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Instability of stretched and twisted soap films in a cylinder

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Abstract. A soap film, or a flexible membrane without bending and torsional stiffness, that is confined in a cylinder is shown to be susceptible to a surface-tension-driven instability when it is stretched or twisted. This leads to its breakdown and places an upper limit on the aspect ratio of such structures. A simple analysis confirms the values for the critical aspect ratio of the stretched film found in both simulations and experiments on soap films, and this threshold decreases with increasing twist of the film.

Keywords: surface tension driven instability; soap films; elastic membranes; Surface Evolver

1. Introduction

Area-minimizing interfaces, such as soap films and elastic membranes without bending and torsional stiffness, are found in a wide variety of man-made and biological structures, and in processes such as liquid confinement in microgravity and enhanced oil recovery. The ability to predict their shape, strength and/or integrity is therefore important.

That soap films are not indefinitely stable is clear from the well-known case of the catenoidal film formed between two closely-spaced parallel rings (Isenberg, 1992). As the two rings are pulled apart, the neck gradually thins. This does not continue indefinitely, and at a critical ring separation $H/R \approx 1.32$, where R is the radius of the rings, the surface becomes unstable and jumps to two discoidal films, one in each ring.

We have implicitly assumed, as we shall do throughout the following, that the amount of surfactant present is sufficiently great that the film doesn't collapse because its tension becomes too high. This assumption is certainly valid in our experiments, described below.

A related instability is the stretching of a cylindrical bubble between parallel walls (Cox et al., 2002). Above a certain wall separation, $H/R = 2\pi$, the bubble undergoes a "wine-bottle", or Rayleigh-Plateau, instability, bulging at one end and thinning at the other. As the stretching continues, the bubble detaches from one wall, and takes a hemispherical shape (Weaire et al., 2007).

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Other soap-bubble instabilities are reviewed in Weaire et al. (2007). Here, we motivate our work by reference to the stability of a single planar film. Consider a single rectangular soap film confined between parallel walls, with its ends held by two wires perpendicular to the walls. This film can be stretched indefinitely by pulling the anchoring wires apart.

However, if the film is confined in a cylinder, oriented parallel to the axis and such that the anchoring wires are parallel diameters of the cylinder, we find that this cylindrical constraint induces a surface-tension-driven instability in sufficiently long films. Once this instability is triggered, at the critical aspect ratio, the film rapidly collapses into two disconnected semicircular regions, one at each end of the cylinder. If the film is twisted, taking the shape of a helicoid, then the instability occurs when the film is shorter.

Our theoretical analysis confirms the values for the critical aspect ratio of the stretched film found in both simulations and experiments on soap films, and this threshold decreases with increasing twist of the film. This implies that there is a fundamental limit to the size of an area-minimizing film or membrane that can exist.

Many physical systems, however, have a more complex arrangement of multiple soap films, and a consequence of area minimization is that these films meet in threes at equal angles of 120° (Plateau, 1873; Taylor, 1976). Thus, it is important in predicting the stability of foams, which are used in many processes including froth flotation and ore separation (Weaire and Hutzler, 1999; Cantat et al., 2010), to determine the degree to which this three-fold arrangement of soap films is susceptible to the same surface-tension-driven instability. We therefore add more soap films parallel to the axis of the cylinder, in both simulation and experiment, and show that these configurations confer no greater stability on the system.

2. Experiments

We used a 10% solution of the commercial detergent Fairy Liquid to create a single soap film between two parallel pieces of pipe-cleaner (refered to here as wires) placed perpendicular to the axis of a perspex cylinder of fixed radius $R = 31$mm and length 300 mm (Figure 1(a)). The upper wire is held in place with a thin perspex ring attached to a handle that allows its position to be controlled. Note that there is no volume constraint on the gas neighbouring the film. The film was then slowly stretched by increasing the distance H between the wires in small steps from zero (the point at which the film is formed). We
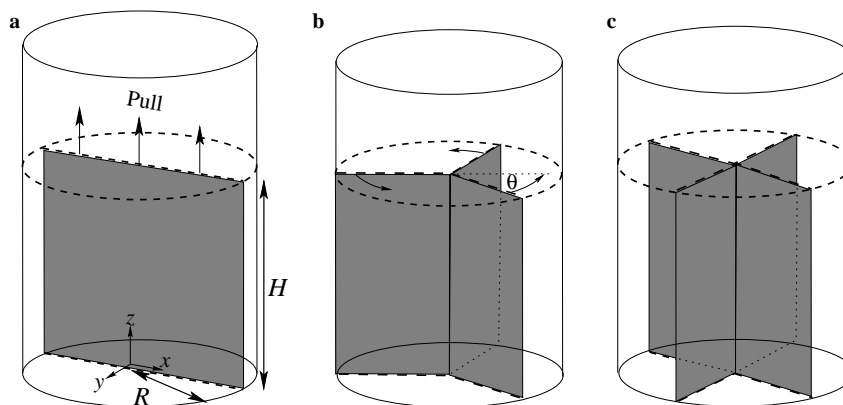
Figure 1. Setup. (a) A rectangular film confined within a perspex cylinder. The two wires on which the film is formed are shown as dashed lines. The film is fixed to these wires, but free to move on the cylinder wall. The instability is seen by pulling apart the two wires along the axis of the cylinder. (b) Three films confined within a cylinder surround a vertical liquid channel (Plateau border). If the upper Y-shaped arrangement of anchoring wires is rotated, at fixed aspect ratio, the instability is again seen. (c) Arrangement of four elastic membranes in a cylinder. Simulations indicate that this configuration is also unstable when stretched or twisted.

define the aspect ratio of the film to be $\alpha = H/R$ (rather than the more conventional definition, which is half this) for consistency with the experiments described below with multiple films. The critical aspect ratio at which the film bends and then collapses into two semicircular regions, one at each end of the cylinder, was found to be $\alpha_{\text{crit}} = 2.76 \pm 0.15$. A photograph of the experiment (Figure 2(a)) shows the way in which the film bows outwards at this point; it then collapses before the next increment in the value of H.

We next formed a film and stretched it to a small value of H, then twisted it into a helicoid by slowly moving the upper constraint clockwise up to an angle θ (Figure 2(e)). We then again stretched the film by increasing H and recorded the value of H/R at which the film became unstable (Figure 2(f)) and collapsed. The critical aspect ratio *decreased* with increasing θ in the manner shown in Figure 4. Equivalently, at given aspect ratio α, the film can only be twisted up to a certain angle until it becomes unstable and then collapses as before.

Both stretching and twisting experiments were repeated with a Y-shaped frame, again made from pipe-cleaners (see Figure 1(b)), to generate data for the collapse of the three-fold film arrangement. Each film is a helicoid (Figure 3(b)), and the instability is first apparent when the triple line along the axis of the cylinder deforms; Figure 3(a)
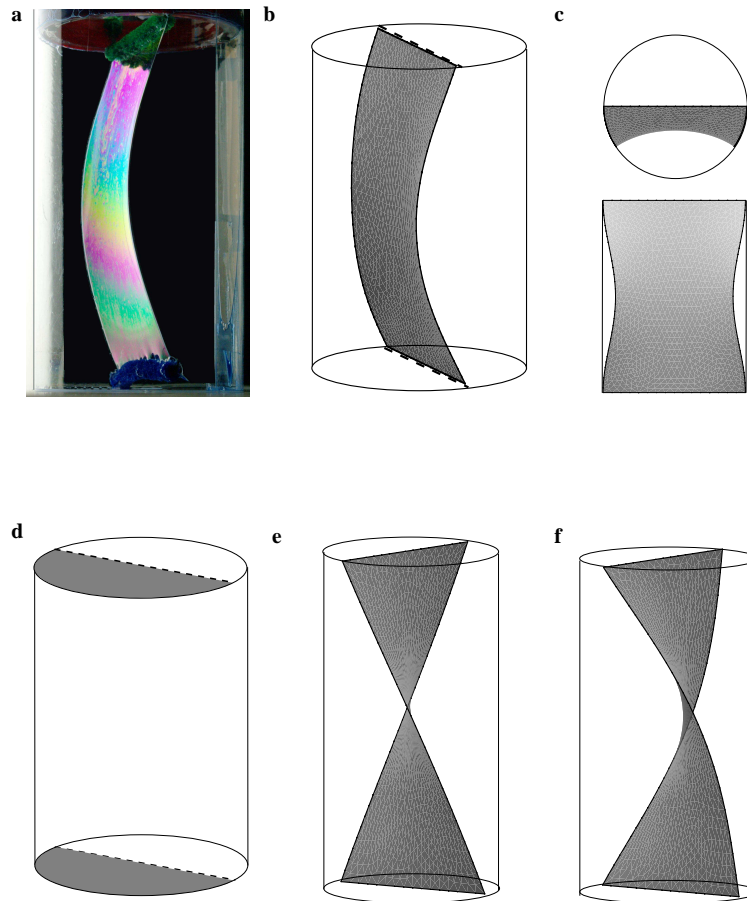
Figure 2. (a-c) The mode of instability of a single film under stretching: the centre of the film bulges out and appears to be pinched where it curves around the cylinder. Oblique view observed in (a) experiment and (b) simulation. (c) View of the film in (b) from in front and above. (d) Sketch of the final collapsed state (under stretching). (e) Single film twisted to an angle of $\theta = 60°$, (f) Mode of instability of a single film under twisting.

shows how this happens under stretching: two of the three films distort in the same way that the single film does (Figure 2(b)). For this three-fold film configuration, both the critical values of the aspect ratio for the stretched planar film case, and the critical angles for the twisted helicoid film case (see Figure 3(c)), are shown in Figure 4. These values agree with those for the single film case, confirming that the same mechanism for instability is involved.
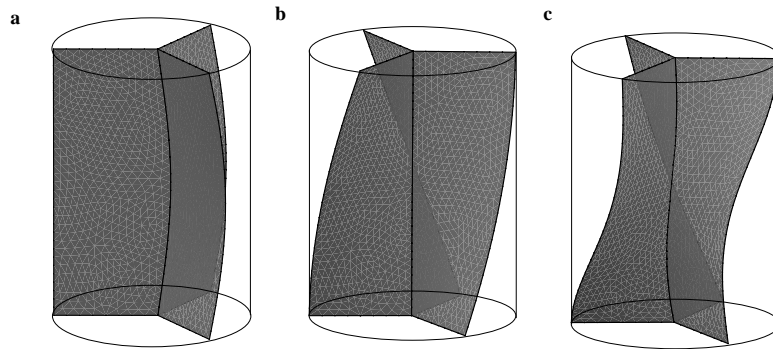
Figure 3. Simulations of the three-film configuration: (a) Mode of instability under stretching ($\theta = 0$). (b) Twisted configuration for $\theta = 60°$. (c) Mode of instability under twisting for $\theta = 60°$.

3. Simulations

We used the Surface Evolver (Brakke, 1992) to determine the film shapes that correspond to the minimum area shape at equilibrium. A surface is tessellated with around 2000 triangles (three levels of refinement); edges that touch the diameters at the end of the cylinder are fixed, and the edges of the surface are constrained to move on the curved walls of the cylinder. For each value of α, we explored whether the shape of the film is restored after a random perturbation was applied to the film and 25000 conjugate gradient steps of area minimization performed. For the single stretched film, we found the value $\alpha_{\mathrm{crit}} = 2.6 \pm 0.05$, in agreement with the experiments. The shape that the film takes just after it becomes unstable is shown in Figure 2(b).

Twisted film structures were examined in the same way as in the experiments by slowly rotating the fixed edge at one end of the film. We also placed two Y-shaped constraints at the ends of the cylinder, with three films between them (Figure 3). In both cases the data is shown in Figure 4, and is almost indistinguishable from the experimental results. Twisting any of the soap film configurations up to an angle of 180° leads to instability at a very small height.

In the case of an elastic membrane, there is no reason why the junction should be restricted to three-fold. A simulation was therefore made with four "fins" (Figure 1(c)) and led to the same critical values for aspect ratio and twist angle (Figure 4).
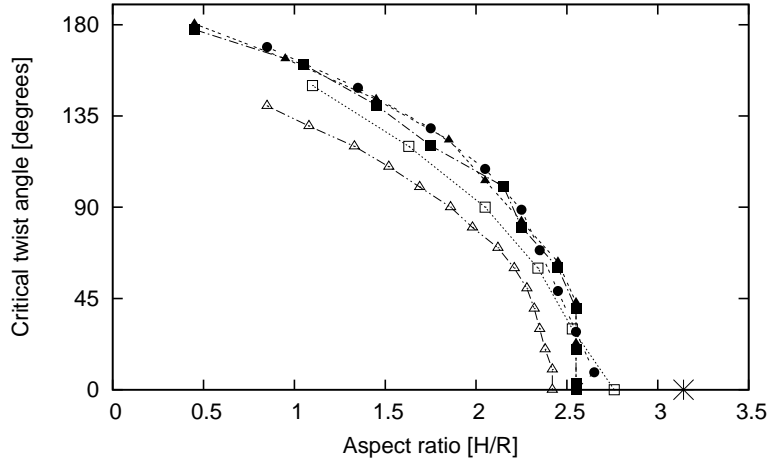
Figure 4. The critical aspect ratio necessary to trigger the instability decreases with increasing twist angle, or, equivalently, the critical twist angle decreases with increasing aspect ratio, and is independent of the number of soap films. Key: simulations are shown with filled symbols, experiments with open symbols; single film: squares, three-fold film arrangement: triangles, and four-fold film arrangement: circles. Measurement errors are not shown, for clarity, but are estimated at: experiment ± 0.15 for aspect ratio, $\pm 10°$ for angle; simulation ± 0.1 for aspect ratio, $\pm 0.01°$ for angle. The theoretical prediction at $\theta = 0$, $\alpha = \pi$, is shown as an asterisk on the horizontal axis.

4. Theory

We explain this effect by examining the way in which the single rectangular film collapses. It could be expected that perturbing the film would induce an increase in area, but the presence of the cylinder walls allows the film to reduce its area due to this perturbation. Beyond a critical aspect ratio, the increase in film area due to the perturbation is more than compensated for by the reduction in area as the film slides round the walls of the cylinder half-way up (Figure 2). Note that this argument is akin to the well-known Rayleigh-Plateau instability of fluid jets, which can also be couched in terms of area minimization (Isenberg, 1992).

To estimate the critical aspect ratio, we assume that an arbitrary perturbation can be represented by $y = \varepsilon \sin(n\pi z/H)$, where $\varepsilon \ll 1$ is a small parameter, y is a coordinate perpendicular to the anchoring wires at the ends, and z is the axial coordinate (Figure 1(a)). The positive integer n labels the possible modes of instability, with wavelengths

$2H/n$. The area of the film can then be written

$$A = 2 \int_0^H \mathrm{d}z \int_0^{\sqrt{R^2-y^2}} \mathrm{d}x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}, \qquad (1)$$

with the upper limit of the x integration representing the lateral pinching of the film due to the cylinder walls. This evaluates to

$$
\begin{aligned}
A &= 2 \int_0^H \mathrm{d}z \sqrt{\left[R^2 - \varepsilon^2 \sin^2\left(\frac{n\pi z}{H}\right)\right]\left[1 + \left(\frac{n\pi\varepsilon}{H}\right)^2 \cos^2\left(\frac{n\pi z}{H}\right)\right]} \\
&\approx 2R \int_0^H \mathrm{d}z \sqrt{1 + \varepsilon^2 \left(\left(\frac{n\pi}{H}\right)^2 \cos^2\left(\frac{n\pi z}{H}\right) - \frac{1}{R^2}\sin^2\left(\frac{n\pi z}{H}\right)\right)} + O(\varepsilon^4) \\
&\approx 2R \int_0^H \mathrm{d}z \left[1 + \frac{1}{2}\varepsilon^2 \left(\left(\frac{n\pi}{H}\right)^2 \cos^2\left(\frac{n\pi z}{H}\right) - \frac{1}{R^2}\sin^2\left(\frac{n\pi z}{H}\right)\right) + O(\varepsilon^4)\right] \\
&= 2RH \left(1 + \frac{1}{4}\varepsilon^2 \left[\left(\frac{n\pi}{H}\right)^2 - \frac{1}{R^2}\right] + O(\varepsilon^4)\right). \qquad (2)
\end{aligned}
$$

The term multiplying ε^2 shows the balance between the increase of surface area due to the perturbation and the decrease of surface area due to the curvature of the cylinder wall; it goes from positive to negataive, indicating a reduction in area and therefore the initiation of the instability, when $\alpha = H/R > n\pi$. The fundamental mode of instability, with $n = 1$, is the first to be triggered, giving $\alpha = H/R > \pi$, in reasonable agreement with the experimental and simulated results, despite the simplifications in the analysis. This value is shown as an asterisk on Figure 4.

5. Conclusions

We have demonstrated that any soap film or flexible membrane, such as can be found in foams and biological tissues, is susceptible to an instability that leads to its breakdown when stretched and/or twisted in a cylindrical tube. Simulations, experiments on soap films, and a simple theoretical analysis all confirm the critical value of the aspect ratio: with zero twist the critical aspect ratio of the film is $\alpha = H/R \leq \pi$, and this threshold decreases with increasing twist. Thus, for example, a tube or capillary of diameter 10μm cannot support a soap film or a planar membrane along its axis of greater than about 16μm in length.

That the simulations agree with the experiments indicates that the liquid present in the central Plateau border and in the meniscus around

the films in the experiments does not affect the triggering of the instability. Simulations of a three-fold film arrangement with a central Plateau border of finite size suggest that the presence of liquid has little effect on its stability. Any bending stiffness, for example, attributed to this Plateau border will not affect the point of instability, only the subsequent time-scale of the dynamics, which we do not explore here. The small discrepancy between the experimental and simulation results may be due to the finite size of the wires and the ring supporting the upper wires, which cause deviations in the shape of the films particularly close to the cylinder, and the effects of gravity-driven liquid drainage, which causes the films to be thicker towards their base. Both effects are more significant at the highest twist angles.

To improve the accuracy of the theoretical prediction, the lateral x dependence of the perturbation should be included, since the film should meet the cylinder walls at $90°$, inducing more curvature and lowering the critical aspect ratio. However, the mathematics becomes analytically intractable, and a numerical solution is unlikely to give more or better information than the simulations described here.

Our theoretical argument for the critical aspect ratio is straightforward to extend to an elliptical tube in the case of pure stretching, by changing the upper limit in the integral in eq. (1), but predicting the critical twist angles again presents a challenging problem for future work.

Finally, we note that if the region between the film(s) and the cylinder walls has a volume constraint, i.e. bubbles (or cells) are present, then this could be accommodated by considering the perturbation $y = \varepsilon \sin(2n\pi z/H)$, resulting in *twice* the critical aspect ratio, $\alpha = H/R \leq 2\pi$, and therefore greater stability.

Acknowledgements

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