JOTA manuscript No.
(will be inserted by the editor)

A Level-Set Approach for Stochastic Optimal Control Problems under Controlled-Loss Constraints

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Received: date / Accepted: date

Abstract We study a family of optimal control problems under a set of controlled-loss constraints holding at different deterministic dates. The characterization of the associated value function by a Hamilton-Jacobi-Bellman equation usually calls for strong assumptions on the dynamics of the processes involved and the set of constraints. To treat this problem in absence of those assumptions, we first convert it into a state-constrained stochastic target problem and then solve the latter by a *level-set approach*. With this approach, state constraints are managed through an exact penalization technique.

Keywords Hamilton-Jacobi-Bellman equations, Viscosity solutions, Optimal control, Expectation constraints

Mathematics Subject Classification (2000) 93E20, 49L20, 49L25, 35K55

1 Introduction

Under general assumptions, the value function associated with unconstrained stochastic optimal control problems can be characterized as the unique continuous viscosity solution of a second-order Hamilton-Jacobi-Bellman (HJB) equation (see e.g. [1, 2]). However, the characterization of the value function associated with stochastic optimal control problems involving state constraints still raises challenges. Such problems arise in many applications and are the object of our study. In particular, we focus on state constraints holding in expectation and on a set of deterministic dates. Constraints of this type often involve loss functions and are referred in the literature as *controlled-loss constraints*. blackIn the finance literature, optimization under risk-measure constraints has been at the cornerstone of

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modern portfolio selection theory since the pioneering work [3]. We refer the interested reader to [4, 5] for an exposition of the different models that have emerged in portfolio selection and their solution methods, and to [6–8] for additional examples of risk-measure constrained portfolio selection problems. In particular, [4] presents a comprehensive analysis of utility-deviation-risk portfolio selection problems. In this study, a deviation-risk-measure term, designed as the expected value of a function of the spread between the underlying portfolio and its mean at the terminal date, appears in the objective function as a penalization to the expected utility. There, under a complete market setting, the necessary and sufficient conditions for optimality can be determined through the derivation of a primitive static problem, the so-called non-linear moment problem, characterizing the optimum. In the aforementioned papers, the constraint holds at the terminal date, the portfolio dynamics is linear in the control variable, the utility function is assumed to be at least strictly increasing and continuously differentiable, while the deviation-risk-measure involves a function satisfying suitable regularity properties. Our paper thus contributes to this existing literature by investigating, under a Markovian and possibly incomplete market framework, general stochastic optimal control problems involving multi-period expected-loss constraints, general dynamics, and possibly non-smooth functions.

Two main approaches can be found in the literature to deal with general stochastic optimal control problems. The first one is the so-called Pontryagin Maximum Principle (PMP) which has been introduced by the seminal work [9] for deterministic problems and provides necessary conditions for optimality. For its application to state constrained stochastic optimal control problems we refer to [10–12] and the references therein. The second approach is based on the Dynamic Programming Principle (DPP). Originally formulated by Richard Bellman in [13], the DPP offers a handy way of tackling a global optimization problem by solving a series of recursive local optimization problems. In particular, the DPP allows to characterize the value function of an optimal control problem as the solution of a non-linear partial differential equation (PDE), the HJB equation. Relying on the notion of viscosity solutions, this approach also applies when the value function only satisfies very mild regularity conditions. There exists a huge literature on stochastic optimal control problems under different types of state-constraints, their DPP and HJB equations, e.g. [14–20]. Recently, results have also been provided in the framework of mean field games (MFG), see [21], to account for the mutual interactions of infinitely many agents having statistically similar behaviors. There, the state constrained HJB equation associated with a representative agent is coupled with a constrained Fokker-Plank equation reflecting the evolution of the density of agents' distribution. We refer to [22] for an overview on (unconstrained) MFG theory. All these references highlight the difficulty to characterize the optimal solution because of the delicate interplay between the dynamics of the processes involved and the set of constraints. In particular, some viability and regularity assumptions on the dynamics are typically required to ensure the finiteness of the value function and its PDE characterization, often rendering the problem not treatable. Our paper thus aims at providing, under a different set of assumptions, an alternative method for dealing with this type of problems in the case of state constraints expressed in expectation and imposed at different discrete times.

This objective is achieved at the price of augmenting the state and control space by additional components and considering unbounded controls. More precisely, following the ideas developed in [23] for the case of a constraint holding pointwise in time almost surely, our approach relies on two main steps. The first one consists in building on the equivalence results developed in [24, 25] to convert, by means of the martingale representation theorem, the original problem into a stochastic target problem involving almost-sure constraints and unbounded controls. Still, because of the presence of state-constraints, a direct HJB characterization of the derived stochastic target problem remains challenging and requires strong viability and regularity assumptions such as those arising in the original problem, see [17, 18, 25]. The second step therefore consists in solving the resulting state constrained stochastic target problem by means of a *level-set approach* where the state constraints are managed via an exact penalization technique. Initially introduced in [26] to model some deterministic front propagation phenomena, the level-set approach has been used in many applications related to (non)linear controlled systems (see e.g. [27–29]). The connection between (unconstrained) stochastic target problems and level-set characterization has been pointed out in [30]. In our case, the level-set approach links the state constrained stochastic target problem to an auxiliary optimal control problem, referred as the *level-set problem*, defined on an augmented state and control space, but without state constraints. The value function associated with this level-set problem can be fully characterized as the solution of a particular HJB equation. However, the HJB equation derived involves unbounded controls raising continuity issues. We solve these issues passing through a compactification of the differential operator. This step is crucial to the derivation of comparison results. We stress that, as in [23] and differently from most of the reference literature such as [14–18], our approach does not aim to provide a HJB characterization for the value function of the original constrained stochastic optimal control problem. Instead, the HJB equation describes the value function of the (unconstrained) level-set problem which we use as an auxiliary tool for characterizing the solution of the original constrained problem. This allows us to work under a set of assumptions different from those in the aforementioned literature. We end this paragraph highlighting the main contributions of this paper to the results derived in [23]. First, this paper extends the preceding results to the case of a time-inconsistent problem. In particular, at a specific time, the number of unbounded controls involved depends on the number of constraints holding in the future and thus on the time interval considered. On the contrary, in [23] the same single unbounded control is involved over the entire duration of the problem. The associated value function is therefore defined differently on each time interval $[t_i, t_{i+1}[$, $0 \leq i \leq n-1$, and shows a discontinuity at each t_{i+1}. Additionally, we rely on a relaxed version of the condition of existence of an optimizer for the level-set problem stated in [23, (H4)], expanding the scope of application of the results. This condition relates to convexity properties of the dynamics, cost functions, and the set of controls, and is independent of the aforementioned viability and regularity assumptions needed to directly characterize the original problem. Finally, our analysis does not need the uniform boundedness in L^2 of the admissible controls, and involves a value function associated with the level-set problem whose regularity cannot be proven a priori.

The rest of the paper is organized as follows. In Section 2, we formally state the problem. In Section 3, we formulate the optimal control problem as a constrained stochastic target problem. The level-set approach is then applied under a suitable assumption on the existence of optimal controls. The latter assumption is investigated in Section 4. A complete characterization of the obtained level-set function is derived in Section 5. An appendix contains proofs of some technical results.

2 Setting and Main Assumptions

In this manuscript, we consider Ω, the space of $\mathbb{R}^q$, $q \geq 1$-valued continuous functions $(\omega_t)_{t\leq T}$ on $[0,T]$ endowed with the Wiener measure $\mathbb{P}$, where, for any integer $q \geq 1$, every element of $\mathbb{R}^q$ is considered as a column vector. We introduce W the coordinate mapping, i.e. $(W(\omega)_t)_{t\leq T}$ for $\omega \in \Omega$ so that W is a q-dimensional Brownian motion on the canonical filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. In particular, $\mathcal{F}$ is the Borel tribe of Ω and $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ is the $\mathbb{P}$-augmentation of the filtration generated by W. We define $\mathcal{U}$ as the collection of progressively measurable processes ν with values in U, a compact subset of $\mathbb{R}^r$, $r \geq 1$. For $t \in [0,T], z \in \mathbb{R}^d$, $d \geq 1$ and for $\nu \in \mathcal{U}$, we define the process $Z^{t,z,\nu}_\cdot$ as the unique (strong) solution on $[t,T]$ to

$$Z^{t,z,\nu}_\cdot = z + \int_t^\cdot \mu(s, Z^{t,z,\nu}_s, \nu_s)\,\mathrm{d}s + \int_t^\cdot \sigma(s, Z^{t,z,\nu}_s, \nu_s)\,\mathrm{d}W_s\,,$$

where $(\mu, \sigma) : (t,z,u) \in [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d \times \mathbb{R}^{d\times q}$ are continuous functions being Lipschitz continuous in z uniformly in (t,ν).

Remark 2.1 Financial applications usually consider the case where $Z^{t,z,\nu}_\cdot := (X^{t,x,\nu}_\cdot, Y^{t,z,\nu}_\cdot)$, with $X^{t,x,\nu}_\cdot := x + \int_t^\cdot \mu_X(s, X^{t,x,\nu}_s, \nu_s)\,\mathrm{d}s + \int_t^\cdot \sigma_X(s, X^{t,x,\nu}_s, \nu_s)\,\mathrm{d}W_s$ on $\mathbb{R}^d$ and $Y^{t,z,\nu}_\cdot := y + \int_t^\cdot \mu_Y(s, Z^{t,z,\nu}_s, \nu_s)\,\mathrm{d}s + \int_t^\cdot \sigma_Y^\top(s, Z^{t,z,\nu}_s, \nu_s)\,\mathrm{d}W_s$ on $\mathbb{R}$ where $(\mu_X, \sigma_X) : (t,x,u) \in [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d \times \mathbb{R}^{d\times q}$ (resp. $(\mu_Y, \sigma_Y) : (t,z,u) \in [0,T] \times \mathbb{R}^{d+1} \times U \to \mathbb{R} \times \mathbb{R}^q$) are continuous functions being Lipschitz continuous in x (resp. z) uniformly in (t,ν). In this form, X models the evolution over time of the price of some underlying assets while Y represents a portfolio process.

To simplify the notations, we assume from now on that the dimensions d, q and r are all equal and we thus disregard the notations q and r.

We now introduce two non-negative Lipschitz continuous maps f and Ψ defined on $\mathbb{R}^d$. We fix $n \in \mathbb{N}$ and consider the time grid $t_0 = 0 \leq \cdots \leq t_i \leq \cdots \leq t_n = T$.

For any $0 \leq i \leq n-1$, we define the set $\mathscr{C}_i := [t_i, t_{i+1}[\times \mathbb{R}^d \times \mathbb{R}^{(n-i)}$, and

$$\begin{aligned} &\mathscr{B}_i := [t_i, t_{i+1}[\times \mathbb{R}^d \times [0,\infty[^{n-i}, \qquad &&\mathrm{int}(\mathscr{B}_i) := [t_i, t_{i+1}[\times\mathbb{R}^d\times]0,\infty[^{n-i}\,,\\ &\mathscr{D}_i := \mathscr{B}_i \times \mathbb{R}_+, &&\mathrm{int}(\mathscr{D}_i) := \mathrm{int}(\mathscr{B}_i) \times \mathbb{R}_+^*\,. \end{aligned}$$

The objective of the paper is to solve for any $0 \leq i \leq n-1$ the following stochastic optimal control problem on $\mathscr{C}_i$,

$$V(t, z, p_{i+1}, ..., p_n) := \inf \left\{ \mathbb{E}\left[f(Z^{t,z,\nu}_T)\right],\, \nu \in \mathcal{U}_{t,z,p_{i+1},\dots,p_n} \right\}, \tag{1}$$

where $\mathcal{U}_{t,z,p_{i+1},\dots,p_n} := \left\{\nu \in \mathcal{U} :\ \mathbb{E}\left[\Psi(Z^{t,z,\nu}_{t_k})\right] \leq p_k,\, i+1 \leq k \leq n\right\}$. On $\{T\}\times\mathbb{R}^d$, we set $V(T,z) = f(z)$. We use the convention $V(t,z,p_{i+1},...,p_n) = \infty$ whenever $\mathcal{U}_{t,z,p_{i+1},\dots,p_n} = \emptyset$. Observe that $\mathcal{U}_{t,z,p_{i+1},\dots,p_n} = \emptyset$ whenever there exists $i+1 \leq k \leq n$ such that $p_k < 0$. This implies that $V = \infty$ on $\mathscr{C}_i \setminus \mathscr{B}_i$. We underline that the problem can be treated similarly if we consider a different loss function at each date.

Remark 2.2 Our approach applies to the more general problem where the objective function is given by $f(Z_T^{t,z,\nu})+\int_t^T \ell(s, Z_s^{t,z,\nu}, \nu_s)\mathrm{d}s$, where ℓ is some continuous non-negative function defined on $[0,T]\times\mathbb{R}^d\times U$, being Lipschitz continuous in the space variable uniformly in the other variables. Indeed, one can always consider the augmented dynamics $\tilde{Z}_\cdot^{t,z,\nu} := (Z_\cdot^{t,z,\nu}, \zeta_\cdot^{t,z,\nu}) \in \mathbb{R}^{d+1}$ for $\zeta_\cdot^{t,z,\nu} := \int_t^\cdot \ell(s, Z_s^{t,z,\nu}, \nu_s)\mathrm{d}s$, together with the terminal condition $\tilde{f}(\tilde{Z}_T^{t,z,\nu}) := f(Z_T^{t,z,\nu}) + \zeta_T^{t,z,\nu}$, and recover the formulation of the problem given in (1).

The following notations will be used throughout the paper. Let $d, q \geq 1$ be integers. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^d$, and by $x^\top$ its transpose. We also set $\mathrm{M}^\top$ the transpose of $\mathrm{M} \in \mathbb{R}^{d\times q}$, while $\mathrm{Tr}[\mathrm{M}]$ is its trace. We denote by $\mathbb{S}^d$ the set of symmetric matrices in $\mathbb{R}^{d\times d}$ and by $I_d \in \mathbb{S}^d$ (resp. $\mathbf{0} \in \mathbb{S}^d$), the identity matrix (resp. the null matrix). Moreover, we define $\mathcal{S}_d$ the unit d-sphere, i.e. $\{b \in \mathbb{R}^{d+1}, |b| = 1\}$, and $\mathcal{D}_d$ the subset of $\mathcal{S}_d$ such that the first component b_1 is null. To alleviate notations we write $\bar{\mathcal{S}}_d := \mathcal{S}_d \setminus \mathcal{D}_d$. For a given set $\mathcal{O}$, we write $\mathrm{cl}\,(\mathcal{O})$ for its closure. We also define $(\cdot)_+ := \max(\cdot, 0)$. The variables $C, \hat{C}, \bar{C} > 0$ are constant terms that we do not keep track of. Finally, the abbreviation "s.t." stands for "such that", and inequalities between random variables hold $\mathbb{P}$-a.s.

3 Problem Reformulation

In the spirit of [23], our approach articulates in two steps. First, we reformulate (1) as a constrained stochastic target problem (see Proposition 3.1 below). Then, this stochastic target problem is described by a level-set approach where the constraints are handled using an exact penalization technique (see Proposition 3.3 below). This links the backward reachable set associated with the stochastic target problem to the zero level-set of a value function associated with a suitable auxiliary unconstrained optimal control problem given by w in (3) below.

3.1 Associated Stochastic Target Problem

We denote by $\mathcal{A}$, the collection of progressively measurable processes in $L^2([0,T]\times\Omega)$, with values in $\mathbb{R}^d$. Let $0 \leq i \leq n-1$ and $i+1 \leq k \leq n$. Before presenting the main result, we define for any $t \in [t_i, t_{i+1}[$, $p_k \in \mathbb{R}$, $\alpha_k \in \mathcal{A}$, $m \in \mathbb{R}$, and $\eta \in \mathcal{A}$ the following new processes on $[t,T]$,

$$P_\cdot^{t,p_k,\alpha_k} := p_k + \int_t^\cdot \alpha_{k_s}^\top \,\mathrm{d}W_s \text{ on } [t,t_k] \quad \text{and} \quad M_\cdot^{t,m,\eta} := m + \int_t^\cdot \eta_s^\top \,\mathrm{d}W_s.$$

We shortly denote $\alpha \equiv (\alpha_{i+1}, \ldots, \alpha_n) \in \mathcal{A} \times \ldots \times \mathcal{A} \equiv \mathcal{A}^{n-i}$. We can now state the following result.

Proposition 3.1 *Let $0 \leq i \leq n-1$. For any $(t, z, p_{i+1}, ..., p_n) \in \mathscr{C}_i$,*

$$V(t,z,p_{i+1},...,p_n) = \inf\Big\{ m \geq 0 : \exists\,(\nu,\alpha,\eta) \in \mathcal{U}\times\mathcal{A}^{n-i}\times\mathcal{A} \text{ s.t.} \quad M_T^{t,m,\eta} \geq f(Z_T^{t,z,\nu}) \text{ and } P_{t_k}^{t,p_k,\alpha_k} \geq \Psi(Z_{t_k}^{t,z,\nu}),\ i+1 \leq k \leq n \Big\}. \tag{2}$$

Proof Let $0 \le i \le n-1$ and $(t,z,p_{i+1},...,p_n) \in \mathscr{C}_i$. One can easily prove that

$$V(t,z,p_{i+1},...,p_n) = \inf\left\{m \ge 0 : \exists\, \nu \in \mathcal{U}_{t,z,p_{i+1},\dots,p_n} \text{ s.t. } m \ge \mathbb{E}\left[f(Z_T^{t,z,\nu})\right]\right\} .$$

We then prove, for any $m \ge 0$, the equivalence between the two following statements

$$(i)\,\exists\, \nu \in \mathcal{U}_{t,z,p_{i+1},\dots,p_n} \text{ s.t. } m \ge \mathbb{E}\left[f(Z_T^{t,z,\nu})\right] ,$$
$$(ii)\,\exists\,(\nu,\alpha,\eta) \in \mathcal{U}\times\mathcal{A}^{n-i}\times\mathcal{A}, \text{ s.t. } \begin{cases} P_{t_k}^{t,p_k,\alpha_k} \ge \Psi(Z_{t_k}^{t,z,\nu}),\ i+1 \le k \le n \\ \text{and } M_T^{t,m,\eta} \ge f(Z_T^{t,z,\nu}) \end{cases} .$$

To this aim we appeal to similar techniques as those exploited in [24,31,32]. The implication $(ii) \Rightarrow (i)$ follows by taking the expectation in (ii) and using the martingale property of the stochastic integrals. On the other hand, the implication $(i) \Rightarrow (ii)$ follows from the martingale representation theorem (see e.g. [33, Theorem 4.15, Chapter 3]). More precisely, from the assumptions on the coefficients of Z and the growth conditions on f and Ψ, there exists, for any $\nu \in \mathcal{U}$, $(\hat\alpha_{i+1},...,\hat\alpha_n,\hat\eta) \in \mathcal{A}^{n-i}\times\mathcal{A}$ such that

$$M_T^{t,m,\hat\eta} = m + \int_t^T \hat\eta_s^\top\, \mathrm{d}W_s \ge f(Z_T^{t,z,\nu}) ,$$

and

$$P_{t_k}^{t,p_k,\hat\alpha_k} = p_k + \int_t^{t_k} \hat\alpha_{k_s}^\top\, \mathrm{d}W_s \ge \Psi(Z_{t_k}^{t,z,\nu}) ,$$

for $i+1 \le k \le n$, leading to the result. □

As intimated in the introduction, a direct treatment of the derived stochastic target problem (2) is challenging and would involve strong regularity assumptions that would considerably restrict the applicability of our study (see e.g. [17,18,25]). Accordingly, a direct resolution of the derived stochastic target problem seems unsatisfactory. We will thus make use of the link between (2) and the auxiliary problem given by w in (3), proved in Section 3.2 below, to solve the former problem.

3.2 Level-Set Approach

For $0 \le i \le n-1$, and $(t,z,p_{i+1},...,p_n,m) \in \mathscr{C}_i \times \mathbb{R}$, we define the following optimal control problem

$$w(t,z,p_{i+1},...,p_n,m) := \inf_{\substack{\nu\in\mathcal{U}\\ \alpha\in\mathcal{A}^{n-i}\\ \eta\in\mathcal{A}}} \mathrm{J}^{\nu,\alpha,\eta}(t,z,p_{i+1},...,p_n,m) , \tag{3}$$

with

$$\mathrm{J}^{\nu,\alpha,\eta}(t,z,p_{i+1},...,p_n,m) := \mathbb{E}\left[\begin{matrix} \left(f(Z_T^{t,z,\nu}) - M_T^{t,m,\eta}\right)_+ \\ +\sum_{k=i+1}^n \left(\Psi(Z_{t_k}^{t,z,\nu}) - P_{t_k}^{t,p_k,\alpha_k}\right)_+ \end{matrix}\right] .$$

On $\{T\}\times\mathbb{R}^d\times\mathbb{R}$, we set $w(T,z,m) = (f(z)-m)_+$.

Observe that the objective function in (3) integrates the constraint of problem (2) by penalizing the upside deviation of $f(Z_T^{t,z,\nu})$ from $M_T^{t,m,\eta}$ and of $\Psi(Z_{t_k}^{t,z,\nu})$ from $P_{t_k}^{t,p_k,\alpha_k}$, $i+1 \le k \le n$.

In what follows, we denote w^* (resp. w_*) the upper (resp. lower) semi-continuous envelope of w on $\mathscr{D}_i$, $0 \le i \le n-1$. The function w satisfies the following regularity properties.

Proposition 3.2 *For any $0 \le i \le n-1$, w is Lipschitz continuous with respect to $(z, p_{i+1}, ..., p_n, m)$ on $\mathscr{C}_i \times \mathbb{R}$ and satisfies on $\mathscr{D}_i$,*

$$0 \le w(t, z, p_{i+1}, ..., p_n, m) \le C\,(1 + |z|) \ , \tag{4}$$

for some $C > 0$. Moreover, on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$,

$$\lim_{t \uparrow T} w(t, z, p_n, m) = (f(z) - m)_+ + (\Psi(z) - p_n)_+ \,. \tag{5}$$

Proof Fix $0 \le i \le n-1$. The Lipschitz continuity of w with respect to $(p_{i+1}, ..., p_n, m)$ (resp. z) is straightforward (resp. follows from the regularity of f, Ψ and of the coefficients of Z). Moreover, by the definition of w, one has on $\mathscr{C}_i \times \mathbb{R}$,

$$0 \le w(t, z, p_{i+1}, ..., p_n, m) \le \inf_{\nu \in \mathcal{U}} \mathbb{E}\left[\left(f(Z_T^{t,z,\nu}) - m\right)_+ + \sum_{k=i+1}^{n} \left(\Psi(Z_{t_k}^{t,z,\nu}) - p_k\right)_+\right] .$$

Therefore, since $m, p_k \ge 0$, $i+1 \le k \le n$, on $\mathscr{D}_i$, (4) follows from the growth conditions on f and Ψ and the assumptions on the coefficients of Z. We now prove (5). Let $0 < h \le T - t_{n-1}$. On $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$, one has for any $\nu \in \mathcal{U}$,

$$w(T-h, z, p_n, m) \le \mathbb{E}\left[\left(f(Z_T^{T-h,z,\nu}) - m\right)_+ + \left(\Psi(Z_T^{T-h,z,\nu}) - p_n\right)_+\right] .$$

On the other hand, by the martingale property of stochastic integrals, one has

$$w(T-h, z, p_n, m) \ge \left(\inf_{\nu \in \mathcal{U}} \mathbb{E}\left[f(Z_T^{T-h,z,\nu})\right] - m\right)_+ + \left(\inf_{\nu \in \mathcal{U}} \mathbb{E}\left[\Psi(Z_T^{T-h,z,\nu})\right] - p_n\right)_+ .$$

Therefore, the Lipschitz continuity of f and Ψ together with classical estimates on the process Z, provide the existence of a uniform $C > 0$ such that

$$|w(T-h, z, p_n, m) - (f(z) - m)_+ - (\Psi(z) - p_n)_+| \le C\sqrt{h}\,(1 + |z|) \ . \tag{6}$$

We finally let h tend to zero in (6) to conclude. □

The following assumption is a weaker condition than [23, (H4)] and is key for proving Proposition 3.3 below.

Assumption 3.1 *On $\mathscr{C}_i \times \mathbb{R}$, $0 \le i \le n-1$, if $w(t, z, p_{i+1}, \ldots, p_n, m) = 0$ at some point $(t, z, p_{i+1}, \ldots, p_n, m)$, then there exists an optimal control for problem (3) at $(t, z, p_{i+1}, \ldots, p_n, m)$.*

Remark 3.1 Assumption 3.1 is weaker than [23, (H4)] since it only requires the existence of an optimal control at those points $(t, z, p_{i+1}, \ldots, p_n, m)$ where w is zero. This is the minimal requirement to obtain the characterization of V in Proposition 3.3 below. Differently from [23, (H4)], Assumption 3.1 can be proved to hold, under a suitable convexity assumption (see next section), without requiring a uniform bound in the L^2-norm of controls.

Proposition 3.3 *Under Assumption 3.1, one has on* $\mathscr{C}_i$, $0 \le i \le n-1$,

$$V(t,z,p_{i+1},...,p_n) = \inf\{m \ge 0 : w(t,z,p_{i+1},...,p_n,m) = 0\}\,.$$

Proof Fix $0 \le i \le n-1$. In virtue of Proposition 3.1 it is sufficient to show that for any point $(t,z,p_{i+1},...,p_n,m) \in \mathscr{C}_i \times \mathbb{R}_+$ the following equivalence holds

$$\exists(\nu,\alpha,\eta) \in \mathcal{U} \times \mathcal{A}^{n-i} \times \mathcal{A} \text{ s.t. } M_T^{t,m,\eta} \ge f(Z_T^{t,z,\nu}) \text{ and } P_{t_k}^{t,p_k,\alpha_k} \ge \Psi(Z_{t_k}^{t,z,\nu}),$$
$$i+1 \le k \le n \Leftrightarrow w(t,z,p_{i+1},...,p_n,m) = 0\,. \tag{7}$$

Step 1. Proof of $\Rightarrow$. The implication follows after observing that, when the left-hand side of the equivalence in (7) is satisfied, one has

$$\left(f(Z_T^{t,z,\nu}) - M_T^{t,m,\eta}\right)_+ = 0 \text{ and } \left(\Psi(Z_{t_k}^{t,z,\nu}) - P_{t_k}^{t,p_k,\alpha_k}\right)_+ = 0, \quad i+1 \le k \le n\,, \tag{8}$$

leading to $w(t,z,p_{i+1},...,p_n,m) = 0$.
Step 2. Proof of $\Leftarrow$. Let $w(t,z,p_{i+1},...,p_n,m) = 0$. We appeal to Assumption 3.1 and consider the optimal control $(\nu,\alpha,\eta) \in \mathcal{U} \times \mathcal{A}^{n-i} \times \mathcal{A}$ such that

$$\mathbb{E}\left[\left(f(Z_T^{t,z,\nu}) - M_T^{t,m,\eta}\right)_+ + \sum_{k=i+1}^{n} \left(\Psi(Z_{t_k}^{t,z,\nu}) - P_{t_k}^{t,p_k,\alpha_k}\right)_+\right] = 0\,.$$

Thus (8) holds by the non-negativity of each term, hence the desired implication. □

Remark 3.2 One can easily verify that $\inf\{m \ge 0 : w(T,z,m) = 0\} = f(z) = V(T,z)$ and thus that the result in Proposition 3.3 extends to $\{T\} \times \mathbb{R}^d$.

Proposition 3.3 is critical here as it allows the reformulation of V in terms of the unconstrained optimal control problem described by w whose associated value function satisfies important regularity properties (recall Proposition 3.2). Therefore a complete PDE characterization for w can be provided in Section 5. We point out that problem (3) is a singular optimal control problem characterized by a discontinuous Hamiltonian. As a result, the HJB characterization must be obtained passing through a reformulation of the differential operator for a comparison result to hold (see e.g. [23, 34]). Observe also that, unlike [23], the cost functional associated with w changes on each time interval $[t_i, t_{i+1}[$, $0 \le i \le n-1$, to adapt to the decreasing number of constraints involved. As a result, a discontinuity at each point t_i, $1 \le i \le n$, arises.

4 Existence Results

We give in this section some sufficient conditions ensuring that Assumption 3.1 is satisfied (recall Remark 3.1).

Proposition 4.1 *Assume that U is a convex set, f and Ψ are convex functions, and the coefficients of the diffusion are of the form $\mu(t,z,u) := A(t)z + B(t)u$ and $\sigma(t,z,u) := C(t)z + D(t)u$, for all $(t,z,u) \in [0,T] \times \mathbb{R}^d \times U$, with A, B, C and D matrices of suitable size. If $(t,z,p_{i+1},\dots,p_n,m) \in \mathscr{C}_i \times \mathbb{R}$, for some $0 \le i \le n-1$, is such that $w(t,z,p_{i+1},\dots,p_n,m) = 0$, then, the optimal control problem (3) admits an optimizer at $(t,z,p_{i+1},\dots,p_n,m)$.*

Proof Fix $0 \le i \le n-1$. Let $(\nu^j, \alpha^j, \eta^j) \in \mathcal{U} \times \mathcal{A}^{n-i} \times \mathcal{A}$ be a minimizing sequence for w at point $(t, z, p_{i+1}, \dots, p_n, m) \in \mathscr{C}_i \times \mathbb{R}$ such that $w(t, z, p_{i+1}, \dots, p_n, m) = 0$. Therefore, for any $\varepsilon > 0$, there exists j_0 such that for all $j \ge j_0$ one has

$$\mathbb{E}\left[\left(f(Z_T^{t,z,\nu^j}) - M_T^{t,m,\eta^j}\right)_+ + \sum_{k=i+1}^{n} \left(\Psi(Z_{t_k}^{t,z,\nu^j}) - P_{t_k}^{t,p_k,\alpha_k^j}\right)_+\right] \le w(t, z, p_{i+1}, \dots, p_n, m) + \varepsilon\,.$$

As ν^j is uniformly bounded in the L^2-norm (since ν takes values in the compact set U), there exists a subsequence (still indexed by j) ν^j that weakly converges in the L^2-norm to some $\hat{\nu} \in \mathcal{U}$. Applying Mazur's theorem, one has the existence of $\tilde{\nu}^j \equiv \sum_{\ell \ge 0} \lambda_\ell \nu^{\ell+j}$ with $\lambda_\ell \ge 0$ and $\sum_{\ell\ge 0} \lambda_\ell = 1$ such that $\tilde{\nu}^j$ strongly converges in the L^2-norm to $\hat{\nu}$. We then consider $(\tilde{\nu}^j, \tilde{\alpha}^j, \tilde{\eta}^j) \equiv \sum_{\ell \ge 0} \lambda_\ell (\nu^{\ell+j}, \alpha^{\ell+j}, \eta^{\ell+j})$. Observe that $(\tilde{\nu}^j, \tilde{\alpha}^j, \tilde{\eta}^j)$ still belongs to $\mathcal{U} \times \mathcal{A}^{n-i} \times \mathcal{A}$ as $\mathcal{U}$ and $\mathcal{A}$ are convex spaces.
Let us now consider $(Z_\cdot^{t,z,\tilde{\nu}^j}, M_\cdot^{t,m,\tilde{\eta}^j}, (P_\cdot^{t,p_k,\tilde{\alpha}_k^j})_{i+1 \le k \le n})$. One has

$$\mathbb{E}\left[\sup_{s\in[t,T]} \left|Z_s^{t,z,\tilde{\nu}^j} - Z_s^{t,z,\hat{\nu}}\right|^2\right] \to 0 \text{ for } j \to +\infty\,.$$

It follows that for any $\varepsilon > 0$ there exists j_1 such that for all $j \ge j_1$,

$$\left|\mathrm{J}^{\tilde{\nu}^j,\tilde{\alpha}^j,\tilde{\eta}^j}(t, z, p_{i+1}, ..., p_n, m) - \mathrm{J}^{\hat{\nu},\tilde{\alpha}^j,\tilde{\eta}^j}(t, z, p_{i+1}, ..., p_n, m)\right| \le \varepsilon\,.$$

Moreover, by the linearity of the dynamics of Z one has $Z_\cdot^{t,z,\tilde{\nu}^j} = \sum_{\ell\ge 0} \lambda_\ell Z_\cdot^{t,z,\nu^{\ell+j}}$, and trivially $M_\cdot^{t,m,\tilde{\eta}^j} = \sum_{\ell \ge 0} \lambda_\ell M_\cdot^{t,m,\eta^{\ell+j}}$ and $P_\cdot^{t,p_k,\tilde{\alpha}_k^j} = \sum_{\ell\ge 0} \lambda_\ell P_\cdot^{t,p_k,\alpha_k^{\ell+j}}$ for $i+1 \le k \le n$. Therefore, using the convexity of f and Ψ, one has for $j \ge \max(j_0, j_1)$,

$$\begin{aligned}
&\mathbb{E}\left[\left(f(Z_T^{t,z,\hat{\nu}}) - M_T^{t,m,\tilde{\eta}^j}\right)_+ + \sum_{k=i+1}^{n} \left(\Psi(Z_{t_k}^{t,z,\hat{\nu}}) - P_{t_k}^{t,p_k,\tilde{\alpha}^j}\right)_+\right] \\
&\le \sum_{\ell\ge 0} \lambda_\ell \mathbb{E}\left[\left(f(Z_T^{t,z,\nu^{\ell+j}}) - M_T^{t,m,\eta^{\ell+j}}\right)_+ + \sum_{k=i+1}^{n} \left(\Psi(Z_{t_k}^{t,z,\nu^{\ell+j}}) - P_{t_k}^{t,p_k,\alpha^{\ell+j}}\right)_+\right] + \varepsilon \\
&\le w(t, z, p_{i+1}, \dots, p_n, m) + 2\varepsilon = 2\varepsilon\,.
\end{aligned}$$

Thanks to the arbitrariness of ε and the martingale property of stochastic integrals, it is immediate to verify that the previous inequality gives

$$\mathbb{E}\left[f(Z_T^{t,z,\hat{\nu}})\right] \le m \qquad \text{and} \qquad \mathbb{E}\left[\Psi(Z_{t_k}^{t,z,\hat{\nu}})\right] \le p_k \quad \text{for any } i+1 \le k \le n\,. \tag{9}$$

By the martingale representation theorem, we define $\hat{\eta}, \hat{\alpha}_k \in \mathcal{A}$, $i+1 \le k \le n$, such that

$$\mathbb{E}\left[f(Z_T^{t,z,\hat{\nu}})\right] = f(Z_T^{t,z,\hat{\nu}}) - \int_t^T \hat{\eta}_s^\top \mathrm{d}W_s \text{ and } \mathbb{E}\left[\Psi(Z_{t_k}^{t,z,\hat{\nu}})\right] = \Psi(Z_{t_k}^{t,z,\hat{\nu}}) - \int_t^{t_k} \hat{\alpha}_{k_s}^\top \mathrm{d}W_s\,.$$

In virtue of (9), this gives

$$\mathrm{J}^{\hat\nu,\hat\alpha,\hat\eta}(t,z,p_{i+1},...,p_n,m) = \mathbb{E}\left[\left(f(Z_T^{t,z,\hat\nu}) - M_T^{t,m,\hat\eta}\right)_+ + \sum_{k=i+1}^{n}\left(\Psi(Z_{t_k}^{t,z,\hat\nu}) - P_{t_k}^{t,p_k,\hat\alpha_k}\right)_+\right] = 0\,,$$

from which the optimality of the control $(\hat\nu,\hat\alpha_{i+1},\dots,\hat\alpha_n,\hat\eta)$ follows. □

For $Z := (X, Y)$ (recall Remark 2.1), the result of Proposition 4.1 also holds true if X is independent of ν and the coefficients of Y are of the form

$$\mu_Y(t,z,u) := A(t,x)y + B(t,x)u \quad \text{and} \quad \sigma_Y(t,z,u) := C(t,x)y + D(t,x)u\,,$$

with A, B, C and D matrices of suitable size.

We point out that Assumption 3.1 is the unique important restriction in our approach. Such requirement is only related to the convexity properties of the dynamics and cost functions defining the original optimal control problem and does not involve any viability assumption usually necessary to deal with state constrained problems.

5 A Complete PDE Characterization for w

In this section, we characterize w as the viscosity solution of a suitable HJB equation with specific boundary conditions. We restrict the characterization to $\cup_{0\le i\le n-1}\mathscr{D}_i$, as outside this set $V = \infty$.

5.1 On the Interior of the Domain

The main ingredient towards the PDE characterization of w is the DPP stated below (see Appendix A for the proof).

Theorem 5.1 (DPP) *Fix $(t,z,p_{i+1},...,p_n,m) \in \mathscr{C}_i \times \mathbb{R}$, $0 \le i \le n-1$, and let $t_i \le \theta < t_{i+1}$ be a stopping time. Then*

$$w(t,z,p_{i+1},\dots,p_n,m) = \inf_{\substack{\nu\in\mathcal{U}\\ \alpha\in\mathcal{A}^{n-i}\\ \eta\in\mathcal{A}}} \mathbb{E}\left[w(\theta, Z_\theta^{t,z,\nu}, P_\theta^{t,p_{i+1},\alpha_{i+1}},\dots,P_\theta^{t,p_n,\alpha_n}, M_\theta^{t,m,\eta})\right]. \tag{10}$$

We now consider two functions $\kappa : \mathbb{R} \mapsto \mathbb{R}_+^*$ and $\lambda : \mathbb{R} \mapsto \mathbb{R}_+^*$. For any $u \in U$, $a := (a_{i+1},\dots,a_n) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$, $e \in \mathbb{R}^d$ and for any $\Theta := (t,z,p_{i+1},...,p_n,m,q,A) \in \mathscr{C}_i \times \mathbb{R} \times \mathbb{R}^{d+n-i+1} \times \mathbb{S}^{d+n-i+1}$, $0 \le i \le n-1$, with $q := \left(q^{z\top}, q^{p_{i+1}},...,q^{p_n},q^m\right)^\top$, for $q^z \in \mathbb{R}^d$, $q^{p_k} \in \mathbb{R}$ $(i+1 \le k \le n)$, $q^m \in \mathbb{R}$, and

$$A := \begin{pmatrix} A^{zz} & A^{zp} & A^{zm} \\ A^{zp^\top} & A^{pp} & A^{pm} \\ A^{zm^\top} & A^{pm^\top} & A^{mm} \end{pmatrix} \in \mathbb{S}^{d+n-i+1}\,,$$

for $A^{zz} \in \mathbb{S}^d$, $A^{pp} \in \mathbb{S}^{n-i}$, $A^{mm} \in \mathbb{R}$, $A^{zp} \in \mathbb{R}^{d\times(n-i)}$, $A^{zm} \in \mathbb{R}^d$ and $A^{pm} \in \mathbb{R}^{n-i}$, we define the operators

$$
\begin{aligned}
\mathrm{L}^{u,a,e}_{\kappa,\lambda}(\Theta) &:= -\mu^\top(t,z,u)q^z - \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top(t,z,u)A^{zz}] \\
&\quad - \lambda(m)e^\top\sigma^\top(t,z,u)A^{zm} - \sum_{k=i+1}^{n}\kappa(p_k)a_k^\top\sigma^\top(t,z,u)A^{zp_k}\,, \\
\mathrm{F}^{a,e}_{\kappa,\lambda}(\Theta) &:= -\frac{1}{2}\lambda(m)^2|e|^2A^{mm} - \frac{1}{2}\sum_{k=i+1}^{n}\kappa(p_k)^2|a_k|^2A^{p_kp_k} \\
&\quad - \lambda(m)\sum_{k=i+1}^{n}\kappa(p_k)e^\top a_kA^{p_km}\,.
\end{aligned}
$$

Hereinafter, we identify each component $a_k \in \mathbb{R}^d$, $i+1 \le k \le n$, of the $(n-i)$-tuple a defined above with the corresponding d-dimensional component of the associated vector in $\mathbb{R}^{d(n-i)}$. To alleviate notations we still denote by a_k, $i+1 \le k \le n$, (resp. a) this component (resp. vector). Moreover, for any $\Theta \in \mathscr{C}_i \times \mathbb{R} \times \mathbb{R}^{d+n-i+1} \times \mathbb{S}^{d+n-i+1}$, $c \in \mathbb{R}$, $u \in U$, $b \in \mathcal{S}_{d(n-i)+d}$, with $b := (b_1, b^{\flat\top}_{i+1}, \ldots, b^{\flat\top}_{n}, b^{\sharp\top})^\top$ for $b_1 \in \mathbb{R}$, $b^\sharp \in \mathbb{R}^d$ and $b^\flat := (b^{\flat\top}_{i+1}, \ldots, b^{\flat\top}_{n})^\top \in \mathbb{R}^{d(n-i)}$, we also introduce the following operator

$$
\mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta,c) := \begin{cases} (b_1)^2\left(-c + \mathrm{L}^{u,\bar{b}^\flat,\bar{b}^\sharp}_{\kappa,\lambda}(\Theta) + \mathrm{F}^{\bar{b}^\flat,\bar{b}^\sharp}_{\kappa,\lambda}(\Theta)\right) & b \in \bar{\mathcal{S}}_{d(n-i)+d} \\ \mathrm{F}^{b^\flat,b^\sharp}_{\kappa,\lambda}(\Theta) & b \in \mathcal{D}_{d(n-i)+d} \end{cases},
$$

where $\bar{b}^\flat := \frac{b^\flat}{b_1} = \frac{1}{b_1}(b^{\flat\top}_{i+1}, \ldots, b^{\flat\top}_{n})^\top \in \mathbb{R}^{d(n-i)}$, $\bar{b}^\sharp := \frac{b^\sharp}{b_1} \in \mathbb{R}^d$.
Whenever $\kappa(p) = \lambda(m) = 1$ for all $m, p \in \mathbb{R}$, we shortly write $\mathrm{F}^{a,e}_{\kappa,\lambda} \equiv \mathrm{F}^{a,e}$, $\mathrm{L}^{u,a,e}_{\kappa,\lambda} \equiv \mathrm{L}^{u,a,e}$ and $\mathrm{H}^{u,b}_{\kappa,\lambda} \equiv \mathrm{H}^{u,b}$.

Remark 5.1 The operator $b \mapsto \mathrm{H}^{u,b}_{\kappa,\lambda}$ is continuous on $\mathcal{S}_{d(n-i)+d}$, and

$$
\sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta,c) = \sup_{\substack{u\in U\\ b\in\bar{\mathcal{S}}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta,c)\,.
$$

In what follows, given a smooth function φ defined on $\mathscr{C}_i \times \mathbb{R}$, $0 \le i \le n-1$, the notation $\mathrm{H}^{u,b}_{\kappa,\lambda}\varphi(\cdot)$ stands for $\mathrm{H}^{u,b}_{\kappa,\lambda}(\cdot, \mathrm{D}\varphi(\cdot), \mathrm{D}^2\varphi(\cdot), \partial_t\varphi(\cdot))$. A similar writing holds for the operators L and F.

Theorem 5.2 *Let $0 \le i \le n-1$ and define $\kappa, \lambda : \mathbb{R} \mapsto \mathbb{R}^*_+$. Then, on $\operatorname{int}(\mathscr{D}_i)$ the function w^* (resp. w_*) is a viscosity sub-solution (resp. super-solution) of*

$$
\sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda} w = 0\;. \tag{11}
$$

Proof Fix $0 \le i \le n-1$ and $(t, z, p_{i+1}, \ldots, p_n, m) \in \operatorname{int}(\mathscr{D}_i)$.
Step 1. We first prove the result for κ, λ identically equal to 1.
We prove the super-solution property. Let φ be a smooth function such that

$$
\text{(strict)} \min_{\operatorname{int}(\mathscr{D}_i)} (w_* - \varphi) = (w_* - \varphi)(t, z, p_{i+1}, ..., p_n, m) = 0\,. \tag{12}
$$

Thanks to Theorem 5.1, it follows by standard arguments (see e.g. [17, Section 6.2]) that at point $(t, z, p_{i+1}, ..., p_n, m)$ one has

$$- \partial_t \varphi + \Big[\sup_{\substack{u \in U \\ a \in \mathbb{R}^{d(n-i)} \\ e \in \mathbb{R}^d}} (\mathrm{L}^{u,a,e}\varphi + \mathrm{F}^{a,e}\varphi) \Big]^* \geq 0 \,, \tag{13}$$

where for a given operator K, $[\mathrm{K}]^*$ denotes its upper-semicontinuous envelope. We then verify that

$$\sup_{\substack{u \in U \\ b \in \mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}\varphi(t, z, p_{i+1}, ..., p_n, m) \geq 0 \,.$$

We adapt the arguments in the proof of [34, Theorem 3.1]. According to (13) and by definition of the upper semi-continuous envelope, we can find a sequence $(t_j, z_j, p_{i+1_j}, ..., p_{n_j}, m_j) \in \mathrm{int}(\mathscr{D}_i)$, $q_j \in \mathbb{R}^{d+(n-i)+1}$, $A_j \in \mathbb{S}^{d+n-i+1}$ such that

$$\begin{aligned} &(t_j, z_j, p_{i+1_j}, ..., p_{n_j}, m_j) \to (t, z, p_{i+1}, ..., p_n, m) \\ \text{and } &|(q_j, A_j) - (\mathrm{D}\varphi, \mathrm{D}^2\varphi)(t, z, p_{i+1}, ..., p_n, m)| \leq j^{-1} \,, \end{aligned} \tag{14}$$

and at $(t_j, z_j, p_{i+1_j}, ..., p_{n_j}, m_j)$,

$$-\partial_t \varphi + \sup_{\substack{u \in U \\ a \in \mathbb{R}^{d(n-i)} \\ e \in \mathbb{R}^d}} (\mathrm{L}^{u,a,e}(\cdot, q_j, A_j) + \mathrm{F}^{a,e}(\cdot, q_j, A_j)) \geq -j^{-1} \,.$$

We can find a maximizing sequence $(u_j, a_j, e_j) \in U \times \mathbb{R}^{d(n-i)} \times \mathbb{R}^d$ such that

$$\left(-\partial_t \varphi + \mathrm{L}^{u_j,a_j,e_j}(\cdot, q_j, A_j) + \mathrm{F}^{a_j,e_j}(\cdot, q_j, A_j)\right)(t_j, z_j, p_{i+1_j}, ..., p_{n_j}, m_j) \geq -2j^{-1} \,.$$

We define $b_j := \frac{1}{\sqrt{1+|a_j|^2+|e_j|^2}} \left(1, a_{j,i+1}^\top, \ldots, a_{j,n}^\top, e_j^\top\right)^\top \in \bar{\mathcal{S}}_{d(n-i)+d}$, and get at point $(t_j, z_j, p_{i+1_j}, ..., p_{n_j}, m_j)$,

$$(b_{j_1})^2 \left(-\partial_t \varphi + \mathrm{L}^{u_j,\bar{b}_j^\flat,\bar{b}_j^\sharp}(\cdot, q_j, A_j) + \mathrm{F}^{\bar{b}_j^\flat,\bar{b}_j^\sharp}(\cdot, q_j, A_j)\right) \geq -2j^{-1}(b_{j_1})^2 \,.$$

Appealing now to the relative compactness of the set $\bar{\mathcal{S}}_{d(n-i)+d}$ we obtain the existence of a subsequence (still indexed by j) such that $\lim_{j\to\infty} b_j = \hat{b}$ with $\hat{b} \in \mathcal{S}_{d(n-i)+d}$. Moreover the compactness of U ensures that, again up to a subsequence, $\lim_{j\to\infty} u_j = \hat{u} \in U$. Therefore, using (14) and the continuity of the coefficients of Z, we obtain, after taking the limit over $j \to \infty$,

$$\mathrm{H}^{\hat{u},\hat{b}}\varphi(t, z, p_{i+1}, ..., p_n, m) \geq 0 \,, \tag{15}$$

leading to the required result. By similar arguments we can prove the sub-solution property. In particular, for any smooth function φ such that

$$(\text{strict}) \max_{\mathrm{int}(\mathscr{D}_i)} (w^* - \varphi) = (w^* - \varphi)(t, z, p_{i+1}, ..., p_n, m) = 0 \,, \tag{16}$$

one has

$$\sup_{\substack{u \in U \\ b \in \mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}\varphi(t, z, p_{i+1}, ..., p_n, m) \leq 0 \,. \tag{17}$$

The proof for the choice of κ and λ identically equal to 1 is then completed.
Step 2. We extend the result to the case of general positive functions κ and λ. One can easily observe that (15) (resp. (17)) is equivalent to writing that at point $(t, z, p_{i+1}, ..., p_n, m)$ and for a smooth function φ satisfying (12) (resp. (16)) one has

$$b^\top G(t, z, u, \partial_t\varphi, \mathrm{D}\varphi, \mathrm{D}^2\varphi)b \geq 0, \quad \text{for some } u \in U, b \in \mathcal{S}_{d(n-i)+d} \tag{18}$$

$$\Big(\text{resp.} \quad b^\top G(t, z, u, \partial_t\varphi, \mathrm{D}\varphi, \mathrm{D}^2\varphi)b \leq 0, \quad \text{for all } u \in U, b \in \mathcal{S}_{d(n-i)+d}\Big), \tag{19}$$

where for any $(t, z, u, c, q, A) \in [t_i, t_{i+1}[\times\mathbb{R}^d \times U \times \mathbb{R} \times \mathbb{R}^{d+n-i+1} \times \mathbb{S}^{d+n-i+1}$ we denote by $G \equiv G(t, z, u, c, q, A)$ the following matrix in $\mathbb{S}^{1+d(n-i)+d}$,

$$G := \begin{pmatrix} D & -\frac{1}{2}\left[\sigma^\top A^{zp_{i+1}}\right]^\top & \cdots & \cdots & \cdots & -\frac{1}{2}\left[\sigma^\top A^{zp_n}\right]^\top & -\frac{1}{2}\left[\sigma^\top A^{zm}\right]^\top \\ -\frac{1}{2}\sigma^\top A^{p_{i+1}z^\top} & -\frac{1}{2}A^{p_{i+1},p_{i+1}}I_d & \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\frac{1}{2}A^{p_{i+1}m}I_d \\ \vdots & \mathbf{0} & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \mathbf{0} & \vdots \\ -\frac{1}{2}\sigma^\top A^{p_n z^\top} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\frac{1}{2}A^{p_n,p_n}I_d & -\frac{1}{2}A^{p_n m}I_d \\ -\frac{1}{2}\sigma^\top A^{mz^\top} & -\frac{1}{2}A^{p_{i+1}m}I_d & \cdots & \cdots & \cdots & -\frac{1}{2}A^{p_n m}I_d & -\frac{1}{2}A^{mm}I_d \end{pmatrix},$$

with $\sigma \equiv \sigma(t, z, u)$ and $D := -c - \mu^\top(t, z, u)q^z - \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top(t, z, u)A^{zz}]$. Define the diagonal matrix

$$Q_{\kappa,\lambda}(p_{i+1}, ..., p_n, m) := \operatorname{diag}\left(1, \kappa(p_{i+1})I_d, ..., \kappa(p_n)I_d, \lambda(m)I_d\right) .$$

A straightforward calculation shows that

$$\det[(Q_{\kappa,\lambda}^\top G Q_{\kappa,\lambda})^{(k)}] = \lambda(m)^{2(k-(n-i)d-1)_+} \prod_{j=1}^{\lfloor\frac{k-1}{d}\rfloor+1} \kappa(p_{i+j})^{2\min(d,(1-j)d+k-1)} \det[G^{(k)}],$$

where for any given matrix $M \in \mathbb{S}^{1+d(n-i)+d}$ and $1 \leq k \leq 1+d(n-i)+d$ we denote by $M^{(k)} \in \mathbb{S}^k$ the k-th leading principal sub-matrix of M. Then, the positivity of the functions κ and λ implies that the quadratic forms associated with G and $Q_{\kappa,\lambda}^\top G Q_{\kappa,\lambda}$ have the same sign. Hence, it follows from (18) (resp. (19)) that

$$b^\top Q_{\kappa,\lambda}^\top G(t, z, u, \partial_t\varphi, \mathrm{D}\varphi, \mathrm{D}^2\varphi)Q_{\kappa,\lambda}b \geq 0, \quad \text{for some } u \in U, b \in \mathcal{S}_{d(n-i)+d}$$

$$\Big(\text{resp.} \quad b^\top Q_{\kappa,\lambda}^\top G(t, z, u, \partial_t\varphi, \mathrm{D}\varphi, \mathrm{D}^2\varphi)Q_{\kappa,\lambda}b \leq 0, \quad \text{for all } u \in U, b \in \mathcal{S}_{d(n-i)+d}\Big),$$

leading to

$$\sup_{\substack{u\in U \\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}_{\kappa,\lambda}^{u,b}\varphi(t, z, p_{i+1}, ..., p_n, m) \geq 0 \ \Big(\text{resp.} \sup_{\substack{u\in U \\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}_{\kappa,\lambda}^{u,b}\varphi(t, z, p_{i+1}, ..., p_n, m) \leq 0\Big),$$

which concludes the proof. □

5.2 On the Space Boundaries

We study here the boundary conditions in m and p_k, $1 \leq k \leq n$. We first divide the boundary of $\mathscr{D}_i$, $0 \leq i \leq n-1$, denoted by $\partial\mathscr{D}_i$, into different regions corresponding to the different boundaries associated with the levels of controlled loss. More precisely, given $0 \leq i \leq n-1$, we define $\mathcal{P}_i := \{I : I \subseteq \{i+1, ..., n\}, I \neq \emptyset\}$. For any $I \in \mathcal{P}_i$, we define $I^c := \{i+1, \ldots, n\} \setminus I$, denote by $\mathrm{Card}(I^c)$ its cardinality, and introduce $B_{i,I} := \{(p_{i+1}, ..., p_n) \in [0, \infty[^{n-i}\colon\ p_k = 0 \text{ for } k \in I \text{ and } p_k > 0 \text{ for } k \in I^c\}$, as well as $\mathscr{B}_{i,I} := [t_i, t_{i+1}[\times\mathbb{R}^d \times B_{i,I}$. In particular, $\mathscr{B}_i = \cup_{I\in\mathcal{P}_i}\mathscr{B}_{i,I} \cup \mathrm{int}(\mathscr{B}_i)$. Then $\mathscr{D}_i = \mathrm{int}(\mathscr{D}_i) \cup \partial\mathscr{D}_i$ where $\partial\mathscr{D}_i := (\cup_{I\in\mathcal{P}_i}\mathscr{B}_{i,I} \times \mathbb{R}_+) \cup (\mathrm{int}(\mathscr{B}_i) \times \{0\})$. For any $0 \leq i \leq n-1$, $I \in \mathcal{P}_i$, we define the following functions

– on $\mathscr{C}_i$,

$$w_0(t, z, p_{i+1}, ..., p_n) := \inf_{\substack{\nu\in\mathcal{U} \\ \alpha\in\mathcal{A}^{n-i}}} \mathbb{E}\left[f(Z_T^{t,z,\nu}) + \sum_{k=i+1}^{n} \left(\Psi(Z_{t_k}^{t,z,\nu}) - P_{t_k}^{t,p_k,\alpha_k}\right)_+\right],$$

– if $I^c \neq \emptyset$, on $[t_i, t_{i+1}[\times\mathbb{R}^d \times \mathbb{R}^{\mathrm{Card}(I^c)} \times \mathbb{R}$,

$$w_{1,I}(t, z, (p_k)_{k\in I^c}, m) := \inf_{\substack{\nu\in\mathcal{U} \\ \alpha\in\mathcal{A}^{\mathrm{Card}(I^c)} \\ \eta\in\mathcal{A}}} \mathbb{E}\left[\begin{array}{c}\left(f(Z_T^{t,z,\nu}) - M_T^{t,m,\eta}\right)_+ + \sum_{k\in I}\Psi(Z_{t_k}^{t,z,\nu}) \\ + \sum_{k\in I^c}\left(\Psi(Z_{t_k}^{t,z,\nu}) - P_{t_k}^{t,p_k,\alpha_k}\right)_+\end{array}\right],$$

– if $I^c = \emptyset$, on $[t_i, t_{i+1}[\times\mathbb{R}^d \times \mathbb{R}$,

$$w_{1,I}(t, z, m) := \inf_{\substack{\nu\in\mathcal{U} \\ \eta\in\mathcal{A}}} \mathbb{E}\left[\left(f(Z_T^{t,z,\nu}) - M_T^{t,m,\eta}\right)_+ + \sum_{k=i+1}^{n} \Psi(Z_{t_k}^{t,z,\nu})\right].$$

We extend the definitions above to $t = T$ by setting $w_0(T, z) = f(z)$ on $\mathbb{R}^d$, and $w_{1,I}(T, z, m) = (f(z) - m)_+$ on $\mathbb{R}^d \times \mathbb{R}$ for all $I \in \mathcal{P}_i$.

Remark 5.2 The functions w_0 and $(w_{1,I})_{I\in\mathcal{P}_i}$, $0 \leq i \leq n-1$ can be fully characterized respectively on $\mathscr{B}_i$, $[t_i, t_{i+1}[\times\mathbb{R}^d \times [0, \infty[^{\mathrm{Card}(I^c)}\times\mathbb{R}_+$ if $I^c \neq \emptyset$, and $[t_i, t_{i+1}[\times\mathbb{R}^d \times \mathbb{R}_+$ if $I^c = \emptyset$. This involves the same techniques as those developed to study the function w (see above and hereinafter) but applied on a lower dimensional state space. For this reason, we consider from now on that these functions are a-priori known and continuous on their domain of definition.

The following proposition gives the natural Dirichlet conditions satisfied at the boundary $m = 0$ and $p_k = 0$, $i+1 \leq k \leq n$.

Proposition 5.1 *Fix* $0 \leq i \leq n-1$. *On* $\mathrm{int}(\mathscr{B}_i) \times \{0\}$,

$$w^*(t, z, p_{i+1}, ..., p_n, m) = w_*(t, z, p_{i+1}, ..., p_n, m) = w_0(t, z, p_{i+1}, ..., p_n). \quad (20)$$

Moreover, on $\mathscr{B}_{i,I} \times \mathbb{R}_+$, $I \in \mathcal{P}_i$,

$$w^*(t, z, p_{i+1}, ..., p_n, m) = w_*(t, z, p_{i+1}, ..., p_n, m) = w_{1,I}(t, z, (p_k)_{k\in I^c}, m). \quad (21)$$

Proof We only prove (20) as (21) can be proved similarly. Fix $0 \le i \le n-1$. On one hand, by the martingale property of stochastic integrals, one has on $\mathscr{C}_i \times \mathbb{R}$,

$$w(t,z,p_{i+1},...,p_n,m) \ge w_0(t,z,p_{i+1},...,p_n) - m\,.$$

On the other hand, on $\mathscr{C}_i \times \mathbb{R}$,

$$w(t,z,p_{i+1},...,p_n,m) \le \inf_{\substack{\nu\in\mathcal{U}\\ \alpha\in\mathcal{A}^{n-i}}} \mathbb{E}\left[\begin{matrix}\left(f(Z_T^{t,z,\nu}) - M_T^{t,m,0}\right)_+ \\ + \sum_{k=i+1}^n \left(\Psi(Z_{t_k}^{t,z,\nu}) - P_{t_k}^{t,p_k,\alpha_k}\right)_+\end{matrix}\right].$$

As $f \ge 0$, we obtain that $w(t,z,p_{i+1},...,p_n,m) \le w_0(t,z,p_{i+1},...,p_n)$ for $m \ge 0$. We conclude by taking the upper/lower limit and recalling Remark 5.2. □

5.3 A Comparison Principle for (11)

We consider $\lambda : m \in \mathbb{R} \mapsto \lambda(m) := 1 \vee m > 0$ and $\kappa : p \in \mathbb{R} \mapsto \kappa(p) := 1 \vee p > 0$. The operator in (11) is non-standard as it involves a non-linearity in the time-derivative. However, thanks to a strict super-solution approach (see e.g. [35,36]), a comparison result can be proved.

Lemma 5.1 (Strict Super-Solution Property) *Fix $0 \le i \le n-1$. Let us consider on $\mathscr{D}_i$ the smooth positive function*

$$\phi(t,p_{i+1},...,p_n,m) := e^{(t_{i+1}-t)}\left(1 + \sum_{k=i+1}^n \ln(1+p_k) + \ln(1+m)\right).$$

Let v be a lower semi-continuous viscosity super-solution of (11). Then the function $v + \xi\phi$, $\xi > 0$, satisfies in the viscosity sense

$$\sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(v+\xi\phi) \ge \xi\frac{1}{8} \ \text{ on } \operatorname{int}(\mathscr{D}_i)\,. \tag{22}$$

Proof Fix $0 \le i \le n-1$ and $(t,z,p_{i+1},...,p_n,m) \in \operatorname{int}(\mathscr{D}_i)$. Let $\xi > 0$ and φ be a smooth function such that $\min_{\operatorname{int}(\mathscr{D}_i)}((v+\xi\phi)-\varphi) = ((v+\xi\phi)-\varphi)(t,z,p_{i+1},...,p_n,m) = 0$. Since ϕ is a smooth function, the function $\psi := \varphi - \xi\phi$ is a test function for v.

Let $b \in \bar{\mathcal{S}}_{d(n-i)+d}$ and $u \in U$. We obtain by definition of $\mathrm{H}^{u,b}_{\kappa,\lambda}$,

$$\mathrm{H}^{u,b}_{\kappa,\lambda}\varphi(t,z,p_{i+1},...,p_n,m) \ge \mathrm{H}^{u,b}_{\kappa,\lambda}\psi(t,z,p_{i+1},...,p_n,m) + \mathfrak{A}\,, \tag{23}$$

where

$$\mathfrak{A} = \xi(b_1)^2\left(-\partial_t\phi - \tfrac{1}{2}\textstyle\sum_{k=i+1}^n \kappa(p_k)^2|\bar{b}^\flat_k|^2\mathrm{D}_{p_kp_k}\phi - \tfrac{1}{2}\lambda(m)^2|\bar{b}^\sharp|^2\mathrm{D}_{mm}\phi\right),$$

at point $(t,z,p_{i+1},...,p_n,m)$. We now provide a lower bound for $\mathfrak{A}$. We thus compute

$$\begin{aligned}\mathfrak{A} =& \xi(b_1)^2 e^{(t_{i+1}-t)}\left(1 + \textstyle\sum_{k=i+1}^n \ln(1+p_k) + \ln(1+m)\right)\\ &+ \xi(b_1)^2 e^{(t_{i+1}-t)}\left(\textstyle\sum_{k=i+1}^n \frac{\kappa(p_k)^2|\bar{b}^\flat_k|^2}{2(1+p_k)^2} + \frac{\lambda(m)^2|\bar{b}^\sharp|^2}{2(1+m)^2}\right) \ge \frac{\xi(b_1)^2}{8}\left(1 + |\bar{b}^\flat|^2 + |\bar{b}^\sharp|^2\right),\end{aligned}$$

since $m, p_k \geq 0$, $i+1 \leq k \leq n$, and for any $l \geq 0$, $\frac{1 \vee l^2}{(1+l)^2} \geq \frac{1}{4}$. Noticing that $(b_1)^2(1+|\bar{b}^\flat|^2+|\bar{b}^\sharp|^2) = 1$, we obtain $\mathfrak{A} \geq \xi \frac{1}{8}$. Thanks to the arbitrariness of u and b and (23), one has, after appealing to the super-solution property of v,

$$\sup_{\substack{u \in U \\ b \in \bar{\mathcal{S}}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}\varphi(t, z, p_{i+1}, ..., p_n, m) \geq \xi \frac{1}{8}\,.$$

We finally conclude the proof recalling Remark 5.1. □

We can now state a comparison result holding for viscosity solutions of (11) whose proof, postponed to Appendix B, is based on Lemma 5.1.

Theorem 5.3 (Comparison Principle) *Fix $0 \leq i \leq n-1$. Let V (resp. U) be a lower semi-continuous (resp. upper semi-continuous) function satisfying*

$$|V(t, z, p_{i+1}, ..., p_n, m)| + |U(t, z, p_{i+1}, ..., p_n, m)| \leq C\,(1+|z|) \ \ on\ \mathscr{D}_i\,.$$

Moreover, assume that on $\mathrm{int}(\mathscr{D}_i)$, V (resp. U) is a viscosity super-solution (resp. sub-solution) of (11), on $\partial\mathscr{D}_i$, $V(\cdot) \geq U(\cdot)$, and on $\mathbb{R}^d \times [0,\infty[^{n-i} \times \mathbb{R}_+$, $V(t_{i+1}, \cdot) \geq U(t_{i+1}, \cdot)$. Then $V \geq U$ on $\mathscr{D}_i$.

5.4 A Complete Characterization of w

Thanks to the results in the previous sections we can now obtain a full characterization of w by the HJB equation. Moreover, we obtain the time-continuity of w on each interval, which completes the result derived in Proposition 3.2.

Theorem 5.4 (Complete Characterization of w) *The function w is the unique viscosity solution of (11) on $\mathrm{int}(\mathscr{D}_i)$ for any $0 \leq i \leq n-1$, in the class of functions being continuous on $\mathscr{D}_i$, and satisfying the growth condition (4) together with the following terminal and boundary conditions*

$$\begin{aligned} &w(T, z, m) = (f(z) - m)_+ \ \ on\ \mathbb{R}^d \times \mathbb{R}_+, \\ &w = w_0 \ \ on\ \mathrm{int}(\mathscr{B}_i) \times \{0\}, \qquad w = w_{1,I} \ \ on\ \mathscr{B}_{i,I} \times \mathbb{R}_+,\ \forall I \in \mathcal{P}_i\,, \end{aligned} \tag{24}$$

and on $\mathbb{R}^d \times [0, \infty[^{n-i} \times \mathbb{R}_+$,

$$\lim_{t \uparrow t_{i+1}} w(t, z, p_{i+1}, ..., p_n, m) = w(t_{i+1}, z, p_{i+2}, ..., p_n, m) + (\Psi(z) - p_{i+1})_+\,. \tag{25}$$

Proof By definition, the condition on $\{T\} \times \mathbb{R}^d \times \mathbb{R}_+$ is satisfied. Additionally, we know from Proposition 3.2 (resp. Proposition 5.1 and Remark 5.2) that w satisfies the linear growth condition (4) (resp. the boundary conditions in (24) and is continuous on $(\mathrm{int}(\mathscr{B}_i) \times \{0\} \cup (\cup_{I \in \mathcal{P}_i} \mathscr{B}_{i,I} \times \mathbb{R}_+))$ for any $0 \leq i \leq n-1$). Moreover, it follows from Theorem 5.2 that for any $0 \leq i \leq n-1$, w^* (resp. w_*) is an upper semi-continuous (resp. lower semi-continuous) viscosity sub-solution (resp. super-solution) to (11) on $\mathrm{int}(\mathscr{D}_i)$. To prove the continuity property on $\mathscr{D}_i$, $0 \leq i \leq n-1$, we proceed by induction on i. The uniqueness property and (25) are a by-product of the proof by induction. Let $i = n-1$. Thanks to Proposition 3.2 and 5.1 as well as Theorem 5.2, the uniqueness of the solution to (11) and continuity of w on $\mathrm{int}(\mathscr{D}_{n-1})$ follow from Theorem 5.3. Let us now assume that w is continuous on $\mathrm{int}(\mathscr{D}_{i+1})$ for some $0 \leq i \leq n-2$, and show its continuity on $\mathrm{int}(\mathscr{D}_i)$. The result follows by the same arguments as above once proved (25), in virtue of the Lipschitz continuity of w in the space variables (recall Proposition 3.2). To this aim we start by introducing on $[0, t_{i+2}[\times\mathbb{R}^d\times\mathbb{R}^{n-i-1}\times\mathbb{R}$, $0 \leq i \leq n-2$, the auxiliary function

$$\hat{w}(t,z,p_{i+2},...,p_n,m) = \inf_{\substack{\nu\in\mathcal{U}\\ \alpha\in\mathcal{A}^{n-i-1}\\ \eta\in\mathcal{A}}} \mathrm{J}^{\nu,\alpha,\eta}(t,z,p_{i+2},...,p_n,m)\,.$$

We observe that on $[t_{i+1}, t_{i+2}[\times\mathbb{R}^d \times \mathbb{R}^{n-i-1} \times \mathbb{R}$, $0 \leq i \leq n-2$,

$$\hat{w}(t,z,p_{i+2},...,p_n,m) = w(t,z,p_{i+2},...,p_n,m)\,.$$

Moreover, $\hat{w}^*$ and $\hat{w}_*$ satisfy a linear growth condition on their respective domain, and the induction assumption implies that $\hat{w}$ is continuous on $\mathscr{D}_{i+1}$. Hence

$$\hat{w}^*(t_{i+2},z,p_{i+2},...,p_n,m) = \hat{w}_*(t_{i+2},z,p_{i+2},...,p_n,m) \text{ on } \mathbb{R}^d\times[0,\infty[^{n-i-1}\times\mathbb{R}_+\,.$$

Moreover, proceeding as in Propositions 5.1 (resp. Theorem 5.2), one can derive continuous boundary conditions on $[0,t_{i+2}[\times\mathbb{R}^d\times]0,\infty[^{n-i-1}\times\{0\}$ and $[0,t_{i+2}[\times\mathbb{R}^d\times B_{i+1,I}\times\mathbb{R}_+$ for all $I\in\mathcal{P}_{i+1}$ (resp. characterize $\hat{w}^*$ and $\hat{w}_*$ on $[0,t_{i+2}[\times\mathbb{R}^d\times]0,\infty[^{n-i-1}\times\mathbb{R}^*_+$). Therefore, appealing to Theorem 5.3, one obtains the continuity of $\hat{w}$ on $[0,t_{i+2}[\times\mathbb{R}^d\times]0,\infty[^{n-i-1}\times\mathbb{R}^*_+$.

As a result, for any $(z,p_{i+2},\ldots,p_n,m)\in\mathbb{R}^d\times[0,+\infty[^{n-i-1}\times\mathbb{R}_+$, one has

$$\lim_{h\to 0}|\hat{w}(t_{i+1}-h,z,p_{i+2},\ldots,p_n,m) - w(t_{i+1},z,p_{i+2},\ldots,p_n,m)| = 0\,. \tag{26}$$

Let $h>0$ be such that $t_{i+1}-h\in[t_i,t_{i+1}[$. On $\mathbb{R}^d\times\mathbb{R}^{n-i}\times\mathbb{R}$, one can easily check that

$$w(t_{i+1}-h,z,p_{i+1},\ldots,p_n,m) - \hat{w}(t_{i+1}-h,z,p_{i+2},\ldots,p_n,m) \leq \sup_{\nu\in\mathcal{U}}\mathbb{E}\left[\left(\Psi(Z^{t_{i+1}-h,z,\nu}_{t_{i+1}}) - P^{t_{i+1}-h,p_{i+1},0}_{t_{i+1}}\right)_+\right]\,.$$

Moreover, from the martingale property of stochastic integrals follows

$$w(t_{i+1}-h,z,p_{i+1},p_{i+2},\ldots,p_n,m) - \hat{w}(t_{i+1}-h,z,p_{i+2},\ldots,p_n,m) \geq \left(\inf_{\nu\in\mathcal{U}}\mathbb{E}\left[\Psi(Z^{t_{i+1}-h,z,\nu}_{t_{i+1}})\right] - p_{i+1}\right)_+\,.$$

Therefore, thanks to the Lipschitz continuity of Ψ together with classical estimates on the process Z, there exists a uniform $C>0$ such that

$$|\,(w(\cdot)-\hat{w}(\cdot))\,(t_{i+1}-h,z,p_{i+2},\ldots,p_n,m) - (\Psi(z)-p_{i+1})_+| \leq C\sqrt{h}\,(1+|z|)\,.$$

Condition (25) thus follows by sending h to zero and appealing to (26). □

The PDE characterization provided by Theorem 5.4 is a crucial point towards the numerical approximation of the level-set function w (and consequently of the original value function V). For HJB equations similar to (11) generalized finite difference schemes have been proposed in [37], while more recently classical finite difference schemes have been coupled with piecewise constant policy approximation in [38]. In our framework, a limitation for the use of these numerical methods is represented by the potential high dimensionality of the level-set problem (3) (a dimension is added for each state constraint) that, under the effect of the so-called "curse of dimensionality", leads to a rapid increase of the computational cost. Powerful tools to overcome this issue could be found in the recent developments of machine learning algorithms for PDEs (see e.g. [39, 40]).

Remark 5.3 The techniques developed in this paper can be applied to the case of *next-period* controlled-loss constraints. This type of constraints has been studied for stochastic target problems under a complete market setting in [32]. For a given $\nu \in \mathcal{U}$, they write on $\mathscr{C}_i$, $0 \leq i \leq n-1$, as $\mathbb{E}\left[\Psi(Z^{t,z,\nu}_{t_{i+1}})|\mathcal{F}_t\right] \leq p_{i+1}$ and $\mathbb{E}\left[\Psi(Z^{t,z,\nu}_{t_k})|\mathcal{F}_{t_{k-1}}\right] \leq p_k$, $i+2 \leq k \leq n$, and appealing to the techniques developed in Section 3, one can prove that they re-write as $\Psi(Z^{t,z,\nu}_{t_{i+1}}) \leq P^{t,p_{i+1},\alpha_{i+1}}_{t_{i+1}}$ and $\Psi(Z^{t,z,\nu}_{t_k}) \leq P^{t_{k-1},p_k,\alpha_k}_{t_k}$, $i+2 \leq k \leq n$, for some $(\alpha_{i+1}, ..., \alpha_n, \eta) \in \mathcal{A}^{n-i} \times \mathcal{A}$.

6 Conclusions

Assuming the existence of an optimizer for the value function w defined in (3), we proved that the original value function V in (1) can be described by means of the zero level-set of w. This result has the great advantage of providing a characterization of the value function associated with a state constrained optimal control problem without requiring any viability or strong regularity assumptions on the coefficients of the diffusion process. However, w is associated with an (unconstrained) optimal control problem involving unbounded controls, which raises additional difficulties in the treatment of the associated PDE. We provided a full characterization of the level-set function w as the unique piecewise continuous viscosity solution of a suitable HJB equation passing through a compactification of the differential operator.

This paper opens new avenues for further research. This includes the study of the numerical approximation of V through the characterization of w and the addition of a mean-field term in the definition of V. It would also be interesting to analyze how MFG theory applies to our framework when the objective function and the constraints are expressed as functionals of the probability measure of the underlying controlled state process. Finally, the study of value-at-risk and general optimal expected utility risk-measure constraints would also merit investigation.

Acknowledgements Authors are grateful to the Young Investigator Training Program and Association of Bank Foundations.

Appendix A

Proof of Theorem 5.1. Fix $(t, z, p_{i+1}, ..., p_n, m) \in \mathscr{C}_i \times \mathbb{R}$, $0 \leq i \leq n-1$, and a stopping time $t \leq \theta < t_{i+1}$. We denote $\widehat{w}$ the right-hand side of (10).

Step 1. Proof of $w \geq \widehat{w}$. By the definition of w in (3) and the Flow property one has

$$w(t,z,p_{i+1},...,p_n,m) \geq \inf_{\substack{\nu\in\mathcal{U}\\ \alpha\in\mathcal{A}^{n-i}\\ \eta\in\mathcal{A}}} \mathbb{E}\left[w\left(\theta, Z_\theta^{t,z,\nu}, P_\theta^{t,p_{i+1},\alpha_{i+1}},...,P_\theta^{t,p_n,\alpha_n}, M_\theta^{t,m,\eta}\right)\right],$$

leading to $w(t,z,p_{i+1},...,p_n,m) \geq \widehat{w}(t,z,p_{i+1},...,p_n,m)$.
Step 2. Proof of $w \leq \widehat{w}$. We fix $(\hat{\nu},\hat{\alpha}_{i+1},...,\hat{\alpha}_n,\hat{\eta}) \in \mathcal{U}\times\mathcal{A}^{n-i}\times\mathcal{A}$, and consider μ, the measure induced by $(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa)$ on $\mathscr{C}_i\times\mathbb{R}$ with $\xi := Z_\theta^{t,z,\hat{\nu}}$, $(\zeta_k)_{i+1\leq k\leq n} := (P_\theta^{t,p_k,\hat{\alpha}_k})_{i+1\leq k\leq n}$, and $\kappa := M_\theta^{t,m,\hat{\eta}}$. We appeal to [41, Proposition 7.50, Lemma 7.27] to prove that, for each $\varepsilon>0$, we can build $(n-i+2)$ Borel-measurable maps ν_μ^ε, $\alpha_\mu^\varepsilon \equiv (\alpha_{i+1,\mu}^\varepsilon,...,\alpha_{n,\mu}^\varepsilon)$ and η_μ^ε such that $(\nu_\mu^\varepsilon,\alpha_\mu^\varepsilon,\eta_\mu^\varepsilon)\in\mathcal{U}\times\mathcal{A}^{n-i}\times\mathcal{A}$, and

$$w(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa) \geq \mathrm{J}^{\nu_\mu^\varepsilon,\alpha_\mu^\varepsilon,\eta_\mu^\varepsilon}(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa) - \varepsilon\,. \tag{27}$$

We now use [42, Lemma 2.1] to obtain ν^ε, α^ε and η^ε such that

$$\begin{aligned}
\nu^\varepsilon\mathbf{1}_{[\theta,T]} &= \nu_\mu^\varepsilon(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa)\mathbf{1}_{[\theta,T]}\,\mathrm{d}t\times\mathrm{d}\mathbb{P}\text{-a.e.}\,,\\
\alpha_k^\varepsilon\mathbf{1}_{[\theta,t_k]} &= \alpha_{k,\mu}^\varepsilon(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa)\mathbf{1}_{[\theta,t_k]}\,\mathrm{d}t\times\mathrm{d}\mathbb{P}\text{-a.e.}, i+1\leq k\leq n\,,\\
\eta^\varepsilon\mathbf{1}_{[\theta,T]} &= \eta_\mu^\varepsilon(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa)\mathbf{1}_{[\theta,T]}\,\mathrm{d}t\times\mathrm{d}\mathbb{P}\text{-a.e.}
\end{aligned}$$

This implies that $\hat{\nu}^\varepsilon := \hat{\nu}\mathbf{1}_{[t,\theta[} + \nu^\varepsilon\mathbf{1}_{[\theta,T]}\in\mathcal{U}$, $\hat{\alpha}_k^\varepsilon := \hat{\alpha}_k\mathbf{1}_{[t,\theta[}+\alpha_k^\varepsilon\mathbf{1}_{[\theta,t_k]}\in\mathcal{A}$, $i+1\leq k\leq n$ and $\hat{\eta}^\varepsilon := \hat{\eta}\mathbf{1}_{[t,\theta[}+\eta^\varepsilon\mathbf{1}_{[\theta,T]}\in\mathcal{A}$ and (27) holds where, according to [17, Remark 6.1],

$$\begin{aligned}
&\mathrm{J}^{(\nu_\mu^\varepsilon,\alpha_\mu^\varepsilon,\eta_\mu^\varepsilon)}(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa)\\
&= \mathbb{E}\left[\left(f(Z_T^{\theta,\xi,\nu^\varepsilon}) - M_T^{\theta,\kappa,\eta^\varepsilon}\right)_+ + \sum_{k=i+1}^n \left(\Psi(Z_{t_k}^{\theta,\xi,\nu^\varepsilon}) - P_{t_k}^{\theta,\zeta_k,\alpha_k^\varepsilon}\right)_+\middle|(\theta,\xi,\zeta_{i+1},...,\zeta_n,\kappa)\right].
\end{aligned}$$

We conclude taking the expectation on both sides in (27), and appealing to the arbitrariness of $(\hat{\nu},\hat{\alpha}_{i+1},...,\hat{\alpha}_n,\hat{\eta})\in\mathcal{U}\times\mathcal{A}^{n-i}\times\mathcal{A}$ and ε. □

Appendix B

We first state the following lemma which is involved in the proof of Theorem 5.3.

Lemma .1 (Modulus of Continuity) *Fix $0\leq i\leq n-1$. There exists $\rho>0$ such that for any $t\in[t_i,t_{i+1}[$, $z,r\in\mathbb{R}^d$, $m,l\in\mathbb{R}_+$, $p,q\in[0,\infty[^{n-i}$ (with $p:=(p_{i+1},...,p_n)^\top$ and $q:=(q_{i+1},...,q_n)^\top$), for any $\mathcal{X},\mathcal{Y}\in\mathbb{S}^{d+n-i+1}$ satisfying*

$$\begin{pmatrix}\mathcal{X} & 0\\ 0 & -\mathcal{Y}\end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix}I & -I\\ -I & I\end{pmatrix} + 2\zeta e^{-\rho t}\begin{pmatrix}\bar{I} & \boldsymbol{0}\\ \boldsymbol{0} & \bar{I}\end{pmatrix}, \tag{28}$$

for $\zeta,\varepsilon>0$, with $I\equiv I_{d+n-i+1}$, and $\bar{I}:=\mathrm{diag}(I_d,0,\dots,0)\in\mathbb{S}^{d+n-i+1}$, and for

$$c_1, c_2 \text{ s.t. } c_1 - c_2 = -\zeta\rho e^{-\rho t}\left(1+|z|^2\right) - \zeta\rho e^{-\rho t}\left(1+|r|^2\right)\in\mathbb{R}_-\,,$$

$$\Delta_1 := \begin{pmatrix}\frac{1}{\varepsilon}(z-r)+2\zeta e^{-\rho t}z\\ \frac{1}{\varepsilon}(p-q)\\ \frac{1}{\varepsilon}(m-l)\end{pmatrix},\quad \Delta_2 := \begin{pmatrix}\frac{1}{\varepsilon}(z-r)-2\zeta e^{-\rho t}r\\ \frac{1}{\varepsilon}(p-q)\\ \frac{1}{\varepsilon}(m-l)\end{pmatrix}\in\mathbb{R}^{d+n-i+1}\,,$$

one has, with $\Theta_1:=(t,z,p,m,\Delta_1,\mathcal{X},c_1)$ and $\Theta_2:=(t,r,q,l,\Delta_2,\mathcal{Y},c_2)$,

$$\sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}_{\kappa,\lambda}^{u,b}(\Theta_2) - \sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}_{\kappa,\lambda}^{u,b}(\Theta_1) \leq \frac{C}{\varepsilon}\left(|z-r|^2+|p-q|^2+(m-l)^2\right),$$

for some $C>0$.

Proof Consider Θ_1 and Θ_2 defined in the theorem. We notice (recall Remark 5.1),

$$\sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \sup_{\substack{u\in U\\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta_1) \leq \sup_{\substack{u\in U\\ b\in\bar{\mathcal{S}}_{d(n-i)+d}}} \left\{\mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta_1)\right\} .$$

For $b \in \bar{\mathcal{S}}_{d(n-i)+d}$ and $u \in U$, we compute by definition of $\mathrm{H}^{u,b}_{\kappa,\lambda}$,

$$\mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta_2) - \mathrm{H}^{u,b}_{\kappa,\lambda}(\Theta_1) \leq (b_1)^2 \left(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}\right) ,$$

where

$$\mathfrak{A} = \frac{1}{\varepsilon} \left(\mu(t,z,u) - \mu(t,r,u)\right)^\top (z-r) ,$$

$$\mathfrak{B} = 2\zeta \mathrm{e}^{-\rho t}\mu^\top(t,z,u)z + 2\zeta\mathrm{e}^{-\rho t}\mu^\top(t,r,u)r - \zeta\rho\mathrm{e}^{-\rho t}\left(2 + |z|^2 + |r|^2\right) ,$$

and

$$\mathfrak{C} = -\frac{1}{2}\operatorname{Tr}\left[\bar\sigma\bar\sigma^\top(t,r,q,l,u,b)\mathcal{Y}\right] + \frac{1}{2}\operatorname{Tr}\left[\bar\sigma\bar\sigma^\top(t,z,p,m,u,b)\mathcal{X}\right] ,$$

where for any $(t,r,q,l) \in [t_i,t_{i+1}[\times\mathbb{R}^d\times[0,\infty[^{n-i}\times\mathbb{R}_+$, $b \in \bar{\mathcal{S}}_{d(n-i)+d}$ and $u \in U$, $\bar\sigma(t,r,q,l,u,b)$ is the matrix with rows $\sigma(t,r,u), \kappa(q_{i+1})\bar b^{\flat\,\top}_{i+1}, \dots, \kappa(q_n)\bar b^{\flat\,\top}_n, \lambda(l)\bar b^{\sharp\,\top}$.

Using the Lipschitz and growth properties of μ, we obtain some $C, \hat C > 0$ such that

$$\mathfrak{A} \leq \frac{C}{\varepsilon}|z-r|^2 \quad\text{and}\quad \mathfrak{B} \leq \zeta\hat C\mathrm{e}^{-\rho t}\left(1+|z|^2+|r|^2\right) - \zeta\rho\mathrm{e}^{-\rho t}\left(1+|z|^2+|r|^2\right) .$$

For $\mathfrak{C}$, we use (28) and the Lipschitz continuity of σ, κ and λ to get some $\bar C > 0$ such that

$$\mathfrak{C} \leq \bar C\left(\tfrac{1}{\varepsilon}\left(1+|\bar b^\flat|^2+|\bar b^\sharp|^2\right)\left(|z-r|^2+|p-q|^2+(m-l)^2\right) + \zeta\mathrm{e}^{-\rho t}\left(1+|z|^2+|r|^2\right)\right) .$$

Taking $\rho \geq \hat C + \bar C + 1$ for instance, we obtain for some $C > 0$,

$$\mathfrak{B} + \mathfrak{C} \leq \frac{C}{\varepsilon}\left(1+|\bar b^\flat|^2+|\bar b^\sharp|^2\right)\left(|z-r|^2+|p-q|^2+(m-l)^2\right) .$$

The proof is concluded by observing that $(b_1)^2(1+|\bar b^\flat|^2+|\bar b^\sharp|^2) = 1$. □

We can now prove Theorem 5.3.

Proof of Theorem 5.3. Fix $0 \leq i \leq n-1$. For $\xi > 0$, we introduce on $\mathcal{O} :=]t_i,t_{i+1}[\times\mathbb{R}^d\times[0,\infty[^{n-i}\times\mathbb{R}_+$ the following auxiliary functions

$$V_\xi(t,z,p,m) := (V+\xi\phi)(t,z,p,m) + \xi\left(\frac{1}{t-t_i}\right), \quad U_\xi(t,z,p,m) := (U-\xi\phi)(t,z,p,m) ,$$

where $p := (p_{i+1},...,p_n)^\top$ and with ϕ defined in Lemma 5.1. Appealing to Lemma 5.1, one can easily check that V_ξ is a strict super-solution of (11) satisfying (22) on the interior of $\mathcal{O}$. Analogously U_ξ can be proved to be a sub-solution of (11) on the interior of $\mathcal{O}$.

We prove that $U - V \leq 0$ on $\mathscr{D}_i$. To this aim we first show arguing by contradiction that for all $\xi > 0$, $(U_\xi - V_\xi) \leq 0$ on $\mathcal{O}$, and the proof is completed sending ξ to zero.

Step 1. We assume to the contrary that we can find $\xi > 0$ such that

$$\sup_{\mathcal{O}}(U_\xi - V_\xi) > 0 . \tag{29}$$

We define on $\mathcal{O}$, $\Phi_{\xi,\zeta}(t,z,p,m) := (U_\xi - V_\xi)(t,z,p,m) - 2\zeta e^{-\rho t}(1+|z|^2)$, for $\zeta > 0$ and with $\rho > 0$ defined in Lemma .1. Using the growth conditions and semi-continuity of U and V as well as (29) we obtain that for $\xi, \zeta > 0$ small enough

$$0 < M := \sup_{\mathcal{O}}\Phi_{\xi,\zeta}(t,z,p,m) = \max_{\bar{\mathcal{O}}}\Phi_{\xi,\zeta}(t,z,p,m) < \infty ,$$

where $\bar{\mathcal{O}}$ is a bounded subset of $\mathcal{O}$. On $\mathcal{O} \times \mathcal{O}$, set

$$\begin{aligned}\Psi_{\xi,\zeta,\varepsilon}(t,z,p,m,t,r,q,l) :=& U_\xi(t,z,p,m) - V_\xi(t,r,q,l) - \zeta \mathrm{e}^{-\rho t}(1+|z|^2) - \zeta \mathrm{e}^{-\rho t}(1+|r|^2) \\ &- \frac{1}{2\varepsilon}\left(|z-r|^2 + |p-q|^2 + (m-l)^2\right),\end{aligned}$$

for $\varepsilon > 0$ and with $q := (q_{i+1}, ..., q_n)^\top$. Again, the growth conditions and semi-continuity of U and V ensure that for $\xi, \zeta, \varepsilon > 0$ the function $\Psi_{\xi,\zeta,\varepsilon}$ admits a maximum M_ε at $(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon, t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon)$, with $p_\varepsilon := (p_{i+1_\varepsilon}, ..., p_{n_\varepsilon})^\top$ and $q_\varepsilon := (q_{i+1_\varepsilon}, ..., q_{n_\varepsilon})^\top$, on $\mathrm{cl}\,(\bar{\mathcal{O}}) \times \mathrm{cl}\,(\bar{\mathcal{O}})$ (we omit the dependency on (ξ, ζ) for the sake of clarity). Using standard arguments (see e.g. [43, Lemma 3.1]), one can prove that there exists $(\bar{t}, \bar{z}, \bar{p}, \bar{m}) \in \bar{\mathcal{O}}$, with $\bar{p} := (\bar{p}_{i+1}, ..., \bar{p}_n)^\top$, such that

$$\begin{cases} \lim_{\varepsilon\downarrow 0} t_\varepsilon = \bar{t},\ \lim_{\varepsilon\downarrow 0} z_\varepsilon, r_\varepsilon = \bar{z},\ \lim_{\varepsilon\downarrow 0} p_\varepsilon, q_\varepsilon = \bar{p},\ \lim_{\varepsilon\downarrow 0} m_\varepsilon, l_\varepsilon = \bar{m}\,, \\ \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(|z_\varepsilon - r_\varepsilon|^2 + |p_\varepsilon - q_\varepsilon|^2 + (m_\varepsilon - l_\varepsilon)^2\right) = 0\,, \\ \lim_{\varepsilon\downarrow 0} M_\varepsilon = M = \Phi_{\xi,\zeta}(\bar{t}, \bar{z}, \bar{p}, \bar{m})\,. \end{cases} \tag{30}$$

Moreover, it follows from the boundaries assumptions on V and U that $(\bar{p}, \bar{m}) \neq 0$, i.e. we can assume that $\bar{\mathcal{O}}$ is an open bounded subset of $\mathcal{O}$. As a consequence we assume that, up to a subsequence, $(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon, t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}}$.

Step 2. Using Ishii's Lemma (see [43, Theorem 8.3]) we obtain the existence of real coefficients $\tilde{c}_{1,\varepsilon}, \tilde{c}_{2,\varepsilon}$, two vectors $\tilde{\Delta}_{1,\varepsilon}, \tilde{\Delta}_{2,\varepsilon}$ and two symmetric matrices $\tilde{\mathcal{X}}_\varepsilon$ and $\tilde{\mathcal{Y}}_\varepsilon$ being such that $(\tilde{c}_{1,\varepsilon}, \tilde{\Delta}_{1,\varepsilon}, \tilde{\mathcal{X}}_\varepsilon) \in \bar{\mathcal{J}}^+_{\bar{\mathcal{O}}}(U_\xi(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon) - \zeta \mathrm{e}^{-\rho t_\varepsilon}(1+|z_\varepsilon|^2))$ and $(\tilde{c}_{2,\varepsilon}, \tilde{\Delta}_{2,\varepsilon}, \tilde{\mathcal{Y}}_\varepsilon) \in \bar{\mathcal{J}}^-_{\bar{\mathcal{O}}}(V_\xi(t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon) + \zeta \mathrm{e}^{-\rho t_\varepsilon}(1+|r_\varepsilon|^2))$, with $\bar{\mathcal{J}}^+$ (resp. $\bar{\mathcal{J}}^-$) the limiting second-order super-jet (resp. sub-jet) of U_ξ (resp. V_ξ) at $(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon) \in \bar{\mathcal{O}}$ (resp. $(t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon) \in \bar{\mathcal{O}}$) and where

$$\tilde{c}_{1,\varepsilon} - \tilde{c}_{2,\varepsilon} = 0,\ \tilde{\Delta}_{1,\varepsilon} = \tilde{\Delta}_{2,\varepsilon} = \begin{pmatrix} \frac{1}{\varepsilon}(z_\varepsilon - r_\varepsilon) \\ \frac{1}{\varepsilon}(p_\varepsilon - q_\varepsilon) \\ \frac{1}{\varepsilon}(m_\varepsilon - l_\varepsilon) \end{pmatrix} \text{ and } \begin{pmatrix} \tilde{\mathcal{X}}_\varepsilon & 0 \\ 0 & -\tilde{\mathcal{Y}}_\varepsilon \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Hence with [43, Remark 2.7 (ii)], one has the existence of $(c_{1,\varepsilon}, \Delta_{1,\varepsilon}, \mathcal{X}_\varepsilon) \in \bar{\mathcal{J}}^+_{\bar{\mathcal{O}}} U_\xi(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon)$ and $(c_{2,\varepsilon}, \Delta_{2,\varepsilon}, \mathcal{Y}_\varepsilon) \in \bar{\mathcal{J}}^-_{\bar{\mathcal{O}}} V_\xi(t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon)$, such that

$$c_{1,\varepsilon} = \tilde{c}_{1,\varepsilon} - \zeta\rho \mathrm{e}^{-\rho t_\varepsilon}(1+|z_\varepsilon|^2) \text{ and } c_{2,\varepsilon} = \tilde{c}_{2,\varepsilon} + \zeta\rho \mathrm{e}^{-\rho t_\varepsilon}(1+|r_\varepsilon|^2)\,, \tag{31a}$$

$$\Delta_{1,\varepsilon} = \tilde{\Delta}_{1,\varepsilon} + 2\zeta \mathrm{e}^{-\rho t_\varepsilon}(z_\varepsilon, 0, 0)^\top \text{ and } \Delta_{2,\varepsilon} = \tilde{\Delta}_{2,\varepsilon} - 2\zeta \mathrm{e}^{-\rho t_\varepsilon}(r_\varepsilon, 0, 0)^\top\,, \tag{31b}$$

$$\begin{pmatrix} \mathcal{X}_\varepsilon & 0 \\ 0 & -\mathcal{Y}_\varepsilon \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{X}}_\varepsilon & 0 \\ 0 & -\tilde{\mathcal{Y}}_\varepsilon \end{pmatrix} + 2\zeta \mathrm{e}^{-\rho t_\varepsilon}\begin{pmatrix} \bar{I} & \mathbf{0} \\ \mathbf{0} & \bar{I} \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\zeta \mathrm{e}^{-\rho t_\varepsilon}\begin{pmatrix} \bar{I} & \mathbf{0} \\ \mathbf{0} & \bar{I} \end{pmatrix}, \tag{31c}$$

with $I \equiv I_{d+n-i+1}$ and $\bar{I}$ defined in Lemma .1. Thus, Lemma .1 and (31) imply

$$\begin{aligned}\sup_{\substack{u\in U \\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon, \Delta_{2,\varepsilon}, \mathcal{Y}_\varepsilon, c_{2,\varepsilon}) - \sup_{\substack{u\in U \\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon, \Delta_{1,\varepsilon}, \mathcal{X}_\varepsilon, c_{1,\varepsilon}) \\ \leq \frac{C}{\varepsilon}\left(|z_\varepsilon - r_\varepsilon|^2 + |p_\varepsilon - q_\varepsilon|^2 + (m_\varepsilon - l_\varepsilon)^2\right),\end{aligned} \tag{32}$$

for some $C > 0$. Sending ε to zero and using (30), the last inequality is non-positive.

Step 3. We also know from the definition of U_ξ and V_ξ that they are respectively sub-/super-solution of (11) on $\bar{\mathcal{O}}$. As a result, appealing to Lemma 5.1 we obtain

$$\begin{aligned}&\sup_{\substack{u\in U \\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(t_\varepsilon, r_\varepsilon, q_\varepsilon, l_\varepsilon, \Delta_{2,\varepsilon}, \mathcal{Y}_\varepsilon, c_{2,\varepsilon}) \\ &\quad - \sup_{\substack{u\in U \\ b\in\mathcal{S}_{d(n-i)+d}}} \mathrm{H}^{u,b}_{\kappa,\lambda}(t_\varepsilon, z_\varepsilon, p_\varepsilon, m_\varepsilon, \Delta_{1,\varepsilon}, \mathcal{X}_\varepsilon, c_{1,\varepsilon}) \geq \xi\frac{1}{8} > 0\,,\end{aligned}$$

contradicting (32). Hence $(U_\xi - V_\xi) \leq 0$ for all $\xi > 0$ on $\mathcal{O}$. □

Conflict of Interest: The authors declare that they have no conflict of interest.

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