The Annals of Mathematical Statistics
1969, Vol. 40, No. 5, 1840–1843

NEAR-CYCLIC REPRESENTATIONS FOR SOME RESOLUTION VI FRACTIONAL FACTORIAL PLANS

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0. Summary. Draper and Mitchell (1968) gave the complete set of even 512-run designs of resolution $\geqq 6$ and the complete set of 256-run designs of resolution $\geqq 5$. The authors showed that each of these designs can be obtained from one or other of five Reference Designs, all of resolution 6. The present note gives near-cyclic representations of the complete sets of interactions in the identity relationships of the Reference Designs. Some of these sets are shown to be related to certain incomplete block designs, including a resolvable balanced incomplete block design of a type that seems to have eluded attention hitherto.

1. The representations. Addelman (1965) gave a 2^{17-9} design of resolution 5, i.e. a design for 17 "factors" or "variables" each at 2 levels, with $2^{17-9} = 256$ "runs" or "plots", and such that each identity (i.e. interaction in the "defining relation" or "identity relationship") contains not less than 5 factors. For brevity, the design may be denoted as 2_{V}^{17-9}, the Roman number subscript indicating the resolution. Preece (1966) showed that, for $k = 5, 6, \cdots, 12$, the set of k-factor identities of Addelman's design can be generated cyclically from a few of its members, and is "equivalent" to the set of blocks of a partially balanced incomplete block design with two associate classes, 17 treatments, and k units per block. By "equivalent" is meant that each identity is in 1–1 correspondence with a block of k units, and each factor is in 1–1 correspondence with a treatment of the incomplete block design.

Draper and Mitchell (1968) gave the complete set of even 512-run designs of resolution $\geqq 6$ and the complete set of 256-run designs of resolution $\geqq 5$. Each of these designs, including Addelman's, was shown to be obtainable from one or other of five blocked "Reference Designs", all of resolution 6. The 512-run designs are constructed and blocked by deleting factors from one of the Reference Designs, i.e. by omitting identities and confounded interactions that contain the factors to be deleted; the 256-run designs are obtained by deleting factors and erasing a single factor, the erased factor simply being removed from any identities and confounded interactions which contain it.

Because of the basic position taken by the Reference Designs, it was decided to find whether their identities could be written as cyclically generated sets in a way similar to that used for Addelman's design. Tables 1 and 2 show that near-cyclic representations are possible for all five designs, but that some of the representations are cumbersome.

Received 11 December 1968; revised 6 March 1969.

TABLE 1

How Draper and Mitchell's factors are renamed for Table 2

No. of Reference Design	Dimensions of design	Draper and Mitchell's numbering of the factors																	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
6.1	2_{VI}^{15-6}	b_1	a_1	d_1	e_3	e_2	d_2	b_2	b_3	d_3	e_1	a_2	a_3	c_3	c_2	c_1	—	—	—
6.3	2_{VI}^{15-6}	a_1	a_2	a_3	c_3	c_2	d_3	d_2	e_3	e_2	c_1	d_1	e_1	b_3	b_2	b_1	—	—	—
7.1	2_{VI}^{16-7}	a_1	C	b_1	A	a_3	d_1	c_1	a_2	B	d_3	D	c_2	b_2	b_3	c_3	d_2	—	—
7.3	2_{VI}^{16-7}	c_4	b_1	a_1	a_3	a_2	c_3	b_3	d_3	d_4	d_1	a_4	c_2	b_4	d_2	c_1	b_2	—	—
9.1	2_{VI}^{18-9}	15	4	2	13	12	9	17	3	1	5	10	7	11	16	14	8	6	$\underline{18}$

In Table 2, the notation

$$a_2a_3b_1c_1d_1e_1 \qquad (123),\ (ab)$$

is used to denote interaction $a_2a_3b_1c_1d_1e_1$ and the five other interactions obtainable from it by repeated independent use of the cyclic substitutions (123) and (ab). The notation for design 9.1 follows that used by, for example, Fisher and Yates (1963) in their table of balanced incomplete block designs; in particular, $\underline{18}$ is used to denote a factor invariant under the cyclic permutation modulo 17.

The various sets of k-factor identities of designs 6.1 and 7.1 are not equivalent to balanced or partially balanced incomplete block designs. The sets of 6-factor and 8-factor identities of 6.3 are equivalent to partially balanced incomplete block designs with three associate classes. The various sets of k-factor identities of 7.3 and 9.1 are all (except the sets consisting solely of the interaction of all factors) equivalent to balanced incomplete block designs.

In Table 2, the same notation is used for both the 6-factor and the 8-factor identities of design 7.3. However, if the factors are renamed as follows:

Names used in Table 2	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4	d_1	d_2	d_3	d_4
New names	2	4	6	1	8	$\underline{16}$	9	5	14	15	7	10	13	3	11	12

the 8-factor identities can be rewritten

$$[(4\ 5\ 6\ 7\ 9\ 11\ 12\ 15),\ (1\ 2\ 3\ 8\ 10\ 13\ 14\ \underline{16})] \quad \text{mod } 15$$

Here, $\underline{16}$ is invariant under the cyclic permutation modulo 15; the square brackets indicate that the identities in them contain one replicate of all the factors. The balanced incomplete block design is thus, in the notation of Rao (1961), the resolvable solution $E(4, 2){:}3$.

The balanced incomplete block design given by the 6-factor identities of design 9.1 (see Table 2) has six generator blocks from which all other blocks are obtained by cyclic substitutions modulo 17, treatment $\underline{18}$ being an invariant. These six blocks are grouped into two sets of three, each set forming a replicate.

TABLE 2

Words (i.e. interactions) in the defining relations of Reference Designs; cyclic notation

Design	Word length	No. of words	Words	
6.1	6	28	$a_1b_1d_1e_1e_2e_3$	(bc)(de), (123)
			$a_2a_3b_2b_3d_2d_3$	(bc)(de), (123)
			$a_2b_2b_3c_3d_1e_3$	(bc)(de), (123)
			$a_3b_2b_3c_2d_1e_2$	(bc)(de), (123)
			$b_1c_1d_2d_3e_2e_3$	(123)
			$b_1b_2b_3c_1c_2c_3$	
	8	21	$a_2a_3b_1c_1c_2c_3d_2d_3$	(bc)(de), (123)
			$a_1a_2a_3b_2b_3c_1d_1e_1$	(bc)(de), (123)
			$a_2a_3b_2c_3d_1d_3e_1e_2$	(bc)(de), (123)
			$b_2b_3c_2c_3d_2d_3e_2e_3$	(123)
	10	12	$a_1b_1b_2b_3c_2c_3d_1d_2d_3e_1$	(bc)(de), (123)
			$a_1a_2b_2b_3c_1c_3d_2d_3e_1e_3$	(bc)(de), (123)
	12	2	$a_1a_2a_3b_1b_2b_3d_1d_2d_3e_1e_2e_3$	(bc)(de)
6.3	6	25	$a_2a_3b_1c_1d_1e_1$	(123), (ab) †
			$a_1a_2a_3c_1c_2c_3$	(cde), (ab)
			$a_1b_1c_2c_3d_1e_1$	(123), (cde)
			$c_1c_2c_3d_1d_2d_3$	(cde)
			$a_1a_2a_3b_1b_2b_3$	
	8	30	$a_1b_2b_3c_1d_2d_3e_2e_3$	(123), (cde), (ab) †
			$a_2a_3b_2b_3c_2c_3d_1e_1$	(123), (cde)
			$a_1b_1c_2c_3d_2d_3e_2e_3$	(123)
	10	3	$a_2a_3b_2b_3c_2c_3d_2d_3e_2e_3$	(123)
	12	5	$b_1b_2b_3c_1c_2c_3d_1d_2d_3e_1e_2e_3$	(abcde)
7.1	6	45	$a_2a_3b_1b_3c_1c_2$	(ac)(bd); (ab)(cd)(CD), (123)
			$b_2c_2c_3d_3AC$	(ac)(bd); (ab)(cd)(CD), (123)
			$a_2a_3b_2c_3BC$	(ac)(bd)(CD); (ab)(cd)(CD), (123)
			$a_2a_3c_2d_3AB$	(ab)(cd)(CD), (123)
			$a_1b_1c_1d_1CD$	(123)
	8	41	$a_1a_2a_3b_2c_1d_3CD$	(ac)(bd); (ab)(cd)(CD), (123)
			$a_1a_3b_2c_2d_1d_3BC$	(cd)(AB); (ab)(cd)(CD), (123)
			$a_1b_2c_1c_2ABCD$	(ab)(cd)(CD), (123)
			$a_1a_3b_1b_3c_2d_2AB$	(123)
			$a_2a_3b_2b_3c_2c_3d_2d_3$	(123)
			$a_1a_2a_3c_1c_2c_3AD$	(cd)(AB), (ab)(cd)(CD)
			$c_1c_2c_3d_1d_2d_3AB$	
	10	34	$a_1a_2b_1b_2b_3c_1d_1d_3BC$	(ac)(bd)(CD); (ab)(cd)(CD), (123)
			$a_1a_3b_1b_2b_3c_1c_2d_1AC$	(ac)(bd); (ab)(cd)(CD), (123)
			$a_1a_2b_1b_3c_1c_2c_3d_1AB$	(ab)(cd)(CD), (123)
			$a_1b_1c_2c_3d_2d_3ABCD$	(123)
			$a_1a_2a_3b_1b_2b_3ABCD$	
	12	6	$a_1b_1b_2b_3c_1c_3d_1d_2ABCD$	(ab)(cd), (123)
	14	1	$a_1a_2a_3b_1b_2b_3c_1c_2c_3d_1d_2d_3CD$	

TABLE 2—*Continued*

Design	Word length	No. of words		
7.3	6	48	$a_1a_2a_3b_1c_4d_1$ / $a_4b_4c_2d_1d_3d_4$ / $b_2b_3c_1c_3d_1d_2$ } $(abcd)$, (1234)	*
	8	30	$a_2a_3b_3b_4c_2c_3d_3d_4$ (1234), $(ab)(cd)$	*
			$b_1b_3c_1c_2c_3c_4d_2d_4$ $(abcd)$, $(12)(34)$	
			$a_1a_2b_1b_2c_3c_4d_3d_4$ (1234), (ac)	
			$a_1a_3b_1b_3c_1c_3d_1d_3$ / $a_1a_3b_2b_4c_1c_3d_2d_4$ } $(12)(34)$	
			$a_1a_2a_3a_4c_1c_2c_3c_4$ $(ab)(cd)$	
	10	48	Complements of words of length 6	*
	16	1	Word of all variables	
9.1	6	102	[(1 2 3 4 7 15), (5 9 10 12 13 17), (6 8 11 14 16 $\underline{18}$)]	
			[(1 2 7 9 11 13), (5 6 8 12 14 15), (3 4 10 16 17 $\underline{18}$)] mod 17	*
	8	153	(1 2 5 6 7 9 15 17), (1 3 4 5 6 8 11 15), (1 2 4 6 8 9 12 15), (1 2 3 8 9 10 13 15), (1 3 4 6 9 10 14 15), (1 3 7 11 13 15 16 $\underline{18}$), (1 3 4 5 14 15 16 $\underline{18}$), (1 2 3 4 5 10 13 $\underline{18}$), (1 2 3 7 10 11 14 $\underline{18}$) mod 17	*
	10	153	Complements of words of length 8	*
	12	102	Complements of words of length 6	*
	18	1	Word of all variables	

* The words are in 1–1 correspondence with the blocks of a balanced incomplete block design, the word length being equal to the block size

† The words are in 1–1 correspondence with the blocks of a partially balanced incomplete block design with three associate classes, the word length being equal to the block size

The design is thus resolvable, but each of its two cyclic resolvable halves is only partially balanced with three associate classes. There seems to be no record of any other cyclic resolvable balanced incomplete block design which splits into cyclic resolvable parts that are not balanced.

A further reason for interest in the near-cyclic representations of this paper is that, if one wishes to know which six-factor interactions are in the defining relation, it is much easier to use the representations than to write out the whole defining relation. (Knowledge of the six-factor interactions permits one to write down the sets of three-factor interactions that are aliases of each other.)

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