A lending scheme for a system of interconnected banks with probabilistic constraints of failure

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Abstract

We derive a closed form solution for an optimal control problem related to an interbank lending schemes subject to terminal probability constraints on the failure of banks which are interconnected through a financial network. The derived solution applies to a real banks network by obtaining a general solution when the aforementioned probability constraints are assumed for all the banks. We also present a direct method to compute the systemic relevance parameter for each bank within the network.

Key words: Hamilton-Jacobi-Bellman Equations, Optimal Control, Terminal Probability Constraints, Bank Failure.

1 Introduction

Starting from the worldwide crisis of 2007-2008, financial analysts, bank practitioners, applied mathematicians and economists, have been pushed to rethink the models they were used to work with, changing a series of assumptions turned out to be too far from real markets. Under this need for more robust mathematical model, a major focus has been put on default probabilities that any financial entity must face. Such *credit risk* analysis has seen an increasing interest in the theoretical financial community, pushing the development of mathematically rigorous models which take into account both the *risk exposure* factor and related *default events*.
Following above mentioned interest, we consider in the present work a network of interconnected financial entities and, following [6], we consider a *financial supervisor*, usually referred as *lender of last resort* (LOLR) aiming at guarantee the *wellness* of the financial network, by lending money to those agents who are near to default. The main novelty of our approach is that we assume fixed probability constraints of non defaulting the banks have to satisfy at a specific terminal time. From a financial point of view, such constraint implies that the LOLR optimal strategy has to be derived satisfying the assumption that each bank is characterized by a probability of bankruptcy. As in [13], we assume that a bank may fail only at a fixed terminal time, namely it goes under bankruptcy if, at terminal time, its wealth is below a given threshold. We remark that we assume the default event to be triggered as soon as the value of the financial entity reaches an endogenous lower threshold, so that the default time results in being a predictable stopping time with respect to the reference filtration, see, e.g., [1,2,13,14].
Using techniques related to stochastic target problems, see, [3–5,17], we will be able to derive an ad hoc *Hamilton-Jacobi-Bellman* (HJB) equation for the related optimal control providing also a closed form solution. Derived results will be applied to a concrete example. We remark that, for the sake of clarity, we will consider a small set of interconnected banks, the case of larger network being of easy derivation being the solution expressed in closed form. Also, a systematic approach to quantify the relative importance of each single player in the network, based on a *page rank* approach first introduced in [15], will be derived. This quantity will be then used to decide the admitted probability of each bank's failure, requiring that *important banks* have larger *non-failure probability*, hence adopting a *too big to fail paradigma*.
The present work is organized as follows: in Section 2 we introduce the main setting; in Section 3 we introduce the optimal control problem with probability constraints and we provide its solution; in Section 4 we present the Pagerank method for the relative importance of the banks in the network and we apply the derived results to a toy example.

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2 The general setting

Following the financial network setting proposed in [11,16], see appendix A for further details, we consider a network composed by n nodes, each of them representing a different financial agent, and we denote by $X^i(t)$ the asset value of the i^{th} agent at time $t \in [0,T]$, being $T < \infty$ a fixed positive terminal time. Each node may have nominal liabilities to other nodes directly connected with it. In this case, we denote by $L_{i,j}(t)$ the payment that the bank i owes to the bank j, at time $t \in [0,T]$. Then, we introduce the time-dependent *liabilities matrix* $\mathcal{L}(t) = (L_{i,j}(t))_{n\times n}$, being $L_{i,j}(t) \neq 0$ for $\iota^+_{i,j} \neq 0$, where, as shown in appendix A, $\iota^+_{i,j}$ is equal to one if i and j are connected, while it equals zero otherwise. In particular, $\mathcal{L}(t)$ explicitly states that there cannot be any cash flow between any two banks which are not edge-connected.

We will denote by $u_i(t)$ the payment made at time $t \in [0,T]$ by the i^{th} bank, whereas $\bar{u}_i(t) = \sum_{j=1}^n L_{i,j}(t)$ is the *total nominal obligation* of node i towards all other nodes. Therefore, if $\bar{u}_i(t) = u_i(t)$, then i has satisfied all its liabilities. We also introduce the *relative liabilities matrix* $\Pi(t) = (\pi_{i,j}(t))$ defined as $L_{i,j}(t)/\bar{u}_i(t)\mathbb{1}_{\bar{u}_i(t)>0}$. The matrix $\Pi(t)$ is row stochastic, in the sense that $\sum_{j=1}^n \pi_{i,j}(t) = 1$, so that $\pi_{i,j}(t)$ represents the proportion of the total debt at time t that the node i owes to the node j.

Similarly, we can define the cash inflow of the node i as the sum of the exogenous cash inflow $F^i(t)$ plus the total payment that node i receives at time t by other nodes, that is $\sum_{j=1}^n \pi^T_{i,j}(t)\, u_j(t)$, where we denoted the transposed of the relative liabilities matrix and its elements as $\Pi^T = (\pi^T_{i,j}(t))$. We thus have that the value of the i^{th} node at time $t \in [0,T]$ is given by

$$\bar{V}^i(t) = \sum_{j=1}^n \pi^T_{i,j}(t)\, u_j(t) + F^i(t) - \bar{u}_i(t)\,, \qquad (1)$$

see Appendix A for a formal financial treatment of the network. Following [12], we assume the liabilities between banks to evolve according the following equation

$$\frac{d}{dt} L_{i,j}(t) = \mu_{ij} L_{i,j}(t)\,, \qquad (2)$$

We also assume that the bank i, at any time t, invests the difference between cash inflow ans cash outflow in an exogenous asset $X^i(t)$ whose dynamic is given by

$$\mathrm{d}X^i(t) = X^i(t)\left(\mu^i\,\mathrm{d}t + \sigma_i\,\mathrm{d}W^i(t)\right)\,, \quad i = 1,\dots,n\,.$$

Moreover, see [12], we introduce continuous (deterministic) default boundaries as $X^i(t) \le v^i(t)$, $\mathbb{P}$ *a.s.* with

$$v^i(t) := \begin{cases} R^i\left(\bar{u}_i(t) - \sum_{j=1}^n \pi^T_{i,j}(t)\,\bar{u}_j(t)\right) & t < T\,, \\ \bar{u}_i(t) - \sum_{j=1}^n \pi^T_{i,j}(t)\,\bar{u}_j(t) & t = T\,, \end{cases} \qquad (3)$$

where $R^i \in (0,1)$, $i = 1,\dots,n$, are suitable constants representing the *recovery rate* of the bank i.

3 The stochastic optimal control with probability constraints

The present Section introduces the mathematical formulation of the main problem, expressing it as an optimal control problem with terminal probability constraint. Furthermore, we provide an analytic solution which allows us to compute the optimal controls.

In what follows we consider a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}\right)$ satisfying usual assumptions, namely right–continuity and saturation by $\mathbb{P}$–null sets. Following [6], we focus our attention on the actions played by a financial supervisor, called *Lender Of Last Resort* (LOLR), connected to any node belonging to the financial network. In particular, the LOLR aims at saving the network from default, ant it is assumed to have *full information* about the network state. Therefore, at any time t the LOLR can lend money to the bank i, $i = 1,\dots,n$, so that the controlled evolution of the bank i satisfies

$$\mathrm{d}X^i_\alpha(t) = \left(\mu^i X^i_\alpha(t) + \alpha^i(t)\right)\mathrm{d}t + \sigma_i X^i_\alpha(t)\mathrm{d}W^i(t)\,, \quad (4)$$

being $\alpha^i(t)$ the loan from the LOLR to the bank i, at time $t \in [0,T]$. Also we assume $\alpha^i(t)$ to have values in a compact set $U^i \subset \mathbb{R}$ and we define $\mathcal{A}^i$ the set of U^i−valued measurable processes. We will also employ the notation $U := U^1 \times \cdots \times U^n$ and $\mathcal{A} := \mathcal{A}^1 \times \cdots \times \mathcal{A}^n$. Accordingly, the LOLR aims at minimizing lend resources

$$J(t,\alpha) = \mathbb{E}\left[\frac{1}{2}\sum_{i=1}^n \int_t^T \alpha_i(s)^2\,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \qquad (5)$$

under the probabilistic constraint

$$\mathbb{P}\left(X^i(T) \ge v^i | \mathcal{F}_t\right) \ge q^i\,, \quad i = 1,\dots,n\,, \qquad (6)$$

for suitable constants $q^i \in (0,1)$, $i = 1,\dots,n$.

3.1 Reduction to a stochastic target problem

The current section formally introduces the *Hamilton–Jacobi–Bellman* (HJB) equation associated to the control problem defined in equation (5), subject to constraint given by equation (6). Let us underline that, in what follows and due to the structure of the optimal control problem, we will focus on a single agent i. In particular, to avoid heavy notation, we will denote for short $X := X^i$.

Recalling that the terminal probability in equation (6) can be rewritten as an expectation, namely

$$\mathbb{P}\left(X(T) \ge v \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}\left[\mathbb{1}_{\{X(T)\ge v\}} \,\middle|\, \mathcal{F}_t\right]\,,$$

then we have the following.

Lemma 1 *Given the stochastic optimal control problem with terminal probability constraint* (6)*, the terminal probability constraints holds if and only if there exists an adapted sub-martingale* $(P(s))_{s\in[t,T]}$ *such that*

$$P(t) = q\,, \quad P(T) \leq \mathbb{1}_{\{X(T)\geq v\}}\,.$$

Proof. We first prove ($\Leftarrow$): since $P(s)$ is a sub-martingale, then

$$\mathbb{E}\left[\mathbb{1}_{\{X(T)\geq v\}}\right] \geq \mathbb{E}\left[P(T)\,|\mathcal{F}_t\right] \geq P(t) = q\,.$$

To prove the converse ($\Rightarrow$), let us denote

$$\begin{aligned} q_0 &:= \mathbb{E}\left[\mathbb{1}_{\{X^s(T)\geq v\}}\right]\,, \\ P(s) &:= \mathbb{E}\left[\mathbb{1}_{\{X^s(T)\geq v\}}\middle|\,\mathcal{F}_s\right] - (q_0 - q)\,, \end{aligned}$$

where X^s represents the solution with initial time $s \in [t,T]$. Then P is an adapted martingale and the claim follows. □

When the probability constraint is *active*, the sub-martingale P is given by

$$P^i(s) = \mathbb{E}\left[\mathbb{1}_{\{X^i(T)\geq v^i\}}\middle|\,\mathcal{F}_s\right]\,,$$

hence P^i turns out to be an adapted martingale, implying the new state variable

$$P^i(s) = q^i + \int_t^T \alpha_P^i(s)\,\mathrm{d}W(s)\,, \tag{7}$$

where α_P^i, taking values in $\mathbb{R}$, is a square integrable $\mathcal{F}_s-$adapted stochastic processes representing the new control. It is worth emphasizing that α_P^i is not bounded *a priori* being derived from the martingale representation theorem. In accordance to the used notation we will denote by α_P the $n-$dimensional vector whose components are α_P^i.
Exploiting the *geometric dynamic programming principle*, see [17], we can define the following value function

$$V(t,x,q) = \inf\left\{\frac{1}{2}\mathbb{E}_t\left[\sum_{i=1}^n\int_t^T \alpha^i(s)^2\,\mathrm{d}s\right] \quad \text{s.t.} \quad \mathbb{1}_{\{X^i(T)\geq v^i\}} - P^i(T) \geq 0 \quad i=1,\ldots,n \quad \mathbb{P}\,\text{a.s.}\right\}, \tag{8}$$

where $\mathbb{E}_t$ is the conditional expectation w.r.t. v$\mathcal{F}_t$.
The Hamiltonian for the related unconstrained optimal control is

$$H^X(x,\alpha,p,Q_x) = (\mu x + \alpha)\cdot p + \frac{1}{2}\sigma^2\,x^2\,Q_x + \frac{1}{2}\|\alpha\|^2\,, \tag{9}$$

where we have denoted by

$$\mu x := (\mu^1 x^1, \ldots, \mu^n x^n)\,,$$

and

$$\sigma^2\,x^2 = diag((\sigma^1 x^1)^2, \ldots, (\sigma^n x^n)^2)\,,$$

being *diag* the $n\times n$ diagonal matrix.
Intuitively, when the terminal constraint is satisfied, the associated HJB equation, whose Hamiltonian is given in equation (9), can be solved, being $\alpha \equiv 0$ the optimal control.
Taking into account the process P along with the original state variable X, the constrained Hamiltonian can be defined as

$$\begin{aligned} &H^{(X,P)}(x,\alpha,p,Q_x,\alpha_P,Q_{xq},Q_q) = \\ &= (\mu x+\alpha)p + \frac{1}{2}\sigma^2x^2Q + \frac{1}{2}\|\alpha\|^2 + \sigma x Q_{xq}\alpha_P + \frac{1}{2}\alpha_P^2 Q_q\,, \end{aligned} \tag{10}$$

which should play the role of the Hamiltonian of the associated problem when the constraint is binding. Therefore, the HJB associated to the optimal control reads as follow

$$\begin{aligned} &-\partial_t V \\ &-\inf_{\alpha\in\mathcal{A}}\inf_{\alpha_P\in\mathbb{R}^n} H^{(X,P)}(x,\alpha,\partial_x V,\partial_x^2 V,\alpha_P,\partial_{xq}^2 V,\partial_q^2 V) = 0\,, \end{aligned}$$

and the following holds

$$H^{(X,P)}(x,\alpha,p,Q_x,\alpha_P,Q_{xq},Q_q) \geq H^X(x,\alpha,p,Q_x)\,,$$

allowing to evaluate the minimum of $H^{(X,P)}$ w.r.t. α_P, by a first order optimality condition as to have

$$\alpha_P = -\sigma\,x\,\frac{Q_{xq}}{Q_q}\,,$$

which, when plugged into equation (10), gives the following minimum for $H^{(X,P)}$

$$\begin{aligned} &\inf_{\alpha_P\in\mathbb{R}^n} H^{(X,P)} = \bar{H}(x,\alpha,p,Q_x,Q_{xq},Q_q) = \\ &= \begin{cases} (\mu x+\alpha)p + \frac{1}{2}\sigma^2x^2Q_x + \frac{1}{2}\|\alpha\|^2 - \frac{1}{2Q_q}\sigma^2x^2Q_{xq}^2 & Q_q > 0\,, \\ (\mu x+\alpha)p + \frac{1}{2}\sigma^2x^2Q_x + \frac{1}{2}\|\alpha\|^2 & Q_q = 0\,, \\ -\infty & \text{otherwise}\,. \end{cases} \end{aligned} \tag{11}$$

It follows that the associated value function, see equation (8), solves the following HJB equation

$$-\partial_t V - \inf_{\alpha\in\mathcal{A}} \bar{H}(x,\alpha,\partial_x V,\partial_x^2 V,\partial_{xq}^2 V,\partial_q^2 V) = 0\,, \tag{12}$$

subject to the terminal condition

$$V(T,x,q) = \begin{cases} 0 & x \geq v\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

where the Hamiltonian $\bar{H}$ is defined as in equation (11).

3.2 The affine control case

In order to obtain a closed form solution for the HJB equation (11) we will further assume that the admissible controls are of the form

$$\alpha^i(t) = \psi^i(t) X^i(t) \,, \tag{13}$$

for $\psi^i(t) \in [0, \Psi]$, $\Psi \in \mathbb{R}_+ \cup \{\infty\}$. We remark that, from a financial perspective, this implies that the LOLR can decide the interest rate at which the banks assets accrues, allowing the bank to have a higher interest rate to lower the probability of failure.

Given the structure of the optimal control problem, we can analyse each node i separately, where we ansatz the value function to be of the form $V(t,x,q) = \sum_{i=1}^n V^i(t,x^i,q^i)$, where each V^i is regarded as the value function for the optimal problem with respect to the element i. Thus, for each player i we compute the solution to the above problem in terms of contour line of a function $\gamma^i(t,x,q)$, defining first the boundaries of the domain for the value function V^i, then computing explicitly the contour line on the interior of the domain.

The first region Γ_0 is the region in which the constraint is not binding, implying that the optimal control is given by $\psi \equiv 0$. Financially speaking, whenever the value of the bank lies within the region Γ_0, the bank satisfies the LOLR requirement regarding survival probability.

In the second region Γ_Ψ the optimal control exceed the maximum rate Ψ the LOLR is willing to grant, implying that the terminal constraint is not satisfied and the value function V diverges. The last domain, denoted by Γ, is characterized by a binding terminal constraint, and here the optimal control $\psi \in (0, \Psi)$ has to be explicitly computed. Similarly, we will denote by γ_0, resp. γ_Ψ, the switching region between Γ_0 and Γ, resp. between Γ and Γ_Ψ.

We thus have the following result.

Proposition 2 *Consider the optimal control problem* (8)*, then the three regions introduced above are defined as*

$$\Gamma_\Psi = \left\{ (t,x) \,:\, \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} > \Psi \right\} \tag{14}$$

$$\Gamma_0 = \left\{ (t,x) \,:\, \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} < 0 \right\} \tag{15}$$

$$\Gamma = \left\{ (t,x) \,:\, 0 < \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} < \Psi \right\} \,, \tag{16}$$

with

$$\rho := \sqrt{2}\, Erf^{-1}\left(1 - 2\bar{q}\right) \,.$$

Proof. Consider first Γ_Ψ and define the *highest reachable probability* for the $i-$th node as

$$\begin{aligned} W^H(t,x) &:= \sup\{q \,:\, V(t,x,q) < \infty\} = \\ &= \sup_{\psi \in [0,\Psi]} \mathbb{P}\left(X^{t,x;\psi}(T) \geq v(T)\right) \,, \end{aligned}$$

where $X^{t,x;\psi}(T)$ denotes the value at time T with initial datum (t,x) and control $\psi \in [0,\Psi]$. It follows that the highest reachable probability is attained when considering the maximum admissible control $\Psi < \infty$. Therefore, by Itô formula and the *Feynman–Kac theorem*, we have that $W^H(t,x)$ solves the parabolic PDE

$$\begin{cases} W^H(t,x)(T,x) &= \mathbb{1}_{\{[v(T),\infty)\}}(x) \,, \\ -\partial_t W^H(t,x) &= \partial_x W^H(t,x)\,(\mu + \Psi)\, x + \\ & \quad + \frac{1}{2}\sigma^2 x^2 \partial_x^2 W^H(t,x) \,, \end{cases}$$

whose solution can be explicitly computed as follows

$$\begin{aligned} W^H(t,x) &= \mathbb{P}\Big(\log X^{t,x;\Psi}(T) \geq \log v(T)\Big) = \\ &= \mathbb{P}\left(W(T-t) \geq \frac{1}{\sigma}\left(\log\frac{v(T)}{x} - \left(\mu + \Psi - \frac{\sigma^2}{2}\right)(T-t)\right)\right) = \\ &= \frac{1}{2}\left(1 - \mathrm{Erf}\left(d(\mu,\Psi,\sigma,T-t)\right)\right) \,, \end{aligned} \tag{17}$$

with

$$d(\mu,\Psi,\sigma,T-t) := \frac{\log\frac{v(T)}{x} - \left(\mu + \Psi - \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{2\sigma^2(T-t)}} \,,$$

and Erf denotes the *error function.* For $W^H(t,x) = \bar{q} \in (0,1)$, solving for Ψ, we obtain the boundary region in implicit form

$$\Psi = \gamma_\Psi(t,x;\bar{q}) = \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log\frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} \,, \tag{18}$$

with

$$\rho := \sqrt{2}\,\mathrm{Erf}^{-1}\left(1 - 2\bar{q}\right) \,.$$

Thus, for a required *success probability* $\bar{q}$, the control problem is not feasible in

$$\Gamma_\Psi = \left\{ (t,x) \,:\, \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log\frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} > \Psi \right\} \,,$$

and, for an initial data (t,x) at the left hand side of $\gamma_\Psi(t,x;\bar{q})$, see equation (18), the terminal constraint cannot be satisfied, see Figure 1. If $\Psi = \infty$, namely the LOLR is willing to give a possibly infinite return rate, any point is controllable, and we can always find an admissible control such that the terminal probability constraint is attained.

As regard Γ_0, computing the no-action region we have

$$\begin{aligned} W^0(t,x) &= \mathbb{P}\left(X^{t,x;\psi_0}(T) \geq v(T)\right) = \\ &= \frac{1}{2}\left(1 - \mathrm{Erf}\left(d(\mu,\psi_0,\sigma,T-t)\right)\right) \,, \end{aligned}$$

then, proceeding as above and solving for ψ_0, we obtain

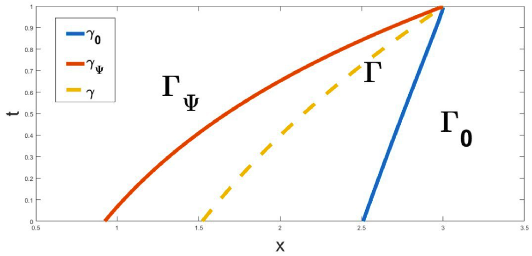

Fig. 1. Representation of different domains for the optimal control problem.

the boundary region

$$0 = \gamma_0(t,x;\bar q) = \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}}\,, \quad (19)$$

where
$$\rho := \sqrt{2}\,\mathrm{Erf}^{-1}\left(1-2\bar q\right)\,,$$
and we are again left with the following no-action region

$$\Gamma_0 = \left\{(t,x)\,:\,\left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} < 0\right\}$$

so that, given a starting value $(t,x) \in \Gamma_0$, the terminal constraint is satisfied and the optimal return is given by the null control $\psi \equiv 0$.
At last the action region Γ is the one delimited by Γ_0 and Γ_Ψ, that is

$$\Gamma = \left\{(t,x)\,:0<\left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} < \Psi\right\}.$$

Thus, being $(t,x) \in \Gamma$, the controller has to find the optimal control so that the terminal probability constraint holds. By computing the reachability set with fixed constant control $\bar\psi$, we eventually obtain with arguments analogous to the ones above,

$$\bar\psi = \gamma_{\bar\psi}(t,x;\bar q) = \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}}\,, \quad (20)$$

see Figure 1 for a representation of the above obtained regions. □

Therefore, if the autonomous process $X^{t,x;0}(T)$ already satisfies the terminal probability constraint, then it is optimal to solve the control problem with no terminal constraint, whose solution is given by the null control in the present case. If instead $(t,x) \in \Gamma$, for a fixed $q \in (0,1)$, the optimal control ψ is given by

$$\gamma_\psi(t,x;q) = \left(\frac{\sigma^2}{2} - \mu\right) + \frac{\log \frac{v(T)}{x}}{T-t} - \frac{\sigma\rho}{\sqrt{T-t}} = \psi\,, \quad (21)$$

The following result holds.

Theorem 3 *The value function for the optimal control problem (8) is given by*

$$V(t,x,W^{\bar\psi}(t,x)) = \sum_{i=1}^n V^i(t,x^i,W^{\bar\psi^i}(t,x^i))\,, \quad (22)$$

where

(i) *if $(t,x^i) \in \Gamma^i$ and $q^i \in (0,1)$ are such that $\gamma_{\bar\psi^i}(t,x^i,q) = \bar\psi^i$, then*

$$V^i\left(t,x^i,W^{\bar\psi^i}\right) = (\bar\psi^i)^2(x^i)^2\left(\frac{e^{(2(\mu^i+\bar\psi^i)+(\sigma^i)^2)(t-T)}-1}{2(\mu^i+\bar\psi^i)+(\sigma^i)^2}\right); \quad (23)$$

(ii) *if $(t,x^i) \in \Gamma_0^i$ and $q^i \in (0,1)$, then it holds $V^i(t,x^i,q^i) = 0$.*

Then V, see equation (22), defines a classical solution to the HJB equation (12) on $\Gamma \cap \Gamma_0$.
Moreover, the optimal control within the class of affine controls is given as in equation (13), where ψ is given as in equation (20)

Proof. The structure of the optimal control problem gives that the contribute of each node can be treated separately, so that the value function is of the form (22), where each V^i can be regarded as the value function for the optimal control for the node i alone. As above, for ease of notation, we will omit the index i.
Fixing the node i, it can be trivially shown that for $(t,x) \in \Gamma_0$, we have $V(t,x,q) = 0$.
Let $(t,x) \in \Gamma$, along the curve $W^{\bar\psi}(t,x)$, the terminal probability of success remains constant, so that the optimal control is given by the constant control $\bar\psi$. Explicit computations shows that V as defined in equation (23) solves the HJB equation (12). Moreover, since the map $q \mapsto V(t,x,q)$ is non–decreasing and $W^{\bar\psi}(t,x) > W^{\psi}(t,x)$ for $\bar\psi > \psi$, we have

$$V\left(t,x,W^{\psi}(t,x)\right) = -\infty\,, \quad \psi < \bar\psi\,,$$

since the terminal constraint in equation (8) is not satisfied. Analogously, if $\psi > \bar\psi$, then $W^{\bar\psi}(t,x) < W^{\psi}(t,x)$. Therefore, the non-decreasing property of V w.r.t. the third argument q, implies

$$V\left(t,x,W^{\psi}(t,x)\right) > V\left(t,x,W^{\bar\psi}(t,x)\right)\,,$$

and the minimum is attained for the control $\bar\psi$ implicitly given by equation (21).
As regard the value function regularity, it is a classical solution in both region Γ and Γ_0. To prove that it is a global classical solution we need to show that it is regular on γ_0. Let $\bar x$ the value on the switching curve γ_0, that is for fixed (t,q), we have $\gamma_0(t,\bar x,q) = 0$, then, since $\bar\psi \to 0$ as $x \to \bar x^-$, we have $\lim_{x\to\bar x^-}\partial_x^2 V = 0 = \lim_{x\to\bar x^+}\partial_x^2 V$ and $\lim_{x\to\bar x^-}\partial_x V = 0 = \lim_{x\to\bar x^+}\partial_x V$, hence the value function is differentiable on $\Gamma \cup \Gamma_0$. □

4 Application to a network of financial banks

In the present section we use previously obtained results to study a real-world application characterized by an interconnected network of banks. In particular, we will show how optimal solutions previously computed can modify the evolution of such a a network. We stress that, for the sake of readability, we will apply our results to a small network, even if, due the fact that the optimal solution is computed in closed form, our results can be easily extended to arbitrary *big* systems.
In order to apply results proved in Section 3, we need to specify for each bank the required probability of non defaulting. A robust and efficient method is proposed in Appendix A.1 and it is based on the *page rank* notion, so that exploiting the network structure the relative importance of each bank with respect to the whole network health is estimated. We are thus considering a LOLR willing to save banks whose failure would cause insolvency and no ability to pay back their liabilities.
We consider a system of banks whose liability matrix and cash vector are as follows, see Figure 2 for the associated graph:

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 10 & 0 \\ 5 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 5.2 \\ 6 \\ 13 \\ 3 \end{bmatrix}.$$

As introduced in Definition 5, the associated -matrix to above network can be computed as

$$\mathcal{G}_d = \begin{bmatrix} 0.0375 & 0.8344 & 0.0375 & 2.3042 \\ 0.0375 & 0.0375 & 0.0375 & 0.9442 \\ 2.9352 & 0.8344 & 0.0375 & 0.0375 \\ 0.0375 & 0.8344 & 0.0375 & 0.0375 \end{bmatrix}.$$

The absolute value of the eigenvector corresponding to the highest eigenvalue is

$$R = v_1 = \begin{bmatrix} 0.3516 & 0.1342 & 0.9177 & 0.1275 \end{bmatrix}^T .$$

Note that the third bank is the one with the highest ranking. Indeed, it is easy to note that its default would cause the default of the first bank yielding a possible insolvency cascade. This is due to the fact that the third bank is systematically more important than the others. Notice that the amount of money due is the most important aspect to be taken into account for the safety of the system. We have reported in Table 1 further considerations. It is worth to underline that, looking at Figure 2 and Table 1, it can be seen that although the first and the third bank are owning the same amount of money, their rankings R are significantly different. This is due to the fact that Bank 3 owns to Bank 1 and its insolvency would probably cause the default of Bank 1. In this example the cascade effect caused by the default of Bank 3 would stop with the default of two banks because of the small dimension of the system, while, on the contrary, such an effect amplifies in big networks.

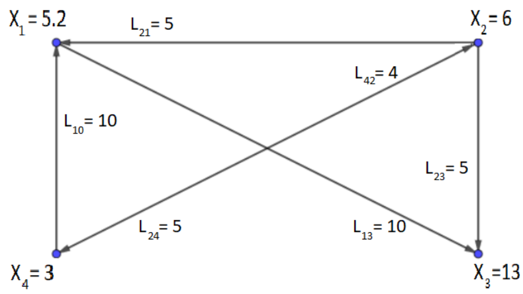

Fig. 2. Graph representing the system of banks: nodes report the cash value of each bank, while the oriented edges represent the amount of money lend from a bank to another.

Banks (i)	1	2	3	4
X_i	5.2	6	13	3
$\sum_{j\sim i} L_{ji}$	15	4	15	5
R_i	0.3516	0.1342	0.9177	0.1275

Table 1
Comparison among the banks rankings. R_i is the rating value associated to each bank as defined in equation (A.4).

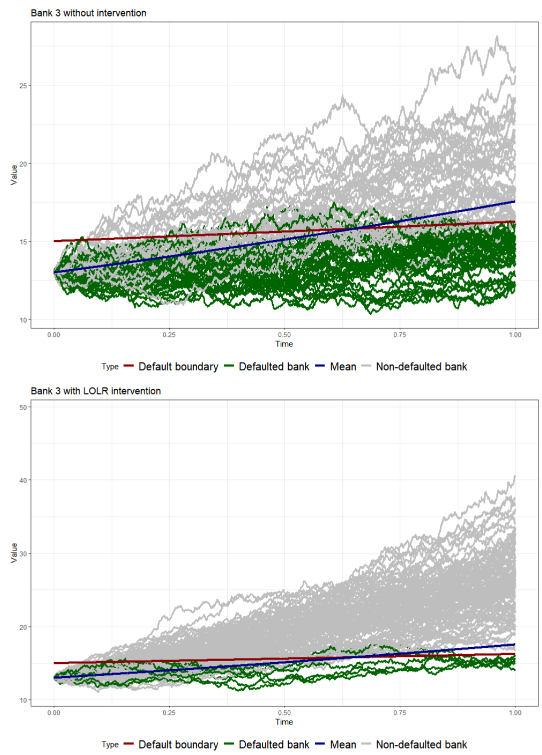

Fig. 3. 100 simulations for the evolution of Bank 3 without LOLR intervention (top panel) and with LOLR intervention (bottom panel).

We recall that the aim of the LOLR is to minimize the cost on banks bailout given by equation (5) constrained by (6), i.e. guaranteeing a probability q^i that the bank i will not default. Fixing an identical probability constraint $q \in [0, 1)$ for all the banks, hence adopt-

ing an equality policy analogous to the *max liquidity* (ML) strategy introduced in [6], see also appendix B. We note that a ML strategy guarantees no privileges to any banks, which would lead the LOLR to lend the same amount of money for systematically important banks as for those banks whose failure would not cause a *cascading effect*.
The main idea of the subsequent analysis is to generalize typical ML strategies to asses different probabilities to each bank in accord to their systemic importance in the network. Consider thus

$$q^i = f(R^i), \quad \text{for } f : \mathbb{R}^+ \to [0,1) \text{ increasing function},$$

where, as seen in Section A.1, R^i is the ranking of the bank i. We remark that that choice of f to be an increasing function leads to a more convenient scenario for the health of networks which have a core-periphery structure, whereas, normally, banks networks have a dense cohesive core, with a periphery less connected, see [6].
Agreeing with above consideration, chose the following probability constraints

$$q^i = 0.9 + 0.05\, \mathbb{1}_{\{R^i > 0.5\}} + 0.04\, \mathbb{1}_{\{R^i > 0.75\}} \,. \tag{24}$$

Figure 2 reports results of the LOLR control with above derived probability of non default. For the sake of clearness, we assuned that all liabilities expire at terminal time T, and that they exponentially increase in time with fixed growth rate $r = 0.05$.
Each bank value evolve according to equations (4), with parameters

$$\begin{aligned}
(\log x_0^1, \mu^1, \sigma^1) &= (1.6, 0.2, 0.1)\,,\\
(\log x_0^2, \mu^2, \sigma^2) &= (1.79.15, 0.25)\,,\\
(\log x_0^3, \mu^3, \sigma^3) &= (2.56, 0.3, 0.2)\,,\\
(\log x_0^4, \mu^4, \sigma^4) &= (1.09, 0.05, 0.4)\,.
\end{aligned}$$

By equation (21), we have that the banks' log-switching regions y^i, $i = 1, \dots, 4$, read as follow

$$\begin{aligned}
y^1 &= 1.622593, & y^2 &= 0, & y^3 &= 2.97332, & y^4 &= 0,\\
q^1 &= 0.9, & q^2 &= 0.9, & q^3 &= 0.99, & q^4 &= 0.9\,.
\end{aligned}$$

The LOLR has not to intervene in banks 2 and 4, since they have more credits than debits, hence they cannot face bankruptcy. Also, according to above derived quantities, bank 1 need no intervention; nonetheless note that choosing $q^1 = 0.95$, would have led to $\widetilde{y}^1 = 1.6589$ with a consequent LOLR intervention injecting money. At last Bank 3 requires the LOLR intervention to be prevented from default.
Figure 3 (top panel) represents 100 simulations for the evolution of Bank 3, with and without LOLR intervention. In particular, since $q^3 = 99\%$ and the default probability of Bank 3 is 0.38, the LOLR is going to inject capital into its cash reserve. After the optimal injection of capital, Bank 3 has probability 0.01 to face default, see lower Figure 3, for the representation of 100 simulations in the case in which the LOLR intervenes.

5 Conclusions

In the present work, we have derived a closed form solution for an optimal control of interbank lending subject to specific terminal probability constraints on the failure. We have also shown a simple and direct method to derive the relative importance of any *node* within the network. It is worth stressing that such a *ranking value* is fundamental in deciding the accepted probability of failure which modifies the final optimal strategy of a financial supervisor aiming at controlling the system to prevent *global crisis* as generalized default.
The results here presented constitute a first step of a wider research program. In particular, in future works we shall consider sequence of *checking times* each of which characterized by possibly different constraints to be considered by the supervisor. In this setting, a solution can be obtained by a backward induction approach, see [7], applied to results here derived. Moreover, as a further development it can be considered a framework where the failure can happen continuously in time, hence imposing strict constraints at any time before the terminal one T.

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A General framework for systemic risk in financial networks

Let us first introduce the mathematical notation needed to properly treat the general financial scenario we are interested in. In particular, we consider a finite connected financial network identified with a graph $\mathbb{G}$ composed by $n \in \mathbb{N}$ vertices $\mathrm{v}_1, \ldots, \mathrm{v}_n$, corresponding to n banks, and $m \in \mathbb{N}$ edges $e_1, \ldots, e_m$ assumed to be normalized on the interval $[0,1]$, which represents interaction between the n banks. In what follows we will use the Greeks letters α, β, $\gamma = 1, \ldots, m$ to denote edges, whereas i, j, $k = 1, \ldots, n$, will denote vertexes. We refer to [9,10,14], for further details

The structure of the graph is based on the *incidence matrix* $\Phi := \Phi^+ - \Phi^-$, where the sum is intended componentwise and $\Phi = (\phi_{i,\alpha})_{n\times m}$, together with the *incoming incidence matrix* $\Phi^+ = \left(\phi^+_{i,\alpha}\right)_{n\times m}$, and the *outgoing incidence matrix* $\Phi^- = \left(\phi^-_{i,\alpha}\right)_{n\times m}$, where

$$\phi^+_{i,\alpha} = \begin{cases} 1 & \mathrm{v}_i = e_\alpha(0)\,, \\ 0 & \text{otherwise} \end{cases}, \quad \phi^-_{i,\alpha} = \begin{cases} 1 & \mathrm{v}_i = e_\alpha(1)\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

In particular, we will say that the edge e_α is *incident* to the vertex v_i if $|\phi_{i,\alpha}| = 1$, so that

$$\Gamma(\mathrm{v}_i) = \{\alpha \in \{1, \ldots, m\} \,:\, |\phi_{i,\alpha}| = 1\}\,,$$

represents the set of incident edges to the vertex v_i. We also introduce the *adjacency matrix* $\mathcal{I} = (\iota_{i,j})_{n\times n}$, defined as $\mathcal{I} := \mathcal{I}^+ + \mathcal{I}^-$, where $\mathcal{I}^+ = \left(\iota^+_{i,j}\right)_{n\times n}$, resp. $\mathcal{I}^- = \left(\iota^-_{i,j}\right)_{n\times n}$, is the *incoming adjacency matrix*, resp. *outgoing adjacency matrix*, defined as

$$\iota^+_{i,j} = \begin{cases} 1 & \text{it exists } \alpha = 1, \ldots, m \,:\, \mathrm{v}_j = e_\alpha(1)\,,\, \mathrm{v}_i = e_\alpha(0)\,, \\ 0 & \text{otherwise}\ , \end{cases}$$

$$\iota^-_{i,j} = \begin{cases} 1 & \text{it exists } \alpha = 1, \ldots, m \,:\, \mathrm{v}_j = e_\alpha(0)\,,\, \mathrm{v}_i = e_\alpha(1)\,. \\ 0 & \text{otherwise}\ . \end{cases}$$

Notice that since $\mathcal{I}^+ = (\mathcal{I}^-)^T$, then we have that $\mathcal{I}$ is symmetric with null entries on the main diagonal.

Above formulation can be used to represent the financial network introduced in Section 2. In particular, to fully specify the flow between each node in the network, we introduce the notion of *clearing vector* representing the payments made by each of the banks in the financial system, see, e.g., [11, Definition 1], [16, Definition 2.6]. In what follows, if not otherwise specified, we will use standard point-wise ordering for vectors in $\mathbb{R}^n$, namely for every $x, y \in \mathbb{R}^n$ it holds $x \le y$ if and only of $x_i \le y_i$, for any $i = 1, \ldots, n$.

Definition 4 *In the aforementioned financial setting, see also appendix A, a clearing vector is a vector $u^*(t) \in [0, \bar{u}(t)]$ satisfying*

- ***Limited liabilities:***

$$u^*_i(t) \le \sum_{j=1}^n \pi^T_{i,j}(t)\, u^*_j(t) + F^i(t)\,, \quad i = 1, \ldots, n\,;$$

- ***Absolute priority:*** *that is either obligations are paid in full, or all value of the node is paid to creditors, i.e.*

$$u^*_i(t) = \begin{cases} \bar{u}_i(t),\ \text{if } \bar{u}_i(t) \le \sum_{j=1}^n \pi^T_{i,j}(t)\, u^*_j(t) + F^i(t) \\ \sum_{j=1}^n \pi^T_{i,j}(t)\, u^*_j(t) + F^i(t),\ \text{otherwise.} \end{cases}$$

Existence and uniqueness of a *clearing vector*, in the sense of Definition 4, is treated in [11,16]. In particular, in [11] it is shown that $u^*(t)$ is a clearing vector if and only if

$$u^*(t) = \bar{u}_i(t) \wedge \left(\sum_{j=1}^n \pi^T_{i,j}(t)\, u^*_j(t) + F^i(t)\right)\,. \quad \text{(A.1)}$$

Equation (A.1) can be interpreted as follows: the term $\bar{u}_i(t)$ specifies which $i - node$ owes to the other nodes at time $t \in [0,T]$, whereas the second term $\left(\sum_{j=1}^n (\pi_{i,j}(t))^T\, u^*_j(t) + F^i(t)\right)$ represents the cash inflow for the node i at time $t \in [0,T]$. Consequently, *clearing vector* represents the payment at time t of each node: each node pays the minimum between what it has and what it owes.

A.1 PageRank

Previously introduced notation allows also to derive an explicit method to address relative importance of a single node in a network. In particular, such an approach has been used in Section 4 to systematically decide the survival probability for each node.

Let us note that, along previous sections, we have stated an optimal control problem which has been then solved deriving its solution under the assumption that the *accepted probability of failure* q^i is a fixed parameter to be chosen endogenously. In what follows we propose a general, automatic, criterion to deduce the global importance of each node in the system. Next computations exploit results on network analysis already used, e.g., to set the functioning logic of the *Google* research engine, see, e.g., [15]. According to the network formulation introduced in Section 2, and using results derived in [15], we show how to score the relative importance of any bank in the network, computing its so called *Page Rank*, allowing us to choose the best survival probability q.

According to the framework described in Section 2, see also Appendix A, let us consider a system of interconnected n banks and related standard *bank enumeration.*

Namely, we take into account the usual *one-to-one* correspondence relation between the set of banks and the set of vertexes $V := \{v_1, v_2, \dots, v_n\}$, referred to as nodes, while $I := \{1, 2, \dots, n\}$ is the associated set of indexes. Moreover, consider a LOLR strategy in which for each $v_i \in V$ the default probability constraint parameter q^i depends on a predetermined rank R^i associated to the i^{th} bank, hence representing its systemic importance in the network.

In what follows we are considering graphs as defined in Section 2. In particular, to each node $v_i \in V$ corresponds a bank, while to edges connecting nodes $(v_i, v_j) \in V \times V$, we associate the following quantities

$$\gamma^+_{(i,j)} = \frac{c^+ L_{i,j} + c^- L_{j,i}}{N_j - \min(N) + 1}, \quad \gamma^-_{(i,j)} = \frac{c^+ L_{j,i} + c^- L_{i,j}}{N_i - \min(N) + 1}, \tag{A.2}$$

where, letting

$$L^+_j = \sum_{i \sim j} L_{ij}, \quad L^-_j = \sum_{i \sim j} L_{ji}, \tag{A.3}$$

$$i \sim j \iff v_i, v_j \text{ are connected},$$

we define N_j as the net amount of money held by bank j if it would pay its debts at the actual time, i.e. $N_j := X_j + L^+_j - L^-_j$. Moreover c^+ and c^- are two non-negative constants chosen to confer more importance to due debts, resp. to owed credits. For the sake of simplicity, since c^+ and c^- are meant to be weight parameters, we set $c^+ + c^- = 1$. Notice that $\gamma^+_{(i,i)} = \gamma^-_{(i,i)} = 0$ and $\gamma^-_{(i,j)} = \gamma^+_{(j,i)}$, for all $i, j \in I$.

Let us introduce the notion of *outdegree* $\deg^+_\gamma$, resp. *indegree* $\deg^-_\gamma$, for any vertex $v_i \in V$, namely

$$\deg^+_\gamma(v_i) = \sum_{j \in \mathcal{I}} \gamma^+_{(i,j)}, \quad \deg^-_\gamma(v_i) = \sum_{j \in \mathcal{I}} \gamma^-_{(i,j)},$$

and normalize the quantities defined in (A.2) associated to any couple (i, j) of edges in the graph

$$\overrightarrow{\mathcal{T}}_{(i,j)} = \frac{\gamma^+_{(i,j)}}{\deg^+_\gamma(v_j)}, \quad \overleftarrow{\mathcal{T}}_{(i,j)} = \frac{\gamma^-_{(i,j)}}{\deg^-_\gamma(v_j)}$$

corresponding to the ratio of a linear combination on the liabilities between bank i and bank j, and the asset value of bank j. Moreover, we define the matrix $\overrightarrow{\mathcal{T}}$ as the matrix whose entries are $\overrightarrow{\mathcal{T}}_{(i,j)}$, for $i, j \in \mathcal{I}$, the quantities $\overrightarrow{\mathcal{T}}_{(i,j)}$ being the weights assigned to each oriented edge. Therefore, the rating value associated to any node/bank v_i is given by the following recursive formula

$$R^i_d = d \sum_{j \sim i} \overrightarrow{\mathcal{T}}_{(i,j)} R^j_d, \tag{A.4}$$

where $d \in (0, 1)$ is a parameter to be chosen, typically $d = 0.85$, see, e.g., [14]. To compute equation (A.4), we introduce the so called *Google-matrix*, see, e.g., [14, ch 2].

We assume that our network is composed by banks not solely owing liabilities if $c^+ = 1$, resp. not solely owning liabilities if $c^+ = 0$, and at least connected for $c^+ \in (0, 1)$. Of course, banks that are non connected to others belonging to the network, are simply not ranked, since their default cannot affect the system. On the other hand, even if the conditions for $c^+ \in \{0, 1\}$ are not required, they guarantee the boundedness of all the elements of the matrix defined in the next Definition 5. We stress that, to avoid above restrictions, one can modify the values assigned to edges by equation (A.2), e.g., as follows: for $c^+ = 1$ and for every $i \sim j$, define $\tilde{\gamma}^+_{(i,j)} = L_{i,j}/(N_j - \min(N) + 1) + \epsilon$ as the modified value assigned to the edges.

Definition 5 (Google-matrix) *Let J be a $n \times n$-matrix whose entries are all ones. A Google-matrix is a $n \times n$-matrix given by*

$$\mathcal{G}_d := \frac{1 - d}{n} J + d \overrightarrow{\mathcal{T}}, \tag{A.5}$$

where $d \in (0, 1)$ can be chosen to guarantee irreducibility of $\mathcal{G}_d$, while J is the $n \times n$ matrix whose all entry are 1.

Since the matrix defined in equation (A.5) is positive we can apply the Perron–Frobenius Theorem which assures us that there exists a *maximum* real eigenvalue $\lambda > 0$ of $\mathcal{G}_d$, indeed λ is the so-called dominant Perron–Frobenius eigenvalue. Moreover, there exists one of the associated eigenvectors, denoted by R_d and usually called *Perron-Frobenius dominant vector*, which is both strictly positive and normalized and whose components represent the rating of each bank. Let us recall that d is usually chosen to be approximately equals to 0.85, see, e.g., [14]. It follows that proposed ranking procedure consists in computing the following series

$$R_d = d \sum_{k=0}^{\infty} (1 - d)^k \, (\mathcal{G}_d)^k \, \mathbf{1},$$

where we denoted by $\mathbf{1}$ a n-dimensional vector whose entries are all equal to one.

B Comparison with the paper by Capponi et al. [6]

As mentioned above, the financial setting has been mainly borrowed by [11] as concerns the lending system formulation, and from [6] for the optimal control problem with an external supervisor aiming at guaranteeing the overall sanity of the system. This section is devoted to a comparison with [6].

We stress that our assumptions on the optimal control are in the spirit of [6], in the sense that we consider failure at discrete times; also we will not consider a global optimal control, deriving a control for the whole time interval but rather we derive a series optimal control and then gluing together the resulting optimal controls. As mentioned we leave the optimal global control to future research being this latter point mathematically more demanding.
This comparison is significant since their work is based on a similar framework, namely a multi-period controlled system of banks, represented by a network, in which an outside entity, named LOLR, provides liquidity assistance loans to financially unstable banks in order to reduce the level of systemic risk within the whole network of banks. To analyze the systemic risk in interbank networks their work follows a clearing system framework consistent with bankruptcy laws. In particular they generalize the single period clearing system in the paper by Eisenberg and Thomas, see [11], by a multi-period controlled clearing payment system assuming limited liability of equity, priority over equity, and proportional repayments of liabilities after the default event. This generalization leads to a better insight in the propagation and aftershocks of defaults.
The main feature in [6] is the comparison between two possible LOLR strategies:

- the *Systemic Importance Driven* (SID) strategy, in which liquidity assistance is available only to banks considered systemically important, i.e. the banks whose default would cause significant losses to the financial system (because of their size, complexity and systemic interconnectedness);
- the *Max-Liquidity* (ML) strategy, in which the regulators aim to maximize the instantaneous total liquidity of the system.

By the analysis of these two different strategies they showed that the SID strategy is preferred when the network has a core-periphery structure, i.e. consisting of a dense cohesive core and a sparse, loosely connected periphery. This is due by the fact that the ML strategy increases the default probability for systematically important banks. Although these two strategies are simplified and do not consider the amount of capital that the LOLR has to inject in the banks network, nonetheless such comparison is useful because the numerical approach fits easily through simulations and systemic risk analysis.
Our work has some important similarities with the one by Capponi et al., in particular we also have considered a finite connected multi-period financial network representing the banks system and the assumptions guaranteeing the consistency with the bankruptcy laws. But, despite this, instead of comparing the two strategies, SID and ML, we considered a LOLR wishing to minimize the square of the lend resources over the probabilistic constraint. Therefore, we did not gave an initial budget at disposal to the LOLR as in [6], but took into consideration regulators aiming to find the loan control $\{\alpha^i(t)\}_{i=1,\dots,N,t\in[t_k,t_{k+1}]}$ minimizing the functional given by equation (5) for each time interval, i.e. $\forall k = 1,\dots,M-1$, ensuring that the probability for each exogenous asset value to be greater than the default boundary is greater than a given constants q^i for each bank $i \in \{1,\dots,N\}$.
Moreover, while [6] is meant to compare two strategies for the LOLR, our approach follows a different path in searching the optimal budget consumption to guarantee a prescribed level of safety of the financial network, given by the parameters q^i $i = 1,\dots,N$. In particular, we do not assume strong constraint over the regulators budget, which depends on the default probability constraint parameters q^i. To switch on a similar comparison as in [6], i.e. considering banks networks of the type *core-periphery and baseline random networks*, and regulator policies of the type *SID and ML*, it suffices to fix the probability constraint depending on the systematic importance of the banks. That is, banks whose failure would cause significant losses to the financial network, because of their size and systemic interconnectedness, should be endorsed with greater default probability parameters q_i. Therefore, our study provides an extension of the admissible policies, through considering an optimal control theory approach.