + j_t\left(\mathcal{L}_{10}^{\text{sys}}X\right)\mathrm{d}B_{\text{in}}(t)^* \\ & + j_t\left(\mathcal{L}_{01}^{\text{sys}}X\right)\mathrm{d}B_{\text{in}}(t) + j_t\left(\mathcal{L}_{11}^{\text{sys}}X\right)\mathrm{d}\Lambda_{\text{in}}(t) \end{aligned} \tag{3}$$

where we have the following superoperators

$$\begin{aligned}
&\mathcal{L}_{00}^{\mathrm{sys}} X = \frac{1}{2}\left[L^*, X\right] L + \frac{1}{2} L^* [X, L] - i [X, H] \\
&\mathcal{L}_{10}^{\mathrm{sys}} X = S^* [X, L], \ \mathcal{L}_{01}^{\mathrm{sys}} X = \left[L^*, X\right] S \\
&\mathcal{L}_{11}^{\mathrm{sys}} X = S^* X S - X
\end{aligned} \tag{4}$$

known as the Evans–Hudson maps (note that $\mathcal{L}_{00}^{\mathrm{sys}}$ is a Lindbladian). The output fields are given by

$$B_{\mathrm{out}}(t) = V(t)^* \left(I_{\mathrm{sys}} \otimes B_{\mathrm{in}}(t)\right) V(t), \tag{5}$$

etc, and we find from the quantum Itō calculus $\mathrm{d}B_{\mathrm{out}}(t) = j_t(S)\,\mathrm{d}B_{\mathrm{in}}(t) + j_t(L)\,\mathrm{d}t$. We remark that we may also write $B_{\mathrm{out}}(t) = V(T)^* \left(I_{\mathrm{sys}} \otimes B_{\mathrm{in}}(t)\right) V(T)$ whenever $T > t$ since we have $V(T) = V(T, t) V(t)$ and the unitary $V(T, t)$ acts non-trivially only on the component of the Fock space generated by fields on the time interval $[t, T]$ and in particular commutes with $I_{\mathrm{sys}} \otimes B_{\mathrm{in}}(t)$, and so can be removed.

Let us fix a state $\eta \in \mathfrak{h}_{\mathrm{sys}}$, then for $t \geqslant 0$ we may define the expectation $\mathbb{E}_{\eta\Omega}[\,\cdot\,] = \langle \eta \otimes \Omega | \cdot | \eta \otimes \Omega \rangle$ and for X a system observable we define $\mathbb{E}_t^{\mathrm{vac}}(\,\cdot\,)$ by

$$\mathbb{E}_t^{\mathrm{vac}}(X) := \mathbb{E}_{\eta\Omega}\left[j_t(X)\right] \equiv tr_{\mathrm{sys}}\left\{\varrho_{\mathrm{sys}} X\right\},$$

which introduces the density matrix ϱ_{sys}. We obtain the *Ehrenfest equation* $\frac{d}{\mathrm{d}t}\mathbb{E}_t^{\mathrm{vac}}(X) = \mathbb{E}_t^{\mathrm{vac}}\left(\mathcal{L}_{00}^{\mathrm{sys}} X\right)$, with equivalent master equation $\frac{d}{\mathrm{d}t}\varrho_{\mathrm{sys}} = \mathcal{L}_{00}^{\mathrm{sys}\star}\varrho_{\mathrm{sys}}$ where $\mathcal{L}_{00}^{\mathrm{sys}\star}\rho = L\rho L^* - \frac{1}{2}\left\{\rho, L^* L\right\}_+ + i[\rho, H]$.

We now consider the following continuous measurements of the output field [5]

(i) Homodyne $Z(t) = B_{\mathrm{in}}(t) + B_{\mathrm{in}}^*(t)$;

(ii) Number counting $Z(t) = \Lambda_{\mathrm{in}}(t)$.

In both cases, the family $\{Z(t)\colon t \geqslant 0\}$ is self-commuting, as is the set of observables

$$Y(t) = V(t)^* \left(I_{\mathrm{sys}} \otimes Z(t)\right) V(t) \tag{6}$$

which constitute the actual measured process. We note the non-demolition property $\left[j_t(X), Y_s\right] = 0$ for all $t \geqslant s$. The aim of filtering theory is to obtain a tractable expression for the least-squares estimate of $j_t(X)$ given the output observations $Y(\,\cdot\,)$ up to time t. Mathematically, this is the conditional expectation

$$\pi_t(X) = \mathbb{E}_{\eta\Omega}\left[j_t(X) \middle| \mathcal{Y}_t\right] \tag{7}$$

onto the measurement algebra $\mathcal{Y}_t$ generated by $Y(s)$ for $s \leqslant t$.

The filters are given respectively by [5]

Homodyne:

$$\begin{aligned}
d\pi_t(X) = {} & \pi_t\left(\mathcal{L}_{00} X\right) \mathrm{d}t \\
& + \left\{\pi_t\left(XL + L^* X\right) - \pi_t(X)\,\pi_t\left(L + L^*\right)\right\}\left[dY(t) - \pi_t\left(L + L^*\right)\mathrm{d}t\right],
\end{aligned}$$

Number counting:

$$d\pi_t(X) = \pi_t\left(\mathcal{L}_{00}X\right)\mathrm{d}t + \left\{\pi_t\left(L^*XL\right)/\pi_t\left(L^*L\right) - \pi_t(X)\right\}\left[dY(t) - \pi_t\left(L^*L\right)\mathrm{d}t\right].$$

Note that there can be other types of measurements performed since a quantum master equation can be 'unravelled' in infinitely many ways [6, 7], leading to infinitely many possible stochastic master equations. However, the two types of measurements that we consider here are the most natural and commonly performed in the laboratory.

3. CMP states

CMP states may likewise be considered as (S, L, H)-model output states. Here we take the initial Hilbert space to be the auxiliary space $\mathfrak{h}_{\mathrm{aux}} = \mathbb{C}^D$. The HP parameters are taken to be $\left(I_{\mathrm{aux}}, R(\cdot), H_{\mathrm{aux}}(\cdot)\right)$, which may be time-dependent. The state (1) is then realized as

$$|\Psi(T)\rangle = V_{\mathrm{aux}}(T)\,|\phi\rangle \otimes |\Omega\rangle \tag{8}$$

where V_{aux} is the associated unitary.

Definition: Suppose that for a fixed auxiliary space $\mathfrak{h}_{\mathrm{aux}}$, a prescribed set of HP parameters, and a fixed unit vector $\phi \in \mathfrak{h}_{\mathrm{aux}}$ as above, the vector states $\Psi(T)$ have a well-defined limit in norm $\Psi(\infty) \in \mathfrak{h}_{\mathrm{aux}} \otimes \mathfrak{F}$, as $T \to \infty$. Then we refer to $\Psi(\infty)$ as an asymptotic CMP state.

For arbitrary operator A on the auxiliary space, we have the expectation

$$\left\langle \phi \otimes \Omega \middle| V_{\mathrm{aux}}(t)^*\left(A \otimes I_{\mathfrak{F}}\right)V_{\mathrm{aux}}(t)\middle|\phi \otimes \Omega\right\rangle \equiv tr_{\mathrm{aux}}\left\{\varrho_{\mathrm{aux}}(t)A\right\}$$

which defines the reduced density matrix $\varrho_{\mathrm{aux}}(t)$. We note that it satisfies the master equation $\frac{d}{\mathrm{d}t}\varrho_{\mathrm{aux}} = \mathcal{L}_{00}^{\mathrm{aux}\star}\varrho_{\mathrm{aux}}$ with $\varrho_{\mathrm{aux}}(0) = |\phi\rangle\langle\phi|$.

4. Filtering for CMP state inputs

We now wish to consider a system with Hilbert space $\mathfrak{h}_{\mathrm{sys}}$ and HP parameters (S, L, H) driven by an input field in a CMP state $\left|\Psi_\infty\right\rangle$ on noise space $\mathfrak{h}_{\mathrm{aux}} \otimes \mathfrak{F}$. To this end we have the expectation

$$\mathbb{E}_t^{\mathrm{cmp}}(X) = \mathbb{E}_{\eta\Psi_\infty}\left[j_t(X) \otimes I_{\mathrm{aux}}\right].$$

Our aim is to derive the master and filtering equations based on the CMP state.

4.1. Cascade realization

As we are effectively working on the joint Hilbert space $\mathfrak{h}_{\mathrm{sys}} \otimes \mathfrak{h}_{\mathrm{aux}} \otimes \mathfrak{F}$, it is convenient to take the initial space to be $\mathfrak{h}_0 = \mathfrak{h}_{\mathrm{sys}} \otimes \mathfrak{h}_{\mathrm{aux}}$. With respect to this decomposition we introduce the pair of HP parameters

$$G_{\mathrm{sys}} = \left(S \otimes I_{\mathrm{aux}}, L \otimes I_{\mathrm{aux}}, H \otimes I_{\mathrm{aux}}\right)$$
$$G_{\mathrm{aux}} = \left(I_{\mathrm{sys}} \otimes I_{\mathrm{aux}}, I_{\mathrm{sys}} \otimes R(t), I_{\mathrm{sys}} \otimes H_{\mathrm{aux}}(t)\right)$$

and denote the associated unitary processes by $\tilde{V}_{\rm sys}(t)$ and $\tilde{V}_{\rm aux}(t)$ respectively. We then have

$$\begin{aligned}\mathbb{E}_t^{\rm cmp}(X) &= \lim_{T\to\infty}\mathbb{E}_{\eta\Psi(T)}\left[\tilde{V}_{\rm sys}(t)^*\left(X\otimes I_{\rm aux}\otimes I_{\mathfrak{F}}\right)\tilde{V}_{\rm sys}(t)\right]\\ &= \mathbb{E}_{\eta\phi\Omega}\left[\tilde{V}(t)^*\left(X\otimes I_{\rm aux}\otimes I_{\mathfrak{F}}\right)\tilde{V}(t)\right],\end{aligned} \tag{9}$$

where $\tilde{V}(t)=\tilde{V}_{\rm sys}(t)\tilde{V}_{\rm aux}(t)$, and we use the fact that $\tilde{V}_{\rm sys}(t)\tilde{V}_{\rm aux}(T)=\tilde{V}_{\rm aux}(T,t)\tilde{V}(t)$. From the quantum Itō calculus it is easy to see that $\tilde{V}$ is associated to the HP parameters (this is a special case of the series product for cascaded network consisting of the auxiliary model fed into the system model [15])

$$\begin{aligned}&\tilde{S}=S\otimes I_{\rm aux},\quad \tilde{L}(t)=L\otimes I_{\rm aux}+S\otimes R(t),\\ &\tilde{H}(t)=H\otimes I_{\rm aux}+I_{\rm sys}\otimes H_{\rm aux}(t)\\ &\qquad+\frac{1}{2i}L^*S\otimes R(t)-\frac{1}{2i}S^*L\otimes R(t)^*.\end{aligned} \tag{10}$$

Let us introduce the more general expectation

$$\tilde{\mathbb{E}}_t(X\otimes A)=\mathbb{E}_{\eta\phi\Omega}\left[\tilde{V}(t)^*\left(X\otimes A\otimes I_{\mathfrak{F}}\right)\tilde{V}(t)\right]$$

for X and A system and auxiliary operators respectively. We then have $\tilde{\mathbb{E}}_t\left(X\otimes I_{\rm aux}\right)=\mathbb{E}_t^{\rm cmp}(X)$. Now

$$\frac{d}{{\rm d}t}\tilde{\mathbb{E}}_t(X\otimes A)=\tilde{\mathbb{E}}_t\left(\tilde{\mathcal{L}}_{00}(X\otimes A)\right)$$

where $\tilde{\mathcal{L}}_{00}$ is the Lindbladian corresponding to the HP parameters $\left(\tilde{S},\tilde{L},\tilde{H}\right)$. We now note the following identity

$$\begin{aligned}\tilde{\mathcal{L}}_{00}(X\otimes A)&=\mathcal{L}_{00}^{\rm sys}X\otimes A\\ &\quad+\mathcal{L}_{01}^{\rm sys}X\otimes AR+\mathcal{L}_{10}^{\rm sys}X\otimes R^*A+\mathcal{L}_{11}^{\rm sys}X\otimes R^*AR\\ &\quad+X\otimes\mathcal{L}_{00}^{\rm aux}A.\end{aligned} \tag{11}$$

4.2. Matrix form of the master and filter equations

Let us fix an orthonormal basis $\{e_n\}$ for the auxiliary space $\mathbb{C}^D$. We introduce the $D\times D$ matrix $\Upsilon_t(X)$ with entries

$$\Upsilon_t^{nm}(X)=\tilde{\mathbb{E}}_t\left(X\otimes E_{mn}\right)$$

where $E_{mn}=\left|e_m\right\rangle\left\langle e_n\right|$. The CMP expectation is then

$$\mathbb{E}_t^{\rm cmp}(X)=tr\,\Upsilon_t(X),$$

and more generally $\tilde{\mathbb{E}}_t(X\otimes A)=tr\left\{\Upsilon_t(X)A\right\}$. Unlike the vacuum case, there is no single closed master equation for $\mathbb{E}_t^{\rm cmp}(X)$, and instead we must solve a system of matrix equations.

4.2.1. CMP master equation: This takes the form

$$\frac{d}{\mathrm{d}t}\Upsilon_t(X) = \Upsilon_t\left(\mathcal{L}_{00}^{\mathrm{sys}}X\right) + R\Upsilon_t\left(\mathcal{L}_{01}^{\mathrm{sys}}X\right) + \Upsilon_t\left(\mathcal{L}_{10}^{\mathrm{sys}}X\right)R^* + \mathcal{L}_{00}^{\mathrm{aux}*}\Upsilon_t(X) + R\Upsilon_t\left(\mathcal{L}_{11}^{\mathrm{sys}}X\right)R^*, \tag{12}$$

where

$$\mathcal{L}_{00}^{\mathrm{aux}*}(A) = \frac{1}{2}\left[LA, L^*\right] + \frac{1}{2}\left[L, AL^*\right]. \tag{13}$$

These are obtained by extending the standard master equation to include the auxiliary system and using (11).

4.2.2. CMP filter equation: Similarly we can consider the total filter

$$\tilde{\Pi}_t(X \otimes A) = \mathbb{E}_{\eta\Psi_\infty}\left[\tilde{V}_{\mathrm{sys}}(t)^*\left(X \otimes A \otimes I_{\mathfrak{F}}\right)\tilde{V}_{\mathrm{sys}}(t)\,\middle|\,\mathcal{Y}_t\right],$$

and introduce the $D \times D$ matrix $\Pi_t(X)$ with entries

$$\Pi_t^{nm}(X) = \tilde{\Pi}_t\left(X \otimes E_{mn}\right).$$

The CMP filter is then $\Pi_t^{\mathrm{cmp}}(X) = \tilde{\Pi}_t(X \otimes I) \equiv \mathrm{tr}_{\mathrm{aux}}\left\{\Pi_t(X)\right\}$, where $\mathrm{tr}_{\mathrm{aux}}(\,\cdot\,)$ denotes the partial trace over the auxiliary Hilbert space. We can again determine the explicit form of the filter equations for both homodyne and number counting cases. We may take the form of the previous filter equations and extend as above. The resulting equations are as follows:

4.2.3. Homodyne CMP state filter. The filter is

$$\begin{aligned} d\Pi_t(X) = &\left\{\Pi_t\left(\mathcal{L}_{00}^{\mathrm{sys}}X\right) + R\Pi_t\left(\mathcal{L}_{01}^{\mathrm{sys}}X\right) + \Pi_t\left(\mathcal{L}_{10}^{\mathrm{sys}}X\right)R^* + R\Pi_t\left(\mathcal{L}_{11}^{\mathrm{sys}}X\right)R^*\right. \\ &\left. + \mathcal{L}_{00}^{\mathrm{aux}\star}\Pi_t(X)\right\}\mathrm{d}t \\ &+ \left\{\Pi_t\left(XL + L^*X\right) + R\Pi_t(XS) + \Pi_t\left(S^*X\right)R^* - \lambda_t\,\Pi_t(X)\right\} \\ &\times\left[dY - \lambda_t\mathrm{d}t\right], \end{aligned} \tag{14}$$

where $\mathcal{L}_{00}^{\mathrm{aux}\star}\Pi = R\Pi R^* - \frac{1}{2}\left\{\Pi, R^*R\right\}_+ + i\left[\Pi, H_{\mathrm{aux}}\right]$ and

$$\lambda_t = tr\left\{\Pi_t\left(L + L^*\right) + R\Pi_t(S) + \Pi_t\left(S^*\right)R^*\right\}.$$

4.2.4. Number counting CMP filter. The filter is

$$\begin{aligned} d\Pi_t(X) = &\left\{\Pi_t\left(\mathcal{L}_{00}^{\mathrm{sys}}X\right) + R\Pi_t\left(\mathcal{L}_{01}^{\mathrm{sys}}X\right) + \Pi_t\left(\mathcal{L}_{10}^{\mathrm{sys}}X\right)R^* + R\Pi_t\left(\mathcal{L}_{11}^{\mathrm{sys}}X\right)R^*\right. \\ &\left. + \mathcal{L}_{00}^{\mathrm{aux}\star}\Pi_t(X)\right\}\mathrm{d}t \\ &+ \nu_t^{-1}\left\{\Pi_t\left(L^*XL\right) + R\Pi_t\left(L^*XS\right) + \Pi_t\left(S^*XL\right)R^*\right. \\ &\left. + R\Pi_t\left(S^*XS\right)R^* - \nu_t\Pi_t(X)\right\}\left[dY - \nu_t\mathrm{d}t\right], \end{aligned} \tag{15}$$

where $\nu_t = tr\left\{\Pi_t\left(L^*L\right) + R\Pi_t\left(L^*S\right) + \Pi_t\left(S^*L\right)R^* + RR^*\right\}$.

4.2.5. Stochastic master equation. We may introduce a density matrix ϱ_t over the system and auxiliary space such that $\Pi_t(X) \equiv tr\left\{\varrho_t X\right\}$. This may be viewed as again as a $D \times D$ matrix whose entries are trace class operators on the system space, and $tr\left\{\varrho_t X\right\}$ denotes taking the trace on the $D \times D$ matrix $\varrho_t X = \left[\varrho_t^{nm} X\right]_{n,m=1,2,\ldots,D}$. The corresponding stochastic master equation for ϱ_t is then

$$d\varrho_t = \mathcal{L}^\star \varrho_t \mathrm{d}t + \begin{cases} \left[\tilde{L}\varrho_t + \varrho_t \tilde{L}^* - \lambda_t \varrho_t\right]\left[dY(t) - \lambda_t \mathrm{d}t\right], & \text{homodyne;} \\ \dfrac{1}{\nu_t}\left[\tilde{L}\varrho_t \tilde{L}_t - \nu_t \varrho_t\right]\left[dY(t) - \nu_t \mathrm{d}t\right], & \text{number counting;} \end{cases}$$

where the dynamical term is

$$\begin{aligned} \mathcal{L}^\star \varrho = &\left(\mathcal{L}_{00}^{\mathrm{sys}\star} \otimes I_{\mathrm{aux}}\right)(\varrho) + \left(\mathcal{L}_{01}^{\mathrm{sys}\star} \otimes I_{\mathrm{aux}}\right)(\varrho R) + \left(\mathcal{L}_{10}^{\mathrm{sys}\star} \otimes I_{\mathrm{aux}}\right)\left(R^* \varrho\right) \\ &+ \left(\mathcal{L}_{11}^{\mathrm{sys}\star} \otimes I_{\mathrm{aux}}\right)\left(R^* \varrho R\right) + \left(I_{\mathrm{sys}} \otimes \mathcal{L}_{00}^{\mathrm{aux}\star}\right)(\varrho) \end{aligned}$$

and $\lambda_t = tr\left\{\varrho_t\left(\tilde{L} + \tilde{L}^*\right)\right\}$, $\nu_t = tr\left\{\varrho_t \tilde{L}^* \tilde{L}\right\}$. Here we use the usual convention of $\mathcal{J}^\star$ for the dual of a superoperator $\mathcal{J}$, and that $\mathcal{J} \otimes I_{\mathrm{aux}}$ acting on a matrix with operator entries ϱ^{nm} yields the matrix with entries $\mathcal{J}\left(\varrho^{nm}\right)$.

5. Examples

5.1. Single photon sources

As a special example of an asymptotic CMP state, let us take $D = 2$ and fix $R(t) = \frac{1}{\sqrt{w(t)}}\xi(t)\sigma_-$, $H_{\mathrm{aux}} = 0$, where ξ is a normalized square-integrable function on $[0, \infty)$, $w(t) = \int_t^\infty |\xi(s)|^2 \mathrm{d}s$, and σ_- is the lowering operator from the upper state $|\uparrow\rangle$ to the ground state $|\downarrow\rangle$ on $\mathfrak{h}_{\mathrm{aux}} = \mathbb{C}^2$. We take the initial state to be $|\phi\rangle = |\uparrow\rangle$. The interpretation is that we have a two-level atom prepared in its excited state $|\uparrow\rangle$ and coupled to the vacuum input field. At some stage the atom decays through spontaneous emission into its ground state $|\downarrow\rangle$ creating a single photon in the output. The Schrödinger equation for $\left|\Psi_t\right\rangle = V_{\mathrm{aux}}(t)\,|\uparrow\rangle \otimes |\Omega\rangle$ is $d\left|\Psi_t\right\rangle = \left[\lambda(t)\sigma_- \mathrm{d}B_t^* - \frac{1}{2}|\lambda(t)|^2 \sigma_+\sigma_- \mathrm{d}t\right]\left|\Psi_t\right\rangle$, where $\lambda(t) = \frac{1}{\sqrt{w(t)}}\xi(t)$ and it is easy to see that this has the exact solution $\left|\Psi_T\right\rangle = \sqrt{w(T)}\,|\uparrow\rangle \otimes |0\rangle + |\downarrow\rangle \otimes B_{\mathrm{in},T}^*(\xi)\,|\Omega\rangle$ where $B_{\mathrm{in},T}^*(\xi) = \int_0^T \xi_t \mathrm{d}B_{\mathrm{in}}^*(t)$. As $w(\infty) = \|\xi\|^2 = 1$, we therefore generate the limit state $\left|\Psi_\infty\right\rangle = |\downarrow\rangle \otimes B_{\mathrm{in}}^*(\xi)\,|\Omega\rangle$. In this way we engineered a single photon with one-particle state $\left|1_\xi\right\rangle = B^*(\xi)\,|\Omega\rangle$. In the single-photon case, we encounter 2×2 systems of equations. As $R(t)^2 = 0$ we have some hierarchical simplification in these systems. The matrix master equation was effectively derived by Gheri *et al* [16] in 1998, however the filtering equations only more recently in [17, 18].

5.2. Time-ordered multi-photon sources

Recently, quantum filters for multiple photon input states have been derived [20], extending the work mentioned in the previous section. The derivation of the multi-photon filters in [20] employed a non-

Markovian embedding technique, generalizing the approach of [17]. Here we indicate briefly that the filter equation for time-ordered multi-photon inputs may be derived using the CMP approach presented in the present paper, generalizing the Markovian embedding approach of [18] for the single photon case.

Consider the n-photon state

$$\begin{aligned}\left|\overrightarrow{\xi_n, \cdots, \xi_1}\right\rangle &= \frac{1}{\prod_{k=1}^{n-1}\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}}\vec{T}B^*(\xi_n)\cdots B^*(\xi_1)\mid\Omega\rangle \\ &= \frac{1}{\prod_{k=1}^{n-1}\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}} \\ &\quad\times\int_{\Delta_n}\xi_n(s_n)\cdots\xi_1(s_1)\,\mathrm{d}B^*(s_n)\cdots\mathrm{d}B^*(s_1)\mid\Omega\rangle, \\ &= \frac{1}{\prod_{k=1}^{n-1}\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}} \\ &\quad\times\int_0^\infty \xi_n(s_n)\,\mathrm{d}B^*(s_n)\int_0^{s_n}\xi_{n-1}(s_{n-1})\,\mathrm{d}B^*(s_{n-1}) \\ &\quad\cdots\int_0^{s_2}\xi_1(s_1)\,\mathrm{d}B^*(s_1)\mid\Omega\rangle,\end{aligned}$$

where $\xi_1, \ldots, \xi_n$ are normalized wave packet shapes (i.e., $\int_0^\infty |\xi_k(s)|^2 ds = 1$) and the integral is over the simplex

$$\Delta_n = \left\{(s_n, \cdots, s_1)\colon s_n > s_{n-1} > \cdots > s_1 > 0\right\}.$$

Note that we may define the time-ordering of a function F_n of n distinct time arguments by

$$\vec{T}F_n(s_n, \cdots, s_1) := F_n\left(s_{\sigma(n)}, \cdots, s_{\sigma(1)}\right),$$

where σ is the permutation such that $\left(s_{\sigma(n)}, \cdots, s_{\sigma(1)}\right) \in \Delta_n$. In this case,

$$\begin{aligned}\left|\overrightarrow{\xi_n, \cdots, \xi_1}\right\rangle &\equiv \frac{1}{n!\prod_{k=1}^{n-1}\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}} \\ &\quad\int_{[0,\infty)^n}\vec{T}\xi_n\otimes\cdots\otimes\xi_1(s_n, \cdots, s_1)\,\mathrm{d}B^*(s_n)\cdots\mathrm{d}B^*(s_1)\mid\Omega\rangle.\end{aligned}$$

We note that when the wavepackets are all identical ($\xi_1 = \cdots\xi_n \equiv \xi$), the state reduces to the n-particle state

$$\left|\overrightarrow{\xi, \cdots, \xi}\right\rangle \equiv \frac{1}{\sqrt{n!}}\,B^*(\xi)^n\mid\Omega\rangle.$$

For these time-ordered multi-photon states, we take $D = n + 1$, so that the auxiliary system will be realized on $\mathfrak{h}_{\mathrm{aux}} = \mathbb{C}^{n+1}$ with basis $\mid 0\rangle, \ldots, \mid n\rangle$. The initial state is taken to be the excited state $\mid n\rangle$. We take $S = I$,

$$R(t) = \sum_{k=1}^{n} \lambda_{n+1-k}(t) \mid k-1\rangle\langle k \mid,$$

and $H_{\text{aux}} = 0$, where

$$\lambda_k(t) = \frac{\xi_k(t)\sqrt{w_{k+1}(t)}}{\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}\sqrt{w_k(t)}},$$

and

$$w_k(t) = \frac{\int_t^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s},$$

for $k = 1, 2\ldots, n$, with

$$w_{n+1}(t) = 1.$$

This auxiliary system acts as the generator of a time-ordered n-photon state, as detailed in appendix A for time-ordered two-photon states, and appendix B for the general time-ordered case with $n > 2$. The treatment of the two-photon generator in appendix A provides a more detailed account of the underlying idea which is subsequently generalized in appendix B. The CMP filter for this time-ordered n-photon input state will be a $(n+1) \times (n+1)$ system of coupled stochastic differential equations obtained from (14) or (15).

For filtering in non-time-ordered multi-photon states of the form $B(\xi_1)^* \cdots B(\xi_n)^* \mid \Omega\rangle$ see [20]. This, however, requires a non-Markovian embedding approach.

6. Conclusion

In this paper we have introduced a new class of CMP states, which includes (continuous-mode) single photon and time-ordered multi-photon states as special cases. We then derive the quantum master equation and quantum filtering (quantum trajectory) equations for Markovian open quantum systems driven by boson fields in the new class of CMP states, that naturally take the form of matrix-valued equations with operator entries. A Markovian generator of time-ordered continuous-mode multi-photon states was also obtained, thus allowing the quantum master and filtering equations for systems driven by time-ordered multi-photon states to be readily derived using the general formulas of this paper, in particular generalizing the Markovian embedding approach in [18].

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Appendix A. Time-ordered continuous-mode two-photon state generator

In this appendix we develop a Markovian generator model for time-ordered continuous-mode two-photon states of the field, and show explicitly that these generators indeed produce time-ordered two-photon states. The ideas in this appendix are extended to the general time-ordered multi-photon case in appendix B.

We consider an open three level system with levels $|0\rangle = (0, 0, 1)^T$, $|1\rangle = (0, 1, 0)^T$, $|2\rangle = (1, 0, 0)^T$ coupled to a vacuum continuum boson field via the (time-varying) coupling operator $L(t) = \lambda_2(t)\,|0\rangle\langle 1| + \lambda_1(t)\,|1\rangle\langle 2|$, for some given functions $\lambda_1(t)$ and $\lambda_2(t)$ that will be specified shortly. We can thus write $L(t)$ as the 3×3 matrix-valued function

$$L(t) = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \end{bmatrix}.$$

For given wave packet shapes $\xi_1(t)$ and $\xi_2(t)$ (they need not be the same shape), define $w_2(t) = \int_t^\infty |\xi_2(s)|^2 \mathrm{d}s$ and

$$w_1(t) = \frac{\int_t^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s},$$

and note that $w_k(0) = 1$ and $w_k(\infty) = 0$ for all k. From these definitions then define $\lambda_1(t)$ and $\lambda_2(t)$ as

$$\lambda_2(t) = \frac{\xi_2(t)}{\sqrt{w_2(t)}}$$

$$\lambda_1(t) = \frac{\xi_1(t)\sqrt{w_2(t)}}{\sqrt{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}\,\sqrt{w_1(t)}}.$$

Before proceeding further, we note that

$$\exp\left(-\frac{1}{2}\int_0^t \left|\lambda_j(s)\right|^2 \mathrm{d}s\right) = \sqrt{w_j(t)}$$

for $j = 1, 2$. Let $|\psi(t)\rangle = \left(\psi_2(t), \psi_1(t), \psi_0(t)\right)^T \otimes \left|\Omega_{[t}\right\rangle$ ($\left|\Omega_{[t}\right\rangle$ denotes the portion of the Fock vacuum on $[t, \infty)$)) be a state vector process solving the QSDE

$$d\,|\psi(t)\rangle = \left(-\frac{1}{2}L^*L\mathrm{d}t + L\mathrm{d}B^*(t) - L^*\mathrm{d}B(t)\right)|\psi(t)\rangle.$$

with initial condition $|\psi(0)\rangle = |2\rangle \otimes |\Omega\rangle$, where $|\Omega\rangle$ denotes the Fock vacuum. Since $|\psi(t)\rangle$ has a tensor product form with a vacuum component on the portion of the Fock space from t onwards, the QSDE can be simplified as

$$d\,|\psi(t)\rangle = \left(-\frac{1}{2}L^*L\mathrm{d}t + L\mathrm{d}B^*(t)\right)|\psi(t)\rangle.$$

$$= \begin{bmatrix} -\frac{1}{2}\left|\lambda_1(t)\right|^2 \mathrm{d}t & 0 & 0 \\ \lambda_1(t)\,\mathrm{d}B^*(t) & -\frac{1}{2}\left|\lambda_2(t)\right|^2 \mathrm{d}t & 0 \\ 0 & \lambda_2(t)\,\mathrm{d}B^*(t) & 0 \end{bmatrix} |\,\psi(t)\,\rangle.$$

This leads to the following set of coupled equations for the component ψ_2, ψ_1 and ψ_0 of ψ:

$$d\begin{bmatrix} \psi_2(t) \\ \psi_1(t) \\ \psi_0(t) \end{bmatrix} \otimes \left|\Omega_{[t}\right\rangle = \begin{bmatrix} -\frac{1}{2}\left|\lambda_1(t)\right|^2 \psi_2(t)\,\mathrm{d}t \\ -\frac{1}{2}\left|\lambda_2(t)\right|^2 \psi_1(t)\,\mathrm{d}t + \lambda_1(t)\,\psi_2(t)\,\mathrm{d}B^*(t) \\ \lambda_2(t)\,\psi_1(t)\,\mathrm{d}B^*(t) \end{bmatrix} \left|\Omega_{[t}\right\rangle$$

with initial conditions $\psi_2(0) = 1$, $\psi_1(0) = 0$, and $\psi_0(0) = 0$. Using the definitions and properties of w_1, w_2, λ_1, and λ_2, the special structure of the coupled equations allow them to be solved explicitly giving the solutions:

$$\psi_2(t) = \exp\left(-\frac{1}{2}\int_0^t \left|\lambda_1(s)\right|^2 \mathrm{d}s\right) = \sqrt{w_1(t)}\ \left|\Omega_{t]}\right\rangle,$$

$$\begin{aligned} \psi_1(t) &= \exp\left(-\frac{1}{2}\int_0^t \left|\lambda_2(s)\right|^2 \mathrm{d}s\right) \\ &\quad \times \int_0^t \lambda_1(s_1)\exp\left(\frac{1}{2}\int_0^{s_1}\left(\left|\lambda_2(s_2)\right|^2 - \left|\lambda_1(s_2)\right|^2\right)ds_2\right)\mathrm{d}B^*(s_1)\ \left|\Omega_{t]}\right\rangle, \\ &= \frac{\sqrt{w_2(t)}}{\sqrt{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}} \int_0^t \xi_1(s_1)\,\mathrm{d}B^*(s_1)\ \left|\Omega_{t]}\right\rangle, \end{aligned}$$

$$\begin{aligned} \psi_0(t) &= \int_0^t \lambda_2(\tau)\,\psi_1(\tau)\,\mathrm{d}B^*(\tau)\ \left|\Omega_{t]}\right\rangle, \\ &= \frac{1}{\sqrt{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}} \int_0^t \lambda_2(\tau)\sqrt{w_2(\tau)} \int_0^\tau \xi_1(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ \left|\Omega_{t]}\right\rangle, \\ &= \frac{1}{\sqrt{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}} \int_0^t \xi_2(\tau)\,\mathrm{d}B^*(\tau) \int_0^\tau \xi_1(s_1)\,\mathrm{d}B^*(s_1)\ \left|\Omega_{t]}\right\rangle. \end{aligned}$$

Note that as $t \to \infty$, $\psi_2(t) \to 0$ and $\psi_1(t) \to 0$ since $\lim_{k\to\infty} w_k(t) = 0$ for $k = 1, 2$. That is, as $t \to \infty$ the atom decays to its ground state and the output field of the system tends to the state $\psi_0(\infty)$. More precisely,

$$\psi_0(t) = \frac{1}{\sqrt{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}} \int_0^t \int_0^\tau \xi_2(\tau)\,\xi_1(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ \left|\Omega_{t]}\right\rangle,$$

which shows that $\psi_0(t)$ converges as $t \to \infty$ to the time-ordered two-photon state

$$\psi_0(\infty) = \frac{1}{\sqrt{\int_0^\infty |\xi_1(s)|^2 w_2(s)\,\mathrm{d}s}} \int_0^\infty \int_0^\tau \xi_2(\tau)\xi_1(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ |\Omega\rangle$$
$$= \left|\overrightarrow{\xi_2, \xi_1}\right\rangle.$$

Moreover, since $\psi_0(\infty)$ is a bona fide pure state vector on the field, we also note that

$$\left\| \int_0^\infty \int_0^\tau \xi_2(\tau)\xi_1(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ |\Omega\rangle \right\| = \sqrt{\int_0^\infty |\xi_1(s)|^2 w_2(s)\,\mathrm{d}s}.$$

If $\xi_1(t) = \xi_2(t) \equiv \xi(t)$ then $\xi_1(s_1)\xi_2(\tau) = \xi(s_1)\xi(\tau)$ becomes symmetric with respect to its arguments s_1 and τ, and so we have the identity:

$$\int_0^\infty \int_0^\tau \xi(\tau)\xi(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ |\Omega\rangle = \frac{1}{2}\int_0^\infty \int_0^\infty \xi(\tau)\xi(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ |\Omega\rangle.$$

Moreover, also we have that $\int_0^\infty |\xi_1(s)|^2 w_2(s)\,\mathrm{d}s = \frac{1}{2}$. It follows that for the special case $\xi_1(t) = \xi_2(t) = \xi(t)$ that

$$\psi_0(\infty) = \sqrt{2}\frac{1}{2}\int_0^\infty \int_0^\infty \xi(\tau)\xi(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ |\Omega\rangle,$$
$$= \frac{1}{\sqrt{2}}\int_0^\infty \int_0^\infty \xi(\tau)\xi(s_1)\,\mathrm{d}B^*(s_1)\,\mathrm{d}B^*(\tau)\ |\Omega\rangle,$$
$$= \frac{1}{\sqrt{2}} B^*(\xi)^2\ |\Omega\rangle.$$

Appendix B. Time-ordered continuous-mode *n*-photon state generator with $n > 2$

In this appendix, we generalize the time-ordered two-photon generator model treated in appendix A to general time-ordered n-photon case with $n > 2$.

Consider an open $n+1$ level system with levels $|0\rangle = (0, 0, \ldots, 0, 1)^T$, $|1\rangle = (0, 0, 0, \ldots, 1, 0)^T, \ldots, |n\rangle = (1, 0, 0, \ldots, 0)^T$, coupled to a vacuum continuum boson field via the (time-varying) coupling operator $L(t) = \sum_{k=1}^n \lambda_{n+1-k}(t)\ |k-1\rangle\langle k|$, for some given functions $\lambda_1(t), \ldots, \lambda_n(t)$ that will be specified shortly. We can thus write $L(t)$ as the $(n+1) \times (n+1)$ matrix-valued function

$$L(t) = \begin{bmatrix} 0_{1\times n} & 0 \\ \mathrm{diag}\left(\lambda_1(\mathrm{t}), \lambda_2(\mathrm{t}), \ldots, \lambda_n(\mathrm{t})\right) & 0_{n\times 1} \end{bmatrix},$$

where $\mathrm{diag}(a_1, a_2, \ldots, a_m)$ denotes a diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_m$ from top left to bottom right.

For given wave packet shapes $\xi_1(t), \xi_2(t), \ldots \xi_n(t)$ (not necessarily identical), define $w_n(t) = \int_t^\infty |\xi_n(s)|^2\, ds$, and

$$w_k(t) = \frac{\int_t^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s},$$

recursively for $k = n-1, n-2, \ldots, 1$. Note, in particular, that $w_k(0) = 1$ and $w_k(\infty) = 0$ for all k. From these definitions then define $\lambda_n(t) = \frac{\xi_n(t)}{\sqrt{w_n(t)}}$, and

$$\lambda_k(t) = \frac{\xi_k(t)\sqrt{w_{k+1}(t)}}{\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}\sqrt{w_k(t)}},$$

recursively for $k = 1, 2, \ldots, n-1$. As was the case for time-ordered two-photon fields, we verify from the definitions that

$$\exp\left(-\frac{1}{2}\int_0^t \left|\lambda_j(s)\right|^2 \mathrm{d}s\right) = \sqrt{w_j(t)},$$

for $j = 1, \ldots, n$. Let $|\psi(t)\rangle = \left(\psi_2(t), \psi_1(t), \psi_0(t)\right)^T \otimes \left|\Omega_{[t}\right\rangle$ ($\left|\Omega_{[t}\right\rangle$ denotes the portion of the Fock vacuum on $[t, \infty)$) be a state vector process solving the QSDE

$$d\,|\psi(t)\rangle = \left(-\frac{1}{2}L^*L\mathrm{d}t + L\mathrm{d}B^*(t) - L^*\mathrm{d}B(t)\right)|\psi(t)\rangle.$$

with initial condition $|\psi(0)\rangle = |n\rangle \otimes |\Omega\rangle$, where $|\Omega\rangle$ denotes the Fock vacuum. Since $|\psi(t)\rangle$ has a tensor product form with a vacuum component on the portion of the Fock space from t onwards, the QSDE can be simplified as

$$\begin{aligned} d\,|\psi(t)\rangle &= \left(-\frac{1}{2}L^*L\mathrm{d}t + L\mathrm{d}B^*(t)\right)|\psi(t)\rangle. \\ &= \left(-\frac{1}{2}\begin{bmatrix} \mathrm{diag}\left(\left|\lambda_1(t)\right|^2, \ldots, \left|\lambda_{n-1}(t)\right|^2, \left|\lambda_n(t)\right|^2\right) & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{bmatrix}\mathrm{d}t \right. \\ &\quad \left. + \begin{bmatrix} 0_{1\times n} & 0 \\ \mathrm{diag}\left(\lambda_1(\mathrm{t}), \ldots, \lambda_{n-1}(\mathrm{t}), \lambda_n(\mathrm{t})\right) & 0_{n\times 1} \end{bmatrix}\mathrm{d}B^*(t)\right)|\psi(t)\rangle. \end{aligned}$$

This leads to the following set of coupled equations for the component $\psi_0, \psi_1, \ldots, \psi_n$ of ψ:

$$\begin{aligned} d\begin{bmatrix} \psi_n(t) \\ \vdots \\ \psi_1(t) \\ \psi_0(t) \end{bmatrix} \otimes \left|\Omega_{[t}\right\rangle = \left(-\frac{1}{2}\begin{bmatrix} \left|\lambda_1(t]\right|^2\psi_n(t) \\ \vdots \\ \left|\lambda_n(t)\right|^2\psi_1(t) \\ 0 \end{bmatrix}\mathrm{d}t \right. \\ \left. + \begin{bmatrix} 0 \\ \lambda_1(t]\psi_n(t) \\ \vdots \\ \lambda_n(t)\psi_1(t) \end{bmatrix}\mathrm{d}B^*(t)\right)\left|\Omega_{[t}\right\rangle, \end{aligned}$$

with initial conditions $\psi_n(0) = 1$, and $\psi_k(0) = 0$ for $k = 0, 1, \ldots, n-1$. Using the definitions and properties of $w_1, \ldots, w_n$, and $\lambda_1, \ldots, \lambda_n$, as with the two-photon case the special structure of

the coupled equations allow them to be solved explicitly, giving the solutions:

$$\psi_n(t) = \sqrt{w_1(t)}\ \left|\Omega_{t]}\right\rangle,$$

$$\psi_{n-1}(t) = \frac{\sqrt{w_2(t)}}{\sqrt{\int_0^\infty \left|\xi_1(s)\right|^2 w_2(s)\,\mathrm{d}s}} \int_0^t \xi_1(s_1)\,\mathrm{d}B^*(s_1)\ \left|\Omega_{t]}\right\rangle,$$

$$\vdots$$

$$\psi_1(t) = \frac{\sqrt{w_n(t)}}{\prod_{k=1}^{n-1}\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}} \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} \xi_{n-1}(s_{n-1}) \cdots \xi_1(s_1)$$
$$\times\, \mathrm{d}B^*(s_1) \ldots \mathrm{d}B^*(s_{n-2})\,\mathrm{d}B^*(s_{n-1})\ \left|\Omega_{t]}\right\rangle,$$

$$\psi_0(t) = \frac{1}{\prod_{k=1}^{n-1}\sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \xi_n(s_n)\,\xi_{n-1}(s_{n-1}) \cdots$$
$$\xi_1(s_1)\,\mathrm{d}B^*(s_1) \cdots \mathrm{d}B^*(s_{n-1})\,\mathrm{d}B^*(s_n)\ \left|\Omega_{t]}\right\rangle.$$

Note that as $t \to \infty$, $\psi_k(t) \to 0$ as $t \to \infty$ for $k = 1, \ldots, n$ since $\lim_{k\to\infty} w_k(t) = 0$. That is, as $t \to \infty$ the atom decays to its ground state and the output field of the system tends to the state $\psi_0(\infty)$. Taking the limit, $\psi_0(t)$ converges as $t \to \infty$ to the time-ordered n-photon state

$$\psi_0(\infty) = \left|\overrightarrow{\xi_n, \cdots, \xi_1}\right\rangle.$$

Moreover, since $\psi_0(\infty)$ is a bona fide pure state vector on the field, we note that

$$\left\|\int_0^\infty \int_0^{s_n} \cdots \int_0^{s_2} \xi_n(s_n)\,\xi_{n-1}(s_{n-1}) \cdots \xi_1(s_1)\,\mathrm{d}B^*(s_1) \cdots \mathrm{d}B^*(s_{n-1})\,\mathrm{d}B^*(s_n)\ |\,\Omega\rangle\right\|$$
$$= \prod_{k=1}^{n-1} \sqrt{\int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s}\,.$$

In the special case where $\xi_1 = \xi_2 = \ldots = \xi_n \equiv \xi$ then $\xi_n(s_n)\,\xi_{n-1}(s_{n-1}) \cdots \xi_1(s_1) = \xi(s_n)\,\xi(s_{n-1}) \cdots \xi(s_1)$ becomes symmetric with respect to the arguments $s_1, s_2, \ldots, s_n$ and we have

$$\int_0^\infty \int_0^{s_n} \cdots \int_0^{s_2} \xi(s_n)\,\xi(s_{n-1}) \cdots \xi(s_1)\,\mathrm{d}B^*(s_1) \cdots \mathrm{d}B^*(s_{n-1})\,\mathrm{d}B^*(s_n)\ |\,\Omega\rangle$$
$$= \frac{1}{n!} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \xi(s_n)\,\xi(s_{n-1}) \cdots \xi(s_1)\,\mathrm{d}B^*(s_1) \cdots \mathrm{d}B^*(s_{n-1})\,\mathrm{d}B^*(s_n)\ |\,\Omega\rangle$$

and

$$\prod_{k=1}^{n-1} \int_0^\infty \left|\xi_k(s)\right|^2 w_{k+1}(s)\,\mathrm{d}s = \frac{1}{n!}.$$

Therefore, in this case

$$\psi_0(\infty) = \frac{1}{\sqrt{n!}} B^*(\xi)^n \mid \Omega\rangle.$$

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