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Alfvén Instability in a Compressible Flow

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A new ideal magnetohydrodynamic instability is presented. It is shown that linear incompressible Alfvénic disturbances can get over-reflected and exponentially amplified in compressible plasma flows. A simple and transparent stability criterion for a two-layer model is derived. The instability does not require a shear in the flow and may arise for rather moderate sub-Alfvénic flow speeds. It is therefore important to be aware of the Alfvén instability when dealing with laboratory or astrophysical plasma flows.

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MHD instabilities occur on large scales and hence have important implications as far as the global plasma structure and dynamics are concerned. They play an important role both in astrophysical plasmas and laboratory devices. Examples of well-known macroscopic instabilities in ideal MHD include the sausage and kink instabilities, the hydromagnetic Rayleigh-Taylor and Kelvin-Helmholtz instabilities.

Since their discovery [1], Alfvén waves have been of great interest in plasma physics. Alfvén waves are notorious for their ability to propagate along the magnetic field covering large distances and have often been invoked as possible candidates for chromospheric and coronal heating. Recent observations indicate possible presence of Alfvén waves in various solar atmospheric structures from chromospheric spicules to x-ray jets [2,3]. It has been argued that the observed waves have enough energy to heat the corona and to power the solar wind. Despite the seeming success it remains unclear how such waves can be generated. A number of difficulties associated with the generation of Alfvén waves at photospheric heights and their dissipation in the corona have been pointed out [4].

Flows commonly exist both in laboratory and astrophysical plasmas. Examples of compressible flows in astrophysical plasmas are the solar wind, siphon flows in coronal loops, Evershed flows, jets, downflows and spicules. The plasma behavior in such flows is often studied using one-dimensional models where motion occurs only along the magnetic field lines. The validity of such models is rarely questioned.

The present work reveals the possibility of an instability associated with transverse Alfvénic motions in compressible plasma flows. One important consequence of the instability could be the generation of large amplitude Alfvén waves which could then rapidly dissipate their energy, for example, through nonlinear coupling to fast and slow mode MHD shocks [5]. It is assumed that the magnetic field $\mathbf{B_0} = B_0\hat{\mathbf{z}}$ is uniform and directed along the z axis. Incompressible flows ($\nabla \cdot \mathbf{u_0} = 0$) and instabilities associated with the flow shear have been extensively studied in the past. The magnetic field usually has a stabilizing effect on the system due to the restoring tension force. However, plasma flows are rarely incompressible.

In a uniform plasma, small-amplitude transverse perturbations propagating along the magnetic field are decoupled from other motions. In the presence of a field-aligned plasma flow $\mathbf{u_0} = u_0\hat{\mathbf{z}}$, these motions are governed by the corresponding components of the linear ideal MHD equations of motion and magnetic induction:

$$\rho_0\frac{\partial v}{\partial t} + \rho_0 u_0\frac{\partial v}{\partial z} = \frac{B_0}{\mu_0}\frac{\partial b}{\partial z}, \qquad \frac{\partial b}{\partial t} + u_0\frac{\partial b}{\partial z} = B_0\frac{\partial v}{\partial z} - b\frac{\partial u_0}{\partial z}, \tag{1}$$

where b, v denote the transverse perturbations of the magnetic field and velocity. Equations (1) describe the propagation of *shear* Alfvén waves in a flowing plasma with a uniform magnetic field. Note that the same set of Eqs. (1) governs the propagation of *torsional* Alfvén waves on magnetic cylinders [5]. In the latter case, b, v represent the azimuthal perturbations. In both cases, the disturbances are Alfvénic in the sense that they are incompressible and propagate along the magnetic field. Thus, the forthcoming discussion is not restricted to a particular geometry or type of disturbances. Using the equilibrium equation of mass continuity, $\rho_0 u_0 = \text{const}$ (or $u_0/c_A^2 = \text{const}$), we reduce the set of Eqs. (1) into a single second order partial differential equation for the magnetic field perturbation b:

$$\frac{\partial^2 b}{\partial t^2} + 2\frac{\partial^2}{\partial t \partial z}(u_0 b) + \frac{\partial}{\partial z}\left(u_0\frac{\partial}{\partial z}\left(\frac{u_0^2 - c_A^2}{u_0}b\right)\right) = 0, \tag{2}$$

where the Alfvén speed $c_A(z) = B_0/(\mu_0\rho_0(z))^{1/2}$ has been introduced. In the particular case of u_0, $c_A = \text{const}$, Eq. (2) describes the propagation of Alfvén waves with Doppler shifted frequencies $\omega = k(u_0 \pm c_A)$. In general, both u_0 and c_A are variable in the z direction.

The JWKB approximation is commonly adopted to study the propagation of Alfvénic disturbances in flowing media with scale heights that are large compared to the wavelengths [6,7]. Non-JWKB aspects of Alfvén wave propagation around a critical layer known as the Alfvén point (a singularity in the governing equations where $u_0 = c_A$) have also been examined [8,9]. Alfvén waves in static media with strong gradients have been studied both analytically [10] and numerically [5,11]. The governing equation cast in the form (2) allows us to derive new connection formulae across an interface. These formulae are applied to the solutions to investigate the spatiotemporal behavior of Alfvénic disturbances in the presence of flows and strong gradients. It is assumed that the flow speed is sub-Alfvénic throughout the domain of interest which is the case in many astrophysical and laboratory applications.

The stability analysis of the solutions of Eq. (2) is not straightforward. It is well known that incompressible flows can be subject to absolute or convective instabilities [12]. Convective instabilities grow only it time being swept away by the flow, whereas absolute instabilities grow both in time and in space. The distinction depends on the choice of a reference frame. One way of distinguishing between different types of growing solutions is to apply a Fourier-Laplace transform and to find the dispersion relation $\omega = \omega(k)$. The roots in the complex k plane are then traced as the real and imaginary parts of ω are altered according to Briggs' recipe [12]. However, the applicability of this method is not clear when the flow is compressible because the coefficients of the governing differential equation are no longer constants.

We find that even for a simple linear flow speed profile the $t \longrightarrow \omega$ Laplace transform of Eq. (2) yields solutions in terms of generalized hypergeometric functions with ω dependent arguments and parameters. The treatment and inversion of such functions is not obvious. In order to make analytical progress, we set the flow speed to have a step function profile. According to the equation of mass continuity, the Alfvén speed c_A and density ρ_0 will have similar profiles. Figure 1 shows a two-layer semi-infinite model in which the equilibrium quantities u_0, c_A, ρ_0 suffer finite jumps at $z = L$. Their constant values in $0 < z < L$ and $z > L$ are denoted by $-$ and $+$, respectively. Small-amplitude Alfvénic (shear or torsional) disturbances launched at $z = 0$ propagate into the region $z > 0$ along the field lines. The driver is arbitrary. It can have a finite duration and does not necessarily represent a monochromatic wave. In what follows, the response of the system is examined in order to determine whether it is stable or not.

Equation (2) is transformed into

$$(u_0^{\pm 2} - c_A^{\pm 2})\frac{\partial^2 \hat{b}}{\partial z^2} + 2\omega u_0^{\pm}\frac{\partial \hat{b}}{\partial z} + \omega^2 \hat{b} = 0, \qquad (3)$$

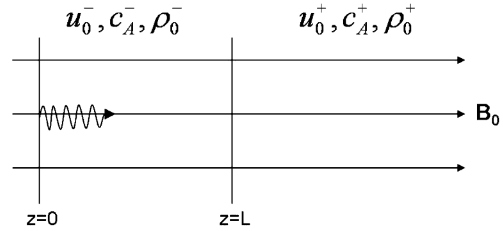

FIG. 1. A semi-infinite two-layer model in which the flow speed u_0, the Alfvén speed c_A and the density ρ_0 are step functions of z. The $-$ and $+$ superscripts denote the constant values of these equilibrium quantities in the intervals $0 < z < L$ and $z > L$. Small-amplitude arbitrary Alfvénic disturbances are launched from the left boundary at $z = 0$.

where the superscripts $\pm$ indicate the equilibrium quantities in the two regions and

$$\hat{b}(\omega, z) = \int_0^\infty b(t, z)\exp(-\omega t)dt \qquad (4)$$

is the Laplace transform of the magnetic field perturbation. In deriving Eqs. (3), it has been assumed that the system is initially unperturbed. Equations (3) are solved separately in the regions $\pm$ shown in Fig. 1:

$$\hat{b} = \begin{cases} A\exp\frac{-\omega z}{u_0^- + c_A^-} + B\exp\frac{-\omega z}{u_0^- - c_A^-}, & 0 < z < L, \\ C\exp\frac{-\omega z}{u_0^+ + c_A^+}, & L < z < \infty, \end{cases} \qquad (5)$$

where the unknown coefficients A, B, C depend on ω. The solution (5) in the region $+$ is chosen to satisfy the requirement of finite energy density. The unknown coefficients are determined by connecting the solutions across the interface $z = L$. First, the magnetic field must be continuous. The second condition follows from Eq. (2) and from the continuity of the magnetic field. Hence the two conditions connecting the solutions on the two sides are

$$\{\hat{b}\} = 0, \qquad \left\{2u_0\omega\hat{b} + (u_0^2 - c_A^2)\frac{\partial \hat{b}}{\partial z}\right\} = 0, \qquad (6)$$

where the braces denote the jump of the enclosed quantity across the interface. From a physical point of view, the second condition expresses the continuity of Poynting flux. Note that, in general, conditions (6) do not imply continuity of the velocity perturbation v because of the presence of flow. A combination of conditions (6) with the boundary condition at $z = 0$ results in

$$\begin{aligned} &A\exp\frac{-\omega L}{u_0^- + c_A^-} + B\exp\frac{-\omega L}{u_0^- - c_A^-} = C\exp\frac{-\omega L}{u_0^+ + c_A^+}, \\ &A(u_0^- + c_A^-)\exp\frac{-\omega L}{u_0^- + c_A^-} + B(u_0^- - c_A^-)\exp\frac{-\omega L}{u_0^- - c_A^-} = C(u_0^+ + c_A^+)\exp\frac{-\omega L}{u_0^+ + c_A^+}, \qquad A + B = I, \end{aligned} \qquad (7)$$

where $I = \hat{b}(\omega, 0)$ represents the Laplace transform of an arbitrary driver at the boundary $z = 0$. The solutions of the algebraic system (7) are

$$A = I\left(1 + \frac{c_A^- + u_0^- - u_0^+ - c_A^+}{c_A^- - u_0^- + u_0^+ + c_A^+} \exp\frac{2\omega L c_A^-}{u_0^{-2} - c_A^{-2}}\right)^{-1},$$

$$B = I\left(1 + \frac{c_A^- - u_0^- + u_0^+ + c_A^+}{c_A^- + u_0^- - u_0^+ - c_A^+} \exp\frac{-2\omega L c_A^-}{u_0^{-2} - c_A^{-2}}\right)^{-1},$$

$$C = \frac{2c_A^- I}{c_A^- - u_0^- + u_0^+ + c_A^+} \exp\left(\frac{\omega L}{u_0^+ + c_A^+} - \frac{\omega L}{u_0^- + c_A^-}\right) \times \left(1 + \frac{c_A^- + u_0^- - u_0^+ - c_A^+}{c_A^- - u_0^- + u_0^+ + c_A^+} \exp\frac{-2\omega L c_A^-}{u_0^{-2} - c_A^{-2}}\right)^{-1}. \quad (8)$$

It is now possible to determine the response of the system to an arbitrary perturbation $b(t, 0)$ by inverting the obtained results into time domain. The inverse Laplace transform is expressed by the integral

$$b(t, z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp(\omega t)\hat{b}(\omega, z) d\omega. \quad (9)$$

The integration in Eq. (9) is to be performed along a vertical line $\omega = \gamma$ in the complex plane ω known as the Bromwich contour: the real number γ is chosen so that $\omega = \gamma$ lies to the right of all singularities of the integrand but is otherwise arbitrary. One way to find the inversion integral (9) is to close the contour of integration using a semicircle Γ of radius R which does not pass through any of the singularities. The semicircle is traversed from $\gamma + i\sqrt{R^2 - \gamma^2}$ to $\gamma - i\sqrt{R^2 - \gamma^2}$ in the positive (counterclockwise) direction. The integral (9) is reduced to

$$b(t, z) = \lim_{R\to\infty} \frac{1}{2\pi i}\left(\oint_{\mathcal{C}} - \int_{\Gamma}\right) \exp(\omega t)\hat{b}(\omega, z) d\omega, \quad (10)$$

where the closed contour $\mathcal{C}$ consists of the Bromwich contour and Γ. For large enough t the integral around the semicircle Γ tends to zero as $R \to \infty$. The remaining integral around the contour $\mathcal{C}$, according to the residue theorem, is equal to the sum of all residues of the integrand. Formulas (5) and (8) show that the function $\hat{b}(\omega, z)$ has poles at

$$\omega_n = \frac{c_A^{-2} - u_0^{-2}}{2Lc_A^-}\left(\ln\frac{c_A^- + u_0^- - u_0^+ - c_A^+}{c_A^- - u_0^- + u_0^+ + c_A^+} + (2n + 1)\pi i\right) \quad (11)$$

with $n = 0, \pm 1, \pm 2, \ldots$ which are of first order (simple) provided none of the singularities of $I(\omega)$ coincide with ω_n. For large enough times the system response is given by the formula

$$b(t, z) = \sum_{n=-\infty}^{\infty} \mathrm{Res}(\exp(\omega t)\hat{b}(\omega, z), \omega_n) + \mathcal{F}(I), \quad (12)$$

where the term $\mathcal{F}(I)$ contains the residues due to $I(\omega)$ and depends on the driver profile. The presence of the factor $\exp(\omega t)$ on the right-hand side of Eq. (12) indicates that the evolution of an arbitrary perturbation is determined by the sign of the real part of ω. The obtained analytical expressions (11) for ω_n enable us to derive a simple and transparent criterion for stability or instability. There are three possibilities: *(i) Damping* ($u_0^- < u_0^+ + c_A^+$).—The poles are in the left-hand plane ($\Re(\omega_n) < 0$) and any disturbance exponentially decays in time due to leakage in the positive z direction. This also includes the case when the flow speed is zero or negative in both $\pm$ regions. The system is stable. *(ii) Undamped oscillations* ($u_0^- = u_0^+ + c_A^+$).—The poles ω_n are on the imaginary axis and the system oscillates with frequencies $\omega_{in} = (c_A^{-2} - u_0^{-2})n\pi/2Lc_A^-$, $n = 1, 3, 5, \ldots$. The amplitudes of these oscillations are determined by the driver strength and remain constant unless the driver is a sinusoidal function with a frequency matching one of the frequencies ω_{in}. This will lead to a second order pole on the imaginary axis and the oscillation will grow proportional to $t\cos(\omega_{in}t)$. From a physical point of view, the latter case represents a resonance. *(iii) Instability* ($u_0^- > u_0^+ + c_A^+$).—The flow speed is large enough to cause exponential growth of an arbitrary perturbation. According to the original assumption, it is nevertheless sub-Alfvénic for any fixed z. Amplification occurs everywhere within $0 < z < \infty$ and the instability is absolute in a static frame of reference. Note that the instability becomes possible only when the flow is compressible ($u_0^- \neq u_0^+$).

For illustration, consider a single pulse launched at $z = 0$ and given by

$$b(t, 0) = t\exp(-\alpha t), \quad (13)$$

where $\alpha > 0$ characterizes the damping rate of the pulse. The Laplace transform of Eq. (13), $I = 1/(\omega + \alpha)^2$, has a second order pole at $\omega = -\alpha$. Using Eq. (12) it can be shown that the response of the system is determined by the expression

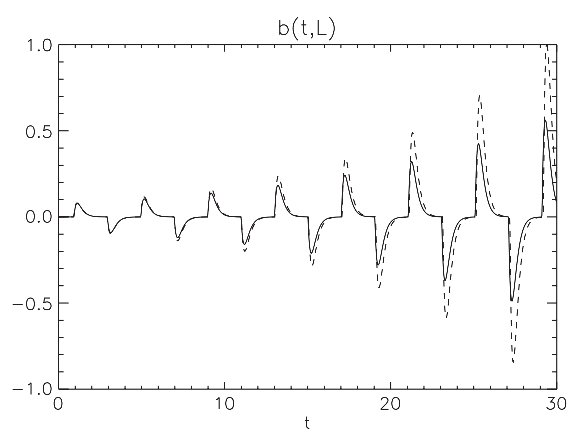

FIG. 2. Evolution of the field perturbation $b(t, z)$ at $z = L$ for two different flow speed profiles which satisfy the instability criterion: the solid and dashed lines represent the cases $u_0^- = 0.08c_A^-$ and $u_0^- = 0.1c_A^-$, respectively. In both cases, the following parameter values have been chosen: $\alpha = 5$, $c_A^+ = 0.01c_A^-$. Time is normalized with respect to c_A^-/L.

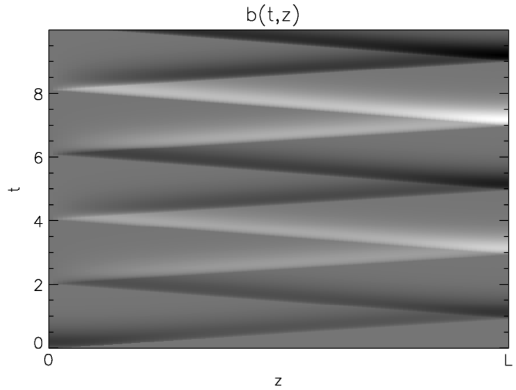

FIG. 3. Time-distance plot for the field perturbation $b(t, z)$ between $0 < z < L$. The flow speed $u_0^- = 0.1 c_A^-$. Other parameters are the same as in Fig. 2. Dark and bright regions represent positive and negative values of $b(t, z)$, respectively.

$$b(t, z) = \frac{\partial}{\partial \omega}\left[\frac{\exp(\omega t)\hat{b}(\omega, z)}{I}\right]_{\omega=-\alpha} + \frac{c_A^{-2} - u_0^{-2}}{2Lc_A} \times \sum_{n=0}^{\infty} \Re\left[\frac{\exp(\frac{-\omega_n z}{u_0^- + c_A^-}) - \exp(\frac{-\omega_n z}{u_0^- - c_A^-})}{(\omega_n + \alpha)^2} \exp(\omega_n t)\right] \quad (14)$$

for $t > z/(u_0^- + c_A^-)$ and $0 < z < L$. The first term on the right-hand side of Eq. (14) represents $\mathcal{F}(I)$ defined in Eq. (12) and ω_n are simple poles determined by formulas (11).

Figure 2 displays the evolution of the field perturbation at $z = L$ following the pulse at the origin given by Eq. (13). Two different values for the flow speed have been chosen: $u_0^- = 0.08 c_A^-$ (solid line) and $u_0^- = 0.1 c_A^-$ (dashed line). In both cases, $c_A^+ = 0.01 c_A^-$ and the instability criterion, $u_0^- > u_0^+ + c_A^+$, is satisfied. Figure 2 indeed shows exponential growth for both cases. The growth rate increases when the flow speed is higher.

A time-distance plot for the magnetic field perturbation is shown in Fig. 3. Dark and bright regions indicate positive and negative values of b. A pulse launched at the origin $z = 0$ reaches the interface $z = L$ and subsequently starts propagating back and forth between these two points. Part of the energy gets transmitted through $z = L$ and propagates further along the flowing plasma. Figure 3 shows that the pulse gets reflected (and transmitted) with an amplified amplitude every time it reaches the interface. It is exactly this effect that eventually leads to an instability.

In order to get further physical insight, we introduce a reflection coefficient $\mathcal{R}$ defined as the ratio of the reflected and incident wave amplitudes. These amplitudes are determined by the two terms with coefficients B and A on the right-hand side of Eq. (12). After some algebra we obtain

$$\mathcal{R} = \left|\frac{c_A^- + u_0^- - u_0^+ - c_A^+}{c_A^- - u_0^- + u_0^+ + c_A^+}\right|. \quad (15)$$

When the flow is absent and the Alfvén speed is constant the pulse propagates along the magnetic field without any reflection ($\mathcal{R} = 0$). There is partial reflection and partial transmission when $u_0^- < u_0^+ + c_A^+$. For $u_0^- = u_0^+ + c_A^+$ total reflection occurs ($\mathcal{R} = 1$). Finally, when $u_0^- > u_0^+ + c_A^+$, the reflection coefficient (15) becomes greater than one. The wave is *over-reflected* which explains the amplification at the interface observed in Fig. 3. The instability is therefore caused by over-reflection. The mechanism of over-reflection has been studied in the past mainly in relation to incompressible shear flows [13].

The above chosen ratio of 0.01 for the Alfvén speeds is representative of the photosphere and corona environment. Using the derived instability criterion, it can be estimated that for a typical coronal Alfvén speed of 1000 km/s the threshold flow speed for over-reflection in the corona is just over 10 km/s. Such conditions are easily satisfied. The critical speeds and ratios are likely to change when applications to more complex systems are considered. Although the model is rather simple to account for processes occurring in various complex magnetic structures, it demonstrates that the above established possibility of over-reflection and instability cannot be ignored in future studies of compressible plasma flows.

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