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Abstract

Candidates to the least perimeter partition of the surface of a sphere into N planar connected regions are calculated for $N \leq 32$. A search procedure based upon random shuffling and combinatorial enumeration is used. It is conjectured that the optimal configuration for each $N > 13$ consists of 12 pentagons and $N-12$ hexagons.

1 Introduction

The surface energy of a two-dimensional foam is simply its perimeter multiplied by surface tension (Weaire and Hutzler, 1999). A foam attains a local minimum of this perimeter, subject to the constraint of fixed bubble volumes. Here, we seek the arrangement of bubbles that gives the global minimum.

The local structure of perimeter-minimizing bubble clusters is well defined: perimeter minimization implies Plateau's rules (Plateau, 1873; Taylor, 1976): three and only three edges meet at a point at 120°. The Laplace Law relating pressure difference and curvatures gives the further condition that each edge is a circular arc.

For bubbles tiling the plane, the hexagonal honeycomb gives the least perimeter (Hales, 2001). Here, we tackle the problem of tiling the sphere, for which, because of the curvature, non-hexagonal bubbles must be introduced. We assume that each bubble is connected (Morgan, 2000) and seek the least perimeter partition of the sphere into N cells of equal area, equivalent to the energetic groundstate for N monodisperse bubbles or the optimal packing of equal-area objects. We examine values of N up to 32 and record the least perimeter and the configuration that realizes it.

2 Results

We consider a sphere of radius $R = 1$, centred at the origin, and tile it with N bubbles of area $A = 4\pi R^2/N$. We use the Surface Evolver (Brakke, 1992), in a "spherical arc mode" that represents each edge as an arc of a great circle, to minimize the total perimeter P.

We commence by covering the sphere with curvilinear triangles that have their base on the equator and their apices at one of the poles. By sequentially allowing neighbour switching topological changes on short edges (Weaire and Rivier, 1984), and converging to a local equilibrium after each one, the perimeter of the pattern is reduced. We continue this process until the perimeter ceases to decrease, and then introduce further topological changes at random to search the

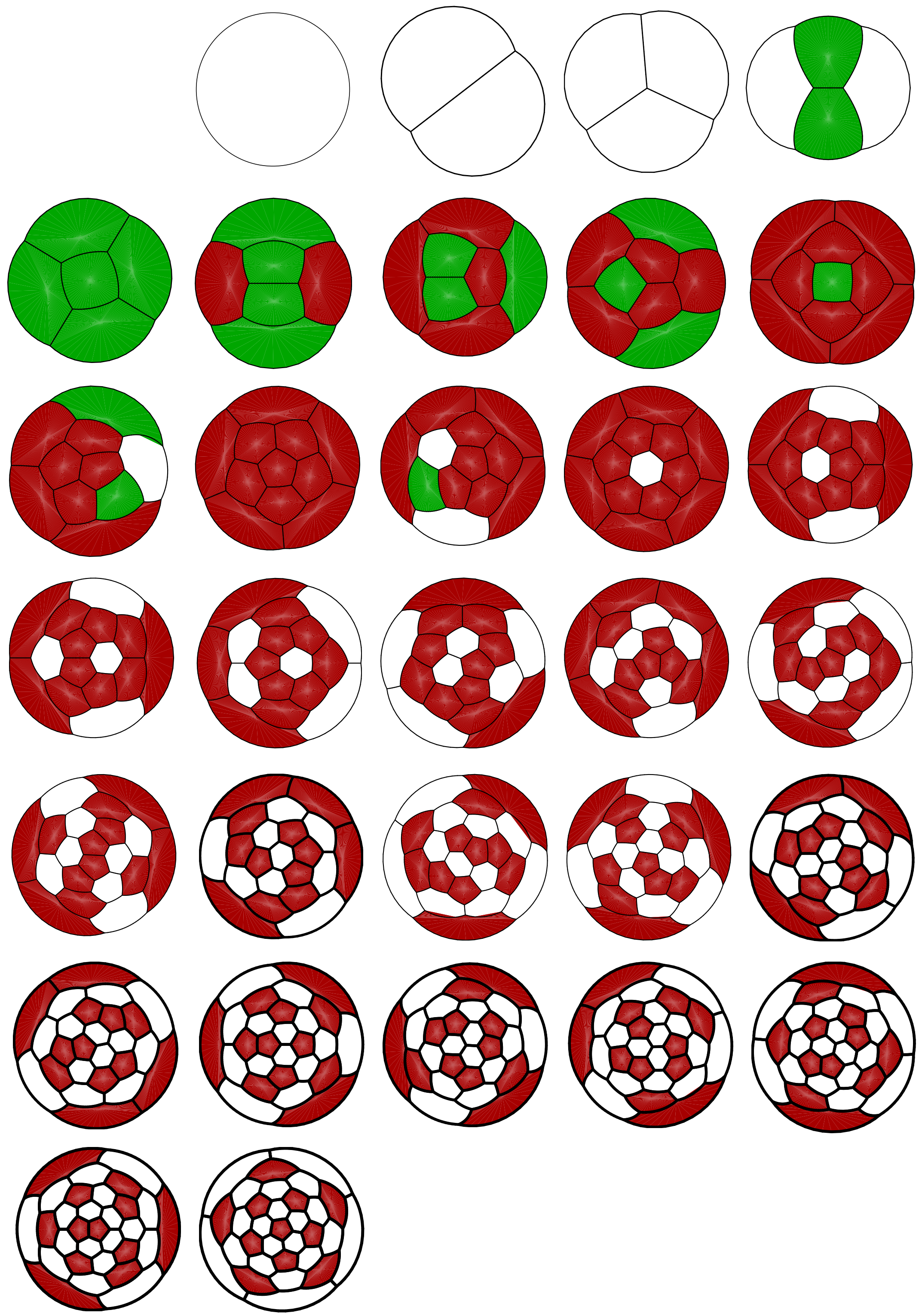

Figure 1: Candidate solutions for the least line length configuration of N bubbles on the surface of a sphere of unit radius, for $2 \leq N \leq 32$. The representation is based upon a Gauss map of the sphere surface to the plane (see text).

nearby energy "landscape". We record the perimeter and the pattern of the topology with the least perimeter.

For small N many of the candidates are as expected. For $N = 2$ there are two hemispheres. For $N = 3$ there are three identical strips joining the poles. $N = 4$ has tetrahedral symmetry, and $N = 6$ cubic. $N = 5$ consists of a pair of triangles covering the poles joined by 3 quadrilaterals, and $N = 7$ consists of a pair of pentagons covering the poles joined by 5 quadrilaterals. $N = 10$ has quadrilaterals at the poles and two rows of four pentagons. $N = 12$ is based upon the pentagonal dodecahdron. For $N = 13$ it is not possible to insert just one hexagon, so this is the highest N for which a quadrilateral bubble appears; in fact, it has the same topology as the Matzke cell, one of the most common types of bubble in 3D monodisperse foams (Matzke, 1946; Kraynik et al., 2003).

For $N \geq 14$ it is apparent that all candidates found consist of 12 pentagons and $N-12$ hexagons. These are fullerenes, now well known from carbon chemistry. We therefore introduced a further refinement: the software CaGe (Brinkmann et al., 1997) was used to enumerate *all* tilings of the sphere by hexagons and pentagons. Each of these was imported into the Surface Evolver and its equilibrium perimeter found. This showed that the random search procedure above was in general only finding optimal candidates for $N \leq 20$. and culminated in the confirmation that for $N = 32$ the optimal candidate is the C_{60} fullerene, in which each pentagon is separated from the other pentagons by hexagons. Thus for $N \geq 14$ we conjecture that the best candidate can be found by finding the optimal location of the 12 pentagons in a partition that otherwise consists of hexagons.

Candidate configurations are shown in figure 1, and the perimeters are tabulated in Table 1. The images are obtained by projecting the vertices and edges to the plane according to

$$x' = \left(\frac{1}{2}\pi + \tan^{-1}\left(\frac{z}{x^2+y^2}\right)\right)\cos(\tan^{-1}\left(\frac{y}{x}\right)) \quad (1)$$

$$y' = \left(\frac{1}{2}\pi + \tan^{-1}\left(\frac{z}{x^2+y^2}\right)\right)\sin(\tan^{-1}\left(\frac{y}{x}\right)) \quad (2)$$

$$z' = -1. \quad (3)$$

3 Conclusion

We have found candidates to the minimal perimeter of partitions of a sphere into N regions of equal area. Equivalently, we have found the global energetic groundstate of a two-dimensional foam confined to the surface of a sphere.

For $N \geq 14$ all candidates are fullerenes. Thus, we conjecture that finding the least perimeter partition of the sphere for large N is equivalent to the problem of finding the fullerene with the largest spacing between pentagonal faces.

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N	P/R	Topology
1	0.000000	–
2	6.283185	1_2
3	9.424777	2_3
4	11.463799	3_4
5	13.451848	$3_2 4_3$
6	14.771513	4_6
7	16.360476	$4_5 5_2$
8	17.710843	$4_4 5_4$
9	18.867143	$4_3 5_6$
10	20.015199	$4_2 5_8$
11	21.162841	$4_2 5_8 6_1$
12	21.891830	5_{12}
13	23.111641	$4_1 5_{10} 6_2$
14	23.964333	$5_{12} 6_2$
15	24.890808	$5_{12} 6_3$
16	25.735884	$5_{12} 6_4$

N	P/R	Topology
17	26.648955	$5_{12} 6_5$
18	27.478297	$5_{12} 6_6$
19	28.290079	$5_{12} 6_7$
20	29.015432	$5_{12} 6_8$
21	29.792431	$5_{12} 6_9$
22	30.528181	$5_{12} 6_{10}$
23	31.246108	$5_{12} 6_{11}$
24	31.932606	$5_{12} 6_{12}$
25	32.639238	$5_{12} 6_{13}$
26	33.289733	$5_{12} 6_{14}$
27	33.918489	$5_{12} 6_{15}$
28	34.574601	$5_{12} 6_{16}$
29	35.230303	$5_{12} 6_{17}$
30	35.843586	$5_{12} 6_{18}$
31	36.416686	$5_{12} 6_{19}$
32	36.951330	$5_{12} 6_{20}$

Table 1: Perimeter P/R and topology of the minimal candidates found here.

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