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Nonexistence results for relaxation spectra with compact support

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Abstract

In this paper we consider the problem of recovering the (transformed) relaxation spectrum h from the (transformed) loss modulus g by inverting the integral equation $g = \text{sech} * h$, where $*$ denotes convolution, using Fourier transforms. We are particularly interested in establishing properties of h, having assumed that the Fourier transform of g has entire extension to the complex plane. In the setting of square integrable functions, we demonstrate that the Paley-Wiener theorem cannot be used to show the existence of non-trivial relaxation spectra with compact support. We prove a stronger result for tempered distributions: there are no non-trivial relaxation spectra with compact support. Finally we establish necessary and sufficient conditions for the relaxation spectrum h to be strictly positive definite.

MSC2010 classification: 42A38, 45Q05, 46F12

Keywords: Fourier transform, inverse problems, Paley-Wiener theorem, relaxation spectrum

1 Introduction

This paper studies an inverse problem arising from the theory of viscoelastic fluids. The (transformed) relaxation spectrum h is found from the (transformed) loss modulus g by inverting the integral equation $g = \text{sech} * h$, where $*$ denotes convolution. Using the notation $\hat{f}$, $\mathcal{F}^{-1}(f)$ for respectively the Fourier transform and inverse Fourier transform of f, we have

$$\hat{g} = \widehat{\text{sech} * h} = \widehat{\text{sech}}\hat{h}.$$

Now a standard calculation yields $\widehat{\text{sech}}(r) = \pi\text{sech}(\pi r/2)$, and writing $\xi_\lambda(r) = \cosh(\lambda r)$ (following the notation of [7], with $\lambda = \pi/2$,) our expression becomes $(1/\pi)\hat{g}\xi_\lambda = \hat{h}$. Taking inverse Fourier transforms, $h = (1/\pi)\mathcal{F}^{-1}(\hat{g}\xi_\lambda)$. (Henceforth we omit the constant.) The

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inverse problem of finding h given g is ill-posed (in the sense of Hadamard); numerical schemes may be producing spurious results. Davies, Anderssen and co-workers [2, 7, 8] proposed a criteria for judging the validity of numerical schemes. The purpose of this paper is to examine whether the relaxation spectrum h can have compact support given conditions on g, a key idea in the above work. There are examples where this is true: if $g = \text{sech}$ then $h = \delta$, the Dirac mass concentrated at zero, which is supported on $\{0\}$. We investigate this problem in the settings considered by [7, 8], and generalisations of these, using Paley-Wiener theorems. We require that $\hat{g}\xi_\lambda$ has entire extension to the complex plane, and make the assumption $\hat{g}$ has entire extension, which excludes the above example. Theorems 1 and 2 are non-existence results for h with compact support; finding the correct theoretical setting for a general existence result for this problem is not only an interesting mathematical problem, it may help in the design of robust numerical techniques to solve the inverse problem (or act as a criteria to judge existing schemes).

Previous work by Loy, Davies, Anderssen and Newbury [7] used a weak formulation: for $f \in L^p(\mathbb{R})$, $1 < p \leq 2$, define

$$\tilde{\kappa}_g(f) = \int_{-\infty}^{\infty} \hat{h}\hat{f} = \int_{-\infty}^{\infty} \hat{g}\xi_\lambda \hat{f}. \tag{1}$$

It is easily seen (via the Hausdorff-Young inequality) that $\tilde{\kappa}_g$ is a bounded linear mapping, and can be represented by $\kappa_g \in L^q(\mathbb{R})$, where q denotes the conjugate exponent of p, with

$$\tilde{\kappa}_g(f) = \int_{-\infty}^{\infty} \kappa_g f.$$

We identify κ_g with h. The basic idea is to exploit $\hat{g}\hat{f} = \widehat{g * f}$ in the latter integral of (1), and change the order of integration, moving the Fourier transform onto ξ_λ. However ξ_λ does not belong to an appropriate function space to allow this, so a regularisation (Gaussian) term is introduced into the latter integrand of (1); as $\varepsilon \to 0$, this term approaches unity. We write $\tilde{\kappa}_{g,\varepsilon}$ for the bounded linear mapping, and $\kappa_g(\varepsilon)$ for its representation. After some manipulations (and an application of the Dominated Convergence theorem) we have for $\delta > 0$,

$$\lim_{\varepsilon\to 0} \tilde{\kappa}_{g,\varepsilon}(f) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \kappa_g(\varepsilon) f = \int_{-\infty}^{\infty} \kappa_g f = \lim_{\varepsilon\to 0} \int_{|s|\leq\lambda+\delta} W_\varepsilon(s)(g * f)(s) ds$$

where

$$W_\varepsilon(s) = \frac{\sqrt{2\pi}}{\varepsilon} \exp\left(\frac{\lambda^2 - s^2}{2\varepsilon^2}\right) \cos\left(\frac{\lambda s}{\varepsilon^2}\right).$$

For $g \in L^1(\mathbb{R})$, supported on $[a, b]$, and $f \in L^p(\mathbb{R})$, supported on the complement of $[-\lambda - \delta - b, \lambda + \delta - a]$,

$$(g * f)(s) = \int_a^b g(t) f(s - t) dt = 0.$$

This was interpreted as $\kappa_g (= h)$ being supported on $[-\lambda - b, \lambda - a]$. However it was noted by Renardy [9] that $\hat{g}\xi_\lambda \in L^p(\mathbb{R})$ coupled with g having compact support constrains g to be the zero function (and hence $h = 0$ also).

A revised calculation by Loy, Davies and Anderssen [8] removed the assumption that g was compactly supported. Instead they introduced $c \in L^1(\mathbb{R})$ with support in $[a, b]$, and noted

for f supported as before

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\kappa_{g+c}(\varepsilon)f=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\kappa_{g}(\varepsilon)f.$$

Loy *et al.* [7] considered a second approach for $p = 2$: the Paley-Wiener theorem states that if an entire function is of exponential type, the inverse Fourier transform of the function restricted to the real line has compact support. If g has compact support, it is easily shown that $\hat{g}\xi_\lambda$ is of exponential type; moreover it can be shown, using Morera's theorem, that $\hat{g}$ has entire extension to the complex plane. (See Dodd [4] for full details.) The Paley-Wiener theorem yields $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda)$ has compact support. However we have previously seen these hypotheses imply $g = 0$ (and $h = 0$). Whilst the assumption that g has compact support is not appropriate, we investigate if weaker hypotheses might be. We use this approach to show non-existence of non-trivial relaxation spectra h with compact support. For a space $\mathcal{T}(\mathbb{R})$, define $F_{[\lambda,\mathcal{T}']} = \{g \in L^1(\mathbb{R}) : \hat{g}\xi_\lambda \in \mathcal{T}'(\mathbb{R})\}$, where $\mathcal{T}'(\mathbb{R})$ denotes the space of bounded linear functionals on $\mathcal{T}(\mathbb{R})$. Our basic strategy is as follows: we show $F_{[\lambda,\mathcal{T}']}$ has no non-trivial elements with compact support; demonstrate that if $\hat{g}\xi_\lambda$ is of exponential type, then so is $\hat{g}$; apply the Paley-Wiener theorem to deduce $g \in F_{[\lambda,\mathcal{T}']}$ with compact support, and therefore $g = 0$. Theorem 1, in the setting of square integrable functions, demonstrates that we cannot use the Paley-Wiener theorem to show existence of non-trivial relaxation spectra. In the setting of tempered distributions, we use a stronger form of the Paley-Wiener theorem which allows us to prove that if h has compact support, $h = 0$. This is the content of Theorem 2; it encompasses the $L^p - L^q$ setting of Loy *et al.* [7], and in Corollary 1 we state the result that no non-trivial relaxation spectra exist. The main work is demonstrating that if $\hat{g}\xi_\lambda$ is of exponential type, then so is $\hat{g}$; when g is one-signed (which is true for the physical problem we are modelling as g is non-negative,) and $\hat{g}$ has an integral representation when extended to the complex plane, an elementary argument is sufficient; for two-signed g, we use properties of meromorphic functions. We assume that $\hat{g}$ has entire extension to the complex plane; Gaussians have this property, so Theorems 1 and 2 apply to non-trivial subsets of $F_{[\lambda,\mathcal{T}']}$ whether $\mathcal{T}'(\mathbb{R})$ is $L^2(\mathbb{R})$ (Theorem 1) or the space of tempered distributions (Theorem 2). For Theorem 2, our assumptions about h, coupled with the Paley-Wiener-Schwartz theorem, yield that $\hat{g}\xi_\lambda$ has entire extension to the complex plane, therefore $\hat{g}$ is analytic except possibly at the zeroes of ξ_λ.

Having established non-existence results, we turn our attention to properties of h: we state a precise condition for h to be strictly positive definite in Proposition 3. Loosely speaking, positive definite functions are "like Gaussians". Finally we discuss properties a space $\mathcal{T}(\mathbb{R})$ would need to have for non-trivial h with compact support to exist.

Our basic problem arises from the study of viscoelastic fluids, that is fluids that have elastic properties usually associated with solids as well as liquid-like properties; roughly speaking, the fluid remembers the flow history (unlike a Newtonian fluid). Memory fades as time elapses. (It would fade instantaneously for a Newtonian fluid.) We can think of the relaxation spectrum $H = H(\tau)$ as describing for how long and to what extent fluid behaviour is affected by the past deformation history. H is related to the storage modulus G' and loss modulus G'' by the following integral equations:

$$G'(\omega)=\int_0^\infty \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}d\tau,\ G''(\omega)=\int_0^\infty \frac{\omega\tau}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}d\tau.$$

Here ω denotes frequency, and τ relaxation time. (See, for example, Davies and Goulding [3] for a derivation of these equations from Boltzmann's linear viscoelastic theory.) Now G'

and G'' can be determined by oscillatory shear experiments; the experiments can only be performed for a limited range of sampled frequencies, therefore the application of inversion formulae based on all positive frequencies is problematic. Whilst in theory (see Renardy [9]) G' and G'' can be recovered on the whole positive half-line by analytic continuation, no practical algorithm exists that can perform this task with sufficient accuracy. Many of the attempts to recover the whole relaxation spectrum are essentially curve-fitting. There are also numerical schemes that produce a relaxation spectrum from a set of experimental results, including the sampling localisation algorithm of Davies and Anderssen [2]. After a change of variables $\omega = \exp(-s)$ and $\tau = \exp(t)$, and writing $h(t) \equiv H(\exp(t))$ and $g(s) \equiv G''(\exp(-s))$, the equation for the loss modulus G'' may be rewritten (after some manipulations)

$$g(s) = \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sech}(s-t)h(t)dt,$$

that is $g = \operatorname{sech} * h$ (ignoring the constant). Solving the inverse problem (finding h given g) is ill-posed; regularisation methods are used, and there have been attempts to put numerical schemes on a sound theoretical footing. Davies and Anderssen [2] have stated results of *sampling localisation*, that is when information about H on some interval $a < \tau < b$ is completely determined (from a practical numerical point of view) by values of G', G'' on some (bounded) frequency interval. Their linear functional strategy relies on results of *relaxation spectrum recovery*, that is showing that if g has certain properties, then h has compact support.

The significance of the theorems in this paper is that under the assumption that $\hat{g}$ can be extended to an entire function in the complex plane, there are no non-trivial results of relaxation spectrum recovery in the settings in which the problem has previously been studied, nor in the more general setting of tempered distributions.

This paper is organised as follows. In section 2, working in the square integrable setting, we prove Theorem 1. Section 3 is devoted to proving Theorem 2, a stronger non-existence result, in Schwartz space. We consider what positive results may be established in Section 4, establishing Proposition 3. Section 5 discusses possible theoretical settings for further study of this problem.

1.1 Definitions and notation

Definitions. For $1 < p \leq 2$, define

$$F_{[\lambda,p]} = \{g \in L^1(\mathbb{R}) : \hat{g}\xi_\lambda \in L^p(\mathbb{R})\}.$$

We will be particularly interested in the case $p = 2$.
We will denote the Schwartz space of functions of rapid algebraic decay by $\mathcal{S}(\mathbb{R})$; let $\mathcal{S}'(\mathbb{R})$ denote the space of bounded linear functionals on $\mathcal{S}(\mathbb{R})$. Define

$$F_{[\lambda,\mathcal{S}']} = \{g \in L^1(\mathbb{R}) : \hat{g}\xi_\lambda \in \mathcal{S}'(\mathbb{R})\}.$$

We use standard definitions of the Fourier transform and inverse Fourier transform:

$$\begin{aligned} \hat{f}(r) &= \int_{-\infty}^{\infty} f(t)\exp(-itr)dt, \\ \mathcal{F}^{-1}(\hat{f})(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(r)\exp(itr)dr. \end{aligned}$$

We extend the definition of Fourier transform to a complex variable z by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-izt}dt. \tag{2}$$

If $t \to f(t)e^{at}$ belongs to $L^1(\mathbb{R})$ for each $a \in \mathbb{R}$, then (2) is well defined and for each $z \in \mathbb{C}$,

$$|\hat{f}(z)| \leq \int_{-\infty}^{\infty} |f(t)e^{-izt}|dt. \tag{3}$$

(Note that this is a condition satisfied by Gaussian functions f.) If we further suppose $\hat{f}(z)$ is analytic for each $z \in \mathbb{C}$, then it is the entire extension of the usual Fourier transform of f. See Friedlander and Joshi [5, Section 10.3] for a discussion of conditions for which the Fourier transform can be defined for complex variables.

1.2 Statement of main results

Our main results are Theorems 1 and 2 which are proved in Sections 2 and 3 respectively. We note that Gaussian functions g belong to the required spaces, with $\hat{g}$ having entire extension to the complex plane, therefore the theorems apply to non-trivial sets.

Theorem 1. *Let $g \in F_{[\lambda,2]}$, and suppose that the extension of $\hat{g}$ to the complex plane is entire, and that there exists $C \geq 0$, $A > 0$ such that $|\hat{g}(z)\cosh(\lambda z)| \leq C\exp(A|z|)$ for all $z \in \mathbb{C}$. Then $g = 0$ (and $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda) = 0$).*

Remark. This is the setting of the Paley-Wiener argument in [7].

Theorem 2. *Suppose that $h \in \mathcal{S}'(\mathbb{R})$ satisfies $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda)$, where $g \in F_{[\lambda,\mathcal{S}']}$, and the extension of $\hat{g}$ to the complex plane is entire. Then if h has compact support, $h = 0$.*

Corollary 1. *Let $1 < p \leq 2$, and let q denote the conjugate exponent of p (i.e. $1/p + 1/q = 1$). Suppose that $h \in L^q(\mathbb{R})$ satisfies $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda)$, where $g \in F_{[\lambda,p]}$, and the extension of $\hat{g}$ to the complex plane is entire. Then if h has compact support, $h = 0$.*

Remark. This is the setting of the direct argument in [7],[8].

2 Non-compactness of support in $L^2(\mathbb{R})$

In this section, we demonstrate that there are no non-trivial elements $g \in F_{[\lambda,2]}$ such that the extension of $\hat{g}$ to the complex plane is entire, and $\hat{g}$ is of exponential type. Our method of proof is to note that $F_{[\lambda,2]}$ has no non-trivial elements with compact support (which follows from the Identity theorem); demonstrate that if $\hat{g}\xi_\lambda$ is of exponential type, then so is $\hat{g}$; apply the Paley-Wiener theorem to deduce $g \in F_{[\lambda,2]}$ has compact support, and therefore $g = 0$. The bulk of the work is concerned with the second step: section 2.1 considers the case when g is one-signed and $\hat{g}$ extended to the complex plane is represented by (2) (with g replacing f), where we have constructed an elementary argument; section 2.2 deals with the general case of two-signed g using properties of meromorphic functions. For the physical problem we are modelling, g is non-negative; the one-signed case is of practical interest.

The following proposition was proved by Loy, Davies and Andersson [8]. (A more detailed version of the proof can be found in Whittle Gruffudd [13].)

Proposition 1. *Let $g \in F_{[\lambda,p]}$ where $1 < p \leq 2$. Then g is an analytic function, and if g has compact support then $g = 0$.*

We state a Paley-Wiener theorem for square integrable functions. It is a trivial modification of Rudin [11, Theorem 19.3]. (We use a change of variables so that $f = \hat{F}$, for F with compact support, and then apply the inverse Fourier transform to both sides, noting the Fourier transform is a Hilbert space isomorphism, see for example [11, Theorem 9.13].)

Theorem (Paley-Wiener in L^2)
Let f be an entire function of exponential type, that is there exist $C \geq 0$, $A > 0$, such that

$$|f(z)| \leq Ce^{A|z|} \text{ for every } z \in \mathbb{C}.$$

If the restriction of f to the real line belongs to $L^2(\mathbb{R})$, then $\mathcal{F}^{-1}(f) \in L^2(\mathbb{R})$ has support in $[-A, A]$.

2.1 One-signed case

Given that $\hat{g}\xi_\lambda$ is of exponential type, we demonstrate that $\hat{g}$ is as well. We can rearrange the inequality $|\hat{g}(z)\cosh(\lambda z)| \leq C_1 e^{A|z|}$ for $|\hat{g}(z)|$ in regions where $|\cosh(\lambda z)|$ is bounded away from zero. The difficulty in the proof arises when dealing with the remainder of the complex plane (as $\cosh(\lambda z)$ vanishes at the odd integers on the imaginary axis); for one-signed g, we compare $|\hat{g}(z)|$ with values from the regions where an exponential type inequality is known to hold. A key result is that for one-signed g, $|\hat{g}(z)| \leq |\hat{g}(iIm(z))|$.

Lemma 1. *Let $g \in F_{[\lambda,2]}$ be one-signed. Suppose $\hat{g}$ has entire extension to the complex plane with*

$$\hat{g}(z) = \int_{-\infty}^{\infty} g(t)e^{-izt}dt$$

for each $z \in \mathbb{C}$, and that $t \to g(t)e^{at}$ belongs to $L^1(\mathbb{R})$ for each $a \in \mathbb{R}$. Then if $\hat{g}\xi_\lambda$ is of exponential type, so is $\hat{g}$.

Proof. Noting $\hat{g}\xi_\lambda$ is of exponential type, we may choose $C_1 \geq 0$ and $A > 0$ such that

$$|\hat{g}(z)\cosh(\lambda z)| \leq C_1 e^{A|z|}$$

for every $z \in \mathbb{C}$. Moreover

$$|e^{\lambda z} + e^{-\lambda z}| \geq |2\cos(\lambda Im(z))|, \tag{4}$$

with equality on the imaginary axis. The right hand side of (4) is zero when $Im(z) = 2k+1$ for $k \in \mathbb{Z}$ (recalling $\lambda = \pi/2$). Choose intervals of uniform $1/2$ width centred on the odd integers on the imaginary axis; for all values z where $iIm(z)$ is outside the intervals, we have that

$$|e^{\lambda z} + e^{-\lambda z}| \geq |2\cos(\lambda Im(z))| \geq 2|\cos(3\pi/8)| = 2\beta > 0.$$

Define $\Omega = \{z \in \mathbb{C} : Im(z) \in (2k + 3/4, 2k + 5/4), \text{ some } k \in \mathbb{Z}\}$, that is extending the intervals of width $1/2$ centred on the odd integers on the imaginary axis to strips in the complex plane. For $z \in \mathbb{C}\backslash\Omega$ we have

$$|\hat{g}(z)| \leq \frac{C_1 e^{A|z|}}{|\cos(\lambda Im(z))|} \leq \frac{C_1 e^{A|z|}}{\beta} = C_2 e^{A|z|}, \tag{5}$$

where $C_2 \geq 0$ is a constant.

Now g is one-signed, so we have that for $z \in \mathbb{C}$,

$$|\hat{g}(z)| = \left| \int_{-\infty}^{\infty} g(t) e^{-izt} dt \right| \leq \int_{-\infty}^{\infty} |g(t)||e^{-izt}| dt = \left| \int_{-\infty}^{\infty} g(t) e^{Im(z)t} dt \right| = |\hat{g}(iIm(z))|,$$

where we have used (3). We note that if $iIm(z)$ is in one of the intervals (on the imaginary axis), then $i(Im(z)+1)$ and $i(Im(z)-1)$ are not (as we chose the intervals sufficiently small). Consequently for $z \in \Omega$ the following calculation holds:

$$\begin{aligned} |\hat{g}(z)| &\leq |\hat{g}(iIm(z))| = \left| \int_{-\infty}^{\infty} g(t) e^{Im(z)t} dt \right| \leq \left| \int_{-\infty}^{\infty} g(t) \left(e^{(Im(z)+1)t} + e^{(Im(z)-1)t} \right) dt \right| \\ &= |\hat{g}(i(Im(z)+1))| + |\hat{g}(i(Im(z)-1))| \leq C_2 e^{A|Im(z)+1|} + C_2 e^{A|Im(z)-1|} \qquad (6) \\ &\leq C_3 e^{A(|Im(z)|+1)} = C_4 e^{A|Im(z)|} \leq C_4 e^{A|z|}, \end{aligned}$$

where $C_3, C_4 \geq 0$ are constants. By way of explanation we have used inequality (5) to obtain the inequality in (6). Combining the above, $\hat{g}$ is of exponential type. □

2.2 General case

For the general case of two-signed g, there is no reason to suppose the estimates of Lemma 1 are valid. Instead we interpret the hypothesis that $\hat{g}\xi_\lambda$ is of exponential type as an upper bound on $|\hat{g}|$, and establish that $\hat{g}$ is a meromorphic function of a given type. Results of Rubel and Taylor [10] yield that $\hat{g}$ is of exponential type. The assumption that $\hat{g}$ is entire simplifies the calculation.

We use (definitions and) results of Rubel and Taylor [10]. These were stated for meromorphic functions, that is a function which is analytic except at a discrete set of isolated points where it has poles. We will be concerned with the simpler case of entire functions.

Definitions. The *Nevanlinna characteristic function* T of a meromorphic function f is given by

$$T(r, f) = m(r, f) + N(r, f),$$

where the *proximity function* m is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where the + superscript denotes positive part, and the *integrated counting function* N, which is only concerned with the poles of f. For an entire function f, $T(r, f) = m(r, f)$.

A meromorphic function f is of *finite ρ-type* if there exist positive constants b, B such that $T(r, f) \leq b\rho(Br)$, where $\rho : [0, \infty) \to \mathbb{R}$ is positive, non-decreasing and continuous.
Remark. Rubel and Taylor use the notation λ-type; λ is already in use in our problem.

Theorem. An entire function f is of finite ρ-type if and only if there exist positive constants α and β such that $|f(z)| \leq \exp(\alpha\rho(\beta|z|))$ for all $z \in \mathbb{C}$.

Lemma 2. *Let $g \in F_{[\lambda,2]}$. Suppose that $\hat{g}\xi_\lambda$ is of exponential type, and that the extension of $\hat{g}$ to the complex plane is entire. Then $\hat{g}$ is of exponential type.*

Proof. Noting that $\hat{g}\xi_\lambda$ is of exponential type, for $z \in \mathbb{C}$ we have that

$$|\hat{g}(z)| \leq \frac{Ce^{A|z|}}{|\cosh(\lambda z)|}, \tag{7}$$

for some constants $A > 0$, $C \geq 0$. Now

$$\begin{aligned} m(r,\hat{g}) &\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{Ce^{A|re^{i\theta}|}}{\cosh(\lambda re^{i\theta})}\right| d\theta = \frac{1}{2\pi \ln d}\int_0^{2\pi} \ln^+ \left|\frac{Ce^{Ar}}{\cosh(\lambda re^{i\theta})}\right| d\theta \qquad (8)\\ &= \frac{1}{2\pi \ln d}\int_{\{\theta\in[0,2\pi]:Ce^{Ar}>|\cosh(\lambda re^{i\theta})|\}} \ln \left|\frac{Ce^{Ar}}{\cosh(\lambda re^{i\theta})}\right| d\theta \\ &\leq \frac{1}{2\pi \ln d}\left\{\int_0^{2\pi} \ln(Ce^{Ar})d\theta - \int_{\{\theta\in[0,2\pi]:|\cosh(\lambda re^{i\theta})|<1\}} \ln|\cosh(\lambda re^{i\theta})|d\theta\right\} \qquad (9)\\ &\leq A_1 r + C_1, \qquad (10) \end{aligned}$$

for some constants $A_1, C_1 > 0$. By way of explanation, the inequality in (8) follows from (7), noting that $\log^+$ is an increasing function; we assume log is to the base d (it is not specified in Rubel and Taylor [10]) to obtain the equality in (8); (10) follows from the fact that the second integral in (9) is bounded (independently of r, see Whittle Gruffudd [13, Lemma 3.12] for full details).

We have shown that $\hat{g}$ is of finite ρ-type where $\rho(r) = r + 1$. It follows from Rubel and Taylor [10] that there exist positive constants α, β such that for $z \in \mathbb{C}$,

$$|\hat{g}(z)| \leq \exp(\alpha\rho(\beta|z|)) = \exp(\alpha(\beta|z| + 1)) = C_2 e^{A_2|z|},$$

for $C_2 = \exp\alpha$, $A_2 = \alpha\beta$. We have demonstrated that $\hat{g}$ is of exponential type. □

Proof of Theorem 1.
Noting that $\hat{g}\xi_\lambda$ is of exponential type, Lemma 2 yields that $\hat{g}$ is of exponential type. Applying the Paley-Wiener theorem to $\hat{g}$ we deduce that $g = \mathcal{F}^{-1}(\hat{g})$ has compact support in $L^2(\mathbb{R})$. Noting that $g \in F_{[\lambda,2]}$, Proposition 1 yields that $g = 0$ (and $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda) = 0$).

3 Non-compactness of support in $\mathcal{S}'(\mathbb{R})$

In this section we demonstrate that there are no non-trivial h with compact support satisfying $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda)$ for $g \in F_{[\lambda,\mathcal{S}']}$ where the extension of $\hat{g}$ to the complex plane is entire. Our method of proof is to suppose h has compact support; apply the Paley-Wiener-Schwartz theorem to h to deduce that $\hat{g}\xi_\lambda$ is of exponential type; demonstrate that if $\hat{g}\xi_\lambda$ is of exponential type, then so is $\hat{g}$; apply the Paley-Wiener-Schwartz theorem to $\hat{g}$ to deduce that $g \in F_{[\lambda,\mathcal{S}']}$ has compact support; note that $F_{[\lambda,\mathcal{S}']}$ has no non-trivial elements with compact support, and conclude that $g = 0$, and hence $h = 0$. In the setting of Schwartz distributions, we use a stronger form of the Paley-Wiener theorem: the inverse Fourier transform of a function has compact support if and only if the function is of exponential type. This enables us to work directly with h, and achieve a better result compared with the square integrable setting.
Most of the work is concerned with proving that if $\hat{g}\xi_\lambda$ is of exponential type, then so is $\hat{g}$. Section 3.1 considers when g is one-signed and $\hat{g}$ extended to the complex plane is

represented by (2) (with g replacing f); the arguments of Lemma 1 are easily adjusted to fit the tempered distribution setting. To deal with two-signed g, we again use properties of meromorphic functions, however extra work is needed to extend the inequality to the whole complex plane. This is the content of section 3.2.

Definitions. Following Rudin [12, Chapter 7], let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of smooth functions of rapid algebraic decay, that is those $f \in C^\infty(\mathbb{R})$ for which

$$\sup_{m \le N} \sup_{x \in \mathbb{R}} (1 + |x|^2)^N |D^m(f)(x)| < \infty$$

for each $N = 0, 1, 2, ...$, where $D^m(f)$ denotes the mth order derivative of f.
We denote the associated space of Schwartz distributions (i.e. the set of bounded linear functionals on $\mathcal{S}(\mathbb{R})$) by $\mathcal{S}'(\mathbb{R})$.
A distribution $T \in \mathcal{S}'(\mathbb{R})$ has support in a closed set K if $T(\varphi) = 0$ for every $\varphi \in \mathcal{S}(\mathbb{R})$ with support in the complement of K.

The Fourier transform maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ and is linear, injective, and bi-continuous.
The Fourier transform $\hat{T}$ in the sense of tempered distribution of $T \in \mathcal{S}'(\mathbb{R})$ is defined by $\hat{T}(\varphi) = T(\hat{\varphi})$ for every $\varphi \in \mathcal{S}(\mathbb{R})$.

The following proposition is easily established by the methods of Loy, Davies and Andersson [8]. Full details can be found in Whittle Gruffudd [13].

Proposition 2. *Let $g \in F_{[\lambda, \mathcal{S}']}$. Then g is an analytic function, and if g has compact support then $g = 0$.*

We state a Paley-Wiener theorem for Schwartz distributions. (See, for example, Friedlander and Joshi [5, Theorem 10.2.1 and Theorem 10.2.2].)

Theorem (Paley-Wiener-Schwartz)
Let $A > 0$. We have that $\mathcal{F}^{-1}(f) \in \mathcal{S}'(\mathbb{R})$ has support in $[-A, A]$ if and only if $f \in \mathcal{S}'(\mathbb{R})$ has entire extension to $\mathbb{C}$ and

$$|f(z)| \le C(1 + |z|)^n e^{A|Im(z)|} \text{ for every } z \in \mathbb{C},$$

for some constants $C, n \ge 0$.

3.1 One-signed case

When g is one-signed, we can modify the arguments of Lemma 1 to fit the definition of exponential type in the tempered distribution setting.

Lemma 3. *Let $g \in F_{[\lambda, \mathcal{S}']}$ be one-signed. Suppose $\hat{g}$ has entire extension to the complex plane with*

$$\hat{g}(z) = \int_{-\infty}^{\infty} g(t) e^{-izt} dt$$

for each $z \in \mathbb{C}$, and that $t \to g(t)e^{at}$ belongs to $L^1(\mathbb{R})$ for each $a \in \mathbb{R}$. Then if $\hat{g}\xi_\lambda$ is of exponential type, so is $\hat{g}$.

Proof. We use the methods of the proof of Lemma 1; let β and Ω be as defined in that lemma. Noting $\hat{g}\xi_\lambda$ is of exponential type, we may choose $C_1 \geq 0$ and $A > 0$ such that

$$|\hat{g}(z)\cosh(\lambda z)| \leq C_1(1+|z|)^n e^{A|Im(z)|}$$

for every $z \in \mathbb{C}$. Now for $z \in \mathbb{C}\backslash\Omega$ we have

$$|\hat{g}(z)| \leq \frac{C_1(1+|z|)^n e^{A|Im(z)|}}{\beta} = C_2(1+|z|)^n e^{A|Im(z)|}$$

where $C_2 \geq 0$ is a constant. Now for $z \in \Omega$,

$$\begin{aligned} |\hat{g}(z)| &\leq |\hat{g}(i(Im(z)+1))| + |\hat{g}(i(Im(z)-1))| \\ &\leq C_2(1+|Im(z)+1|)^n e^{A|Im(z)+1|} + C_2(1+|Im(z)-1|)^n e^{A|Im(z)-1|} \\ &\leq C_3(2+|Im(z)|)^n e^{A(|Im(z)|+1)} \leq C_4(1+|Im(z)|)^n e^{A|Im(z)|} \leq C_4(1+|z|)^n e^{A|Im(z)|}, \end{aligned}$$

where $C_3, C_4 \geq 0$ are constants. Combining the above, $\hat{g}$ is of exponential type. □

3.2 General case

As in section 2.2, we make use of results of Rubel and Taylor [10]. These yield a bound on the size of $\hat{g}$ in terms of $e^{A|z|}$ rather than $e^{A|Im(z)|}$. Accordingly we split our argument into two cases: for z satisfying $|Im(z)| \geq |Re(z)|$, we have $|z| \leq \sqrt{2}|Im(z)|$; for z satisfying $|Im(z)| < |Re(z)|$ we can rearrange the exponential type inequality for $\hat{g}\xi_\lambda$ into one for $\hat{g}$ as $|\cosh(\lambda z)|$ is bounded away from zero.

Lemma 4. *Let $g \in F_{[\lambda,\mathcal{S}']}$. Suppose that $\hat{g}\xi_\lambda$ is of exponential type, and that the extension of $\hat{g}$ to the complex plane is entire. Then $\hat{g}$ is of exponential type.*

Proof. Noting that $\hat{g}\xi_\lambda$ is of exponential type, for $z \in \mathbb{C}$ we have

$$|\hat{g}(z)| \leq \frac{C(1+|z|)^n e^{A|Im(z)|}}{|\cosh(\lambda z)|} \equiv \eta(z) \tag{11}$$

for some constants $A > 0$, $C \geq 0$, some $n \in \mathbb{N}$. Now, by making similar estimates to those in the proof of Lemma 2,

$$\begin{aligned} m(r,\hat{g}) &\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \eta(re^{i\theta})d\theta = \frac{1}{2\pi \ln d}\int_0^{2\pi} \ln^+ \eta(re^{i\theta})d\theta \\ &= \frac{1}{2\pi \ln d}\int_{\{\theta\in[0,2\pi]:\eta(re^{i\theta})>1\}} \ln \eta(re^{i\theta})d\theta \\ &\leq \frac{1}{2\pi \ln d}\int_0^{2\pi} \ln C(1+|re^{i\theta}|)^n e^{A|Im(re^{i\theta})|}d\theta \\ &- \frac{1}{2\pi \ln d}\int_{\{\theta\in[0,2\pi]:|\cosh(\lambda re^{i\theta})|<1\}} \ln|\cosh(\lambda re^{i\theta})|d\theta \\ &\leq \frac{1}{2\pi \ln d}\int_0^{2\pi} \ln(C(1+r)^n e^{Ar})d\theta + C_0 \\ &\leq A_2 n\ln(1+r) + A_1 r + C_1, \end{aligned}$$

for some constants $A_1, A_2, C_1, C_0 > 0$. We have shown that $\hat{g}$ is of finite ρ_1-type (where $\rho_1(r) = n\ln(1+r) + r + 1$). It follows from Rubel and Taylor [10] that there exist positive constants α and β such that for $z \in \mathbb{C}$,

$$|\hat{g}(z)| \leq \exp(\alpha\rho_1(\beta|z|)) \leq C_3(1+|z|)^{\alpha n} e^{A_3|z|}, \tag{12}$$

where $C_3 = C_3(\alpha, \beta) > 0$, $A_3 = \alpha\beta$. For $z \in \mathbb{C}$ satisfying $|Im(z)| \geq |Re(z)|$, we have $|z| \leq \sqrt{2}|Im(z)|$, and from (12) we have

$$|\hat{g}(z)| \leq C_3(1+|z|)^{\alpha n} e^{\sqrt{2}A_3|Im(z)|}. \tag{13}$$

For $z \in \mathbb{C}$ satisfying $|Im(z)| < |Re(z)|$, we can choose $\gamma > 0$ such that $|\cosh(\lambda z)| \geq \gamma$. In this region (11) yields that

$$|\hat{g}(z)| \leq \frac{C}{\gamma}(1+|z|)^n e^{A|Im(z)|}. \tag{14}$$

Combining (13) and (14) yields that $\hat{g}$ is of exponential type. □

Proof of Theorem 2
Applying the Paley-Wiener-Schwartz theorem to h yields that $\hat{g}\xi_\lambda$ is of exponential type. Now Lemma 4 implies that $\hat{g}$ is of exponential type. Applying the Paley-Wiener-Schwartz theorem to $\hat{g}$ we deduce that $g = \mathcal{F}^{-1}(\hat{g})$ has compact support in $\mathcal{S}'(\mathbb{R})$. Noting $g \in F_{[\lambda,\mathcal{S}']}$, Proposition 2 yields that $g = 0$, and $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda) = 0$.

Proof of Corollary 1
Noting that $\mathcal{S}'(\mathbb{R})$ contains $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, h satisfies the hypotheses of Theorem 2. Therefore $h = 0$.

4 Strictly positive definite relaxation spectra

In this section we establish conditions for the relaxation spectrum to be strictly positive definite; Gaussians are archetypal examples of such functions. (See, for example, Chang [1, Theorem 1.2] for elementary properties satisfied by (strictly) positive definite functions.)
Definition. Let f be a (real or) complex continuous function defined on $\mathbb{R}$. Then f is *strictly positive definite* if the matrix A defined by $A_{ij} = f(x_i - x_j)$ is strictly positive definite for all sets of points $x_1, ..., x_n$.

Bochner's theorem yields that every (strictly) positive definite function f is the Fourier transform of a positive finite Borel measure. Recalling that the *carrier* (or *support*) of a positive finite Borel measure is the smallest closed set with full measure, the following theorem (see Chang [1]) may be considered a converse:

Theorem. Let μ be a non-zero, finite Borel measure on $\mathbb{R}$ such that the carrier of μ is not a discrete set. Then the generalised Fourier transform $\hat{\mu}$ of μ is strictly positive definite on $\mathbb{R}$.

Proposition 3. *Let non-zero $g \in F_{[\lambda,p]}$ for some $1 < p \leq 2$. Define*

$$\mu(B) = \int_B \hat{g}\cosh(\lambda x)dx$$

for B belonging to the Borel field of $\mathbb{R}$. Then μ defines a finite Borel measure on $\mathbb{R}$ if and only if $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda) \in L^q(\mathbb{R})$ is strictly positive definite, where q denotes the conjugate exponent of p.

Proof. Suppose μ defines a finite Borel measure on $\mathbb{R}$. The carrier of $\hat{g}\xi_\lambda$ as a measure is synonymous with the support of $\hat{g}$, noting cosh is strictly positive on $\mathbb{R}$. Now $\hat{g}$ is non-zero (as g is non-zero), that is it is non-zero on a set of positive size; the support of $\hat{g}$ (and the carrier of μ) is not discrete. The arguments in the above theorem are equally applicable to the inverse Fourier transform; we deduce that $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda)$ is strictly positive definite.

Conversely suppose that $h = \mathcal{F}^{-1}(\hat{g}\xi_\lambda)$ is strictly positive definite. Then, noting that the arguments of Bochner's theorem apply equally well to the inverse Fourier transform, we have that $h = \mathcal{F}^{-1}(\nu)$ for some finite Borel measure ν; we identify ν with μ. □

5 Conclusion and Discussion

An outstanding question is whether we can establish general results of relaxation spectrum recovery, that is conditions on g which imply h has compact support. There are examples (not satisfying the conditions of Theorems 1 and 2) which indicate such results should exist (under different hypotheses). As noted in the introduction, if $g = \text{sech}$ then $\hat{g}(r) = \pi\text{sech}(\lambda r)$ and $h = \delta$, where δ denotes the Dirac mass (concentrated at zero). Further study in the settings of square integrable functions or tempered distributions should consider $\hat{g}$ which fail to be analytic at (some of) the zeroes of ξ_λ (in the complex plane).

To prove a result of relaxation recovery by Paley-Wiener methods, it seems natural to assume $\hat{g}$ has entire extension to the complex plane, ensuring $\hat{g}\xi_\lambda$ is entire, but work in a larger class of functions. Let $\mathcal{T}(\mathbb{R})$ be a space of functions, $\mathcal{T}'(\mathbb{R})$ the associated space of bounded linear functionals on $\mathcal{T}(\mathbb{R})$. Define

$$F_{[\lambda,\mathcal{T}']} = \{g \in L^1(\mathbb{R}) : \hat{g}\xi_\lambda \in \mathcal{T}'(\mathbb{R})\}.$$

Suppose we have a Plancherel theorem on $\mathcal{T}(\mathbb{R})$ (that is the Fourier transform is a well behaved mapping from $\mathcal{T}(\mathbb{R})$ to itself) and a Paley-Wiener theorem on $\mathcal{T}'(\mathbb{R})$. A minimum requirement for the correct setting is that $F_{[\lambda,\mathcal{T}']}$ contains non-zero elements with compact support. Smooth functions with Gaussian decay, as described in Gindikin and Volevich [6], would be a candidate.

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References

[1] Chang K F 1996 Strictly positive definite functions *Journal of Approximation Theory* **87** 148–158

[2] Davies A R and Anderssen R S 1997 Sampling localization in determining the relaxation spectrum *J. Non-Newt. Fluid Mech.* **73** 163–179

[3] Davies A R and Goulding N J 2012 Wavelet regularization and the continuous relaxation spectrum *J. Non-Newt. Fluid Mech.* **189** 19–30

[4] Dodd T J 2003 *Inverse Problems in Relaxation Spectrum Recovery* (MSc Thesis, University of Wales, Aberystwyth)

[5] Friedlander G and Joshi M 1998 *Introduction to the Theory of Distributions, 2nd edition* (Cambridge: Cambridge University Press)

[6] Gindikin S G and Volevich L R 1992 *Distributions and Convolution Equations* (Philadelphia: Gordon and Breach Science Publishers)

[7] Loy R J, Newbury C M, Anderssen R S and Davies A R 2001 A duality proof of sampling localization in relaxation spectrum recovery *Bull. Aust. Math. Soc.* **64** 265–269

[8] Loy R J, Davies A R and Anderssen R S 2009 A Revised Duality Proof of Sampling Localization in Relaxation Spectrum Recovery *Bull. Aust. Math. Soc.* **79** 79–83

[9] Renardy M 2008 On the use of Laplace transform inversion for reconstruction of relaxation spectra *J. Non-Newt. Fluid Mech.* **154** 47–51

[10] Rubel L A and Taylor B A 1966 A Fourier Series Method for Meromorphic and Entire Functions *Bull. Amer. Math. Soc.* **72** 858–860

[11] Rudin W 1987 *Real and Complex Analysis, 3rd edition* (Singapore: McGraw-Hill)

[12] Rudin W 1991 *Functional Analysis, 2nd edition* (Singapore: McGraw-Hill)

[13] Whittle Gruffudd H R 2012 *Relaxation Spectrum Recovery using Fourier Transforms* (PhD Thesis, Aberystwyth University)