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# Fuzzy Complex Numbers and their Application for Classifiers Performance Evaluation

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**Abstract**

There are a variety of measures to describe classification performance with respect to different criteria and they are often represented by numerical values. Psychologists have commented that human beings can only reasonably manage to process seven or-so items of information at any one time. Hence, selecting the best classifier amongst a number of alternatives whose performances are represented by similar numerical values is a difficult problem faced by end users. To alleviate such difficulty, this paper presents a new method of linguistic evaluation of classifiers performance. In particular, an innovative notion of fuzzy complex numbers (FCNs) is developed in an effort to represent and aggregate different evaluation measures conjunctively without necessarily integrating them. Such an approach well maintains the underlying semantics of different evaluation measures, thereby ensuring that the resulting ranking scores are readily interpretable and the inference easily explainable. The utility and applicability of this research are illustrated by means of an experiment which evaluates the performance of 16 classi-

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fiers using different benchmark datasets. The effectiveness of the proposed approach is compared to conventional statistical approach. Experimental results show that the FCN-based performance evaluation provides an intuitively reliable and consistent means in assisting end users to make informed choices of available classifiers.
*Keywords:* Fuzzy complex numbers, performance evaluation, feature selection, pattern classification

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## 1. Introduction

Pattern classification has been successfully applied to many application domains. For instance, classifiers have been developed in conjunction with feature selection approaches [1, 2, 3] to perform tasks such as image analysis [4], face recognition, and remote sensing. However, classifiers which are applied to different problem domains and trained by various learning algorithms can perform quite differently. In fact, evaluating classifier performance is perhaps one of the most deceiving and tricky problems in classifier design [5]. To tackle this important problem, a variety of measures have been proposed to describe the classification performance with respect to different criteria, ranging from classification accuracy and error rate, through storage complexity and computation time to sensitivity and robustness [6, 7, 8].

In principle, performance measures can be qualitative or quantitative. Quantitative measures are naturally expressed by numerical values. However, using such seemingly precise measures to compare a number of classifiers, their performances may turn out to be very close in value. Such pure numerical values with small differences may not make much sense to the user who would like to make an informed choice of available classifiers. It would be more appropriate and often desirable to describe the relative performance of the classifiers using linguistic terms, such as *good*, *average* and *bad*. The assessment in qualitative measures often reflects the knowledge of domain experts and such measures are usefully represented by linguistic terms. Compared to numerical values, linguistic terms make it easier for users to understand the evaluation outcome. Indeed, human beings appear to use qualitative reasoning when initially attempting to gain an understanding of a problem.

It is worth noting that in order to obtain a fair evaluation of classification performance, several measures may need to be taken into account concurrently. For example, *precision* and *recall* are two widely used statistical measures which jointly provide a common indication of classifier performance. However, for many classification tasks, these two statistical measures should not be utilised in isolation, as neither measure alone contains sufficient information to assess the performance. It can be trivial to achieve a *recall* score of 1.0 by simply assigning all instances to a certain class. Similarly, *precision* may remain high by classifying only a few instances. To combat this, *precision* and *recall* are usually combined into a single measure, such as the *F-Measure* which is the weighted harmonic mean of these two measures [9]. Unfortunately, in so doing, the underlying semantics associated with these two base measures may be destroyed, even if a qualitative version of the *precision* and *recall* measures are used. Thus, it is of great interest and potentially beneficial to establish a new mechanism which can maintain the associated semantics when performing evaluation without necessarily using just one transformed measure. Inspired by this observation, this paper proposes a novel framework of fuzzy complex numbers (FCNs) that will entail effective and efficient representation of different types of evaluation measures concurrently and explicitly.

Note that the term FCN is not new; the concept of complex numbers has been proposed in the literature. For example, a form of *fuzzy complex numbers* has been defined in [10] as a mapping from the conventional complex number plane to the real-valued interval $[0, 1]$. Such an FCN is therefore, simply a type-1 fuzzy set [11]. Work on the differentiation and integration of this type of FCNs has been proposed in [12, 13], with more advanced follow-on research on their mathematical properties reported in [14, 15, 16, 17]. Recently, in combining fuzzy complex analysis and statistical learning theory, important theorems (of a learning process) based on fuzzy complex random samples were developed [18]. This work further demonstrates the interesting properties of so-called *rectangular fuzzy complex numbers*, which are a special type of FCN as proposed in [10]. Another relevant development is the notion that relates real complex numbers to fuzzy sets [19]. It introduces a new type of set, named *complex fuzzy sets*, to allow the membership value of a standard fuzzy set to be represented using a classical complex number. However, as discussed in [19], it may be difficult to identify suitable real-world problems for the use of such complex-valued memberships. Despite this obstacle, work has continued along this theme of research. This is evident in that *complex fuzzy sets* have been integrated with propositional logic to construct a specific instance of fuzzy reasoning systems [20].

Existing research regarding the concept of FCNs is all framed by either giving conventional complex numbers a real-valued membership or assigning a fuzzy set element to a complex number as its membership value. These approaches are rather different from what is proposed in this paper, where both the real and imaginary values of an FCN are in general, themselves fuzzy numbers; each with an embedded semantic meaning. By extending the initial definition and calculus of the proposed FCNs as given in [21], important algebraic properties, including closure, associativity, commutativity and distributivity of such FCN are established in the present work. This helps to support the aggregation process of FCNs. This new aggregation approach enhances the original work of [21] by allowing an arbitrary number of components of an FCN to be integrated in a random order. Further, the newly derived modulus of this type of FCN is introduced to impose an order over a given set of FCNs. Apart from these theoretical contributions, this work is applied to a completely new problem domain to gauge the performance of classifiers. This differs significantly from what is reported in [21]. The underlying development of this new approach to FCNs is general. It offers great potential for other application problems which exhibit similar characteristics as those of multi-criteria performance evaluation (e.g. student performance evaluation [22]).

The rest of this paper is organized as follows. Section 2 proposes the novel approach to the notion of FCNs, which extends real-valued complex numbers to representing two-dimensional linguistic-valued measures concurrently. In Section 3, this approach is utilised to construct a general linguistic evaluation method which effectively ranks the overall performance of different classifiers. For computational simplicity, such a general evaluation method is specified using the linear triangular fuzzy sets. Details of the implemented classifier evaluator are also presented in this section. Section 4 describes the experimentation carried out on standard benchmark datasets and discusses the evaluation results. The paper is concluded in Section 5, with the perspective of further work pointed out.

## 2. Fuzzy complex numbers

### *2.1. Prerequisites*

#### *2.1.1. Fuzzy numbers*

Fuzzy numbers are a special type of fuzzy sets which can be used to represent imprecise quantities such as *about 0.6*. Fuzzy numbers map real values from $\mathbb{R}$ on to a closed interval $[0,1]$.

**Definition 1.** *(Fuzzy numbers [23]) A fuzzy number, $\tilde{a}$, is defined as:*

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x)) \mid \mu_{\tilde{a}}(x) \in [0,1], x \in \mathbb{R}\},$$

*and satisfies the following properties:*

*a) Continuity: $\mu_{\tilde{a}}(x)$ is a continuous function mapping from $\mathbb{R}$ to a closed interval $[0,1]$.*

*b) Normality: i.e. $\exists x \in \mathbb{R}$ and $\mu_{\tilde{a}}(x) = 1$.*

*c) Convexity: i.e. $\forall x, y, z \in \mathbb{R}$, if $x \leq y \leq z$ then $\mu_{\tilde{a}}(y) \geq \min(\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(z))$.*

*d) Boundness of support: i.e. $\exists S \in \mathbb{R}$ and $\forall x \in \mathbb{R}$, if $|x| \geq S$ then $\mu_{\tilde{a}}(x) = 0$.*

### 2.1.2. *Extension principle*

The extension principle [24] provides a fundamental mechanism to translate conventional boolean set-based concepts into their fuzzy-set counterparts. In this work, it forms the foundation to derive the arithmetic operations of the proposed FCNs.

**Definition 2.** *Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $A_1, \ldots, A_n$ be fuzzy sets. Then $B = f(A_1, \ldots, A_n)$ is a fuzzy set with the following membership function:*

$$\mu_B(y) = \bigvee_{y=f(x_1,\ldots,x_n)} (\mu_{A_1}(x_1) \wedge \cdots \wedge \mu_{A_n}(x_n)). \tag{1}$$

Note that the operators $\wedge$ and $\vee$ above denote a given $t$-norm and $s$-norm respectively. Throughout this paper, they are interpreted using the min and max operators.

### 2.2. *Definition of FCNs*

Inherit from the real complex numbers, an FCN, $\tilde{z}$, is defined in the form of:

$$\tilde{z} = \tilde{a} + i\tilde{b}, \tag{2}$$

where both $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, regarding a given domain variable $x$. $\tilde{a}$ is the real part of $\tilde{z}$ while $\tilde{b}$ represents the imaginary part, i.e. $Re(\tilde{z}) = \tilde{a}$ and $Im(\tilde{z}) = \tilde{b}$.

An FCN can be visually shown as in Figure. 1. Importantly, in general, for a given $\tilde{z}$, both $Re(\tilde{z})$ and $Im(\tilde{z})$ are fuzzy. If $\tilde{b}$ does not exist, $\tilde{z}$ degenerates to a fuzzy number. Further, if $\tilde{b}$ does not exist and $\tilde{a}$ itself degenerates to a real number, then $\tilde{z}$ degenerates to a real number.

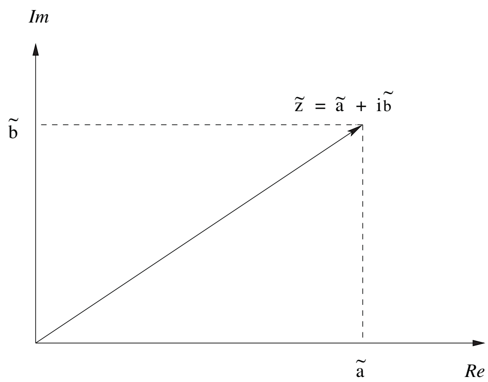


Figure 1: A fuzzy complex number

*2.3. Operations on FCNs*

The operations on the proposed FCNs are a straightforward extension of those on real complex numbers. Let $\widetilde{z}_1 = \tilde{a} + i\tilde{b}$ and $\widetilde{z}_2 = \tilde{c} + i\tilde{d}$ be two FCNs, where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$, $\mu_{\tilde{b}}(x)$, $\mu_{\tilde{c}}(x)$ and $\mu_{\tilde{d}}(x)$, respectively. The basic arithmetic operations on $\widetilde{z}_1$ and $\widetilde{z}_2$ are defined as follows:

- Addition

$$\widetilde{z}_1 + \widetilde{z}_2 = (\tilde{a} + \tilde{c}) + i(\tilde{b} + \tilde{d}), \tag{3}$$

  where $\tilde{a}+\tilde{c}$ and $\tilde{b}+\tilde{d}$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{a}+\tilde{c}}(y) &= \bigvee_{y=x_1+x_2} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{c}}(x_2)), \\ \mu_{\tilde{b}+\tilde{d}}(y) &= \bigvee_{y=x_1+x_2} (\mu_{\tilde{b}}(x_1) \wedge \mu_{\tilde{d}}(x_2)). \end{aligned} \tag{4}$$

- Subtraction

$$\widetilde{z}_1 - \widetilde{z}_2 = (\tilde{a} - \tilde{c}) + i(\tilde{b} - \tilde{d}), \tag{5}$$

where $\tilde{a}-\tilde{c}$ and $\tilde{b}-\tilde{d}$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{a}-\tilde{c}}(y) &= \bigvee_{y=x_1-x_2} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{c}}(x_2)), \\ \mu_{\tilde{b}-\tilde{d}}(y) &= \bigvee_{y=x_1-x_2} (\mu_{\tilde{b}}(x_1) \wedge \mu_{\tilde{d}}(x_2)). \end{aligned} \tag{6}$$

- Multiplication

$$\widetilde{z}_1 \times \widetilde{z}_2 = (\tilde{a}\tilde{c} - \tilde{b}\tilde{d}) + i(\tilde{b}\tilde{c} + \tilde{a}\tilde{d}), \tag{7}$$

where $\tilde{a}\tilde{c} - \tilde{b}\tilde{d}$ and $\tilde{b}\tilde{c} + \tilde{a}\tilde{d}$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{a}\tilde{c}-\tilde{b}\tilde{d}}(y) &= \bigvee_{y=x_1x_2-x_3x_4} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{c}}(x_2) \wedge \mu_{\tilde{b}}(x_3) \wedge \mu_{\tilde{d}}(x_4)), \\ \mu_{\tilde{b}\tilde{c}+\tilde{a}\tilde{d}}(y) &= \bigvee_{y=x_1x_2+x_3x_4} (\mu_{\tilde{b}}(x_1) \wedge \mu_{\tilde{c}}(x_2) \wedge \mu_{\tilde{a}}(x_3) \wedge \mu_{\tilde{d}}(x_4)). \end{aligned} \tag{8}$$

- Division

$$\frac{\widetilde{z}_1}{\widetilde{z}_2} = \left( \frac{\tilde{a}\tilde{c} + \tilde{b}\tilde{d}}{\tilde{c}^2 + \tilde{d}^2} \right) + i \left( \frac{\tilde{b}\tilde{c} - \tilde{a}\tilde{d}}{\tilde{c}^2 + \tilde{d}^2} \right). \tag{9}$$

For notational simplicity, let $\tilde{t}_1 = \frac{\tilde{a}\tilde{c}+\tilde{b}\tilde{d}}{\tilde{c}^2+\tilde{d}^2}$ and $\tilde{t}_2 = \frac{\tilde{b}\tilde{c}-\tilde{a}\tilde{d}}{\tilde{c}^2+\tilde{d}^2}$, where $\tilde{t}_1$ and $\tilde{t}_2$ are newly derived fuzzy numbers with the following membership functions:

$$\mu_{\tilde{t}_1}(y) = \bigvee_{y=\frac{x_1x_3+x_2x_4}{x_3^2+x_4^2}, x_3^2+x_4^2 \neq 0} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{b}}(x_2) \wedge \mu_{\tilde{c}}(x_3) \wedge \mu_{\tilde{d}}(x_4)),$$

$$\mu_{\tilde{t}_2}(y) = \bigvee_{y=\frac{x_2x_3-x_1x_4}{x_3^2+x_4^2}, x_3^2+x_4^2 \neq 0} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{b}}(x_2) \wedge \mu_{\tilde{c}}(x_3) \wedge \mu_{\tilde{d}}(x_4)). \quad (10)$$

- Modulus

  Given $\tilde{z} = \tilde{a} + i\tilde{b}$, the modulus of $\tilde{z}$ is defined:

$$|\tilde{z}| = \sqrt{\tilde{a}^2 + \tilde{b}^2}. \quad (11)$$

  It is obvious that $|\tilde{z}|$ is a newly derived fuzzy number with the following membership function:

$$\mu_{|\tilde{z}|}(y) = \bigvee_{y=\sqrt{x_1^2+x_2^2}} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{b}}(x_2)). \quad (12)$$

Note that the *rectangular FCNs* as proposed in [10] are represented in a form that is similar to the present work (but have different interpretation). In performing addition and subtraction on these two types of FCN, the same results are obtained. However, for multiplication and division this is not the case. The *rectangular FCNs* are defined as type-1 fuzzy sets, and the basic arithmetic operations on them are developed using the extension principle.

The algebraic properties of the proposed FCNs are presented in the Appendix. These properties, including closure, associativity, commutativity and distributivity are important for the further exploration and application of this novel framework. In particular, the associativity and commutativity of FCNs are used to derive the aggregation of components of an FCN in the next section.

### 2.4. Aggregation of Components of an FCN

Importantly, if there are more than two components involved in the problem domain, a hierarchical aggregation approach can be taken due to the commutativity of FCNs. That is, any two fuzzy numbers can be selected to construct a working FCN first. Then, the newly derived modulus of this FCN, together with a third fuzzy number can be used to construct another FCN. This process continues until all the involved fuzzy numbers are aggregated. For notation simplicity, arbitrary $n$ components can be represented in one single FCN and each component is denoted as a fuzzy number.

**Definition 3.** *Let $\tilde{a}_1, \ldots, \tilde{a}_n$ be $n$ fuzzy numbers, an aggregation operator $\tau$ is defined as:*

$$\tau(\tilde{a}_1, \ldots, \tilde{a}_n) = \sqrt{\tilde{a}_1^2 + \cdots + \tilde{a}_n^2}. \tag{13}$$

Note that this aggregation results in a new fuzzy number. This can be obtained by directly applying Theorems 2 - 3 (see Appendix). Since the multiplication and addition on fuzzy numbers are commutative, different components can be aggregated in a random order with this aggregation operator.

## 3. Evaluation of classifiers performance

### 3.1. System overview

The problem considered herein is that of classifier performance evaluation. In particular, the classification task considered consists of two phases: feature selection and classification. Thus, this work helps to assist the end user to determine what combinations of feature selectors and classifiers may outperform the others with respect to a variety of given criteria. Obviously, a system implemented for such experimentation involves two main processes: *Data Processing* and *Evaluation*, with each carrying out certain subtasks as outlined in Figure 2.

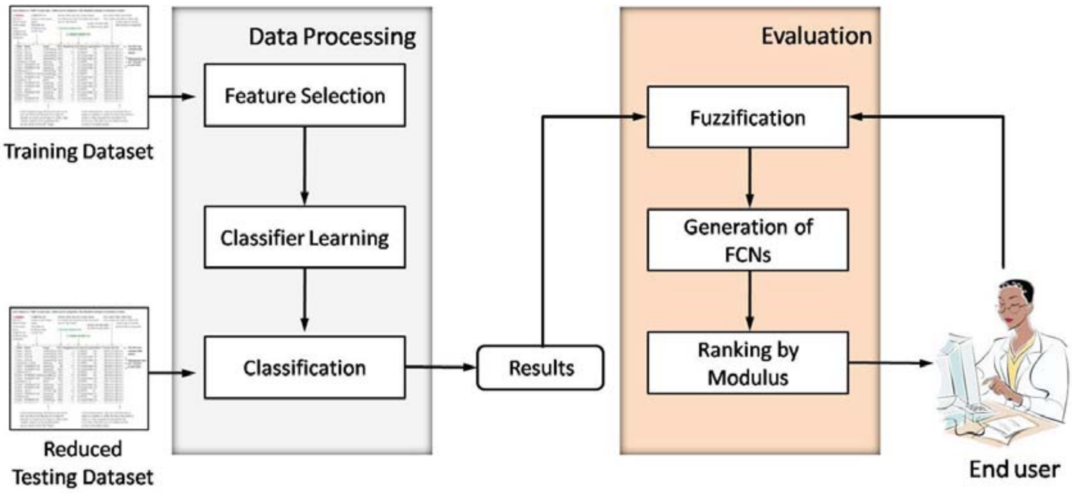


Figure 2: System overview

Initially, feature selection is utilised as the dimensionality reduction technique to extract a minimal feature subset from a given dataset while preserving the semantics of the original information. After that, a classifier learning step is employed to build a model or models which represent the relationship between the input data and the class labels from the training dataset. Then, the generated model is applied to predict the class labels of objects in the reduced testing dataset, using those selected features only.

The classification results obtained are recorded and fed to the *Evaluation* component for performance evaluation. The proposed FCNs notion is employed herein to enable such evaluation concurrently and explicitly. Finally, ranked classifiers are provided to the end user to support decision making. Based on the resulting evaluation scores, the user may adjust the partition of the quantity space in an effort to improve the evaluation results. In this work, the focus lies on the implementation of the proposed FCNs for the *Evaluation* step. Technical details for accomplishing the relevant subtasks are described in the following subsections.

### *3.2. Fuzzification*

In general, the results of classification may be either qualitative or quantitative. As argued previously, the absolute numerically-valued results (e.g. the probability associated with the class indices) of a given classifier may not be easily understood by the user. This is especially true when more than one value needs to be considered and whilst these numerical values are very close to each other. Instead of relying on the absolute real values, the relative performance of the classifiers may bear more information for the user. Thus, the numerical classification outcomes may be more humanly interpretable with fuzzy numbers or linguistic terms such as *High* and *Low* which help to capture the relative performance of a given classifier. In order to accomplish such subjective evaluation, a fuzzification process [25] is often employed, converting an observed input space into fuzzy sets defined in certain universes of discourse. Fuzzification of numerical real values also helps to “hide” the confidential context of performance, as the user may not be willing to disclose the actual values for certain sensitive evaluation metrics.

### 3.2.1. *Fuzzy quantity space*

Within the present work, the (relative) performance of a given classifier takes a value from a pre-defined fuzzy quantity space [26]. A fuzzy quantity space is simply a collection of all the membership functions defining the fuzzy sets that jointly partition a given universe of discourse. In particular, if defined on real numbers, the elements of a fuzzy quantity space are fuzzy numbers and such a fuzzy quantity space is denoted as $Q_{FN}$ (see later for example). Fuzzy quantity spaces provide an intuitive way to represent a qualitative value through the use of gradual membership functions, enabling a flexible representation of domain knowledge. This is because a fuzzy quantity space consists of a finite number of fuzzy sets, different cardinalities reflect different detailed abstractions of the modelled variables.

### 3.2.2. *Transforming a numerical value into a fuzzy set in $Q_{FN}$*

In general, the classification results may take values from different physical dimensions and scales. Thus, a normalization process is firstly applied. This process normalises the range of the resultant values to the interval $[0, 1]$. Given a class attribute, its classification results obtained by using different classification approaches are represented by: $v_1, v_2, \ldots, v_n$, where $n$ is the total number of combinations of feature selectors and classifiers. The normalized values of these results are defined as follows:

$$Nor(v_i) = \frac{v_i - min(v_1, v_2, \ldots, v_n)}{max(v_1, v_2, \ldots, v_n) - min(v_1, v_2, \ldots, v_n)}, \tag{14}$$

where $i = 1, 2, \ldots, n$.

The normalised results need to be mapped onto a pre-defined fuzzy quantity space $Q_{FN}$. A matching mechanism is employed to determine which element of $Q_{FN}$ best represents the given value.

**Definition 4.** *Given two fuzzy numbers $A$ and $B$, the matching degree, $S$, between them is:*

$$S(A, B) = 1 - S_d(A, B), \tag{15}$$

where $S_d(A, B)$ denotes a distance measure between two normalized fuzzy sets.

Suppose that $Q_{FN} = \{V_{FN_1}, V_{FN_2}, \ldots, V_{FN_n}\}$. Given a fuzzy set $A$, it is obvious that the larger the matching degree $S(V_{FN_i}, A)$ is, the more similar $V_{FN_i}$ and $A$ are, where $i \in \{1, 2, \ldots, n\}$. $V_{FN}$ is selected to represent $A$ if $V_{FN} = argmaxS(V_{FN_i}, A)$, $i \in \{1, 2, \ldots, n\}$.

### *3.3. Performance representation using FCNs*

From the application point of view, an FCN is capable of representing two-dimensional inexact information concurrently. To support the task of classification performance evaluation, both the real and imaginary parts of a proposed FCN can be assigned with their embedding meanings (e.g. evaluation outcomes using two different measures). Having carried out the fuzzification step as described above, the classification value of a certain measure is represented by one of the pre-specified fuzzy numbers. Thus, the outcomes can be readily applied to construct the FCNs. For example, the real part can be utilised to represent the *precision* measure while the imaginary part represents the *recall* measure. As such, the modulus of the corresponding FCN offers a good indication of the overall system performance.

Importantly, if there are more than two evaluation measures involved, a hierarchical approach can be taken due to the commutativity of FCNs. That is, any two fuzzy numbers can be selected to construct a working FCN first. Then, the newly derived modulus of this FCN, together with a third fuzzy number can be used to construct another FCN. This process continues until all the involved fuzzy numbers are aggregated. Mathematically, the aggregation result of these fuzzy numbers is obviously equivalent to that which is obtained by applying Eq. (13) to all individual performance measures.

### *3.4. Overall performance ranking*

This is a fairly straightforward sub-task: the overall performance of a classifier is decided on the basis of the modulus of the corresponding FCN. According to Theorem 3, the modulus of an FCN is a fuzzy number. The relevant position of individual outcomes can be readily compared and hence ranked using the conventional partial ordering relation holding amongst fuzzy numbers. If, however, it is desirable to have an absolute ordering, the resulting fuzzy numbers can be compared by defuzzifying these fuzzy numbers. There exist many defuzzification operators that may be applied for this purpose (see later for an example).

### *3.5. An implemented classifier evaluator*

This subsection presents an implemented classifier evaluator which specifies the general evaluation approach. It is assumed that the relative performance is represented using fuzzy numbers which take values from the following fuzzy quantity space, $Q_{FN} = \{Worst, VL, L, M, H, VH, Best\}$, as shown in Figure 3. Although a wide range of membership functions may be adopted to represent fuzzy sets, for computational efficiency, triangular membership functions are used here. A fuzzy set $A$ with the triangular membership is denoted by $A = [a, b, c]$, where $a$ is its left most element, $b$ is the element whose membership value is 1, and $c$ is its right most element. This is shown in Figure 4.

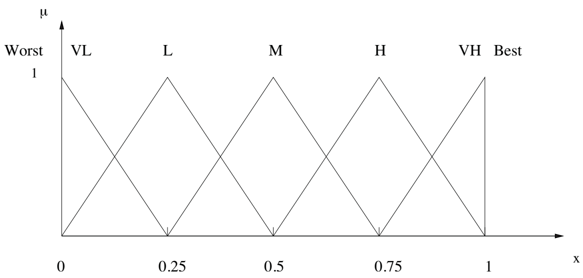


Figure 3: Fuzzy quantity space for implemented classifier evaluator

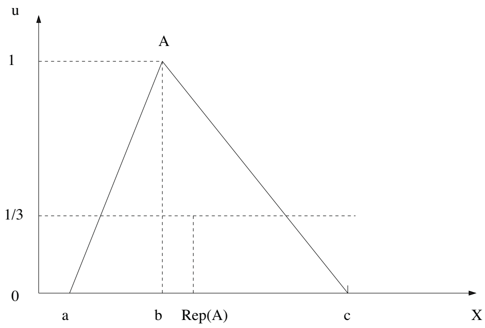


Figure 4: Representative value of a triangular fuzzy set $A = [a, b, c]$

In general, quantitative attributes which are represented by crisp values are often involved in the process of classifier performance evaluation. A fuzzification process is therefore needed when transforming a numerical value into a fuzzy set in $Q_{FN}$. This is achieved by: 1) normalising a crisp value $x$, $x \in D$ (the domain of the attribute in question), 2) treating the normalised value $\bar{x}$, $\bar{x} \in [0, 1]$ as a special case of triangular fuzzy numbers i.e. $[\bar{x}, \bar{x}, \bar{x}]$, and

3) calculating the degree of similarity between such a specific fuzzy number and an element of $Q_{FN}$. The degree of matching of two fuzzy numbers is a value within the range of $[0, 1]$. In this work, without losing generality, the *Hausdorff* distance [27] is employed to measure the fuzzy set matching degrees. This is defined below.

**Definition 5.** *Given two triangular fuzzy sets $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$, $A \neq B$, the Hausdorff distance between them is defined as:*

$$S_d(A, B) = \max\{d(A, B), d(B, A)\} = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}, \tag{16}$$

where $d(a, b)$ is the normalised absolute distance between parameters $a$ and $b$, i.e.

$$d(a, b) = \frac{|a - b|}{\max\{|b_3 - a_1|, |a_3 - b_1|\}}.$$

Once the distance between $A$ and $B$ is obtained, Eq. (15) can be applied to derive their matching degree. The fuzzification of a given fuzzy set $A$ is deemed to be the element of $Q_{FN}$ which receives the largest matching degree.

In order to derive an absolute overall performance ranking (amongst possible classifiers for use), the moduli of the constructed FCNs are defuzzified using their representative values [28]. For computational simplicity, the representative value of a triangular membership function $A = [a, b, c]$ (as shown in Figure 4) is defined as [29]:

$$Rep(A) = \frac{a + b + c}{3}. \tag{17}$$

Note that, this representative value also happens to be the centre of gravity (CoG) of the area under the triangular membership function.

In this work, the number of involved evaluation measures can vary. When calculating the moduli of the constructed FCNs, the resulting representative values may be larger than 1 (e.g. the representative value of $|Best + iBest|$ is 1.414). A normalisation is therefore employed to make all the derived representative values lie within a common scale $[0, 1]$, in which 1 represents the classifier performs best, whereas 0 represents the classifier performs worst.

## 4. Experimentation and discussions

To verify the applicability and utility of the proposed method for FCN-based classifier evaluation, a set of experiments are carried out in this section. Four feature selectors are used in this investigation in conjunction with four different classifiers, creating 16 combinations. These are each evaluated over five standard benchmark datasets.

### *4.1. Datasets*

Three benchmark datasets from the UCI Machine Learning Repository [30] (namely *glass*, *vehicle* and *sonar*) and two mammographic datasets (namely the Mammographic Image Analysis Society (MIAS) dataset [31] and the Digital Database of Screening Mammography (DDSM) dataset [32]) are used in this paper for experimental investigation.

The *glass* dataset consists of 10 attributes (including the index number) and all these attributes are continuously valued. There are 214 instances in total and they can be classified into 7 different classes. The *vehicle* dataset aims to classify a silhouette into one of the four given types of vehicle. The dataset records 946 instances together with 18 features which are extracted from the silhouette images. The *sonar* dataset contains 111 mine patterns and 97 rock patterns which are obtained by bouncing sonar signals off a metal cylinder at different angles and conditions. Each pattern is represented by 60 attributes which take values from $[0, 1]$.

The MIAS dataset contains 322 digitised mammograms images which are taken from 161 women by the UK National Breast Screening Programme. Each image is represented by 281 features extracted using the process described in [33]. Similarly, the DDSM dataset consists of 281 features obtained in the same manner as with MIAS datset but from 832 mammograms images. Three mammographic experts are invited to classify all the images in the above two datasets into the following four BIRADS categories, according to their density [34]:

- BIRADS 1: the breast is almost entirely fatty.
- BIRADS 2: there is some fibro-glandular tissue.
- BIRADS 3: the breast is heterogeneously dense.
- BIRADS 4: the breast is extremely dense.

Note that, if three experts classified the image into different classes, the consensus opinion is achieved by applying the method reported in [33].

### *4.2. Combined use of feature selection and classification*

#### *4.2.1. Feature selection*

In many real-world applications of feature-based pattern classification, due to the involvement of noisy, irrelevant or misleading features, it is likely that not all the input features are useful [35]. When considering learning tasks, it is evident that using an increasing number of features requires an exponentially increasing number of training objects and this is called the curse of dimensionality in the literature. Thus, it is important to apply feature selection (FS) to remove noisy and redundant features, while tackling the curse of dimensionality. In addition to the benefits of gaining computational efficiency and removing noisy inputs, FS also helps to reduce the costs associated with collecting large amounts of unnecessary (feature) measurements. However, choosing the most informative features is not an easy task as there may be many inter-dependencies between subsets of features. Amongst many FS techniques designed to support classification tasks [1, 3], the following four feature selectors are employed in this experimentation due to their availability:

- Consistency subset evaluator (CS) [36]: This is a probabilistic approach to feature selection. A subset of features is evaluated by means of the consistency criterion which specifies to what extent the subset can be accepted. Consistency of any subset can never be lower than that of the full set of features. Therefore, this subset evaluator is used in conjunction with a search algorithm which looks for the smallest subset with consistency equal to that of the full set of features.

- Fuzzy-rough feature selection (FRFS) [37]: Fuzzy-rough sets encapsulate the related but distinct concepts of vagueness (fuzzy sets) and indiscernibility (rough sets), offering a high degree of flexibility when dealing with real-valued data. Conventional fuzzy-rough sets [38] extend the rough set concepts [39, 40, 41] through the use of fuzzy equivalence classes [42], but process several problems (e.g. complexity of calculating the Cartesian product of fuzzy equivalence classes, and fuzzy lower approximation not being a subset of fuzzy upper approximation) that render them ineffective for large datasets. Recently, FRFS [37] proposed three new approaches based on the use of fuzzy $T$-transitive similarity relations - fuzzy lower approximation-based FS, fuzzy boundary region-based FS and, in particular, fuzzy discernibility matrix-based FS - to effectively address the above stated issues. No user-defined thresholds are required in any of these three new methods, although a choice must be made regarding fuzzy similarity relations and connectives.

- Correlation-based feature subset evaluator (CFS) [43]: This is a simple filtering algorithm that ranks feature subsets according to a correlation based heuristic evaluation function. Those subsets which contain features that are highly correlated with the class and uncorrelated with each other are searched for.

- Distance metric-assisted tolerance rough set feature selection (DM-TRS) [44]: This is an extension of the tolerance rough set (TRS) approach as described in [45], which is capable of dealing with real-valued data. It marries TRS with the distance metric assisted rough set approaches [46]. The information of the TRS boundary region that is otherwise ignored is examined and used to guide feature selection.

#### *4.2.2. Classifiers and classifier learning*

In this work, in order to demonstrate the challenge in classifiers performance evaluation and hence in their selection, four classifier which share similar underlying theoretical foundations are employed to classify the given data. Each classifier learning algorithm is briefly discussed below.

- Fuzzy k-nearest neighbours classifier (FNN) [47]: The classical (crisp) k-nearest neighbour (kNN) algorithm was introduced to classify objects based on their similarity to each of $k$ clusters created with the training data. However, each sample object is considered equally important in the assignment of the cluster label and once an object is assigned to a cluster, there is no indication of its strength of membership in that cluster. This work has been extended by assigning partial membership of an object to different clusters [48]. FNN also takes into account the relative importance of any test object with respect to each neighbouring cluster.

- Fuzzy-rough k-nearest neighbours classifier (FRNN_FRS) [49]: This approach combines the fuzzy-rough approximations [3, 37] with the underlying ideas of FNN. Given a test object, the nearest neighbours of this object are employed to construct the lower and upper approximations of each decision class. These derived approximations provide a clue for determining the class membership of the test object. This approach has the ability to handle real-valued data and is proven to be efficient in improving classification accuracy as well as considerably removing redundant, irrelevant, and noisy features.

- Fuzzy-rough ownership k-nearest neighbours classifier (FRNN_O) [50]: This approach combines conventional kNN algorithms with both fuzzy and rough uncertainties to generate class confidence values using a fuzzy-rough ownership function. Unlike conventional kNN algorithms, this approach does not need to specify the number of neighbours with all training objects considered. Initially, a parameter that determines the bandwidth of the fuzzy-rough ownership function is calculated for each attribute and all confidence values of decision classes for the test object $y$ are set to 0. Next, the squared weighted distance of $y$ from all objects is computed in order to update the fuzzy-rough ownership value of $y$. Finally, when all training objects have been applied, the algorithm outputs the class with the highest fuzzy-rough ownership value.

- Vaguely-quantified k-nearest neighbours classifier (VQNN) [51]: This method takes a similar approach to FRNN_FRS. However, it applies the vaguely quantified rough set (VQRS) model [52] to derive the fuzzy-rough upper and lower approximations, as this model may be more robust in the presence of noisy data.

### *4.3. Experimental setup*

In this work, the best-first search algorithm [53] is employed to perform CS and CFS-based feature selection. The FRFS feature selector uses the following similarity measure:

$$\mu_{R_a}(x, y) = 1 - \frac{|a(x) - a(y)|}{|a_{max} - a_{min}|} \tag{18}$$

where $\mu_{R_a}(x, y)$ is the degree to which object $x$ and $y$ are similar for feature $a$, along with the Łukasiewicz t-norm ($T(x, y) = \max(x + y - 1, 0)$) and the Łukasiewicz fuzzy implicator ($I(x, y) = \min(1, 1 - x + y)$). For the DM-TRS feature selector, the weighting of the distance measure is set to 0.2, while the weighting of the rough set dependency is set to 0.8. In addition, the tolerance value is set to 0.97 for all the experimental datasets. These parameters were empirically demonstrated to achieve the best level of dimensionality reduction for the given datasets [49].

For each of the classifiers, the value of $k$ is set to 10 initially and then decremented by 1 for each experiment. Thus, a set of 10 results are obtained for each dataset. Importantly, cross validation of 10 $\times$ 10-fold cross-validation is performed for each experiment. As with feature selection, in implementing each of the four classifiers, the similarity measure used is the same as $\mu_{R_a}(x, y)$ as specified above. For the FRNN_FRS approach, the Łukasiewicz t-norm and the Kleene-Dienes implicator ($I(x, y) = \max(1 - x, y)$) are chosen. The choice of this implicator is based on empirical studies [49]. In addition, the VQNN approach softens the universal and existential quantifier by means of vague quantifiers. In implementing the VQNN approach, the upper and lower fuzzy quantifiers are specified as $Q_l = Q_{(0.1,0.6)}$ and $Q_u = Q_{(0.2,1.0)}$ to reflect the vague quantifiers *some* and *most* from natural language respectively.

### *4.4. Experimental results*

Four measures, namely correct classification percentage (CCP), cardinality of surviving feature subset (i.e. number of selected features or classifier inputs), average precision and average recall, are adopted to evaluate the

Table 1: Evaluation results of using CCP and reduction capability for *glass* classification

| Classifier | CCP | | No. of features | | FCN | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | |
| CS + FRNN_FRS | 0.6869 | 0.6667 | 7 | 0.4 | H + iM | 0.9035 | 0.6163 | Joint 5 |
| CS + FNN | 0.6729 | 0.5 | 7 | 0.4 | M + iM | 0.7071 | 0.4687 | 9 |
| CS + FRNN_O | 0.6916 | 0.7222 | 7 | 0.4 | H + iM | 0.9035 | 0.6163 | Joint 5 |
| CS + VQNN | 0.6355 | 0.0556 | 7 | 0.4 | VL + iM | 0.5135 | 0.3232 | 12 |
| FRFS + FRNN_FRS | 0.6682 | 0.4444 | 9 | 0.0 | M + iWorst | 0.5 | 0.3131 | Joint 13 |
| FRFS + FNN | 0.6729 | 0.5 | 9 | 0.0 | M + iWorst | 0.5 | 0.3131 | Joint 13 |
| FRFS + FRNN_O | 0.6963 | 0.7778 | 9 | 0.0 | H + iWorst | 0.75 | 0.501 | 8 |
| FRFS + VQNN | 0.6355 | 0.0556 | 9 | 0.0 | VL + iWorst | 0.0833 | 0.0 | 16 |
| CFS + FRNN_FRS | 0.6869 | 0.6667 | 8 | 0.2 | H + iL | 0.8029 | 0.5407 | 7 |
| CFS + FNN | 0.6729 | 0.5 | 8 | 0.2 | M + iL | 0.5701 | 0.3658 | Joint 10 |
| CFS + FRNN_O | 0.6822 | 0.6111 | 8 | 0.2 | M + iL | 0.5701 | 0.3658 | Joint 10 |
| CFS + VQNN | 0.6355 | 0.0556 | 8 | 0.2 | VL + iL | 0.2697 | 0.14 | 15 |
| DMTRS + FRNN_FRS | 0.7149 | 1.0 | 4 | 1.0 | Best + iBest | 1.4142 | 1.0 | 1 |
| DMTRS + FNN | 0.6308 | 0.0 | 4 | 1.0 | Worst + iBest | 1.0 | 0.6888 | 4 |
| DMTRS + FRNN_O | 0.6589 | 0.3333 | 4 | 1.0 | L + iBest | 1.0496 | 0.7261 | 3 |
| DMTRS + VQNN | 0.6682 | 0.4444 | 4 | 1.0 | M + iBest | 1.1329 | 0.7886 | 2 |

performance of classifiers. Combining the aforementioned 4 feature selectors and 4 classifiers results in 16 different integrated approaches to classification using reduced input pattern dimensionality. Given the five datasets, a set of experimentations were carried out by using these 16 approaches, along with the corresponding parameters specified in Section 4.3.

### *4.4.1. Evaluation measures: CCP and reduction capability*

In this sub-section, the CCP and reduction capability are jointly considered for each combined feature pattern classifier. The evaluation results by using the 16 different approaches with respect to the given datasets are shown in Table 1 - Table 5, respectively.

Prior to the joint evaluation, the obtained absolute value of CCP and the number of selected features are normalized to $[0, 1]$. Intuitively, the normalised value of 1 is assigned to the classifier which achieves the largest

Table 2: Evaluation results of using CCP and reduction capability for *vehicle* classification

| Classifier | CCP | | No. of features | | FCN | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | |
| CS + FRNN_FRS | 0.6678 | 0.7478 | 18 | 0.0 | H + iWorst | 0.75 | 0.2909 | Joint 14 |
| CS + FNN | 0.6489 | 0.6087 | 18 | 0.0 | M + iWorst | 0.5 | 0.0 | 16 |
| CS + FRNN_O | 0.6891 | 0.9043 | 18 | 0.0 | VH + iWorst | 0.9167 | 0.4848 | 7 |
| CS + VQNN | 0.7021 | 1.0 | 18 | 0.0 | Best + iWorst | 1.0 | 0.5817 | 6 |
| FRFS + FRNN_FRS | 0.6076 | 0.3043 | 9 | 0.8182 | L + iH | 0.8029 | 0.3524 | Joint 10 |
| FRFS + FNN | 0.6158 | 0.3652 | 9 | 0.8182 | L + iH | 0.8029 | 0.3524 | Joint 10 |
| FRFS + FRNN_O | 0.6843 | 0.8696 | 9 | 0.8182 | H + iH | 1.0607 | 0.6524 | 3 |
| FRFS + VQNN | 0.6312 | 0.4783 | 9 | 0.8182 | M + iH | 0.9035 | 0.4695 | Joint 8 |
| CFS + FRNN_FRS | 0.591 | 0.1826 | 11 | 0.6364 | L + iH | 0.8029 | 0.3524 | Joint 10 |
| CFS + FNN | 0.5887 | 0.1652 | 11 | 0.6364 | L + iH | 0.8029 | 0.3524 | Joint 10 |
| CFS + FRNN_O | 0.6371 | 0.5217 | 11 | 0.6364 | M + iH | 0.9035 | 0.4695 | Joint 8 |
| CFS + VQNN | 0.5662 | 0.0 | 11 | 0.6364 | Worst + iH | 0.75 | 0.2909 | Joint 14 |
| DMTRS + FRNN_FRS | 0.6064 | 0.2957 | 7 | 1.0 | L + iBest | 1.0496 | 0.6394 | Joint 4 |
| DMTRS + FNN | 0.695 | 0.9478 | 7 | 1.0 | VH + iBest | 1.3595 | 1.0 | 1 |
| DMTRS + FRNN_O | 0.6726 | 0.7826 | 7 | 1.0 | H + iBest | 1.2607 | 0.885 | 2 |
| DMTRS + VQNN | 0.6135 | 0.3478 | 7 | 1.0 | L + iBest | 1.0496 | 0.6394 | Joint 4 |

Table 3: Evaluation results of using CCP and reduction capability for *sonar* classification

| Classifier | CCP | | No. of features | | FCN | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | |
| CS + FRNN_FRS | 0.7885 | 0.5172 | 14 | 0.3571 | M + iL | 0.5701 | 0.1108 | Joint 13 |
| CS + FNN | 0.8173 | 0.7241 | 14 | 0.3571 | H + iL | 0.8029 | 0.4786 | Joint 9 |
| CS + FRNN_O | 0.851 | 0.9655 | 14 | 0.3571 | VH + iL | 0.9663 | 0.7368 | 6 |
| CS + VQNN | 0.8077 | 0.6552 | 14 | 0.3571 | H + iL | 0.8029 | 0.4786 | Joint 9 |
| FRFS + FRNN_FRS | 0.8558 | 1.0 | 13 | 0.4286 | Best + iM | 1.1329 | 1.0 | Joint 1 |
| FRFS + FNN | 0.7596 | 0.3103 | 13 | 0.4286 | L + iM | 0.5701 | 0.1108 | Joint 13 |
| FRFS + FRNN_O | 0.8317 | 0.8276 | 13 | 0.4286 | H + iM | 0.9035 | 0.6375 | 8 |
| FRFS + VQNN | 0.7788 | 0.4483 | 13 | 0.4286 | M + iM | 0.7071 | 0.3272 | 12 |
| CFS + FRNN_FRS | 0.7788 | 0.4483 | 19 | 0.0 | M + iWorst | 0.5 | 0.0 | Joint 15 |
| CFS + FNN | 0.8221 | 0.7586 | 19 | 0.0 | H + iWorst | 0.75 | 0.395 | 11 |
| CFS + FRNN_O | 0.8462 | 0.931 | 19 | 0.0 | VH + iWorst | 0.9167 | 0.6584 | 7 |
| CFS + VQNN | 0.7692 | 0.3793 | 19 | 0.0 | M + iWorst | 0.5 | 0.0 | Joint 15 |
| DMTRS + FRNN_FRS | 0.7452 | 0.2069 | 5 | 1.0 | L + iBest | 1.0496 | 0.8684 | 3 |
| DMTRS + FNN | 0.7163 | 0.0 | 5 | 1.0 | Worst + iBest | 1.0 | 0.79 | 5 |
| DMTRS + FRNN_O | 0.7308 | 0.1034 | 5 | 1.0 | VL + iBest | 1.0103 | 0.8063 | 4 |
| DMTRS + VQNN | 0.774 | 0.4138 | 5 | 1.0 | M + iBest | 1.1329 | 1.0 | Joint 1 |

Table 4: Evaluation results of using CCP and reduction capability for MIAS classification

| Classifier | CCP | | No. of features | | FCN | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | |
| CS + FRNN_FRS | 0.6646 | 0.6632 | 11 | 0.807 | H + iH | 1.0607 | 0.8144 | Joint 4 |
| CS + FNN | 0.5559 | 0.1893 | 11 | 0.807 | L + iH | 0.8029 | 0.4399 | 14 |
| CS + FRNN_O | 0.6615 | 0.6488 | 11 | 0.807 | H + iH | 1.0607 | 0.8144 | Joint 4 |
| CS + VQNN | 0.7019 | 0.8246 | 11 | 0.807 | H + iH | 1.0607 | 0.8144 | Joint 4 |
| FRFS + FRNN_FRS | 0.6149 | 0.4460 | 7 | 0.961 | M + iVH | 1.0528 | 0.8029 | Joint 7 |
| FRFS + FNN | 0.5124 | 0.0 | 7 | 0.961 | Worst + iVH | 0.9167 | 0.6052 | Joint 12 |
| FRFS + FRNN_O | 0.6584 | 0.6353 | 7 | 0.961 | H + iVH | 1.1885 | 1.0 | 1 |
| FRFS + VQNN | 0.6522 | 0.6084 | 7 | 0.961 | M + iVH | 1.0529 | 0.803 | Joint 7 |
| CFS + FRNN_FRS | 0.7236 | 0.9191 | 32 | 0.0 | VH + iWorst | 0.9167 | 0.6052 | Joint 12 |
| CFS + FNN | 0.6025 | 0.3921 | 32 | 0.0 | M + iWorst | 0.5 | 0.0 | 16 |
| CFS + FRNN_O | 0.6988 | 0.8111 | 32 | 0.0 | H + iWorst | 0.75 | 0.3631 | 15 |
| CFS + VQNN | 0.7422 | 1.0 | 32 | 0.0 | Best + iWorst | 1.0 | 0.7262 | 11 |
| DMTRS + FRNN_FRS | 0.5528 | 0.1758 | 6 | 1.0 | L + iBest | 1.0496 | 0.7983 | Joint 9 |
| DMTRS + FNN | 0.6273 | 0.5 | 6 | 1.0 | M + iBest | 1.1329 | 0.9192 | Joint 2 |
| DMTRS + FRNN_O | 0.6211 | 0.473 | 6 | 1.0 | M + iBest | 1.1329 | 0.9192 | Joint 2 |
| DMTRS + VQNN | 0.59 | 0.338 | 6 | 1.0 | L + iBest | 1.0496 | 0.7983 | Joint 9 |

Table 5: Evaluation results of using CCP and reduction capability for DDSM classification

| Classifier | CCP | | No. of features | | FCN | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | |
| CS + FRNN_FRS | 0.4838 | 0.3333 | 21 | 0.45 | L + iM | 0.5701 | 0.3626 | Joint 10 |
| CS + FNN | 0.5018 | 0.4524 | 21 | 0.45 | M + iM | 0.7071 | 0.5177 | Joint 8 |
| CS + FRNN_O | 0.5259 | 0.6111 | 21 | 0.45 | M + iM | 0.7071 | 0.5177 | Joint 8 |
| CS + VQNN | 0.5572 | 0.8175 | 21 | 0.45 | H + iM | 0.9035 | 0.7402 | 6 |
| FRFS + FRNN_FRS | 0.5150 | 0.5397 | 23 | 0.36 | M + iL | 0.5701 | 0.3626 | Joint 10 |
| FRFS + FNN | 0.4789 | 0.3016 | 23 | 0.36 | L + iL | 0.3536 | 0.1173 | 15 |
| FRFS + FRNN_O | 0.4561 | 0.4789 | 23 | 0.36 | M + iL | 0.5701 | 0.3626 | Joint 10 |
| FRFS + VQNN | 0.5848 | 1.0 | 23 | 0.36 | Best + iL | 1.0496 | 0.9057 | Joint 3 |
| CFS + FRNN_FRS | 0.4838 | 0.3333 | 31 | 0.0 | L + iWorst | 0.25 | 0.0 | 16 |
| CFS + FNN | 0.5078 | 0.4921 | 31 | 0.0 | M + iWorst | 0.5 | 0.2832 | Joint 13 |
| CFS + FRNN_O | 0.5199 | 0.5714 | 31 | 0.0 | M + iWorst | 0.5 | 0.2832 | Joint 13 |
| CFS + VQNN | 0.5548 | 0.8016 | 31 | 0.0 | H + iWorst | 0.75 | 0.5663 | 7 |
| DMTRS + FRNN_FRS | 0.4332 | 0.0 | 9 | 1.0 | Worst + iBest | 1.0 | 0.8495 | 5 |
| DMTRS + FNN | 0.4862 | 0.3492 | 9 | 1.0 | L + iBest | 1.0496 | 0.9057 | Joint 3 |
| DMTRS + FRNN_O | 0.5067 | 0.4841 | 9 | 1.0 | M + iBest | 1.1329 | 1.0 | Joint 1 |
| DMTRS + VQNN | 0.5235 | 0.5952 | 9 | 1.0 | M + iBest | 1.1329 | 1.0 | Joint 1 |

absolute CCP, whereas the value of 0 is given to the one with smallest CCP value. However, for the reduction capability measure, the normalised value of 1 conversely reflects the classifier having the smallest cardinality of surviving feature subset, whereas the value of 0 represents the classifier having the most selected features. The normalized values are then used to match with the predefined $Q_{FN}$ (of Figure 3) to derive the corresponding linguistic terms which are in turn, applied to construct the corresponding FCNs. In particular, the real part of such an FCN represents the CCP of the classifier, while the imaginary part represents the reduction capability of the feature selectors. Note that the less features survived, the better reduction capability a feature selector has. These approaches are then ranked according to the representative values of the derived FCNs modulus.

The classifiers using the DM-TRS feature selector achieve a better performance on *glass*, *vehicle*, *sonar* and DDSM datasets. This is likely because DM-TRS significantly reduces the number of original features which are noisy. The classifier learning methods can therefore benefit from a high quality feature subset to produce a more accurate classification. In particular, for the *glass* dataset, the combination of DM-TRS and FRNN_FRS achieves the *Best* performance when considering just two criteria: CCP and reduction capability. For the *vehicle* dataset, although the combination of CS and VQNN results in the *Best* CCP, its reduction capability is relatively the *Worst*. Thus, it only ranks in the $6^{th}$ place. For the MIAS dataset, the combination of FRFS and FNN_O outperforms the others, because it achieves a *High* CCP when employing a feature selector with *Very_High* reduction capability. For the DDSM dataset, due to the *Medium* correct classification rate and *Best* reduction capability, the combination of DM-TRS and FRNN_O together with that of DM-TRS and VQNN rank jointly in the first place.

Clearly, with the use of linguistic terms, the evaluation results are transparent and can be readily understood by the user. Note that there are certain approaches that have different absolute performance values but receive the same linguistic ranking using the proposed FCN approach. This reflects the reality well. Due to the involvement of noisy data, it may be difficult and even unfair to distinguish the overall performances of those approaches which receive very similar numerical outcome values.

*4.4.2. Evaluation measures: precision and recall*

As previously mentioned, precision and recall are not discussed in isolation. In this sub-section, each combined approach is assessed from a different point of view, using the performance criteria of precision and recall. This forms a useful basis upon which to compare against the F-measure, which is defined by:

$$F = 2 \times \frac{precision \times recall}{precision + recall}. \tag{19}$$

Table 6 - Table 10 show the evaluation results from applying different classifiers to the *glass*, *vehicle*, *sonar*, MIAS and DDSM datasets, respectively. Note that, the absolute precisions and recalls, together with their associated FCNs are also included in this experiment. This helps non-expert users to gain a better understanding of the overall quality of a given classifier. In the event where none of the involved classifiers achieves an acceptable result in terms of absolute FCNs, users can abandon all these classifiers without taking the risk of picking up a poorly performing classifier for use. However,

if a classifier must be chosen whilst knowing that the absolute performances of all available classifiers are poor, the normalised FCNs still help to suggest a relative ranking.

Considering each given dataset, both the FCN-based measure and the F-measure obtain the same result for the first and the last three ranked classifiers. Similar ranking scores of different classifiers are obtained for the rest of the cases also. The only difference is that certain classifiers have the same ranking when using the FCN-based approach. A closer examination into the results reveals that, in addition to the inherent comprehensibility, the performance assessment outcomes by the FCN-based approach is intuitively more reliable and consistent.

For example, the "CS + VQNN" and "CFS + VQNN" classifiers on the MIAS dataset jointly rank the first when employing the FCN-based approach. However, the use of F-measure concludes that the "CFS + VQNN" classifier performs better than "CS + VQNN". Yet, it is clear that "CS + VQNN" achieves a higher absolute precision, whereas "CFS + VQNN" obtains a higher absolute recall. When combining these two measures, the resulting F-measure values of these two classifiers are actually extremely close to each other (0.7372 and 0.7404 respectively). It appears rather artificial to say one is better than the other overall just based on such a minute numerical difference (that may well result from noise in data). It is the relative performance to other classifiers that may be of more interest to the user. In FCN-based ranking, the precision of "CS + VQNN" is ranked the *Best* amongst all classifers, while its recall achieves a relatively *High* value. Conversely, the recall value of "CFS + VQNN" is ranked the *Best*, while its precision achieves a

Table 6: Evaluation results of using precision and recall for *glass* classification

| Classifier | Average Precision | | Average Recall | | F-measure | FCN | FCN | Rep(Modulus) | | Ranking | Ranking(F) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | Absolute | Normalised | | |
| CS + FRNN_FRS | 0.552 | 0.5875 | 0.5264 | 0.868 | 0.5389 | M + iM | M + iH | 0.9035 | 0.6427 | Joint 6 | 6 |
| CS + FNN | 0.4616 | 0.0119 | 0.4833 | 0.4987 | 0.4722 | M + iM | VL + iM | 0.5135 | 0.3371 | 9 | 9 |
| CS + FRNN_O | 0.6168 | 1.0 | 0.5288 | 0.8886 | 0.5694 | M + iM | Best + iVH | 1.3594 | 1.0 | Joint 1 | 2 |
| CS + VQNN | 0.4772 | 0.1119 | 0.4557 | 0.2622 | 0.4662 | M + iM | VL + iL | 0.2697 | 0.146 | Joint 13 | 13 |
| FRFS + FRNN_FRS | 0.528 | 0.4348 | 0.5027 | 0.665 | 0.515 | M + iM | M + iH | 0.9034 | 0.6427 | Joint 6 | 7 |
| FRFS + FNN | 0.526 | 0.422 | 0.479 | 0.4619 | 0.5014 | M + iM | M + iM | 0.7071 | 0.4889 | 8 | 8 |
| FRFS + FRNN_O | 0.5636 | 0.6614 | 0.5382 | 0.9692 | 0.5506 | M + iM | H + iVH | 1.1885 | 0.866 | Joint 3 | 3 |
| FRFS + VQNN | 0.4899 | 0.1922 | 0.4339 | 0.0754 | 0.4602 | M + iM | L + iVL | 0.2697 | 0.146 | Joint 13 | 14 |
| CFS + FRNN_FRS | 0.558 | 0.6257 | 0.5384 | 0.9709 | 0.548 | M + iM | H + iVH | 1.1885 | 0.866 | Joint 3 | 4 |
| CFS + FNN | 0.4597 | 0.0 | 0.4834 | 0.4996 | 0.4713 | M + iM | Worst + iM | 0.5 | 0.3265 | 10 | 10 |
| CFS + FRNN_O | 0.6106 | 0.9605 | 0.5418 | 1.0 | 0.5741 | M + iM | VH + iBest | 1.3594 | 1.0 | Joint 1 | 1 |
| CFS + VQNN | 0.491 | 0.1992 | 0.4446 | 0.1668 | 0.4666 | M + iM | L + iL | 0.3536 | 0.2118 | Joint 11 | 11 |
| DMTRS + FRNN_FRS | 0.5564 | 0.6155 | 0.5308 | 0.9057 | 0.5433 | M + iM | M + iVH | 1.0529 | 0.7598 | 5 | 5 |
| DMTRS + FNN | 0.491 | 0.1992 | 0.4446 | 0.1668 | 0.4666 | M + iM | L + iL | 0.3536 | 0.2118 | Joint 11 | 12 |
| DMTRS + FRNN_O | 0.491 | 0.1992 | 0.4327 | 0.0651 | 0.46 | M + iM | L + iVL | 0.2697 | 0.146 | Joint 13 | 15 |
| DMTRS + VQNN | 0.4678 | 0.0516 | 0.4251 | 0.0 | 0.4454 | M + iM | VL + iWorst | 0.0833 | 0.0 | 16 | 16 |

Table 7: Evaluation results of using precision and recall for *vehicle* classification

| Classifier | Average Precision | | Average Recall | | F-measure | FCN | FCN | Rep(Modulus) | | Ranking | Ranking(F) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | Absolute | Normalised | | |
| CS + FRNN_FRS | 0.6705 | 0.8379 | 0.669 | 0.7459 | 0.6697 | H + iH | H + iH | 1.0607 | 0.7522 | Joint 5 | 6 |
| CS + FNN | 0.6368 | 0.5893 | 0.653 | 0.628 | 0.6448 | H + iH | M + iH | 0.9035 | 0.6408 | 7 | 7 |
| CS + FRNN_O | 0.6835 | 0.9337 | 0.6923 | 0.9171 | 0.6878 | H + iH | VH + iVH | 1.2964 | 0.9194 | Joint 2 | 3 |
| CS + VQNN | 0.6925 | 1.0 | 0.7035 | 1.0 | 0.698 | H + iH | Best + iBest | 1.41 | 1.0 | 1 | 1 |
| FRFS + FRNN_FRS | 0.6143 | 0.424 | 0.6088 | 0.302 | 0.6115 | M + iM | M + iL | 0.5701 | 0.4043 | Joint 10 | 10 |
| FRFS + FNN | 0.6008 | 0.3241 | 0.6208 | 0.3904 | 0.6106 | M + iM | L + iM | 0.5701 | 0.4043 | Joint 10 | 11 |
| FRFS + FRNN_O | 0.6828 | 0.9282 | 0.6873 | 0.8803 | 0.685 | H + iH | VH + iVH | 1.2964 | 0.9194 | Joint 2 | 4 |
| FRFS + VQNN | 0.621 | 0.4733 | 0.633 | 0.4807 | 0.6269 | M + iH | M + iM | 0.7071 | 0.5015 | Joint 8 | 9 |
| CFS + FRNN_FRS | 0.6068 | 0.3683 | 0.5913 | 0.1731 | 0.5989 | M + iM | L + iL | 0.3556 | 0.2522 | Joint 13 | 14 |
| CFS + FNN | 0.5725 | 0.116 | 0.5923 | 0.1805 | 0.5822 | M + iM | VL + iL | 0.2697 | 0.1913 | 15 | 15 |
| CFS + FRNN_O | 0.6345 | 0.5727 | 0.6403 | 0.534 | 0.6374 | H + iH | M + iM | 0.7071 | 0.5015 | Joint 8 | 8 |
| CFS + VQNN | 0.5568 | 0.0 | 0.5678 | 0.0 | 0.5622 | M + iM | Worst + iWorst | 0.0 | 0.0 | 16 | 16 |
| DMTRS + FRNN_FRS | 0.6128 | 0.4125 | 0.6068 | 0.2873 | 0.6097 | M + iM | M + iL | 0.5701 | 0.4043 | Joint 10 | 12 |
| DMTRS + FNN | 0.6898 | 0.9797 | 0.6983 | 0.9613 | 0.694 | H + iH | VH + iVH | 1.2964 | 0.9194 | Joint 2 | 2 |
| DMTRS + FRNN_O | 0.6693 | 0.8287 | 0.6748 | 0.7882 | 0.672 | H + iH | H + iH | 1.0607 | 0.7523 | Joint 5 | 5 |
| DMTRS + VQNN | 0.6035 | 0.3444 | 0.6148 | 0.3462 | 0.6091 | M + iM | L + iL | 0.3536 | 0.2508 | Joint 13 | 13 |

relatively *High* value. Therefore, it is difficult to tell the difference between these two with regard to the precision and recall measures. They should intuitively be assigned the same ranking. This matches well the resulting ranking score obtained from the proposed FCN approach.

Considering the classification of the DDSM dataset as another example, similar evaluation results are obtained overall, when using the FCN-based measure and the F-measure. For instance, they both suggest that the

Table 8: Evaluation results of using precision and recall for *sonar* classification

| Classifier | Average Precision | | Average Recall | | F-measure | FCN | FCN | Rep(Modulus) | | Ranking | Ranking(F) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | Absolute | Normalised | | |
| CS + FRNN_FRS | 0.791 | 0.5054 | 0.7845 | 0.5263 | 0.7877 | H + iH | M + iM | 0.7071 | 0.5015 | Joint 8 | 8 |
| CS + FNN | 0.8255 | 0.7527 | 0.8115 | 0.7158 | 0.8184 | H + iH | H + iH | 1.0607 | 0.7523 | Joint 4 | 6 |
| CS + FRNN_O | 0.8595 | 0.9964 | 0.8455 | 0.9544 | 0.8524 | H + iH | VH + iVH | 1.2964 | 0.9194 | Joint 2 | 2 |
| CS + VQNN | 0.809 | 0.6344 | 0.8045 | 0.6667 | 0.8067 | H + iH | H + iH | 1.0607 | 0.7523 | Joint 4 | 7 |
| FRFS + FRNN_FRS | 0.86 | 1.0 | 0.852 | 1.0 | 0.856 | H + iH | Best + iBest | 1.41 | 1.0 | 1 | 1 |
| FRFS + FNN | 0.7665 | 0.3297 | 0.7525 | 0.3018 | 0.7594 | H + iH | L + iL | 0.3536 | 0.2508 | Joint 13 | 13 |
| FRFS + FRNN_O | 0.8425 | 0.8746 | 0.8255 | 0.814 | 0.8339 | H + iH | H + iH | 1.0607 | 0.7523 | Joint 4 | 4 |
| FRFS + VQNN | 0.7895 | 0.4946 | 0.7715 | 0.4351 | 0.7804 | H + iH | M + iM | 0.7071 | 0.5015 | Joint 8 | 9 |
| CFS + FRNN_FRS | 0.778 | 0.4122 | 0.778 | 0.4807 | 0.778 | H + iH | M + iM | 0.7071 | 0.5015 | Joint 8 | 10 |
| CFS + FNN | 0.8245 | 0.7455 | 0.8185 | 0.7649 | 0.8215 | H + iH | H + iH | 1.0607 | 0.7523 | Joint 4 | 5 |
| CFS + FRNN_O | 0.85 | 0.9283 | 0.8425 | 0.9333 | 0.8462 | H + iH | VH + iVH | 1.2964 | 0.9194 | Joint 2 | 3 |
| CFS + VQNN | 0.768 | 0.3405 | 0.769 | 0.4175 | 0.7685 | H + iH | L + iM | 0.5701 | 0.4043 | 12 | 12 |
| DMTRS + FRNN_FRS | 0.749 | 0.204 | 0.739 | 0.207 | 0.744 | H + iH | L + iL | 0.3536 | 0.2508 | Joint 13 | 14 |
| DMTRS + FNN | 0.7205 | 0.0 | 0.7095 | 0.0 | 0.715 | H + iH | Worst + iWorst | 0.0 | 0.0 | 16 | 16 |
| DMTRS + FRNN_O | 0.736 | 0.1111 | 0.724 | 0.1018 | 0.73 | H + iH | VL + iVL | 0.1179 | 0.0836 | 15 | 15 |
| DMTRS + VQNN | 0.791 | 0.5054 | 0.765 | 0.3895 | 0.7778 | H + iH | M + iM | 0.7071 | 0.5015 | Joint 8 | 11 |

Table 9: Evaluation results of using precision and recall for MIAS classification

| Classifier | Average Precision | | Average Recall | | F-measure | FCN | FCN | Rep(Modulus) | | Ranking | Ranking(F) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | Absolute | Normalised | | |
| CS + FRNN_FRS | 0.654 | 0.4839 | 0.672 | 0.7922 | 0.6628 | H + iH | M + iH | 0.9035 | 0.7167 | 5 | 5 |
| CS + FNN | 0.539 | 0.072 | 0.514 | 0.1082 | 0.5262 | M + iM | VL + iVL | 0.1179 | 0.0935 | 15 | 15 |
| CS + FRNN_O | 0.682 | 0.5842 | 0.63 | 0.6104 | 0.655 | H + iH | M + iM | 0.7071 | 0.5609 | Joint 7 | 7 |
| CS + VQNN | 0.798 | 1.0 | 0.685 | 0.8485 | 0.7372 | H + iH | Best + iH | 1.2607 | 1.0 | Joint 1 | 2 |
| FRFS + FRNN_FRS | 0.64 | 0.434 | 0.606 | 0.5065 | 0.6225 | H + iM | M + iM | 0.7071 | 0.5609 | Joint 7 | 11 |
| FRFS + FNN | 0.519 | 0.0 | 0.489 | 0.0 | 0.5036 | M + iM | Worst + iWorst | 0.0 | 0.0 | 16 | 16 |
| FRFS + FRNN_O | 0.618 | 0.3548 | 0.637 | 0.6407 | 0.6274 | M + iH | L + iH | 0.8029 | 0.6369 | 6 | 9 |
| FRFS + VQNN | 0.688 | 0.6057 | 0.632 | 0.619 | 0.6588 | H + iH | M + iM | 0.7071 | 0.5609 | Joint 7 | 6 |
| CFS + FRNN_FRS | 0.728 | 0.7491 | 0.699 | 0.9091 | 0.7132 | H + iH | H + iVH | 1.1885 | 0.9427 | 3 | 3 |
| CFS + FNN | 0.627 | 0.3871 | 0.56 | 0.3074 | 0.5916 | M + iM | M + iL | 0.5701 | 0.4522 | Joint 12 | 13 |
| CFS + FRNN_O | 0.73 | 0.7563 | 0.654 | 0.7143 | 0.6899 | H + iH | H + iH | 1.0607 | 0.8414 | 4 | 4 |
| CFS + VQNN | 0.762 | 0.8710 | 0.72 | 1.0 | 0.7404 | H + iH | H + iBest | 1.2607 | 1.0 | Joint 1 | 1 |
| DMTRS + FRNN_FRS | 0.5573 | 0.1371 | 0.555 | 0.2857 | 0.5561 | M + iM | L + iL | 0.3536 | 0.2805 | 14 | 14 |
| DMTRS + FNN | 0.647 | 0.4588 | 0.6225 | 0.5779 | 0.6345 | H + iM | M + iM | 0.7071 | 0.5609 | Joint 7 | 8 |
| DMTRS + FRNN_O | 0.6528 | 0.4794 | 0.6023 | 0.4903 | 0.6265 | H + iM | M + iM | 0.7071 | 0.5609 | Joint 7 | 10 |
| DMTRS + VQNN | 0.6545 | 0.4857 | 0.57 | 0.3506 | 0.6093 | H + iM | M + iL | 0.5701 | 0.4522 | Joint 12 | 12 |

Table 10: Evaluation results of using precision and recall for MIAS classification

| Classifier | Average Precision | | Average Recall | | F-measure | FCN | FCN | Rep(Modulus) | | Ranking | Ranking(F) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Absolute | Normalised | Absolute | Normalised | | Absolute | Normalised | Absolute | Normalised | | |
| CS + FRNN_FRS | 0.4608 | 0.3493 | 0.4555 | 0.6888 | 0.4581 | M+ iM | L + iH | 0.8029 | 0.5407 | Joint 5 | 8 |
| CS + FNN | 0.431 | 0.1863 | 0.3876 | 0.0686 | 0.4082 | M+ iM | L + iVL | 0.2697 | 0.1401 | 14 | 15 |
| CS + FRNN_O | 0.5015 | 0.5726 | 0.4353 | 0.5034 | 0.466 | M+ iM | M + iM | 0.7071 | 0.4687 | Joint 7 | 5 |
| CS + VQNN | 0.5473 | 0.8233 | 0.4675 | 0.7986 | 0.5042 | M+ iM | H + iH | 1.0607 | 0.7344 | Joint 2 | 2 |
| FRFS + FRNN_FRS | 0.4865 | 0.4904 | 0.4778 | 0.8924 | 0.4821 | M+ iM | M + iVH | 1.0529 | 0.7286 | 4 | 4 |
| FRFS + FNN | 0.4728 | 0.4151 | 0.402 | 0.1991 | 0.4345 | M+ iM | M + iL | 0.5701 | 0.3658 | Joint 10 | 12 |
| FRFS + FRNN_O | 0.4358 | 0.2123 | 0.4133 | 0.3021 | 0.4242 | M+ iM | L + iL | 0.3536 | 0.2031 | 13 | 13 |
| FRFS + VQNN | 0.5795 | 1.0 | 0.4895 | 1.0 | 0.5307 | M+ iM | Best + iBest | 1.4142 | 1.0 | 1 | 1 |
| CFS + FRNN_FRS | 0.4615 | 0.3534 | 0.4683 | 0.8055 | 0.4649 | M+ iM | L + iH | 0.8029 | 0.5407 | Joint 5 | 6 |
| CFS + FNN | 0.4828 | 0.4699 | 0.3955 | 0.1396 | 0.4348 | M+ iM | M + iL | 0.5701 | 0.3658 | Joint 10 | 11 |
| CFS + FRNN_O | 0.4785 | 0.4466 | 0.443 | 0.5744 | 0.4601 | M+ iM | M + iM | 0.7071 | 0.4687 | Joint 7 | 7 |
| CFS + VQNN | 0.54 | 0.7836 | 0.4713 | 0.833 | 0.5033 | M+ iM | H + iH | 1.0607 | 0.7344 | Joint 2 | 3 |
| DMTRS + FRNN_FRS | 0.387 | 0.0 | 0.3875 | 0.0664 | 0.3922 | M+ iM | Worst + iVL | 0.0833 | 0.0 | 16 | 16 |
| DMTRS + FNN | 0.443 | 0.2521 | 0.3803 | 0.0 | 0.4092 | M+ iM | L + iWorst | 0.25 | 0.1253 | 15 | 14 |
| DMTRS + FRNN_O | 0.481 | 0.4603 | 0.422 | 0.3822 | 0.4496 | M+ iM | M + iM | 0.7071 | 0.4687 | Joint 7 | 10 |
| DMTRS + VQNN | 0.5043 | 0.5877 | 0.4183 | 0.3478 | 0.4572 | M+ iM | M + iL | 0.5701 | 0.3658 | Joint 10 | 9 |

"FRFS + VQNN" classifier is the best in dealing with this dataset (with respect to combined precision and recall criteria), whereas the "DMTRS + FRNN_FRS" performs the worst. However, the use of F-measure ranks "CS + VQNN" and "CFS + VQNN" differently, preferring the former to the latter, just because they receive F-measure values of 0.5042 and 0.5033, respectively. Again, it has a natural appeal to assign the same ranking score to these two classifiers, rather than to distinguish them due to such a tiny numerical difference. Hence, the results obtained by the FCN approach seems to be more reasonable.

*4.4.3. Overall evaluation*

The aforementioned four performance measures, namely CCP, reduction capability, precision and recall, are jointly taken into account in this subsection. For each measure, the relative performance of each classifier is represented by linguistic terms. In order to obtain the ranking score of the overall performance, these linguistic terms are aggregated by applying Eq. (13). The overall ranking of different classifiers are shown in Table 11 - Table 15, with

Table 11: Overall evaluation for *glass* classification

| Classifier | Precision | Recall | CCP | No. of features | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|
| | | | | | Absolute | Normalised | |
| CS + FRNN_FRS | M | H | H | M | 1.1792 | 0.6122 | 6 |
| CS + FNN | VL | M | M | M | 0.887 | 0.3743 | 11 |
| CS + FRNN_O | Best | VH | H | M | 1.514 | 0.8849 | 2 |
| CS + VQNN | VL | L | VL | M | 0.6585 | 0.1883 | 14 |
| FRFS + FRNN_FRS | M | H | M | Worst | 0.9761 | 0.4469 | 10 |
| FRFS + FNN | M | H | M | Worst | 0.87 | 0.3605 | 12 |
| FRFS + FRNN_O | H | VH | H | Worst | 1.2616 | 0.6793 | 5 |
| FRFS + VQNN | L | VL | VL | Worst | 0.4273 | 0.0 | 16 |
| CFS + FRNN_FRS | H | VH | H | L | 1.304 | 0.7139 | 4 |
| CFS + FNN | Worst | M | M | L | 0.7873 | 0.2931 | 13 |
| CFS + FRNN_O | VH | Best | M | L | 1.4019 | 0.7936 | 3 |
| CFS + VQNN | L | L | VL | L | 0.6058 | 0.1453 | 15 |
| DMTRS + FRNN_FRS | M | VH | Best | Best | 1.6554 | 1.0 | 1 |
| DMTRS + FNN | L | L | Worst | Best | 1.0692 | 0.5227 | 9 |
| DMTRS + FRNN_O | L | VL | L | Best | 1.0845 | 0.5351 | 8 |
| DMTRS + VQNN | VL | Worst | M | Best | 1.0954 | 0.544 | 7 |

respect to the given five datasets.

It is interesting to note that this experimentation proposes five different *Best* classifiers for the five datasets. This implies that different classifiers may only be suitable for a certain type of problem. For example, in the MIAS dataset, the "CS + VQNN" classifier reaches the *Best* classification precision, and *High* classification recall, CCP and reduction capability. Although the DM-TRS feature selector achieves the *Best* reduction capability, its combination with any other learning method only obtains *Medium* or *Low* classification precision, recall and CCP. Also, the "CFS" feature selector performs *Worst* in feature reduction. As a result, "CS + VQNN" is rated as the best amongst these classifier overall. For the DDSM dataset, owing to the individually *Best* performance in precision, recall and CCP measures, it

Table 12: Overall evaluation for *vehicle* classification

| Classifier | Precision | Recall | CCP | No. of features | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|
| | | | | | Absolute | Normalised | |
| CS + FRNN_FRS | H | H | H | Worst | 1.1579 | 0.4551 | 7 |
| CS + FNN | M | H | M | Worst | 0.9761 | 0.2809 | 11 |
| CS + FRNN_O | VH | VH | VH | Worst | 1.446 | 0.7311 | 5 |
| CS + VQNN | Best | Best | Best | Worst | 1.6385 | 0.9156 | 2 |
| FRFS + FRNN_FRS | M | L | L | H | 0.961 | 0.2665 | Joint 12 |
| FRFS + FNN | L | M | L | H | 0.961 | 0.2665 | Joint 12 |
| FRFS + FRNN_O | VH | VH | H | H | 1.5095 | 0.792 | 3 |
| FRFS + VQNN | M | M | M | H | 1.0936 | 0.3935 | Joint 9 |
| CFS + FRNN_FRS | L | L | L | H | 0.887 | 0.1956 | 14 |
| CFS + FNN | VL | L | L | H | 0.8429 | 0.1533 | 15 |
| CFS + FRNN_O | M | M | M | H | 1.0936 | 0.3935 | Joint 9 |
| CFS + VQNN | Worst | Worst | Worst | H | 0.6829 | 0.0 | 16 |
| DMTRS + FRNN_FRS | M | L | L | Best | 1.1786 | 0.4749 | 6 |
| DMTRS + FNN | VH | VH | VH | Best | 1.7266 | 1.0 | 1 |
| DMTRS + FRNN_O | H | H | H | Best | 1.4951 | 0.7782 | 4 |
| DMTRS + VQNN | L | L | L | Best | 1.1198 | 0.4186 | 8 |

Table 13: Overall evaluation for *sonar* classification

| Classifier | Precision | Recall | CCP | No. of features | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|
| | | | | | Absolute | Normalised | |
| CS + FRNN_FRS | M | M | M | L | 0.9291 | 0.1722 | 13 |
| CS + FNN | H | H | H | L | 1.2038 | 0.4622 | Joint 6 |
| CS + FRNN_O | VH | VH | VH | L | 1.4836 | 0.7575 | 2 |
| CS + VQNN | H | H | H | L | 1.2038 | 0.4622 | Joint 6 |
| FRFS + FRNN_FRS | Best | Best | Best | M | 1.7134 | 1.0 | 1 |
| FRFS + FNN | L | L | L | M | 0.7659 | 0.0 | 16 |
| FRFS + FRNN_O | H | H | H | M | 1.2587 | 0.5201 | 5 |
| FRFS + VQNN | M | M | M | M | 1.0 | 0.2471 | 11 |
| CFS + FRNN_FRS | M | M | M | Worst | 0.87 | 0.1099 | 14 |
| CFS + FNN | H | H | H | Worst | 1.1579 | 0.4137 | 8 |
| CFS + FRNN_O | VH | VH | VH | Worst | 1.446 | 0.7178 | 3 |
| CFS + VQNN | L | M | M | Worst | 0.7873 | 0.0226 | 15 |
| DMTRS + FRNN_FRS | L | L | L | Best | 1.1198 | 0.3735 | 9 |
| DMTRS + FNN | Worst | Worst | Worst | Best | 0.958 | 0.2027 | 12 |
| DMTRS + FRNN_O | VL | VL | VL | Best | 1.0096 | 0.2572 | 10 |
| DMTRS + VQNN | M | M | M | Best | 1.2881 | 0.5511 | 4 |

Table 14: Overall evaluation for MIAS classification

| Classifier | Precision | Recall | CCP | No. of features | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|
| | | | | | Absolute | Normalised | |
| CS + FRNN_FRS | M | H | H | H | 1.3936 | 0.7091 | 5 |
| CS + FNN | VL | VL | L | H | 0.8211 | 0.069 | 15 |
| CS + FRNN_O | M | M | H | H | 1.2777 | 0.5796 | 9 |
| CS + VQNN | Best | H | H | H | 1.6541 | 1.0 | 1 |
| FRFS + FRNN_FRS | M | M | M | VH | 1.2761 | 0.5779 | Joint 10 |
| FRFS + FNN | Worst | Worst | Worst | VH | 0.9167 | 0.1765 | 14 |
| FRFS + FRNN_O | L | H | H | VH | 1.4375 | 0.7581 | 4 |
| FRFS + VQNN | M | M | M | VH | 1.2761 | 0.5778 | Joint 10 |
| CFS + FRNN_FRS | H | VH | VH | Worst | 1.5018 | 0.8299 | 3 |
| CFS + FNN | M | L | M | Worst | 0.7587 | 0.0 | 16 |
| CFS + FRNN_O | H | H | H | Worst | 1.299 | 0.6034 | 8 |
| CFS + VQNN | H | Best | Best | Worst | 1.6109 | 0.9518 | 2 |
| DMTRS + FRNN_FRS | L | L | L | Best | 1.1375 | 0.4231 | 13 |
| DMTRS + FNN | M | M | M | Best | 1.3507 | 0.6612 | Joint 6 |
| DMTRS + FRNN_O | M | M | M | Best | 1.3507 | 0.6612 | Joint 6 |
| DMTRS + VQNN | M | L | L | Best | 1.2132 | 0.5076 | 12 |

Table 15: Overall evaluation for DDSM classification

| Classifier | Precision | Recall | CCP | No. of features | Rep(Modulus) | | Ranking |
|---|---|---|---|---|---|---|---|
| | | | | | Absolute | Normalised | |
| CS + FRNN_FRS | L | H | L | M | 0.9878 | 0.2808 | 10 |
| CS + FNN | L | VL | M | M | 0.7675 | 0.076 | 13 |
| CS + FRNN_O | M | M | M | M | 1.0 | 0.2921 | 9 |
| CS + VQNN | H | H | H | M | 1.3936 | 0.658 | 2 |
| FRFS + FRNN_FRS | M | VH | M | L | 1.2068 | 0.4843 | 6 |
| FRFS + FNN | M | L | L | L | 0.6857 | 0.0 | Joint 15 |
| FRFS + FRNN_O | L | L | M | L | 0.6857 | 0.0 | Joint 15 |
| FRFS + VQNN | Best | Best | Best | L | 1.7616 | 1.0 | 1 |
| CFS + FRNN_FRS | L | H | L | Worst | 0.8513 | 0.1539 | 12 |
| CFS + FNN | M | L | M | Worst | 0.7587 | 0.0679 | 14 |
| CFS + FRNN_O | M | M | M | Worst | 0.866 | 0.1676 | 11 |
| CFS + VQNN | H | H | H | Worst | 1.299 | 0.57 | 4 |
| DMTRS + FRNN_FRS | Worst | VL | Worst | Best | 1.0103 | 0.3017 | 8 |
| DMTRS + FNN | L | Worst | L | Best | 1.0951 | 0.3805 | 7 |
| DMTRS + FRNN_O | M | M | M | Best | 1.3507 | 0.6181 | 3 |
| DMTRS + VQNN | M | L | M | Best | 1.2839 | 0.556 | 5 |

is not surprising that "FRFS + VQNN" reaches the first place in terms of overall performance.

Another point to note is that the DM-TRS feature selector produces the greatest data reduction on all of the experimental datasets. However, the impact of feature selection may have different effects upon different classifiers. Combining DM-TRS with each of the five classifiers does not necessarily lead to the best overall performance all the time. It only works extremely well in conjunction with FRNN_FRS and FNN classifiers on the *glass* and *vehicle* dataset, respectively. Thus, one of the most important conclusions that can be drawn from these results is that it is difficult to choose a clear "winner". There is no such a thing as the best combination of feature selector and classifier with regard to different performance criteria and different datasets. This gives rise to the need for more careful selection of what feature selectors to combine with what classifiers in general when facing a new problem. The work developed herein offers such a helpful means to linguistically evaluate such combinations.

## 5. Conclusion

This paper has proposed a novel notion of fuzzy complex numbers (FCNs) and introduced the calculus for the proposed FCNs. This is achieved by substantially extending the seminal ideas proposed in [21]. In particular, the algebraic properties of the FCNs, including closure, associativity, commutativity and distributivity are discussed and added on to this work. Further, the closure, associativity and commutativity properties are utilised to form the basis upon which to establish a new hierarchical approach of aggregating different components of an FCN. An arbitrary number of FCN components can now be aggregated in a random order.

This notion of FCNs is capable of representing and aggregating a variety of inexact knowledge and data in a unified manner. This ability is demonstrated by exploiting the framework to support the performance evaluation of classifiers. The effectiveness of the approach is compared to the traditional F-measure-based approach. Experimental results demonstrates that the FCN-based performance evaluation is intuitively reliable and consistent. Importantly, unlike semantics-destroying approaches (e.g. the F-measure) for classifier performance assessment, the proposed work maintains the underlying semantics of different evaluation measures. This ensures that the resulting ranking and hence selection process of choosing (what combination of feature selector and) pattern classifier is interpretable and explainable to the user. This is essential in assisting the user to make informed decisions when given a challenging classification task.

Although the proposed approach is promising, much may be done through further research. One such work is to extend the FCN aggregation process by considering the relative significance of the real and imaginary parts of the FCNs when deriving the modulus (e.g. by introducing weights to these parts). This may lead to the development of a new OWA operator [54, 55]. Also, a more general mechanism can be built to automatically generate the corresponding fuzzy quantity spaces (including the fuzzy partitions) from the training data. This would help to avoid the need for prefixing just one common quantity space which the real and imaginary parts of FCNs may take values from. Finally, as the proposed notion of FCNs is mathematically generic, it would be very interesting to investigate how it might be applied to other application domains, such as fuzzy compositional modelling [56, 57] and student academic performance evaluation [22].

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## Appendix

### Algebraic properties of FCNs

Given the aforementioned arithmetic operators of FCNs, the algebraic properties of the proposed FCNs can be established. This appendix addresses these properties.

**Definition 6.** *An $n-ary$ function/operation $f$ from $\mathbb{R}^n$ to $\mathbb{R}$ is said increasing if and only if*

$$\begin{aligned}&(x_1 > y_1) \wedge (x_2 > y_2) \wedge \cdots \wedge (x_n > y_n) \rightarrow \\ &f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n).\end{aligned} \tag{.1}$$

*An $n-ary$ function/operation $f$ from $\mathbb{R}^n$ to $\mathbb{R}$ is said decreasing if and only if*

$$\begin{aligned}&(x_1 > y_1) \wedge (x_2 > y_2) \wedge \cdots \wedge (x_n > y_n) \rightarrow \\ &f(x_1, x_2, \ldots, x_n) < f(y_1, y_2, \ldots, y_n).\end{aligned} \tag{.2}$$

**Theorem 1.** *([58]) Let $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ be continuous fuzzy numbers with membership functions mapping from $\mathbb{R}$ to $[0, 1]$. Let $f$ be a continuous and monotonic function (increasing or decreasing), then $f(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ is a continuous fuzzy number.*

**Theorem 2.** *If $\tilde{z}_1 = \tilde{a} + i\tilde{b}$ and $\tilde{z}_2 = \tilde{c} + i\tilde{d}$ are FCNs, then so are $\tilde{z}_1 + \tilde{z}_2$, $\tilde{z}_1 - \tilde{z}_2$, $\tilde{z}_1\tilde{z}_2$.*

**Proof:** (a) According to Eq. (3), $Re(\tilde{z}_1 + \tilde{z}_2) = f(\tilde{a}, \tilde{c}) = \tilde{a} + \tilde{c}$ and $Im(\tilde{z}_1 + \tilde{z}_2) = f(\tilde{b}, \tilde{d}) = \tilde{b} + \tilde{d}$. Because the addition function is always increasing, Theorem 1 can always be applied to it. Hence, the addition of fuzzy numbers gives a fuzzy number and both the real and imagery parts of $\tilde{z}_1 + \tilde{z}_2$ are fuzzy numbers. Therefore, $\tilde{z}_1 + \tilde{z}_2$ is an FCN.

(b) It has been proven in [58] that the subtraction between fuzzy numbers also gives a fuzzy number. According to Eq. (6), $\tilde{z}_1 - \tilde{z}_2$ is an FCN.

(c) It has been proven in [58] that the multiplication of fuzzy numbers gives a fuzzy number. In addition, a fuzzy number adding or subtracting another always yields a new fuzzy number. In Eq. (7), $Re(\tilde{z}_1\tilde{z}_2)$ and $Im(\tilde{z}_1\tilde{z}_2)$ are both fuzzy numbers. Hence, $\tilde{z}_1\tilde{z}_2$ is an FCN. ■

**Theorem 3.** *If $\tilde{z} = \tilde{a} + i\tilde{b}$ is an FCN, then $|\tilde{z}|$ is a fuzzy number.*

**Proof:** In Eq. (12), let $f(\tilde{a}, \tilde{b}) = \sqrt{\tilde{a}^2 + \tilde{b}^2}$: when $\tilde{a} > 0$ and $\tilde{b} > 0$, $f$ is increasing and continuous, when $\tilde{a} < 0$ and $\tilde{b} < 0$, $f$ is decreasing and continuous, then Theorem 1 can be directly applied to both cases. When $\tilde{a} > 0$ and $\tilde{b} < 0$, $f$ can be rewritten as $f(\tilde{a}, \tilde{b}) = \sqrt{\tilde{a}^2 + (-\tilde{b})(-\tilde{b})}$, Theorem 1 also applies in this case. Similarly, when $\tilde{a} < 0$ and $\tilde{b} > 0$, the same conclusion can be derived. Therefore, $|\tilde{z}|$ is a newly derived fuzzy number. ■

To investigate the basic arithmetic properties of the proposed FCNs, let $\tilde{z}_1 = \tilde{a} + i\tilde{b}$, $\tilde{z}_2 = \tilde{c} + i\tilde{d}$ and $\tilde{z}_3 = \tilde{e} + i\tilde{f}$ be three FCNs, where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$ and $\tilde{f}$ are fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$, $\mu_{\tilde{b}}(x)$, $\mu_{\tilde{c}}(x)$, $\mu_{\tilde{d}}(x)$, $\mu_{\tilde{e}}(x)$ and $\mu_{\tilde{f}}(x)$, respectively. From this, the following properties hold:

- Associativity

**Theorem 4.** *Associativity, $\tilde{z}_1 * (\tilde{z}_2 * \tilde{z}_3) = (\tilde{z}_1 * \tilde{z}_2) * \tilde{z}_3$, holds if $* = +$.*

**Proof:** For $* = +$, since

$$\begin{aligned} \tilde{z}_1 + (\tilde{z}_2 + \tilde{z}_3) &= (\tilde{a} + i\tilde{b}) + ((\tilde{c} + \tilde{e}) + i(\tilde{d} + \tilde{f})) &= (\tilde{a} + \tilde{c} + \tilde{e}) + i(\tilde{b} + \tilde{d} + \tilde{f}), \\ (\tilde{z}_1 + \tilde{z}_2) + \tilde{z}_3 &= ((\tilde{a} + \tilde{c}) + i(\tilde{b} + \tilde{d})) + (\tilde{e} + i\tilde{f}) &= (\tilde{a} + \tilde{c} + \tilde{e}) + i(\tilde{b} + \tilde{d} + \tilde{f}). \end{aligned}$$

Hence, $Re(\tilde{z}_1 + (\tilde{z}_2 + \tilde{z}_3)) = Re((\tilde{z}_1 + \tilde{z}_2) + \tilde{z}_3)$ and $Im(\tilde{z}_1 + (\tilde{z}_2 + \tilde{z}_3)) = Im((\tilde{z}_1 + \tilde{z}_2) + \tilde{z}_3)$. Thus, if $* = +$, FCNs is associative. ■

However, for $* = \times$,

$$\begin{aligned} \tilde{z}_1 \times (\tilde{z}_2 \times \tilde{z}_3) &= (\tilde{a} + i\tilde{b}) \times ((\tilde{c}\tilde{e} - \tilde{d}\tilde{f}) + i(\tilde{d}\tilde{e} + \tilde{c}\tilde{f})) \\ &= (\tilde{a}(\tilde{c}\tilde{e} - \tilde{d}\tilde{f}) - \tilde{b}(\tilde{d}\tilde{e} + \tilde{c}\tilde{f})) + i(\tilde{b}(\tilde{c}\tilde{e} - \tilde{d}\tilde{f}) + \tilde{a}(\tilde{d}\tilde{e} + \tilde{c}\tilde{f})), \\ (\tilde{z}_1 \times \tilde{z}_2) \times \tilde{z}_3 &= ((\tilde{a}\tilde{c} - \tilde{b}\tilde{d}) + i(\tilde{b}\tilde{c} + \tilde{a}\tilde{d})) \times (\tilde{e} + i\tilde{f}) \\ &= ((\tilde{a}\tilde{c} - \tilde{b}\tilde{d})\tilde{e} - (\tilde{b}\tilde{c} + \tilde{a}\tilde{d})\tilde{f}) + i((\tilde{b}\tilde{c} + \tilde{a}\tilde{d})\tilde{e} + (\tilde{a}\tilde{c} - \tilde{b}\tilde{d})\tilde{f}). \end{aligned}$$

Unfortunately, the distributivity of $\times$ over $+$ for fuzzy numbers does not always hold (see later): Take $\tilde{b}(\tilde{d}\tilde{e} + \tilde{c}\tilde{f})$ for example, the distributivity is only valid if $\tilde{b}$ is either a positive or negative fuzzy number, and if $\tilde{d}\tilde{e}$ and $\tilde{c}\tilde{f}$ are both either a positive or negative fuzzy number [58].

- Commutativity

**Theorem 5.** *Commutativity,* $\tilde{z}_1 * \tilde{z}_2 = \tilde{z}_2 * \tilde{z}_1$*, holds for* $* \in \{+, \times\}$.

**Proof:** For $* = +$, given

$$\begin{aligned}\tilde{z}_1 + \tilde{z}_2 &= (\tilde{a} + \tilde{c}) + i(\tilde{b} + \tilde{d}),\\ \tilde{z}_2 + \tilde{z}_1 &= (\tilde{c} + \tilde{a}) + i(\tilde{d} + \tilde{b}).\end{aligned}$$

Since addition on fuzzy numbers is commutative, i.e. $\tilde{a}+\tilde{c} = \tilde{c}+\tilde{a}$ and $\tilde{b}+\tilde{d} = \tilde{d}+\tilde{b}$, $Re(\tilde{z}_1+\tilde{z}_2) = Re(\tilde{z}_2+\tilde{z}_1)$ and $Im(\tilde{z}_1+\tilde{z}_2) = Im(\tilde{z}_2+\tilde{z}_1)$. Thus, for $* = +$, FCNs is commutative.

For $* = \times$, given

$$\begin{aligned}\tilde{z}_1 \times \tilde{z}_2 &= (\tilde{a}\tilde{c} - \tilde{b}\tilde{d}) + i(\tilde{b}\tilde{c} + \tilde{a}\tilde{d}),\\ \tilde{z}_2 \times \tilde{z}_1 &= (\tilde{c}\tilde{a} - \tilde{d}\tilde{b}) + i(\tilde{c}\tilde{b} + \tilde{d}\tilde{a}).\end{aligned}$$

Since multiplication on fuzzy numbers is commutative, i.e. $\tilde{a}\tilde{c} = \tilde{c}\tilde{a}$, $\tilde{b}\tilde{d} = \tilde{d}\tilde{b}$, $\tilde{b}\tilde{c} = \tilde{c}\tilde{b}$ and $\tilde{a}\tilde{d} = \tilde{d}\tilde{a}$. Thus, $Re(\tilde{z}_1 \times \tilde{z}_2) = Re(\tilde{z}_2 \times \tilde{z}_1)$ and $Im(\tilde{z}_1 \times \tilde{z}_2) = Im(\tilde{z}_2 \times \tilde{z}_1)$. Therefore, $* = \times$ is commutative. ■

- Distributivity

**Theorem 6.** *Given* $\tilde{a} > 0$ *or* $\tilde{a} < 0$ *and* $\tilde{b} > 0$ *or* $\tilde{b} < 0$*, when* $\tilde{c}$ *and* $\tilde{e}$ *have the same sign (they are both either a positive or negative fuzzy number), also* $\tilde{d}$ *and* $\tilde{f}$ *have the same sign, then* $\tilde{z}_1 \times (\tilde{z}_2 + \tilde{z}_3) = \tilde{z}_1 \times \tilde{z}_2 + \tilde{z}_1 \times \tilde{z}_3$.

**Proof:**

$$
\begin{aligned}
\tilde{z}_1 \times (\tilde{z}_2 + \tilde{z}_3) &= (\tilde{a} + i\tilde{b}) \times ((\tilde{c} + \tilde{e}) + i(\tilde{d} + \tilde{f})) \\
&= (\tilde{a}(\tilde{c} + \tilde{e}) - \tilde{b}(\tilde{d} + \tilde{f})) + i(\tilde{b}(\tilde{c} + \tilde{e}) + \tilde{a}(\tilde{d} + \tilde{f})), \\
\tilde{z}_1 \times \tilde{z}_2 + \tilde{z}_1 \times \tilde{z}_3 &= ((\tilde{a}\tilde{c} - \tilde{b}\tilde{d}) + i(\tilde{b}\tilde{c} + \tilde{a}\tilde{d})) + ((\tilde{a}\tilde{e} - \tilde{b}\tilde{f}) + i(\tilde{b}\tilde{e} + \tilde{a}\tilde{f})) \\
&= (\tilde{a}\tilde{c} - \tilde{b}\tilde{d} + \tilde{a}\tilde{e} - \tilde{b}\tilde{f}) + i(\tilde{b}\tilde{c} + \tilde{b}\tilde{e} + \tilde{a}\tilde{d} + \tilde{a}\tilde{f}).
\end{aligned}
$$

Given $\tilde{a} > 0$ or $\tilde{a} < 0$ and $\tilde{b} > 0$ or $\tilde{b} < 0$, when $\tilde{c}$ and $\tilde{e}$ have the same sign, also $\tilde{d}$ and $\tilde{f}$ have the same sign, the distributivity of $\times$ over $+$ for fuzzy numbers can be applied:

$$\tilde{z}_1 \times (\tilde{z}_2 + \tilde{z}_3) = (\tilde{a}\tilde{c} + \tilde{a}\tilde{e} - (\tilde{b}\tilde{d} + \tilde{b}\tilde{f})) + i(\tilde{b}\tilde{c} + \tilde{b}\tilde{e} + \tilde{a}\tilde{d} + \tilde{a}\tilde{f}).$$

Note that owning to $-\tilde{b}(\tilde{d} + \tilde{f}) = -\tilde{b}\tilde{d} - \tilde{b}\tilde{f}$, it can be derived that $Re(\tilde{z}_1 \times (\tilde{z}_2 + \tilde{z}_3)) = Re(\tilde{z}_1 \times \tilde{z}_2 + \tilde{z}_1 \times \tilde{z}_3)$ and $Im(\tilde{z}_1 \times (\tilde{z}_2 + \tilde{z}_3)) = Im(\tilde{z}_1 \times \tilde{z}_2 + \tilde{z}_1 \times \tilde{z}_3)$. Hence, $\tilde{z}_1 \times (\tilde{z}_2 + \tilde{z}_3) = \tilde{z}_1 \times \tilde{z}_2 + \tilde{z}_1 \times \tilde{z}_3$. ■

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