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Random Fuzzy Alternating Renewal Processes

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Abstract

Random fuzzy theory offers an appropriate mechanism to model random fuzzy phenomena, with a random fuzzy variable defined as a function from a credibility space to a collection of random variables. Based on this theory, this paper presents the results of an investigation into the representation of properties of alternating renewal processes that are described by sequences of positive random fuzzy vectors. It provides a theorem on the limit value of the average chance of a given random fuzzy event in terms of "system being on at time t". The resultant model coincides with that attainable by stochastic analysis when the random fuzzy vectors degenerate to random vectors.

Keywords: Fuzzy variable; Stochastic process; Renewal process; Random fuzzy variable; Interarrival time

1 Introduction

Alternating renewal process is one of the most common and popular processes in renewal theory. In classical alternating renewal processes, one of assumptions is that the process behaviour can be fully characterized by probability theory with parameters such as interarrival times or system lifetimes assumed to be random variables. Details of classical alternating renewal processes can be found in reference texts [2], [3], [5], [23], [25] and [27].

However, estimation of the probability distributions of such parameters can be very difficult in many processes due to the uncertainties and imprecision of data. Fuzzy set theory has been introduced to develop the renewal theory by several authors, in an effort to avoid or minimise this difficulty. In fuzzy renewal theory, the interarrival times and other variables are often deemed to be known inexactly and characterized as fuzzy variables. For example, Dozzi *et al* provided a limit theorem for counting renewal processes indexed by fuzzy sets [4]. Zhao and Liu discussed a fuzzy renewal process depicted by a sequence of positive fuzzy variables and established the fuzzy elementary renewal theorem and renewal reward theorem [28]. Hong considered a renewal process in which the inter-arrival times and rewards are characterized as fuzzy variables which are operated on with t norm-based fuzzy operators [6].

Nevertheless, in practice, fuzziness and randomness may jointly occur within one process. Two distinct approaches can be applied to deal with this. One is to employ the fuzzy random theory [8] [9] which informally speaking describes process variables by a measurable function from a probability space to a collection of fuzzy sets. Based on this theory, models for fuzzy random renewal processes have been proposed in the literature. In particular, Hwang investigated a renewal process in which the interarrival times are considered as independent and identically distributed (iid) fuzzy random variables, providing a usful theorem on the fuzzy rate of a fuzzy random renewal process [7]. Popova and Wu considered a renewal reward process with fuzzy random interarrival times and rewards, focusing their attention on the long-run average fuzzy reward per unit time [22]. Zhao *et al* discussed two kinds of process—fuzzy random renewal processes and fuzzy random renewal reward processes [31]. In modelling the former, they presented the fuzzy random elementary renewal theorem on the limit value of the expected renewal rate of the process; and in dealing with the latter, they proved the fuzzy random renewal reward theorem on the limit value of the long-run expected reward per unit time. Also, further interesting properties of fuzzy random renewal processes, e.g. fuzzy random Blackwell's renewal theorem and Smith's key renewal theorem were given in [30]. In addition, fuzzy random homogeneous Poisson processes, fuzzy random compound Poisson processes and fuzzy random delayed renewal processes have been introduced in [10] [11].

The other approach is based on the random fuzzy theory [13]. Briefly, a random fuzzy variable is a function from a credibility space to a collection of random variables. This theory is applied to extend the scope of renewal process modelling and to enhance the utility of such models for more complex applications. In particular, a random fuzzy renewal process model has been built [29] with a random fuzzy elementary renewal theorem and the respective version of Blackwell's theorem established. To further develop such work, this paper is set to describe a theory on the representation of alternating renewal processes that are each depicted underlyingly by a sequence of random fuzzy vectors. Throughout the paper emphasis is put on the limit value of the average chance of the event that "the system is on at time t".

The remainder of this paper is organized as follows. In Sections 2 and 3, basic concepts and properties regarding fuzzy variables and random fuzzy variables are briefly reviewed, in terms of their features as defined on a credibility space [13][15][17]. In Section 4, random fuzzy alternating renewal processes are formalised, including a theorem on the limit value of the average chance of the event that a system under consideration is deemed working at a given time.

2 Fuzzy Variables on Credibility Space

Let ξ be a fuzzy variable on a credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ (for the concept of the credibility space, see [13][15][17]), where Θ is a universe, $\mathcal{P}(\Theta)$ is the power set of Θ and Cr is a credibility measure defined on $\mathcal{P}(\Theta)$.

Definition 1 *([16]) Let ξ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then its membership function is derived from the credibility measure by*

$$\mu_\xi(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \Re.$$

Definition 2 *Let ξ be a fuzzy variable and $\alpha \in (0, 1]$. Then*

$$\xi_\alpha^{'} = \inf\{r \mid \mu_\xi(r) \geq \alpha\} \quad \text{and} \quad \xi_\alpha^{''} = \sup\{r \mid \mu_\xi(r) \geq \alpha\} \tag{1}$$

are called the α-pessimistic value and α-optimistic value of ξ, respectively.

Definition 3 *([17]) Let ξ be a fuzzy variable. The expected value $E[\xi]$ is defined as*

$$E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r \tag{2}$$

provided that at least one of the two integrals is finite. Especially, if ξ is a nonnegative fuzzy variable, then $E[\xi] = \int_0^\infty \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r$.

Proposition 1 *([19]) Let ξ be a fuzzy variable with finite expected value $E[\xi]$. Then*

$$E[\xi] = \frac{1}{2}\int_0^1 \left[\xi_\alpha^{'} + \xi_\alpha^{''}\right] \mathrm{d}\alpha, \tag{3}$$

where $\xi_\alpha^{'}$ *and* $\xi_\alpha^{''}$ *are the* α*-pessimistic value and the* α*-optimistic value of* ξ*, respectively.*

Definition 4 *([20]) The fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be independent if and only if*

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} = \min_{1\le i\le n} \mathrm{Cr}\{\xi_i \in B_i\} \tag{4}$$

for any sets $B_1, B_2, \cdots, B_n$ *of* $\Re$*.*

Proposition 2 *([19]) Let* ξ_1 *and* ξ_2 *be two independent fuzzy variables with finite expected values. Then for any real numbers* a *and* b*,* $E[a\xi_1 + b\xi_2] = aE[\xi_1] + bE[\xi_2]$*.*

Definition 5 *([14]) The fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be identically distributed if and only if*

$$\mathrm{Cr}\{\xi_i \in B\} = \mathrm{Cr}\{\xi_j \in B\},\ i, j = 1, 2, \cdots, n \tag{5}$$

for any set B *of* $\Re$*.*

Proposition 3 *([14]) Let* $(\Theta_i, P_i(\Theta), \mathrm{Cr}_i)$*,* $i = 1, 2, \cdots$*, be an arbitrary sequence of credibility spaces and* $\Theta = \prod_{i=1}^{\infty} \Theta_i$*. Define* Cr *on* $P(\Theta)$ *such that*

$$\mathrm{Cr}\{(\theta_1, \theta_2, \cdots)\} = \mathrm{Cr}_1\{\theta_1\} \wedge \mathrm{Cr}_2\{\theta_2\} \wedge \cdots.$$

Then the function Cr *is a fuzzy measure on* $P(\Theta)$ *and* $(\Theta, P(\Theta), \mathrm{Cr})$ *is a credibility space (called an infinite product credibility space of* $(\Theta_i, P_i(\Theta), \mathrm{Cr}_i)$*,* $i = 1, 2, \cdots$*).*

Remark 1 *Let* $(\Theta, P(\Theta), \mathrm{Cr})$ *be an infinite product credibility space of* $(\Theta_i, P_i(\Theta), \mathrm{Cr}_i)$*,* $i = 1, 2, \cdots$ *For any* $A \in P(\Theta)$*,*

$$\mathrm{Cr}\{A\} = \begin{cases} \sup\limits_{(\theta_1,\theta_2,\cdots)\in A} \min\limits_{1\le i} \mathrm{Cr}_i\{\theta_i\}, \\ \qquad \text{if} \sup\limits_{(\theta_1,\theta_2,\cdots)\in A} \min\limits_{1\le i} \mathrm{Cr}_i\{\theta_i\} < 0.5 \\ 1 - \sup\limits_{(\theta_1,\theta_2,\cdots)\in A^c} \min\limits_{1\le i} \mathrm{Cr}_i\{\theta_i\}, \\ \qquad \text{if} \sup\limits_{(\theta_1,\theta_2,\cdots)\in A} \min\limits_{1\le i} \mathrm{Cr}_i\{\theta_i\} \ge 0.5. \end{cases}$$

3 Random Fuzzy Variables on Credibility Space

In the presentation below, it is assumed that $(\Omega, \mathcal{A}, \Pr)$ is a probability space, and that $\mathcal{F}$ is a collection of random variables defined on probability space $(\Omega, \mathcal{A}, \Pr)$.

Definition 6 *([13]) A random fuzzy variable is a function from a credibility space* $(\Theta, P(\Theta), \mathrm{Cr})$ *to a collection of random variables* $\mathcal{F}$*.*

Example 1 *Let* X *be the lifetime of a system under consideration. Then, it might be known that the lifetime* X *is an exponentially distributed random variable with an unknown mean* λ*,*

$$\phi(x) = \begin{cases} \dfrac{1}{\lambda} e^{-x/\lambda}, & \text{if } 0 \leq x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

In statistic theory, an interval estimate or point estimate of the value of λ *is provided by sufficient experiment data. In many practical situations, however, there often lacks such data. If the value of* λ *is provided as a fuzzy variable defined on the credibility space* $(\Theta, P(\Theta), \mathrm{Cr})$*, then* X *is a random fuzzy variable defined as*

$$X(\lambda(\theta)) \sim \mathcal{EXP}(\lambda(\theta)),$$

where λ *is a fuzzy variable on* $(\Theta, P(\Theta), \mathrm{Cr})$*, and* $\mathcal{EXP}(\cdot)$ *stands for exponential distribution.*

Remark 2 *([29]) If* Θ *consists of a single element, then a random fuzzy variable degenerates to a random variable. For instance, in (1), the fuzzy variable* $\lambda(\theta)$ *becomes a crisp number when there is only a single element* $\theta \in \Theta$*. In such case, the random fuzzy system lifetime* X *degenerates to an exponentially distributed random variable. If* $\mathcal{F}$ *is a collection of real numbers (rather than random variables), then Definition 6 superposes the definition of a fuzzy variable. In such a case, a random fuzzy variable also degenerates to a fuzzy variable.*

Definition 7 *A random fuzzy variable* ξ *defined on the credibility space* $(\Theta, P(\Theta), \mathrm{Cr})$ *is said to be positive if and only if* $\Pr\{\xi(\theta) \leq 0\} = 0$ *for each* $\theta \in \Theta$ *with* $\mathrm{Cr}\{\theta\} > 0$*.*

Proposition 4 *([14]) Let* ξ *be a random fuzzy variable defined on the credibility space* $(\Theta, P(\Theta), \mathrm{Cr})$*. Then, for* $\theta \in \Theta$*,*

(i) $\Pr\{\xi(\theta) \in B\}$ *is a fuzzy variable for any Borel set* B *of* $\Re$*; and*

(ii) $E[\xi(\theta)]$ *is a fuzzy variable provided that* $E[\xi(\theta)]$ *is finite for each* $\theta \in \Theta$*.*

Definition 8 *([19]) Let* ξ *be a random fuzzy variable defined on the credibility space* $(\Theta, P(\Theta), \mathrm{Cr})$*. The expected value* $E[\xi]$ *is defined by*

$$E[\xi] = \int_0^{\infty} \mathrm{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite. Especially, if ξ *is a nonnegative random fuzzy variable, then* $E[\xi] = \int_0^\infty \mathrm{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \geq r\}\mathrm{d}r$.

Remark 3 *([13]) If the random fuzzy variable* ξ *degenerates to a random variable, then the expected value operator becomes*

$$E[\xi] = \int_0^\infty \mathrm{Pr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Pr}\{\xi \leq r\}\mathrm{d}r,$$

which is just the conventional mathematical expectation of random variable ξ. *If a random fuzzy variable* ξ *degenerates to a fuzzy variable, then the expected value operator becomes*

$$E[\xi] = \int_0^\infty \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r,$$

which is just the expected value of fuzzy variable ξ.

Definition 9 *([18]) Let* ξ *be a random fuzzy variable. Then the average chance, denoted by* Ch, *of a random fuzzy event characterized by* $\{\xi \leq 0\}$ *is defined as*

$$\mathrm{Ch}\{\xi \leq 0\} = \int_0^1 \mathrm{Cr}\left\{\theta \in \Theta \,\middle|\, \mathrm{Pr}\left\{\xi(\theta) \leq 0\right\} \geq p\right\} \mathrm{d}p. \tag{6}$$

Remark 4 *If* ξ *degenerates to a random variable, then the average chance degenerates to*

$$\mathrm{Pr}\left\{\xi \leq 0\right\},$$

which is just the probability of random event.

Remark 5 *If* ξ *degenerates to a fuzzy variable, then the average chance degenerates to*

$$\mathrm{Cr}\left\{\xi \leq 0\right\},$$

which is just the credibility of fuzzy event.

Proposition 5 *([18]) Let* ξ *be a random fuzzy variable. Then*

$$\mathrm{Ch}\{\xi > 0\} = 1 - \mathrm{Ch}\{\xi \leq 0\}. \tag{7}$$

Definition 10 ([12]) *Random fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent if*

(1) $\xi_1(\theta)$, $\xi_2(\theta)$, $\cdots, \xi_n(\theta)$ *are independent random variables for each* θ; *and*

(2) $E[\xi_1(\cdot)]$, $E[\xi_2(\cdot)]$, $\cdots$, $E[\xi_n(\cdot)]$ *are independent fuzzy variables.*

Definition 11 ([12]) *The random fuzzy variables* ξ *and* η *are identically distributed if*

$$\sup_{\mathrm{Cr}\{A\}\geq\alpha}\inf_{\theta\in A}\{\Pr\{\xi(\theta)\in B\}\}=\sup_{\mathrm{Cr}\{A\}\geq\alpha}\inf_{\theta\in A}\{\Pr\{\eta(\theta)\in B\}\}$$

for any $\alpha\in(0,1]$ *and Borel set* B *of real numbers.*

Finally, the following shows useful observations regarding stochastic ordering which is usually employed for comparison of renewal processes.

Definition 12 *A collection of random variables* $\mathcal{F}$ *is said to be a totally ordered set with stochastic ordering if and only if, for any given* ζ_1,$\zeta_2\in\mathcal{F}$ *and* $r\in\Re$, *either*

$$\Pr\{\zeta_1\leq r\}\leq\Pr\{\zeta_2\leq r\}\quad(\textit{denoted by }\zeta_2\leq_d\zeta_1)$$

or

$$\Pr\{\zeta_1\leq r\}\geq\Pr\{\zeta_2\leq r\}\quad(\textit{denoted by }\zeta_1\leq_d\zeta_2).$$

Remark 6 *It follows from Definition 12 that, for any given* $\zeta_1,\zeta_2\in\mathcal{F}$,

$$E[\zeta_1]\leq E[\zeta_2]\Leftrightarrow\zeta_1\leq_d\zeta_2. \tag{8}$$

A number of common families of random variables, which satisfy (8), have been discussed in [24]; these include such as families of exponential distributions, Poisson distributions, and normal distributions as well as other families of nonnegative random variables.

Lemma 1 *([29]) Assume that* $\{\zeta_i,i\geq1\}$ *and* $\{\zeta_i^{'},i\geq1\}$ *are two mutually independent sequences of random variables. Let* $S_n=\sum\limits_{i=1}^{n}\zeta_i$ *and* $S_n^{'}=\sum\limits_{i=1}^{n}\zeta^{'}$, $n=1,2\cdots$ *If* $\zeta_i\leq_d\zeta_i^{'}$, $i=1,2,\cdots$, *then* $S_n\leq_d S_n^{'}$, $n=1,2,\cdots$

Lemma 2 *Let* F_1 *and* F_2 *be the probability distribution functions of random variables* ζ_1 *and* ζ_2, *respectively. Then,* $\zeta_1\leq_d\zeta_2$ *if and only if*

$$\int_{-\infty}^{\infty}f(x)\mathrm{d}F_1(x)\leq_d\int_{-\infty}^{\infty}f(x)\mathrm{d}F_2(x)$$

for any monotonic nondecreasing function $f(x)$, $x\in\Re$.

4 Modelling Random Fuzzy Alternating Renewal Processes

Consider a system that can be in one of two operating states: *on* or *off*. Initially it is on and it remains on for a time ξ_1; it then goes off and remains off for a time η_1; it then goes on for a time ξ_2; then off for a time η_2, and so forth. Without losing generality, suppose that ξ_i, $i = 1, 2, \cdots$, are random fuzzy variables defined on credibility space $(\Theta_i, \mathrm{P}(\Theta_i), \mathrm{Cr}_i)$ and η_i are random fuzzy variables defined on credibility space $(\Gamma_i, \mathrm{P}(\Gamma_i), \mathrm{Cr}_i^{'})$. If the random fuzzy vectors (ξ_i, η_i), $i = 1, 2\cdots$ are iid, then the process depicting by the sequence $\{(\xi_i, \eta_i), i \geq 1\}$ is called a *random fuzzy alternating renewal process*, defined on the credibility space $(\Theta, \mathrm{P}(\Theta), \mathrm{Cr})$, where $(\Theta, \mathrm{P}(\Theta), \mathrm{Cr})$ is the infinite product credibility space characterized by

$$\Theta = \prod_{i=1}^{\infty} (\Theta_i, \Gamma_i)$$

and

$$\mathrm{Cr}\{((\theta_1, \gamma_1), (\theta_2, \gamma_2), \cdots)\} = \mathrm{Cr}_1\{\theta_1\} \wedge \mathrm{Cr}_1^{'}\{\gamma_1\} \wedge \mathrm{Cr}_2\{\theta_2\} \wedge \mathrm{Cr}_2^{'}\{\gamma_2\} \wedge \cdots$$

for any $\theta = ((\theta_1, \gamma_1), (\theta_2, \gamma_2), \cdots) \in \Theta$.

The rest of this research is developed on the basis of the following assumptions;

Assumptions

(a) The random fuzzy vectors (ξ_i, η_i), $i = 1, 2, \cdots$ are iid, especially, ξ_i and η_i are independent.

(b) The image sets of ξ_i and η_i are totally ordered sets with stochastic ordering, $i = 1, 2, \cdots$.

For any fixed $\theta \in \Theta$, by Definition 6, $\xi_i(\theta)$ and $\eta_i(\theta)$ are the random variables and $E[\xi_i(\theta)]$ and $E[\eta_i(\theta)]$ are just the expected values of $\xi_i(\theta)$ and $\eta_i(\theta)$. However, when θ is varied all over in Θ, $E[\xi_i(\theta)]$ and $E[\eta_i(\theta)]$, as functions of θ, are fuzzy variables and their α-pessimistic values and α-optimistic values can be written as

$$E[\xi_i(\theta)]_{\alpha}^{'} = \inf\left\{r \mid \mathrm{Pos}\{E[\xi_i(\theta)] \leq r\} \geq \alpha\right\}, \tag{9}$$

$$E[\xi_i(\theta)]_{\alpha}^{''} = \sup\left\{r \mid \mathrm{Pos}\{E[\xi_i(\theta)] \geq r\} \geq \alpha\right\}, \tag{10}$$

$$E[\eta_i(\theta)]_{\alpha}^{'} = \inf\left\{r \mid \mathrm{Pos}\{E[\eta_i(\theta)] \leq r\} \geq \alpha\right\}, \tag{11}$$

$$E[\eta_i(\theta)]_{\alpha}^{''} = \sup\left\{r \mid \mathrm{Pos}\{E[\eta_i(\theta)] \geq r\} \geq \alpha\right\}, \tag{12}$$

where $\alpha \in (0,1]$.

Stochastic alternating renewal processes focus on the limit of probability of the random event that the system under consideration is on at time t. However, in random fuzzy alternating renewal processes, the event {system is on at time t} is complicated with not only randomness but also fuzziness. Thus, it is interesting to the average chance of such as event.

It follows from Definition 9 that

$$\text{Ch}\{\text{system being on at time } t\} = \int_0^1 \text{Cr}\left\{\theta \in \Theta \,\middle|\, \Pr\{\text{system being on at time } t\} \geq p\right\} \mathrm{d}p. \quad (13)$$

For convenience, let

$$P(t) = \Pr\{\text{system being on at time } t\}. \quad (14)$$

From Proposition 4 it is know that $P(t)$ is a fuzzy variable.

Definition 13 *A positive random variable ζ is said to be lattice if and only if there exists $d \geq 0$ such that $\sum_{n=0}^{\infty} \Pr\{\zeta = nd\} = 1$.*

Theorem 1 *Let $\{(\xi_i, \eta_i), i \geq 1\}$ be a sequence of iid positive random fuzzy vectors which satisfies Assumptions (a) and (b). Assume that the distribution functions of $\xi_i(\theta)$ and $\eta_i(\theta)$, for any given $\theta \in \Theta$, are nonlattice, the α-pessimistic values and the α-optimistic values of the fuzzy variables $E[\xi_i(\theta)]$ and $E[\eta_i(\theta)]$, $i = 1, 2, \cdots$ are continuous at the point α_0, $\alpha_0 \in (0,1]$, and $E[\xi_1 + \eta_1] < \infty$, then*

$$\lim_{t \to \infty} P'_{\alpha_0}(t) = \frac{E[\xi_1(\theta)]'_{\alpha_0}}{E[\xi_1(\theta)]'_{\alpha_0} + E[\eta_1(\theta)]''_{\alpha_0}} \quad (15)$$

and

$$\lim_{t \to \infty} P''_{\alpha_0}(t) = \frac{E[\xi_1(\theta)]''_{\alpha_0}}{E[\xi_1(\theta)]''_{\alpha_0} + E[\eta_1(\theta)]'_{\alpha_0}}, \quad (16)$$

where $E[\xi_1(\theta)]'_{\alpha_0}$, $E[\xi_1(\theta)]''_{\alpha_0}$, $E[\eta_1(\theta)]'_{\alpha_0}$, $E[\eta_1(\theta)]''_{\alpha_0}$, $P'_{\alpha_0}(t)$ and $P''_{\alpha_0}(t)$ are the α_0-pessimistic values and the α_0-optimistic values of $\xi_1(\theta)$, $\eta_1(\theta)$ and $P(t)$, respectively.

Proof. Let $A_i = \{\theta_i \in \Theta_i \mid \mu(\theta_i) \geq \alpha\}$ and $B_i = \{\vartheta_i \in \Gamma_i \mid \mu(\vartheta_i) \geq \alpha\}$. Since the α-pessimistic and α-optimistic values of the fuzzy variables $E[\xi_i(\theta)]$ and $E[\eta_i(\theta)]$, $\theta \in \Theta$, $i = 1, 2, \cdots$ are continuous at the point α_0, there must exist points $\theta_{i_1}, \theta_{i_2} \in A_i$ and $\vartheta_{i_1}, \vartheta_{i_2} \in B_i$ such that

$$E[\xi_i(\theta_{i_1})] = E[\xi_i(\theta)]^{'}_{\alpha_0}, \quad E[\xi_i(\theta_{i_2})] = E[\xi_i(\theta)]^{''}_{\alpha_0}$$
$$E[\eta_i(\vartheta_{i_1})] = E[\eta_i(\theta)]^{'}_{\alpha_0}, \quad E[\eta_i(\vartheta_{i_2})] = E[\eta_i(\theta)]^{''}_{\alpha_0}. \tag{17}$$

It follows from (9) to (12) that

$$E[\xi_i(\theta_{i_1})] \leq E[\xi_i(\theta_i)] \leq E[\xi_i(\theta_{i_2})], \quad \forall \theta_i \in A_i$$

and

$$E[\eta_i(\vartheta_{i_1})] \leq E[\eta_i(\vartheta_i)] \leq E[\eta_i(\vartheta_{i_2})], \quad \forall \vartheta_i \in B_i.$$

Hence, it follows from (8) and Assumption (b) that

$$\xi_i(\theta_{i_1}) \leq_d \xi_i(\theta_i) \leq_d \xi_i(\theta_{i_2}), \quad \forall \theta_i \in A_i \tag{18}$$

and

$$\eta_i(\vartheta_{i_1}) \leq_d \eta_i(\vartheta_i) \leq_d \eta_i(\vartheta_{i_2}) \quad \forall \vartheta_i \in B_i. \tag{19}$$

Given that $E[\xi_i(\theta)]$, $i = 1, 2, \cdots$ are iid fuzzy variables, taking $\theta_{1_1} = \theta_{2_1} = \cdots$ and $\theta_{1_2} = \theta_{2_2} = \cdots$ from A_i such that $\xi_i(\theta_{i_1})$ and $\xi_i(\theta_{i_2})$ are iid random variables, $i = 1, 2, \cdots$, and $\vartheta_{1_1} = \vartheta_{2_1} = \cdots$ and $\vartheta_{1_2} = \vartheta_{2_2} = \cdots$ from B_i such that $\eta_i(\vartheta_{i_1})$ and $\eta_i(\vartheta_{i_2})$ are iid random variables, $i = 1, 2, \cdots$, then the following three processes can be obtained:

(1) Process A characterized by $\{(\xi_i(\theta_{i_1}), \eta_i(\vartheta_{i_2})), i \geq 1\}$;

(2) Process B characterized by $\{(\xi_i(\theta_{i_2}), \eta_i(\vartheta_{i_1})), i \geq 1\}$; and

(3) Process C characterized by $\{(\xi_i(\theta_i), \eta_i(\vartheta_i)), i \geq 1\}$.

It is obvious that both Processes A and B are standard stochastic alternating renewal processes.

Let

$$P_1(t) = \Pr\{\text{process A being on at time } t\}, \tag{20}$$

$$P_2(t) = \Pr\{\text{process B being on at time } t\}, \tag{21}$$

$$P_3(t) = \Pr\{\text{process C being on at time } t\}. \tag{22}$$

Then,

$$P_1(t) \leq P_3(t) \leq P_2(t). \tag{23}$$

Without the loss of generality, it is sufficient to prove the left inequation of (23). To do so, it is presumed that a renewal takes place each time the system goes on. Conditioning on the time of that last renewal prior to or at time t yields

$$\begin{aligned} P_1(t) &= \Pr\left\{\text{process A being on at time } t \mid S_A = 0\right\} \Pr\{S_A = 0\} \\ &+ \int_0^\infty \Pr\left\{\text{process A being on at time } t \mid S_A = y\right\} \mathrm{d}F_{S_A}(y) \end{aligned}$$

and

$$\begin{aligned} P_3(t) &= \Pr\left\{\text{process C being on at time } t \mid S_C = 0\right\} \Pr\{S_C = 0\} \\ &+ \int_0^\infty \Pr\left\{\text{process C being on at time } t \mid S_C = y\right\} \mathrm{d}F_{S_C}(y), \end{aligned}$$

where S_A represents the time of the last renewal prior to or at time t in process A, S_C the time of the last renewal prior to or at time t in process C, $F_{S_A}(y)$ the distribution function of S_A and $F_{S_C}(y)$ the distribution function of S_C. Furthermore,

$$\Pr\left\{\text{process A being on at time } t \mid S_A = 0\right\} = \Pr\left\{\xi_{1,\alpha}^{'}(\omega) > t \mid \xi_{1,\alpha}^{'}(\omega) + \eta_{1,\alpha}^{''}(\omega) > t\right\},$$

$$\Pr\left\{\text{process C being on at time } t \mid S_C = 0\right\} = \Pr\left\{\xi_{\beta_1}(\omega) > t \mid \xi_{\beta_1}(\omega) + \eta_{\gamma_1}(\omega) > t\right\},$$

and for $y < t$,

$$\Pr\left\{\text{process A being on at time } t \mid S_A = y\right\} = \Pr\left\{\xi_{1,\alpha}^{'}(\omega) > t - y \mid \xi_{1,\alpha}^{'}(\omega) + \eta_{1,\alpha}^{''}(\omega) > t - y\right\},$$

$$\Pr\left\{\text{process C being on at time } t \mid S_C = y\right\} = \Pr\left\{\xi_{\beta}(\omega) > t - y \mid \xi_{\beta}(\omega) + \eta_{\gamma}(\omega) > t - y\right\},$$

where $\xi_\beta(\omega)$ and $\eta_\gamma(\omega)$ are the β-pessimistic (or β-optimistic) and γ-pessimistic (or γ-optimistic) values of the last on and off times prior to or at time t in process C.

By Lemma 1 and Lemma 2, it is obvious that

$$P_1(t) \le P_3(t). \tag{24}$$

Similarly, $P_3(t) \le P_2(t)$. Since points θ_i and ϑ_i are arbitrary in A_i and B_i, respectively, the process characterized by $\{(\xi_i(\theta_i), \eta_i(\vartheta_i)), i \ge 1\}$ is an arbitrary process defined on the σ-algebra containing all rectangles of Cartesian product $\prod_{i=1}^{\infty}(A_i, B_i) = A_1 \otimes B_1 \otimes A_2 \otimes B_2 \otimes \cdots$. Hence,

$$P_1(t) = P_{\alpha_0}^{'}(t) \text{ and } P_2(t) = P_{\alpha_0}^{''}(t) \tag{25}$$

where $P_{\alpha_0}^{'}(t)$ and $P_{\alpha_0}^{'}(t)$ are the α_0-pessimistic and α_0-optimistic values of the fuzzy variable $P(t)$.

Furthermore, using the result of stochastic alternating renewal processes (see [23]),

$$\lim_{t\to\infty} P_1(t) = \frac{E\left[\xi_1(\theta_{1_1})\right]}{E\left[\xi_1(\theta_{1_1})\right] + E\left[\eta_1(\vartheta_{1_2})\right]} \tag{26}$$

and

$$\lim_{t\to\infty} P_2(t) = \frac{E\left[\xi_1(\theta_{1_2})\right]}{E\left[\xi_1(\theta_{1_2})\right] + E\left[\eta_1(\vartheta_{1_1})\right]} \tag{27}$$

provided that $E\left[\xi_1(\theta_{1_1})\right] + E\left[\eta_1(\vartheta_{1_2})\right] < \infty$ and $E\left[\xi_1(\theta_{1_2})\right] + E\left[\eta_1(\vartheta_{1_1})\right] < \infty$. It follows from (17) that

$$\lim_{t\to\infty} P_1(t) = \frac{E\left[\xi_1(\theta)\right]^{'}_{\alpha_0}}{E\left[\xi_1(\theta)\right]^{'}_{\alpha_0} + E\left[\eta_1(\theta)\right]^{''}_{\alpha_0}} \tag{28}$$

and

$$\lim_{t\to\infty} P_2(t) = \frac{E\left[\xi_1(\theta)\right]^{''}_{\alpha_0}}{E\left[\xi_1(\theta)\right]^{''}_{\alpha_0} + E\left[\eta_1(\theta)\right]^{'}_{\alpha_0}}. \tag{29}$$

Finally, it follows from (25) that the results of (15) and (16) hold. The theorem is proved.

Remark 7 *If $\{(\xi_i, \eta_i), i \geq 1\}$ degenerates to a sequence of iid random vectors, then the results of (15) and (16) in Theorem 1 degenerate to the form*

$$\lim_{t\to\infty} \Pr\{\text{system being on at time } t\} = \frac{E\left[\xi_1\right]}{E\left[\xi_1\right] + E\left[\eta_1\right]},$$

which is just the conventional result in stochastic case (see [23]).

Remark 8 *If $\{(\xi_i, \eta_i), i \geq 1\}$ degenerates to a sequence of fuzzy vectors with the same membership function, then, for each $\alpha_0 \in (0, 1]$, the α_0-pessimistic and α_0-optimistic values of ξ_1 and η_1 degenerate to four real numbers (denoted by $\xi^{'}_{\alpha_0}$, $\xi^{''}_{\alpha_0}$, $\eta^{'}_{\alpha_0}$, $\eta^{''}_{\alpha_0}$). The results of (15) and (16) in Theorem 1 respectively degenerate to the form*

$$\lim_{t\to\infty} P^{'}_{\alpha_0}(t) = \frac{\xi^{'}_{\alpha_0}}{\xi^{'}_{\alpha_0} + \eta^{''}_{\alpha_0}}$$

and

$$\lim_{t\to\infty} P^{''}_{\alpha_0}(t) = \frac{\xi^{''}_{\alpha_0}}{\xi^{''}_{\alpha_0} + \eta^{'}_{\alpha_0}}. \tag{30}$$

Remark 9 *Let*

$$Q(t) = \Pr\{\text{system being off at time } t\} \tag{31}$$

which is a fuzzy variable by Proposition 4. Also, let $Q'_{\alpha_0}(t)$ *and* $Q''_{\alpha_0}(t)$ *be the* α_0*-pessimistic and* α_0*-optimistic values of* $Q(t)$*, respectively. Then, under the conditions of Theorem 1,*

$$\lim_{t\to\infty} Q'_{\alpha_0}(t) = \frac{E\left[\eta_1(\theta)\right]'_{\alpha_0}}{E\left[\xi_1(\theta)\right]''_{\alpha_0} + E\left[\eta_1(\theta)\right]'_{\alpha_0}} \tag{32}$$

and

$$\lim_{t\to\infty} Q''_{\alpha_0}(t) = \frac{E\left[\eta_1(\theta)\right]''_{\alpha_0}}{E\left[\xi_1(\theta)\right]'_{\alpha_0} + E\left[\eta_1(\theta)\right]''_{\alpha_0}}. \tag{33}$$

Theorem 2 *Let* $\{(\xi_i, \eta_i), i \geq 1\}$ *be a sequence of iid positive random fuzzy vectors which satisfies Assumptions (a) and (b). Assume that the distribution functions of* $\xi_i(\theta)$ *and* $\eta_i(\theta)$*, for any given* $\theta \in \Theta$*, are nonlattice,* $E\left[\xi_1 + \eta_1\right] < \infty$ *and* $E\left[\dfrac{\xi_1}{\xi_1+\eta_1}\right] < \infty$*, then*

$$\lim_{t\to\infty} \mathrm{Ch}\{\text{system being on at time } t\} = E\left[\frac{\xi_1}{\xi_1+\eta_1}\right]. \tag{34}$$

Proof. It follows from Definition 9 and Proposition 1 that

$$\begin{aligned}\mathrm{Ch}\{\text{system being on at time } t\} &= \int_0^1 \mathrm{Cr}\left\{\theta \in \Theta \;\middle|\; \Pr\{\text{system being on at time } t\} \geq p\right\} \mathrm{d}p \\ &= \frac{1}{2}\int_0^1 \left(P'_\alpha(t) + P''_\alpha(t)\right) \mathrm{d}\alpha.\end{aligned} \tag{35}$$

By Theorem 1,

$$\lim_{t\to\infty} P'_\alpha(t) = \frac{E\left[\xi_1(\theta)\right]'_\alpha}{E\left[\xi_1(\theta)\right]'_\alpha + E\left[\eta_1(\theta)\right]''_\alpha} \tag{36}$$

and

$$\lim_{t\to\infty} P''_\alpha(t) = \frac{E\left[\xi_1(\theta)\right]''_\alpha}{E\left[\xi_1(\theta)\right]''_\alpha + E\left[\eta_1(\theta)\right]'_\alpha} \tag{37}$$

provided that $E\left[\xi_1(\theta)\right]'_\alpha$, $E\left[\xi_1(\theta)\right]''_\alpha$, $E\left[\eta_1(\theta)\right]'_\alpha$ and $E\left[\eta_1(\theta)\right]''_\alpha$ are continuous at point α, $\alpha \in (0, 1]$. Furthermore, it follows from the definition of the limit that there exist two real numbers t_1 and t_2 with $t_1 > 0$ and $t_2 > 0$ such that

$$0 \leq P'_\alpha(t) \leq 1 + \frac{E\left[\xi_1(\theta)\right]'_\alpha}{E\left[\xi_1(\theta)\right]'_\alpha + E\left[\eta_1(\theta)\right]''_\alpha}, \qquad \forall t \geq t_1 \tag{38}$$

and

$$0 \leq P''_\alpha(t) \leq 1 + \frac{E\left[\xi_1(\theta)\right]''_\alpha}{E\left[\xi_1(\theta)\right]''_\alpha + E\left[\eta_1(\theta)\right]'_\alpha}, \qquad \forall t \geq t_2. \tag{39}$$

Therefore, for any $t \geq \max(t_1, t_2)$,

$$0 \leq P'_\alpha(t) + P''_\alpha(t) \leq 2 + \frac{E\left[\xi_1(\theta)\right]'_\alpha}{E\left[\xi_1(\theta)\right]'_\alpha + E\left[\eta_1(\theta)\right]''_\alpha} + \frac{E\left[\xi_1(\theta)\right]''_\alpha}{E\left[\xi_1(\theta)\right]''_\alpha + E\left[\eta_1(\theta)\right]'_\alpha}. \tag{40}$$

Note that

$$E\left[\frac{\xi_1}{\xi_1+\eta_1}\right]=\frac{1}{2}\int_0^1\left(\frac{E\left[\xi_1(\theta)\right]_\alpha^{'}}{E\left[\xi_1(\theta)\right]_\alpha^{'}+E\left[\eta_1(\theta)\right]_\alpha^{''}}+\frac{E\left[\xi_1(\theta)\right]_\alpha^{''}}{E\left[\xi_1(\theta)\right]_\alpha^{''}+E\left[\eta_1(\theta)\right]_\alpha^{'}}\right)\mathrm{d}\alpha. \tag{41}$$

Since $E\left[\frac{\xi_1}{\xi_1+\eta_1}\right]$ is finite, then $\frac{E[\xi_1(\theta)]_\alpha^{'}}{E[\xi_1(\theta)]_\alpha^{'}+E[\eta_1(\theta)]_\alpha^{''}}+\frac{E[\xi_1(\theta)]_\alpha^{''}}{E[\xi_1(\theta)]_\alpha^{''}+E[\eta_1(\theta)]_\alpha^{'}}$ is an integrable function of $\alpha\in(0,1]$. Hence, it can be deduced that $2+\left(\frac{E[\xi_1(\theta)]_\alpha^{'}}{E[\xi_1(\theta)]_\alpha^{'}+E[\eta_1(\theta)]_\alpha^{''}}+\frac{E[\xi_1(\theta)]_\alpha^{''}}{E[\xi_1(\theta)]_\alpha^{''}+E[\eta_1(\theta)]_\alpha^{'}}\right)$ is an integrable function of $\alpha\in(0,1]$.

It follows from Fatou's lemma that

$$\liminf_{t\to\infty}\int_0^1\left(P_\alpha^{'}(t)+P_\alpha^{''}(t)\right)\mathrm{d}\alpha\geq\int_0^1\liminf_{t\to\infty}\left(P_\alpha^{'}(t)+P_\alpha^{''}(t)\right)\mathrm{d}\alpha$$

and

$$\limsup_{t\to\infty}\int_0^1\left(P_\alpha^{'}(t)+P_\alpha^{''}(t)\right)\mathrm{d}\alpha\leq\int_0^1\limsup_{t\to\infty}\left(P_\alpha^{'}(t)+P_\alpha^{''}(t)\right)\mathrm{d}\alpha.$$

Since

$$\lim_{t\to\infty}\left(P_\alpha^{'}(t)+P_\alpha^{''}(t)\right)=\frac{E\left[\xi_1(\theta)\right]_\alpha^{'}}{E\left[\xi_1(\theta)\right]_\alpha^{'}+E\left[\eta_1(\theta)\right]_\alpha^{''}}+\frac{E\left[\xi_1(\theta)\right]_\alpha^{''}}{E\left[\xi_1(\theta)\right]_\alpha^{''}+E\left[\eta_1(\theta)\right]_\alpha^{'}}$$

and $E\left[\xi_1(\theta)\right]_\alpha^{'}$, $E\left[\xi_1(\theta)\right]_\alpha^{''}$, $E\left[\eta_1(\theta)\right]_\alpha^{'}$ and $E\left[\eta_1(\theta)\right]_\alpha^{''}$ are almost surely continuous functions (see [26]) of α, $\alpha\in(0,1]$, it follows that

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Ch}\{\text{system being on at time } t\}&=\frac{1}{2}\int_0^1\lim_{t\to\infty}\left(P_\alpha^{'}(t)+P_\alpha^{''}(t)\right)\mathrm{d}\alpha\\&=\frac{1}{2}\int_0^1\left(\frac{E\left[\xi_1(\theta)\right]_\alpha^{'}}{E\left[\xi_1(\theta)\right]_\alpha^{'}+E\left[\eta_1(\theta)\right]_\alpha^{''}}+\frac{E\left[\xi_1(\theta)\right]_\alpha^{''}}{E\left[\xi_1(\theta)\right]_\alpha^{''}+E\left[\eta_1(\theta)\right]_\alpha^{'}}\right)\mathrm{d}\alpha\\&=E\left[\frac{\xi_1}{\xi_1+\eta_1}\right].\end{aligned}$$

The proof is completed.

Remark 10 *If $\{(\xi_i,\eta_i),i\geq 1\}$ degenerates to a sequence of iid random vectors, then the result in Theorem 2 degenerates to the form*

$$\lim_{t\to\infty}\Pr\{\text{system being on at time } t\}=\frac{E\left[\xi_1\right]}{E\left[\xi_1\right]+E\left[\eta_1\right]},$$

which is just the conventional result in stochastic case (see [23]).

Remark 11 *If $\{(\xi_i,\eta_i),i\geq 1\}$ degenerates to a sequence of fuzzy vectors, then, the result in Theorem 2 degenerates to the form*

$$\lim_{t\to\infty}\mathrm{Cr}\{\text{system being on at time } t\}=E\left[\frac{\xi_1}{\xi_1+\eta_1}\right].$$

Remark 12 *Under the conditions of Theorem 2, it follows from Proposition 5 that*

$$\begin{aligned} \lim_{t\to\infty} \mathrm{Ch}\{\text{system being off at time } t\} &= 1 - \lim_{t\to\infty} \mathrm{Ch}\{\text{system being on at time } t\} \\ &= E\left[\frac{\eta_1}{\xi_1+\eta_1}\right]. \end{aligned} \tag{42}$$

Theorem 2 extends the asymptotic result in the stochastic case to the random fuzzy case, enabling more complex alternating renewal processes to be modelled.

5 Conclusions

It is well known that probability theory provides the mathematical foundation for stochastic renewal processes, while possibility theory provides the mathematical foundation for fuzzy renewal processes. In this paper, random fuzzy theory is introduced to further develop the alternating renewal theory so that more complex processes, especially, those with fuzziness and randomness, can be modelled. In a random fuzzy alternating renewal process, the event "system being on at time t" is a random fuzzy event and thus the average chance can be used to measure it. By considering the α-pessimistic and α-optimistic values of random fuzzy vectors, useful properties of random fuzzy alternating renewal processes can be readily elicited from the foregone results of the conventional alternating renewal processes. Such techniques can be extended to coping with other processes and applications. For example, in the domain of crime prevention, when analyzing forensic data, it might be known that the concentration of aluminium in c-glass may be an exponentially distributed variable associated with an unknown parameter [1]. If this parameter is itself regarded as a fuzzy variable, then the aluminium concentration is a random fuzzy variable.

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