The Series Product and Its Application to Quantum Feedforward and Feedback Networks

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***Abstract*—The purpose of this paper is to present simple and general algebraic methods for describing series connections in quantum networks. These methods build on and generalize existing methods for series (or cascade) connections by allowing for more general interfaces, and by introducing an efficient algebraic tool, the series product. We also introduce another product, which we call the concatenation product, that is useful for assembling and representing systems without necessarily having connections. We show how the concatenation and series products can be used to describe feedforward and feedback networks. A selection of examples from the quantum control literature are analyzed to illustrate the utility of our network modeling methodology.**

Keywords: Quantum control, quantum networks, series, cascade, feedforward, feedback, quantum noise.

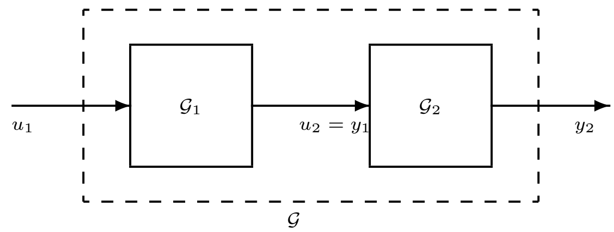

Fig. 1. Series connection of two (classical, or non-quantum) linear systems, denoted $\mathcal{G} = \mathcal{G}_2 \triangleleft \mathcal{G}_1$.

I. Introduction

Engineers routinely use a wide range of methods and tools to help them analyze and design control systems. For instance, control engineers often use block diagrams to represent feedforward and feedback systems, Figure 1. Among the methods that have been developed to assist engineers are those concerning the connection of components or subsystems to form a network. One of the most basic connections is the series connection, where the output of one component is fed into the input of another, as in Figure 1. When the components are (classical, or non-quantum) linear systems, the connected system can be described in the frequency domain by a transfer function $\mathcal{G}(s) = \mathcal{G}_2(s)\mathcal{G}_1(s)$ which is the product of the transfer functions of the components. The description can also be expressed in the time domain in terms of the state space parameters $\mathcal{G} = (A, B, C, D)$ (as we briefly review in section II). The series connection has an algebraic character, and can be regarded as a product, $\mathcal{G} = \mathcal{G}_2 \triangleleft \mathcal{G}_1$. Because of new imperatives concerning quantum network analysis and design, in particular, quantum feedback control, [24], [25], [18], [23], [26], [4], [17], [12] the purpose of this paper is to present simple and general algebraic methods for describing series connections in quantum networks.

The types of quantum networks we consider include those arising in quantum optics, such as the optical network shown in Figure 2. This network consists of a pair of optical cavities (discussed in subsection III-B) connected in series by a light beam which serves as an optical interconnect or quantum "wire". In this paper (section V) we show how series connections of quantum components such as this may be described as a *series product* $\mathbf{G} = \mathbf{G}_2 \triangleleft \mathbf{G}_1$. This product is defined in terms of system parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$, where H specifies the internal energy of the system, and $\mathcal{I} = (\mathbf{S}, \mathbf{L})$ specifies the interface of the system to external field channels (as explained in subsection III-D and section IV).

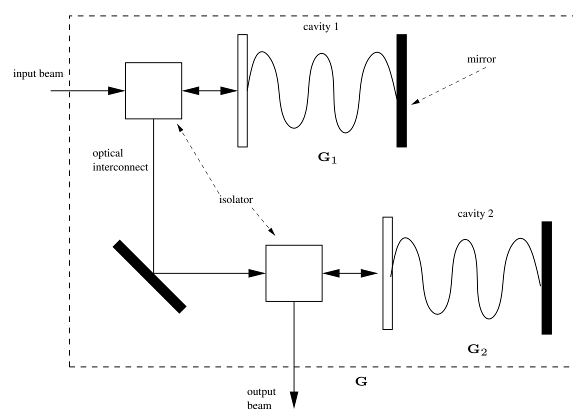

Fig. 2. Series connection of two optical cavities via an optical interconnect (light beam) or quantum "wire", denoted $\mathbf{G} = \mathbf{G}_2 \triangleleft \mathbf{G}_1$. Each cavity consists of a pair of mirrors, one of which is perfectly reflecting (shown solid) while the other is partially transmitting (shown unfilled). The partially transmitting mirror enables the light mode inside the cavity to interact with an external light field, such as a laser beam. The external field is separated into input and output components by a Faraday isolator. The optical interconnect is formed when light from the output of one cavity is directed into the input of the other, here using an additional mirror.

Series (also called cascade) connections of quantum optical components were first considered in the papers [6], [3], and certain linear feedback networks were considered in [26]. Our results extend the series connection results in these works by including more general interfaces, and by introducing an efficient algebraic tool, the series product. We also introduce

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another product, which we call the *concatenation product* $\mathbf{G} = \mathbf{G}_1 \boxplus \mathbf{G}_2$, that is useful for assembling and representing systems without necessarily having connections. Both products may be used to describe a wide range of open quantum physical systems (including those with physical variables that evolve nonlinearly) and networks of such systems (with boson field interconnects such as optical beams or phonon vibrations in materials). We believe our modeling framework is of fundamental system-theoretic interest. The need for general and efficient methods for describing networks of quantum components has been recognized to some extent and has begun to emerge in the quantum optics and quantum information and computing literature, e.g. [27], [6], [3], [8, Chapter 12], [20, Chapter 4], [26], [4]. It is expected that an effective quantum network theory will assist the design of quantum technologies, just as electrical network theory and block diagram manipulations help engineers design filters, control systems, and many other classical electrical systems.

Series connections provide the foundation for some important developments in quantum feedback control, e.g. [24], [25], [23], [26], [17], [12], [13]. To illustrate the power and utility of our quantum network modeling methodology, we analyze several examples from this literature. The series and concatenation products allow us to express these quantum feedback control and quantum filtering examples in a simple, transparent way (there are some subtle technical issues in some of the examples for which we provide explanation and references). We hope this will help open up some of the quantum feedback control literature to control engineers, which at present is largely unknown outside the physics community. A number of articles and books are available to help readers with the background material on which the present paper is based. The papers [26] and [22] provide excellent introductions to aspects of the quantum models we use. The paper [2] is a tutorial article written to assist control theorists and engineers by providing introductory discussions of quantum mechanics, open quantum stochastic models, and quantum filtering. The book [8] is an invaluable resource for quantum noise models and quantum optics, while the book [21] provides a detailed mathematical treatment of the Hudson-Parthasarathy theory of the quantum stochastic calculus. The book [19] is a standard textbook on quantum mechanics.

We begin in section II by discussing an analog of our results in the context of classical linear systems theory, elaborating further on the discussion at the beginning of this section. In section III we provide a review of some example quantum components (including the cavity mentioned above) and connections. This section includes a brief discussion of quantum mechanics, introduces examples of parametric representations, and provides a glimpse of how the general theory can be used. Open quantum stochastic models are described in more detail in section IV. The main definitions and results concerning the concatenation and series products are given in section V; in particular, the *principle of series connections*, Theorem 5.5. In general the series product is not commutative, but we are able to show how the order can be interchanged by modifying one of the components, Theorem 5.6. A selection of examples from the quantum control literature are analyzed in section VI. The appendices contain proofs of some of the results and some additional technical material.

Notation. In this paper we use matrices $\mathbf{M} = \{m_{ij}\}$ with entries m_{ij} that are operators on an underlying Hilbert space. The asterisk $*$ is used to indicate the Hilbert space adjoint A^* of an operator A, as well as the complex conjugate $z^* = x - iy$ of a complex number $z = x + iy$ (here, $i = \sqrt{-1}$ and x, y are real). Real and imaginary parts are denoted $\mathrm{Re}(z) = (z + z^*)/2$ and $\mathrm{Im}(z) = -i(z - z^*)/2$ respectively. The conjugate transpose $\mathbf{M}^\dagger$ of a matrix $\mathbf{M}$ is defined by $\mathbf{M}^\dagger = \{m_{ji}^*\}$. Also defined are the conjugate $\mathbf{M}^\sharp = \{m_{ij}^*\}$ and transpose $\mathbf{M}^T = \{m_{ji}\}$ matrices, so that $\mathbf{M}^\dagger = (\mathbf{M}^T)^\sharp = (\mathbf{M}^\sharp)^T$. In the physics literature, it is common to use the dagger $\dagger$ to indicate the Hilbert space adjoint. The commutator of two operators A, B is defined by $[A, B] = AB - BA$. $\delta(\cdot)$ is the Dirac delta function, and δ_{jk} is the Kronecker delta. The tensor product of operators A, B defined on Hilbert spaces H, G is an operator $A \otimes B$ defined on the Hilbert space $\mathsf{H} \otimes \mathsf{G}$ (tensor product of Hilbert spaces) defined by $(A \otimes B)(\psi \otimes \phi) = (A\psi) \otimes (B\phi)$ for $\psi \in \mathsf{H}$, $\phi \in \mathsf{G}$; we usually follow the standard shorthand and write simply $AB = A \otimes B$ for the tensor product, and also $A = A \otimes I$ and $B = I \otimes B$.

II. Classical Linear Systems

As mentioned in the Introduction (section I), it is common practice in classical linear control theory to perform manipulations of block diagrams. Such manipulations, of course, greatly assist the analysis and design of control systems. To assist readers in interpreting the main quantum results concerning series and concatenation products (section V), we describe concatenation and series products for familiar classical linear systems in algebraic terms.

Consider two classical deterministic linear state space models

$$\begin{aligned} \dot{x}_j &= A_j x_j + B_j u_j \\ y_j &= C_j x_j + D_j u_j \end{aligned} \qquad (1)$$

where $j = 1, 2$. As usual, x_j, u_j and y_j are vectors and A_j, B_j, C_j and D_j are appropriately sized matrices. These systems are often represented by the matrix

$$\mathcal{G}_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}, \qquad (2)$$

or the transfer function $\mathcal{G}_j(s) = C_j(sI - A_j)^{-1}B_j + D_j$.

In modeling networks of such systems, one may form the *concatenation product*

$$\mathcal{G} = \mathcal{G}_1 \boxplus \mathcal{G}_2 = \begin{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} & \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \\ \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} & \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \end{pmatrix},$$

see Figure 3. In terms of transfer functions, the concatenation of two systems is $\mathcal{G}(s) = \mathrm{diag}\{\mathcal{G}_1(s), \mathcal{G}_2(s)\}$. The concatenation product simply assembles the two components together, without making any connections between them. It is *not* a parallel connection.

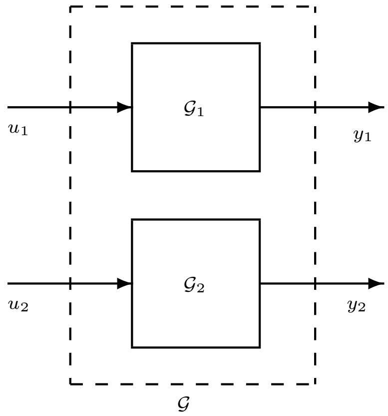

Fig. 3. Concatenation product.

Of considerable importance is the series connection, described by *series product*

$$\mathcal{G} = \mathcal{G}_2 \triangleleft \mathcal{G}_1 = \left(\begin{array}{cc} \begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix} & \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix} \\ \begin{pmatrix} D_2C_1 & C_2 \end{pmatrix} & D_2D_1 \end{array} \right),$$

see Figure 1. Here the connection is specified by $u_2 = y_1$, and so we require $\dim u_2 = \dim y_1$. In the frequency domain, the series product is given by the matrix transfer function product $\mathcal{G}(s) = \mathcal{G}_2(s)\mathcal{G}_1(s)$. This product describes a series (or cascade) connection which is fundamental to feedforward and feedback control.

Notice that both products are defined in terms of system parameters (state space parameters or transfer function matrices).

III. Example Components and Connections

A. Some Introductory Quantum Mechanics

Central to quantum mechanics are the notions of observables X, which are mathematical representations of physical quantities that can (in principle) be measured, and state vectors ψ, which summarize the status of physical systems and permit the calculation of expectations of observables. State vectors may be described mathematically as elements of a Hilbert space H, while observables are self-adjoint operators on H. The expected value of an observable X when in state ψ is given by the inner product $\langle \psi, X\psi \rangle$.

A basic example is that of a particle moving in a potential well, [19, Chapter 14]. The position and momentum of the particle are represented by observables Q and P, respectively, defined by

$$(Q\psi)(q) = q\psi(q), \quad (P\psi)(q) = -i\hbar \frac{d}{dq}\psi(q)$$

for $\psi \in \mathsf{H} = L^2(\mathbf{R})$. Here, $i = \sqrt{-1}$, $\hbar = h/2\pi$, h is Planck's constant, and $q \in \mathbf{R}$ represents position values. In following subsections we use units such that $\hbar = 1$, but retain it in our expressions in this subsection. The position and momentum operators satisfy the commutation relation $[Q, P] = i\hbar$. The dynamics of the particle is given by *Schrödinger's equation*

$$i\hbar \frac{d}{dt} V(t) = HV(t),$$

with initial condition $V(0) = I$, where $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$ is the *Hamiltonian* (here, m is the mass of the particle, and ω is the frequency of oscillation). The operator $V(t)$ is unitary ($V^*(t)V(t) = V(t)V^*(t) = I$, where I is the identity operator, and the asterisk denotes Hilbert space adjoint)—it is analogous to the transition matrix in classical linear systems theory. State vectors and observables evolve according to

$$\psi_t = V(t)\psi \in \mathsf{H}, \quad X(t) = V^*(t)XV(t).$$

These expressions provide two equivalent descriptions (dual), the former is referred to as the *Schrödinger picture*, while the latter is the *Heisenberg picture*. In this paper we use the Heisenberg picture, which is more closely related to models used in classical control theory and classical probability theory. In the Heisenberg picture, observables (and more generally other operators on H) evolve according to

$$\frac{d}{dt}X(t) = -\frac{i}{\hbar}[X(t), H(t)], \tag{3}$$

where $H(t) = V^*(t)HV(t)$.

Energy eigenvectors ψ_n are defined by the equation $H\psi_n = E_n\psi_n$ for real numbers E_n. The system has a discrete energy spectrum $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$. The state ψ_0 corresponding to E_0 is called the *ground state*. The *annihilation operator*

$$a = \sqrt{\frac{m\omega}{2\hbar}}(Q + i\frac{P}{2m\omega})$$

and the creation operator a^* lower and raise energy levels, respectively: $a\psi_n = \sqrt{n}\psi_{n-1}$, and $a^*\psi_n = \sqrt{n+1}\psi_{n+1}$. They satisfy the canonical commutation relation $[a, a^*] = 1$. In terms of these operators, the Hamiltonian can be expressed as $H = \hbar\omega(a^*a + \frac{1}{2})$. Using (3), the annihilation operator evolves according to

$$\frac{d}{dt}a(t) = -i\omega a(t) \tag{4}$$

with solution $a(t) = e^{-i\omega t}a$. Note that also $a^*(t) = e^{i\omega t}a^*$, and so commutation relations are preserved by the unitary dynamics: $[a(t), a^*(t)] = [a, a^*] = 1$. Because of the oscillatory nature of the dynamics, this system is often refereed to as the *quantum harmonic oscillator*.

It can be seen that the Hamiltonian H is a key "parameter" of the quantum physical system, specifying its energy.

B. Optical Cavities

A diagram of an optical cavity is shown in Figures 4, 5, together with a simplified representation. It consists of a pair of mirrors; the left one is partially transmitting (shown unfilled), while the right mirror is assumed perfectly reflecting (shown solid). Between the mirrors a trapped electromagnetic (optical) mode is set up, whose frequency depends on the separation between the mirrors. This mode is described by a harmonic oscillator with annihilation operator a (an operator acting on a Hilbert space H (as in subsection III-A), called the initial space). The partially transmitting mirror affords the opportunity for this mode to interact with an external free field, represented by a quantum stochastic process $b(t)$ (to be discussed shortly). When the external field is in the vacuum state, energy initially inside the cavity mode may leak out, in which case the cavity system is a damped harmonic oscillator, [8].

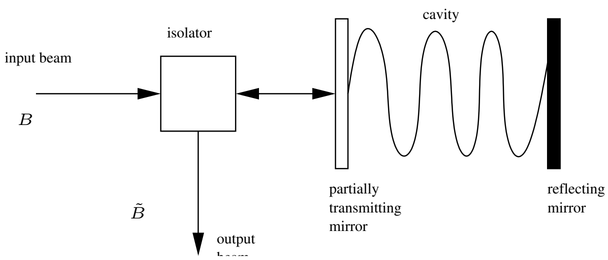

Fig. 4. A cavity consists of a pair of mirrors, one of which is perfectly reflecting (shown solid) while the other is partially transmitting (shown unfilled). The partially transmitting mirror enables the light mode inside the cavity to interact with an external light field, such as a laser beam. The external field is separated into input and output components by a Faraday isolator.

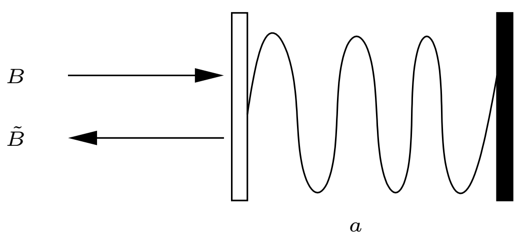

Fig. 5. A simplified representation of the cavity from Figure 4 which omits the Faraday isolator. It shows input B and output $\tilde{B}$ fields and the cavity mode annihilation operator a. This representation will be used for the remainder of this paper.

Quantization of a (free) electromagnetic field leads to an expression for the vector potential

$$\mathbf{A}(x,t) = \int \kappa(\omega)[b(\omega)e^{-i\omega t + i\omega x/c} + b^*(\omega)e^{i\omega t - i\omega x/c}]d\omega,$$

for a suitable coefficients $\kappa(\omega)$, and annihilation operators $b(\omega)$. Such a field can be considered as an infinite collection of harmonic oscillators, satisfying the singular canonical commutation relations

$$[b(\omega), b^*(\omega')] = \delta(\omega - \omega'),$$

where δ is the Dirac delta function.

An optical signal, such as a laser beam, is a free field with frequency content concentrated at a very high frequency $\omega_0 \approx 10^{14}$ rad/sec. The fluctuations about this nominal frequency can be considered as a quantum stochastic process consisting of signal plus noise, where the noise is of high bandwidth relative to the signal. Indeed, a coherent field is a good, approximate, model of a laser beam, and can be considered as the sum $b(t) = s(t) + b_0(t)$, where $s(t)$ is a signal, and $b_0(t)$ is quantum (vacuum) noise. Such "signal plus noise" models are of course common in engineering.

The cavity mode-free field system has a natural input-output structure, where the free field is decomposed as a superposition of right and left traveling fields. The right traveling field component is regarded as the *input*, while the left traveling component is an *output*, containing information about the cavity mode after interaction. The interaction facilitated by the partially transmitting mirror provides a boundary condition for the fields. The two components can be separated in the laboratory using a Faraday isolator. This leads to idealized models based on rotating wave and Markovian approximations, where, in the time domain, the input optical field (when in the ground or vacuum state) is described by *quantum white noise* $b(t) = b_0(t)$ [8, Chapters 5 and 11], which satisfies the singular canonical commutation relations

$$[b(t), b^*(t')] = \delta(t - t'). \tag{5}$$

In order to accommodate such singular processes, rigorous white noise and Itō frameworks have been developed, where in the Itō theory one uses the integrated noise, informally written

$$B(t) = \int_0^t b(s)ds.$$

The operators $B(t)$ are defined on a particular Hilbert space called a *Fock space*, F, [21, sec. 19]. When the field is in the vacuum (or ground) state, this is the quantum Wiener process which satisfies the Itō rule

$$dB(t)dB^*(t) = dt$$

(all other Itō products are zero). Field quadratures, such as $B(t) + B^*(t)$ and $-i(B(t) - B^*(t))$ are each equivalent to classical Wiener processes, but do not commute. A field quadrature can be measured using homodyne detection, [8, Chapter 8].

The cavity mode-free field system can be described by the Hamiltonian

$$H = \Delta a^* a - i\hbar \int k(\omega)(a^* b(\omega) - b^*(\omega) a) d\omega, \tag{6}$$

where the first term represents the self-energy of the cavity mode (the number Δ is called the "detuning", and represents the difference between the nominal external field frequency and the cavity mode frequency), while the remaining two terms describe the energy flow between the cavity mode and the free field (a photon in the free field may be created by a loss of a photon from the cavity mode, and vice versa). This Hamiltonian is defined on the composite Hilbert space, the tensor product $\mathsf{H}\otimes\mathsf{F}$; the tensor product is not written explicitly in the expression (6).

The Schrödinger equation for the cavity-free field system is derived from (6) under certain assumptions [8], and is given by the Itō quantum stochastic differential equation (QSDE)

$$\begin{aligned} dV(t) = \{ &\sqrt{\gamma} a dB^*(t) - \sqrt{\gamma} a^* dB(t) \\ &- \frac{\gamma}{2} a^* a dt - i\Delta a^* a dt \} V(t), \end{aligned} \tag{7}$$

with vacuum input and initial condition $V(0) = I$, so that $V(t)$ is unitary. The complete cavity mode-free field system thus has a unitary model. In the Heisenberg picture, cavity mode operators X (operators on the initial space H) evolve according the quantum Itō equation

$$\begin{aligned} dX(t) &= -i\Delta[X(t), a^*(t)a(t)]dt \\ &+ \frac{\gamma}{2}(a^*(t)[X(t), a(t)] + [a^*(t), X(t)]a(t))dt \\ &+ \sqrt{\gamma}dB^*(t)[X(t), a(t)] + \sqrt{\gamma}[a^*(t), X(t)]dB(t). \end{aligned} \quad (8)$$

Here, $\gamma > 0$ is a parameter specifying the coupling strength, and is related an approximation of the function $k(\omega)$ in the Hamiltonian (6). In particular, for $X = a$, the cavity mode annihilation operator, we have

$$da(t) = -(\frac{\gamma}{2} + i\Delta)a(t)dt - \sqrt{\gamma}\, dB(t); \quad (9)$$

cf. (4). The output field $\tilde{B}(t)$ is given by

$$d\tilde{B}(t) = \sqrt{\gamma}\, a(t)dt + dB(t), \quad (10)$$

where one can see the "signal plus noise" form of the field.

This is an example of an *open quantum system*, characterized by the parameters $\sqrt{\gamma}a$ and $\Delta a^* a$; the latter being the cavity mode Hamiltonian (specifying internal energy), and the former being the operator coupling the cavity mode to the external field (specifying the interface). These parameters are operators defined on the initial space H. These parameters specify a simpler, idealized model employing quantum noise, in place of the more basic but complicated Hamiltonian (6).

C. Optical Beamsplitters

A beamsplitter is a device that effects the interference of incoming optical fields A_1, A_2 and produces outgoing optical fields $\tilde{A}_1, \tilde{A}_2$, Figure 6. The relationship between these fields is

$$\tilde{A}_1(t) = \beta A_1(t) - \alpha A_2(t), \quad \tilde{A}_2(t) = \alpha A_1(t) + \beta A_2(t), \quad (11)$$

where α and β are complex numbers describing the beamsplitter relations, and they satisfy $\alpha^*\alpha + \beta^*\beta = 1$, $\alpha^*\beta = \alpha\beta^*$ (here the asterisk indicates the conjugate of a complex number).

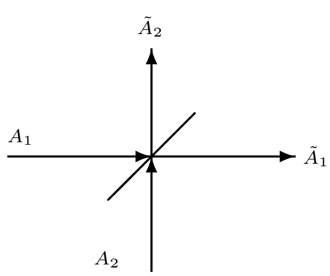

Fig. 6. Diagram of an optical beamsplitter showing inputs A_1, A_2 and outputs $\tilde{A}_1, \tilde{A}_2$ fields.

The initial space is trivial, $\mathsf{H} = \mathbf{C}$, the complex numbers; nevertheless, the Schrödinger equation for the beamsplitter is

$$dV(t) = \{(\mathbf{S} - \mathbf{I})d\mathbf{\Lambda}\}V(t), \quad (12)$$

with initial condition $V(0) = I$, where $\mathbf{S}$ is the unitary matrix defined by (14) below, $\mathbf{I}$ is the identity matrix, and $\mathbf{\Lambda}$ is the matrix of gauge processes

$$\mathbf{\Lambda} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (13)$$

Here, A_{ij} describes the destruction of a photon in channel j and the creation of a photon in channel i. In terms of their formal derivatives, $A_{ij}(t) = \int_0^t a_i^*(s)a_j(s)ds$, where $A_i(t) = \int_0^t a_i(s)ds$. The self-adjoint processes A_{jj} are equivalent to classical Poisson processes when the channels are in coherent states (signal plus quantum noise). These counting processes may be observed by a photodetector, [8, Chapters 8 and 11].

This open system is characterized by the unitary parameter matrix

$$\mathbf{S} = \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix}, \quad (14)$$

which describes scattering among the field channels. The matrix $\mathbf{S}$ specifies the interface for the beamsplitter.

D. Open Quantum Systems

In general, as we shall explain in more detail in section IV, open quantum systems with multiple field channels are characterized by the parameter list

$$\mathbf{G} = (\mathbf{S}, \mathbf{L}, H) \quad (15)$$

where $\mathbf{S}$ is a square matrix with operator entries such that $\mathbf{S}^\dagger\mathbf{S} = \mathbf{S}\mathbf{S}^\dagger = \mathbf{I}$ (recall the notational conventions mentioned at the end of section I), $\mathbf{L}$ is a column vector with operator entries, and H is a self-adjoint operator. The matrix $\mathbf{S}$ is called a *scattering matrix*, the vector $\mathbf{L}$ is a *coupling vector*; together, these parameters specify the *interface* between the system and the fields. The parameter H is the Hamiltonian describing the self-energy of the system. Thus the parameters describe the system by specifying energies—internal energy, and energy exchanged with the fields. All operators in the parameter list are defined on the initial Hilbert space H for the system.

The closed, undamped, harmonic oscillator of subsection III-A is specified by the parameters

$$\mathbf{H} = (\_, \_, \omega a^* a) \quad (16)$$

(the blanks _ indicate the absence of field channels), while the open, damped oscillator (cavity) of subsection III-B has parameters

$$\mathbf{C} = (I, \sqrt{\gamma}\, a, \Delta a^* a). \quad (17)$$

The beamsplitter, described in subsection III-C has parameters

$$\mathbf{M} = (\begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix}, 0, 0). \quad (18)$$

E. Series Connection Example

Consider the feedforward network shown in Figure 7, where one of the beamsplitter output beams is fed into an optical

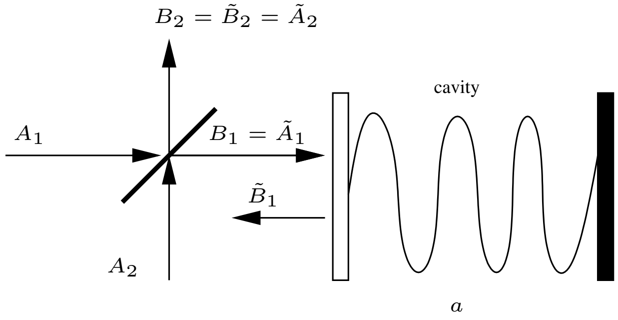

Fig. 7. Beam splitter (left) and cavity (right) network.

cavity. From the previous subsections, we see that the quantum stochastic differential equations describing the network are

$$\begin{aligned} da(t) &= (-\frac{\gamma}{2} + i\Delta)a(t)dt - \sqrt{\gamma}\, dB_1(t) && (19) \\ \tilde{A}_1(t) &= \beta A_1(t) - \alpha A_2(t) && (20) \\ \tilde{A}_2(t) &= \alpha A_1(t) + \beta A_2(t) && (21) \\ B_1(t) &= \tilde{A}_1(t) && (22) \\ B_2(t) &= \tilde{A}_2(t) && (23) \\ d\tilde{B}_1(t) &= \sqrt{\gamma} a(t)dt + dB_1(t) && (24) \\ d\tilde{B}_2(t) &= dB_2(t). && (25) \end{aligned}$$

It can be seen that algebraic manipulations are required to describe the complete system (in general such manipulations may be simple in principle, but complicated in practice). The key motivation for this paper is more efficient algebraic methods for describing such networks.

We now describe how the parameters for the complete network may be obtained. We first assemble the field channels into vectors as follows:

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tilde{\mathbf{A}} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix}, \tilde{\mathbf{B}} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}.$$

The beamsplitter acts on the input vector $\mathbf{A}$, and is described by the parameters $\mathbf{M}$ given in equation (18)). Now the beamsplitter output has two channels, while the cavity has one channel (described by the parameters $\mathbf{C}$, equation (17)), and so we augment the cavity to accept a second channel in a trivial way. This is achieved by forming the concatenation $\mathbf{C} \boxplus \mathbf{N}$, where $\mathbf{N} = (1, 0, 0)$ represents a trivial component (pass-through). The augmented cavity $\mathbf{C} \boxplus \mathbf{N}$ can now accept the output of the beamsplitter, so that the complete network is described as a series connection as follows:

$$\mathbf{G} = (\mathbf{C} \boxplus \mathbf{N}) \triangleleft \mathbf{M}. \qquad (26)$$

The definition of the concatenation $\boxplus$ and series $\triangleleft$ products will be explained below in section V (Definitions 5.1 and 5.3, and the principle of series connections, Theorem 5.5). By applying these definitions, we obtain the network parameters

$$\mathbf{G} = \left(\begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix}, \begin{pmatrix} \sqrt{\gamma}\, a \\ 0 \end{pmatrix}, \Delta a^* a \right). \qquad (27)$$

A schematic representation of the network is shown in Figure 8, which illustrates the important point that *components, parts of components, as well as the complete network, are described by parameters of the form (15).*

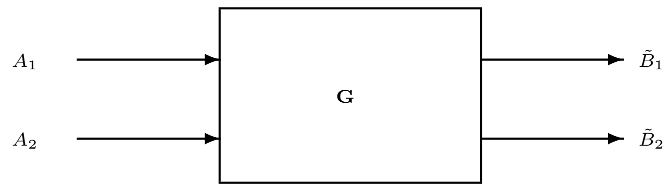

Fig. 8. Beam splitter-cavity network representation illustrating the network model given by (27).

For the purposes of network modeling and design, it can be useful to perform manipulations of the network to yield equivalent networks; this, of course, is common practice in classical electrical circuit theory and control engineering. For instance, in our example we could move the beam splitter to the output, but the cavity should be modified (to have two partially transmitting mirrors) as follows (see Remark 5.7):

$$\mathbf{G} = (\mathbf{C} \boxplus \mathbf{N}) \triangleleft \mathbf{M} = \mathbf{M} \triangleleft (\mathbf{C}' \boxplus \mathbf{N}'). \qquad (28)$$

Here, the modified cavity $\mathbf{C}' \boxplus \mathbf{N}'$ (see Figure 9) is described by the subsystems

$$\mathbf{C}' = (I, \beta^* \sqrt{\gamma}\, a, \Delta a^* a)\,, \quad \mathbf{N}' = (I, -\alpha^* \sqrt{\gamma}\, a, 0)\,. \qquad (29)$$

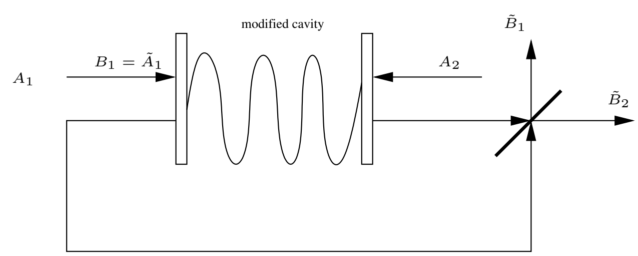

Fig. 9. Equivalent beam splitter and cavity network.

The connections described here so far are unidirectional *field mediated connections*. Components interact indirectly via a quantum field, which acts as a quantum "wire". One can also consider bidirectional *direct connections*, which can be accommodated by using interaction Hamiltonian terms in the models. Our emphasis in this paper will be on field mediated connections, with direct connections readily available in the modeling framework if required. See subsection V-D.

IV. Open Quantum Stochastic Models

In this section we describe in more detail the open quantum models of the type encountered in section III. Specifically, we consider models specified by the parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$ (recall (15)), where

$$\mathbf{S} = \begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & \vdots & \vdots \\ S_{n1} & \dots & S_{nn} \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix},$$

are respectively a scattering matrix with operator entries satisfying $\mathbf{S}^\dagger\mathbf{S} = \mathbf{S}\mathbf{S}^\dagger = \mathbf{I}$, and coupling vector with operator entries, and H is a self-adjoint operator called the Hamiltonian (this parameterization is due to Hudson-Parthasarathy, [15], and is closely related to a standard form of the Lindblad generator, given in (33) below). The operators constituting these parameters are assumed to be defined on an underlying Hilbert space H, called the *initial space*. These parameters specify an open quantum system coupled to n field channels with corresponding gauge processes:

$$\mathbf{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \vdots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix}.$$

All differentials shall be understood in the Itō sense - that is, $dX(t) \equiv X(t+dt) - X(t)$. We assume that these processes are *canonical*, meaning that we have the following non-vanishing second order Itō products: $dA_j(t)\, dA_k(t)^* = \delta_{jk}dt$, $dA_{jk}(t)\, dA_l(t)^* = \delta_{kl}dA_j(t)^*$, $dA_j(t)\, dA_{kl}(t) = \delta_{jk}dA_l(t)$ and $dA_{jk}(t)\, dA_{lm}(t) = \delta_{kl}dA_{jm}(t)$.

If we consider the open system specified by $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$ with canonical inputs, the Schrödinger equation

$$\begin{aligned} dV(t) &= \{\mathrm{tr}[(\mathbf{S}-\mathbf{I})d\mathbf{\Lambda}] + d\mathbf{A}^\dagger\mathbf{L} \\ &- \mathbf{L}^\dagger\mathbf{S}d\mathbf{A} - \frac{1}{2}\mathbf{L}^\dagger\mathbf{L}dt - iHdt\}V(t) \equiv dG(t)\, V(t) \end{aligned} \quad (30)$$

with initial condition $V(0) = I$ determines the unitary motion of the system. Equation (30) serves as the definition of the time-dependent generator $dG(t)$. Given an operator X defined on the initial space H, its Heisenberg evolution is defined by

$$X(t) = \mathrm{j}_t(X) = V(t)^* XV(t) \quad (31)$$

and satisfies

$$\begin{aligned} dX(t) &= (\mathcal{L}_{\mathbf{L}(t)}(X(t)) - i[X(t), H(t)])dt \\ &+ d\mathbf{A}^\dagger(t)\mathbf{S}^\dagger(t)[X(t), \mathbf{L}(t)] + [\mathbf{L}^\dagger(t), X(t)]\mathbf{S}(t)d\mathbf{A}(t) \\ &+ \mathrm{tr}[(\mathbf{S}^\dagger(t)X(t)\mathbf{S}(t) - X(t))d\mathbf{\Lambda}(t)]. \end{aligned} \quad (32)$$

In this expression, all operators evolve unitarily according to (31) (e.g. $\mathbf{L}(t) = \mathrm{j}_t(\mathbf{L})$) (commutators of vectors and matrices of operators are defined component-wise), and tr denotes the trace of a matrix. We also employ the notation

$$\begin{aligned} \mathcal{L}_{\mathbf{L}}(X) &= \frac{1}{2}\mathbf{L}^\dagger[X, \mathbf{L}] + \frac{1}{2}[\mathbf{L}^\dagger, X]\mathbf{L} \\ &= \sum_{j=1}^n (\frac{1}{2}L_j^*[X, L_j] + \frac{1}{2}[L_j^*, X]L_j); \end{aligned} \quad (33)$$

this is called the *Lindblad superoperator* in the physics literature (it is analogous to the transition matrix for a classical Markov chain, or the generator of a classical diffusion process). The dynamics is unitary, and hence preserves commutation relations. The output fields are defined by

$$\tilde{\mathbf{A}}(t) = V^*(t)\mathbf{A}(t)V(t), \quad \tilde{\mathbf{\Lambda}}(t) = V^*(t)\mathbf{\Lambda}(t)V(t), \quad (34)$$

and satisfy the quantum stochastic differential equations

$$\begin{aligned} d\tilde{\mathbf{A}}(t) &= \mathbf{S}(t)d\mathbf{A}(t) + \mathbf{L}(t)dt \\ d\tilde{\mathbf{\Lambda}}(t) &= \mathbf{S}^\sharp(t)d\mathbf{\Lambda}(t)\mathbf{S}^T(t) + \mathbf{S}^\sharp(t)d\mathbf{A}^\sharp(t)\mathbf{L}^T(t) \\ &\quad + \mathbf{L}^\sharp(t)d\mathbf{A}^T(t)\mathbf{S}^T(t) + \mathbf{L}^\sharp(t)\mathbf{L}^T(t)dt, \end{aligned}$$

where $\mathbf{L}(t) = \mathrm{j}_t(\mathbf{L})$, etc, as above. The output processes also have canonical quantum Itō products.

In the physics literature, it is common practice to describe open systems using a *master equation* (analogous to the Kolmogorov equation for the density of a classical diffusion process) for a density operator ρ, a convex combination of outer products $\psi\psi^*$ (here ψ is a state vector). Master equations can easily be obtained from the parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$; indeed, we have

$$\frac{d}{dt}\rho = i[\rho, H(t)] + \mathcal{L}'_{\mathbf{L}(t)}(\rho), \quad (35)$$

where $\mathcal{L}'_{\mathbf{L}}(\rho) = \mathbf{L}^T\rho\mathbf{L}^\sharp - \frac{1}{2}\mathbf{L}^\sharp\mathbf{L}^T\rho - \frac{1}{2}\rho\mathbf{L}^\sharp\mathbf{L}^T$ is the adjoint of the Lindbladian: $\mathrm{tr}[\rho(t)\mathcal{L}_{\mathbf{L}}(X)] = \mathrm{tr}[\mathcal{L}'_{\mathbf{L}}(\rho)\ X]$. Note that while the master equation does not depend on the scattering matrix $\mathbf{S}$, this matrix plays an important role in describing the architecture of the input channels, as in subsections III-E and VI-B. We also mention that if an observable of one or more output channels is continuously monitored, then a quantum filter (also called a stochastic master equation) for the conditional density operator can be written down in terms of the parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$; an example of this is discussed in subsection VI-C, see [2].

Open systems specified by parameters $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$ preserve the canonical nature of the quantum signals. However, if the inputs are not canonical, one will need to modify the equations for the unitary, the Heisenberg dynamics, and the outputs, etc, to accommodate non-canonical correlations; we do not pursue this matter further here, and in this paper we will always use canonical quantum signals.

V. The Concatenation and Series Products and their Application to Quantum Networks

This section contains the main results of the paper. The concatenation and series products are defined in subsection V-A, and applied to a feedback arrangement in Theorem 5.5, the *principle of series connections* (subsection V-B). This is followed in subsection V-C with a specialization to cascade networks, and a consideration in subsection V-D of reducible networks. These results are applied to a range of examples in section VI.

A. Definitions

In this subsection we define two products between system parameters. It is assumed that both systems are defined on the same underlying initial Hilbert space, enlarging if necessary by using a tensor product.

Definition 5.1: (Concatenation product) Given two systems $\mathbf{G}_1 = (\mathbf{S}_1, \mathbf{L}_1, H_1)$ and $\mathbf{G}_2 = (\mathbf{S}_2, \mathbf{L}_2, H_2)$, we define their *concatenation* to be the system $\mathbf{G}_1 \boxplus \mathbf{G}_2$ by

$$\mathbf{G}_1 \boxplus \mathbf{G}_2 = (\begin{pmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix}, H_1 + H_2). \quad (36)$$

The concatenation product is useful for combining distinct systems, or for decomposing a given system into subsystems. It does not describe interconnections via field channels, but does allow for direct connections via the Hamiltonian parameters.

Systems without field channels are included by employing blanks; set $(\_,\_,H) \boxplus (\_,\_,H') := (\_,\_,H+H')$ and more generally $(\_,\_,H) \boxplus (S',\mathbf{L}',H') = (S',\mathbf{L}',H') \boxplus (\_,\_,H) := (S',\mathbf{L}',H+H')$.

Definition 5.2: (Reducible system) We say that a system $\mathbf{G} = (\mathbf{S},\mathbf{L},H)$ is *reducible* if it can be expressed as

$$\mathbf{G} = \mathbf{G}_1 \boxplus \mathbf{G}_2 \tag{37}$$

for two systems $\mathbf{G}_1$ and $\mathbf{G}_2$. In particular, the parameters of a reducible system have the form

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix}, \quad H = H_1 + H_2. \tag{38}$$

Such decompositions are not unique. Furthermore, if one or more of the subsystems is reducible, the reduction process may be iterated to obtain a decomposition $\mathbf{G} = \boxplus_j \mathbf{G}_j$.

Definition 5.3: (Series product) Given two systems $\mathbf{G}_1 = (\mathbf{S}_1,\mathbf{L}_1,H_1)$ and $\mathbf{G}_2 = (\mathbf{S}_2,\mathbf{L}_2,H_2)$ with the same number of field channels, the *series product* $\mathbf{G}_2 \triangleleft \mathbf{G}_1$ defined by

$$\begin{aligned} \mathbf{G}_2 \triangleleft \mathbf{G}_1 \quad = \quad & (\mathbf{S}_2\mathbf{S}_1, \mathbf{L}_2 + \mathbf{S}_2\mathbf{L}_1, \\ & H_1 + H_2 + \frac{1}{2i}(\mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 - \mathbf{L}_1^\dagger \mathbf{S}_2^\dagger \mathbf{L}_2)). \end{aligned}$$

As will be explained in the following subsection, the series product specifies the parameters for a system formed by feeding the output channel of the first system into the input channel of the second. Both of these products are powerful tools for describing quantum networks.

Remark 5.4: Let $dG_j(t)$ denote the infinitesimal Itō generators corresponding to parameters $\mathbf{G}_j = (\mathbf{S}_j,\mathbf{L}_j,H_j)$, for $j = 1,2$ respectively, as constructed in (30). The generator corresponding to $\mathbf{G}_2 \triangleleft \mathbf{G}_1$ is then

$$dG(t) = dG_1(t) + dG_2(t) + dG_2(t)dG_1(t). \tag{39}$$

The last term is to be computed using the Itō table for second order products of differentials.

B. Feedback

Let us consider a reducible system $\mathbf{G} = \mathbf{G}_1 \boxplus \mathbf{G}_2$ (recall Definition 5.2), where number of channels in the factors is the same (i.e. dim $\mathbf{L}_1$ = dim $\mathbf{L}_2$). The setup is sketched in Figure 10. We investigate what will happen if we feed one of the outputs, say $\tilde{\mathbf{A}}_1$ back in as the input $\mathbf{A}_2$. Either of the two diagrams in Figure 11 may serve to describe the resulting feedback system. Note that the outputs will be different after the feedback connection has been made.

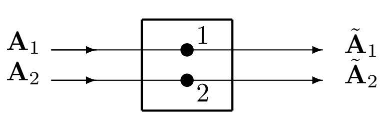

Fig. 10. Reducible system $\mathbf{G}_1 \boxplus \mathbf{G}_2$ with inputs $\mathbf{A}_1,\mathbf{A}_2$ and outputs $\tilde{\mathbf{A}}_1,\tilde{\mathbf{A}}_2$.

We now state our main result applying the series product to feedback.

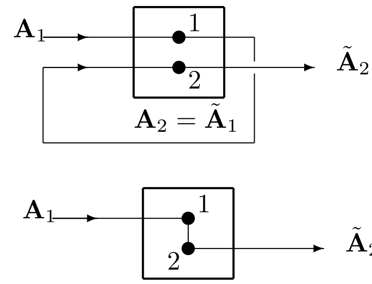

Fig. 11. Direct feedback system $\mathbf{G}_2 \triangleleft \mathbf{G}_1$, with input $\mathbf{A}_1$ and output $\tilde{\mathbf{A}}_2$.

Theorem 5.5: **(Principle of Series Connections)** *The parameters $\mathbf{G}_{2\leftarrow 1}$ for the feedback system obtained from $\mathbf{G}_1 \boxplus \mathbf{G}_2$ when the output of the first subsystem is fed into the input of the second is given by the series product $\mathbf{G}_{2\leftarrow 1} = \mathbf{G}_2 \triangleleft \mathbf{G}_1$.*

A proof of this theorem is given in Appendix B.

C. Cascade

In our treatment of series connections, we nowhere assumed that the matrix entries commuted, and this of course facilitated feedback. However, the principle of series connections also applies to the special case where the subsystems commute, as in a *cascade* of independent systems, as shown in Figure 12. [1]

To formulate the cascade arrangement, we first consider the concatenation of the two systems $\mathbf{G}_1 \boxplus \mathbf{G}_2$. The system $\mathbf{G} = \mathbf{G}_1 \boxplus \mathbf{G}_2$ is reducible with components $\mathbf{G}_j$.

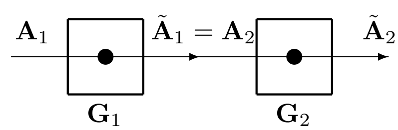

Fig. 12. Cascade of independent quantum components, $\mathbf{G}_2 \triangleleft \mathbf{G}_1$.

The notion of cascaded quantum systems goes back to Carmichael [3], who used a quantum trajectory analysis, and Gardiner [6] who used (scalar) quantum noise models of the form $\mathbf{G}_j = (1, L_j, H_j)$ (no scattering). As a special case of the series principle, we see that the cascaded generator for this type of setup is $\mathbf{G}_{\text{cascade}} = \mathbf{G}_2 \triangleleft \mathbf{G}_1 = (1, L_1 + L_2, H_1 + H_2 + \text{Im}\{L_2^* L_1\})$. This is entirely in agreement with Gardiner's analysis, cf. [8, Chapter 12] with $L_j = \sqrt{\gamma_j} c_j$ where we have $L_{2\leftarrow 1} = \sqrt{\gamma_1} c_1 + \sqrt{\gamma_2} c$ and $H_{2\leftarrow 1} = H_1 + H_2 + \frac{1}{2i}\sqrt{\gamma_1\gamma_2}\,(c_2^* c_1 - c_1^* c_2)$.

We now consider cascade arrangements and ask what happens if we try to swap the order of the components. Since the series product is not in general commutative, we cannot expect to be able to swap the order without, say, modifying one of the components. We now make this precise as follows.

We say that two systems are *parametrically equivalent* if their parameters are identical. This implies that, for the same

[1] Indeed, the reason we use the term "series" is to indicate that it applies more generally than to cascades of independent components.

input, they produce the same internal dynamics and output. Consider the cascaded systems shown in Figure 13.

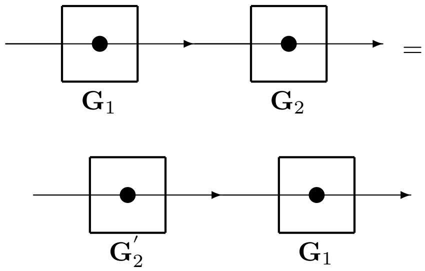

Fig. 13. Equivalent Systems.

We assume that the initial inputs are canonical in both cases and ask, for fixed choices of $\mathbf{G}_1$ and $\mathbf{G}_2$, what we should take for $\mathbf{G}_2'$ so that the setups are parametrically equivalent.

Theorem 5.6: The two cascaded systems shown in Figure 13 are parametrically equivalent if and only if

$$\mathbf{G_2} \triangleleft \mathbf{G}_1 = \mathbf{G_1} \triangleleft \mathbf{G}_2'. \tag{40}$$

Furthermore, if $(\mathbf{S}_j, \mathbf{L}_j, H_j)$ are the parameters for $\mathbf{G}_1$ and $\mathbf{G}_2$ $(j = 1, 2)$, then the parameters $(\mathbf{S}_2', \mathbf{L}_2', H_2')$ of $\mathbf{G}_2'$ are uniquely determined by

$$\begin{aligned} \mathbf{S}_2' &= \mathbf{S}_1^\dagger \mathbf{S}_2 \mathbf{S}_1, \\ \mathbf{L}_2' &= \mathbf{S}_1^\dagger (\mathbf{S}_2 - \mathbf{I}) \mathbf{L}_1 + \mathbf{S}_1^\dagger \mathbf{L}_2, \\ H_2' &= H_2 + \mathrm{Im}\left\{ \mathbf{L}_2^\dagger (\mathbf{S}_2 + \mathbf{I}) \mathbf{L}_1 - \mathbf{L}_1^\dagger \mathbf{S}_2 \mathbf{L}_1 \right\}. \end{aligned} \tag{41}$$

The proof of this theorem is given in Appendix C.

Remark 5.7: A useful special case of this result is moving a scattering matrix from the input to the output of a modified system:

$$(\mathbf{S}, \mathbf{L}, H) = (\mathbf{I}, \mathbf{L}, H) \triangleleft (\mathbf{S}, 0, 0) = (\mathbf{S}, 0, 0) \triangleleft (\mathbf{I}, \mathbf{S}^\dagger \mathbf{L}, H). \tag{42}$$

This is illustrated in subsection III-E. □

D. Reducible Networks

Networks can be formed by combining components with the concatenation and series products. Within this framework, components may interact directly, or indirectly via fields. This framework is useful for modeling existing systems, as we have seen above, as well as for designing new systems.

Let $\{\mathbf{G}_j\}$ be a collection of components, which we may combine together to form an unconnected system $\mathbf{G} = \boxplus_j \mathbf{G}_j$. The components may interact directly via bidirectional exchanges of energy, and this may be specified by a direct connection Hamiltonian K of the form

$$K = i \sum_k (N_k^* M_k - M_k^* N_k), \tag{43}$$

where M_k, N_k are operators defined on the initial Hilbert space for $\mathbf{G}$. The components may also interact via field interconnects, specified by a list of series connections

$$\mathscr{S} = \{\mathbf{G}_{j_1} \triangleleft \mathbf{G}_{k_1}, \ldots, \mathbf{G}_{j_n} \triangleleft \mathbf{G}_{k_n}\} \tag{44}$$

such that (i) the field dimensions of the members of each pair are the same, and (ii) each input and each output (relative to the decomposition $\mathbf{G} = \boxplus_j \mathbf{G}_j$) has at most one connection.

A *reducible network* $\mathbf{N}$ is the system formed from $\mathbf{G}$ by implementing the connections (43) and (44). The parameters of the network $\mathbf{N}$ may be obtained as follows. A *series chain* is a system of the form

$$\mathbf{C} = \mathbf{G}_{j_l} \triangleleft \mathbf{G}_{k_l} \triangleleft \cdots \triangleleft \mathbf{G}_{j_m} \triangleleft \mathbf{G}_{k_m}.$$

Let $\mathscr{C}$ denote the set of maximal-length chains drawn from the list of series connections (44), and let $\mathscr{U}$ denote the set of components not involved in any series connection. Then the reducible network is given by

$$\mathbf{N} = (\boxplus_{\mathbf{G}_k \in \mathscr{U}} \mathbf{G}_k) \boxplus \left(\boxplus_{\mathbf{C}_j \in \mathscr{C}} \mathbf{C}_j\right) \boxplus (1, 0, K). \tag{45}$$

An example of a reducible network is shown in Figure 14.

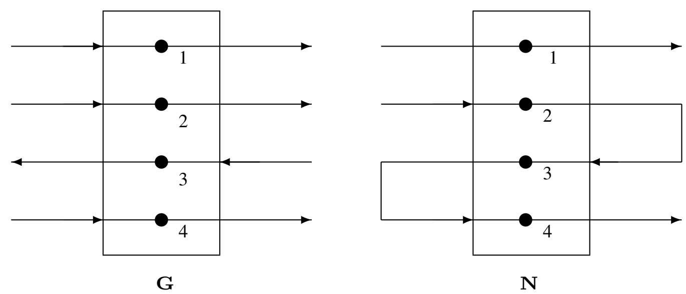

Fig. 14. A reducible network $\mathbf{N} = \mathbf{G}_1 \boxplus (\mathbf{G}_4 \triangleleft \mathbf{G}_3 \triangleleft \mathbf{G}_2)$ formed from the collection $\mathbf{G} = \mathbf{G}_1 \boxplus \mathbf{G}_2 \boxplus \mathbf{G}_3 \boxplus \mathbf{G}_4$ of components with connections specified by the list of series connections $\mathscr{S} = \{\mathbf{G}_3 \triangleleft \mathbf{G}_2, \mathbf{G}_4 \triangleleft \mathbf{G}_3\}$.

Remark 5.8: The examples considered in section VI below are all important examples of reducible networks that have appeared in the literature. However, we mention that there are important examples of quantum networks that are not reducible. An example of a non-reducible network was considered by Yanagisawa and Kimura, [26, Fig. 4], which consists of two systems in a feedback arrangement formed by a beam splitter, as occurs if in Figure 7 we connect the output $\tilde{B}_1$ to the input A_2 (i.e. setting $A_2 = \tilde{B}_1$). The feedback loop formed in this way is "algebraic", and the resulting in-loop field is not a free field in general. A general theory of quantum feedback networks, both reducible and non-reducible, is given in [11]. □

VI. Examples

In this section we look at a number of examples from the literature which can be represented by reducible networks.

A. All-Optical Feedback

We consider a simple situation first introduced by Wiseman and Milburn as an example of all-optical feedback, [25, section II.B. A]. Referring to Figure 15, vacuum light field A_1 is reflected off mirror 1 to yield an output beam $\tilde{A}_1$ which results from interaction with the internal cavity mode a. This beam is reflected onto mirror 2, as shown, where it constitutes the input A_2. It is assume that both mirrors have the same transmittivity, so that we can model the coupling operators for the two field channels as $L_1 = L_2 = \sqrt{\gamma}\,a$, where γ is the damping rate. We may also assume that the light picks up a phase $S = e^{i\theta}$ when reflected by the cavity mirror.

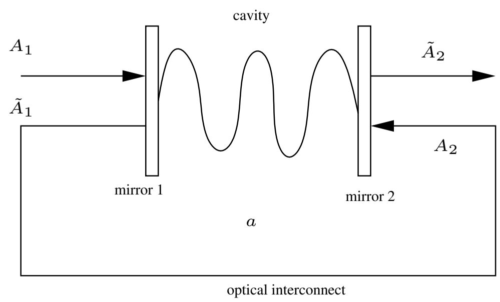

Fig. 15. All-optical feedback for a cavity. The feedback path is a light beam from mirror 1 to mirror 2, both of which are partially transmitting). There is a phase shift θ along the feedback path.

Before feedback, the cavity is described by

$$\mathbf{G} = (\mathbf{I}, \left(\begin{array}{c} L_1 \\ L_2 \end{array} \right), 0) = (1, L_1, 0) \boxplus (1, L_2, 0).$$

The phase shift between the mirrors is described by the system $(S, 0, 0)$.

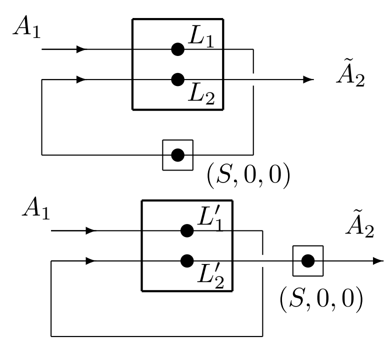

Fig. 16. Representations of the all-optical feedback scheme of Figure 15 as reducible networks.

Two equivalent reducible network representations are shown in Figure 16. From the left diagram in Figure 16, we see that the closed loop system is described by

$$\begin{aligned} \mathbf{G}_{cl} &= (1, L_2, 0) \triangleleft (S, 0, 0) \triangleleft (1, L_1, 0) \\ &= (S, SL_1 + L_2, \frac{1}{2i}(L_2^* S L_1 - L_1^* S^* L_2)). \end{aligned}$$

Here we have twice applied the formulas (39) given in Definition 5.3.

Alternatively, we may use our theory of equivalent components (Theorem 5.6) to move the phase change $(S, 0, 0)$ to the very end, as shown in the right diagram in Figure 16. Then

$$\begin{aligned} \mathbf{G}_{cl} &= (S, 0, 0) \triangleleft (1, S^* L_2, 0) \triangleleft (1, L_1, 0) \\ &= (S, SL_1 + L_2, \frac{1}{2i}(L_2^* S L_1 - L_1^* S^* L_2)), \end{aligned}$$

as before. Either way, the closed loop feedback system is described by $\mathbf{G}_{cl} = (S_{\rm cl}, L_{\rm cl}, H_{\rm cl})$ where

$$\begin{aligned} S_{\rm cl} &= S \equiv e^{i\theta}, \\ L_{\rm cl} &= SL_1 + L_2 \equiv \left(1 + e^{i\theta}\right)\sqrt{\gamma} a, \\ H_{\rm cl} &= \mathrm{Im}\left\{L_2^* S L_1\right\} \equiv \gamma \sin\theta\, a^\dagger a. \end{aligned}$$

From this we obtain the Heisenberg dynamical equation for the cavity mode

$$\begin{aligned} da &= -\left[a, \left(1 + e^{i\theta}\right)\sqrt{\gamma} a^\dagger\right] dA_1 \\ &\quad -\frac{\gamma}{2}\left(1 + e^{i\theta}\right)\left(1 + e^{-i\theta}\right) a dt - i\gamma \sin\theta\, a dt \\ &\equiv -\left(1 + e^{i\theta}\right)\left(\sqrt{\gamma} dA_1 + \gamma a dt\right), \end{aligned}$$

and the input/output relation, in agreement with [25, eq. (2.29)],

$$d\tilde{A}_2 = e^{i\theta} dA_1 + \left(1 + e^{i\theta}\right)\sqrt{\gamma} a dt.$$

B. Direct Measurement Feedback

In the paper [24], Wiseman considers two types of measurement feedback, one involving photon counting, and another based on quadrature measurement using homodyne detection (which is a diffusive limit of photon counts). In both cases proportional feedback involving an electrical current was used. We describe these feedback situations in the following subsections using our network theory.

Consider the measurement feedback arrangement shown in Figure 17, which shows a vacuum input field A, a control signal c, a photodetector PD, and a proportional feedback gain k.

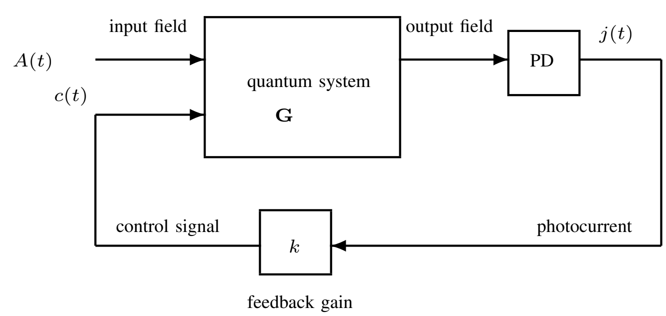

Fig. 17. Direct feedback of photocurrent obtained by photon counting using a photodetector (PD).

Before feedback, the quantum system is described by

$$\mathbf{G} = (1, L, H_0 + Fc), \tag{46}$$

where H_0 and F are self-adjoint, and c represent a classical control variable. The photocurrent $j(t)$ resulting from ideal photodetection of the output field is given by

$$\text{“}j(t)dt\text{”} = d\Lambda + L dA^\dagger + L^\dagger dA + L^\dagger L dt, \tag{47}$$

where, mathematically, the photocurrent $j(t)$ is the formal derivative of a field observable (a self-adjoint commutative jump stochastic process) $\tilde{\Lambda}(t)$ (the output gauge process) whose Itō differential is given by the RHS of (47). The feedback is given by

$$c(t) = kj(t), \tag{48}$$

where k is a (real, scalar) proportional gain. The feedback gain can be absorbed into F, and so we assume $k = 1$ in what follows.

An alternative is to again consider the quantum system $\mathbf{G}$ given by (46), but replace the photodetector PD in Figure 17 with a homodyne detector HD.[2] The homodyne detector then produces a photocurrent $j(t)$ given by

$$\text{“}j(t)dt\text{”} = dJ(t) = (L(t) + L^\sharp(t))dt + dA(t) + dA^\sharp(t).$$

The feedback is given by (48) as above, with feedback gain absorbed into F, as above. The measurement result $J(t)$ is a field observable (here a self-adjoint commutative diffusive process).

In order to describe these types of direct measurement feedback within our framework, we view the setup before feedback as being described by

$$\mathbf{G} = (1, L, H_0) \boxplus (S_{fb}, L_{fb}, H_{fb}) \equiv \mathbf{G}_0 \boxplus \mathbf{G}_{fb}.$$

Here, $\mathbf{G}_0$ describes the internal energy of the system and its coupling to the input field A. The second term, $\mathbf{G}_{fb}$, describes the way in which the classical input signal is determined from a second quantum input field (which will be replaced by the output $\tilde{A}$ when the feedback loop is closed). The idea is that by appropriate choice of the coupling operator L_{fb}, the relevant observable of the field can be selected. In this way, the photodection and homodyne detection measurements are accommodated. The singular nature of the feedback signal (which contains white noise in the homodyne case) means that care must be taken to describe it correctly. The correct form of the parameters is given by the Holevo parameterization (Appendix A, equation (55)) rather than the expression arising from the implicit-explicit formalism of [24], since the later does not capture correctly gauge couplings, see Appendix A. We shall interpret the feedback interaction as being due to a Holevo generator $K_{fb}(t) = H_{00}t + H_{01}A(t) + H_{10}A^*(t) + H_{11}\Lambda(t)$, see Appendix A, equation (54). The closed loop system after feedback is given by the series connection $\mathbf{G}_{cl} = \mathbf{G}_{fb} \triangleleft \mathbf{G}_0 = \Big(S_{fb}, L_{fb} + S_{fb}L, H_0 + H_{fb} + \mathrm{Im}\left(L_{fb}^* S_{fb} L_0\right)\Big)$.

1) Photon Counting: Here we take $K_{fb}(t) = F\Lambda(t)$, so that $S_{fb} = e^{-iF}$, see Appendix A, equation (55). Note that this coupling picks out the required photon number observable of the field. We then have $\mathbf{G}_{fb} = (e^{-iF}, 0, 0)$ and so

$$\mathbf{G}_{cl} = (e^{-iF}, e^{-iF}L, H_0).$$

This is illustrated in Figure 18. The resulting Heisenberg equation agrees with the results obtained by Wiseman, [24, eq. (3.44)], which we write in our notation as

$$\begin{aligned} dX &= (-i[X, H_0] + \mathcal{L}_{e^{-iF}L}(X))dt + (e^{iF}Xe^{-iF} - X)d\Lambda \\ &\quad + e^{iF}[X, e^{-iF}L]dA^* + [L^* e^{iF}, X]e^{-iF}dA. \end{aligned} \tag{49}$$

[2]An ideal homodyne detector HD takes an input field A and produces a quadrature, say $A + A^*$ (real quadrature), thus effecting a measurement. This is achieved routinely to good accuracy in optics laboratories, [8, Chapter 8].

(*Technical aside.* Note that if we set $E(t) = E\Lambda(t)$, with E self-adjoint, then the Stratonovich equation $dV(t) = -idE(t) \circ V(t) \equiv -idE(t)V(t) - \frac{i}{2}dE(t)dV(t)$ is equivalent to $dV(t) = S_{fb}d\Lambda(t)V(t)$ where $S_{fb} = \frac{1-\frac{i}{2}E}{1+\frac{i}{2}E}$. Therefore the implicit form [24] is not the Stratonovich form [10].)

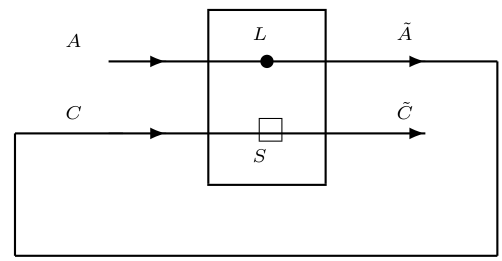

Fig. 18. Representation of the direct photocount feedback scheme of Figure 17 as a reducible network.

2) Quadrature Measurement: Here we take $K_{fb}(t) = F(A^*(t) + A(t))$ in which case $\mathbf{G}_{fb} = (1, -iF, 0)$, see Appendix A, equation (55). The skew-symmetry of $-iF$ ensures that the coupling selects the desired field quadrature observable. After feedback, the closed loop system is

$$\mathbf{G}_{cl} = (1, L - iF, H_0 + \frac{1}{2}(FL + L^*F))$$

using (39). This is illustrated in Figure 19. The resulting Heisenberg equation then agrees with [24, eq. (4.21)], which we write as

$$\begin{aligned} dX &= (-i[X, H_0 + \frac{1}{2}(FL + L^*F)] + \mathcal{L}_{L-iF}(X))dt \\ &\quad + [X, (L - iF)]dA^* + [(L - iF)^*, X])dA. \end{aligned} \tag{50}$$

(*Technical aside.* Note that for diffusions (that is, no gauge terms) the Holevo generator and Stratonovich generator coincide: that is, $dV(t) = (e^{-idK_{fb}(t)} - 1)V(t)$ is the same as $dV(t) = -idK_{fb}(t) \circ V(t)$, Appendix A.)

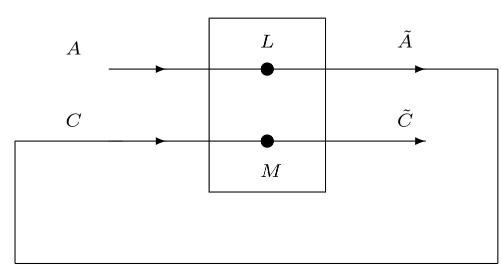

Fig. 19. Representation of the direct homodyne feedback scheme (Figure 17 with HD replacing PD) as a reducible network.

C. Realistic Detection

Consider a quantum system $\mathbf{G}_q$ continuously monitored by observing the real quadrature $\tilde{B} + \tilde{B}^*$ of an output field $\tilde{B}$. This measurement can ideally be carried out by homodyne detection, but due to finite bandwidth of the electronics and electrical noise, this measurement could be more accurately modeled by introducing a classical system (low pass filter) and additive noise as shown in Figure 20, as analyzed in [23]. Here,

B is a vacuum field, I is the output of the ideal homodyne detector (HD), v is a standard Wiener process, and Y is the (integral of) the electric current providing the measurement information.

We wish to derive a filter to estimate quantum system variables X_q from the information available in the measurement Y.

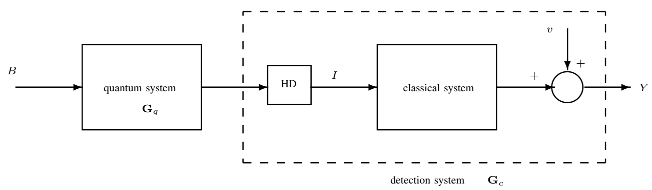

Fig. 20. Model of a realistic detection scheme for a quantum system, showing ideal homodyne detection followed by a classical system (e.g. low pass filter) and additive classical noise.

The quantum system is given by $\mathbf{G}_q = (1, L_q, H_q)$, and the classical detection system is given by the classical stochastic equations

$$\begin{aligned} dx(t) &= \tilde{f}(x(t))dt + g(x(t))dw(t), \\ dY(t) &= h(x(t))dt + dv(t), \end{aligned} \qquad (51)$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $\tilde{f}, g$ are smooth vector fields, h is a smooth real-valued function, and w and v are independent standard classical Wiener processes. As described in the Appendix D, this classical system is equivalent to a commutative subsystem of $\mathbf{G}_c = (1, L_{c1}, H_c) \boxplus (1, L_{c2}, 0)$, where $L_{c1} = -ig^T p - \frac{1}{2}\nabla^T g$, $L_{c2} = \frac{1}{2}h$ and $H_c = \frac{1}{2}(f^T p + p^T f)$. We represent the system of Figure 20 as a redicible network, as shown in Figure 21.

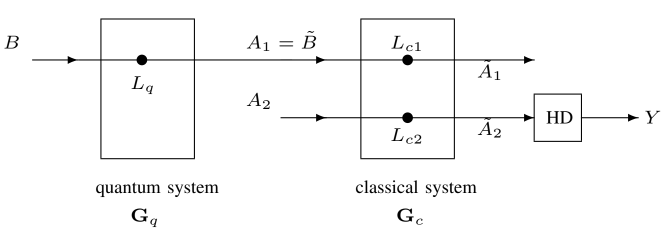

Fig. 21. Representation of the realistic detection scheme of Figure 20 as a reducible network.

Here, the classical noises are represented as real quadratures $w = A_1 + A_1^*$, $v = A_2 + A_2^*$. Note that since L_{c1} is skew-symmetric, only the real quadrature $w = A_1 + A_1^* = \tilde{B} + \tilde{B}^*$ affects the classical system (this captures the ideal homodyne detection). The complete cascade system is

$$\begin{aligned} \mathbf{G} &= ((1, L_{c1}, H_c) \triangleleft (1, L_q, H_q)) \boxplus (1, L_{c2}, 0) \qquad (52) \\ &= (\mathbf{I}, \begin{pmatrix} L_1 + L_{c1} \\ L_{c2} \end{pmatrix}, H_q + H_c + \frac{1}{2i}(L_{c1}^* L_q - L_q^* L_{c1})) \end{aligned}$$

Applying quantum filtering [1], [2], the unnormalized quantum filter for the cascade system $\mathbf{G}$ is

$$\begin{aligned} d\sigma_t(X) &= \sigma_t(-i[X, H_q + H_c + \frac{1}{2i}(L_{c1}^* L_q - L_q^* L_{c1})] \\ &+ \mathcal{L}_{\begin{pmatrix} L_1 + L_{c1} \\ L_{c2} \end{pmatrix}}(X))dt + \sigma_t(L_{c2}^* X + X L_{c2})dy. \end{aligned} \qquad (53)$$

Here, X is any operator defined on the quantum-classical cascade system. For instance, $X = X_q \otimes \varphi$, where φ is a smooth real valued function on $\mathbb{R}^n$. In particular, if $X = X_q$ is a quantum system operator, one can compute the desired estimate of X_q from $\pi_t(X_q) = \sigma_t(X_q)/\sigma_t(1)$.

Equation (53) can be normalized, and compared with [23, eq. (17)]. In the case that the quantum system is a linear gaussian system, and the filter is a linear system, the complete filter reduces to a Kalman filter from which the desired quantum system variables can be estimated.

VII. Conclusion

In this paper we have presented algebraic tools for modeling quantum networks. The tools include a parametric representation for open quantum systems, and the concatenation and series products. The concatenation product allows us to form a larger system from components, without necessarily including connections. The series product, through the principle of series connections (Theorem 5.5), provides a mechanism for combining systems via field mediated connections. We demonstrated how to model a class of quantum networks, called reducible networks, using our theory and we illustrated our results by examining some examples from the literature.

Future work will involve further development of the network theory described here, and applying the theory to develop control engineering tools and to applications in quantum technology, e.g. [16].

Appendix

A. Time-Ordered Exponentials in the sense of Holevo

Holevo [14] developed a parameterization of open system dynamics that is different to the Hudson-Parthasarathy parameters $\mathbf{G} = (S, L, H)$. Holevo's parameterization is defined as follows. Let

$$K(t) = H_{00}t + H_{01}A(t) + H_{10}A^*(t) + H_{11}\Lambda(t), \qquad (54)$$

where $\{H_{\alpha\beta}\}$ consists of bounded operators with $H_{\alpha\beta} = H_{\beta\alpha}$, and the indices α, β range from 0 to 1 (here we are considering a single field channel for simplicity). The *time-ordered exponential with Holevo generator* $\{H_{\alpha\beta}\}$ is the unitary adapted process U satisfying the quantum stochastic differential equation

$$dU(t) = \left(e^{-idK(t)} - 1\right) U(t)$$

with $U(0) = 1$, [14], [9]. Expanding the differential $e^{-idK(t)} - 1$ we obtain

$$dU(t) = \sum_{n \geq 1} \frac{(-i)^n}{n!} (dK)^n U(t).$$

Now for a system with parameters $\mathbf{G} = (S, L, H)$ we have

$$dU(t) = \{(S - I)d\Lambda(t) + LdA^*(t) - L^*dA - (iH + \frac{1}{2}L^*L)dt\}U(t).$$

Comparing these expressions, we find that

$$S = \exp(-iH_{11}), L = \frac{\exp(-iH_{11}) - 1}{H_{11}} H_{10},$$
$$H = H_{00} - H_{01} \frac{H_{11} - \sin(H_{11})}{(H_{11})^2} H_{10}. \tag{55}$$

The relationship between the generating coefficients $H_{\alpha\beta}$ and the parameters $\mathbf{G} = (S, L, H)$ are exactly as occur in the implicit-explicit formalism of [24], however, this formalism only coincides with the Stratonovich-Itō correspondence in the case where $H_{11} = 0$ [10].

B. *Proof of Theorem 5.5*

There are a number of independent derivations of the series product. For instance it can be derived from a purely Hamiltonian formalism for quantum networks [11], alternatively Gardiner's arguments in the Heisenberg picture can be extended to include the scattering terms [12]. Here we present a discretization argument for the input/output fields based on [9]. Rather than considering a continuous noise source, we take a beam consisting of qubits (spin one-half particles) with a rate of one qubit every τ seconds. A qubit has the Hilbert space $\mathsf{H} = \mathbb{C}^2$ spanned by a pair of orthogonal vectors e_0 and e_1. We define raising/lowering operators $\sigma^{\pm}$ for each qubit by $\sigma^+(\alpha e_0 + \beta e_1) = \alpha e_1$ and $\sigma^-(\alpha e_0 + \beta e_1) = \beta e_0$. In our model of the interaction of a qubit with a given plant, we shall assume that the interaction is much shorter than τ so that at most one qubit may interacting with a given plant at any instant of time. For two plants in cascade, we shall take them to be separated so that the time of flight of the qubits is exactly τ seconds. This is purely for convenience and can be easily relaxed. For definiteness, we assume that each qubit is prepared independently in the "ground state" e_0 and we denote by $\sigma_k^{\pm}$ the raising/lowering operators for the kth qubit: the operators corresponding to different qubits commute, while we have $\sigma_k^- \sigma_k^+ + \sigma_k^+ \sigma_k^- = 1$, $\left(\sigma_k^+\right)^2 = 0 = \left(\sigma_k^-\right)^2$. At time $t_k = k\tau$ $(k \in \mathbb{N})$, we take the most recent qubit to interact with the first system to be the kth qubit, and the most recent to interact with the second to be the $(k-1)$st qubit.

Let us denote the value of $x > 0$ rounded down to the nearest whole number by $\lfloor x \rfloor$ and set

$$\sigma_\tau^{\alpha\beta}(k) := \left[\frac{\sigma_k^+}{\sqrt{\tau}}\right]^\alpha \left[\frac{\sigma_k^-}{\sqrt{\tau}}\right]^\beta$$

where α, β may take the values zero and one and where $[B]^0 = 1$, $[B]^1 = B$ for any operator B. In the following, we shall denote by $O(\tau^n)$ any expression which is norm-convergent to zero as $\tau \to 0$ as fast as τ^n. The identity $\tau\sigma_\tau^{\alpha 1}(k)\sigma_\tau^{1\beta}(k) = \sigma_\tau^{\alpha\beta}(k) + O(\tau)$ will be important in what follows and will correspond to the discrete version of the second order Itō products. For $t > 0$ fixed, the processes

$$A_\tau^{\alpha\beta}(t) := \tau \sum_{k=1}^{\lfloor t/\tau \rfloor} \sigma_\tau^{\alpha\beta}(k)$$

are well-known approximations to the fundamental processes $A^{\alpha\beta}(t)$ in the limit $\tau \to 0^+$, [9].

We shall fix bounded operators $H_j^{\alpha\beta}$ on the jth system such that $H_j^{\alpha\beta\dagger} = H_j^{\beta\alpha}$ and set $\mathcal{H}_\tau^{(j)}(k) = H_j^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k)$. We shall first recall some well known results [9] for the situation where the qubits interact with only the first system (that is, set $H_2^{\alpha\beta} = 0$). The discrete time evolution is described by unitary kicks every τ seconds according to $U_\tau(t) = \mathcal{U}_{\lfloor t/\tau \rfloor} \cdots \mathcal{U}_2 \mathcal{U}_1$ where $\mathcal{U}_k = \exp\left\{-\mathrm{i}\tau \mathcal{H}_\tau^{(1)}(k)\right\}$. Expanding the exponential yields $\mathcal{U}_k = 1 + \tau G_1^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k) + O(\tau^2)$ with the $G_1^{\alpha\beta}$ forming the coefficients of the unitary QSDE with parameters $\mathbf{G}_1$ related to $\mathbf{H}_1 = \left\{H_{\alpha\beta}^{(1)}\right\}$ as in Appendix A.

In the limit $\tau \to 0^+$, the discrete time process $U_\tau(t)$ converges weakly in matrix elements to the solution of the QSDE

$$dU(t) = G_1^{\alpha\beta} \otimes dA^{\alpha\beta}(t)\, U(t).$$

We now turn to the case of a cascaded system. This time the discrete time dynamics is given by $V_\tau(t) = \mathcal{V}_{\lfloor t/\tau \rfloor} \cdots \mathcal{V}_2 \mathcal{V}_1$ where $\mathcal{V}_k = \exp\left\{-\mathrm{i}\tau \mathcal{H}_\tau^{(1)}(k) - \mathrm{i}\tau \mathcal{H}_\tau^{(2)}(k-1)\right\}$. Expanding the exponential now yields

$$\mathcal{V}_k = 1 + \tau G_1^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k) + \tau G_2^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k-1) + O(\tau^2).$$

with the $G_2^{\alpha\beta}$ forming the coefficients of the unitary QSDE with parameters $\mathbf{G}_2$ related to $\mathbf{H}_2$ as in Appendix A.

To better understand what is going on, we compute

$$\begin{aligned}\mathcal{V}_k \mathcal{V}_{k-1} &= 1 + \tau G_1^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k) \\ &+ \tau \left\{G_2^{\alpha\beta} + G_1^{\alpha\beta} + G_2^{\alpha 1} G_1^{1\beta}\right\} \otimes \sigma_\tau^{\alpha\beta}(k-1) \\ &+ \tau G_2^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k-2) + O(\tau^2).\end{aligned}$$

This may be iterated to give

$$\begin{aligned}\mathcal{V}_k \mathcal{V}_{k-1} \cdots \mathcal{V}_l &= \\ 1 + \tau &\left\{G_2^{\alpha\beta} + G_1^{\alpha\beta} + G_2^{\alpha 1} G_1^{1\beta}\right\} \otimes \sum_{j=l}^{k-1} \sigma_\tau^{\alpha\beta}(k-1) \\ &+ \tau G_1^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(k) + \tau G_2^{\alpha\beta} \otimes \sigma_\tau^{\alpha\beta}(l-1) + O(\tau^2).\end{aligned}$$

Under the same mode of convergence as before, we obtain the limit QSDE

$$dV_t = G_{\alpha\beta}^{(2\leftarrow 1)} \otimes dA^{\alpha\beta}(t)\, V(t)$$

where we recognize $G_{(2\leftarrow 1)}^{\alpha\beta} = G_2^{\alpha\beta} + G_1^{\alpha\beta} + G_2^{\alpha 1} G_1^{1\beta}$ as the coefficients the unitary QSDE with the series product parameters $\mathbf{G}_2 \triangleleft \mathbf{G}_1$, see (39). Therefore $\mathbf{G}_{2\leftarrow 1} \equiv \mathbf{G}_2 \triangleleft \mathbf{G}_1$. The generalization to multi-dimensional noise is straightforward.

C. *Proof of Theorem 5.6*

Clearly, if (40) is satisfied, then both cascade systems are described by the same parameters, which implies that they are equivalent. Now suppose the two systems are parametrically equivalent, with $\mathbf{S}_2'$ undetermined. Now by Definition 5.3 we may obtain expressions for $\mathbf{G}_2 \triangleleft \mathbf{G}_1$ and $\mathbf{G}_1 \triangleleft \mathbf{G}_2'$. Equating the first terms, we have $\mathbf{S}_2\mathbf{S}_1 = \mathbf{S}_1\mathbf{S}_2'$, and solving for $\mathbf{S}_2'$ one obtains $\mathbf{S}_2' = \mathbf{S}_1^\dagger \mathbf{S}_2 \mathbf{S}_1$, as in (41). Next, equating the second terms gives $\mathbf{L}_2 + \mathbf{S}_2\mathbf{L}_1 = \mathbf{L}_1 + \mathbf{S}_1\mathbf{L}_2'$. This expression can be solved for $\mathbf{L}_2'$, as in (41). Similarly, the Hamiltonian term H_2' in (41) can be found by equating the third terms.

D. Classical Systems as Commutative Quantum Subsystems

In this subsection we explain how to model the classical system (51), shown in Figure 22, as a commutative subsystem of a larger quantum system. This representation is used in subsection VI-C. In equation (51), $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $\tilde{f}, g$ are smooth vector fields, h is a smooth real-valued function, and w and v are independent standard classical Wiener processes.

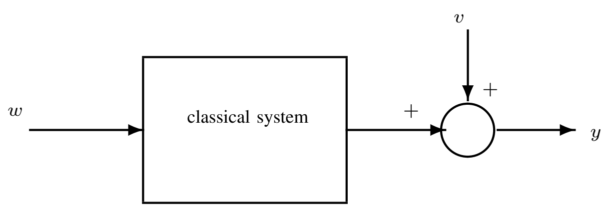

Fig. 22. Block diagram of the classical system (51).

To model this classical system, we take the underlying Hilbert space of the system to be $\mathfrak{h} = L_2(\mathbb{R}^n)$ with q^j, p_j being the usual canonical position and momentum observables: $q^j\psi(x) = X_j\psi(x)$ and $p_j\psi(x) = -i\partial_j\psi(\vec{x})$. We write $q = (q^1, \ldots, q^n)^T$, $p = (p_1, \ldots, p_n)^T$, and $\nabla = (\partial_1, \ldots, \partial_n)^T$. If φ is a smooth function of x, then we find that, by Itō's rule, for $\varphi_t = \varphi(x(t))$,

$$d\varphi = \mathfrak{L}_{\text{classical}}(\varphi)\, dt + g^T\nabla\varphi\, dw, \tag{56}$$

where $\mathfrak{L}_{\text{classical}}(\varphi) = f^T\nabla\varphi + \frac{1}{2}g^T\nabla\left(g^T\nabla\varphi\right)$ is the (classical) generator of the diffusion process $x(t)$ in (51).

We seek a quantum network representation $\mathbf{G}_c$, as shown in Figure 23.

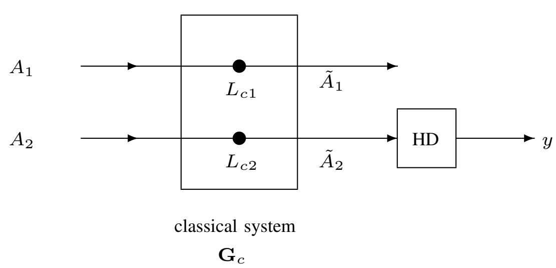

Fig. 23. Network representation of the classical system (51) shown in Figure 22.

The classical noises are viewed as real quadratures of quantum noises $w = A_1 + A_1^*$, $v = A_2 + A_2^*$. Now define port operators $L_{c1} = -ig^Tp - \frac{1}{2}\nabla^T g$, $L_{c2} = \frac{1}{2}h$ and internal Hamiltonian $H_c = \frac{1}{2}\left(f^Tp + p^Tf\right)$, where $f = \tilde{f} - \frac{1}{2}[\nabla g]g$ (the Stratonovich drift) and g are n-vectors whose components are viewed as functions of q and $h = h(q)$ is viewed as a self-adjoint observable function of q. We claim that the classical system (51) behaves as an invariant commutative subsystem of the open quantum system $\mathbf{G}_c = (1, L_{c1}, H_c) \boxplus (1, L_{c2}, 0)$.

To verify this assertion, we examine the dynamics. From (31) we have

$$\begin{aligned} dX_c &= (-i[X_c, H_c] + \mathcal{L}_{L_{c1}}(X_c) + \mathcal{L}_{L_{c2}}(X_c))dt \\ &+ [X_c, L_{c1}](dA_1^* + dA_1) + [X_c, L_{c2}](dA_2^* - dA_2) \end{aligned} \tag{57}$$

Now set $X_c = \varphi = \varphi(q)$, a smooth function of the position operator. Then (57) gives

$$\begin{aligned} d\varphi &= (-i[\varphi, H_c] + \mathcal{L}_{L_{c1}}(\varphi) + \mathcal{L}_{L_{c2}}(\varphi))dt \\ &\quad + [\varphi, L_{c1}](dA_1^* + dA_1) + [\varphi, L_{c2}](dA_2^* - dA_2) \\ &= (f^T\nabla\varphi + \frac{1}{2}g^T\nabla\left(g^T\nabla\varphi\right))dt + g^T\nabla\varphi\, dw, \end{aligned} \tag{58}$$

where, we have used $-i[\varphi, H_c] = f^T\nabla\varphi$, $\mathcal{L}_{L_{c1}}(\varphi) = \frac{1}{2}g^T\nabla(g^T\nabla\varphi)$, $\mathcal{L}_{L_{c2}}(\varphi) = 0$, $[\varphi, L_{c1}] = g^T\nabla\varphi$, and $[\varphi, L_{c2}] = 0$. Hence the classical dynamics (56) is embedded in the dynamics of the position observable q only in the quantum system $\mathbf{G}_q$ (independent of momentum dynamics). Note that only the real quadrature of the input field affects these dynamics, and they are unaffected by the field A_2.

Next we look at the outputs. The first output is not of interest, so we focus on the second one. The output $y(t)$ of the homodyne detector HD in Figure 23 is

$$dy = d\tilde{A}_2 + d\tilde{A}_2^* = (L_{c2} + L_{c2}^*)dt + dA_2 + dA_2^* = hdt + dv \tag{59}$$

which agrees with (51), as required. The unnormalized quantum filter for $\mathbf{G}_c$ is

$$\begin{aligned} d\sigma_t(X_c) &= \sigma_t(-i[X_c, H_c] + \mathcal{L}_{L_{c1}}(X_c) + \mathcal{L}_{L_{c2}}(X_c))dt \\ &\quad + \sigma_t(L_{c2}^*X_c + X_cL_{c2})dy. \end{aligned} \tag{60}$$

When $X_c = \varphi$, this reduces to

$$d\sigma_t(\phi) = \sigma_t(\mathfrak{L}_{\text{classical}}(\varphi))dt + \sigma_t(h\varphi)dy, \tag{61}$$

which is the usual Duncan-Mortensen-Zakai equation of classical nonlinear filtering, [5, Chapter 18].

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