**Aberystwyth University**

# *An asymptotic method of factorization of a class of matrix-functions*

Mishuris, Gennady; Rogosin, Sergei

*Published in:*
Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences

*DOI:*
10.1098/rspa.2014.0109

*Publication date:*
2014

*Citation for published version (APA):*
Mishuris, G., & Rogosin, S. (2014). An asymptotic method of factorization of a class of matrix-functions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *470*(2166), [20140109]. https://doi.org/10.1098/rspa.2014.0109

PROCEEDINGS OF THE ROYAL SOCIETY A

rspa.royalsocietypublishing.org

Research

Cite this article: Mishuris G, Rogosin S. 2014 An asymptotic method of factorization of a class of matrix functions. *Proc. R. Soc. A* **470**: 20140109. http://dx.doi.org/10.1098/rspa.2014.0109

Received: 10 February 2014
Accepted: 18 March 2014

**Subject Areas:**
analysis

**Keywords:**
factorization of matrix functions, asymptotic method, non-rational matrices

**Author for correspondence:**
Gennady Mishuris
e-mail: ggm@aber.ac.uk

# An asymptotic method of factorization of a class of matrix functions

Gennady Mishuris[1] and Sergei Rogosin[1,2]

[1]Institute of Mathematics, Physics and Computer Science, Aberystwyth University, Penglais, Aberystwyth SY23 2BZ, UK
[2]Department of Economics, Belarusian State University, Nezavisimosti Avenue 4, Minsk 220030, Belarus

A novel method of asymptotic factorization of $n \times n$ matrix functions is proposed. The considered class of matrices is motivated by certain problems originated from the elasticity theory. An example is constructed to illustrate efficiency of the proposed procedure. The quality of approximation and the role of the chosen small parameter are discussed.

## 1. Introduction

We consider here the problem of factorization of continuous matrix functions of the real variable. This means the representation of a given invertible square matrix $G \in (\mathcal{C}(\mathbb{R}))^{n\times n}$ in the following form:

$$G(x) = G^{-}(x)\Lambda(x)G^{+}(x), \tag{1.1}$$

where continuous invertible matrices $G^{-}(x)$ and $G^{+}(x)$ possess an analytic continuation in the lower $\Pi^{-} = \{z = x + iy : \mathrm{Im} < 0\}$ and upper $\Pi^{+} = \{z = x + iy : \mathrm{Im} > 0\}$ half-planes, respectively, and

$$\Lambda(x) = \mathrm{diag}\left(\left(\frac{x-i}{x+i}\right)^{\kappa_1}, \ldots, \left(\frac{x-i}{x+i}\right)^{\kappa_n}\right), \quad \kappa_1, \ldots, \kappa_n \in \mathbb{Z}. \tag{1.2}$$

The representation (1.1) is called *right (continuous or standard) factorization* and can be considered for any oriented curve $\Gamma$ of a certain classes which divides the complex plane into two domain $D^{-}$ and $D^{+}$ with a changing of diagonal entries in $\Lambda(x)$ for $((x - t^{+})/(x - t^{-}))^{\kappa_j}$, $t^{\mp} \in D^{\mp}$, or for $x^{\kappa_j}$ (if $0 \in D^{+}$). A similar representation

$$G(x) = G^{+}(x)\Lambda(x)G^{-}(x)$$

is called *left (continuous or standard) factorization*. If the right- (left-) factorization exists, then the integer numbers

Royal Society Publishing

$\kappa_1, \ldots, \kappa_n$, called *partial indices*, are determined uniquely up to the order. In particular, there exist constant transformations of factors, such that $\kappa_1 \geq \cdots \geq \kappa_n$. The factors $G^-$ and $G^+$ are not unique. Relations of pairs of factors are described, for example, in [1]. The right- (left-) factorization is called *canonical factorization* if all partial indices are equal to 0, i.e. $\kappa_1 = \cdots = \kappa_n = 0$.

Factorization of matrix functions was first studied in relation to the vector–matrix Riemann (or Riemann–Hilbert) boundary value problem [2]; this was formulated by Riemann in his work on construction of complex differential equations with algebraic coefficients having a prescribed monodromy group [3] (see also [4]). By using the method of the Cauchy type integral, the vector–matrix Riemann boundary value problem was reduced in [5,6] to a system of the Fredholm integral equations. A part of the theory of the factorization problem is based on the study of such systems (see also [7]), though this approach does not answer, in particular, the questions of when it is possible to get factorization, how to construct factors and how to determine partial indices.

Among other sources of interest to the factorization problems, one can point out the vector-valued Wiener–Hopf equations on a half-line [8–11] and their discrete analogies, namely the block Toeplitz equations [12,13]. The developed technique found several applications in diffraction theory, fracture mechanics, geophysics, financial mathematics, etc. (see a brief description given, for example, in [14] and references therein).

Theoretical background for the study of the matrix factorization and its numerous generalizations is presented in [1,3,6,15–17] (see also [18]).

The theory of the factorization is more or less complete [15], but the above-mentioned constructive questions about existence, factors and partial indices (which are very important for practical applications) have been answered only in a number of special cases. Among them, one can mention rational matrix functions [7], functional commutative matrix functions (those satisfying $G(t)G(s) = G(s)G(t), \forall t, s \in \Gamma$ [19]), upper- (lower-) triangular matrices with factorizable diagonal elements [20–23], certain classes of meromorphic matrix functions [24–26], special cases of $2 \times 2$ Daniele–Khrapkov matrix functions (with a small degree of deviator polynomial) [27–30], special cases of $2 \times 2$ matrix functions with three rationally independent entries (see [28,31,32] and references therein), special cases of $n \times n$ generalization of the Daniele–Khrapkov matrix functions [14,33–35] and special classes of matrices possessing certain symmetry property (see [36] and references therein). Several approximate and asymptotic methods for matrix factorization have been developed too [37–39]. In particular, in Abrahams [40], the method of Padé approximation has been developed and applied (see also [18] and references therein).

In this paper, we propose a new asymptotic method of construction of factors for a special class of $n \times n$ non-rational matrix functions. The essential property of the considered matrices is that they become close (after suitable transformation) to a unit matrix (a similar assumption is used in [31,41]). The idea to use such representation in factorization is similar to that in general operator theory and has been exploited since the seminal work by Gohberg & Krein [8] (see also [1,42,43]).

To the best of authors' knowledge, our class does not coincide with any of the above-mentioned classes. This class contains matrix functions which appears in the study of certain problems in fracture mechanics related to perturbation of the crack propagation [44–47]. Another motivation is the use of such matrix functions in the study, the inverse scattering problem [41]. In [41], it is considered the generalized factorization of $2 \times 2$ matrix functions which are similar to that in our special case in §3*b*. An analysis of these authors is based on the equivalence theorem, relating the studied matrices with a product of triangular matrices, followed by solution of the corresponding Riemann boundary value problems. The conditions required to realize this method, as well as the used techniques differ essentially from ours. The main idea of our study is to reduce determination of factors at each step of approximation to the solution of a so-called vector–matrix jump boundary value problem. The paper is organized as follows. In §2, we introduce necessary notation and formulate the problem. A constructive algorithm is presented in §3. We also find it interesting to present here the realization of the algorithm in a special case of matrices of practical importance. The method is illustrated by an example given in §4. We conclude our study by showing the quality of the factorization approximation by restricting ourselves only to the first asymptotic term and discuss the role of the chosen small parameter.

## 2. A class of matrices: problem formulation

Let us introduce the following class of invertible continuous $n \times n, n \geq 2$, matrix functions $\mathcal{G}K_n$ depending on a real parameter $\varphi \in \mathbb{R}$, satisfying the following conditions:

(1) $G_\varphi \in (\mathcal{C}(\mathbb{R}))^{n\times n}$ belongs to $\mathcal{G}K_n$ if it can be represented in the form

$$G_\varphi = R_\varphi F R_\varphi^{-1}, \tag{2.1}$$

where bounded locally Hölder-continuous on $\mathbb{R}$ (in general non-rational) invertible matrix $R_\varphi$ is such that

(2)

$$R_0 = R_\varphi|_{\varphi=0} = I, \tag{2.2}$$

(3) matrix function $F$ does not depend on parameter $\varphi$, has Hölder-continuous entries $f_{kl}$ on the extended real line $\bar{\mathbb{R}}$, i.e. $\forall k, l = 1, \ldots, n$,

$$|f_{kl}(x_1) - f_{kl}(x_2)| \leq C \left| \frac{1}{x_1 + i} - \frac{1}{x_2 + i} \right|^\mu, \quad \forall x_1, x_2 \in \bar{\mathbb{R}}, \ 0 < \mu < 1, \tag{2.3}$$

satisfies the following asymptotic estimate at infinity

(4)

$$F(x) \to I \quad \text{and} \quad |x| \to \infty, \tag{2.4}$$

(5) $F$ admits a right canonical factorization, i.e.

$$F(x) = F^-(x)F^+(x), \tag{2.5}$$

where Hölder-continuous on $\bar{\mathbb{R}}$ matrix functions $F^-(x)$ and $F^+(x)$ possess an analytic continuation in the lower $\Pi^-$ and the upper $\Pi^+$ half-plane, respectively.

The matrices of the following form constitute a simple subclass of class $\mathcal{G}K_2$:

$$G_\varphi(x) = \begin{pmatrix} p(x) & q(x)\,e^{i\varphi x} \\ q(x)\,e^{-i\varphi x} & p(x) \end{pmatrix}. \tag{2.6}$$

This function appears after Fourier transforms of the Wiener–Hopf equation, describing a problem in fracture mechanics.

We note that the matrix functions of this type do not belongs to any known class of matrix functions which admit explicit factorization.

It can be seen below that in our algorithm, the factors retain one of the properties of the matrix function $G_\varphi$, namely,

$$G_\varphi^-(z), \quad G_\varphi^+(z) \to I, \ \text{as } z \to \infty, \quad \mp \operatorname{Im} z > 0. \tag{2.7}$$

## 3. An algorithm

### (a) General construction

By assumption, any matrix $G_\varphi(x) \in \mathcal{G}K_n$ can be written in the form

$$G_\varphi(x) = F^-(x)(F^-(x))^{-1}G_\varphi(F^+(x))^{-1}F^+(x) = F^-(x)G_{1,\varphi}F^+(x), \tag{3.1}$$

where $F^-(x)$ and $F^+(x)$ are components of the canonical factorization of the corresponding matrix $F$, and (see (2.2) and (2.4)) the matrix $G_{1,\varphi}(x)$ is represented in the form

$$G_{1,\varphi}(x) = (F^-(x))^{-1}G_\varphi(F^+(x))^{-1} = (F^-(x))^{-1}R_\varphi F R_\varphi^{-1}(F^+(x))^{-1}.$$

In addition to (1)–(5), we assume that

(6) There exist a small parameter $\varepsilon = \varepsilon(\varphi)$ (more exactly its value will be described later), such that for all $x \in \mathbb{R}$ and any finite $\varphi$

$$G_{1,\varphi}(x) = I + \varepsilon N_{\varphi}(x), \tag{3.2}$$

and matrix $N_{\varphi}(x)$ is bounded and locally Hölder continuous on $\mathbb{R}$.

Note that by assumption, each entry of the matrix $N_{\varphi}(x)$ has a limit as $|x| \to +\infty$, i.e. there exists the value $N_{\varphi}(\infty)$. Note also that the commutativity of the involved matrices is not assumed.

Let us look for the first-order factorization of the matrix $G_{1,\varphi}(x)$ in the form

$$G_{1,\varphi}(x) = I + \varepsilon N_{\varphi}(x) = (I + \varepsilon N^{-}_{1,\varphi}(x))(I + \varepsilon N^{+}_{1,\varphi}(x)). \tag{3.3}$$

Comparing terms for different powers of $\varepsilon$ we get, in particular, the following relation for determination of factors $N^{-}_{1,\varphi}$ and $N^{+}_{1,\varphi}$:

$$N^{-}_{1,\varphi}(x) + N^{+}_{1,\varphi}(x) = N_{\varphi}(x), \quad x \in \mathbb{R}. \tag{3.4}$$

It is customary to denote

$$M_{0,\varphi}(x) \equiv N_{\varphi}(x).$$

The jump boundary value problem (3.4) has a solution represented in terms of a slight modification of the matrix-valued Cauchy type integral [1,5]

$$N^{\mp}_{1,\varphi}(z) = \frac{1}{2} M_{0,\varphi}(\infty) \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M_{0,\varphi}(t) - M_{0,\varphi}(\infty)}{t - z} \, \mathrm{d}t \equiv \frac{1}{2} M_{0,\varphi}(\infty) \mp (\mathbf{C}_0 M_{0,\varphi})(z). \tag{3.5}$$

The above modification is proposed in order to avoid extra discussion of the convergence of the above integrals. In this form, the integrals are convergent automatically. Moreover, its boundary values

$$(\mathbf{C}_0 M_{0,\varphi})^{\mp}(x) = \lim_{\operatorname{Im} z \to \mp 0} (\mathbf{C}_0 M_{0,\varphi})(z)$$

satisfy Sokhotsky–Plemelj formulae, i.e.

$$\begin{aligned}(\mathbf{C}_0 M_{0,\varphi})^{\mp}(x) &= \frac{1}{2} M_{0,\varphi}(x) - \frac{1}{2} M_{0,\varphi}(\infty) \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M_{0,\varphi}(t) - M_{0,\varphi}(\infty)}{t - x} \, \mathrm{d}t \\ &= \frac{1}{2} M_{0,\varphi}(x) - \frac{1}{2} M_{0,\varphi}(\infty) \mp \frac{1}{2} (\mathbf{S}_0 M_{0,\varphi})(x)\end{aligned} \tag{3.6}$$

or

$$N^{\mp}_{1,\varphi}(x) = \tfrac{1}{2} M_{0,\varphi}(x) \mp \tfrac{1}{2} (\mathbf{S}_0 M_{0,\varphi})(x), \tag{3.7}$$

where $\mathbf{S}_0$ is the singular integral operator along the real line with density $M_{0,\varphi}(t) - M_{0,\varphi}(\infty)$. It follows from [2, n. 4.6] that both matrices $N^{-}_{1,\varphi}(x)$ and $N^{+}_{1,\varphi}(x)$ satisfy Hölder conditions on $\bar{\mathbb{R}}$, are bounded there with $N^{\mp}_{1,\varphi}(\infty) = \frac{1}{2} M_{0,\varphi}(\infty)$ and possess an analytic continuation into lower $\Pi^{-}$ and upper $\Pi^{+}$ half-planes, respectively. Surely, its product is also Hölder conditions on $\bar{\mathbb{R}}$ and bounded.

Let us refine the factorization of the matrix $G_{1,\varphi}(x)$, i.e. look for a presentation of $G_{1,\varphi}(x)$ in the form

$$G_{1,\varphi}(x) = I + \varepsilon N_{\varphi}(x) = (I + \varepsilon N^{-}_{1,\varphi}(x) + \varepsilon^2 N^{-}_{2,\varphi}(x))(I + \varepsilon N^{+}_{1,\varphi}(x) + \varepsilon^2 N^{-}_{2,\varphi}(x)), \tag{3.8}$$

where $N^{-}_{1,\varphi}(x)$ and $N^{+}_{1,\varphi}(x)$ are those found at the previous step.

Comparing terms at different powers of $\varepsilon$, we get, in particular, the following relation for determination of factors $N^-_{2,\varphi}$ and $N^+_{2,\varphi}$:

$$N^-_{1,\varphi}(x)N^+_{1,\varphi}(x) + N^-_{2,\varphi}(x) + N^+_{2,\varphi}(x) = 0. \tag{3.9}$$

Denoting

$$M_{1,\varphi}(x) \equiv -N^-_{1,\varphi}(x)N^+_{1,\varphi}(x),$$

we arrive at the following jump boundary value problem

$$N^-_{2,\varphi}(x) + N^+_{2,\varphi}(x) = M_{1,\varphi}(x), \quad x \in \mathbb{R}, \tag{3.10}$$

where the right-hand side is already known. The solution to this problem is given by the formula similar to (3.5)

$$N^{\mp}_{2,\varphi}(z) = \tfrac{1}{2}M_{1,\varphi}(\infty) \mp (\mathbf{C}_0 M_{1,\varphi})(z). \tag{3.11}$$

It has the same properties as the solution of (3.5), in particular, its boundary values satisfy the relation

$$N^{\mp}_{2,\varphi}(x) = \tfrac{1}{2}M_{1,\varphi}(x) \mp \tfrac{1}{2}(\mathbf{S}_0 M_{1,\varphi})(x) \tag{3.12}$$

and $N^{\mp}_{2,\varphi}(\infty) = -\frac{1}{8}N^2_{\varphi}(\infty)$.

One can proceed in the same manner. Thus on the $k$th step, we use the representation

$$\begin{aligned} G_{1,\varphi}(x) = I + \varepsilon N_{\varphi}(x) &= (I + \varepsilon N^-_{1,\varphi}(x) + \cdots + \varepsilon^k N^-_{k,\varphi}(x)) \\ &\quad \times (I + \varepsilon N^+_{1,\varphi}(x) + \cdots + \varepsilon^k N^-_{k,\varphi}(x)), \end{aligned} \tag{3.13}$$

where $N^-_{1,\varphi}(x), \ldots, N^-_{k-1,\varphi}(x)$ and $N^+_{1,\varphi}(x), \ldots, N^+_{k-1,\varphi}(x)$ are found in the previous steps. This leads to the jump boundary value problem

$$N^-_{k,\varphi}(x) + N^+_{k,\varphi}(x) = M_{k-1,\varphi}, \quad x \in \mathbb{R}, \tag{3.14}$$

where

$$M_{k-1,\varphi} = -[N^-_{1,\varphi}(x)N^+_{k-1,\varphi}(x) + N^-_{2,\varphi}(x)N^+_{k-2,\varphi}(x) + \cdots + N^-_{k-1,\varphi}(x)N^+_{1,\varphi}(x)].$$

The solution to this problem is given by a formula similar to (3.5) (or to (3.11)).

Thus, the factorization of the matrix function $G_{1,\varphi}(x)$ is given in the form of an asymptotic series

$$G_{1,\varphi}(x) = \left(I + \sum_{k=1}^{\infty} \varepsilon^k N^-_{k,\varphi}(x)\right)\left(I + \sum_{k=1}^{\infty} \varepsilon^k N^+_{k,\varphi}(x)\right), \tag{3.15}$$

where the pair $N^-_{k,\varphi}(x)$ and $N^+_{k,\varphi}(x)$ is the unique solution to the jump problem (3.14) for any $k \in \mathbb{N}$.

The following theorem gives conditions when this asymptotic factorization becomes an explicit one, i.e. gives convergence conditions for the asymptotic series involved.

**Theorem 3.1.** *Let $G_{\varphi}$ be a matrix which meets conditions (1) to (6). Let the parameter $\varepsilon$ (defined in (6)) satisfies the inequality*

$$|\varepsilon| \leq \frac{1}{A} \tag{3.16}$$

*with the constant $A = A(\varphi)$ being equal to*

$$A = \|N_{\varphi}(\cdot)\|_{\mu}(1 + C_{\mu})^2, \tag{3.17}$$

*$\|N_\varphi(\cdot)\|_\mu$ being the norm of the matrix function $N_\varphi(x)$ in the Hölder space $H_\mu$ equal to the maximum of the norms of its entries, and $C_\mu$ being the norm of the singular integral operator $\mathbf{S}_0 : H_\mu \to H_\mu$.*

*Then both series in the right-hand side of* (3.15) *converge for all $x \in \mathbb{R}$.*

*Proof.* It follows (e.g. [2, p. 48]) that singular integral operator $\mathbf{S}_0$ is bounded in Hölder spaces since the 'standard' singular integral operator $\mathbf{S}$ is. The later is well-known, see [48] for the exact value of the norm of $\|\mathbf{S}\|_{H_\mu\to H_\mu}$. Let us denote the norm of $\mathbf{S}_0$ in Hölder space $H_\mu(\mathbb{R})$ by $C_\mu$, i.e.

$$C_\mu = \|\mathbf{S}_0\|_{H_\mu\to H_\mu}.$$

Then, we have the following series of estimates

$$\|N^{\mp}_{1,\varphi}(\cdot)\|_\mu \le \alpha_1 \|N_\varphi(\cdot)\|_\mu (1 + C_\mu), \quad \text{where } \alpha_1 = \tfrac{1}{2},$$

$$\|N^{\mp}_{2,\varphi}(\cdot)\|_\mu \le \tfrac{1}{2} \|M_{1,\varphi}(\cdot)\|_\mu (1 + C_\mu)$$

and

$$\|M_{1,\varphi}(\cdot)\|_\mu \le (\alpha_1 \|N_\varphi(\cdot)\|_\mu (1 + C_\mu))^2,$$

i.e.

$$\|N^{\mp}_{2,\varphi}(\cdot)\|_\mu \le \alpha_2 \|N_\varphi(\cdot)\|^2_\mu (1 + C_\mu)^3, \quad \text{where } \alpha_2 = \tfrac{1}{2}\alpha_1^2.$$

Finally, for each $k \ge 2$

$$\|N^{\mp}_{k,\varphi}(\cdot)\|_\mu \le \tfrac{1}{2} \|M_{k-1,\varphi}(\cdot)\|_\mu (1 + C_\mu)$$

and

$$\|M_{k-1,\varphi}(\cdot)\|_\mu \le (\alpha_1\alpha_{k-1} + \alpha_2\alpha_{k-2} + \cdots + \alpha_{k-1}\alpha_1) \|N_\varphi(\cdot)\|^k_\mu (1 + C_\mu)^{2k-2},$$

i.e.

$$\|N^{\mp}_{k,\varphi}(\cdot)\|_\mu \le \alpha_k \|N_\varphi(\cdot)\|^k_\mu (1 + C_\mu)^{2k-1}, \quad \text{where } \alpha_k = \tfrac{1}{2}(\alpha_1\alpha_{k-1} + \cdots + \alpha_{k-1}\alpha_1).$$

We can calculate explicitly few first coefficients $\alpha_k$, namely, $\alpha_1 = 1/2$, $\alpha_2 = 1/8$ and $\alpha_3 = 1/16$. As for coefficients with large enough indices, we can proof by induction that

$$\alpha_k < \frac{1}{16(k-3)}, \quad \forall k \ge 12.$$

Therefore,

$$\|\varepsilon^k N^{\mp}_{k,\varphi}(\cdot)\|_\mu \le |\varepsilon|^k \frac{1}{32(k-3)(1+C_\mu)} (\|N_\varphi(\cdot)\|_\mu (1 + C_\mu)^2)^k, \quad \forall k \ge 12.$$

As the sequence $\sqrt[k]{1/(32(k-3)(1+C_\mu))} \le 1$ is increasing for sufficiently large $k$ and

$$\lim_{k\to\infty} \sqrt[k]{\frac{1}{32(k-3)(1+C_\mu)}} = 1,$$

then the convergence of the series

$$\left(I + \sum_{k=1}^{\infty} \varepsilon^k N^-_{k,\varphi}(x)\right) \quad \text{and} \quad \left(I + \sum_{k=1}^{\infty} \varepsilon^k N^+_{k,\varphi}(x)\right)$$

for all $x \in \mathbb{R}$ follows from (3.16). ■

**Remark 3.2.** It follows from the standard properties of the Cauchy type integral and singular integral with Cauchy kernel that conditions of theorem 3.1 guarantee convergence of the series in the right-hand side of (3.15) in the half-planes $\Pi^-$, $\Pi^+$, respectively.

**Remark 3.3.** In fact, the decay of the second term in the right-hand side of (3.2) at infinity follows from the asymptotic relations (2.7) which in turn follows from the properties of matrices of the considered class and the proposed construction.

**Remark 3.4.** If the number $A = A(\varphi)$ in theorem 3.1 is small enough, i.e.

$$A = \|N_\varphi(\cdot)\|_\mu (1 + C_\mu)^2 < 1, \tag{3.18}$$

then the results remain valid for $\varepsilon = 1$ and the described procedure will work without any changes.

## (b) Special case

Let us consider the problem of factorization of $2 \times 2$ invertible matrices from a subclass of $\mathcal{G}K_2$, namely

$$G_\varphi(x) = \begin{pmatrix} p(x) & q(x)\,\mathrm{e}^{\mathrm{i}x\varphi} \\ q(x)\,\mathrm{e}^{-\mathrm{i}x\varphi} & p(x) \end{pmatrix}, \tag{3.19}$$

given on the real line ($x \in \mathbb{R}$) and depending on the real parameter $\varphi \in \mathbb{R}$.

We assume that the following assumptions hold:

(1) the entries $p(x)$ and $q(x)$ are real-valued Hölder continuous functions on $\bar{\mathbb{R}}$, i.e. $p, q \in H_\mu(\bar{\mathbb{R}})$;
(2) the combinations of the functions $p$ and $q$ are positive

$$p(x) \pm q(x) > 0, \quad x \in \mathbb{R}; \quad \text{and} \tag{3.20}$$

(3) the following limits exist

$$\lim_{|x|\to+\infty} p(x) = 1 \quad \text{and} \quad \lim_{|x|\to+\infty} q(x) = 0. \tag{3.21}$$

As in the general case, the factorization of matrices of type (3.19) is motivated by certain problem of fracture mechanics. The considered matrices are similar to those which are studied and explicitly factorized in [31,41,49], but certain conditions of the above-cited papers are not satisfied in our case.

Note that even if one supposes that the functions $p(x)$ and $q(x)$ are meromorphically continued into semi-planes $\Pi^-$ and $\Pi^+$, then it does not mean that these extended functions have a finite number of zeroes and poles there.

**Remark 3.5.** Under conditions (1)–(3), matrix (3.19) admits the canonical factorization

$$G_\varphi(x) = G_\varphi^-(x) G_\varphi^+(x). \tag{3.22}$$

By (3.20), matrix (3.19) is positive definite and thus admits the canonical factorization [1].

Let us start with factorization of an auxiliary matrix $F(x) = G_0(x)$ having no exponential term in their entries

$$F(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & p(x) \end{pmatrix}. \tag{3.23}$$

Note that

$$F(x) = P \begin{pmatrix} p(x) + q(x) & 0 \\ 0 & p(x) - q(x) \end{pmatrix} P, \tag{3.24}$$

where the projector $P$ is defined

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It follows from conditions (1)–(3) that both diagonal elements of the middle diagonal matrix have index equal to zero. Hence, they admit the representation

$$\left.\begin{aligned} p(x) + q(x) &= (p(x) + q(x))^- \cdot (p(x) + q(x))^+ \\ \text{and} \quad p(x) - q(x) &= (p(x) - q(x))^- \cdot (p(x) - q(x))^+, \end{aligned}\right\} \tag{3.25}$$

with the factors of the form [2],

$$\left.\begin{aligned} (p(z) + q(z))^\mp &= \exp\left\{\mp \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log[p(\tau) + q(\tau)]}{\tau - z}\, \mathrm{d}\tau\right\}, \quad z \in \Pi^\mp \\ \text{and} \quad (p(z) - q(z))^\mp &= \exp\left\{\mp \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\log[p(\tau) - q(\tau)]}{\tau - z}\, \mathrm{d}\tau\right\}, \quad z \in \Pi^\mp, \end{aligned}\right\} \tag{3.26}$$

and boundary values $(p(x)+q(x))^{\mp}$ of the functions $(p(z)+q(z))^{\mp}$ (and $(p(x)-q(x))^{\mp}$ of the functions $(p(z)-q(z))^{\mp}$ are determined by using Sokhotsky–Plemelj formulae [2].

Therefore, we immediately obtain the right canonical factorization of the matrix $G_0(x)$

$$G_0(x) = F(x) = F^-(x)F^+(x), \tag{3.27}$$

where

$$F^-(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}(p(x)+q(x))^- & (p(x)-q(x))^- \\ (p(x)+q(x))^- & -(p(x)-q(x))^-\end{pmatrix} \tag{3.28}$$

and

$$F^+(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}(p(x)+q(x))^+ & (p(x)+q(x))^+ \\ (p(x)-q(x))^+ & -(p(x)-q(x))^+\end{pmatrix}. \tag{3.29}$$

Now we can represent the initial matrix $G_\varphi(x)$ in the form

$$G_\varphi(x) = F^-(x)(F^-(x))^{-1}G_\varphi(x)(F^+(x))^{-1}F^+(x) \equiv F^-(x)G_{1,\varphi}(x)F^-(x) \tag{3.30}$$

and proceed with factorization of the matrix $G_{1,\varphi}(x)$.

The inverse matrices $(F^-(x))^{-1}$ and $(F^+(x))^{-1}$ are equal, respectively,

$$\left.\begin{aligned}(F^-(x))^{-1} &= \frac{1}{\Delta_-}\begin{pmatrix}(p(x)-q(x))^- & (p(x)-q(x))^- \\ (p(x)+q(x))^- & -(p(x)+q(x))^-\end{pmatrix}\\ \text{and}\qquad \Delta_- &= \frac{2}{\sqrt{2}}(p(x)+q(x))^-(p(x)-q(x))^-,\end{aligned}\right\} \tag{3.31}$$

and

$$\left.\begin{aligned}(F^+(x))^{-1} &= \frac{1}{\Delta_+}\begin{pmatrix}(p(x)-q(x))^+ & (p(x)+q(x))^+ \\ (p(x)-q(x))^+ & -(p(x)+q(x))^+\end{pmatrix}\\ \text{and}\qquad \Delta_+ &= \frac{2}{\sqrt{2}}(p(x)+q(x))^+(p(x)-q(x))^+.\end{aligned}\right\} \tag{3.32}$$

Hence (for shortness, we omit the argument $x$ for functions $p$ and $q$),

$$\left.\begin{aligned}G_{1,\varphi} &= \frac{1}{\Delta}\begin{pmatrix}(p-q)(2p+q(\mathrm{e}^{\mathrm{i}\varphi k}+\mathrm{e}^{-\mathrm{i}\varphi k})) & (p-q)^-(p+q)^+q(\mathrm{e}^{-\mathrm{i}\varphi k}-\mathrm{e}^{\mathrm{i}\varphi k}) \\ (p-q)^+(p+q)^-q(\mathrm{e}^{\mathrm{i}\varphi k}-\mathrm{e}^{-\mathrm{i}\varphi k}) & (p+q)(2p-q(\mathrm{e}^{\mathrm{i}\varphi k}-\mathrm{e}^{-\mathrm{i}\varphi k}))\end{pmatrix}\\ \text{and}\qquad \Delta &= 2(p^2-q^2).\end{aligned}\right\} \tag{3.33}$$

It is not hard to see that $G_{1,\varphi}$ is a sum of the unit matrix and the matrix which is 'small' for appropriate choice of $\varphi$. Hence, following remark 3.4, we rewrite the right-hand side of (3.33) as the following sum:

$$G_{1,\varphi} = I + N_\varphi, \tag{3.34}$$

where

$$N_\varphi = \begin{pmatrix}-\dfrac{2q\sin^2(\varphi x/2)}{(p+q)} & -\dfrac{\mathrm{i}q(p-q)^-(p+q)^+\sin\varphi x}{(p^2-q^2)} \\ \dfrac{\mathrm{i}q(p-q)^+(p+q)^-\sin\varphi x}{(p^2-q^2)} & \dfrac{2q\sin^2(\varphi x/2)}{(p-q)}\end{pmatrix}. \tag{3.35}$$

Note that the latter matrix can be written as the sum of two diagonal matrices, namely

$$N_\varphi = N_1 + N_2 \tag{3.36}$$

$$N_1 = 2\sin^2\frac{\varphi x}{2}\begin{pmatrix}-\dfrac{q}{(p+q)} & 0 \\ 0 & \dfrac{q}{(p-q)}\end{pmatrix} \tag{3.37}$$

and

$$N_2 = \mathrm{i}\sin\varphi x \begin{pmatrix} 0 & -\dfrac{q(p-q)^-(p+q)^+}{(p^2-q^2)} \\ \dfrac{q(p-q)^+(p+q)^-}{(p^2-q^2)} & 0 \end{pmatrix}. \tag{3.38}$$

Let us denote

$$\varepsilon_1 = \varepsilon_1(\varphi) \equiv \max_{x\in\mathbb{R}} \left| q(x)\sin\frac{\varphi x}{2} \right|. \tag{3.39}$$

**Lemma 3.6.** *The parameter $\varepsilon_1$ can be taken smaller than any positive number $\delta$ by an appropriate choice of $\varphi$.*

*Proof.* Indeed, taking into account condition $(3.21)_2$, one concludes that there exists $x_\delta > 0$ such that for any $|x| \geq x_\delta$

$$\left| q(x)\sin\frac{\varphi x}{2} \right| \leq |q(x)| \leq \delta.$$

On the other hand, as $xq(x)$ belongs to the space $H_\mu[-x_\delta, x_\delta] \subset C[-x_\delta, x_\delta]$ there exists a constant $q_h > 0$ such that

$$|xq(x)| \leq q_h, \quad \text{for all } |x| \leq x_\delta.$$

Finally, this means

$$\left| q(x)\sin\frac{\varphi x}{2} \right| \leq |q(x)|\frac{\varphi x}{2} \leq \frac{\varphi}{2}q_h.$$

Choosing $\varphi = 2\delta/q_h$, we finish the proof. ∎

It follows from the structure of the matrix $N_\varphi$ from (3.35) that parameter $A = A(\varphi)$ discussed in theorem 3.1 is smaller than 1 for an appropriate choice of $\varphi$. It guarantees applicability of the general procedure in this special case.

## 4. An example

We present here an example of $2\times 2$ matrix function $G_\varphi \in \mathcal{G}K_2$ for which the above-discussed factorization does not involve Cauchy type integration for the components and auxiliary matrices.

Let,

$$G_\varphi(x) = \begin{pmatrix} \dfrac{x^2+10}{x^2+1} & \dfrac{6}{x^2+1}\mathrm{e}^{\mathrm{i}x\varphi} \\ \dfrac{6}{x^2+1}\mathrm{e}^{-\mathrm{i}x\varphi} & \dfrac{x^2+10}{x^2+1} \end{pmatrix}, \quad x \in \mathbb{R}. \tag{4.1}$$

It is a special case of the matrix functions discussed in §3*b*. Here, $p(x) = (x^2+10)/(x^2+1)$ and $q(x) = 6/(x^2+1)$, and an auxiliary matrix $F(x) = G_0(x)$ has the form

$$G_0(x) = \begin{pmatrix} \dfrac{x^2+10}{x^2+1} & \dfrac{6}{x^2+1} \\ \dfrac{6}{x^2+1} & \dfrac{x^2+10}{x^2+1} \end{pmatrix}, \quad x \in \mathbb{R}. \tag{4.2}$$

The above matrix satisfies all conditions (1)–(3) of §3*b*, moreover, the functions $p$ and $q$ are not only Hölder continuous on $\mathbb{R}$ but infinitely differentiable.

It follows from condition (2) that

$$\mathrm{Ind}\ \det G_\varphi(x) = \mathrm{Ind}\ \det G_0(x) = 0,$$

besides, owing to condition (2), the matrix $G_\varphi(x)$ is positive definite and thus admits the canonical factorization (if it exists).

Canonical factorization of the matrix $G_0(x)$

$$G_0(x) = G_0^-(x)G_0^+(x)$$

can be found in an explicit form

$$G_0^-(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{x-4i}{x-i} & \dfrac{x-2i}{x-i} \\ \dfrac{x-4i}{x-i} & -\dfrac{x-2i}{x-i} \end{pmatrix} \tag{4.3}$$

and

$$G_0^+(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \dfrac{x+4i}{x+i} & \dfrac{x+4i}{x+i} \\ \dfrac{x+2i}{x+i} & -\dfrac{x+2i}{x+i} \end{pmatrix}. \tag{4.4}$$

Hence, one can calculate an auxiliary matrix

$$G_{1,\varphi}(x) = (G_0^-(x))^{-1}G_\varphi(x)(G_0^+(x))^{-1}$$

in the following form:

$$G_{1,\varphi}(x) = \frac{(x^2+1)^2}{2(x^2+16)(x^2+4)}\begin{pmatrix} g_{1,1}(x) & g_{1,2}(x) \\ g_{2,1}(x) & g_{2,2}(x) \end{pmatrix}, \tag{4.5}$$

where

$$g_{1,1}(x) = \frac{2(x^2+4)(x^2+10) + 6(x^2+4)(\mathrm{e}^{\mathrm{i}\varphi x} + \mathrm{e}^{-\mathrm{i}\varphi x})}{(x^2+1)^2}, \tag{4.6}$$

$$g_{1,2}(x) = \frac{12(x-2i)(x+4i)(-\mathrm{e}^{\mathrm{i}\varphi x} + \mathrm{e}^{-\mathrm{i}\varphi x})}{(x^2+1)^2}, \tag{4.7}$$

$$g_{2,1}(x) = \frac{12(x+2i)(x-4i)(\mathrm{e}^{\mathrm{i}\varphi x} - \mathrm{e}^{-\mathrm{i}\varphi x})}{(x^2+1)^2} \tag{4.8}$$

and

$$g_{2,2}(x) = \frac{2(x^2+16)(x^2+10) + 6(x^2+16)(-\mathrm{e}^{\mathrm{i}\varphi x} - \mathrm{e}^{-\mathrm{i}\varphi x})}{(x^2+1)^2}. \tag{4.9}$$

It leads to the following representation of the matrix:

$$G_{1,\varphi}(x) = I + N_1 + N_2, \tag{4.10}$$

where we can take $\varepsilon = 1$ due to theorem 3.1 (see also remark 3.4), and

$$N_1 = 6i\sin^2\frac{\varphi x}{2}\begin{pmatrix} \dfrac{1}{x^2+16} & 0 \\ 0 & -\dfrac{1}{x^2+4} \end{pmatrix}$$

and $$N_2 = 12i\sin\varphi x\begin{pmatrix} 0 & -\dfrac{1}{(x+2i)(x-4i)} \\ \dfrac{1}{(x-2i)(x+4i)} & 0 \end{pmatrix}.$$

This follows from the consideration presented in the previous section.

Following general scheme of §3*a*, we factorize $G_{1,\varphi}(x)$ at first in the form (3.3) where the first terms of factorization should be computed by the formula (3.4). Following the asymptotic procedure described above, it is sufficient at the first stage to factorize the matrix

$$M_{0,\varphi}(x) = 6i\begin{pmatrix} \dfrac{\sin^2(\varphi x/2)}{x^2+16} & -\dfrac{2\sin\varphi x}{(x+2i)(x-4i)} \\ \dfrac{2\sin\varphi x}{(x-2i)(x+4i)} & -\dfrac{\sin^2(\varphi x/2)}{x^2+4} \end{pmatrix}, \tag{4.11}$$

For this particular case, instead of using the Cauchy integrals, one can factorize each entry $n_{ij}(x), i=1,2,$ of the matrix by using decomposition in simple fractions and the Taylor formula. Combining the obtained results, we get the following representation of matrix $N^-_{1,\varphi}(x)$ and $N^+_{1,\varphi}(x)$ (first components of asymptotic factorization)

$$N^-_{1,\varphi}(x) = 6i\begin{pmatrix} \dfrac{-\mathrm{e}^{-\mathrm{i}\varphi x}-\mathrm{e}^{-4\varphi}+2}{32i(x-4i)} + \dfrac{\mathrm{e}^{-\mathrm{i}\varphi x}-\mathrm{e}^{-4\varphi}}{32i(x+4i)} & \dfrac{-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi x}+i\,\mathrm{e}^{-4\varphi}}{6i(x-4i)} + \dfrac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi x}-\mathrm{i}\,\mathrm{e}^{-2\varphi}}{6i(x+2i)} \\ \dfrac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi x}-\mathrm{i}\,\mathrm{e}^{-2\varphi}}{6i(x-2i)} - \dfrac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi x}-\mathrm{i}\,\mathrm{e}^{-4\varphi}}{6i(x+4i)} & \dfrac{\mathrm{e}^{-\mathrm{i}\varphi x}+\mathrm{e}^{-2\varphi}-2}{16i(x-2i)} - \dfrac{\mathrm{e}^{-\mathrm{i}\varphi x}-\mathrm{e}^{-2\varphi}}{16i(x+2i)} \end{pmatrix} \tag{4.12}$$

and

$$N^+_{1,\varphi}(x) = 6i\begin{pmatrix} \dfrac{\mathrm{e}^{\mathrm{i}\varphi x}+\mathrm{e}^{-4\varphi}-2}{32i(x+4i)} - \dfrac{\mathrm{e}^{\mathrm{i}\varphi x}-\mathrm{e}^{-4\varphi}}{32i(x-4i)} & \dfrac{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi x}+i\,\mathrm{e}^{-2\varphi}}{6i(x+2i)} + \dfrac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi x}-\mathrm{i}\,\mathrm{e}^{-4\varphi}}{6i(x-4i)} \\ \dfrac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi x}-i\,\mathrm{e}^{-4\varphi}}{6i(x+4i)} - \dfrac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\varphi x}-i\,\mathrm{e}^{-2\varphi}}{6i(x-2i)} & \dfrac{-\mathrm{e}^{\mathrm{i}\varphi x}-\mathrm{e}^{-2\varphi}+2}{16i(x+2i)} + \dfrac{\mathrm{e}^{\mathrm{i}\varphi x}-\mathrm{e}^{-2\varphi}}{16i(x-2i)} \end{pmatrix}. \tag{4.13}$$

Note that both plus ($N^+_{1,\varphi}$) and minus ($N^-_{1,\varphi}$) matrix functions vanish at infinity in the corresponding half-plane. The procedure can be performed further as it was described in the general case.

## 5. Outlook and discussions

To illuminate the efficiency of the proposed procedure, we present here the numerical results related to the previous example (§4) showing the quality of the factorization if one decides to restrict the approximation to the first asymptotic term only.

In figures 1 and 2, we present the normalized absolute error of the factorization using only the first asymptotic term for different values of the parameter $\varphi$. We compute the errors for each component of remainder, $\Delta K$, that is difference between the exact factorization and its asymptotic approximation along the real axis

$$\Delta K(x) = G_{1,\varphi}(x) - (I + N^-_{1,\varphi})(I + N^+_{1,\varphi}) = N^-_{1,\varphi}N^+_{1,\varphi}.$$

As it follows from the properties of the Cauchy type integral [5], the obtained estimates are valid also into the upper and lower half-planes.

In figure 1, we depict the error related to the diagonal elements of the remainder while in figure 2, we show the result for the off-diagonal element. Note that as they are complex conjugate with respect to each other, it is enough to discuss only one of them. Unexpectedly, even for rather large value of the parameter $\varphi = 1$, the error is not too high, while for smaller magnitudes of $\varphi$, it decays quickly with the argument and its larger value is concentrated only near the centre of the coordinate. Moreover, one can observe that for $j = 1, 2,$

$$\Delta k_{jj}(0) = 4\varphi^2 + O(\varphi^3), \quad \varphi \to 0.$$

This result can also be verified analytically.

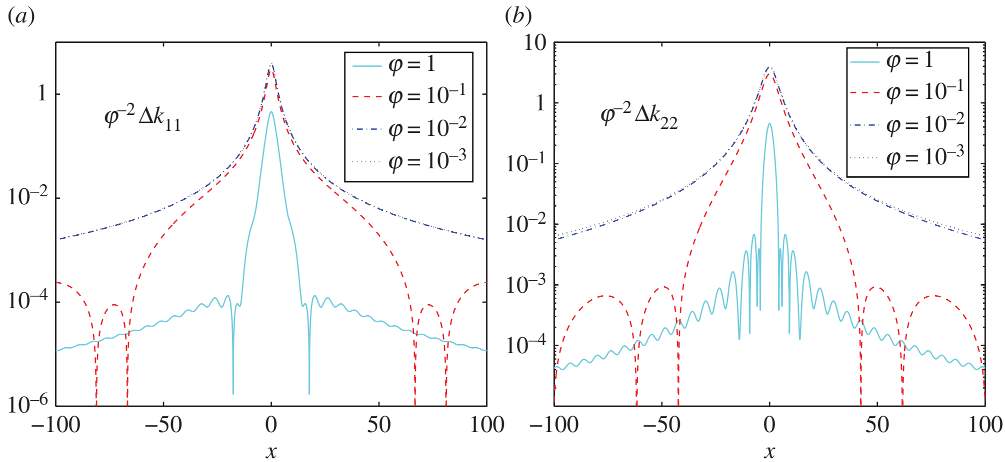


**Figure 1.** (*a*,*b*) Absolute error in the diagonal elements appeared by replacing the exact factorization with the only first asymptotic terms (4.12) and (4.13) for various values of the parameter $\varphi$. (Online version in colour.)

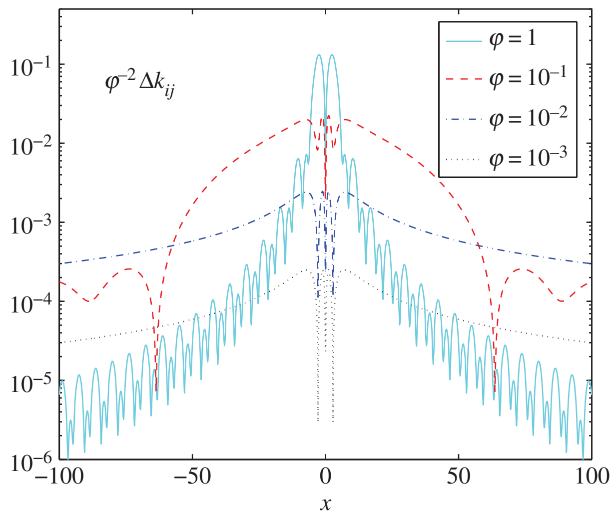


**Figure 2.** Absolute error in the off-diagonal elements appeared by replacing the exact factorization with the only first terms (4.12) and (4.13) for various values of the parameter $\varphi$. (Online version in colour.)

With a decrease in the parameter $\varphi$, the amplitude of the error oscillations increases but their support moves out from the coordinate centre to infinity. Thus, such oscillations play a minor role and the numerical computation of the Cauchy integrals in the described procedure would not affect the accuracy of the computations.

It is not correct to say, however, that $\varphi$ is an optimal small parameter in the problem under consideration. As it follows from theorem 3.1, not only the parameter $\varphi$ but the decay of the corresponding functions (e.g. the function $q(x)$ in the special case) play an important role in the analysis and it is rather $\delta$ which should be taken as the appropriate small parameter.

To clarify this, let us note that two terms $N_1$ and $N_2$ are of different orders with respect to the small parameter $\varphi$. Indeed: $N_1(\varphi) = O(\varphi^2)$ and $N_2(\varphi) = O(\varphi)$, as $\varphi \to 0$. As a result, one could construct another first-order approximation of the factorization based only on the term $N_2$ instead of $N_1 + N_2$. This gives the same first-order estimate in terms of the small parameter $\varphi$. Simple

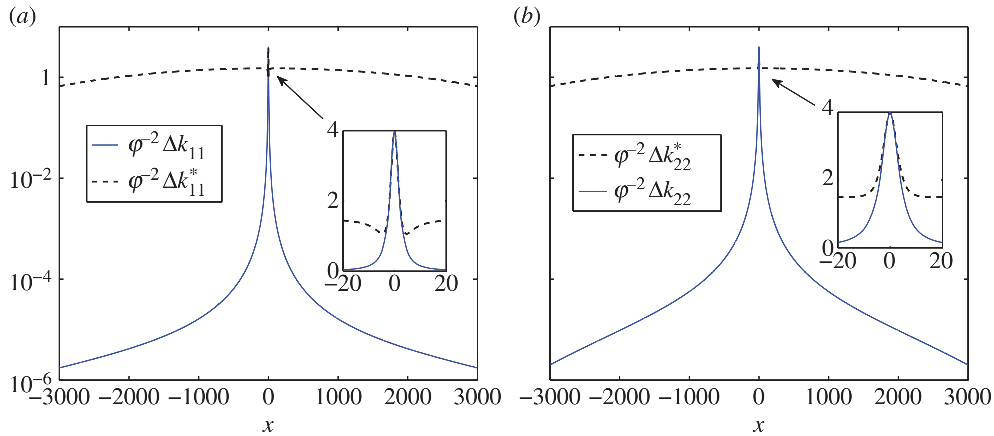


**Figure 3.** (*a*,*b*) Comparison of the errors in the diagonal elements and different approximations for the value of the parameter $\varphi = 10^{-3}$. The errors corresponding to the factorizations of matrix functions (5.1) and (5.2) are given by doted lines, whereas those related to (4.12) and (4.13) are depicted by the solid line. (Online version in colour.)

calculations give the following (first-order) factorization terms in this case:

$$N^{*-}_{1,\varphi}(x) = \begin{pmatrix} 0 & -\dfrac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}\varphi x} - \mathrm{e}^{-4\varphi})}{x - 4i} + \dfrac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}\varphi x} - \mathrm{e}^{-2\varphi})}{x + 2i} \\ \dfrac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}\varphi x} - \mathrm{e}^{-2\varphi})}{x - 2i} - \dfrac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}\varphi x} - \mathrm{e}^{-4\varphi})}{x + 4i} & 0 \end{pmatrix} \tag{5.1}$$

and

$$N^{*+}_{1,\varphi}(x) = \begin{pmatrix} 0 & -\dfrac{\mathrm{i}(\mathrm{e}^{\mathrm{i}\varphi x} - \mathrm{e}^{-2\varphi})}{x + 2i} + \dfrac{\mathrm{i}(\mathrm{e}^{\mathrm{i}\varphi x} - \mathrm{e}^{-4\varphi})}{x - 4i} \\ \dfrac{\mathrm{i}(\mathrm{e}^{\mathrm{i}\varphi x} - \mathrm{e}^{-4\varphi})}{x + 4i} - \dfrac{\mathrm{i}(\mathrm{e}^{\mathrm{i}\varphi x} - \mathrm{e}^{-2\varphi})}{x - 2i} & 0 \end{pmatrix}. \tag{5.2}$$

We estimate then the quality of the approximation for the factorization of the matrix function $G_{1,\varphi}(x)$ based now solely on the first-order term with respect to the parameter $\varphi$. The new reminder is

$$\Delta K^*(x) = G_{1,\varphi}(x) - (I + N^{*-}_{1,\varphi})(I + N^{*+}_{1,\varphi}) = N_1(x) - N^{*-}_{1,\varphi} N^{*+}_{1,\varphi}.$$

Note that the off-diagonal terms give exact (identical) results for the terms (5.1) and (5.2) as opposite to the factorization provided by the terms (4.12) and (4.13). Thus at the first glance, the latter approach looks less beneficial than the former.

In figure 3, we compare the errors related to the diagonal elements of the remainders for the value of the parameter $\varphi = 10^{-3}$. The errors corresponding to matrix functions (5.1) and (5.2) are given by doted lines while those related to (4.12) and (4.13) are depicted by the solid line. One can observe a striking difference in the accuracy of the two approximations. While for a small values of the variable $|x| < 1$ they are identical, for larger values of the argument, the approximation given by the general procedure provides much better accuracy than that based on the only first-order term, $N_2$, with respect to the small parameter $\varphi$. Moreover, if one decides to continue asymptotic expansion further, it may become a real issue in numerical computations of the next asymptotic terms as the decay is very slow. Thus, the procedure suggested here is in a sense optimal.

Note that proposed procedure is working not only in the case described in §3*b*, but also in the case when entries of matrices are quasi-polynomials, i.e. the sum of different exponentials with meromorphic coefficients, provided that all conditions of the class $\mathcal{G}K_n$ are satisfied.

In the case of non-canonical factorization, our algorithm becomes more cumbersome. This situation is the subject of further publication.

Funding statement. The work is supported by the FP7 PEOPLE IAPP project no. 251475 'HYDROFRAC'.





## References


1. Litvinchuk GS, Spitkovsky IM. 1987 *Factorization of measurable matrix functions*. Basel, Switzerland: Birkhäuser.
2. Gakhov FD. 1977 *Boundary value problems*, 3rd edn. Moscow, Russia: Nauka. [In Russian.]
3. Ehrhardt T, Spitkovsky IM. 2001 Factorization of piece-wise constant matrix-functions and systems of differential equations (transl. from 2001 *Algebra Anal.* **13**, no. 6, 56–123; Engl. transl. in *St. Petersbg. Math. J.* **13**, no. 6, 939–991.)
4. Spitkovskij IM, Tashbaev AM. 1991 Factorization of certain piecewise constant matrix functions and its applications. *Math. Nachr.* **151**, 241–261. (doi:10.1002/mana.19911510115)
5. Muskhelishvili NI. 1968 *Singular integral equations*, 3rd edn. Moscow, Russia: Nauka. [In Russian.]
6. Vekua NP. 1967 *Systems of singular integral equations*. Groningen, The Netherlands: P. Noordhoff.
7. Gakhov FD. 1952 Riemann boundary value problem for a system of $n$ pairs of functions. *Uspekhi mat. nauk.* **VII**, no. 4 (50), 3–54. [In Russian.]
8. Gohberg ITs, Krein MG. 1958 Systems of integral equations on a half-line with kernels depending on the difference of arguments. *Uspekhi mat. nauk* **XIII**, no. 2 (80), 3–72. [In Russian.]
9. Speck F-O. 1985 *General Wiener–Hopf factorization methods*. Boston, MA: Pitman.
10. Noble B. 1988 *Methods based on the Wiener–Hopf technique*, 2nd edn. New York, NY: Chelsea Publishing Company.
11. Rawlins AD, Williams WE. 1981 Matrix Wiener–Hopf factorisation. *Q. J. Mech. Appl. Math.* **34**, 1–8. (doi:10.1093/qjmam/34.1.1)
12. Böttcher A, Silberman B. 1990 *Analysis of toeplitz operators*. Heidelberg, Germany: Springer.
13. Gohberg IC, Feldman IA. 1974 *Convolution equations and and projection methods for their solution* (Transl. Math. Monographs, vol. 41). Providence, RI: American Mathematical Society.
14. Veitch BH, Abrahams ID. 2007 On the commutative factorization of $n \times n$ matrix Wiener–Hopf kernels with distinct eigenvalues. *Proc. R. Soc. A* **463**, 613–639. (doi:10.1098/rspa.2006.1780)
15. Böttcher A, Spitkovsky IM. 2013 The factorization problem: some known results and open questions. In *Advances in harmonic analysis and operator theory*, vol. 229 (eds A Almeida, L Castro, F-O Speck), pp. 101–122. Operator theory: advances and applications. Basel, Switzerland: Birkäuser.
16. Markus A, Matsaev V. 1994 The failure of factorization of positive matrix functions on noncircular contours. *Linear Algebra Appl.* 208–209, 231–237. (doi:10.1016/0024-3795(94)90441-3)
17. Simonenko IB. 1968 Some general questions of the theory of Riemann boundary value problem. *Mat. USSR Izv.* **2**, 1091–1099. [Transl. from 1968 *Izv. AN SSSR, ser. mat.* **32**, 1138–1146.] (doi:10.1070/IM1968v002n05ABEH000706)
18. Lawrie JB, Abrahams ID. 2007 A brief historical perspective of the Wiener–Hopf technique. *J. Eng. Math.* **59**, 351–358. (doi:10.1007/s10665-007-9195-x)
19. Chebotarev GN. 1956 To the solution in an explicit form of the Riemann boundary value problem for a system of $n$ pairs of functions. *Uch. zapiski Kazan State University. Mathematika* **116**, no. 4, 31–58. [In Russian.]
20. Chebotarev GN. 1956 Partial indices of the Riemann boundary value problem with a triangular matrix of the second order. *Uspekhi mat. nauk* **II** , 192–202. [In Russian.]
21. Feldman I, Gohberg I, Krupnik N. 1994 A method of explicit factorization of matrix functions and applications. *Integr. Equ. Oper. Theory* **18**, 277–302. (doi:10.1007/BF01206294)
22. Kiyasov SN. 1995 Linear fractional Riemann boundary value problem and its application to factorization of certain classes of Hölder matrix functions of the second order. *Russ. Math.* **39**, 21–27. [Transl. from 1995 *Izv. Vyssh. Uchebn. Zaved., Mat.* **9**, 23–29.]
23. Kiyasov SN. 2008 Effective factorization of some classes of third-order matrix-functions. *Uch. Zap. Kazan. Gos. Univ. Ser. Fiz.-Mat. Nauki.* **150**, 65–70. [In Russian.]
24. Adukov VM. 1992 Wiener–Hopf factorization of meromorphic matrix-functions. *Algebra Anal.* **4**, no. 1, 54–74. [In Russian.]
25. Amirjanyan HA, Kamalyan AG. 2007 Factorization of meromorphic matrix functions. *J. Contemp. Math. Anal. Armen. Acad. Sci.* **42**, 303–319. [Transl. from 2007 *Izv. Nats. Akad. Nauk Armen. Mat.* **6**, 1–24.]

26. Kiyasov SN. 2012 Some cases of efficient factorization of second-order matrix functions. *Russ. Math.* (*Izv. VUZ*) **56**, 30–36. [Transl. from 2012 *Izv. vyssh. uch. zav. Matematika* **6**, 36–43.]
27. Daniele VG. 1978 On the factorization of Wiener–Hopf matrices in problems solvable with Hurd's method. *IEEE Trans. Antennas Propag.* **26**, 614–616. (doi:10.1109/TAP.1978.1141895)
28. Ehrhardt T, Speck F-O. 2002 Transformation techniques towards the factorization of non-rational $2 \times 2$ matrix functions. *Linear Algebra Appl.* **353**, 53–90. (doi:10.1016/S0024-3795(02)00288-4)
29. Khrapkov AA. 1971 Certain cases of the elastic equilibrium of an infinite wedge with a non-symmetric notch at the vertex, subjected to concentrated forces. *Appl. Math. Mech.* (*PMM*) **35**, 625–637. (doi:10.1016/0021-8928(71)90056-6)
30. Khrapkov AA. 1971 Closed form solutions of problems on the elastic equilibrium of an infinite wedge with nonsymmetric notch at the apex. *Appl. Math. Mech.* (*PMM*) **35**, 1009–1016. (doi:10.1016/0021-8928(71)90105-5)
31. Aktosun T, Klaus M, van der Mee C. 1992 Explicit Wiener–Hopf factorization for certain non-rational matrix functions. *Integral Equ. Oper. Theory* **15**, 879–900. (doi:10.1007/BF01203119)
32. Prössdorf S, Speck F-O. 1990 A factorisation procedure for two by two matrix fanctions on the circle with two rationally independent entries. *Proc. R. Soc. Edinb. A* **115**, 119–138.
33. Camara MC, dos Santos AF, Manojlovic N. 2001 Generalized factorization for $n \times n$ Daniele–Khrapkov matrix functions. *Math. Mech. Appl. Sci.* **393**, 993–1020. (doi:10.1002/mma.253)
34. Jones DS. 1984 Commutative Wiener–Hopf factorization of a matrix. *Proc. R. Soc. Lond. A* **393**, 185–192. (doi:10.1098/rspa.1984.0053)
35. Rawlins AD. 1985 A note on Wiener–Hopf matrix factorization. *Q. J. Mech. Appl. Math.* **38**, 433–437. (doi:10.1093/qjmam/38.3.433)
36. Voronin AF. 2011 A method of determination of partial indices of symmetric matrix-functions. *Sib. Math. J.* **52**, 41–53. [Transl. from 2011 *Sib. mat. zh.* **52**, no. 1, 54–68.] (doi:10.1134/S0037446606010058)
37. Crighton DG. 2001 Asymptotic factorization of Wiener–Hopf kernel. *Wave Motion* **33**, 51–65. (doi:10.1016/S0165-2125(00)00063-9)
38. Janashia GA, Lagvilava ET, Ephremidze LN. 1999 The approximate factorization of positive-definite matrix functions. *Russ. Math. Surv.* **54**, 1246–1247. [Transl. from 1999 *Usp. Mat. Nauk* **54**, no. 6, 161–162.] (doi:10.1070/RM1999v054n06ABEH000238)
39. Koiter WT. 1954 Approximate solution of Wiener–Hopf integral equations with applications. I–III. *Koninkl. Ned. Akad. Wetenschap. Proc. B.* **57**, 558–579.
40. Abrahams ID. 2000 The application of Padé approximants to Wiener–Hopf factorization. *IMA J. Appl. Math.* **65**, 257–281. (doi:10.1093/imamat/65.3.257)
41. Camara MC, dos Santos AF. 2000 Wiener–Hopf factorization for a class of oscillatory symbols. *Integr. Equ. Oper. Theory* **36**, 409–432. (doi:10.1007/BF01232738)
42. Clancey K, Gohberg I. 1987 *Factorization of matrix functions and singular integral operators. Operator theory: advances and applications*, vol. 25. Basel, Switzerland: Birkäuser.
43. Feldman I, Gohberg I, Krupnik N. 2000 Convolution equations on finite intervals and factorization of matrix functions. *Integral Equ. Oper. Theory.* **36**, 201–211. (doi:10.1007/BF01202095)
44. Mishuris GS, Movchan NV, Movchan AB. 2010 Dynamic mode-III interface crack in a bi-material strip. *Int. J. Fract.* **166**, 121–133. (doi:10.1007/s10704-010-9507-4)
45. Piccolroaz A, Mishuris G, Movchan A. 2007 Evaluation of the Lazarus–Leblond constants in the asymptotic model of the interfacial wavy crack. *J. Mech. Phys. Solids* **55**, 1575–1600. (doi:10.1016/j.jmps.2007.02.001)
46. Piccolroaz A, Mishuris G, Movchan A. 2009 Symmetric and skew-symmetric weight functions in 2D perturbation models for semi-infinite interfacial cracks. *J. Mech. Phys. Solids* **57**, 1657–1682. (doi:10.1016/j.jmps.2009.05.003)
47. Piccolroaz A, Mishuris G, Movchan A, Movchan N. 2012 Perturbation analysis of Mode III interfacial cracks advancing in a dilute heterogeneous material. *Int. J. Solids Struct.* **49**, 244–255. (doi:10.1016/j.ijsolstr.2011.10.006)
48. Aleksandrov AB. 1975 Norm of the Hilbert transformation in a space of Hölder functions. *Funct. Anal. Appl.* **9**, 94–96. (doi:10.1007/BF01075444)
49. Feldman I, Gohberg I, Krupnik N. 2004 An explicit factorization algorithm. *Integral Equ. Oper. Theory.* **49**, 149–164. (doi:10.1007/s00020-003-1277-1)