Diophantine Approximation as Cosmic Censor for AdS Black Holes

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Declaration

This dissertation is based on research done while a graduate student at the Department of Pure Mathematics and Mathematical Statistics and the Cambridge Centre for Analysis of the University of Cambridge, in the period between October 2016 and May 2020. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the following and specified in the text. This dissertation has not been submitted for any other degree or qualification. The thesis consists of three chapters.

Chapter 1 is based on joint work with Yakov Shlapentokh-Rothman (Princeton University) to which both authors have contributed equally. The work is published as follows.

> Kehle, C., Shlapentokh-Rothman, Y. "A Scattering Theory for Linear Waves on the Interior of Reissner–Nordström Black Holes."*Ann. Henri Poincaré* **20**, 1583–1650 (2019).

Chapter 2 and Chapter 3 are my own work and not outcome of work done in collaboration.

Chapter 2 is based on the paper

> Kehle, C. "Uniform Boundedness and Continuity at the Cauchy Horizon for Linear Waves on Reissner–Nordström–AdS Black Holes." *Commun. Math. Phys.* **376**, 145–200 (2020).

Chapter 3 consists of novel research which has not been published.

Christoph Kehle
Munich
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Summary

The present thesis reveals a novel connection of *Diophantine approximation* arising from small divisors to *general relativity*, more precisely, the Strong Cosmic Censorship conjecture. The main results provide theorems which resolve a linear scalar analog of the Strong Cosmic Censorship conjecture in general relativity for $\Lambda < 0$. The proofs are intimately tied to small divisors and the resolution crucially depends on suitable Diophantine conditions. A further ingredient is the novel scattering theory on black hole interiors established at first. The thesis consists of three parts.

In the first part we develop a scattering theory for the linear wave equation $\Box_g \psi = 0$ on the interior of Reissner–Nordström black holes. The main result shows the existence, uniqueness and asymptotic completeness of finite energy scattering states on the interior of Reissner–Nordström. The past and future scattering states are represented as suitable traces of the solution ψ on the bifurcate event and Cauchy horizons. Finally, we prove that, in contrast to the above, on the Reissner–Nordström–(Anti-)de Sitter interior, there is no analogous finite T energy scattering theory for either the linear wave equation or the Klein–Gordon equation with conformal mass. This part is joint work with Yakov Shlapentokh-Rothman (Princeton University).

The second and third parts are motivated by the Strong Cosmic Censorship Conjecture for asymptotically AdS spacetimes. We consider smooth linear perturbations governed by the conformal wave equation $\Box_g \psi - \frac{2}{3}\Lambda\psi = 0$ on Reissner–Nordström–AdS and Kerr–AdS black holes, respectively. We prescribe initial data on a spacelike hypersurface of a Reissner–Nordström–AdS and Kerr–AdS black hole and impose Dirichlet (reflecting) boundary conditions at infinity. It was known previously by work of Holzegel–Smulevici that such waves only decay at a sharp logarithmic rate (in contrast to the polynomial rate in the asymptotically flat regime) in the black hole exterior. In view of this slow decay the question of uniform boundedness or blow-up at the Cauchy horizon in the black hole interior (and thus the validity of the linear scalar analog of the C^0-formulation of the Strong Cosmic Censorship conjecture) has remained up to now open.

In the second part of the thesis, we answer the question of uniform boundedness in the affirmative for Reissner–Nordström–AdS: We show that $|\psi| \leq C$ in the black hole interior. In this setting, this corresponds to the statement that the linear scalar analog of the C^0-formulation of Strong Cosmic Censorship is false. The proof follows a new approach, combining physical space estimates with Fourier based estimates exploited in the scattering theory developed in the first part.

In the third part of the thesis, we show that $|\psi| \to \infty$ at the Cauchy horizon of

Kerr–AdS if the dimensionless black hole parameters mass $\mathfrak{m} = M\sqrt{-\Lambda}$ and angular momentum $\mathfrak{a} = a\sqrt{-\Lambda}$ satisfy certain Diophantine properties. This is in stark contrast to the second part as well as previous works on Strong Cosmic Censorship for $\Lambda \geq 0$. In particular, as a result of the Diophantine conditions, we show that these resonant black hole parameters form a Baire-generic but Lebesgue-exceptional subset of parameters below the Hawking–Reall bound. On the other hand, we conjecture that, as is the case for Reissner–Nordström–AdS, linear waves remain bounded at the Cauchy horizon $|\psi| \leq C$ for a set of black hole parameters which is Baire-exceptional but Lebesgue-generic. This means that the answer to the above question concerning uniform boundedness or blow-up on the Kerr–AdS interior is either negative or affirmative depending on the parameters considered. Thus, in this setting, the validity of the linear scalar analog of the C^0-formulation of Strong Cosmic Censorship depends in an unexpected way on the notion of genericity imposed.

Contents

Prologue

> Diese Coefficienten erscheinen aber in Bruchform, und es werden die Nenner unendlich klein, wenn die Summe der absoluten Beträge der ganzen Zahlen $\nu_1, \ldots, \nu_r$ unendlich gross wird. Es muss also gezeigt werden, dass auch die Zähler unendlich klein werden, und ebenso die Brüche selbst, was bei der complizierten Zusammensetzung der Ausdrücke unmöglich erscheint.
>
> *Karl Weierstraß, 1878*

In this excerpt of a letter addressed to Sofya Kovalevskaya in 1878, Weierstraß describes his ongoing attempts at constructing quasiperiodic solutions to the n-body problem in celestial mechanics [103, p. 31]. Motivated by the quest for a rigorous proof of the stability of the solar system within ***Newtonian gravity***, he tried to prove the existence of such quasiperiodic solutions via successive approximation, the so-called Lindstedt series. As apparent from his notes to Kovalevskaya [103], he struggled showing the convergence of the formal expansion due to the inevitable occurrence of ***small divisors***, which Laplace and Lagrange had already encountered previously [88]. Although lacking a proof, Weierstraß was convinced that the Lindstedt series can be shown to converge. His hopes were founded in a remark of Dirichlet to Kronecker in 1858, in which Dirichlet claimed to have shown such a series expansion [103, p. 48]. Unfortunately, Dirichlet died shortly after without leaving any written work supporting his claims. Being intrigued by this problem of convergence, Weierstraß also reached out to Mittag-Leffler who persuaded the Swedish King Oscar II to sponsor a prize for a resolution of the problem on the occasion of the King's 60th birthday in 1889. The prize was awarded—after some famous corrections—to Henri Poincaré for his groundbreaking work [126]. Instead of proving the convergence of the series, Poincaré's revolutionary final submission and subsequent work [127] actually suggested something unexpected—instability and chaos! This made Poincaré very doubtful, yet still cautious ("Les raisonnements de ce Chapitre ne me permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invraisemblable." [127, Vol. II, p. 104]), whether such Lindstedt series can ever be convergent. He saw the problem of small divisors as unavoidable and at the very nature of things ("à la nature même des choses" [125, p. 217]).

It was only several decades later in the mid 20th century that the existence of quasiperiodic orbits and a definitive resolution (see also [136]) of the prize question in honor of King Oscar II was found in a conglomerate of works by Siegel [135], Kolmogorov [85], Arnold [3] and Moser [109] in terms of the celebrated KAM theorem. The principle result of the KAM theorem shows the existence of quasiperiodic orbits for a general class of dynamical systems including the n-body problem in Newtonian gravity. These sets of orbits arise from ***Diophantine conditions*** to "avoid" the small divisors and have a bizarre Cantor set-like structure. This is consistent with non-existence results proved by Poincaré for sufficiently nice sets with nonempty interior. The question of stability or instability becomes even more peculiar as its answer crucially depends on the notion of genericity imposed. Indeed, a consequence of the KAM theory is that these quasiperiodic orbits are generic in the sense of the Lebesgue measure, while being exceptional in the sense of Baire.

The main result of the present thesis reveals a novel connection of the small divisor problem and Diophantine approximation to ***general relativity***. General relativity generalizes Newtonian gravity and is the mathematical theory formulated by Albert Einstein [43] in 1915 upon which our contemporary understanding of gravitational physics rests. One of the most celebrated yet wonderfully simple predictions of general relativity is the existence of *black holes*. The mathematical formulation of general relativity considers space and time as part of a four-dimensional geometric entity—*spacetime*. Its evolution is governed by the Einstein equations which determine how spacetime is curved. This thesis is concerned with the problem of where and how this determinism breaks down in the context of black holes. This foundational question of determinism and the statement that general relativity is a deterministic theory were first mathematically formulated by Roger Penrose [123] as the ***Strong Cosmic Censorship (SCC)*** Conjecture.

Conjecture (Strong Cosmic Censorship [123])**.** *For generic initial data for the Einstein equations, the maximal Cauchy development is inextendible as a suitable regular spacetime.*

A crucial ingredient [44] in the Einstein equations is the so-called cosmological constant Λ. For $\Lambda \geq 0$, remarkable progress in proving and disproving different formulations (see already the discussion around Conjecture 1 and Conjecture 2) of the Strong Cosmic

Censorship conjecture has been made over the last two decades. For $\Lambda < 0$, despite its prominent role in theoretical physics [97, 146], the Strong Cosmic Censorship conjecture and its various versions have remained open up until now. The main results in Chapter 2 and Chapter 3 of this thesis provide theorems which resolve the linear scalar analog of the Strong Cosmic Censorship conjecture in its strongest (C^0-) formulation for $\Lambda < 0$:

(1) The amplitude of linear waves perturbing a charged, non-rotating black hole remains finite in its interior.

(2) On the other hand, if the black hole is rotating, the amplitude of such waves blows up in black hole interiors if a suitable ratio of the black hole parameters mass, angular momentum and cosmological constant satisfies certain Diophantine conditions.

We further conjecture that the amplitude of such waves remains bounded if the Diophantine conditions are not fulfilled. Thus we see that which of the scenarios happens—boundedness or blow-up—crucially depends on the notion of genericity imposed. We prove that instability and blow-up occurs for Baire-generic but Lebesgue-exceptional parameters, whereas we now conjecture boundedness for Baire-exceptional but Lebesgue-generic parameters. This resembles some of the key aspects and insights of KAM theory within the framework of general relativity. Even if Lebesgue-genericity is imposed, we argue that stability is somewhat weak as such a parameter set has empty interior and any quantitative bound on the amplitude of ψ can in principle be arbitrarily large. In this sense, the fate of observers falling into such black holes, though bleak, is determined—answering the linear scalar analog of the Strong Cosmic Censorship conjecture in the affirmative. ***Thus, surprisingly, small divisors and Diophantine approximation, the "villains" of the stability of the solar system in Newtonian gravity, may turn out to be the elusive "Cosmic Censor" which Penrose was searching for in order to protect determinism in general relativity*** [123]. We see in our results that the pervasive nature of instability in dynamical systems can sometimes be conscripted to do good—a vindication, perhaps, of Poincaré's faith in instabilities caused by small divisors.

> L'instabilité est donc la règle et la stabilité est l'exception.
>
> *Henri Poincaré, 1885*
> *[124, p. 172]*

Introduction

In this thesis, we study linear scalar perturbations ψ solving the conformal wave equation

$$\Box_g \psi - \frac{2}{3}\Lambda\psi = 0 \tag{WE}$$

on the interior of black hole spacetimes. We will first consider Reissner–Nordström black holes [132, 113] which are asymptotically flat ($\Lambda = 0$) and spherically symmetric solutions to the Einstein equations

$$\mathrm{Ric}(g)_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{EE}$$

coupled to the Maxwell equations through the energy-momentum tensor $T_{\mu\nu}$, see already (1.2.1). Reissner–Nordström black holes are parameterized by their charge Q and mass M and we will focus on the subextremal range $0 < |Q| < M$. Then, in the main part of the thesis we will study asymptotically Anti-de Sitter (AdS) black holes which are solutions to (EE) for $\Lambda < 0$. We will consider subextremal Reissner–Nordström–AdS black holes [13], which solve the Einstein–Maxwell system in spherical symmetry for cosmological constant $\Lambda < 0$. Moreover, we will study the rotating Kerr–AdS black holes [83] which solve the Einstein equations for $\Lambda < 0$ in vacuum, i.e. $T_{\mu\nu} = 0$. Kerr–AdS black holes are parameterized by their mass M and angular momentum a and we consider subextremal parameters below the Hawking–Reall bound. All of the above black holes violate determinism in the sense that they all posses a smooth Cauchy horizon. Thus, they play an essential role in the Strong Cosmic Censorship conjecture discussed later.

In Chapter 1, we develop the first scattering theory for linear waves solving (WE) on the interior of Reissner–Nordström black holes. We also prove that in contrast to Reissner–Nordström, there is no analogous finite T energy scattering theory (WE) on the Reissner–Nordström–(Anti-)de Sitter interior. A detailed introduction to the scattering problem will given in Chapter 1. The insights from Chapter 1 will also play a key role for the second and third part.

Chapter 2 and Chapter 3 constitute the main part of the thesis. We consider perturbations ψ solving (WE) arising from initial data on a spacelike hypersurface on Reissner–Nordström–AdS and Kerr–AdS black holes [13]. We also consider reflecting boundary conditions at infinity. We treat the cases of Reissner–Nordström–AdS and Kerr–AdS in Chapter 2 and Chapter 3, respectively.

Our main result of Chapter 2 is Theorem 2.1 which shows that perturbations ψ solving (WE) remain ***uniformly bounded*** $|\psi| \leq C$ in the black hole interior and extend ***continuously*** across the Cauchy horizon of Reissner–Nordström–AdS. This corresponds to the statement that the linear analog of the C^0-formulation of Strong Cosmic Censorship is false. Our result is surprising because in contrast to black hole backgrounds with non-negative cosmological constants ($\Lambda \geq 0$), the decay of ψ in the exterior region for asymptotically AdS black holes ($\Lambda < 0$) is only *logarithmic* as shown by Holzegel–Smulevici [75] (cf. polynomial [129, 29, 2] ($\Lambda = 0$) and exponential [10, 42] ($\Lambda > 0$)). Indeed, the logarithmic decay is too slow to adapt the mechanism exploited in previous studies of black hole interiors [23, 51, 26]. The proof of Theorem 2.1 will now follow a new approach, combining physical space estimates with Fourier based estimates exploited in the scattering theory developed in Chapter 1.

Our main result Theorem 3.1 of Chapter 3 shows that perturbations ψ solving (WE) ***blow up*** everywhere at the Cauchy horizon on Kerr–AdS if the dimensionless black hole's mass $\mathfrak{m} = M\sqrt{-\Lambda}$ and angular momentum $\mathfrak{a} = a\sqrt{-\Lambda}$ satisfy certain Diophantine properties. We show that such black hole parameters are ***Baire-generic*** but ***Lebesgue-exceptional***. This is in sharp contrast to the result in Chapter 2 on Reissner–Nordström–AdS black holes. We also conjecture that, if the parameters $\mathfrak{m}$ and $\mathfrak{a}$ do not satisfy the Diophantine conditions, linear perturbations remain ***bounded*** at the Cauchy horizon. This would be the case for ***Lebesgue-generic*** but ***Baire-exceptional*** black hole parameters.

In the rest of the introduction we will focus on Chapter 2 and Chapter 3 which constitute the main results of the thesis. The above results can be viewed as providing a surprising—mixed—resolution of the linear scalar analog of the C^0-formulation of the Strong Cosmic Censorship conjecture. We will briefly present the various formulations of the Strong Cosmic Censorship conjecture, review relevant previous work and give an outline of the main results and difficulties. This will be complemented with a detailed discussion in the introductions of Chapter 2 and Chapter 3, respectively. We will briefly mention how the scattering theory developed in Chapter 1 fits into the above but postpone the discussion and motivation of the scattering problem to the introduction of Chapter 1.

The Strong Cosmic Censorship Conjecture

Our main motivation for studying linear perturbations on black hole interiors is to shed light on one of the most fundamental problems in general relativity: The Kerr (–de Sitter or –Anti-de Sitter) and Reissner–Nordström (–de Sitter or –Anti-de Sitter) black holes share the property that in addition to the *event horizon* $\mathcal{H}$, they hide another horizon, the so-called *Cauchy horizon* $\mathcal{CH}$ in their interiors. This Cauchy horizon defines the boundary beyond which initial data on a spacelike hypersurface (together with boundary conditions at infinity in the asymptotically AdS case) no longer uniquely determine the spacetime as a solution of (EE). In particular, these spacetimes admit infinitely many smooth extensions beyond their Cauchy horizons solving (EE). This severe violation of determinism is conjectured to be an artifact of the high degree of symmetry in those explicit spacetimes and generically, due to blue-shift instabilities, it is expected that some sort of singularity ought to form at or before the Cauchy horizon. The presence of this singularity is paradoxically "good" because—if sufficiently strong—it can be argued that it restores determinism as the fate of any observer, though bleak, is determined. Making this precise gives rise to various formulations of what is known as the *Strong Cosmic Censorship Conjecture* (SCC) [123, 17]. A full resolution of the SCC conjecture would also include a precise description of the breakdown of regularity at or before the Cauchy horizon.

We begin with the C^0-formulation of the SCC conjecture which can be seen as the strongest inextendability statement in this context. Formally, this formulation can be regarded as the statement that observers crossing the Cauchy horizon are torn apart by infinite tidal deformations [26].

Conjecture 1 (C^0-formulation of Strong Cosmic Censorship)**.** *For generic compact, asymptotically flat or asymptotically Anti-de Sitter vacuum initial data, the maximal Cauchy development of* (EE) *is inextendible as a Lorentzian manifold with C^0 metric.*

Surprisingly, the C^0-formulation (Conjecture 1) was recently proved to be false for both cases $\Lambda = 0$ and $\Lambda > 0$ (see discussion later, [26]). However, the following weaker, yet still well-motivated, formulation introduced by Christodoulou in [17] is still expected to hold true (at least) in the asymptotically flat case ($\Lambda = 0$).

Conjecture 2 (Christodoulou's reformulation of Strong Cosmic Censorship)**.** *For generic asymptotically flat vacuum initial data, the maximal Cauchy development of* (EE) *is inextendible as a Lorentzian manifold with C^0 metric and locally square integrable Christoffel symbols.*

Unlike the C^0-formulation of Conjecture 1, the statement of Conjecture 2 does not guarantee the complete destruction of observers approaching the boundary of spacetime. However, it restores determinism in the sense that even just weak solutions must break down at the boundary of spacetime. Nonetheless, one may always be worried about what notion of weak solution is finally the correct one [122, 93, 96]. In this sense it is a pity that Conjecture 1 is false in the $\Lambda \geq 0$ cases as it would have provided a much more definitive resolution of the Strong Cosmic Censorship conjecture.

Linear scalar analog of the Strong Cosmic Censorship conjecture

The aforementioned formulations of SCC have linear scalar analogs on the level of (WE). Indeed, under the identification $\psi \sim g$, equation (WE) can be seen as a naive linearization of the Einstein equations (EE) after neglecting the nonlinearities and the tensorial structure. Moreover, many phenomena and difficulties for the full Einstein equations (EE) are already present at the level of (WE). The linear scalar analog of Conjecture 1 in a neighborhood of Kerr and Kerr–(Anti-)de Sitter corresponds to the statement that linear perturbations arising from smooth data on a spacelike hypersurface solving (WE) blow up (in L^∞) at the Cauchy horizon.

Conjecture 3 (Linear scalar analog of the C^0-formulation of SCC (Conjecture 1))**.** *Linear perturbations ψ solving* (WE) *of subextremal Reissner–Nordström–(dS/AdS) or Kerr–(dS/AdS) black holes blow up in amplitude*

$$|\psi| \to +\infty$$

at the Cauchy horizon.

Remark that in order to show Conjecture 3, it suffices to show that there exists one solution ψ which blows up at the Cauchy horizon. Indeed, since (WE) is linear, any solution which does not blow up would be manifestly exceptional in that case. The reformulation due to Christodoulou (Conjecture 2) finds its linear scalar analog in the H^1_{loc} blow up of ψ at the Cauchy horizon in view of the identification $\partial\psi \sim \Gamma$.

Conjecture 4 (Linear scalar analog of Christodoulou's reformulation of SCC (Conjecture 2))**.** *Linear perturbations ψ solving* (WE) *of subextremal Reissner–Nordström–(dS/AdS) or Kerr–(dS/AdS) black holes blow up in local energy*

$$\|\psi\|_{H^1_{\mathrm{loc}}} \to +\infty$$

at the Cauchy horizon.

The state of the art for $\Lambda = 0$ and $\Lambda > 0$

Linear level for $\Lambda = 0$***.*** In the asymptotically flat case ($\Lambda = 0$) it was shown in [51, 52] (see also [68]) that solutions of (WE) arising from data on a spacelike hypersurface remain continuous and uniformly bounded (no C^0 blow-up) at the Cauchy horizon of subextremal Kerr or Reissner–Nordström black hole interiors, hence disproving Conjecture 3 for $\Lambda = 0$. (For the extremal case see [56, 57].) The key method for the proof is to use the polynomial decay on the event horizon proved in [29] (with rate $|\psi| \lesssim v^{-p}$ and $p > 1$) and propagate it into the interior. The boundedness and continuity of ψ at the Cauchy horizon was then concluded from red-shift estimates, energy estimates associated to the novel vector field $S = |u|^p \partial_u + |v|^p \partial_v$ and commuting with angular momentum operators followed by Sobolev embeddings. Here u, v are Eddington–Finkelstein-type null coordinates in the interior.

Besides the above C^0 boundedness, it was proved that the (non-degenerate) local energy at the Cauchy horizon blows up for a generic set of solutions ψ in Reissner–Nordström [89] and Kerr [30] black holes. (Note that this blow-up is compatible with the finiteness of the flux associated to S because ∂_v and ∂_u degenerate at the Cauchy horizons $\mathcal{CH}_A$ and $\mathcal{CH}_B$, respectively.) A similar blow-up behavior was obtained for Kerr in [94] assuming lower bounds (which were shown afterwards in [69]) on the energy decay rate of a solution along the event horizon. These results prove Conjecture 4 in the affirmative for $\Lambda = 0$ and support the validity of Conjecture 2.

Another type of result proved in Chapter 1 (see also [82]) is a finite energy scattering theory for solutions of (WE) from the event horizon $\mathcal{H}_R \cup \mathcal{H}_L$ to the Cauchy horizon $\mathcal{CH}_R \cup \mathcal{CH}_L$ in the interior of Reissner–Nordström black holes. In this scattering theory a linear isomorphism between the degenerate energy spaces (associated to the Killing field $T = \partial_v - \partial_u$) corresponding to the event and Cauchy horizon is established. The question reduces to obtaining uniform control over transmission and reflection coefficients $\mathfrak{T}(\omega, \ell)$ and $\mathfrak{R}(\omega, \ell)$ corresponding to fixed frequency solutions. Intuitively, for a purely incoming wave at the event horizon $\mathcal{H}_R$, the transmission and reflection coefficients correspond to the amount of T-energy scattered to $\mathcal{CH}_L$ and $\mathcal{CH}_R$, respectively. Indeed, the theory also carries over to $\Lambda \neq 0$ and Klein–Gordon masses $\mu \neq 0$ *except* for the $\omega = 0$ frequency. (Again, these results are compatible with the blow-up of the local energy at the Cauchy horizon because of the degeneracy of the T-energy.) This will turn out to be important for the results in Chapter 2 and Chapter 3. We refer to the introduction of Chapter 1 for a more detailed motivation and discussion.

Linear level for $\Lambda > 0$***.*** For Kerr(and Reissner–Nordström)–de Sitter ($\Lambda > 0$) it was shown in [70] that solutions of (WE) also remain bounded up to and including the Cauchy horizon—thus disproving Conjecture 3 for $\Lambda > 0$. Note that in both cases, $\Lambda = 0$ and $\Lambda > 0$, the proofs crucially rely on quantitative decay along the event horizon (polynomial for $\Lambda = 0$ and exponential for $\Lambda > 0$).

On the other hand the exponential convergence on the event horizon of a Reissner–Nordström–de Sitter and Kerr–de Sitter black hole is in direct competition with the exponential blue-shift instability. Thus, the question of the validity of Conjecture 4 becomes even more subtle for $\Lambda > 0$ and has received lots of attention in the recent literature. We refer to the conjecture in [24], the survey article [131] and the recent work [31, 34, 33, 32, 20].

Nonlinear level. Now we turn to the full nonlinear problem for (EE). As mentioned before, for the Einstein vacuum equations Dafermos–Luk showed that the Kerr Cauchy horizon is C^0 stable [26], i.e. the spacetime is extendible as a C^0 Lorentzian manifold. Note that this definitively falsifies Conjecture 1 for $\Lambda = 0$ (subject only to the completion of a proof of the nonlinear stability of the Kerr exterior). In principle, their proof of C^0 extendibility also applies to the interior of Kerr–de Sitter black holes, where the exterior has been proved to be stable for slowly rotating Kerr–de Sitter black holes [71], thus falsifying Conjecture 1 for $\Lambda > 0$.

Nonlinear inextendibility results at the Cauchy horizon have been proved only in spherical symmetry: Coupling the Einstein equation (EE) to a Maxwell–Scalar field system, it is proved in [23] that the Cauchy horizon is C^0 stable, yet C^2 unstable [90, 91, 23] for a generic set of spherically symmetric initial data. See also the pioneering work in [128, 120] and the more general results on the Maxwell–Charged–Scalar field system in [139, 140, 141]. This shows the C^2 formulation of SCC (but not yet Conjecture 2) in spherical symmetry. See [21, 22] for work in the $\Lambda > 0$ case. The question of any type of nonlinear instability of the Cauchy horizon without symmetry assumptions and the validity of Conjecture 2 (even restricted to a neighborhood of Kerr) have yet to be understood.

SCC for asymptotically AdS spacetimes $\Lambda < 0$

The situation is changed radically if one considers asymptotically Anti-de Sitter ($\Lambda < 0$) spacetimes. Due to the timelike nature of null infinity $\mathcal{I}$, see for example Fig. 1, these spacetimes are not globally hyperbolic. For well-posedness of (EE) and (WE) it is required to impose also boundary conditions at infinity [54, 50]. The most natural conditions are Dirichlet (reflecting) boundary conditions, see [54]. Before we address the question of stability of the Cauchy horizon, it is essential to understand the behavior in the exterior

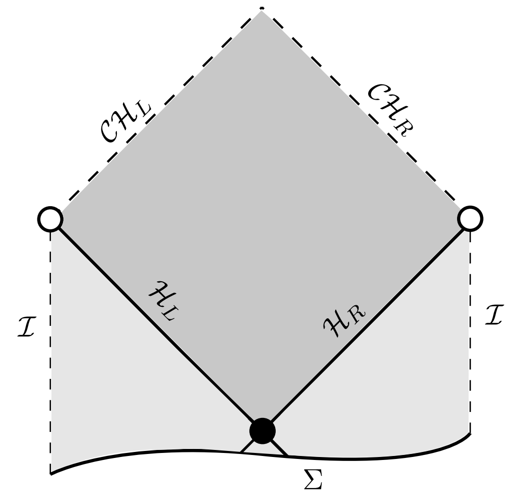

Figure 1: Penrose diagram of the maximal Cauchy development of Reissner–Nordström–AdS or Kerr–AdS data on a spacelike surface Σ with Dirichlet (reflecting) boundary conditions prescribed on null infinity $\mathcal{I}$.

region of Kerr–AdS and Reissner–Nordström–AdS.

Logarithmic decay for linear waves on the exterior of Kerr–AdS and Reissner–Nordström–AdS. For the massive linear wave equation (WE) on Kerr–AdS and Reissner–Nordström–AdS, Holzegel–Smulevici showed in [75] stability in the exterior region. Indeed, they proved that solutions decay at least at logarithmic rate towards i^+ (cf. polynomial ($\Lambda = 0$) and exponential ($\Lambda > 0$)) assuming the Hawking–Reall [67] bound[1] $r_+ > |a|l$. Moreover, they showed that solutions of (WE) with fixed angular momentum actually decay exponentially on the exterior of Reissner–Nordström–AdS. (This is in contrast to the asymptotically flat case, in which fixed angular momentum solutions of (WE) decay polynomially on the exterior of Reissner–Nordström.) However, their main insight was that a suitable infinite sum of such rapidly decaying fixed angular momentum solutions, possessing finite energy in some weighted norm, indeed achieves the logarithmic decay rate [77]. This is due to the presence of stable trapping. Note that this sharpness can also be concluded from later work showing the existence of quasinormal modes converging to the real axis at an exponential rate as the real part of the frequency and angular momentum tend to infinity [145, 59]. (For some asymptotically flat five dimensional black holes a similar inverse logarithmic lower bound was shown in [6].)

Strong Cosmic Censorship for AdS black holes. With the logarithmic decay on the exterior at hand, we turn to the question of the stability of the Cauchy horizon. We first

[1] Note that otherwise exponentially growing mode solutions can be constructed as shown in [40].

recall from the discussion before that Conjecture 4 holds true for the cases $\Lambda \geq 0$. Indeed, our methods developed in Chapter 2 and Chapter 3 in principle also show Conjecture 4 for $\Lambda < 0$. However, in view of the slower decay in the case $\Lambda < 0$, we even expect a stronger instability at the Cauchy horizon for $\Lambda < 0$. This raises the attractive possibility of the validity of the C^0-formulation of Conjecture 1 which would be a more definitive resolution than Conjecture 2. Thus, at the level of (WE), it is the validity of Conjecture 3, the linear scalar analog of Conjecture 1, which remains the unsolved puzzle for the $\Lambda < 0$ case.

First, attempting to disprove Conjecture 3 as was done in the cases $\Lambda \geq 0$, we note that the logarithmic decay rate on the exterior is too slow to follow the methods involving the red-shift vector field and the vector field S (see discussion before) to prove uniform boundedness and continuous extendibility at the Cauchy horizon of solutions to (WE). More specifically, after propagating the logarithmic decay through the red-shift region, the energy flux associated to S is infinite on a $\{r = const.\}$ hypersurface in the black hole interior due to the slow logarithmic decay towards i^+. (Contrast this with the work [35, 121] in 2+1 dimensions.) Thus, the question of whether to expect the validity of Conjecture 3 for asymptotically AdS black holes appears to be completely open.

In the present thesis we provide a surprising—mixed—resolution of Conjecture 3 for $\Lambda < 0$.

In Chapter 2 we will show (Theorem 2.1) that, *despite the slow decay on the exterior*, *boundedness*, $|\psi| \leq C$, in the interior and continuous extendibility to the Cauchy horizon still holds for solutions of (WE) on Reissner–Nordström–AdS black holes. The additional phenomenon which we exploit to prove boundedness is that the trapped frequencies responsible for slow decay have high energy with respect to the T vector field and can be bounded using the scattering theory developed in [82]. ***Thus, for Reissner–Nordström–AdS, Conjecture 3 is falsified, just as in the*** $\Lambda \geq 0$ ***cases.***

In Chapter 3 we show that linear perturbations ψ of Kerr–AdS *blow up* $|\psi| \to +\infty$ for a set of normalized Kerr–AdS parameters of mass $\mathfrak{m} := M\sqrt{-\Lambda}$ and angular momentum $\mathfrak{a} := a\sqrt{-\Lambda}$ which is *Baire-generic* but *Lebesgue-exceptional.* These blow-up parameters are defined through a Diophantine condition. This condition arises from a suitable Diophantine approximation related to a coupling of stable trapping in the exterior and Cauchy horizon resonances in the black hole interior. This will be discussed in more details in the introduction of Chapter 3. ***Thus, for Kerr–AdS, Conjecture 3 holds true if Baire-genericity is imposed.*** We also complement our main result with the conjecture that linear perturbations *remain bounded* at the Cauchy horizon for black holes which are *Baire-exceptional* but *Lebesgue-generic.*

Instability of asymptotically AdS spacetimes?

In the above sense, the results of the present thesis leave determinism in better shape for $\Lambda < 0$ compared to the $\Lambda \geq 0$ cases. However, turning to the fully nonlinear dynamics, there is another scenario which could happen. Recall that Minkowski space ($\Lambda = 0$) and de Sitter space ($\Lambda > 0$) have been proved to be nonlinearly stable [53, 18]. Anti-de Sitter space ($\Lambda < 0$), however, is expected to be nonlinearly unstable with Dirichlet conditions imposed at infinity. This was recently proved in [106, 105, 108, 107] for appropriate matter models. See also the original conjecture in [25] and the numerical results in [9]. Similarly, for Kerr–AdS (or Reissner–Nordström–AdS), the slow logarithmic decay on the linear level proved in [77] could in fact give rise to nonlinear instabilities in the exterior. (Note that in contrast, nonlinear stability for spherically symmetric perturbations of Schwarzschild–AdS was shown for Einstein–Klein–Gordon systems [76].) If indeed the exterior of Kerr–AdS was nonlinearly unstable, linear analysis like that in the present thesis would be manifestly inadequate and the question of the validity of Strong Cosmic Censorship would be thrown even more open!

Chapter 1

A scattering theory for linear waves on the interior of Reissner–Nordström black holes

1.1 Introduction

In this chapter, we initiate the mathematical study of the finite energy scattering problem on black hole interiors. Specifically, we will consider solutions of the wave equation on what can be viewed as the most elementary interior, that of Reissner–Nordström. The Reissner–Nordström metrics constitute a family of spacetimes, parametrized by mass M and charge Q, which satisfy the Einstein–Maxwell system in spherical symmetry [132, 113] and admit an additional Killing vector field T. For vanishing charge $Q = 0$, the family reduces to Schwarzschild. We shall moreover restrict in the following to the subextremal case where $0 < |Q| < M$. Our past and future scattering states will be defined as suitable traces of the solution on the bifurcate event horizon and bifurcate Cauchy horizon, respectively, restricted to have finite T energy flux on each component of the horizons.

In the rest of the introduction we will state our main results for the scattering problem on the interior of Reissner–Nordström (Theorems 1.1 – 1.5), relate them to existing literature in fixed frequency scattering, and draw links to various recent results in the interior and exterior of black holes. Finally, we will see that the existence of a bounded scattering map for the wave equation on Reissner–Nordström turns out to be a very fragile property; we shall show that there does not exist an analogous scattering theory in the presence of a cosmological constant (Theorem 1.6) or Klein–Gordon mass (Theorem 1.7).

The scattering problem on Reissner–Nordström interior. In this chapter, we will establish a scattering theory for finite energy solutions of the linear wave equation,

$$\Box_g \psi = 0, \tag{1.1.1}$$

on the interior of a Reissner–Nordström black hole, from the bifurcate event horizon $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B \cup \mathcal{B}_-$ to the bifurcate Cauchy horizon $\mathcal{CH} = \mathcal{CH}_A \cup \mathcal{CH}_B \cup \mathcal{B}_+$, as depicted in Fig. 1.1. The first main result of this chapter is **Theorem 1.1** (see Section 1.3.1) in

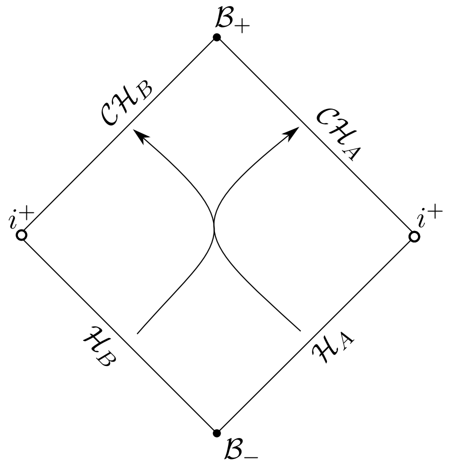

Figure 1.1: Penrose diagram of the interior of the Reissner–Nordström black hole and visualization of the scattering map.

which we will show *existence*, *uniqueness* and *asymptotic completeness* of finite energy scattering states. In this context, *existence* and *uniqueness* mean that for given finite energy data ψ_0 on the event horizon $\mathcal{H}$, there exist unique finite energy data on the Cauchy horizon $\mathcal{CH}$ arising from ψ_0 as the evolution of (1.1.1). With *asymptotic completeness* we denote the property that all finite energy data on the Cauchy horizon $\mathcal{CH}$ can indeed be achieved from finite energy data on the event horizon $\mathcal{H}$. This provides a way to construct solutions with desired asymptotic properties which is a necessary first step to properly understand quantum theories in the interior of a Reissner–Nordström black hole (cf. [143, 65, 41]). The energy spaces on the event and Cauchy horizon are associated to the Killing field and generator of the time translation T. Indeed, T is null on the horizons and, in particular, is the generator of the event and Cauchy horizon $\mathcal{H}$ and $\mathcal{CH}$. Because of the sign-indefiniteness of the energy flux of the vector field T on the bifurcate event (resp. Cauchy) horizon (see already (1.1.4)), we define our energy space by requiring the finiteness of the T energy on both components separately of the event (resp. Cauchy) horizon. These define Hilbert spaces with respect to which the scattering map is proven to be bounded.

Finally, it is instructive to draw a comparison between the interior of Reissner–Nordström and the interior of Schwarzschild ($Q = 0$). As opposed to Reissner–Nordström discussed above, the Schwarzschild interior terminates at a singular boundary at which solutions to (1.1.1) generically blow-up (see [49]). In contrast, the non-singular and, moreover, Killing, Cauchy horizons (see Fig. 1.1) of Reissner–Nordström immediately yield natural Hilbert spaces of finite energy data to consider. In view of this, Reissner–Nordström with $Q \neq 0$ can be considered the most elementary interior on which to study the scattering problem. Furthermore, in view of the recent work [26], we have that the causal structure of Reissner–Nordström is stable in a weak sense (see the discussion below about related works in the interior).

Fixed frequency scattering. It is well known that the wave equation (1.1.1) on Reissner–Nordström spacetime allows separation of variables which reduces it to the radial o.d.e.

$$u'' - V_\ell u + \omega^2 u = 0, \tag{1.1.2}$$

with potential V_ℓ (see already (1.2.37)), where $\omega \in \mathbb{R}$ is the time frequency and $\ell \in \mathbb{N}_0$ is the angular parameter. Indeed, most of the existing literature concerning scattering of waves in the interior of Reissner–Nordström mainly considers fixed frequency solutions, e.g. [100, 101, 14, 64, 99, 63, 147]. For a purely *incoming* (i.e. supported only on $\mathcal{H}_A$) fixed frequency solution with parameters (ω, ℓ), we can associate transmission and reflection coefficients $\mathfrak{T}(\omega, \ell)$ and $\mathfrak{R}(\omega, \ell)$. The transmission coefficient $\mathfrak{T}(\omega, \ell)$ measures what proportion of the incoming wave is transmitted to $\mathcal{CH}_B$, whereas the reflection coefficient specifies the proportion reflected to $\mathcal{CH}_A$. An essential step to go from fixed frequency scattering to physical space scattering is to prove ***uniform boundedness*** of $\mathfrak{T}(\omega, \ell)$ and $\mathfrak{R}(\omega, \ell)$. This is non-trivial in view of the discussion of the energy identity (1.1.4) below. In this chapter, we indeed obtain this uniform bound in **Theorem 1.2** (see Section 1.3.2). In particular, the regime $\omega \to 0, \ell \to \infty$ is the most involved frequency range to prove uniform boundedness. As we shall see, the proof relies on an explicit computation at $\omega = 0$ (see [64]) together with a careful analysis of special functions and perturbations thereof.

The uniform boundedness of the scattering coefficients is the main ingredient to prove the boundedness of the scattering map in Theorem 1.1. Moreover, it allows us to connect the separated picture to the physical space picture by means of a Fourier representation formula. This is stated as **Theorem 1.3** (see Section 1.3.3). A somewhat surprising, direct consequence of the Fourier representation of the scattered data on the Cauchy horizon is that purely incoming compactly supported data lead to a solution which vanishes at the future bifurcation sphere $\mathcal{B}_+$. This is moreover shown to be a necessary condition for the existence of a bounded scattering map (**Corollary 1.3.1**).

Comparison to scattering on the exterior of black holes. On the exterior of black holes, the scattering problem has been studied more extensively; see the pioneering works [36, 38, 37, 4, 5], the book [55] and related results on conformal scattering in [98, 112, 104, 137]. Note that for the exterior of a Schwarzschild or Reissner–Nordström black hole, the uniform boundedness of the scattering coefficients or equivalently, the boundedness of the scattering map, can be viewed a posteriori[1] as a consequence of the global T energy identity

$$\int_{\mathcal{H}^-} |T\psi|^2 + \int_{\mathcal{I}^-} |T\psi|^2 = \int_{\mathcal{H}^+} |T\psi|^2 + \int_{\mathcal{I}^+} |T\psi|^2. \tag{1.1.3}$$

Considering only incoming radiation from $\mathcal{I}^-$, this statement translates into $|\mathfrak{R}|^2+|\mathfrak{T}|^2 = 1$ for the reflection coefficient $\mathfrak{R}$ and transmission coefficients $\mathfrak{T}$. In the interior, however, due to the different orientations of the T vector field on the horizons (cf. Fig. 1.2), boundedness of the scattering map is not at all manifest. In particular, the global T energy identity on

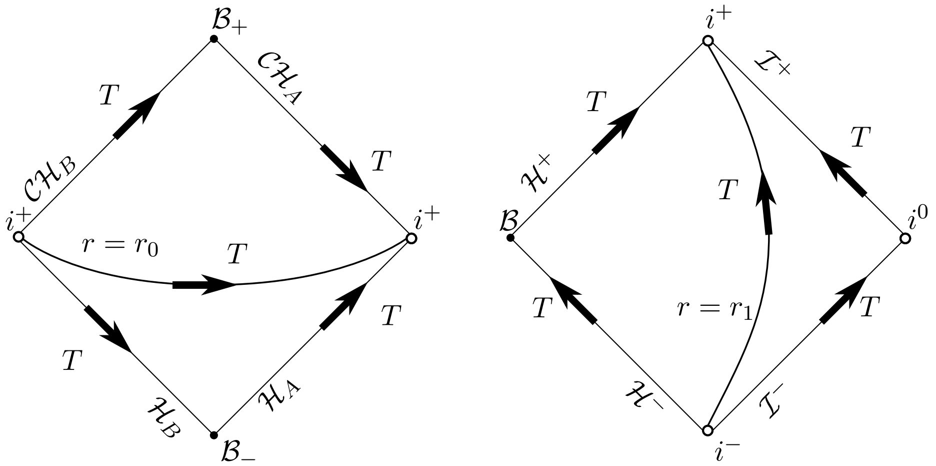

Figure 1.2: Interior of Reissner–Nordström (left) and exterior of Schwarzschild or Reissner–Nordström (right).
In both diagrams the arrows denote the direction of the T Killing vector field. Note that we have the identifications $\mathcal{H}_A = \mathcal{H}^+$ and $\mathcal{B}_- = \mathcal{B}$.

the interior of a Reissner–Nordström black hole reads

$$\int_{\mathcal{H}_A} |T\psi|^2 - \int_{\mathcal{H}_B} |T\psi|^2 = \int_{\mathcal{CH}_B} |T\psi|^2 - \int_{\mathcal{CH}_A} |T\psi|^2 \tag{1.1.4}$$

[1]Note that proving (1.1.3) requires first establishing some form of qualitative decay towards i^+ and i^-.

from which we cannot deduce boundedness of the scattering map even a posteriori. (Indeed, identity (1.1.4) corresponds only to the "pseudo-unitarity" statement of Theorem 1.1.) Again, considering only ingoing radiation from $\mathcal{H}_A$, this translates to

$$|\mathfrak{T}(\omega,\ell)|^2 - |\mathfrak{R}(\omega,\ell)|^2 = 1 \tag{1.1.5}$$

for the reflection coefficient $\mathfrak{R}$ and the transmission coefficient $\mathfrak{T}$. Hence, while for fixed $|\omega| > 0$ and ℓ, it is straightforward to show that $\mathfrak{T}$ and $\mathfrak{R}$ are finite, there is no a priori obvious obstruction from (1.1.5) for these scattering coefficients to blow up in the limits $\omega \to 0, \pm\infty$ and $\ell \to \infty$.

Moreover, note that in the exterior, the Killing field T is timelike, so the radial o.d.e. (1.1.2) should be considered as an equation for a fixed time frequency wave on a constant time slice. In the interior, however, the Killing field T is spacelike so the radial o.d.e. (1.1.2) is rather an evolution equation for a constant spatial frequency.

The Schwarzschild family can be viewed as a special case ($a = 0$) of the two parameter Kerr family, describing rotating black holes with mass parameter M and rotation parameter a [83].[2] On the exterior of Kerr many other difficulties arise: superradiance, intricate trapping, presence of ergoregion, etc. [29]. Nevertheless, using the decay results in [29], a definitive physical space scattering theory for Kerr black holes has been established in [28] (see also the earlier [60]). The proof on the exterior of Kerr involved first establishing a scattering map from past null infinity $\mathcal{I}^-$ to a constant time slice Σ and then concatenating that map with a second scattering map from Σ to the future event horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$. In the interior, however, we will directly show the existence of a "global" scattering map from the event horizon $\mathcal{H}$ to the Cauchy horizon $\mathcal{CH}$. Indeed, due to blue-shift instabilities (see [30]), we do not expect that the analogous concatenation of scattering maps (event horizon $\mathcal{H}$ to spacelike hypersurface Σ and then from Σ to the Cauchy horizon $\mathcal{CH}$) as in the Kerr exterior, to be bounded in the interior of Reissner–Nordström.

Injectivity of the reflection map and blue-shift instabilities. We can also conclude from our work that there is always non-vanishing reflection to the Cauchy horizon $\mathcal{CH}_A$ arising from non-vanishing purely ingoing radiation at $\mathcal{H}_A$. This follows from the fact that in the separated picture and for fixed ℓ, the reflection coefficient $\mathfrak{R}(\omega,\ell)$ can be analytically continued to the strip $|\operatorname{Im}(\omega)| < \kappa_+$ and hence, only vanishes on a discrete set of points on the real axis. This is shown in **Theorem 1.4** (see Section 1.3.4).

[2] Both Kerr and Reissner–Nordström can be viewed as special cases of the Kerr–Newman spacetime. For decay results on Kerr–Newman see [19].

We will also deduce from the Fourier representation of the scattered data on the Cauchy horizon $\mathcal{CH}$, and a suitable meromorphic continuation of the transmission coefficient, that there exist purely incoming compactly supported data on the event horizon $\mathcal{H}$ leading to solutions which fail to be C^1 on the Cauchy horizon $\mathcal{CH}$. This C^1-blow-up of linear waves puts on a completely rigorous footing the pioneering work of Chandrasekhar and Hartle [14]. We state this as **Theorem 1.5** (see Section 1.3.5).

For generic solutions arising from localized data on an asymptotically flat hypersurface, one expects a stronger instability, namely, non-degenerate energy blow-up at the Cauchy horizon $\mathcal{CH}$. Such non-degenerate energy blow-up was proven in [89] for generic compactly supported data on an asymptotically flat Cauchy hypersurface. Later, for the more difficult Kerr interior, non-degenerate energy blow-up was proven in [95] assuming certain polynomial lower bounds on the tail of incoming data on the event horizon $\mathcal{H}$ and in [30] for solutions arising from generic initial data along past null infinity $\mathcal{I}^-$ with polynomial tails.

Finally, we mention the forthcoming work [92] which studies the problem of non-degenerate energy blow-up from a scattering theory perspective and also uses the non-triviality of reflection to establish results related to mass inflation for the spherically symmetric Einstein–Maxwell–scalar field system (cf. [90, 91]).

Breakdown of T energy scattering for $\Lambda \neq 0$ or $\mu \neq 0$. If a cosmological constant $\Lambda \in \mathbb{R}$ is added to the Einstein–Maxwell system, we can consider the analogous (anti-) de Sitter–Reissner–Nordström family of solutions whose interiors have the same Penrose diagram as depicted in Fig. 1.1. For very slowly rotating Kerr–de Sitter and Reissner–Nordström–de Sitter spacetimes, boundedness, continuity, and regularity up to and including the Cauchy horizon has been shown for solutions to (1.1.1) arising from smooth and compactly supported data on a Cauchy hypersurface [70]. However, remarkably, there is no analogous T energy scattering theory for either the linear wave equation (1.1.1) or the Klein–Gordon equation with conformal mass. This is the statement of **Theorem 1.6** (see Section 1.3.6). The reason for this failure is the unboundedness of the transmission coefficient $\mathfrak{T}$ and reflection coefficients $\mathfrak{R}$ in the vanishing frequency limit $\omega \to 0$. To be more precise, we will prove that there does not exist a T energy scattering theory of the wave or Klein–Gordon equation in the interior of a (anti-) de Sitter–Reissner–Nordström black hole for generic subextremal black hole parameters (M, Q, Λ). In particular, for fixed $0 < |Q| < M$, there is an $\epsilon > 0$ such that there does not exist a T energy scattering theory for all $0 \neq |\Lambda| < \epsilon$.

Similarly, we prove in **Theorem 1.7** (see Section 1.3.7) that there does not exist a T energy scattering theory for the Klein–Gordon equation $\Box_g \psi - \mu\psi = 0$ on the Reissner–

Nordström interior for a generic set of masses μ. This is in contrast to the exterior, where T energy scattering theories were established for massive fields in [5, 102].

It remains an open problem to formulate an appropriate scattering theory in the cosmological setting and for the Klein–Gordon equation as well as for the interior of Kerr.

Outline. This chapter is organized as follows. In Section 1.2, we shall set up the spacetime, introduce the relevant energy spaces, and define the scattering coefficients in the separated picture. In Section 1.3 we state the main results of this chapter, Theorems 1.1 – 1.7. Section 1.4 is devoted to the proof of Theorem 1.2. Then, the statement of Theorem 1.2 allows us to prove Theorem 1.1 in Section 1.5. Finally, in the last two sections are show our non-existence results: In Section 1.6, we prove Theorem 1.6 and in Section 1.7, we give the proof of Theorem 1.7.

1.2 Preliminaries

In this section we will define the background differentiable structure and metric for the Reissner–Nordström spacetime and introduce some convenient coordinate systems.

1.2.1 Interior of the subextremal Reissner–Nordström black hole

The global geometry of Reissner–Nordström was first described in [62]. For completeness, we will explicitly construct in this section the coordinates for the region considered. We start, in Section 1.2.1.1, by defining a Lorentzian manifold corresponding to the interior of the Reissner–Nordström black hole without the horizons. Then, in Section 1.2.1.2, we will attach the boundaries which will correspond to the event horizon and Cauchy horizon.

1.2.1.1 The interior without boundary

We will now give an explicit description of the differential structure and metric. The Reissner–Nordström solutions [132, 113] are a two-parameter family of spherically symmetric spacetimes with mass parameter $M \in \mathbb{R}$ and electric charge parameter $Q \in \mathbb{R}$ solving the Einstein–Maxwell system

$$\begin{aligned} &Ric_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} := 8\pi\left(\frac{1}{4\pi}\left(F_{\mu}{}^{\lambda}F_{\lambda\nu} - \frac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}\right)\right), \qquad (1.2.1)\\ &\nabla^{\mu}F_{\mu\nu} = 0, \nabla_{[\mu}F_{\nu\lambda]} = 0. \end{aligned}$$

In this chapter, we consider the subextremal family of black holes with parameter range $0 < |Q| < M$. Define the manifold $\mathcal{M}$ by

$$\mathcal{M} = \mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2, \tag{1.2.2}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2} > 0$. The manifold is parametrized by the standard coordinates $t \in \mathbb{R}$, $r \in (r_-, r_+)$, and a choice of spherical coordinates (θ, ϕ) on the sphere $\mathbb{S}^2$. We denote the global coordinate vector field ∂_t by T:

$$T := \frac{\partial}{\partial t}. \tag{1.2.3}$$

Using the above coordinates, we equip $\mathcal{M}$ with the Lorentzian metric

$$g_{Q,M} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 \not{g}_{\mathbb{S}^2}, \tag{1.2.4}$$

where $\not{g}_{\mathbb{S}^2}$ is the round metric on the 2-sphere. We also define the quantities

$$\Delta := r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-) \text{ and } h := \frac{\Delta}{r^2}. \tag{1.2.5}$$

Furthermore, define r_* by

$$\mathrm{d}r_* := \frac{r^2}{\Delta} \mathrm{d}r, \tag{1.2.6}$$

where we choose $r_*(\frac{r_+ + r_-}{2}) = 0$ for definiteness. Thus,

$$r_*(r) = r + \frac{1}{2\kappa_+} \log |r - r_+| + \frac{1}{2\kappa_-} \log |r - r_-| + C \tag{1.2.7}$$

for a constant C only depending on the black hole parameters and

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}. \tag{1.2.8}$$

We also introduce null coordinates

$$v = r_* + t \text{ and } u = r_* - t \tag{1.2.9}$$

on $\mathcal{M}$. The Penrose diagram of $\mathcal{M}$ is depicted in Fig. 1.3.

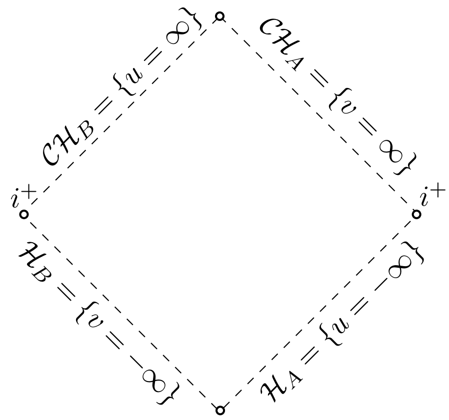

Figure 1.3: Penrose diagram of $\mathcal{M}$; formally we have denoted the boundary (not part of the manifold) by $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$ and $\mathcal{CH} = \mathcal{CH}_A \cup \mathcal{CH}_B$.

1.2.1.2 Attaching the event and Cauchy horizon

Now, we will attach the Cauchy and event horizon to the manifold. The Cauchy horizon characterizes the future boundary up to which the spacetime is uniquely determined as a solution to the Einstein–Maxwell system arising from data on the event horizon. Although the metric is smoothly extendible beyond the Cauchy horizon, any such extension fails to be uniquely determined from initial data, leading to a severe failure of determinism.

Attaching the event and Cauchy horizon gives rise to a manifold with corners. To do so, we first define the following two pairs of null coordinates.

Let $U_{\mathcal{H}} \colon \mathbb{R} \to (0, \infty)$ and $V_{\mathcal{H}} \colon \mathbb{R} \to (0, \infty)$ be smooth and strictly increasing functions such that

- $U_{\mathcal{H}}(u) = u$ for $u \geq 1$, $V_{\mathcal{H}}(v) = v$ for $v \geq 1$,
- $U_{\mathcal{H}}(u) \to 0$ as $u \to -\infty$, $V_{\mathcal{H}}(v) \to 0$ as $v \to -\infty$,
- there exists a $u_+ \leq 1$ such that $\frac{\mathrm{d}U_{\mathcal{H}}}{\mathrm{d}u} = \exp(\kappa_+ u)$ for $u \leq u_+$,
- there exists a $v_+ \leq 1$ such that $\frac{\mathrm{d}V_{\mathcal{H}}}{\mathrm{d}v} = \exp(\kappa_+ v)$ for $v \leq v_+$.

This defines – in mild abuse of notation – the null coordinates $U_{\mathcal{H}} \colon \mathcal{M} \to (0, \infty)$ via $U_{\mathcal{H}}(u)$ and $V_{\mathcal{H}} \colon \mathcal{M} \to (0, \infty)$ via $V_{\mathcal{H}}(v)$, where u, v are the null coordinates defined in (1.2.9).

Similarly, let $U_{\mathcal{CH}} \colon \mathbb{R} \to (-\infty, 0)$ and $V_{\mathcal{CH}} \colon \mathbb{R} \to (-\infty, 0)$ be smooth and strictly increasing functions such that

- $U_{\mathcal{CH}}(u) = u$ for $u \leq -1$, $V_{\mathcal{CH}}(v) = v$ for $v \leq -1$,
- $U_{\mathcal{CH}}(u) \to 0$ as $u \to \infty$, $V_{\mathcal{CH}}(v) \to 0$ as $v \to \infty$,

- there exists a $u_+ \geq -1$ such that $\frac{\mathrm{d}U_{\mathcal{CH}}}{\mathrm{d}u} = \exp(\kappa_- u)$ for $u \geq u_+$,

- there exists a $v_+ \geq -1$ such that $\frac{\mathrm{d}V_{\mathcal{CH}}}{\mathrm{d}v} = \exp(\kappa_- v)$ for $v \geq v_+$.

As above, this defines null coordinates $U_{\mathcal{CH}} : \mathcal{M} \to (0, \infty)$ via $U_{\mathcal{CH}}(u)$ and $V_{\mathcal{CH}} \colon \mathcal{M} \to (0, \infty)$ via $V_{\mathcal{CH}}(v)$, where u, v are the null coordinates defined in (1.2.9).

To define the event horizon, we consider the coordinate chart $(U_{\mathcal{H}}, V_{\mathcal{H}}, \theta, \phi)$. We now define the event horizon without the bifurcation sphere as the union

$$\mathcal{H}_0 := \mathcal{H}_A \cup \mathcal{H}_B, \tag{1.2.10}$$

where

$$\mathcal{H}_A := \{U_{\mathcal{H}} = 0\} \times (0, \infty) \times \mathbb{S}^2 \text{ and } \mathcal{H}_B := (0, \infty) \times \{V_{\mathcal{H}} = 0\} \times \mathbb{S}^2. \tag{1.2.11}$$

Analogously, we also define the Cauchy horizon without the bifurcation sphere in the coordinate chart $(U_{\mathcal{CH}}, V_{\mathcal{CH}}, \theta, \phi)$ as the union

$$\mathcal{CH}_0 := \mathcal{CH}_A \cup \mathcal{CH}_B, \tag{1.2.12}$$

where

$$\mathcal{CH}_A := (0, \infty) \times \{V_{\mathcal{CH}} = 0\} \times \mathbb{S}^2 \text{ and } \mathcal{CH}_B := \{U_{\mathcal{CH}} = 0\} \times (0, \infty) \times \mathbb{S}^2. \tag{1.2.13}$$

Then, we define the interior of the Reissner–Nordström spacetime without the bifurcation sphere as the manifold with boundary

$$\tilde{\mathcal{M}} := \mathcal{M} \cup \mathcal{H} \cup \mathcal{CH}. \tag{1.2.14}$$

The Lorentzian metric on $\mathcal{M}$ extends smoothly to $\tilde{\mathcal{M}}$. In particular, the boundary of $\tilde{\mathcal{M}}$ consists of four disconnected null hypersurfaces. In Fig. 1.4 we have depicted its Penrose diagram. In mild abuse of notation we shall also use the coordinate systems

$$\begin{aligned}
&(U_{\mathcal{H}}, v, \theta, \phi) \text{ on } \mathcal{M} \cup \mathcal{H}_A, && (1.2.15)\\
&(u, V_{\mathcal{H}}, \theta, \phi) \text{ on } \mathcal{M} \cup \mathcal{H}_B, && (1.2.16)\\
&(u, V_{\mathcal{CH}}, \theta, \phi) \text{ on } \mathcal{M} \cup \mathcal{CH}_A, && (1.2.17)\\
&(U_{\mathcal{CH}}, v, \theta, \phi) \text{ on } \mathcal{M} \cup \mathcal{CH}_B. && (1.2.18)
\end{aligned}$$

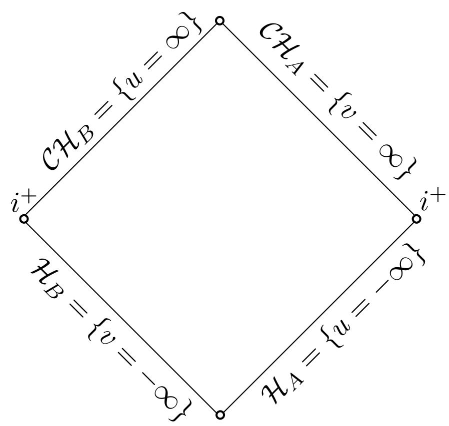

Figure 1.4: Penrose diagram of $\tilde{\mathcal{M}}$.

In particular, we can write

$$\mathcal{H}_A = \{U_{\mathcal{H}} = 0\} \times \{v \in \mathbb{R}\} \times \mathbb{S}^2, \tag{1.2.19}$$

$$\mathcal{H}_B = \{u \in \mathbb{R}\} \times \{V_{\mathcal{H}} = 0\} \times \mathbb{S}^2, \tag{1.2.20}$$

$$\mathcal{CH}_A = \{u \in \mathbb{R}\} \times \{V_{\mathcal{CH}} = 0\} \times \mathbb{S}^2, \tag{1.2.21}$$

$$\mathcal{CH}_B = \{U_{\mathcal{CH}} = 0\} \times \{v \in \mathbb{R}\} \times \mathbb{S}^2. \tag{1.2.22}$$

Note also that the vector field T, initially defined on $\mathcal{M}$ in (1.2.3), extends to a smooth vector field on $\tilde{\mathcal{M}}$ with

$$T \restriction_{\mathcal{H}_A} = \frac{\partial}{\partial v} \restriction_{\mathcal{H}_A}, \tag{1.2.23}$$

where $\frac{\partial}{\partial v}$ is the coordinate derivative with respect to local chart defined in (1.2.15). Similarly, we have

$$T \restriction_{\mathcal{H}_B} = -\frac{\partial}{\partial u} \restriction_{\mathcal{H}_B} \text{ w.r.t. to the local chart (1.2.16)}, \tag{1.2.24}$$

$$T \restriction_{\mathcal{CH}_A} = -\frac{\partial}{\partial u} \restriction_{\mathcal{CH}_A} \text{ w.r.t. to the local chart (1.2.17)}, \tag{1.2.25}$$

$$T \restriction_{\mathcal{CH}_B} = \frac{\partial}{\partial v} \restriction_{\mathcal{CH}_B} \text{ w.r.t. to the local chart (1.2.18)}. \tag{1.2.26}$$

Note that T is a Killing null generator of the Killing horizons $\mathcal{H}_A, \mathcal{H}_B, \mathcal{CH}_A$, and $\mathcal{CH}_B$. Recall also that $\nabla_T T \restriction_{\mathcal{CH}} = \kappa_- T \restriction_{\mathcal{CH}}$ and $\nabla_T T \restriction_{\mathcal{H}} = \kappa_+ T \restriction_{\mathcal{H}}$, where $\kappa_\pm$ is defined by (1.2.8).

At this point, we note that we can attach corners to $\mathcal{H}_0$ and $\mathcal{CH}_0$ to extend $\tilde{\mathcal{M}}$ smoothly to a Lorentzian manifold with corners. To be more precise, we attach the past bifurcation sphere $\mathcal{B}_-$ to $\mathcal{H}_0$ as the point $(U_{\mathcal{H}}, V_{\mathcal{H}}) = (0,0)$. Then, define $\mathcal{H} := \mathcal{H}_0 \cup \mathcal{B}_-$. Similarly, we can attach the future bifurcation sphere $\mathcal{B}_+$ to the Cauchy horizon which will be denoted by $\mathcal{CH}$. We call the resulting manifold $\mathcal{M}_{\mathrm{RN}}$. Further details are not given since the precise construction is straightforward and the metric extends smoothly. Moreover, the T vector field extends smoothly to $\mathcal{B}_+$ and $\mathcal{B}_-$ and vanishes there. Its Penrose diagram is depicted in Fig. 1.5.

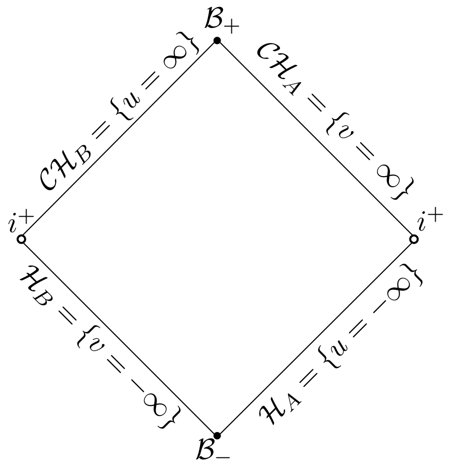

Figure 1.5: Penrose diagram of $\mathcal{M}_{\mathrm{RN}}$ which includes the bifurcate spheres $\mathcal{B}_+$ and $\mathcal{B}_-$.

Further details about the coordinate systems can be found in [115]. From a dynamical point of view, we can also consider the spacetimes $(\mathcal{M}_{\mathrm{RN}}, g_{M,Q})$ as being contained in the Cauchy development of a spacelike hypersurface with two asymptotically flat ends solving the Einstein–Maxwell system in spherical symmetry.

1.2.2 The characteristic initial value problem for the wave equation

In the context of scattering theory we will be interested in solutions to the wave equation (1.1.1) arising from suitable characteristic initial data. Recall the following well-posedness result for (1.1.1) with characteristic initial data.

Proposition 1.2.1. *Let $\Psi \in C_c^\infty(\mathcal{H})$ be smooth compactly supported data on the event horizon $\mathcal{H}$. Then, there exists a unique smooth solution ψ to* (1.1.1) *on $\mathcal{M}_{\mathrm{RN}} \setminus \mathcal{CH}$ such that $\psi \restriction_{\mathcal{H}} = \Psi$.*

The previous proposition is well known, see [111, 133]. Analogously, we have the following for the backward evolution.

Proposition 1.2.2. *Let $\Psi \in C_c^\infty(\mathcal{CH})$ be smooth compactly supported data on the Cauchy horizon $\mathcal{CH}$. Then, there exists a unique smooth solution ψ to* (1.1.1) *on $\mathcal{M}_{\mathrm{RN}} \setminus \mathcal{H}$ such that $\psi \restriction_{\mathcal{CH}} = \Psi$.*

1.2.3 Hilbert spaces of finite T energy on both horizon components

Now, we are in the position to define the Hilbert spaces on the event $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B \cup \mathcal{B}_-$ and Cauchy horizon $\mathcal{CH} = \mathcal{CH}_A \cup \mathcal{CH}_B \cup \mathcal{B}_+$, respectively.

We will start with constructing the Hilbert space on the event horizon. Roughly speaking, it will be defined by requiring finiteness of the T energy flux on $\mathcal{H}_A$ minus the T energy flux on $\mathcal{H}_B$. More precisely, let $C_c^\infty(\mathcal{H})$ be the vector space of smooth compactly supported functions on $\mathcal{H}$. Recall that the Killing vector field T is also a null generator of $\mathcal{H}$ and vanishes at the past bifurcation sphere $\mathcal{B}_-$. This allows us to define the norm $\|\cdot\|^2_{\mathcal{E}^T_{\mathcal{H}}}$ on the vector space $C_c^\infty(\mathcal{H})$ as

$$\|\psi\|^2_{\mathcal{E}^T_{\mathcal{H}}} := \int_{\mathcal{H}_A} J^T_\mu[\psi] n^\mu_{\mathcal{H}_A} \,\mathrm{dvol}_{n_{\mathcal{H}_A}} - \int_{\mathcal{H}_B} J^T_\mu[\psi] n^\mu_{\mathcal{H}_B} \,\mathrm{dvol}_{n_{\mathcal{H}_B}}, \tag{1.2.27}$$

where $\psi \in C_c^\infty(\mathcal{H})$, $\mathbf{T}[\psi]$ is the energy momentum tensor

$$\mathbf{T}[\psi]_{\mu\nu} := \mathrm{Re}(\partial_\mu\psi\overline{\partial_\nu\psi}) - \frac{1}{2} g_{\mu\nu}\partial_\alpha\psi\overline{\partial^\alpha\psi}, \tag{1.2.28}$$

and $J^T[\psi] := \mathbf{T}[\psi](T,\cdot)$. In (1.2.27), the fluxes are defined with respect to future directed normal vector fields $n_{\mathcal{H}_A}$ and $n_{\mathcal{H}_B}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.[3] Moreover, recall from Fig. 1.2 that T is future (resp. past) directed on $\mathcal{H}_A$ (resp. $\mathcal{H}_B$). Thus, the terms arising in (1.2.27) satisfy $\int_{\mathcal{H}_A} J^T_\mu[\psi] n^\mu_{\mathcal{H}_A} \mathrm{dvol} \geq 0$ and $-\int_{\mathcal{H}_B} J^T_\mu[\psi] n^\mu_{\mathcal{H}_B} \mathrm{dvol} \geq 0$. Again, in view of the fact that on the component $\mathcal{H}_B$ the normal vector field T is past directed, we can choose $n_{\mathcal{H}_A} := T \restriction_{\mathcal{H}_A}$ and $n_{\mathcal{H}_B} := -T \restriction_{\mathcal{H}_B}$ as the future directed normal vector fields on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that we can realize the norm (1.2.27) as (using the coordinate charts (1.2.15) and (1.2.16))

$$\|\psi\|^2_{\mathcal{E}^T_{\mathcal{H}}} = \int_{\mathbb{R}\times\mathbb{S}^2} |\partial_v\psi \restriction_{\mathcal{H}_A}|^2 \mathrm{d}v \sin\theta \mathrm{d}\theta\mathrm{d}\varphi + \int_{\mathbb{R}\times\mathbb{S}^2} |\partial_u\psi \restriction_{\mathcal{H}_B}|^2 \mathrm{d}u \sin\theta\mathrm{d}\theta\mathrm{d}\varphi. \tag{1.2.29}$$

The norm (1.2.27) defines an inner product, hence its completion is a Hilbert space.

Definition 1.2.1. *We define the Hilbert space of finite T energy $\mathcal{E}^T_{\mathcal{H}}$ on both components of*

[3] A choice of such normal vectors fixes the volume form. Also note that this is the natural setup for energy estimates.

the event horizon as the completion of smooth and compactly supported functions $C_c^\infty(\mathcal{H})$ *on the event horizon* $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B \cup \mathcal{B}_-$ *with respect to the norm* (1.2.27).

Analogously, we can consider the vector space $C_c^\infty(\mathcal{CH})$ and define the norm $\|\cdot\|^2_{\mathcal{E}^T_{\mathcal{CH}}}$ as the T energy flux on the component $\mathcal{CH}_B$ minus the T energy flux on the component $\mathcal{CH}_A$:

$$\|\psi\|^2_{\mathcal{E}^T_{\mathcal{CH}}} := \int_{\mathcal{CH}_B} J^T_\mu[\psi] n^\mu_{\mathcal{CH}_B} \,\mathrm{dvol}_{n_{\mathcal{CH}_B}} - \int_{\mathcal{CH}_A} J^T_\mu[\psi] n^\mu_{\mathcal{CH}_A} \,\mathrm{dvol}_{n_{\mathcal{CH}_A}}. \tag{1.2.30}$$

Again, in view of the orientation of the T vector field (cf. Fig. 1.2), this norm can be represented as (using the coordinate charts (1.2.17) and (1.2.18))

$$\|\psi\|^2_{\mathcal{E}^T_{\mathcal{CH}}} = \int_{\mathbb{R}\times\mathbb{S}^2} |\partial_v \psi \restriction_{\mathcal{CH}_B}|^2 \mathrm{d}v \sin\theta \mathrm{d}\theta \mathrm{d}\varphi + \int_{\mathbb{R}\times\mathbb{S}^2} |\partial_u \psi \restriction_{\mathcal{CH}_A}|^2 \mathrm{d}u \sin\theta \mathrm{d}\theta \mathrm{d}\varphi. \tag{1.2.31}$$

Definition 1.2.2. *We define the Hilbert space of finite T energy $\mathcal{E}^T_{\mathcal{CH}}$ on both components of the Cauchy horizon as the completion of smooth and compactly supported functions $C_c^\infty(\mathcal{CH})$ the Cauchy horizon $\mathcal{CH} = \mathcal{CH}_A \cup \mathcal{CH}_B \cup \mathcal{B}_+$ with respect to the norm* (1.2.30).

1.2.4 Separation of variables

In this section we show how the radial o.d.e. (1.1.2) arises from decomposing a general solution in spherical harmonics and Fourier modes. For concreteness, let ψ be a smooth solution to $\Box_g \psi = 0$ such that on each $\{r = \mathit{const.}\}$ slice, ψ is compactly supported in the t variable.[4] Then, we can define its Fourier transform in the t variable and also decompose ψ in spherical harmonics to end up with

$$\hat{\psi}_{m\ell}(r,\omega) := \int_{\mathbb{R}\times\mathbb{S}^2} e^{-i\omega t} Y_{m\ell}(\theta,\phi)\psi(t,r,\theta,\phi) \sin\theta \mathrm{d}\theta \mathrm{d}\phi \frac{\mathrm{d}t}{\sqrt{2\pi}}. \tag{1.2.32}$$

Due to the compact support on constant r slices, the wave equation $\Box_g \psi = 0$ implies that

$$\hat{\psi}_{m\ell}(r,\omega) =: R^{(\omega)}_{m\ell}(r) =: R(r) \tag{1.2.33}$$

satisfies the radial o.d.e.

$$\Delta \frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta \frac{\mathrm{d}}{\mathrm{d}r} R\right) - \Delta\ell(\ell+1)R + r^4\omega^2 R = 0. \tag{1.2.34}$$

[4]Note that we will prove later that such solutions arise from data which are dense in $\mathcal{E}^T_{\mathcal{H}}$.

In Section 1.4 we will analyze solutions to (1.2.34) and denote a solution thereof with $R(r)$. Moreover, it is useful to introduce the function u defined as

$$u(r) := rR(r) \tag{1.2.35}$$

and consider $u = u(r(r_*))$ as a function of r_*, which is defined in (1.2.7). Using the r_* variable, the o.d.e. (1.2.34) finally reduces to

$$u'' + (\omega^2 - V_\ell)u = 0 \tag{1.2.36}$$

on the real line with potential

$$V = V_\ell = \Delta \left(\frac{r(r_+ + r_-) - 2r_+ r_-}{r^3} + \frac{\ell(\ell+1)}{r^4} \right). \tag{1.2.37}$$

In Lemma 1.8.3 in the appendix it is proven that, as a function of r_*, the potential V_ℓ decays exponentially as $r_* \to \pm\infty$. In particular, this indicates that we have asymptotic free waves (asymptotic states) near the event and Cauchy horizon of the form $e^{\pm i\omega r_*}$ as $|r_*| \to \infty$. In order to construct these solutions we use the following proposition for Volterra integral equations (see Lemma 2.4 of [134]).

Proposition 1.2.3. *Let $x_0 \in \mathbb{R} \cup \{+\infty\}$ and $g \in L^\infty(-\infty, x_0)$. Suppose the integral kernel K satisfies*

$$\alpha := \int_{-\infty}^{x_0} \sup_{\{x: y < x < x_0\}} |K(x,y)| \mathrm{d}y < \infty. \tag{1.2.38}$$

Then, the Volterra integral equation

$$f(x) = g(x) + \int_{-\infty}^{x} K(x,y) f(y) \mathrm{d}y \tag{1.2.39}$$

has a unique solution f satisfying

$$\|f\|_{L^\infty(-\infty,x_0)} \leq e^\alpha \|g\|_{L^\infty(-\infty,x_0)}. \tag{1.2.40}$$

If in addition K is smooth in both variables and

$$\int_{-\infty}^{x_0} \sup_{\{x: y < x < x_0\}} |\partial_x^k K(x,y)| \mathrm{d}y < \infty \tag{1.2.41}$$

for all $k \in \mathbb{N}$, then the solution f is smooth on $(-\infty, x_0)$ and the derivatives can be *computed by formal differentiation of* (1.2.39).

Remark 1.2.1. *Analogous results as in Proposition 1.2.3 also hold true for Volterra integral equations on intervals of the form* (x_0, x_1) *or* $(x_0, +\infty)$.

This allows us to define the following fundamental pairs of solutions to the o.d.e. (1.2.36). In view of the exponential decay of the potential, it is straightforward to check that the assumptions of Proposition 1.2.3 are satisfied.

Definition 1.2.3. *Let* $\omega \in \mathbb{R}$ *and* $\ell \in \mathbb{N}_0$ *be fixed. Define asymptotic state solutions* u_1 *and* u_2 *of the radial o.d.e.* (1.2.36) *as the unique solutions to the Volterra integral equations*

$$u_1(\omega, r_*) = e^{i\omega r_*} + \int_{-\infty}^{r_*} \frac{\sin(\omega(r_* - y))}{\omega} V(y) u_1(\omega, y) \mathrm{d}y, \tag{1.2.42}$$

$$u_2(\omega, r_*) = e^{-i\omega r_*} + \int_{-\infty}^{r_*} \frac{\sin(\omega(r_* - y))}{\omega} V(y) u_2(\omega, y) \mathrm{d}y. \tag{1.2.43}$$

Analogously, define v_1 *and* v_2 *as the unique solutions to the Volterra integral equations*

$$v_1(\omega, r_*) = e^{i\omega r_*} - \int_{r_*}^{\infty} \frac{\sin(\omega(r_* - y))}{\omega} V(y) v_1(\omega, y) \mathrm{d}y, \tag{1.2.44}$$

$$v_2(\omega, r_*) = e^{-i\omega r_*} - \int_{r_*}^{\infty} \frac{\sin(\omega(r_* - y))}{\omega} V(y) v_2(\omega, y) \mathrm{d}y. \tag{1.2.45}$$

For $\omega = 0$, *we set* $\frac{\sin(\omega(r_* - y))}{\omega} \restriction_{\omega=0} = r_* - y$ *in the integral kernel in which case* u_1 *and* u_2 *coincide. We define*

$$\tilde{u}_1(r_*) := u_1(0, r_*) = u_2(0, r_*) \tag{1.2.46}$$

and similarly,

$$\tilde{v}_1(r_*) := v_1(0, r_*) = v_2(0, r_*). \tag{1.2.47}$$

Since $u_1(0, r_*) = u_2(0, r_*)$ *for* $\omega = 0$, *there exists another linearly independent fundamental solution* $\tilde{u}_2$ *solving the Volterra integral equation*

$$\tilde{u}_2(r_*) = r_* + \int_{-\infty}^{r_*} (r_* - y) V(y) \tilde{u}_2(y) \mathrm{d}y. \tag{1.2.48}$$

Similarly, we also have another fundamental solution, which is linearly independent from

$\tilde{v}_1$, solving

$$\tilde{v}_2(r_*) = r_* - \int_{r^*}^{\infty} (r_* - y)V(y)\tilde{v}_2(y)\mathrm{d}y. \tag{1.2.49}$$

Since r_* is not uniformly bounded, we cannot apply Proposition 1.2.3 to construct $\tilde{u}_2$ and $\tilde{v}_2$. Nevertheless, after switching to coordinates which are regular at $\mathcal{H}$ or $\mathcal{CH}$, the existence of the desired solutions follows immediately from the usual local theory of regular singularities (see [119]).

Remark 1.2.2. *Due to the exponential decay of the potential V_ℓ (see Lemma 1.8.3 in the appendix), it follows from standard theory that the solutions $u_1(\omega, r_*), u_2(\omega, r_*)$, $v_1(\omega, r_*)$ and $v_2(\omega, r_*)$ can be continued to holomorphic functions of ω in the strip $|\operatorname{Im}(\omega)| < \min(\kappa_+, |\kappa_-|) = \kappa_+$ for fixed $r_* \in \mathbb{R}$. Indeed, in [14] it is shown that $u_1(\omega, r_*)$ is analytic in $\mathbb{C} \setminus \{im\kappa_+ : m \in \mathbb{N}\}$ with possible poles at $\{im\kappa_+ : m \in \mathbb{N}\}$ and analogously for u_2, v_1, and v_2. See also the proof of Proposition 1.8.2 in the appendix.*

Formally, the solution u_1 represents a fixed frequency incoming wave from the right event horizon $\mathcal{H}_A$. This wave will scattering in the black hole interior and some portion will be transmitted to the left Cauchy horizon $\mathcal{CH}_B$ with corresponding solution v_1 and some other portion will be reflected to the right Cauchy horizon $\mathcal{CH}_A$ with corresponding solution v_2. The transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$ will be defined as the transmitted and reflected parts of the incoming wave v_1 to the left and right Cauchy horizon, respectively. This leads us to

Definition 1.2.4. *Let $\omega \neq 0$. Then we define the transmission coefficient $\mathfrak{T}(\omega, \ell)$ and reflection coefficient $\mathfrak{R}(\omega, \ell)$ as the unique coefficients such that*

$$u_1 = \mathfrak{T} v_1 + \mathfrak{R} v_2. \tag{1.2.50}$$

Using the fact that the Wronskian

$$\mathfrak{W}(f, g) := fg' - f'g \tag{1.2.51}$$

of two solutions f and g is independent of r_*, we can equivalently define the scattering coefficients as

$$\mathfrak{T} := \frac{\mathfrak{W}(u_1, v_2)}{\mathfrak{W}(v_1, v_2)} = \frac{\mathfrak{W}(u_1, v_2)}{-2i\omega} \tag{1.2.52}$$

and

$$\mathfrak{R} := \frac{\mathfrak{W}(u_1, v_1)}{\mathfrak{W}(v_2, v_1)} = \frac{\mathfrak{W}(u_1, v_1)}{2i\omega}. \tag{1.2.53}$$

In contrast to the black hole exterior, there is no conservation law which gives a priori bound on the scattering coefficients $\mathfrak{T}$ and $\mathfrak{R}$. In particular, the conservation law associated to the vector field ∂_t is degenerate. For fixed frequency and on the level of the transmission and reflection coefficients, this leads to the following pseudo-unitarity property.

Proposition 1.2.4 (Pseudo-unitarity in the separated picture)**.** *The transmission and reflection coefficients satisfy*

$$1 = |\mathfrak{T}|^2 - |\mathfrak{R}|^2. \tag{1.2.54}$$

Proof. First, note that any solution to the o.d.e. (1.2.36) satisfies the identity

$$\operatorname{Im}(\bar{u}u') = const. \tag{1.2.55}$$

Applying this to the solution $u_1 = \mathfrak{T}v_1 + \mathfrak{R}v_2$ shows the claim. □

In the following we shall see that the reflection and transmission coefficients are regular at $\omega = 0$.

Proposition 1.2.5. *Let $\ell \in \mathbb{N}_0$ be fixed. Then the scattering coefficients $\mathfrak{R}(\omega, \ell)$ and $\mathfrak{T}(\omega, \ell)$ are analytic functions of ω in the strip $\{\omega \in \mathbb{C}\colon |\operatorname{Im}(\omega)| < \kappa_+\}$ with values for $\omega = 0$ given by*

$$\mathfrak{R}(0, \ell) = \frac{(-1)^\ell}{2}\left(\frac{r_-}{r_+} - \frac{r_+}{r_-}\right), \tag{1.2.56}$$

$$\mathfrak{T}(0, \ell) = \frac{(-1)^\ell}{2}\left(\frac{r_-}{r_+} + \frac{r_+}{r_-}\right). \tag{1.2.57}$$

In particular, the reflection coefficient $\mathfrak{R}(\omega, \ell)$ only vanishes on a discrete set of points ω.

Moreover, the reflection and transmission coefficients $\mathfrak{R}(\omega, \ell)$ and $\mathfrak{T}(\omega, \ell)$ are analytic functions on $\mathbb{C} \setminus \mathbb{P}$ with possible poles at $\mathbb{P} = \{im\kappa_+\colon m \in \mathbb{N}\} \cup \{ik\kappa_-\colon k \in \mathbb{Z} \setminus \{0\}\}$.

Proof. From the analyticity of $u_1, u_2, v_1,$ and v_2 in the strip $|\operatorname{Im}(\omega)| < \kappa_+$ (cf. Remark 1.2.2), we conclude that $\mathfrak{T}$ and $\mathfrak{R}$ are holomorphic in $\{\omega \neq 0 \in \mathbb{C} : |\operatorname{Im}(\omega)| < \kappa_+\}$ with a possible pole at $\omega = 0$. In the following we shall show that $\{\omega = 0\}$ is a removable singularity. Indeed, we will compute the explicit value of the reflection and transmission coefficient at $\omega = 0$ and deduce that for fixed $\ell \in \mathbb{N}_0$, the transmission coefficient $\mathfrak{T}(\omega, \ell)$ and the reflection coefficient $\mathfrak{R}(\omega, \ell)$ are analytic functions on the strip $\{\omega \in \mathbb{C} \colon \operatorname{Im}(\omega)| < \kappa_+\}$ (cf. unpublished work of McNamara cited in [64]). To do so, note that from Proposition 1.4.2 in Section 1.4.1.3 we conclude the pointwise limits

$$u_1(\omega, r_*) \to \tilde{u}_1(r_*), \tag{1.2.58}$$

$$v_1(\omega, r_*) \to \tilde{v}_1(r_*) = (-1)^\ell \frac{r_+}{r_-} \tilde{u}_1(r_*), \tag{1.2.59}$$

$$v_2(\omega, r_*) \to \tilde{v}_1(r_*) = (-1)^\ell \frac{r_+}{r_-} \tilde{u}_1(r_*) \tag{1.2.60}$$

as $|\omega| \to 0$. Using the definition in (1.2.50) of $\mathfrak{T}(\omega, \ell)$, $\mathfrak{R}(\omega, \ell)$, and the condition $1+|\mathfrak{R}|^2 = |\mathfrak{T}|^2$ (cf. Proposition 1.2.4), we deduce that the limits $\lim_{\omega \to 0} \mathfrak{R}(\omega, \ell)$ and $\lim_{\omega\to 0} \mathfrak{T}(\omega, \ell)$ exist and moreover can be computed to be (1.2.56) and (1.2.57). Note that (1.2.56) and (1.2.57) have been established in [63]. Also note that in view of the analyticity properties of u_1, v_1, and v_2, the $\mathfrak{R}(\omega, \ell)$ and $\mathfrak{T}(\omega, \ell)$ are analytic functions on $\mathbb{C} \setminus \mathbb{P}$ with possible poles at $\mathbb{P} = \{im\kappa_+ \colon m \in \mathbb{N}\} \cup \{ik\kappa_- \colon k \in \mathbb{Z} \setminus \{0\}\}$. $\square$

1.2.5 Conventions

Let X be a point set with a limit point c (e.g. $X = \mathbb{R}, [a, b], \mathbb{C}$). Throughout this chapter we will use the symbols $\lesssim$ and $\gtrsim$, where the implicit constants might depend on the black hole parameters M and Q. In particular, for functions (or constants) $a(x), b(x) > 0$ the notation $a \lesssim b$ means that there is a constant $C = C(M, Q) > 0$ such that $a(x) \leq Cb(x)$ for all $x \in X$. We will also make use of the notation $\lesssim_\ell$ or $\gtrsim_\ell$ which means that the constant may additionally also depend on ℓ. We also write $a \sim b$ if there are constants $C(M, Q), \tilde{C}(M, Q) > 0$ such that $Ca(x) \leq b(x) \leq \tilde{C}a(x)$ for all $x \in X$.

We shall also make use of the standard Landau notation O and o [39, 119]. To be more precise, as $x \to c$ in X

$$f(x) = O(g(x)) \text{ means } \left| \frac{f(x)}{g(x)} \right| \leq C(M, Q) \tag{1.2.61}$$

and

$$f(x) = o(g(x)) \text{ means } \frac{f(x)}{g(x)} \to 0. \tag{1.2.62}$$

We will also use the notation O_ℓ if the constant C in (1.2.61) may additionally depend on ℓ.

1.3 Main theorems

In this section we will formulate our main theorems.

Theorem 1.1, which we state in Section 1.3.1, establishes the existence of a scattering map S^T of the form

$$S^T : \mathcal{E}^T_{\mathcal{H}} \to \mathcal{E}^T_{\mathcal{CH}}, \tag{1.3.1}$$

which is a Hilbert space isomorphism, i.e. a bounded and invertible map with bounded inverse. Theorem 1.1 will be proven in Section 1.5. In the separated picture, the boundedness of S^T corresponds to the uniform boundedness of the transmission and reflection coefficients which is stated as Theorem 1.2 in Section 1.3.2. Theorem 1.2 will be proven in Section 1.4 (and later used in the proof of Theorem 1.1).

Section 1.3.3 is then devoted to Theorem 1.3, which connects our physical space scattering theory to the fixed frequency scattering theory. (We will infer Theorem 1.3 as a consequence of Theorem 1.1.) In Section 1.3.4, this connection allows us to prove that the reflection map is injective, which is the content of Theorem 1.4. In Theorem 1.5, which is stated and proven in Section 1.3.5, we construct data which are incoming and compactly supported but nevertheless, lead to a solution which fails to be in C^1 on the Cauchy horizon.

We end this section with the statement of our two non-existence results. In Section 1.3.6 we formulate Theorem 1.6, the non-existence of the T energy scattering theory for the Klein–Gordon equation with conformal mass on the interior of (anti-) de Sitter–Reissner–Nordström black holes. The proof of Theorem 1.6 is given in Section 1.6. Finally, in Theorem 1.7, stated in Section 1.3.7, we show the non-existence of the T energy scattering map for the Klein–Gordon equation on the interior of Reissner–Nordström. The proof of Theorem 1.7 is given in Section 1.7.

1.3.1 Existence and boundedness of the T energy scattering map

First, we define the forward (resp. backward) evolution on a dense domain.

Definition 1.3.1. *The domains of the forward and backward evolution are defined as*

$$\mathcal{D}^T_{\mathcal{H}} := \{\psi \in C^\infty_c(\mathcal{H}) \subset \mathcal{E}^T_{\mathcal{H}} \text{ s.t. the Cauchy evolution of } \psi \text{ has compact support on constant } r = const. \text{ hypersurfaces}\} \tag{1.3.2}$$

and

$$\mathcal{D}^T_{\mathcal{CH}} := \{\psi \in C^\infty_c(\mathcal{CH}) \subset \mathcal{E}^T_{\mathcal{CH}} \text{ s.t. the backward evolution of } \psi \text{ has compact support on constant } r = const. \text{ hypersurfaces}\}, \quad (1.3.3)$$

respectively. Here, we consider $r_- < r < r_+$ and note that if ψ is compactly supported on one $\{r = const.\}$ slice, then, as a direct consequence of the domain of dependence, its evolution will be compactly supported on all other $\{r = const.\}$ hypersurfaces for $r_- < r < r_+$.

We will prove in Lemma 1.5.1 in Section 1.5 that $\mathcal{D}^T_{\mathcal{H}} \subset \mathcal{E}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}} \subset \mathcal{E}^T_{\mathcal{CH}}$ are dense domains.

These definitions of the domains are motivated by the following observation.

Remark 1.3.1. *Suppose we are given data in $\mathcal{D}^T_{\mathcal{H}}$ on the event horizon $\mathcal{H}$. Consider now the unique Cauchy development (cf. Proposition 1.2.1) and observe that its restriction to the Cauchy horizon $\mathcal{CH}$ will lie in $\mathcal{D}^T_{\mathcal{CH}}$. This holds true since we can first smoothly extend the metric beyond the Cauchy horizon $\mathcal{CH}$ and then use the compact support on a constant r_* hypersurface to solve an equivalent Cauchy problem in an appropriate region which extends the Cauchy horizon $\mathcal{CH}$, includes the support of the solution, but does not include i^+. The smoothness of the solution up to and including the Cauchy horizon $\mathcal{CH}$ follows now from Cauchy stability.*

In view of Remark 1.3.1 we can define the forward and backward map on the domains $\mathcal{D}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}}$, respectively.

Definition 1.3.2. *Define the forward map $S^T_0 \colon \mathcal{D}^T_{\mathcal{H}} \subset \mathcal{E}^T_{\mathcal{H}} \to \mathcal{D}^T_{\mathcal{CH}} \subset \mathcal{E}^T_{\mathcal{CH}}$ as the unique forward evolution from data on the event horizon to data on the Cauchy horizon. More precisely, let ψ be the solution to* (1.1.1) *arising from initial data $\Psi \in \mathcal{D}^T_{\mathcal{H}} \subset \mathcal{E}^T_{\mathcal{H}}$. Then, define $S^T_0(\Psi)$ as the restriction of ψ to the Cauchy horizon, i.e. $S^T_0(\Psi) := \psi \restriction_{\mathcal{CH}} \in \mathcal{D}^T_{\mathcal{CH}}$.*

Similarly, let ϕ be the unique backward evolution of (1.1.1) *arising from $\Phi \in \mathcal{D}^T_{\mathcal{CH}}$. Then, define the backward map by $B^T_0(\Phi) := \phi \restriction_{\mathcal{H}} \in \mathcal{D}^T_{\mathcal{H}}$.*

Remark 1.3.2. *Note that by the uniqueness of the Cauchy evolution we have that S^T_0 and B^T_0 are inverses of each other, i.e. $B^T_0 \circ S^T_0 = \mathrm{Id}_{\mathcal{D}^T_{\mathcal{H}}}$, $S^T_0 \circ B^T_0 = \mathrm{Id}_{\mathcal{D}^T_{\mathcal{CH}}}$.*

Now, we are in the position to state our main theorem.

Theorem 1.1. *The map* $S_0^T\colon \mathcal{D}_{\mathcal{H}}^T \subset \mathcal{E}_{\mathcal{H}}^T \to \mathcal{D}_{\mathcal{CH}}^T \subset \mathcal{E}_{\mathcal{CH}}^T$ *is bounded and uniquely extends to*

$$S^T\colon \mathcal{E}_{\mathcal{H}}^T \to \mathcal{E}_{\mathcal{CH}}^T, \tag{1.3.4}$$

called the "scattering map". The scattering map S^T *is a Hilbert space isomorphism, i.e. a* <u>*bounded*</u> *and* <u>*invertible*</u> *linear map with* <u>*bounded inverse*</u> $B^T\colon \mathcal{E}_{\mathcal{CH}}^T \to \mathcal{E}_{\mathcal{H}}^T$ *satisfying*

$$B^T \circ S^T = \mathrm{Id}_{\mathcal{E}_{\mathcal{H}}^T},\ S^T \circ B^T = \mathrm{Id}_{\mathcal{E}_{\mathcal{CH}}^T}. \tag{1.3.5}$$

Here, $B^T\colon \mathcal{E}_{\mathcal{CH}}^T \to \mathcal{E}_{\mathcal{H}}^T$ *is the "backward map", which is the unique bounded extension of* B_0^T.

In addition, the scattering map S^T *is pseudo-unitary, meaning that for* $\psi \in \mathcal{E}_{\mathcal{H}}^T$*, we have*

$$\int_{\mathcal{H}_A} |T\psi|^2 - \int_{\mathcal{H}_B} |T\psi|^2 = \int_{\mathcal{CH}_B} |TS^T\psi|^2 - \int_{\mathcal{CH}_A} |TS^T\psi|^2. \tag{1.3.6}$$

In more traditional language, Theorem 1.1 yields existence, uniqueness, and asymptotic completeness of scattering states.

The proof of Theorem 1.1 is given in Section 1.5. Let us note already that Theorem 1.1 is a posteriori the physical space equivalent of the uniform boundedness of the scattering coefficients proven in Theorem 1.2 (see Section 1.3.2). This equivalence is made precise in Theorem 1.3 (see Section 1.3.3).

Remark 1.3.3. *Note that in general, neither initial data nor scattered data have to be bounded in* L^∞ *or continuous. Indeed, we have that* $\Phi_A(u,\theta,\varphi) = \log(u)\chi_{u\geq 1} \in \mathcal{E}_{\mathcal{CH}_A}^T$*, where* $\chi_{u\geq 1}$ *is a smooth cutoff. Similarly,* $\Phi_A(u,\theta,\varphi) = f(u)g(\theta,\varphi) \in \mathcal{E}_{\mathcal{CH}_A}^T$*, where* $f \in C_c^\infty(\mathbb{R})$ *and* $g \in L^2(\mathbb{S}^2) \setminus L^\infty(\mathbb{S}^2)$*. Thus, there exist initial data* $B^T(\Phi_A) \in \mathcal{E}_{\mathcal{H}}^T$ *such that its image under the scattering map is not in* L^∞ *and not continuous. We emphasize the contrast with the estimates from [51] for which more regularity and decay along the event horizon* $\mathcal{H}$ *are necessary.*

1.3.2 Uniform boundedness of the transmission and reflection coefficients

On the level of the o.d.e. (1.2.36) in the separated picture, the problem of boundedness of the scattering map reduces to proving that the transmission coefficient $\mathfrak{T}$ and the reflection coefficient $\mathfrak{R}$ are uniformly bounded over all parameter ranges of $\omega \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$. This is stated as Theorem 1.2 below.

Theorem 1.2. *The reflection and transmission coefficients* $\mathfrak{R}(\omega,\ell)$ *and* $\mathfrak{T}(\omega,\ell)$ *are uniformly bounded, i.e. they satisfy*

$$\sup_{\omega\in\mathbb{R},\ell\in\mathbb{N}_0}\left(|\mathfrak{R}(\omega,\ell)|+|\mathfrak{T}(\omega,\ell)|\right)\lesssim 1. \tag{1.3.7}$$

Theorem 1.2 is proved in Section 1.4. As discussed in the introduction, the proof relies on an explicit calculation for $\omega = 0$ together with a careful analysis of the radial o.d.e. (1.2.36), involving properties of special functions and perturbations thereof.

Let us note that, *given* Theorem 1.1, we could infer Theorem 1.2 as a corollary (using the theory to be described in Section 1.3.3). We emphasize, however, that in the present chapter we *use* Theorem 1.2 to prove Theorem 1.1 in Section 1.5.

1.3.3 Connection between the separated and the physical space picture

In this section, we will make the connection of the separated and physical space picture precise.

First, let us note that we have natural Hilbert space decompositions $\mathcal{E}^T_{\mathcal{H}}\cong\mathcal{E}^T_{\mathcal{H}_A}\oplus\mathcal{E}^T_{\mathcal{H}_B}$ and $\mathcal{E}^T_{\mathcal{CH}}\cong\mathcal{E}^T_{\mathcal{CH}_B}\oplus\mathcal{E}^T_{\mathcal{CH}_A}$.

Proposition 1.3.1. *The Hilbert spaces* $\mathcal{E}^T_{\mathcal{H}}$ *and* $\mathcal{E}^T_{\mathcal{CH}}$ *of finite* T *energy on the event horizon* $\mathcal{H}$ *and on the Cauchy horizon* $\mathcal{CH}$ *admit the orthogonal decomposition*

$$\mathcal{E}^T_{\mathcal{H}}\cong\mathcal{E}^T_{\mathcal{H}_A}\oplus\mathcal{E}^T_{\mathcal{H}_B}\ \text{ and }\ \mathcal{E}^T_{\mathcal{CH}}\cong\mathcal{E}^T_{\mathcal{CH}_A}\oplus\mathcal{E}^T_{\mathcal{CH}_B}. \tag{1.3.8}$$

Proof. Clearly, the embedding $i\colon \mathcal{E}^T_{\mathcal{H}_A}\oplus\mathcal{E}^T_{\mathcal{H}_B}\hookrightarrow\mathcal{E}^T_{\mathcal{H}}$ is well-defined and isometric. It remains to show that i is surjective. Let $\psi\in C^\infty_c(\mathcal{H})$. First, we show that we can approximate (in T-energy) $\psi\restriction_{\mathcal{H}_A}$ on $\mathcal{H}_A$ with functions $\psi_\epsilon\in C^\infty_c(\mathcal{H}_A)$ which are supported away from the past bifurcation sphere. On $\mathcal{H}_A$ choose non-degenerate coordinates $(V,\theta,\varphi):=(V_{\mathcal{H}},\theta,\varphi)$ as in Section 1.2.1.2 and recall that the past bifurcation sphere is $\{V=0\}$. Then, for small $\epsilon>0$, set

$$\psi_\epsilon(V,\theta,\varphi):=\psi(U=0,V,\theta,\varphi)\chi(-\epsilon\log(V)), \tag{1.3.9}$$

where $\chi\colon\mathbb{R}\to[0,1]$ is smooth and such that $\operatorname{supp}(\chi)\subseteq(-\infty,2]$ and $\chi\restriction_{(-\infty,1]}=1$. Then, it is straightforward to check that $\psi_\epsilon \in C_c^\infty(\mathcal{H}_A)$ and

$$\int_{\mathcal{H}_A} J^T[\psi-\psi_\epsilon]_\mu n^\mu \mathrm{dvol} \lesssim \int_{\mathbb{S}^2}\int_0^\infty V(\partial_V(\psi-\psi_\epsilon))^2 \mathrm{d}V \sin\theta \mathrm{d}\theta\mathrm{d}\varphi \to 0 \tag{1.3.10}$$

as $\epsilon \to 0$. Analogously, we can do this for $\mathcal{H}_B$ from which the claim follows. □

We will use this identification to represent the scattering map also in the Fourier picture and show how these pictures connect. To do so we define the following.

Definition 1.3.3. *For $(\Psi_A, \Psi_B) \in \mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B}$ note that $\partial_v \Psi_A(v,\theta,\phi) \in L^2(\mathbb{R}\times\mathbb{S}^2;\mathbb{C})$ and analogously for Ψ_B. Hence, in mild abuse of notation, we can define the Fourier and spherical harmonics coefficients $\mathcal{F}_{\mathcal{H}_A}(\Psi_A)$ and $\mathcal{F}_{\mathcal{H}_B}(\Psi_B)$ as*

$$i\omega\mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega,m,\ell) := r_+ \int_{\mathbb{R}}\int_{\mathbb{S}^2} \partial_v\Psi_A(v,\theta,\varphi)e^{-i\omega v}Y_{\ell m}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\frac{\mathrm{d}v}{\sqrt{2\pi}} \tag{1.3.11}$$

and

$$-i\omega\mathcal{F}_{\mathcal{H}_B}(\Psi_B)(\omega,m,\ell) := r_+ \int_{\mathbb{R}}\int_{\mathbb{S}^2} \partial_u\Psi_B(u,\theta,\varphi)e^{i\omega u}Y_{\ell m}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\frac{\mathrm{d}u}{\sqrt{2\pi}}. \tag{1.3.12}$$

Similarly, for $(\Phi_A,\Phi_B) \in \mathcal{E}^T_{\mathcal{CH}_A} \oplus \mathcal{E}^T_{\mathcal{CH}_B}$ set

$$-i\omega\mathcal{F}_{\mathcal{CH}_A}(\Phi_A)(\omega,m,\ell) := r_- \int_{\mathbb{R}}\int_{\mathbb{S}^2} \partial_u\Phi_A(u,\theta,\varphi)e^{i\omega u}Y_{\ell m}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\frac{\mathrm{d}u}{\sqrt{2\pi}} \tag{1.3.13}$$

and

$$i\omega\mathcal{F}_{\mathcal{CH}_B}(\Phi_B)(\omega,m,\ell) := r_- \int_{\mathbb{R}}\int_{\mathbb{S}^2} \partial_v\Phi_B(v,\theta,\varphi)e^{-i\omega v}Y_{\ell m}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\frac{\mathrm{d}v}{\sqrt{2\pi}}. \tag{1.3.14}$$

Also, recall the Hilbert space decomposition $\mathcal{E}^T_{\mathcal{H}} \cong \mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B}$ and $\mathcal{E}^T_{\mathcal{CH}} \cong \mathcal{E}^T_{\mathcal{CH}_B} \oplus \mathcal{E}^T_{\mathcal{CH}_A}$. Thus, the scattering matrix can be also decomposed as

$$S^T = \begin{pmatrix} S^T_{BA} & S^T_{BB} \\ S^T_{AA} & S^T_{AB} \end{pmatrix}, \tag{1.3.15}$$

where

$$S^T_{ij}\colon \mathcal{E}^T_{\mathcal{H}_j} \to \mathcal{E}^T_{\mathcal{CH}_i} \tag{1.3.16}$$

is a bounded linear map for $i,j \in \{A,B\}$.[5]

[5] Note that T does not denote the transpose but the fact that it is the scattering map associated with

Definition 1.3.4. *Define the Hilbert spaces*

$$\hat{\mathcal{E}}^T_{\mathcal{H}_A} := \ell^2(Z; L^2(r_+^{-2}\omega^2 \mathrm{d}\omega)),\ \ \hat{\mathcal{E}}^T_{\mathcal{H}_B} := \ell^2(Z; L^2(r_+^{-2}\omega^2 \mathrm{d}\omega)),$$
$$\hat{\mathcal{E}}^T_{\mathcal{CH}_A} := \ell^2(Z; L^2(r_-^{-2}\omega^2 \mathrm{d}\omega)),\ \ \hat{\mathcal{E}}^T_{\mathcal{CH}_B} := \ell^2(Z; L^2(r_-^{-2}\omega^2 \mathrm{d}\omega)),$$

where $Z = \{(m,\ell) \in \mathbb{Z} \times \mathbb{N}_0 : |m| \leq \ell\}$.

The Hilbert spaces defined in Definition 1.3.4 are unitary isomorphic to their corresponding physical energy spaces. This is captured in

Proposition 1.3.2. *The linear maps defined in* (1.3.11)–(1.3.14)

$$\mathcal{F}_{\mathcal{H}_A} \oplus \mathcal{F}_{\mathcal{H}_B} \colon \mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B} \to \hat{\mathcal{E}}^T_{\mathcal{H}_A} \oplus \hat{\mathcal{E}}^T_{\mathcal{H}_B} \tag{1.3.17}$$
$$\mathcal{F}_{\mathcal{CH}_B} \oplus \mathcal{F}_{\mathcal{CH}_A} \colon \mathcal{E}^T_{\mathcal{CH}_B} \oplus \mathcal{E}^T_{\mathcal{CH}_A} \to \hat{\mathcal{E}}^T_{\mathcal{CH}_B} \oplus \hat{\mathcal{E}}^T_{\mathcal{CH}_A} \tag{1.3.18}$$

are unitary.

Proof. This follows from the fact that the Fourier transform and the decomposition into spherical harmonics are unitary maps. □

Now, we will define the scattering map in the separated picture and show that it is bounded.

Proposition 1.3.3. *The scattering map in the separated picture*

$$\hat{S^T} \colon \hat{\mathcal{E}}^T_{\mathcal{H}_A} \oplus \hat{\mathcal{E}}^T_{\mathcal{H}_B} \to \hat{\mathcal{E}}^T_{\mathcal{CH}_B} \oplus \hat{\mathcal{E}}^T_{\mathcal{CH}_A}, \tag{1.3.19}$$

defined as the multiplication operator

$$\hat{S^T} = \begin{pmatrix} S^{\hat{T}}_{BA} & S^{\hat{T}}_{BB} \\ S^{\hat{T}}_{AA} & S^{\hat{T}}_{AB} \end{pmatrix} := \begin{pmatrix} \mathfrak{T}(\omega,\ell) & \bar{\mathfrak{R}}(\omega,\ell) \\ \mathfrak{R}(\omega,\ell) & \bar{\mathfrak{T}}(\omega,\ell) \end{pmatrix}, \tag{1.3.20}$$

is bounded. Moreover, the map $\hat{S}^T$ *is invertible with bounded inverse given by*

$$\hat{S^T}^{-1} = \begin{pmatrix} \bar{\mathfrak{T}}(\omega,\ell) & -\bar{\mathfrak{R}}(\omega,\ell) \\ -\mathfrak{R}(\omega,\ell) & \mathfrak{T}(\omega,\ell) \end{pmatrix}. \tag{1.3.21}$$

Proof. Indeed, $\hat{S}^T$ is bounded in view of the uniform boundedness of the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$ (cf. Theorem 1.2). Also note that $|\mathfrak{T}|^2 = 1 + |\mathfrak{R}|^2$ implies

the T vector field.

that

$$\det\left(\hat{S}^T\right) = 1 \tag{1.3.22}$$

which shows (1.3.21). The boundedness of $\hat{S^T}^{-1}$ is again immediate since the scattering coefficients are uniformly bounded. $\square$

Using the previous definitions, we obtain the following connection for the scattering map between the physical space and the separated picture.

Theorem 1.3. *The following diagram commutes and each arrow is a Hilbert space isomorphism:*

$$\begin{array}{ccc}
\mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B} & \xrightarrow{S^T} & \mathcal{E}^T_{\mathcal{CH}_B} \oplus \mathcal{E}^T_{\mathcal{CH}_A} \\
{\scriptstyle \mathcal{F}_{\mathcal{H}_A}\oplus\mathcal{F}_{\mathcal{H}_B}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{F}_{\mathcal{CH}_B}\oplus\mathcal{F}_{\mathcal{CH}_A}} \\
\hat{\mathcal{E}}^T_{\mathcal{H}_A} \oplus \hat{\mathcal{E}}^T_{\mathcal{H}_B} & \xrightarrow{\hat{S}^T} & \hat{\mathcal{E}}^T_{\mathcal{CH}_B} \oplus \hat{\mathcal{E}}^T_{\mathcal{CH}_A}.
\end{array}$$

Moreover, the maps S^T and $\hat{S}^T$ are pseudo-unitary satisfying (1.3.6) *and* (1.2.54)*, respectively. More concretely, for $(\Psi_A, \Psi_B) \in \mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B}$, we can write*

$$\begin{pmatrix} \Phi_B \\ \Phi_A \end{pmatrix} = S^T \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix}, \tag{1.3.23}$$

where $\partial_u \Phi_A \in L^2(\mathcal{CH}_A)$ and $\partial_v \Phi_B \in L^2(\mathcal{CH}_B)$ can be represented by

$$\begin{aligned}
\partial_u \Phi_A(u,\theta,\varphi) =& \frac{1}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \sum_{|m|\leq \ell} -i\omega \mathfrak{R}(\omega,\ell)\, \mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega,m,\ell) Y_{m\ell}(\theta,\varphi) e^{-i\omega u} \mathrm{d}\omega \\
&+ \frac{1}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \sum_{|m|\leq \ell} -i\omega \bar{\mathfrak{T}}(\omega,\ell)\, \mathcal{F}_{\mathcal{H}_B}(\Psi_B)(\omega,m,\ell) Y_{m\ell}(\theta,\varphi) e^{-i\omega u} \mathrm{d}\omega
\end{aligned} \tag{1.3.24}$$

and

$$\begin{aligned}
\partial_v \Phi_B(v,\theta,\varphi) =& \frac{1}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \sum_{|m|\leq \ell} i\omega \mathfrak{T}(\omega,\ell)\, \mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega,m,\ell) Y_{m\ell}(\theta,\varphi) e^{i\omega v} \mathrm{d}\omega \\
&+ \frac{1}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \sum_{|m|\leq \ell} i\omega \bar{\mathfrak{R}}(\omega,\ell)\, \mathcal{F}_{\mathcal{H}_B}(\Psi_B)(\omega,m,\ell) Y_{m\ell}(\theta,\varphi) e^{i\omega v} \mathrm{d}\omega
\end{aligned} \tag{1.3.25}$$

as well as $\Phi_A \in \mathcal{E}^T_{\mathcal{CH}_A} \cong \dot{H}^1(\mathbb{R}; L^2(\mathbb{S}^2)), \Phi_B \in \mathcal{E}^T_{\mathcal{CH}_B} \cong \dot{H}^1(\mathbb{R}; L^2(\mathbb{S}^2))$ *can be represented by regular distributions as*

$$\begin{aligned}\Phi_A(u,\theta,\varphi) =& \frac{1}{\sqrt{2\pi r_-}}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\sum_{|m|\leq\ell}\mathfrak{R}(\omega,\ell)\,\mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega,m,\ell)Y_{m\ell}(\theta,\varphi)e^{-i\omega u}\mathrm{d}\omega\\ &+\frac{1}{\sqrt{2\pi r_-}}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\sum_{|m|\leq\ell}\bar{\mathfrak{T}}(\omega,\ell)\,\mathcal{F}_{\mathcal{H}_B}(\Psi_B)(\omega,m,\ell)Y_{m\ell}(\theta,\varphi)e^{-i\omega u}\mathrm{d}\omega \end{aligned} \tag{1.3.26}$$

and

$$\begin{aligned}\Phi_B(v,\theta,\varphi) =& \frac{1}{\sqrt{2\pi r_-}}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\sum_{|m|\leq\ell}\mathfrak{T}(\omega,\ell)\,\mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega,m,\ell)Y_{m\ell}(\theta,\varphi)e^{i\omega v}\mathrm{d}\omega\\ &+\frac{1}{\sqrt{2\pi r_-}}\,\mathrm{p.\,v.}\int_{\mathbb{R}}\sum_{|m|\leq\ell}\bar{\mathfrak{R}}(\omega,\ell)\,\mathcal{F}_{\mathcal{H}_B}(\Psi_B)(\omega,m,\ell)Y_{m\ell}(\theta,\varphi)e^{i\omega v}\mathrm{d}\omega. \end{aligned} \tag{1.3.27}$$

Proof. This is a direct consequence of Theorem 1.1, Theorem 1.2 and (1.5.30), (1.5.31) in the proof of Proposition 1.5.1. □

From the previous representation of the scattered solution we can draw a link between the boundedness of the scattering map and the fact that compactly supported incoming data will lead to solutions which vanish on the future bifurcation sphere $\mathcal{B}_+$. This is the content of the following

Corollary 1.3.1. *Let* $\Psi = (\Psi_A, 0) \in \mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B}$ *be purely incoming smooth data. Assume further that* Ψ_A *is supported away from the past bifurcation sphere* $\mathcal{B}_-$ *and future timelike infinity* i^+.

Then, the Cauchy evolution ψ *arising from* Ψ_A *vanishes at the future bifurcation sphere* $\mathcal{B}_+$.

On the other hand, if Ψ*, as above, led to a solution which does not vanish at the future bifurcation sphere* $\mathcal{B}_+$*, then the scattering map* $S^T : \mathcal{E}^T_{\mathcal{H}} \to \mathcal{E}^T_{\mathcal{CH}}$ *could not be bounded.*

Proof. The first claim is a direct consequence of (1.3.27) in Theorem 1.3.

For the second statement let Ψ_A be compactly supported data on the event horizon and assume that its Cauchy evolution ψ does not vanish at the future bifurcation sphere $\mathcal{B}_+$. Now take data $\tilde{\Psi}_A$ which is supported away from the past bifurcation sphere $\mathcal{B}_-$ and satisfies $T\tilde{\Psi}_A = \Psi_A$. Then, $\tilde{\Psi}_A \in \mathcal{E}^T$ but its Cauchy evolution $\tilde{\psi}$ satisfies $\tilde{\psi} \restriction_{\mathcal{CH}} \notin \mathcal{E}^T_{\mathcal{CH}}$

since

$$\|\tilde{\psi} \restriction_{\mathcal{CH}_B} \|^2_{\mathcal{E}^T_{\mathcal{CH}_B}} = \int_{\mathbb{R}\times\mathbb{S}^2} |\psi \restriction_{\mathcal{CH}_B} (v,\theta,\varphi)|^2 \mathrm{d}v \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = \infty, \tag{1.3.28}$$

as $\psi \restriction_{\mathcal{CH}_B} = T\tilde{\psi} \restriction_{\mathcal{CH}_B}$ does not vanish at the future bifurcation sphere $\mathcal{B}_+$. By cutting off smoothly, one can construct normalized (in $\mathcal{E}^T_{\mathcal{H}}$-norm) smooth compactly supported initial data on $\mathcal{E}^T_{\mathcal{H}}$ such that its Cauchy evolution has arbitrary large norm $\mathcal{E}^T_{\mathcal{CH}}$-norm at the Cauchy horizon. $\square$

Remark 1.3.4. *For convenience we have stated the second statement of Corollary 1.3.1 only for the interior of Reissner–Nordström. However, note that it holds true for more general black hole interiors (e.g. subextremal (anti-) de Sitter–Reissner–Nordström) with Penrose diagram as depicted in Fig. 1.5.*

1.3.4 Injectivity of the reflection map

In this section, we define the reflection operator of purely incoming radiation (cf. Fig. 1.6) and prove that it is has trivial kernel as an operator from $\mathcal{E}^T_{\mathcal{H}_A} \to \mathcal{E}^T_{\mathcal{CH}_A}$.

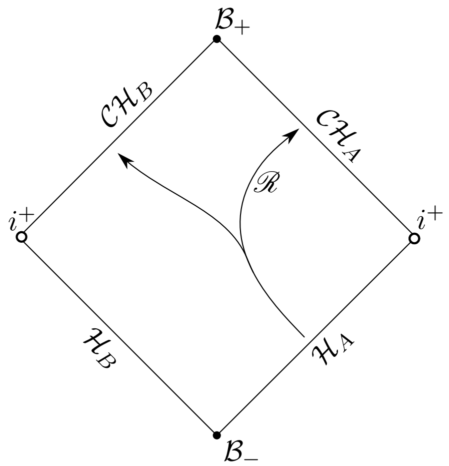

Figure 1.6: Reflection $\mathscr{R}$ of purely incoming radiation.

Definition 1.3.5 (Reflection operator)**.** *For purely incoming radiation $(\Psi_A, 0) \in \mathcal{E}^T_{\mathcal{H}_A} \oplus \mathcal{E}^T_{\mathcal{H}_B}$, define the reflection operator*

$$\mathscr{R} : \mathcal{E}^T_{\mathcal{H}_A} \to \mathcal{E}^T_{\mathcal{CH}_A} \tag{1.3.29}$$

as

$$\mathscr{R}(\Psi_A) = \Phi_A := \mathrm{pr}_A \left(S^T \begin{pmatrix} \Psi_A \\ 0 \end{pmatrix} \right), \tag{1.3.30}$$

where $\mathrm{pr}_A \colon \mathcal{E}^T_{\mathcal{CH}_B} \oplus \mathcal{E}^T_{\mathcal{CH}_A} \to \mathcal{E}^T_{\mathcal{CH}_A}$ *is the orthogonal projection.*

Theorem 1.4. *The reflection operator* $\mathscr{R}$ *defined in Definition 1.3.5 has trivial kernel.*

Proof. Assume $\mathscr{R}(\Psi_A) = 0$ for some $\Psi_A \in \mathcal{E}^T_{\mathcal{H}_A}$. Then, in view of Theorem 1.3,

$$\mathfrak{R}(\omega, \ell) \mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega, m, \ell) = 0 \tag{1.3.31}$$

for all m, ℓ, and a.e. $\omega \in \mathbb{R}$. Moreover, since $\mathfrak{R}(\omega, \ell)$ only vanishes on a discrete set (cf. Proposition 1.2.5), we obtain that $\mathcal{F}_{\mathcal{H}_A}(\Psi_A)(\omega, m, \ell) = 0$ for all m, ℓ, and a.e. $\omega \in \mathbb{R}$. Again, in view of Theorem 1.3, we conclude $\Psi_A = 0$ as an element of $\mathcal{E}^T_{\mathcal{H}_A}$. □

1.3.5 C^1-blow-up on the Cauchy horizon

In this section, we shall revisit and discuss the seminal work [14] of Chandrasekhar and Hartle. The Fourier representation of the scattered data on the Cauchy horizon in Theorem 1.3 serves as a good framework to provide a completely rigorous framework for the C^1-blow-up at the Cauchy horizon studied in [14].

Theorem 1.5 (C^1-blow-up on the Cauchy horizon [14])**.** *There exist smooth, compactly supported and purely incoming data* Ψ_A *on the event horizon* $\mathcal{H}_A$ *for which the Cauchy evolution of* (1.1.1) *fails to be* C^1 *at the Cauchy horizon* $\mathcal{CH}$*. More precisely, the solution* ψ *arising from* Ψ_A *fails to be* C^1 *at every point on the Cauchy horizon* $\mathcal{CH}_A \cup \mathcal{B}_+$*. Moreover, the incoming radiation can be chosen to be only supported on any angular parameter* ℓ_0 *which satisfies* $\ell_0(\ell_0+1) \neq r_+^2(r_+ - 3r_-)$*.*

Proof. Let ℓ_0 be fixed and satisfy $\ell_0(\ell_0+1) \neq r_+^2(r_+ - 3r_-)$. Define purely incoming smooth data $\Psi_A(v, \theta, \varphi) = f(v) Y_{\ell_0 0}(\theta, \varphi)$ on $\mathcal{H}_A$, where $f(v)$ is smooth and compactly supported. Moreover, assume that the entire function $\hat{f}$ satisfies $\hat{f}(i\kappa_+) \neq 0$. In view of the representation formula (1.3.27) from Theorem 1.3, the degenerate derivative of its Cauchy evolution Φ_B on the Cauchy horizon $\mathcal{CH}_B$ reads

$$\partial_v \Phi_B(v, \theta, \varphi) = \frac{r_+}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} i\omega \mathfrak{T}(\omega, \ell_0) \hat{f}(\omega) e^{i\omega v} \mathrm{d}\omega Y_{\ell_0 0}(\theta, \varphi). \tag{1.3.32}$$

Since $\mathfrak{T}(\omega, \ell)$ has a simple pole at $\omega = i\kappa_+$ (cf. Proposition 1.8.2 in the appendix), we pick up the residue at $i\kappa_+$ when we deform the contour of integration in (1.3.32) from the real line to the line $\mathrm{Im}(\omega) = \kappa_+ + \delta$ for some $\kappa_+ > \delta > 0$. Here we use that the compact support of $f(v)$ implies the bound $|\hat{f}(\omega)| \leq e^{|\operatorname{Im}(\omega)| \sup|\operatorname{supp}(f)|} \hat{f}(\operatorname{Re}(\omega))$ and that, in addition, by Proposition 1.8.2, the transmission coefficient $\mathfrak{T}$ remains bounded as $|\operatorname{Re}(\omega)| \to \infty$. This ensures that the deformation of the integration contour is valid. Hence,

$$
\begin{aligned}
\partial_v \Phi_B(v, \theta, \varphi) = & \frac{i r_+}{\sqrt{2\pi} r_-} 2\pi i (i\kappa_+) \hat{f}(i\kappa_+) e^{-\kappa_+ v} Y_{\ell_0 0}(\theta, \varphi) \operatorname{Res}(\mathfrak{T}(\omega, \ell_0), i\kappa_+) \\
& + i \frac{r_+ e^{-(\kappa_+ + \delta) v}}{\sqrt{2\pi} r_-} \int_{\mathbb{R}} \Big[(\omega_R + i(\kappa_+ + \delta)) \mathfrak{T}(\omega_R + i(\kappa_+ + \delta)) \\
& \hat{f}(\omega_R + i(\kappa_+ + \delta)) e^{i\omega_R v} Y_{\ell_0 0}(\theta, \varphi) \Big] \mathrm{d}\omega_R \\
& = C e^{-\kappa_+ v} Y_{\ell_0 0}(\theta, \varphi) + o\left(e^{-(\kappa_+ + \delta) v}\right)
\end{aligned}
\tag{1.3.33}
$$

as $v \to \infty$, where

$$
C = -i\kappa_+ \frac{r_+}{r_-} \sqrt{2\pi} \hat{f}(i\kappa_+) \operatorname{Res}(\mathfrak{T}(\omega, \ell_0), \omega = i\kappa_+) \neq 0 \tag{1.3.34}
$$

by construction. Thus, Φ_B is not in C^1 at the future bifurcation sphere as the non-degenerate derivative diverges as $v \to \infty$:

$$
\frac{\partial}{\partial V_{\mathcal{CH}}} \Phi_B = e^{-\kappa_- v} \partial_v \Psi_B(v, \theta, \varphi) = C e^{-(\kappa_+ + \kappa_-) v} (1 + o(1)), \tag{1.3.35}
$$

where we recall that $\kappa_- < -\kappa_+ < 0$. Finally, propagation of regularity gives that the solution is not in C^1 at each point on the Cauchy horizon $\mathcal{CH}_A$. More precisely, expressing (1.1.1) is (u, v) coordinates gives

$$
\partial_u \partial_v \psi = \frac{-\Delta}{2r^3} (\partial_v \psi + \partial_u \psi) + \frac{\Delta}{4r^4} \ell_0 (\ell_0 + 1) \psi, \tag{1.3.36}
$$

where Δ is as in (1.2.5) and where we have used that $\Delta_{\mathbb{S}^2} \psi = -\ell_0(\ell_0 + 1)\psi$. Now, note that $|\psi|, |\partial_u \psi|$ and $|\partial_v \psi|$ are uniformly bounded in the interior by a higher order norm of Ψ_A. This follows from [51], commuting with T and angular momentum operators as well as elliptic estimates. Finally, integrating (1.3.36) in u, using the estimate $|\Delta| \lesssim e^{\kappa_-(u+v)}$ for $r_* \geq 0$ (see (1.8.7)) and using the non-degenerate coordinate $V_{\mathcal{CH}}$ gives the C^1 blow-up also everywhere on $\mathcal{CH}_A$. $\square$

1.3.6 Breakdown of T energy scattering for cosmological constants $\Lambda \neq 0$

Interestingly, the analogous result to Theorem 1.1 on the interior of a subextremal (anti-) de Sitter–Reissner–Nordström black hole does not hold for almost all cosmological constants Λ. In the presence of a cosmological constant it is also natural to consider the Klein–Gordon equation with conformal mass $\mu = \frac{3}{2}\Lambda$. We will consider in fact a general mass term of the form $\mu = \nu\Lambda$, where $\nu \in \mathbb{R}$. Note that $\nu = \frac{3}{2}$ corresponds to the conformal invariant Klein–Gordon equation. To be more precise, we prove that for generic subextremal black hole parameters (M, Q, Λ), there exists a normalized (in $\mathcal{E}^T_{\mathcal{H}}$-norm) sequence of Schwartz initial data on the event horizon for which the $\mathcal{E}^T_{\mathcal{CH}}$-norm of the evolution restricted to the Cauchy horizon blows up.

We define a black hole parameter triple (M, Q, Λ) to be *subextremal* if

$$(M, Q, \Lambda) \in \mathcal{P}_{\mathrm{se}} := \mathcal{P}^{\Lambda=0}_{\mathrm{se}} \cup \mathcal{P}^{\Lambda>0}_{\mathrm{se}} \cup \mathcal{P}^{\Lambda<0}_{\mathrm{se}}, \tag{1.3.37}$$

where

$$\begin{aligned}
\mathcal{P}^{\Lambda=0}_{\mathrm{se}} :=& \{(M,Q,\Lambda) \in \mathbb{R}_+ \times \mathbb{R} \times \{0\} \colon \\
& \quad \Delta(r) := r^2 - 2Mr + Q^2 \text{ has two positive simple roots satisfying } 0 < r_- < r_+ .\},
\end{aligned} \tag{1.3.38}$$

$$\begin{aligned}
\mathcal{P}^{\Lambda>0}_{\mathrm{se}} :=& \{(M,Q,\Lambda) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \colon \\
& \quad \Delta(r) := r^2 - 2Mr - \frac{1}{3}\Lambda r^4 + Q^2 \text{ has three positive simple roots satisfying} \\
& \qquad 0 < r_- < r_+ < r_c\},
\end{aligned} \tag{1.3.39}$$

$$\begin{aligned}
\mathcal{P}^{\Lambda<0}_{\mathrm{se}} :=& \{(M,Q,\Lambda) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_- \colon \\
& \quad \Delta(r) := r^2 - 2Mr - \frac{1}{3}\Lambda r^4 + Q^2 \text{ has two positive roots satisfying } 0 < r_- < r_+\}.
\end{aligned} \tag{1.3.40}$$

Theorem 1.6. *Let $\nu \in \mathbb{R}$ be a fixed Klein–Gordon mass parameter. (In particular, we may choose $\nu = \frac{3}{2}$ to cover the conformal invariant case or $\nu = 0$ for the wave equation* (1.1.1)*.) Consider the interior of a subextremal (anti-) de Sitter–Reissner–Nordström black hole with generic parameters $(M, Q, \Lambda) \in \mathcal{P}_{\mathrm{se}} \setminus D(\nu)$. (Here, $D(\nu) \subset \mathcal{P}_{\mathrm{se}}$ is a set with measure zero defined in Proposition 1.6.1 (see Section 1.6). Moreover $D(\nu)$ satisfies $\mathcal{P}^{\Lambda=0}_{\mathrm{se}} \subset D(\nu)$ and $U \cap D(\nu) = \mathcal{P}^{\Lambda=0}_{\mathrm{se}}$ for some open set $U \subset \mathcal{P}_{\mathrm{se}}$.)*

Then, there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of purely ingoing and compactly supported data *on $\mathcal{H}_A$ with*

$$\|\Psi_n\|_{\mathcal{E}^T_{\mathcal{H}}} = 1 \text{ for all } n \tag{1.3.41}$$

such that the solution ψ_n to the Klein–Gordon equation with mass $\mu = \nu\Lambda$

$$\Box_{g_{M,Q,\Lambda}}\psi - \mu\psi = 0 \tag{1.3.42}$$

arising from Ψ_n has unbounded T energy at the Cauchy horizon

$$\|\psi_n \restriction_{\mathcal{CH}} \|_{\mathcal{E}^T_{\mathcal{CH}}} \to \infty \text{ as } n \to \infty. \tag{1.3.43}$$

Proof. See Section 1.6. □

Remark 1.3.5. *Note that from Theorem 1.6 it also follows that for fixed $0 < |Q| < M$, the T energy scattering breaks down (in sense of Theorem 1.6) for all cosmological constants $0 < |\Lambda| < \epsilon$, where $\epsilon = \epsilon(M, Q) > 0$ is small enough.*

1.3.7 Breakdown of T energy scattering for the Klein–Gordon equation

Finally, we will also prove that the T energy scattering theory does not hold for the Klein–Gordon equation for a generic set of masses μ, even in the case of vanishing cosmological constant $\Lambda = 0$.

Theorem 1.7. *Consider the interior of a subextremal Reissner–Nordström black hole. There exists a discrete set $\tilde{D}(M,Q) \subset \mathbb{R}$ with $0 \in \tilde{D}$ such that the following holds true. For any $\mu \in \mathbb{R} \setminus \tilde{D}$ there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of purely ingoing and compactly supported data on $\mathcal{H}_A$ with*

$$\|\Psi_n\|_{\mathcal{E}^T_{\mathcal{H}}} = 1 \text{ for all } n \tag{1.3.44}$$

such that the solution ψ_n to the Klein–Gordon equation with mass μ

$$\Box_{g_{M,Q,\Lambda}}\psi - \mu\psi = 0 \tag{1.3.45}$$

arising from Ψ_n has unbounded T energy at the Cauchy horizon

$$\|\psi_n \restriction_{\mathcal{CH}} \|_{\mathcal{E}^T_{\mathcal{CH}}} \to \infty \text{ as } n \to \infty. \tag{1.3.46}$$

Proof. See Section 1.7. □

The above Theorem 1.6 and Theorem 1.7 show that the existence of a T energy scattering theory for the wave equation (1.1.1) on the interior of Reissner–Nordström is in retrospect a surprising property. Implications of the non-existence of a T energy scattering map and in particular, the unboundedness of the scattering map in the cosmological setting $\Lambda \neq 0$, are yet to be understood.

1.4 Proof of Theorem 1.2: Uniform boundedness of the transmission and reflection coefficients

This section is devoted to the proof of Theorem 1.2. We will analyze solutions to the o.d.e. (recall from (1.2.34))

$$\Delta \frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta \frac{\mathrm{d}}{\mathrm{d}r} R\right) - \Delta \ell(\ell+1) R + r^4 \omega^2 R = 0.$$

This o.d.e. can be written equivalently (recall from (1.2.36)) as

$$u'' + (\omega^2 - V_\ell) u = 0,$$

in the r_* variable, where $u = rR$.

For the convenience of the reader we recall the statement of Theorem 1.2.

Theorem 1.2. *The reflection and transmission coefficients $\mathfrak{R}(\omega,\ell)$ and $\mathfrak{T}(\omega,\ell)$ are uniformly bounded, i.e. they satisfy*

$$\sup_{\omega\in\mathbb{R},\ell\in\mathbb{N}_0} \left(|\mathfrak{R}(\omega,\ell)| + |\mathfrak{T}(\omega,\ell)|\right) \lesssim 1. \tag{1.3.7}$$

The proof of Theorem 1.2 will involve different arguments for different regimes of parameters. Also, note that in view of (1.2.56) and (1.2.57) it is enough to assume $\omega \neq 0$.

The first regime for bounded frequencies ($|\omega| \leq \omega_0$, ℓ arbitrary) requires the most work. One of its main difficulties is to obtain estimates which are uniform in the limit $\ell \to \infty$. We shall use that the o.d.e. (1.2.36) with $\omega = 0$, which reads

$$u'' - V_\ell u = 0, \tag{1.4.1}$$

can be solved explicitly in terms of Legendre polynomials and Legendre functions of second kind. The specific algebraic structure of the Legendre o.d.e. leads to the feature that solutions which are bounded at $r_* = -\infty$ are also bounded at $r_* = +\infty$. For generic perturbations of the potential this property fails to hold. Nevertheless, for perturbations of the form as in (1.2.36) for $\omega \neq 0$ and $|\omega| \leq |\omega_0|$, this behavior survives and most importantly, can be quantified. To prove this we will essentially divide the real line $\mathbb{R} \ni r_*$ into three regions.

The first region will be near the event horizon ($r_* = -\infty$), where we will consider the potential V_ℓ as a perturbation. The second region will be the intermediate region, where we will consider the term involving ω as a perturbation. Finally, in the third region near the Cauchy horizon ($r_* = +\infty$), we consider the potential V_ℓ as a perturbation again. This eventually allows us to prove the uniform boundedness of the reflection and transmission coefficients $\mathfrak{R}$ and $\mathfrak{T}$ in the bounded frequency regime $|\omega| < \omega_0$.

The second regime will be bounded angular momenta and ω-frequencies bounded from below ($|\omega| \geq \omega_0, \ell \leq \ell_0$). For this parameter range we will consider V_ℓ as a perturbation of the o.d.e. since V_ℓ might only grow with ℓ, which is, however, bounded in that range. Again, this allows us to show uniform boundedness for the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$.

The third regime will be angular momenta and frequencies both bounded from below ($|\omega| \geq \omega_0$, $\ell \geq \ell_0$). To prove boundedness of reflection and transmission coefficients $\mathfrak{R}$ and $\mathfrak{T}$, we will consider $\frac{1}{\ell}$ as a small parameter to perform a WKB-approximation.

1.4.1 Low frequencies ($|\omega| \leq \omega_0$)

We first analyze the o.d.e. for the special case of vanishing frequency. Then, we will summarize properties of special functions, which we will need to finally prove the boundedness of reflection and transmission coefficients in the low frequency regime. Let

$$0 < \omega_0 \leq \frac{1}{2} \tag{1.4.2}$$

be a fixed constant.

1.4.1.1 Explicit solution for vanishing frequency ($\omega = 0$)

For $\omega = 0$ we can explicitly solve the o.d.e. with special functions. In that case the o.d.e. reads

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta \frac{\mathrm{d}R}{\mathrm{d}r}\right) - \ell(\ell+1)R = 0. \tag{1.4.3}$$

We define the coordinate $x(r)$ as

$$x(r) := -\frac{2r}{r_+ - r_-} + \frac{r_+ + r_-}{r_+ - r_-} \tag{1.4.4}$$

or equivalently,

$$r(x) = -\frac{r_+ - r_-}{2}x + \frac{r_+ + r_-}{2}. \tag{1.4.5}$$

Then, we can write

$$\Delta(x) = \left(\frac{r_+ - r_-}{2}\right)^2 (x+1)(x-1) = \left(\frac{r_+ - r_-}{2}\right)^2 (x^2 - 1). \tag{1.4.6}$$

Hence, Eq. (1.4.3) reduces to the Legendre o.d.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}R}{\mathrm{d}x}\right) + \ell(\ell+1)R = 0. \tag{1.4.7}$$

We will denote by $P_\ell(x)$ and $Q_\ell(x)$ the two independent solutions, the Legendre polynomials and the Legendre functions of second kind, respectively [119, 39]. We will prove later in Proposition 1.4.2 that $\tilde{u}_1$ and $\tilde{u}_2$ from Definition 1.2.3 satisfy

$$\tilde{u}_1(r_*) = w_1(r_*) := (-1)^\ell \frac{r(r_*)}{r_+} P_\ell(x(r_*)), \tag{1.4.8}$$

$$\tilde{u}_2(r_*) = w_2(r_*) := (-1)^\ell \frac{r(r_*)}{k_+ r_+} Q_\ell(x(r_*)). \tag{1.4.9}$$

These are a fundamental pair of solutions for the o.d.e. in the case $\omega = 0$. We will perturb these explicit solutions for the regime of low frequencies ($|\omega| \leq \omega_0$). To do so, we will need properties about special functions which will be considered first.

In view of the fact that ω_0 is fixed, constants appearing in $\lesssim$ and $\gtrsim$ may also depend on ω_0. Before we begin, we shall summarize the special functions we will use and list their relevant properties in the case $|\omega| \leq \omega_0$.

1.4.1.2 Special functions

Good references for the following discussion are [1, 119, 39]. First, we shall recall the definition of the Gamma and Digamma function.

Definition 1.4.1. *For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ we denote the Gamma function with $\Gamma(z)$*

and will also make use of the Digamma function $\digamma(z)$ defined as

$$\digamma(z) := \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-zx}}{1-e^{-x}} \right) \mathrm{d}x. \tag{1.4.10}$$

Note that

$$\digamma(z+1) - \digamma(z) = \frac{1}{z} \tag{1.4.11}$$

and

$$\digamma(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma = \log(n) + O(n^{-1}), \tag{1.4.12}$$

where γ is the Euler–Mascheroni constant.

As we mentioned above, we shall use the Legendre polynomials and the Legendre functions of second kind. We will express them in terms of the hypergeometric function $\mathbf{F}(a, b; c; x)$ for $x \in (-1, 1)$, $a, b, c \in \mathbb{R}$ as defined in [119, Equation (9.3)].

Definition 1.4.2 (Legendre functions of first and second kind)**.** *We use the standard conventions which are used in [119, 39].*

For $x \in (-1, 1)$, we define the associated Legendre polynomials by

$$P_\ell^m(x) = \left(\frac{1+x}{1-x} \right)^{\frac{m}{2}} \mathbf{F}\left(\ell+1, -\ell; 1-m; \frac{1-x}{2} \right) \tag{1.4.13}$$

and the associated Legendre functions of second kind by

$$Q_\ell^m(x) = -\frac{1}{2}\pi \sin\left(\frac{1}{2}\pi(\ell+m) \right) w_1(\ell, x) + \frac{1}{2}\pi \cos\left(\frac{1}{2}(\ell+m)\pi \right) w_2(\ell, x). \tag{1.4.14}$$

Here,

$$w_1(\ell, x) = \frac{2^m \Gamma(\frac{\ell+m+1}{2})}{\Gamma(1+\frac{\ell}{2})} (1-x^2)^{-\frac{m}{2}} \mathbf{F}\left(-\frac{\ell+m}{2}, \frac{1+\ell-m}{2}; \frac{1}{2}; x^2 \right), \tag{1.4.15}$$

$$w_2(\ell, x) = \frac{2^m \Gamma(1+\frac{\ell+m}{2})}{\Gamma(\frac{\ell-m+1}{2})} x(1-x^2)^{-\frac{m}{2}} \mathbf{F}\left(\frac{1-\ell-m}{2}, 1+\frac{\ell-m}{2}; \frac{3}{2}; x^2 \right). \tag{1.4.16}$$

We shall also use the convention $P_\ell = P_\ell^0$ and $Q_\ell^m = Q_\ell^0$. Also, recall the symmetry

$$P_\ell(x) = (-1)^\ell P_\ell(-x), \tag{1.4.17}$$

$$Q_\ell(x) = (-1)^{\ell+1} Q_\ell(-x). \tag{1.4.18}$$

In the asymptotic expansion in the parameter ℓ for the Legendre polynomials and functions we will make use of Bessel functions which we define in the following.

Definition 1.4.3 (Bessel functions of first and second kind)**.** *Recall the Bessel functions of first kind*

$$J_0(x) := \sum_{k=0}^{\infty} \frac{x^{2k}}{(-4)^k k!^2}, \tag{1.4.19}$$

$$J_1(x) := \frac{x}{2} \sum_{k=0}^{\infty} \frac{x^{2k}}{(-4)^k k!(k+1)!}, \tag{1.4.20}$$

and the Bessel functions of second kind

$$Y_0(x) := \frac{2}{\pi} J_0(x) \left(\log\left(\frac{x}{2}\right) + \gamma \right) - \frac{2}{\pi} \sum_{k=1}^{\infty} H_k \frac{x^{2k}}{(-4)^k (k!)^2}, \tag{1.4.21}$$

$$\begin{aligned} Y_1(x) := &-\frac{1}{2\pi x} + \frac{2}{\pi} \log\left(\frac{x}{2}\right) J_1(x) \\ &- \frac{x}{2\pi} \sum_{k=0}^{\infty} (\digamma(k+1) + \digamma(k+2)) \frac{x^{2k}}{(-4)^k k!(k+1)!}, \end{aligned} \tag{1.4.22}$$

where $H_k = \sum_{n=1}^k n^{-1}$ is the k-the harmonic number. We have the asymptotic expansions

$$J_0(x) = 1 + O(x^2), \tag{1.4.23}$$

$$J_1(x) = \frac{x}{2} + O(x^3), \tag{1.4.24}$$

$$Y_0(x) = \frac{2}{\pi} \log\left(\frac{x}{2}\right) + O(1), \tag{1.4.25}$$

$$Y_1(x) = -\frac{1}{2\pi x} + o(1) \text{ as } x \to 0. \tag{1.4.26}$$

Note that bounds deduced from (1.4.23) – (1.4.26) *hold uniformly on any interval* $(0, a]$ *of finite length. We shall also use the bounds*

$$|J_0(x)| \leq 1, \; |Y_0(x)| \lesssim 1 + |\log(x)| \tag{1.4.27}$$

for $0 < x \leq 1$ *and*

$$|J_0(x)| \lesssim \frac{1}{\sqrt{x}}, \; |Y_0(x)| \lesssim \frac{1}{\sqrt{x}} \tag{1.4.28}$$

for $x \geq 1$ *[1, p. 360, p. 364].*

In the proof we will also use the following asymptotic formulae for P_ℓ and Q_ℓ for large ℓ in terms of Bessel functions.

Lemma 1.4.1. *[39, §14.15(iii)] We have*

$$P_\ell(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} \left(J_0\left(\frac{\theta(2\ell+1)}{2}\right) + e_{1,\ell}(\theta)\right), \tag{1.4.29}$$

$$Q_\ell(\cos\theta) = -\frac{\pi}{2}\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} \left(Y_0\left(\frac{\theta(2\ell+1)}{2}\right) + e_{2,\ell}(\theta)\right), \tag{1.4.30}$$

$$Q_\ell^1(\cos\theta) = -\frac{\pi}{2\ell}\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} \left(Y_1\left(\frac{\theta(2\ell+1)}{2}\right) + e_{3,\ell}(\theta)\right), \tag{1.4.31}$$

where the error terms can be estimated by

$$|e_{1,\ell}(\theta)|, \, |e_{2,\ell}(\theta)| \lesssim \frac{1}{1+\ell}\left[\left|J_0\left(\frac{\theta(2\ell+1)}{2}\right)\right| + \left|Y_0\left(\frac{\theta(2\ell+1)}{2}\right)\right|\right], \tag{1.4.32}$$

$$|e_{3,\ell}(\theta)| \lesssim \frac{1}{1+\ell}\left[\left|J_1\left(\frac{\theta(2\ell+1)}{2}\right)\right| + \left|Y_1\left(\frac{\theta(2\ell+1)}{2}\right)\right|\right] \tag{1.4.33}$$

for $\theta \in (0, \pi - \delta)$ *and for any fixed* $\delta > 0$*. In particular, this holds uniformly as* $\theta \to 0$*.*

We shall use the following asymptotic formulae for the Legendre functions at the singular endpoints.

Lemma 1.4.2. *[39, §14.8] For* $0 < x < 1$ *we have*

$$P_\ell(x) = 1 + f_1(x), \tag{1.4.34}$$

$$Q_\ell(x) = \frac{1}{2}(\log(2) - \log(1-x)) - \gamma - \digamma(\ell+1) + f_1(x), \tag{1.4.35}$$

where $|f_1(x)| \lesssim_\ell (1-x)$*. Moreover, analogous results hold true for* $-1 < x < 0$ *due to symmetry.*

Now, we will estimate the derivatives of the Legendre polynomials and Legendre functions of second kind.

Lemma 1.4.3. *For $x \in (-1,1)$ we have*

$$\left|\frac{\mathrm{d}P_\ell}{\mathrm{d}x}\right| \leq \ell^2. \tag{1.4.36}$$

For $x_{\alpha,\ell} := 1 - \frac{\alpha}{1+\ell^2}$ with $0 < \alpha < 1$ and $\ell \in \mathbb{N}$ we have

$$(1-(\pm x_{\alpha,\ell})^2)\left|\frac{\mathrm{d}Q_\ell}{\mathrm{d}x}(\pm x_{\alpha,\ell})\right| \lesssim 1. \tag{1.4.37}$$

Proof. Inequality (1.4.36) is known as Markov's inequality and is proven in [11, Theorem 5.1.8]. We only have to prove (1.4.37) for $x = +x_{\alpha,\ell}$ due to symmetry. From the recursion relation [39, §14.10] we have

$$\begin{aligned}(\ell+1)^{-1}(1-x_{\alpha,\ell}^2)\frac{\mathrm{d}Q_\ell}{\mathrm{d}x}(x_{\alpha,\ell}) &= x_{\alpha,\ell}Q_\ell(x_{\alpha,\ell}) - Q_{\ell+1}(x_{\alpha,\ell})\\ &= (x_{\alpha,\ell}-1)Q_\ell(x_{\alpha,\ell}) + (Q_\ell(x_{\alpha,\ell}) - Q_{\ell+1}(x_{\alpha,\ell})).\end{aligned} \tag{1.4.38}$$

We will consider both summands separately.

Part 1: Summand $(x_{\alpha,\ell}-1)Q_\ell(x_{\alpha,\ell})$

First, consider $1 - x_{\alpha,\ell} = \frac{\alpha}{1+\ell^2}$, where we implicitly define $\cos(\theta_{\alpha,\ell}) = x_{\alpha,\ell}$. Note that we have

$$\begin{aligned}\theta_{\alpha,\ell}(x) = \sqrt{2(1-x_{\alpha,\ell})} + O((1-x_{\alpha,\ell})^{\frac{3}{2}}) &= \sqrt{\frac{2\alpha}{1+\ell^2}} + O\left(\left(\frac{\alpha}{1+\ell^2}\right)^{\frac{3}{2}}\right)\\ &= \sqrt{\frac{2\alpha}{1+\ell^2}}\left(1 + O\left(\frac{\alpha}{1+\ell^2}\right)\right).\end{aligned} \tag{1.4.39}$$

In particular, we have $\theta_{\alpha,\ell}\ell \lesssim 1$. This gives

$$-Q_\ell(x_{\alpha,\ell}) = -Q_\ell(\cos\theta_{\alpha,\ell}) = \frac{\pi}{2}\left(\frac{\theta_{\alpha,\ell}}{\sin\theta_{\alpha,\ell}}\right)^{\frac{1}{2}}\left(Y_0\left(\frac{\theta_{\alpha,\ell}(2\ell+1)}{2}\right) + e_{2,\ell}(\theta_{\alpha,\ell})\right). \tag{1.4.40}$$

Again, we will look at the two terms independently. First, note that

$$
\begin{aligned}
&\frac{\pi}{2}\left(\frac{\theta_{\alpha,\ell}}{\sin\theta_{\alpha,\ell}}\right)^{\frac{1}{2}}\left(Y_0\left(\theta_{\alpha,\ell}\left(\ell+\frac{1}{2}\right)\right)\right)\\
&\quad=\frac{\pi}{2}\left(\frac{\theta_{\alpha,\ell}}{\sin\theta_{\alpha,\ell}}\right)^{\frac{1}{2}}\left(\frac{2}{\pi}\log\left(\frac{\theta_{\alpha,\ell}(2\ell+1)}{4}\right)+O(1)\right)\\
&\quad=\left(1+O(\theta_{\alpha,\ell}^2)\right)\left(\log(\theta_{\alpha,\ell})+\log\left(\ell+\frac{1}{2}\right)+O(1)\right)\\
&\quad=\left(1+O\left(\frac{\alpha}{1+\ell^2}\right)\right)\left(\frac{1}{2}\log\left(\frac{\alpha}{1+\ell^2}\right)+\log\left(\ell+\frac{1}{2}\right)+O(1)\right)\\
&\quad=\left(1+O\left(\frac{\alpha}{1+\ell^2}\right)\right)\left(\frac{1}{2}\log(\alpha)+\frac{1}{2}\log\left(1+\frac{\ell-\frac{3}{4}}{\ell^2+1}\right)+O(1)\right)\\
&\quad=\frac{1}{2}\log(\alpha)+O(1).
\end{aligned}
\tag{1.4.41}
$$

In order to estimate $e_{2,\ell}(\theta_{\alpha,\ell})$ we shall recall inequality (1.4.32). It works analogously to the previous estimate up to a good term of $\frac{1}{1+\ell}$. In particular, this shows

$$
|Q_\ell(x_{\alpha,\ell})|\lesssim|\log(\alpha)|+1
\tag{1.4.42}
$$

and

$$
|(x_{\alpha,\ell}-1)Q_\ell(x_{\alpha,\ell})|\lesssim\frac{\alpha}{1+\ell^2}(|\log(\alpha)|+1)\lesssim\frac{1}{1+\ell^2}.
\tag{1.4.43}
$$

Part 2: Summand $(Q_\ell(x_{\alpha,\ell})-Q_{\ell+1}(x_{\alpha,\ell}))$

Using the recursion relation for the difference of two Legendre function [39, §14.10], we have

$$
(\ell+1)(Q_\ell(x_{\alpha,\ell})-Q_{\ell+1}(x_{\alpha,\ell})=-(1-x_{\alpha,\ell}^2)^{\frac{1}{2}}Q_\ell^1(x_{\alpha,\ell})+(1-x_{\alpha,\ell})Q_\ell(x_{\alpha,\ell}).
\tag{1.4.44}
$$

We estimate the term $(1-x_{\alpha,\ell})Q_\ell(x_{\alpha,\ell})$ by what we have done above as

$$
|(1-x_{\alpha,\ell})Q_\ell(x_{\alpha,\ell})|\lesssim\frac{\alpha}{1+\ell^2}(|\log(\alpha)|+1)\lesssim 1.
\tag{1.4.45}
$$

For the term $-(1-x_{\alpha,\ell}^2)^{\frac{1}{2}}Q_\ell^1(x_{\alpha,\ell})$ we use (1.4.31) to get

$$\begin{aligned}
&\left|-(1-x_{\alpha,\ell}^2)^{\frac{1}{2}}Q_\ell^1(x_{\alpha,\ell})\right| \\
&\lesssim \sqrt{\frac{\alpha}{\ell^2+1}}\frac{1}{1+\ell}\left(1+O\left(\frac{\alpha}{1+\ell^2}\right)\right)\left(Y_1\left(\left(\ell+\frac{1}{2}\right)\theta_{\alpha,\ell}\right)+e_{2,\ell}(\theta_{\alpha,\ell})\right). \qquad (1.4.46)
\end{aligned}$$

As before, we shall start estimating the first term using (1.4.26) and (1.4.39) to obtain

$$\begin{aligned}
&\sqrt{\frac{\alpha}{\ell^2+1}}\frac{1}{1+\ell}\left(1+O\left(\frac{\alpha}{1+\ell^2}\right)\right)Y_1\left(\left(\ell+\frac{1}{2}\right)\theta_{\alpha,\ell}\right) \\
&= \sqrt{\frac{\alpha}{\ell^2+1}}\frac{1}{1+\ell}\left(1+O\left(\frac{\alpha}{1+\ell^2}\right)\right)\left(-\frac{1}{\pi(2\ell+1)\theta_{\alpha,\ell}}+O(1)\right) \\
&\lesssim \sqrt{\frac{\alpha}{\ell^2+1}}\frac{1}{1+\ell}\left(\frac{1}{\sqrt{\alpha}}+1\right)\lesssim 1. \qquad (1.4.47)
\end{aligned}$$

We estimate the second term using (1.4.33), (1.4.24), (1.4.26), and (1.4.39) to obtain

$$\begin{aligned}
&\left|\sqrt{\frac{\alpha}{\ell^2+1}}\frac{1}{1+\ell}\left(1+O\left(\frac{\alpha}{1+\ell^2}\right)\right)e_{2,\ell}(\theta_{\alpha,\ell})\right| \\
&\lesssim \sqrt{\frac{\alpha}{\ell^2+1}}\frac{1}{1+\ell^2}\left(\frac{1}{\sqrt{\alpha}}+1\right)\lesssim 1. \qquad (1.4.48)
\end{aligned}$$

We have estimated that $|Q_\ell(x_{\alpha,\ell})-Q_{\ell+1}(x_{\alpha,\ell})|\lesssim \frac{1}{1+\ell}$ which proves the claim in view of (1.4.38). □

Finally, we prove asymptotics for the derivatives of the Legendre of functions of second kind near the singular points.

Lemma 1.4.4. *For $0<x<1$ and $x\to 1$ we have*

$$(1-x^2)\frac{\mathrm{d}Q_\ell}{\mathrm{d}x} = 1+O_\ell((1-x)\log(1-x)). \qquad (1.4.49)$$

By symmetry this also yields for $-1<x<0$ and $x\to -1$

$$(1-x^2)\frac{\mathrm{d}Q_\ell}{\mathrm{d}x} = (-1)^\ell+O_\ell((1+x)\log(1+x)). \qquad (1.4.50)$$

Proof. From the recursion relation [39, §14.10] and (1.4.35) we obtain

$$
\begin{aligned}
&(1-x^2)\frac{\mathrm{d}Q_\ell}{\mathrm{d}x} = (\ell+1)(xQ_\ell - Q_{\ell+1}) \\
&= (\ell+1)(x-1)Q_\ell + (\ell+1)(Q_\ell - Q_{\ell+1}) \\
&= (\ell+1)(Q_\ell - Q_{\ell+1}) + O_\ell((1-x)\log(1-x)) \\
&= (\ell+1)(\digamma(\ell+2) - \digamma(\ell+1)) + O_\ell((1-x)\log(1-x)) \\
&= 1 + O_\ell((1-x)\log(1-x)).
\end{aligned}
\tag{1.4.51}
$$

$\square$

Having reviewed the required facts about special functions, we shall now proceed to prove the uniform boundedness of the reflection and transmission coefficients.

1.4.1.3 Boundedness of the reflection and transmission coefficients

As mentioned before, we will consider three different regions: a region near the event horizon, an intermediate region, and a region near the Cauchy horizon. In r_* coordinates we separate these regions at

$$R_1^*(\omega,\ell) := \frac{1}{2\kappa_+}\log\left(\frac{\omega^2}{1+\ell^2}\right) \tag{1.4.52}$$

and

$$R_2^*(\omega,\ell) := \frac{1}{2\kappa_-}\log\left(\frac{\omega^2}{1+\ell^2}\right) \tag{1.4.53}$$

for $0<|\omega|<\omega_0$ and $\ell\in\mathbb{N}_0$. Note that $-\infty < R_1^*(\omega,\ell) < 0 < R_2^*(\omega,\ell) < \infty$.

Region near the event horizon.

Proposition 1.4.1. *Let $0<|\omega|<\omega_0$ and $\ell\in\mathbb{N}_0$. Then, we have*

$$\|u_1'\|_{L^\infty(-\infty,R_1^*)} \lesssim |\omega|, \tag{1.4.54}$$

$$\|u_1\|_{L^\infty(-\infty,R_1^*)} \lesssim 1. \tag{1.4.55}$$

Proof. Recall the defining Volterra integral equation for u_1 from Definition 1.2.3

$$u_1(r_*) = e^{i\omega r_*} + \int_{-\infty}^{r_*} \frac{\sin(\omega(r_*-y))}{\omega} V(y)u_1(y)\mathrm{d}y. \tag{1.4.56}$$

with integral kernel

$$K(r_*, y) := \frac{\sin(\omega(r_* - y))}{\omega} V(y). \tag{1.4.57}$$

From Lemma 1.8.3 in the appendix, we obtain for $r_* \leq R_1^*$

$$|V(r_*)| \lesssim e^{2k_+ r_*}(1 + \ell^2) \tag{1.4.58}$$

and in particular,

$$|V(R_1^*)| \lesssim e^{2k_+ R_1^*}(1 + \ell^2) = \omega^2. \tag{1.4.59}$$

This implies for $r_* \leq R_1^*$

$$|K(r_*, y)| \leq \frac{1}{|\omega|}|V(y)| \lesssim \frac{1}{|\omega|}(1 + \ell^2)e^{2k_+ y} \tag{1.4.60}$$

and thus,

$$\int_{-\infty}^{R_1^*} \sup_{y < r_* < R_1^*} |K(r_*, y)| \mathrm{d}y \lesssim \frac{\ell^2 + 1}{|\omega|} e^{2k_+ R_1^*} \lesssim 1. \tag{1.4.61}$$

The claim follows now from Proposition 1.2.3. □

Now, we would like to consider ω as a small parameter and perturb the explicit solutions for the $\omega = 0$ case in order to propagate the behavior of the solution through the intermediate region, where V_ℓ is large compared to ω. In particular, V_ℓ can be arbitrarily large since ℓ is not bounded above in the considered parameter regime.

Intermediate region. First, recall the following fundamental pair of solutions which is based on the Legendre functions of first and second kind

$$w_1(r_*) := (-1)^\ell \frac{r(r_*)}{r_+} P_\ell(x(r_*)), \tag{1.4.62}$$

$$w_2(r_*) := (-1)^\ell \frac{r(r_*)}{k_+ r_+} Q_\ell(x(r_*)), \tag{1.4.63}$$

where P_ℓ and Q_ℓ are the Legendre polynomials and Legendre functions of second kind, respectively. Our first claim is that we have constructed this fundamental pair (w_1, w_2) to have unit Wronskian and moreover $\tilde{u}_1 = w_1$ and $\tilde{u}_2 = w_2$ holds true.

Proposition 1.4.2. *We have $w_1 = \tilde{u}_1$ and $w_2 = \tilde{u}_2$ and the Wronskian of u_1 and u_2*

satisfies

$$\mathfrak{W}(w_1, w_2) = \mathfrak{W}(\tilde{u}_1, \tilde{u}_2) = 1. \tag{1.4.64}$$

Similarly, we also have $\tilde{v}_1 = (-1)^\ell \frac{r_+}{r_-} w_1 = (-1)^\ell \frac{r_+}{r_-} \tilde{u}_1$.

Proof. We first prove that $\mathfrak{W}(w_1, w_2) = 1$. Since the Wronskian is independent of r_*, we will compute its value in the limit $r_* \to -\infty$. In this proposition ℓ is fixed and we shall allow implicit constants in $\lesssim$ to depend on ℓ. Clearly,

$$w_1(r_*) \to 1 \text{ as } r_* \to -\infty. \tag{1.4.65}$$

Moreover, we have that for $r_* \leq 0$

$$\left| \frac{\mathrm{d}}{\mathrm{d}r_*} w_1(r_*) \right| \lesssim e^{2k_+ r_*} |P_\ell(x(r_*))| + \left| \frac{\mathrm{d}P_\ell(x)}{\mathrm{d}x}(r_*) \frac{\mathrm{d}x}{\mathrm{d}r_*}(r_*) \right| \lesssim e^{2k_+ r_*}, \tag{1.4.66}$$

where we have used (1.4.36). This, in particular, also shows that w_1 satisfies the same boundary conditions ($w_1 \to 1$, $w_1' \to 0$ as $r_* \to -\infty$) as $\tilde{u}_1$ defined in Definition 1.2.3 and thus, w_1 and $\tilde{u}_1$ have to coincide. Similarly, we can deduce $\tilde{v}_1 = (-1)^\ell \frac{r_+}{r_-} w_1$.

For w_2, we use (1.4.35) to obtain

$$|w_2(r_*) - r_*| \lesssim \left(-\frac{r(r_*)}{k_+ r_+} \left(\frac{1}{2} \log \left(\frac{2}{1 + x(r_*)} \right) - \gamma - \digamma(\ell + 1) \right) - r_* \right) + e^{2k_+ r_*}. \tag{1.4.67}$$

For an intermediate step, we compute $\log(1 + x(r_*))$ from (1.4.4) near $r_* = -\infty$. In particular, for the limit $r_* \to -\infty$, we can assume that $r_* \leq 0$ and thus, $r - r_- \gtrsim r_+ - r_-$. Hence,

$$\begin{aligned} \log(1 + x(r_*)) &= \log \left(1 + \frac{(r_+ - r) + (r_- - r)}{r_+ - r_-} \right) \\ &= \log \left(1 + \frac{f(r_*)}{r_+ - r_-} e^{2k_+ r_*} + \frac{r_- - r}{r_+ - r_-} \right) \\ &= \log \left(\frac{2f(r_*)}{r_+ - r_-} e^{2k_+ r_*} \right) \\ &= 2k_+ r_* + \log(2f(r_*)(r_+ - r_-)^{-1}), \end{aligned} \tag{1.4.68}$$

where f is defined in (1.8.11). Thus, this directly implies

$$|w_2(r_*) - r_*| \lesssim r_* e^{2k_+ r_*} + 1 \lesssim 1. \tag{1.4.69}$$

Finally, we claim that $w_2' \to 1$ as $r_* \to -\infty$. We shall use estimate (1.4.50) near $x(r_*) = -1$ to obtain

$$
\begin{aligned}
|w_2'(r_*) - 1| &\lesssim e^{2k_+r_*}(|r_*|+1) + \left|(-1)^\ell \frac{r(r_*)}{k_+r_+}\frac{\mathrm{d}Q_\ell(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}r_*} - 1\right| \\
&\lesssim e^{2k_+r_*} + \left|\frac{r(r_*)}{k_+r_+}\left[1 + O\left((1+x(r_*))\log(1+x(r_*))\right)\right]\frac{1}{1-x^2(r_*)}\frac{\mathrm{d}x}{\mathrm{d}r_*} - 1\right|. \qquad (1.4.70)
\end{aligned}
$$

Now, in order to conclude that

$$
|w_2'(r_*) - 1| \to 0 \text{ as } r_* \to -\infty, \qquad (1.4.71)
$$

it suffices to check that

$$
\frac{1}{1-x^2(r_*)}\frac{\mathrm{d}x}{\mathrm{d}r_*} \to k_+ \text{ as } r_* \to -\infty. \qquad (1.4.72)
$$

But this holds true because

$$
\frac{1}{1-x^2(r_*)}\frac{\mathrm{d}x}{\mathrm{d}r_*} = \frac{1}{1-x^2(r_*)}\frac{-2}{r_+-r_-}\frac{\Delta}{r^2} = \frac{r_+-r_-}{2r^2} \to k_+ \text{ as } r_* \to -\infty. \qquad (1.4.73)
$$

Now, this implies that

$$
\mathfrak{W}(w_1, w_2) = \lim_{r_*\to-\infty}\left(w_1w_2' - w_1'w_2\right) = 1, \qquad (1.4.74)
$$

and moreover, that $w_2 = \tilde{u}_2$ as they satisfy the same boundary conditions at $r_* = -\infty$. $\square$

Having proved the Wronskian condition we are in the position to define the perturbations of $\tilde{u}_1$ and $\tilde{u}_2$ to non-zero frequencies.

Definition 1.4.4. *Define perturbations $\tilde{u}_{1,\omega}$ and $\tilde{u}_{2,\omega}$ of $\tilde{u}_1$ and $\tilde{u}_2$ (cf.* (1.4.8) *and* (1.4.9)*) in the intermediate region by the unique solutions to the Volterra equations*

$$
\tilde{u}_{1,\omega}(r_*) = \tilde{u}_1(r_*) + \omega^2\int_{R_1^*}^{r_*}\left(\tilde{u}_1(r_*)\tilde{u}_2(y) - \tilde{u}_1(y)\tilde{u}_2(r_*)\right)\tilde{u}_{1,\omega}(y)\mathrm{d}y \qquad (1.4.75)
$$

and

$$
\tilde{u}_{2,\omega}(r_*) = \tilde{u}_2(r_*) + \omega^2\int_{R_1^*}^{r_*}\left(\tilde{u}_1(r_*)\tilde{u}_2(y) - \tilde{u}_1(y)\tilde{u}_2(r_*)\right)\tilde{u}_{2,\omega}(y)\mathrm{d}y. \qquad (1.4.76)
$$

Proposition 1.4.3. *Let $0 < |\omega| < \omega_0$ and $\ell \in \mathbb{N}_0$, then we have for $r_* \in [R_1^*, R_2^*]$*

$$u_1(\omega, r_*) = A(\omega, \ell)\tilde{u}_{1,\omega}(r_*) + B(\omega, \ell)\omega\tilde{u}_{2,\omega}(r_*), \tag{1.4.77}$$

where

$$|A(\omega, \ell)| + |B(\omega, \ell)| \lesssim 1. \tag{1.4.78}$$

Proof. First, note that by construction in Definition 1.4.4 we have

$$\tilde{u}_{1,\omega}(R_1^*) = \tilde{u}_1(R_1^*), \tag{1.4.79}$$
$$\tilde{u}'_{1,\omega}(R_1^*) = \tilde{u}'_1(R_1^*), \tag{1.4.80}$$
$$\tilde{u}_{2,\omega}(R_1^*) = \tilde{u}_2(R_1^*), \tag{1.4.81}$$
$$\tilde{u}'_{2,\omega}(R_1^*) = \tilde{u}'_2(R_1^*). \tag{1.4.82}$$

Now, we want to estimate the previous terms. By construction, we directly have that

$$|\tilde{u}_1(R_1^*)| \leq 1. \tag{1.4.83}$$

Then, note that

$$\frac{\omega^2}{\ell^2+1} \lesssim 1 + x(R_1^*) \lesssim \frac{\omega^2}{\ell^2+1}. \tag{1.4.84}$$

Hence, from (1.4.35), we obtain

$$|\tilde{u}_2(R_1^*)| \lesssim 1 + \left| -\frac{1}{2}\log(1 + x(R_1^*)) - \digamma(\ell+1) \right| \lesssim 1 + |\log(|\omega|)| \lesssim \log\left(\frac{1}{|\omega|}\right), \tag{1.4.85}$$

where we have used that for $\ell \geq 1$ we have $\digamma(\ell+1) = \log(\ell) + \gamma + O(\ell^{-1})$. For $\tilde{u}'_2(R_1^*)$ we have the estimate

$$|\tilde{u}'_2(R_1^*)| \lesssim |\Delta(R_1^*)Q_\ell(x(R_1^*))| + \left| \frac{\mathrm{d}Q_\ell}{\mathrm{d}x}(R_1^*)\frac{\mathrm{d}x}{\mathrm{d}r_*}(R_1^*) \right| \lesssim 1, \tag{1.4.86}$$

where we have used (1.4.37) and (1.4.84) as well as the fact that

$$\frac{\mathrm{d}x}{\mathrm{d}r_*}(1 - x(r_*)^2)^{-1} \lesssim 1. \tag{1.4.87}$$

Now, we can express A via the Wronskian as

$$|A| = \left| \frac{\mathfrak{W}(u_1, \tilde{u}_{2,\omega})}{\mathfrak{W}(\tilde{u}_{1,\omega}, \tilde{u}_{2,\omega})} \right|. \tag{1.4.88}$$

By construction, we have $\mathfrak{W}(\tilde{u}_{1,\omega}, \tilde{u}_{2,\omega}) = \mathfrak{W}(\tilde{u}_1, \tilde{u}_2) = 1$. Hence, using Proposition 1.4.1 we conclude

$$|A| \leq |u_1(R_1^*)\tilde{u}_{2,\omega}'(R_1^*)| + |u_1'(R_1^*)\tilde{u}_{2,\omega}(R_1^*)| \lesssim |\tilde{u}_2'(R_1^*)| + |\omega \tilde{u}_2(R_1^*)|. \tag{1.4.89}$$

Thus, we conclude

$$|A| \lesssim 1. \tag{1.4.90}$$

Note that from (1.4.36), we have

$$|\tilde{u}_1'(R_1^*)| \lesssim \left| \left(1 + \frac{\mathrm{d}P_\ell}{\mathrm{d}x}\right) \frac{\mathrm{d}x}{\mathrm{d}r_*} \right| \lesssim (1+\ell^2)\frac{\omega^2}{1+\ell^2} \leq \omega^2. \tag{1.4.91}$$

Hence, we can also estimate B by

$$\begin{aligned} |B| &= \frac{1}{|\omega|}|\mathfrak{W}(u_1, \tilde{u}_{1,\omega})| \lesssim \frac{1}{|\omega|}\left(|\tilde{u}_1'(R_1^*)| + |\omega \tilde{u}_1(R_1^*)|\right) \\ &\lesssim 1 + \frac{1}{|\omega|}|\tilde{u}_1'(R_1^*)| \lesssim 1, \end{aligned} \tag{1.4.92}$$

where we used Proposition 1.4.1 again. □

For the intermediate region we will need the following result in order to get uniform bounds for the Volterra iteration.

Lemma 1.4.5. *Let* $0 < |\omega| < \omega_0$ *and* $\ell \in \mathbb{N}_0$*, then*

$$\int_{R_1^*}^{R_2^*} |\tilde{u}_1(r_*)| \mathrm{d}r_* \lesssim \log^2\left(\frac{1}{|\omega|}\right), \tag{1.4.93}$$

$$\int_{R_1^*}^{R_2^*} |\tilde{u}_2(r_*)| \mathrm{d}r_* \lesssim \log^2\left(\frac{1}{|\omega|}\right). \tag{1.4.94}$$

Proof. We first prove (1.4.93). We shall split the integral in two regions. The first region is from $r_* = R_1^*$ to $r_* = 0$. In that region we define $\theta \in (0, \frac{\pi}{2}]$ such that $\cos(\theta) = -x(r_*)$.

Using also Lemma 1.4.1 we obtain

$$\begin{aligned}|\tilde{u}_1(r_*)| &\lesssim |P_\ell(x(r_*))| = |P_\ell(-x(r_*))| = |P_\ell(\cos\theta)| \\ &\lesssim \left|\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} J_0((\ell+\frac{1}{2})\theta)\right| + |e_{1,\ell}(\theta)|. \end{aligned} \tag{1.4.95}$$

The last term shall be treated as an error term. Thus,

$$\begin{aligned}\int_{R_1^*}^0 |\tilde{u}_1(r_*)|\mathrm{d}r_* &\lesssim \int_{x(R_1^*)}^0 |P_\ell(x)|\frac{1}{1+x}\mathrm{d}x \le \int_{-1+C\frac{\omega^2}{1+\ell^2}}^0 |P_\ell(-x)|\frac{1}{1+x}\mathrm{d}x \\ &\lesssim \int_{\arccos(1-C\frac{\omega^2}{1+\ell^2})}^{\frac{\pi}{2}} |P_\ell(\cos\theta)|\frac{1}{1-\cos\theta}\sin\theta\,\mathrm{d}\theta \\ &\le \int_{C_1\frac{|\omega|}{1+\ell}}^{\frac{\pi}{2}} |P_\ell(\cos\theta)|\frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta. \end{aligned} \tag{1.4.96}$$

Here, C and C_1 are positive constants only depending on the black hole parameters. We further estimate using equation (1.4.95)

$$\begin{aligned}&\int_{R_1^*}^0 |\tilde{u}_1(r_*)|\mathrm{d}r_* \\ &\lesssim \int_{C_1\frac{\omega}{1+\ell}}^{\frac{\pi}{2}} \left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} \left|J_0((\ell+\frac{1}{2})\theta)\right| \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta + \text{Error}, \end{aligned} \tag{1.4.97}$$

where we will take care of the term

$$\text{Error} = \int_{C_1\frac{\omega}{1+\ell}}^{\frac{\pi}{2}} |e_{1,\ell}(\theta)| \tag{1.4.98}$$

later. First, we look at the term

$$
\begin{aligned}
&\int_{C_1\frac{\omega}{1+\ell}}^{\frac{\pi}{2}} \left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}} \left|J_0\left(\left(\ell+\frac{1}{2}\right)\theta\right)\right| \frac{\sin\theta}{1-\cos\theta}\mathrm{d}\theta \\
&\lesssim \int_{C_1\frac{\omega}{1+\ell}}^{\frac{\pi}{2}} \frac{1}{\theta}\left|J_0\left(\left(\ell+\frac{1}{2}\right)\theta\right)\right|\mathrm{d}\theta \\
&\lesssim \int_{C_1\omega}^{\frac{\pi}{2}(\ell+1)} \frac{1}{\theta}\left|J_0\left(\frac{\ell+\frac{1}{2}}{\ell+1}\theta\right)\right|\mathrm{d}\theta \\
&\lesssim \int_{C_1\omega}^{1} \frac{\left|J_0\left(\frac{\ell+\frac{1}{2}}{\ell+1}\theta\right)\right|}{\theta}\mathrm{d}\theta + \int_1^\infty \frac{\left|J_0\left(\frac{\ell+\frac{1}{2}}{\ell+1}\theta\right)\right|}{\theta}\mathrm{d}\theta \\
&\lesssim \int_{C_1\omega}^{1} \frac{1}{\theta}\mathrm{d}\theta + \int_1^\infty \frac{1}{\theta^{\frac{3}{2}}}\mathrm{d}\theta \lesssim |\log(|\omega|)|, \qquad (1.4.99)
\end{aligned}
$$

where we have used equation (1.4.27) and (1.4.28). Now, we are left with the error term

$$
\begin{aligned}
\text{Error} &\leq \frac{1}{1+\ell}\int_{C_1\frac{\omega}{\ell+1}}^{\frac{\pi}{2}} \frac{\sin\theta}{1-\cos\theta}(|J_0((\ell+\frac{1}{2})\theta)| + |Y_0((\ell+\frac{1}{2})\theta)|)\mathrm{d}\theta \\
&\lesssim \frac{1}{1+\ell}\int_{C_1\frac{\omega}{\ell+1}}^{\frac{\pi}{2}} \frac{\sin\theta}{1-\cos\theta}(1+|\log(|\omega|)|)\mathrm{d}\theta \lesssim \frac{|\log(|\omega|)|}{1+\ell}\int_{C_1\frac{\omega}{\ell+1}}^{\frac{\pi}{2}}\frac{1}{\theta}\mathrm{d}\theta \\
&\lesssim \frac{\log^2(|\omega|)+\log(1+\ell)}{1+\ell} \lesssim \log^2\left(\frac{1}{|\omega|}\right). \qquad (1.4.100)
\end{aligned}
$$

Thus,

$$
\int_{R_1^*}^{0} |\tilde{u}_1(r_*)|\mathrm{d}r_* \lesssim \log^2\left(\frac{1}{|\omega|}\right). \qquad (1.4.101)
$$

Completely analogously, we can compute

$$
\int_{0}^{R_2^*} |\tilde{u}_1(r_*)|\mathrm{d}r_* \lesssim \log^2\left(\frac{1}{|\omega|}\right). \qquad (1.4.102)
$$

The proof of equation (1.4.93) is completely similar up to a term which involves

$$
\int_{C_1\omega}^{1} \frac{\left|Y_0\left(\frac{\ell+\frac{1}{2}}{\ell+1}\theta\right)\right|}{\theta}\mathrm{d}\theta \lesssim \log^2\left(\frac{1}{|\omega|}\right) \qquad (1.4.103)
$$

appearing in the estimate analogous to (1.4.99). □

With the help of the previous lemma we can now bound our solution u_1 at R_2^*. This results in

Proposition 1.4.4. *Let* $0 < |\omega| < \omega_0$ *and* $\ell \in \mathbb{N}_0$*, then*

$$\|u_1\|_{L^\infty(R_1^*, R_2^*)} \lesssim 1 \text{ and } |u_1'|(R_2^*) \lesssim |\omega|. \tag{1.4.104}$$

Proof. Recall that we have from Proposition 1.4.3 for $r_* \in [R_1^*, R_2^*]$

$$u_1(\omega, r_*) = A(\omega, \ell)\tilde{u}_{1,\omega}(r_*) + \omega B(\omega, \ell)\tilde{u}_{2,\omega}(r_*) \tag{1.4.105}$$

for some uniformly bounded (in $|\omega| \le \omega_0$ and ℓ) constants A, B. In particular, from Proposition 1.2.3 and Remark 1.2.1 we obtain the bound

$$\|\tilde{u}_{1,\omega}\|_{L^\infty(R_1^*, R_2^*)} \le e^\alpha \|\tilde{u}_1\|_{L^\infty(R_1^*, R_2^*)} \tag{1.4.106}$$

for

$$\alpha = \omega^2 \int_{R_1^*}^{R_2^*} \sup_{\{r_* | y \le r_* \le R_2^*\}} |\tilde{u}_1(r_*)\tilde{u}_2(y) - \tilde{u}_1(y)\tilde{u}_2(r_*)| \mathrm{d}y. \tag{1.4.107}$$

First, we have the bound

$$\|\tilde{u}_1\|_{L^\infty(R_1^*, R_2^*)} \le 1. \tag{1.4.108}$$

Secondly, for $r_* \in [R_1^*, R_2^*]$ we have

$$1 - x(r_*) \gtrsim \frac{\omega^2}{1 + \ell^2} \tag{1.4.109}$$

and

$$1 + x(r_*) \gtrsim \frac{\omega^2}{1 + \ell^2}. \tag{1.4.110}$$

Consider the case $x(r_*) \ge 0$ first and implicitly define $\theta(r_*)$ by $\cos\theta(r_*) = x(r_*)$. Then, in view of (1.4.30) and $\theta(x(r_*)) = \sqrt{2 - 2x(r_*)} + O((1 - x(r_*)^{\frac{3}{2}}))$, we estimate

$$|\tilde{u}_2(r_*)| \lesssim |Q_\ell(\cos(\theta(r_*)))| \lesssim \left| Y_0\left(\frac{\theta(r_*)(2\ell + 1)}{2}\right)\right| \lesssim |Y_0(C|\omega|)| \tag{1.4.111}$$

for a $C = C(M, Q) > 0$. Analogously, this also holds for $x(r_*) < 0$ such that (1.4.27) and

(1.4.28) imply

$$\|\tilde{u}_2\|_{L^\infty(R_1^*,R_2^*)} \lesssim \log\left(\frac{1}{|\omega|}\right). \tag{1.4.112}$$

Together with Lemma 1.4.5 we obtain

$$\alpha \lesssim 1. \tag{1.4.113}$$

Hence,

$$\|\tilde{u}_{1,\omega}\|_{L^\infty(R_1^*,R_2^*)} \lesssim 1 \tag{1.4.114}$$

and similarly,

$$\|\tilde{u}_{2,\omega}\|_{L^\infty(R_1^*,R_2^*)} \lesssim \log\left(\frac{1}{|\omega|}\right). \tag{1.4.115}$$

This shows $\|u_1\|_{L^\infty(R_1^*,R_2^*)} \lesssim 1$ in view of (1.4.105).

Now, we are left with the derivative $u_1'(R_2^*)$. To do so, we start by estimating $\tilde{u}_1'(R_2^*)$ and $\tilde{u}_2'(R_2^*)$. Using the analogous estimate as we did for R_1^* in (1.4.86) and (1.4.91), we obtain

$$|\tilde{u}_2'(R_2^*)| \lesssim 1 \text{ and } |\tilde{u}_1'(R_2^*)| \lesssim \omega^2. \tag{1.4.116}$$

Note that

$$\tilde{u}_{2,\omega}'(R_2^*) = \tilde{u}_2'(R_2^*) + \omega^2 \int_{R_1^*}^{R_2^*} \left(\tilde{u}_1'(R_2^*)\tilde{u}_2(y) - \tilde{u}_1(y)\tilde{u}_2'(R_2^*)\right) \tilde{u}_{2,\omega}(y)\mathrm{d}y \tag{1.4.117}$$

and thus in view of Lemma 1.4.5, (1.4.116), (1.4.115), (1.4.112), and (1.4.108) we estimate

$$\begin{aligned} |\tilde{u}_{2,\omega}'(R_2^*)| &\leq |\tilde{u}_2'(R_2^*)| + \omega^2 \log\left(\frac{1}{|\omega|}\right) \int_{R_1^*}^{R_2^*} |\tilde{u}_1'(R_2^*)\tilde{u}_2(y)| + |\tilde{u}_1(y)\tilde{u}_2'(R_2^*)|\mathrm{d}y \\ &\lesssim 1 + \omega^2 \,|\log(|\omega|)|\, (\omega^2 \log^2(|\omega|) + \log^2(|\omega|)) \lesssim 1. \end{aligned} \tag{1.4.118}$$

Similarly, we obtain

$$
\begin{aligned}
|\tilde{u}_{1,\omega}'(R_2^*)| &\leq |\tilde{u}_1'(R_2^*)| + \omega^2 \int_{R_1^*}^{R_2^*} |\tilde{u}_1'(R_2^*)\tilde{u}_2(y)| + |\tilde{u}_1(y)\tilde{u}_2'(R_2^*)| \mathrm{d}y \\
&\lesssim \omega^2 + \omega^2(\omega^2 \log^2(|\omega|) + \log^2(|\omega|)) \lesssim |\omega| \qquad (1.4.119)
\end{aligned}
$$

which concludes the proof in the light of (1.4.105). $\square$

Region near the Cauchy horizon. Completely analogously to Proposition 1.4.1, we have

Proposition 1.4.5. *Let $0 < |\omega| < \omega_0$ and $\ell \in \mathbb{N}_0$. Then, we have*

$$
\|v_1'\|_{L^\infty(R_2^*,\infty)} \lesssim |\omega|, \quad \|v_1\|_{L^\infty(R_2^*,\infty)} \lesssim 1 \qquad (1.4.120)
$$

and

$$
\|v_2'\|_{L^\infty(R_2^*,\infty)} \lesssim |\omega|, \quad \|v_2\|_{L^\infty(R_2^*,\infty)} \lesssim 1. \qquad (1.4.121)
$$

Boundedness of the scattering coefficients. Finally, we conclude that the reflection and transmission coefficients are uniformly bounded for parameters $0 < |\omega| < \omega_0$ and $\ell \in \mathbb{N}_0$.

Proposition 1.4.6. *We have*

$$
\sup_{0<|\omega|<\omega_0, \ell \in \mathbb{N}_0} (|\mathfrak{R}(\omega,\ell)| + |\mathfrak{T}(\omega,\ell)|) \lesssim 1. \qquad (1.4.122)
$$

Proof. Let $0 < |\omega| < \omega_0$ and $\ell \in \mathbb{N}_0$ and recall Definition 1.2.4. Then, Proposition 1.4.4 and Proposition 1.4.5 imply

$$
|\mathfrak{T}| \lesssim \frac{|\mathfrak{W}(u_1, v_2)|}{|\omega|} \leq \frac{|u_1(R_2^*)v_2'(R_2^*)| + |u_1'(R_2^*)v_2(R_2^*)|}{|\omega|} \lesssim 1 \qquad (1.4.123)
$$

and

$$
|\mathfrak{R}| \lesssim \frac{|\mathfrak{W}(u_1, v_1)|}{|\omega|} \leq \frac{|u_1(R_2^*)v_1'(R_2^*)| + |u_1'(R_2^*)v_1(R_2^*)|}{|\omega|} \lesssim 1. \qquad (1.4.124)
$$

$\square$

1.4.2 Frequencies bounded from below and bounded angular momenta ($|\omega| \geq \omega_0, \ell \leq \ell_0$)

Now, we will consider parameters of the form $|\omega| \geq \omega_0$ and $\ell \leq \ell_0$, where ω_0 is small and determined from Section 1.4.1. Also, the upper bound on the angular momentum ℓ_0 will be determined from Section 1.4.3. As before, constants appearing in $\lesssim$ and $\gtrsim$ may depend on ω_0.

Proposition 1.4.7. *We have*

$$\sup_{\omega_0 \leq |\omega|, \ell \leq \ell_0} (|\mathfrak{R}(\omega, \ell)| + |\mathfrak{T}(\omega, \ell)|) \lesssim 1. \tag{1.4.125}$$

Proof. Recall the definition of u_1 as the unique solution to

$$u_1(\omega, r_*) = e^{i\omega r_*} + \int_{-\infty}^{r_*} \frac{\sin(\omega(r_* - y))}{\omega} V(y) u_1(\omega, y) \mathrm{d}y. \tag{1.4.126}$$

Note that in the regime $\ell \leq \ell_0$ we have a bound of the form

$$|V(r_*)| \lesssim e^{-2\min(k_+, |k_-|)|r_*|} \tag{1.4.127}$$

which implies the following bound on the integral kernel of the perturbation in (1.4.126)

$$|K(r_*, y)| = \left| \frac{\sin(\omega(r_* - y))}{\omega} V(y) \right| \lesssim |V(y)| \tag{1.4.128}$$

in view of $|\omega| \geq \omega_0$. Thus,

$$\int_{-\infty}^{\infty} \sup_{r_* \in \mathbb{R}} |K(r_*, y)| \mathrm{d}y \lesssim \int_{-\infty}^{\infty} |V(y)| \mathrm{d}y \lesssim 1. \tag{1.4.129}$$

Hence, from Proposition 1.2.3 we deduce

$$\|u_1\|_{L^\infty(\mathbb{R})} \lesssim 1 \tag{1.4.130}$$

and

$$\|u_1'\|_{L^\infty(\mathbb{R})} \lesssim |\omega|. \tag{1.4.131}$$

Note that we have obtained similar, indeed even stronger bounds for u_1 as in Proposition 1.4.4. An argument completely similar to Proposition 1.4.6 allows us to conclude. $\square$

1.4.3 Frequencies and angular momenta bounded from below ($|\omega| \geq \omega_0$, $\ell \geq \ell_0$)

In this regime we assume $\omega \geq \omega_0$ and $\ell \geq \ell_0$, where we choose ℓ_0 large enough such that $V_\ell < 0$ everywhere. Note that such an ℓ_0 can be chosen only depending on the black hole parameters.

We write the o.d.e. as

$$u'' = -(\omega^2 - V_\ell)u \tag{1.4.132}$$

and will represent the solution of the o.d.e. via a WKB approximation. For concreteness we will use the following theorem which is a slight modification of [116, Theorem 4].

Lemma 1.4.6 (Theorem 4 of [116])**.** *Let $p \in C^2(\mathbb{R})$ be a positive function such that*

$$F(x) = \left| \int_{-\infty}^{x} p^{-\frac{1}{4}} \left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(p^{-\frac{1}{4}}\right) \right| \mathrm{d}y \right| \tag{1.4.133}$$

satisfies $\sup_{x\in\mathbb{R}} F(x) < \infty$. *Then, the differential equation*

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = -p(x)u(x) \tag{1.4.134}$$

has conjugate solutions u and $\bar{u}$ such that

$$u(x) = p^{-\frac{1}{4}} \left(\exp\left(i \int_0^x \sqrt{p}(y)\mathrm{d}y \right) + \epsilon \right), \tag{1.4.135}$$

$$u'(x) = ip^{\frac{1}{4}} \left[\exp\left(i \int_0^x \sqrt{p}(y)\mathrm{d}y \right) - i\eta + \frac{ip'}{4p^{\frac{3}{2}}} \left(\exp\left(-i \int_0^x \sqrt{p}(y)\mathrm{d}y \right) + \epsilon \right) \right], \tag{1.4.136}$$

where

$$|\eta(x)|, |\epsilon(x)| \leq \exp\left(F(x)\right) - 1. \tag{1.4.137}$$

Proposition 1.4.8. *Let $\omega_0 \leq |\omega|$ and $\ell \geq \ell_0$. Assume without loss of generality that*

$\omega > 0$. *Then,*

$$u_1(\omega, r_*) = A\omega^{\frac{1}{2}}(\omega^2 - V(r_*))^{-\frac{1}{4}} \left(\exp\left(i \int_0^{r_*} (\omega^2 - V_\ell(y))^{\frac{1}{2}} \mathrm{d}y \right) + \epsilon(r_*) \right), \tag{1.4.138}$$

$$\begin{aligned} u_1'(\omega, r_*) = A\omega^{\frac{1}{2}} i(\omega^2 - V(r_*))^{\frac{1}{4}} &\left[\exp\left(i \int_0^{r_*} (\omega^2 - V_\ell(y))^{\frac{1}{2}} \mathrm{d}y \right) - i\eta(r_*) \right. \\ &\left. - \frac{iV'(r_*)}{4(\omega^2 - V)^{\frac{3}{2}}(r_*)} \left(\exp\left(i \int_0^{r_*} (\omega^2 - V_\ell(y))^{\frac{1}{2}} \mathrm{d}y \right) + \epsilon(r_*) \right) \right], \end{aligned} \tag{1.4.139}$$

where

$$|A| = 1, \sup_{r_* \in \mathbb{R}} (|\epsilon|(r_*) + |\eta|(r_*)) \lesssim 1 \tag{1.4.140}$$

and

$$\lim_{r_* \to -\infty} \eta(r_*) = \lim_{r_* \to -\infty} \epsilon(r_*) = 0. \tag{1.4.141}$$

In particular, this proves

$$\limsup_{r_* \to \infty} |u(r_*)| \lesssim 1, \tag{1.4.142}$$

$$\limsup_{r_* \to \infty} |u'(r_*)| \lesssim |\omega|, \tag{1.4.143}$$

and uniform bounds on the reflection and transmission coefficients

$$\sup_{\omega_0 \le |\omega|, \ell \ge \ell_0} (|\mathfrak{R}(\omega, \ell)| + |\mathfrak{T}(\omega, \ell)|) \lesssim 1. \tag{1.4.144}$$

Proof. We will apply Lemma 1.4.6. First, we set

$$p = (\omega^2 - V_\ell) \tag{1.4.145}$$

which is positive and smooth. Then, the o.d.e. reads

$$u'' = -pu. \tag{1.4.146}$$

Now we have to show that F is uniformly bounded on the real line. Note that we have the following bounds on the potential and its derivatives

$$|V_\ell(r_*)|, |V_\ell'(r_*)|, |V_\ell''(r_*)| \lesssim \ell^2 e^{2\kappa_+ r_*} \text{ and } \ell^2 e^{2\kappa_+ r_*} \lesssim |V_\ell(r_*)| \text{ for } r_* \leq 0, \tag{1.4.147}$$

$$|V_\ell(r_*)|, |V_\ell'(r_*)|, |V_\ell''(r_*)| \lesssim \ell^2 e^{2\kappa_- r_*} \text{ and } \ell^2 e^{2\kappa_- r_*} \lesssim |V_\ell(r_*)| \text{ for } r_* \geq 0. \tag{1.4.148}$$

Here, we might have to choose $\ell_0(M,Q)$ even larger ($r_+^2(r_+ - 3r_-) + \ell(\ell+1) > 0$, cf. (1.8.16)) in order to assure the lower bounds on the potential. Finally, we can estimate F by

$$\begin{aligned}
\sup_{r_* \in \mathbb{R}} F(r_*) &\leq \left| \int_{-\infty}^{\infty} p^{-\frac{1}{4}} \left| \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(p^{-\frac{1}{4}} \right| \right) \mathrm{d}y \right| \\
&= \int_{-\infty}^{\infty} p^{-\frac{1}{4}} \left(p^{-\frac{9}{4}} p'^2 + p^{-\frac{5}{4}} |p''| \right) \mathrm{d}y \\
&\lesssim \frac{1}{\ell} \int_0^\infty \left(\frac{e^{4\kappa_- y}}{(\ell^{-2} + e^{2\kappa_- y})^{\frac{5}{2}}} + \frac{e^{2\kappa_- y}}{(\ell^{-2} + e^{2\kappa_- y})^{\frac{3}{2}}} \right) \mathrm{d}y \\
&+ \frac{1}{\ell} \int_{-\infty}^0 \left(\frac{e^{4\kappa_+ y}}{(\ell^{-2} + e^{2\kappa_+ y})^{\frac{5}{2}}} + \frac{e^{2\kappa_+ y}}{(\ell^{-2} + e^{2\kappa_+ y})^{\frac{3}{2}}} \right) \mathrm{d}y,
\end{aligned} \tag{1.4.149}$$

where we have used the bounds from (1.4.147) and (1.4.148). We shall estimate both terms independently. After a change of variables $y \mapsto \frac{1}{2\kappa_-} \log(y)$, we can estimate the first term by

$$\begin{aligned}
&\frac{1}{\ell} \int_0^\infty \left(\frac{e^{4\kappa_- y}}{(\ell^{-2} + e^{2\kappa_- y})^{\frac{5}{2}}} + \frac{e^{2\kappa_- y}}{(\ell^{-2} + e^{2\kappa_- y})^{\frac{3}{2}}} \right) \mathrm{d}y \\
&\lesssim \frac{1}{\ell} \int_0^1 \left(\frac{y}{(\ell^{-2} + y)^{\frac{5}{2}}} + \frac{1}{(\ell^{-2} + y)^{\frac{3}{2}}} \right) \mathrm{d}y \\
&\lesssim \ell^2 \int_0^1 \frac{\ell^2 y}{(1 + \ell^2 y)^{\frac{5}{2}}} + \frac{1}{(1 + \ell^2 y)^{\frac{3}{2}}} \mathrm{d}y \\
&\lesssim \int_0^\infty \frac{y}{(1+y)^{\frac{5}{2}}} + \frac{1}{(1+y)^{\frac{3}{2}}} \mathrm{d}y \lesssim 1.
\end{aligned} \tag{1.4.150}$$

Completely analogously, we get the bound for the second integral. In particular, this shows

$$\sup_{\mathbb{R}} F \lesssim 1. \tag{1.4.151}$$

This implies the bounds on η and ϵ in the statement of the theorem (cf. (1.4.140)) using

(1.4.137).

The limits in equation (1.4.141) follow from the fact that $F(r_*) \to 0$ as $r_* \to -\infty$ by construction.

The bound on the reflection and transmission coefficients follows now from

$$|\mathfrak{R}| \lesssim \left| \frac{\mathfrak{W}(u_1, v_1)}{\omega} \right| \leq \frac{1}{|\omega|} \limsup_{r_* \to \infty} \left(|u_1' v_1| + |u_1 v_1'| \right) \lesssim 1 \tag{1.4.152}$$

and analogously for $\mathfrak{T}$.

Finally, A can be determined from the asymptotic behaviour $u \to e^{i\omega r_*}$ as $r_* \to -\infty$ and it is given by

$$\begin{aligned} A &= \lim_{r_* \to -\infty} \exp\left(i\omega r_* - i \int_0^{r_*} (\omega^2 - V(y))^{\frac{1}{2}} \mathrm{d}y \right) \\ &= \lim_{r_* \to -\infty} \exp\left(-i \int_0^{r_*} \left((\omega^2 - V(y))^{\frac{1}{2}} - \omega \right) \mathrm{d}y \right) \end{aligned} \tag{1.4.153}$$

which converges since V tends to zero exponentially fast. In particular, this also shows that $|A| = 1$. □

Finally, Theorem 1.2 is a consequence of Proposition 1.4.6, Proposition 1.4.7, and Proposition 1.4.8.

1.5 Proof of Theorem 1.1: Existence and boundedness of the T energy scattering map

Having performed the analysis of the radial o.d.e. and having in particular proven uniform boundedness of the transmission coefficient $\mathfrak{T}$ and the reflection coefficients $\mathfrak{R}$, we shall prove Theorem 1.1 in this section.

1.5.1 Density of the domains $\mathcal{D}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}}$

We start by proving that the domains $\mathcal{D}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}}$ are dense.

Lemma 1.5.1. *The domains of the forward and backward evolution $\mathcal{D}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}}$ are dense in $\mathcal{E}^T_{\mathcal{H}}$ and $\mathcal{E}^T_{\mathcal{CH}}$, respectively.*

Proof. We will only prove that the domain of the forward evolution is dense since the other claim is analogous.

Recall that by definition $C_c^\infty(\mathcal{H})$ is dense in $\mathcal{E}^T_{\mathcal{H}}$. Now, let $\Psi \in C_c^\infty(\mathcal{H})$ be arbitrary and denote by ψ its forward evolution. We will show that we can approximate Ψ with functions of $\mathcal{D}^T_{\mathcal{H}}$ arbitrarily well. To do so, fix $r_{red} < r_0 < r_+$. Then, using the red-shift effect (see Lemma 1.8.1 in the appendix) the N energy of $\psi \restriction_{r=r_0}$ will have exponential decay towards i_+. Hence, it can be approximated with smooth functions ϕ_n of compact support on the hypersurface $r = r_0$ w.r.t. the norm induced by the non-degenerate N energy (see Remark 1.8.1 in the appendix). More precisely, on $\Sigma_{r_0} = \{r = r_0\}$ define a sequence $\phi_n \in C_c^\infty(\Sigma_{r_0})$ by

$$\phi_n(t,\theta,\phi) = \psi \restriction_{r=r_0} (t,\theta,\phi)\chi(n^{-1}t), \tag{1.5.1}$$

where $(\theta,\phi) \in \mathbb{S}^2$ and $\chi\colon \mathbb{R} \to [0,1]$ is smooth with $\operatorname{supp}\chi \subseteq [-2,2]$, $\chi \restriction_{[-1,1]}= 1$. Then, we obtain that $\int_{\Sigma_{r_0}} J^N_\mu[\psi - \phi_n] n^\mu_{\Sigma_{r_0}} \mathrm{dvol} \to 0$ as $n \to \infty$. By construction, the restriction to the event horizon of the backward evolution, Φ_n of each ϕ_n will lie in $\mathcal{D}^T_{\mathcal{H}}$. Finally, we can conclude the proof by applying Lemma 1.8.2 from the appendix, which yields

$$\|\Psi - \Phi_n\|^2_{\mathcal{E}^T_{\mathcal{H}}} = \int_{\mathcal{H}} J^T_\mu[\Psi - \Phi_n] T^\mu \mathrm{dvol} \lesssim \int_{r=r_0} J^N_\mu[\psi - \phi_n] n^\mu_{\Sigma_{r_0}} \mathrm{dvol} \to 0 \tag{1.5.2}$$

as $n \to \infty$. □

1.5.2 Boundedness of the scattering and backward map on $\mathcal{D}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}}$

In the following proposition we shall lift the boundedness of the transmission and reflection coefficients (Theorem 1.2) to the physical space picture on the dense domains $\mathcal{D}^T_{\mathcal{H}}$ and $\mathcal{D}^T_{\mathcal{CH}}$.

Proposition 1.5.1. *Let ψ be a smooth solution to* (1.1.1) *on $\mathcal{M}_{\mathrm{RN}}$ such that $\psi \restriction_{\mathcal{H}} \in \mathcal{D}^T_{\mathcal{H}}$ (or equivalently, $\psi \restriction_{\mathcal{CH}} \in \mathcal{D}^T_{\mathcal{CH}}$). Then,*

$$\|\psi \restriction_{\mathcal{CH}_A}\|^2_{\mathcal{E}^T_{\mathcal{CH}_A}} + \|\psi \restriction_{\mathcal{CH}_B}\|^2_{\mathcal{E}^T_{\mathcal{CH}_B}} \leq B \left(\|\psi \restriction_{\mathcal{H}_A}\|^2_{\mathcal{E}^T_{\mathcal{H}_A}} + \|\psi \restriction_{\mathcal{H}_B}\|^2_{\mathcal{E}^T_{\mathcal{H}_B}} \right) \tag{1.5.3}$$

and

$$\|\psi \restriction_{\mathcal{H}_A}\|^2_{\mathcal{E}^T_{\mathcal{H}_A}} + \|\psi \restriction_{\mathcal{H}_B}\|^2_{\mathcal{E}^T_{\mathcal{H}_B}} \leq \tilde{B} \left(\|\psi \restriction_{\mathcal{CH}_A}\|^2_{\mathcal{E}^T_{\mathcal{CH}_A}} + \|\psi \restriction_{\mathcal{CH}_B}\|^2_{\mathcal{E}^T_{\mathcal{CH}_B}} \right) \tag{1.5.4}$$

for constants B and $\tilde{B}$ only depending on the black hole parameters.

Proof. Set $\phi := T\psi$ and note that $\phi \restriction_{\mathcal{H}} \in \mathcal{D}^T_{\mathcal{H}}$ and ϕ also solves (1.1.1). Since $\psi \in \mathcal{D}^T_{\mathcal{H}} \subset \mathcal{E}^T_{\mathcal{H}}$, we have that $\phi \restriction_{\mathcal{H}_A} = T\psi \restriction_{\mathcal{H}_A} \in L^2(\mathcal{H}_A)$ with respect to the unique volume form induced by the normal vector field T. Analogously, we also have $\phi\restriction_{\mathcal{H}_B} = T\psi\restriction_{\mathcal{H}_B} \in L^2(\mathcal{H}_B)$. Thus, we can define the Fourier transform on the event horizon with the charts (1.2.15) and (1.2.16) as

$$a_{\mathcal{H}_A}(\omega,\theta,\phi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi\restriction_{\mathcal{H}_A}(v,\theta,\phi) e^{-i\omega v} \mathrm{d}v \tag{1.5.5}$$

and

$$a_{\mathcal{H}_B}(\omega,\theta,\phi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi\restriction_{\mathcal{H}_B}(u,\theta,\phi) e^{i\omega u} \mathrm{d}u. \tag{1.5.6}$$

We can further decompose the Fourier coefficients in spherical harmonics to obtain

$$a^{\ell,m}_{\mathcal{H}_A}(\omega) = \langle Y_{\ell m}, a_{\mathcal{H}_A}\rangle_{L^2(\mathbb{S}^2)} \text{ and } a^{\ell,m}_{\mathcal{H}_B}(\omega) = \langle Y_{\ell m}, a_{\mathcal{H}_B}\rangle_{L^2(\mathbb{S}^2)}. \tag{1.5.7}$$

From Plancherel's theorem, we obtain

$$\|\psi\restriction_{\mathcal{H}_A}\|^2_{\mathcal{E}^T_{\mathcal{H}_A}} = \sum_{|m|\le\ell,\ell\ge0} \int_{\mathbb{R}} |a^{\ell,m}_{\mathcal{H}_A}(\omega)|^2 \mathrm{d}\omega, \tag{1.5.8}$$

$$\|\psi\restriction_{\mathcal{H}_B}\|^2_{\mathcal{E}^T_{\mathcal{H}_b}} = \sum_{|m|\le\ell,\ell\ge0} \int_{\mathbb{R}} |a^{\ell,m}_{\mathcal{H}_B}(\omega)|^2 \mathrm{d}\omega. \tag{1.5.9}$$

Similarly, since $\phi\restriction_{\mathcal{CH}} \in \mathcal{D}^T_{\mathcal{CH}}$, we define

$$b_{\mathcal{CH}_A}(\omega,\theta,\phi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi\restriction_{\mathcal{CH}_A}(v,\theta,\phi) e^{-i\omega v} \mathrm{d}v \tag{1.5.10}$$

and

$$b_{\mathcal{CH}_B}(\omega,\theta,\phi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi\restriction_{\mathcal{CH}_B}(u,\theta,\phi) e^{i\omega u} \mathrm{d}u. \tag{1.5.11}$$

We can further decompose the Fourier coefficients in spherical harmonics to obtain

$$b^{\ell,m}_{\mathcal{CH}_A}(\omega) = \langle Y_{\ell m}, b_{\mathcal{CH}_A}\rangle_{L^2(\mathbb{S}^2)} \text{ and } b^{\ell,m}_{\mathcal{CH}_B}(\omega) = \langle Y_{\ell m}, b_{\mathcal{CH}_B}\rangle_{L^2(\mathbb{S}^2)}. \tag{1.5.12}$$

Again, in view of Plancherel's theorem

$$\|\psi \restriction_{\mathcal{CH}_A} \|^2_{\mathcal{E}^T_{\mathcal{CH}_A}} = \sum_{|m|\leq \ell, \ell \geq 0} \int_{\mathbb{R}} |b^{\ell,m}_{\mathcal{CH}_A}(\omega)|^2 \mathrm{d}\omega, \tag{1.5.13}$$

$$\|\psi \restriction_{\mathcal{CH}_B} \|^2_{\mathcal{E}^T_{\mathcal{CH}_B}} = \sum_{|m|\leq \ell, \ell \geq 0} \int_{\mathbb{R}} |b^{\ell,m}_{\mathcal{CH}_B}(\omega)|^2 \mathrm{d}\omega. \tag{1.5.14}$$

and similarly for $\mathcal{CH}_B$. We shall also decompose ϕ on a constant r slice. Fix $r \in (r_-, r_+)$, then set

$$\hat{\phi}_{m\ell}(\omega, r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{S}^2} Y_{m\ell}(\theta, \phi) \phi(t, r, \theta, \phi) e^{-i\omega t} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}t \tag{1.5.15}$$

such that

$$\phi(t, r, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \sum_{|m|\leq \ell, \ell\geq 0} \int_{\mathbb{R}} \hat{\phi}_{m\ell}(\omega, r) Y_{m\ell}(\theta, \phi) e^{i\omega t} \mathrm{d}\omega. \tag{1.5.16}$$

This is well-defined since $\phi(t, r, \theta, \phi)$ is compactly supported on each $r = \mathit{const.}$ slice.

Since ϕ is smooth, we also know that $\hat{\phi}_{m\ell}$ satisfies the radial o.d.e. (1.2.34) and can be expanded as

$$\hat{\phi}_{m\ell}(\omega, r(r_*)) = \alpha^{\ell,m}_{\mathcal{H}_A}(\omega) \frac{r_+}{r} u_1(\omega, r_*) + \alpha^{\ell,m}_{\mathcal{H}_B}(\omega) \frac{r_+}{r} u_2(\omega, r_*), \tag{1.5.17}$$

where

$$|u_1 - e^{i\omega r_*}| \lesssim_\ell e^{2\kappa_+ r_*} \sim (r_+ - r), \tag{1.5.18}$$

$$|u_2 - e^{-i\omega r_*}| \lesssim_\ell e^{2\kappa_+ r_*} \sim (r_+ - r) \tag{1.5.19}$$

for $r_* \leq 0$. Note that this holds uniformly in ω. We shall show in the following that indeed $\alpha^{\ell,m}_{\mathcal{H}_A} = a^{\ell,m}_{\mathcal{H}_A}$ and $\alpha^{\ell,m}_{\mathcal{H}_B} = a^{\ell,m}_{\mathcal{H}_B}$. To do so, note that for $r(r_*)$ with $r_* \leq 0$ we have for fixed (m, ℓ) that

$$\begin{aligned} \phi^{\ell,m}(t, r) &= \langle \phi, Y_{m\ell} \rangle_{L^2(\mathbb{S}^2)} \\ &= \int_{\mathbb{R}} \left(\alpha^{\ell,m}_{\mathcal{H}_A}(\omega) \frac{r_+}{r} u_1(\omega, r_*(r)) + \alpha^{\ell,m}_{\mathcal{H}_B}(\omega) \frac{r_+}{r} u_2(\omega, r_*(r)) \right) e^{i\omega t} \frac{\mathrm{d}\omega}{\sqrt{2\pi}}. \end{aligned} \tag{1.5.20}$$

We want to interchange the limit $r \to r_+$ with the integral. In order to use Lebesgue's dominated convergence theorem we will estimate $\alpha^{\ell,m}_{\mathcal{H}_A}$ and $\alpha^{\ell,m}_{\mathcal{H}_B}$. Note that

$$|\alpha^{\ell,m}_{\mathcal{H}_A}| = \left|\frac{\mathfrak{W}(\frac{r}{r_+}\hat{\phi}_{m\ell},u_2)}{\mathfrak{W}(u_1,u_2)}\right| = \left|\frac{\mathfrak{W}(\frac{r}{r_+}T\hat{\psi}_{m\ell},u_2)}{\mathfrak{W}(u_1,u_2)}\right| \leq \frac{|\omega\mathfrak{W}(\frac{r}{r_+}\hat{\psi}_{m\ell},u_2)|}{2|\omega|} \leq \left|\mathfrak{W}\left(\frac{r}{r_+}\hat{\psi}_{m\ell},u_2\right)\right|, \tag{1.5.21}$$

which is independent of $r(r_*)$ and integrable since $\omega \mapsto \hat{\psi}_{m\ell}(\omega,r_*)$ is a Schwartz function. Now, we shall fix $v = r_* + t$ and let $r \to r_+$ such that $r_* \to -\infty$. Then, using Lebesgue's dominated convergence theorem, we obtain

$$\phi^{\ell,m} = \int_{\mathbb{R}} \left(\alpha^{\ell,m}_{\mathcal{H}_A}(\omega)e^{i\omega v} + \alpha^{\ell,m}_{\mathcal{H}_B}(\omega)e^{-2i\omega r_*}e^{i\omega v}\right)\frac{\mathrm{d}\omega}{\sqrt{2\pi}} + O(r_+ - r)$$

as $r \to r_+$. Finally, for v fixed and letting $r \to r_+$ (or $r_* \to -\infty$), we obtain

$$\phi^{\ell,m}\restriction_{\mathcal{H}_A}(v) = \int_{\mathbb{R}} \alpha^{\ell,m}_{\mathcal{H}_A}(\omega)e^{i\omega v}\frac{\mathrm{d}\omega}{\sqrt{2\pi}} \tag{1.5.22}$$

in view of the Riemann–Lebesgue lemma. Also, by definition of $a^{\ell,m}_{\mathcal{H}_A}$,

$$\phi\restriction_{\mathcal{H}_A}(v,\theta,\phi) = \sum_{|m|\leq\ell,\ell\geq 0}\int_{\mathbb{R}} a^{\ell,m}_{\mathcal{H}_A}(\omega,\theta,\phi)e^{i\omega v}Y_{\ell m}(\theta,\phi)\frac{\mathrm{d}v}{\sqrt{2\pi}}. \tag{1.5.23}$$

In view of the Fourier inversion theorem and the fact that the spherical harmonics form a basis we conclude that

$$\alpha^{\ell,m}_{\mathcal{H}_A} = a^{\ell,m}_{\mathcal{H}_A} \text{ and analogously, } \alpha^{\ell,m}_{\mathcal{H}_B} = a^{\ell,m}_{\mathcal{H}_B}. \tag{1.5.24}$$

Similarly to (1.5.17), we can expand $\hat{\psi}_{m\ell}$ in a fundamental pair of solutions corresponding to both Cauchy horizons $\mathcal{CH}_A$ and $\mathcal{CH}_B$. In particular, we can write

$$\hat{\phi}_{m\ell}(\omega,r(r_*)) = \beta^{\ell,m}_{\mathcal{CH}_A}(\omega)\frac{r_+}{r}v_1(\omega,r_*) + \beta^{\ell,m}_{\mathcal{CH}_A}(\omega)\frac{r_+}{r}v_2(\omega,r_*), \tag{1.5.25}$$

where

$$|v_1 - e^{-i\omega r_*}| \lesssim_\ell e^{2\kappa_- r_*} \sim (r - r_-), \tag{1.5.26}$$

$$|v_2 - e^{i\omega r_*}| \lesssim_\ell e^{2\kappa_- r_*} \sim (r - r_-). \tag{1.5.27}$$

for $r_* \geq 0$. Similarly to (1.5.24), we can prove

$$\frac{r_+}{r_-}\beta^{\ell,m}_{\mathcal{CH}_A}(\omega) = b^{\ell,m}_{\mathcal{CH}_A}(\omega) \text{ and } \frac{r_+}{r_-}\beta^{\ell,m}_{\mathcal{CH}_B}(\omega) = b^{\ell,m}_{\mathcal{CH}_B}(\omega). \tag{1.5.28}$$

Moreover, from the uniform boundedness of the reflection and transmission coefficients (cf. Theorem 1.2) we have the estimate

$$\begin{aligned} &|b^{\ell,m}_{\mathcal{CH}_A}(\omega)| + |b^{\ell,m}_{\mathcal{CH}_B}(\omega)| \\ &= \frac{r_+}{r_-}|\beta^{\ell,m}_{\mathcal{CH}_A}(\omega)| + \frac{r_+}{r_-}|\beta^{\ell,m}_{\mathcal{CH}_B}(\omega)| = \frac{r_+}{r_-}\left(\left|\mathfrak{R}\alpha^{\ell,m}_{\mathcal{H}_A} + \bar{\mathfrak{T}}\alpha^{\ell,m}_{\mathcal{H}_B}\right| + \left|\bar{\mathfrak{R}}\alpha^{\ell,m}_{\mathcal{H}_B} + \mathfrak{T}\alpha^{\ell,m}_{\mathcal{H}_A}\right|\right) \\ &\leq C(|\alpha^{\ell,m}_{\mathcal{H}_A}(\omega)| + |\alpha^{\ell,m}_{\mathcal{H}_B}(\omega)|) = C(|a^{\ell,m}_{\mathcal{H}_A}(\omega)| + |a^{\ell,m}_{\mathcal{H}_B}(\omega)|) \end{aligned} \tag{1.5.29}$$

for a constant C which only depends on the black hole parameters. Here, we have used the fact that

$$\begin{pmatrix} \beta^{\ell,m}_{\mathcal{CH}_B} \\ \beta^{\ell,m}_{\mathcal{CH}_A} \end{pmatrix} = \begin{pmatrix} \mathfrak{T} & \bar{\mathfrak{R}} \\ \mathfrak{R} & \bar{\mathfrak{T}} \end{pmatrix} \begin{pmatrix} \alpha^{\ell,m}_{\mathcal{H}_A} \\ \alpha^{\ell,m}_{\mathcal{H}_B}. \end{pmatrix}. \tag{1.5.30}$$

In view of $1 = |\mathfrak{T}|^2 - |\mathfrak{R}|^2$, we also have

$$\begin{pmatrix} \alpha^{\ell,m}_{\mathcal{H}_A} \\ \alpha^{\ell,m}_{\mathcal{H}_B} \end{pmatrix} = \begin{pmatrix} \bar{\mathfrak{T}} & -\bar{\mathfrak{R}} \\ -\mathfrak{R} & \mathfrak{T} \end{pmatrix} \begin{pmatrix} \beta^{\ell,m}_{\mathcal{CH}_B} \\ \beta^{\ell,m}_{\mathcal{CH}_A} \end{pmatrix} \tag{1.5.31}$$

from which we deduce

$$|a^{\ell,m}_{\mathcal{H}_A}(\omega)| + |a^{\ell,m}_{\mathcal{H}_B}(\omega)| \lesssim |b^{\ell,m}_{\mathcal{CH}_A}(\omega)| + |b^{\ell,m}_{\mathcal{CH}_B}(\omega)|. \tag{1.5.32}$$

Estimate (1.5.29) and (1.5.32) show the claim in view of (1.5.8), (1.5.9), (1.5.13), and (1.5.14). Finally, in view of the Fourier inversion theorem, note that the previous also justifies the Fourier representation of scattering map (1.3.20), and the Fourier representations (1.3.24) and (1.3.25). □

1.5.3 Completing the proof

Having proven Lemma 1.5.1 and Proposition 1.5.1, we can finally show Theorem 1.1 in the following.

Proof of Theorem 1.1. Since $\mathcal{D}^T_{\mathcal{H}} \subset \mathcal{E}^T_{\mathcal{H}}$ is dense (Lemma 1.5.1) and $S^T_0 \colon \mathcal{D}^T_{\mathcal{H}} \subset \mathcal{E}^T_{\mathcal{H}} \to \mathcal{D}^T_{\mathcal{CH}} \subset \mathcal{E}^T_{\mathcal{CH}}$ is a bounded injective map (Remark 1.3.2, Proposition 1.5.1), we can uniquely extend S_0^T to the bounded injective scattering map

$$S^T : \mathcal{E}^T_{\mathcal{H}} \to \mathcal{E}^T_{\mathcal{CH}}. \tag{1.5.33}$$

Analogously, in view of Proposition 1.2.2, Remark 1.3.1, Remark 1.3.2, and Proposition 1.5.1, we can uniquely extend the bounded injective map $B_0^T : \mathcal{D}^T_{\mathcal{CH}} \subset \mathcal{E}^T_{\mathcal{CH}} \to \mathcal{D}^T_{\mathcal{CH}} \subset \mathcal{E}^T_{\mathcal{H}}$ to the bounded injective backward map $B^T : \mathcal{E}^T_{\mathcal{CH}} \to \mathcal{E}^T_{\mathcal{H}}$ (Lemma 1.5.1).

Since $B_0^T \circ S_0^T = \mathrm{Id}_{\mathcal{D}^T_{\mathcal{H}}}$ and $S_0^T \circ B_0^T = \mathrm{Id}_{\mathcal{D}^T_{\mathcal{CH}}}$ on dense sets, it also extends to $\mathcal{E}^T_{\mathcal{H}}$ and $\mathcal{E}^T_{\mathcal{CH}}$ from which (1.3.5) follows. Similarly, it suffices to check (1.3.6) for $\psi \in \mathcal{D}^T_{\mathcal{H}}$. Indeed, (1.3.6) holds true for $\psi \in \mathcal{D}^T_{\mathcal{H}}$ in view of the T energy identity. □

1.6 Proof of Theorem 1.6: Breakdown of T energy scattering for cosmological constants $\Lambda \neq 0$

In the presence of a cosmological constant Λ, the situation regarding the T energy scattering problem is changed radically. In this section we will consider the subextremal (anti-) de Sitter–Reissner–Nordström black hole interior $(\mathcal{M}_{\text{(a)dSRN}}, g_{Q,M,\Lambda})$ which is completely analogous to $(\mathcal{M}_{\text{RN}}, g_{Q,M})$. We will assume that $(M, Q, \Lambda) \in \mathcal{P}_{\text{se}}$ as defined in Section 1.3.6. Also, recall that in the presence of a cosmological constant it is natural to look at the Klein–Gordon equation

$$\Box_g \psi - \mu \psi = 0 \tag{1.6.1}$$

with mass $\mu = \frac{3}{2}\Lambda$ for the conformal invariant equation or more general $\mu = \nu \Lambda$ for fixed $\nu \in \mathbb{R}$.

This section is devoted to prove Theorem 1.6 which relies on the fact that solutions of the corresponding radial o.d.e. in the vanishing frequency limit $\omega = 0$ generically map bounded solutions at $r_* = -\infty$ to unbounded solutions at $r_* = +\infty$. More precisely, for $\Lambda \neq 0$ we obtain—after separation of variables for (1.6.1) and setting $\mathrm{d}r_* = h^{-1}\mathrm{d}r$—the o.d.e.

$$-u'' + V_{\ell,\Lambda} u = \omega^2 u \tag{1.6.2}$$

for $u(r_*) = r(r_*) R(r_*)$, where

$$V_{\ell,\Lambda} = h \left(\frac{h h'}{r} + \frac{\ell(\ell+1)}{r^2} - \mu \right) = h \left(\frac{\frac{\mathrm{d}h}{\mathrm{d}r}}{r} + \frac{\ell(\ell+1)}{r^2} - \mu \right) \tag{1.6.3}$$

and

$$h = \frac{\Delta}{r^2} = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2 + \frac{Q^2}{r^2}. \tag{1.6.4}$$

Here, consider $r(r_*)$ as a function r_* and recall that $'$ denotes the derivative with respect to r_*. The presence of the mass and the cosmological constant leads to a modification of the potential $V_{\ell,\Lambda}$.

Nevertheless, the potential $V_{\ell,\Lambda}$ still decays exponentially at $\pm\infty$ and we can define asymptotic states $u_1^{(\Lambda)}, u_2^{(\Lambda)}$, and $v_1^{(\Lambda)}, v_2^{(\Lambda)}$ for $\omega \neq 0$ and $\tilde{u}_1^{(\Lambda)}$, $\tilde{u}_2^{(\Lambda)}$, and $\tilde{v}_1^{(\Lambda)}$, $\tilde{v}_2^{(\Lambda)}$ for $\omega = 0$ just as in the case where $\Lambda = \mu = 0$ in Definition 1.2.3. In particular, $\tilde{u}_1^{(\Lambda)}$ and $\tilde{v}_1^{(\Lambda)}$ remain bounded as $r_* \to -\infty$ and $r_* \to +\infty$, respectively. In contrast to that, $\tilde{u}_2^{(\Lambda)}$ and $\tilde{v}_2^{(\Lambda)}$ grow linearly in their respective limits. The next proposition states that in the presence of a cosmological constant, solutions to (1.6.1) in the case $\omega = 0$ which are bounded at $r_* = -\infty$ do not need to be bounded at $r_* = +\infty$.

Proposition 1.6.1. *Fix $\nu \in \mathbb{R}$ (e.g. $\nu = \frac{3}{2}$ for the conformal invariant mass) and fix subextremal black hole parameters $(M, Q, \Lambda) \in \mathcal{P}_{\mathrm{se}}$. Assume moreover that $(M, Q, \Lambda) \notin D(\nu)$, where $D(\nu) \subset \mathcal{P}_{\mathrm{se}}$ is defined in the proof and has measure zero. Then, there exists an $\ell_0 = \ell_0(\nu) \in \mathbb{N}_0$ such that we have*

$$\tilde{u}_1^{(\Lambda)} = A(\ell_0, \Lambda, M, Q)\tilde{v}_1^{(\Lambda)} + B(\ell_0, \Lambda, M, Q)\tilde{v}_2^{(\Lambda)}, \tag{1.6.5}$$

with $B = B(\ell_0, \Lambda, M, Q) \neq 0$. Moreover, $\mathcal{P}_{\mathrm{se}}^{\Lambda=0} \subset D(\nu)$ for all $\nu \in \mathbb{R}$ and there exists an open subset U with $\mathcal{P}_{\mathrm{se}}^{\Lambda=0} \subset U \subset \mathcal{P}_{\mathrm{se}}$ and $\mathcal{P}_{\mathrm{se}} \cap U = \mathcal{P}_{\mathrm{se}}^{\Lambda=0}$.

Proof. Let $\nu \in \mathbb{R}$ be fixed. In the case $\Lambda = 0$ we can represent $\tilde{u}_1$ with Legendre polynomials and in particular we have that $B(\ell, \Lambda = 0, M, Q) = 0$ for all ℓ and $0 < |Q| < M$. Note that we can write B as

$$B(\Lambda, \ell, M, Q) = \frac{\mathfrak{W}(\tilde{v}_2^{(\Lambda)}, \tilde{u}_1^{(\Lambda)})}{\mathfrak{W}(\tilde{v}_1^{(\Lambda)}, \tilde{v}_2^{(\Lambda)})} = \mathfrak{W}(\tilde{v}_2^{(\Lambda)}, \tilde{u}_1^{(\Lambda)}) \tag{1.6.6}$$

for all Λ such that $(M, Q, \Lambda) \in \mathcal{P}_{\mathrm{se}}$.

***Step 1:* $\mathcal{P}_{\mathrm{se}} \subset \mathbb{R}^3$ *is open and has two connected components where either* $Q > 0$ *or* $Q < 0$.** For the sake of completeness we will give a proof of *Step 1*, although this seems a quite well-known fact. Note that $\mathcal{P}_{\mathrm{se}} = \mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cup \mathcal{P}_{\mathrm{se}}^{\Lambda<0} \cup \mathcal{P}_{\mathrm{se}}^{\Lambda=0}$ is open which can be inferred from its definition.

For the second statement, first note that $\{Q = 0\} \cap \mathcal{P}_{\mathrm{se}} = \emptyset$. We will now show that $\{Q > 0\} \cap \mathcal{P}_{\text{se}}$ is connected. In Proposition 1.8.3 in the appendix we show that $\mathcal{P}_{\text{se}}^{\Lambda>0} \cap \{Q > 0\}$ and $\mathcal{P}_{\text{se}}^{\Lambda<0} \cap \{Q > 0\}$ are path-connected. To conclude, note that for every $(M_0, Q_0, \Lambda_0 = 0) \in \mathcal{P}_{\text{se}}^{\Lambda=0}$, there exist paths from (M_0, Q_0, Λ_0) to both $(M_0, Q_0, \epsilon) \in \mathcal{P}_{\text{se}}^{\Lambda>0}$ and $(M_0, Q_0, -\epsilon) \in \mathcal{P}_{\text{se}}^{\Lambda<0}$ for some $\epsilon(M_0, Q_0) > 0$. Together with the fact that $\mathcal{P}_{\text{se}}^{\Lambda=0} \cap \{Q > 0\}$ is path-connected, this shows that $\{Q > 0\} \cap \mathcal{P}_{\text{se}}$ is path-connected and similarly that $\{Q < 0\} \cap \mathcal{P}_{\text{se}}$ is path-connected which proves the claim.

Step 2: $\mathcal{P}_{\text{se}} \ni (M, Q, \Lambda) \mapsto B(\ell, \Lambda, M, Q)$ ***is real analytic.*** To show *Step 2* we first express (1.6.5) in r coordinates. Note that for $(M, Q, \Lambda) \in \mathcal{P}_{\text{se}}$ equation (1.6.5) is equivalent to

$$\frac{r_+}{r_-}(-1)^\ell P_\ell^{(\Lambda)}(x(r)) = A(\ell, \Lambda)\tilde{P}_\ell^{(\Lambda)}(x(r)) + B(\ell, \Lambda)\tilde{Q}_\ell^{(\Lambda)}(x(r)), \tag{1.6.7}$$

where $r \in (r_-, r_+)$,

$$x(r) := -\frac{2r}{r_+ - r_-} + \frac{r_+ + r_-}{r_+ - r_-}, \tag{1.6.8}$$

$$r(x) = -\frac{r_+ - r_-}{2}x + \frac{r_+ + r_-}{2} \tag{1.6.9}$$

and $0 < r_- < r_+$. Now, note that $\mathcal{P}_{\text{se}} \ni (M, Q, \Lambda) \mapsto r_-$ and $\mathcal{P}_{\text{se}} \ni (M, Q, \Lambda) \mapsto r_+$ are real analytic. Moreover, we can write $\Delta = (r - r_-)(r - r_+)p(r)$ for a second order polynomial $p(r)$, where $\mathcal{P}_{\text{se}} \ni \Lambda \mapsto p(r)$ is also real analytic for fixed r. Now, $P_\ell^{(\Lambda)}$, $\tilde{P}_\ell^{(\Lambda)}$ and $\tilde{Q}_\ell^{(\Lambda)}$ appearing in (1.6.7) are defined as the unique solutions of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1 - x^2)p(r(x))\frac{\mathrm{d}R}{\mathrm{d}x}\right) + \ell(\ell+1)R - r(x)^2\nu\Lambda R = 0 \tag{1.6.10}$$

satisfying

$$P_\ell^{(\Lambda)} = (-1)^\ell + O_\ell(1+x) \text{ as } x \to -1, \tag{1.6.11}$$

$$\frac{\mathrm{d}P_\ell^{(\Lambda)}}{\mathrm{d}x} = O_\ell(1) \text{ as } x \to -1, \tag{1.6.12}$$

$$\tilde{P}_\ell^{(\Lambda)} = 1 + O_\ell(1-x) \text{ as } x \to 1, \tag{1.6.13}$$

$$\frac{\mathrm{d}\tilde{P}_\ell^{(\Lambda)}}{\mathrm{d}x} = O_\ell(1) \text{ as } x \to 1, \tag{1.6.14}$$

$$\tilde{Q}_\ell^{(\Lambda)} = -\frac{1}{2}\log(1-x) + O_\ell(1) \text{ as } x \to 1, \tag{1.6.15}$$

$$\frac{\mathrm{d}\tilde{Q}_\ell^{(\Lambda)}}{\mathrm{d}x} = \frac{1}{2(1-x)} + O_\ell((1-x)\log(1-x)) \text{ as } x \to 1. \tag{1.6.16}$$

Note that (1.6.10) depends real analytically on $(M, Q, \Lambda) \in \mathcal{P}_{\mathrm{se}}$ such that $P_\ell^{(\Lambda)}(x)$, $\tilde{P}_\ell^{(\Lambda)}(x)$, $\tilde{Q}_\ell^{(\Lambda)}(x)$ are real analytic functions of $(M, Q, \Lambda) \in \mathcal{P}_{\mathrm{se}}$ for $x \in (-1, 1)$. Hence, $\mathcal{P}_{\mathrm{se}} \ni (M, Q, \Lambda) \mapsto B(\ell, \Lambda, M, Q)$ is real analytic.

Step 3: $B(\ell_0(\nu), \Lambda, M, Q)$ ***only vanishes on a set*** $D(\nu) \subset \mathcal{P}_{\mathrm{se}}$ ***of measure zero.*** The claim follows from

$$\frac{\partial B(\ell, \Lambda, M_0, Q_0)}{\partial \Lambda}\bigg|_{\Lambda=0} \neq 0 \tag{1.6.17}$$

for some $0 < |Q_0| < M_0$. Throughout *Step 2* we fix $0 < |Q_0| < M_0$ and avoid writing their explicit dependence. First note that that for $\Lambda = 0$ we obtain the Legendre functions of first and second kind, i.e. $P_\ell^{(0)} = \tilde{P}_\ell^{(0)} = P_\ell$ and $\tilde{Q}_\ell^{(0)} = Q_\ell$ and $B(0, \ell) = 0$. Now, define coefficients $\tilde{A}(\ell, \Lambda)$ and $\tilde{B}(\ell, \Lambda)$ to satisfy

$$P_\ell^{(\Lambda)} = \tilde{A}(\ell, \Lambda)\tilde{P}_\ell^{(\Lambda)} + \tilde{B}(\ell, \Lambda)\tilde{Q}_\ell^{(\Lambda)}, \tag{1.6.18}$$

and note that (1.6.17) is equivalent (use that $B(\ell, 0) = \tilde{B}(\ell, 0) = 0$) to

$$\frac{\partial \tilde{B}(\ell, \Lambda)}{\partial \Lambda}\bigg|_{\Lambda=0} \neq 0. \tag{1.6.19}$$

By construction, $P_\ell^{(\Lambda)}$ solves (1.6.10). Multiplying

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)p(r(x))\frac{\mathrm{d}P_\ell^{(\Lambda)}}{\mathrm{d}x}\right) + \ell(\ell+1)P_\ell^{(\Lambda)} - r(x)^2\nu\Lambda P_\ell^{(\Lambda)} = 0 \tag{1.6.20}$$

by $P_\ell^{(0)}$ and integrating from $x=-1$ to $x=1$ yields

$$0=\int_{-1}^{1} P_\ell^{(0)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)p(r(x))\frac{\mathrm{d}P_\ell^{(\Lambda)}}{\mathrm{d}x}\right)+\ell(\ell+1)P_\ell^{(\Lambda)}-r(x)^2\nu\Lambda P\ell^{(\Lambda)}\right)\mathrm{d}x. \tag{1.6.21}$$

Using the expansion (1.6.18) and the properties (1.6.11) – (1.6.16) at the end points $x=-1$ and $x=1$ gives after an integration by parts

$$\begin{aligned}0=&\int_{-1}^{1} P_\ell^{(\Lambda)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)p(r(x))\frac{\mathrm{d}P_\ell^{(0)}}{\mathrm{d}x}\right)+\ell(\ell+1)P_\ell^{(0)}-r(x)^2\nu\Lambda P_\ell^{(0)}\right)\mathrm{d}x\\&+p(r(1))\tilde{B}(\ell,\Lambda).\end{aligned} \tag{1.6.22}$$

Now, taking $\partial_\Lambda\big|_{\Lambda=0}$ and integrating by parts once again yields

$$\begin{aligned}&p(r(1))\partial_\Lambda\big|_{\Lambda=0}\tilde{B}(\ell,\Lambda)\\&=\int_{-1}^{1}\left[\left|\frac{\mathrm{d}P_\ell^{(0)}}{\mathrm{d}x}\right|^2(1-x^2)\partial_\Lambda\big|_{\Lambda=0}(p(r(x)))+\left|P_\ell^{(0)}\right|^2\partial_\Lambda\big|_{\Lambda=0}(\nu r(x)^2\Lambda)\right]\mathrm{d}x\\&=\int_{-1}^{1}\left[\left|\frac{\mathrm{d}P_\ell^{(0)}}{\mathrm{d}x}\right|^2(1-x^2)\partial_\Lambda\big|_{\Lambda=0}(p(r(x)))+\nu\left|P_\ell^{(0)}\right|^2 r(x)^2|_{\Lambda=0}\right]\mathrm{d}x.\end{aligned} \tag{1.6.23}$$

Recall that we are in the subextremal range which guarantees that $p(r(1))\neq 0$. We will now distinguish two cases, $\nu=0$ and $\nu\neq 0$.

***Part I:* $\nu=0$.** In the case $\nu=0$ we have

$$p(r(1))\partial_\Lambda|_{\Lambda=0}\tilde{B}(\ell,\Lambda)=\partial_\Lambda|_{\Lambda=0}\int_{-1}^{1}\left|\frac{\mathrm{d}P_\ell}{\mathrm{d}x}\right|^2(1-x^2)p(r(x))\mathrm{d}x \tag{1.6.24}$$

In the case $\nu = 0$ we will choose $\ell = 1$ such that

$$
\begin{aligned}
&p(r(1))\partial_\Lambda|_{\Lambda=0}\tilde{B}(1,\Lambda)\\
&\qquad=\partial_\Lambda|_{\Lambda=0}\int_{-1}^{1}(1-x^2)p(r(x))\mathrm{d}x\\
&\qquad=\partial_\Lambda|_{\Lambda=0}\int_{-1}^{1}-\Delta(r(x))\frac{4}{(r_+-r_-)^2}\mathrm{d}x\\
&\qquad=\partial_\Lambda|_{\Lambda=0}\left(\frac{-8}{(r_+-r_-)^3}\int_{r_-}^{r_+}\Delta(r)\mathrm{d}r\right)\\
&\qquad=-8\,\partial_\Lambda|_{\Lambda=0}\left(\frac{\frac{r_+^3-r_-^3}{3}-M_0(r_+^2-r_-^2)+Q_0^2(r_+-r_-)-\frac{1}{15}\Lambda(r_+^5-r_-^5)}{(r_+-r_-)^3}\right)\\
&\qquad=\frac{8(r_+^5-r_-^5)}{15(r_+-r_-)^3}\Big|_{\Lambda=0}\\
&\qquad\quad+8\frac{\frac{r_+^3-r_-^3}{3}-M_0(r_+^2-r_-^2)+Q_0^2(r_+-r_-)}{(r_+-r_-)^5}(r_+^4+r_-^4)\Big|_{\Lambda=0}\\
&\qquad\quad-\frac{8}{3}\frac{r_+^6+r_-^6-2M_0(r_+^5+r_-^5)+Q_0^2(r_+^4+r_-^4)}{(r_+-r_-)^4}\Big|_{\Lambda=0}\\
&\qquad=\frac{-8}{15}\left(3r_+^3+3r_-^2+4r_+r_-\right)\Big|_{\Lambda=0}\\
&\qquad=\frac{-8}{15}\left(6M_0^2-Q_0^2\right)<-24M_0^2.
\end{aligned}
$$

The last step is a long but direct computation using that $\Delta = r^2 - 2M_0 r + Q_0^2 - \frac{\Lambda}{3}r^4$ and $r_\pm|_{\Lambda=0} = M_0 \pm \sqrt{M_0^2 - Q_0^2}$, i.e. $Q_0^2 = r_+r_-|_{\Lambda=0}$ and $2M_0 = r_+|_{\Lambda=0} + r_-|_{\Lambda=0}$. Moreover, in view of the inverse function theorem we have

$$\partial_\Lambda|_{\Lambda=0}r_+ = \frac{r_+^4}{3(r_+-r_-)}\Big|_{\Lambda=0} \tag{1.6.25}$$

and

$$\partial_\Lambda|_{\Lambda=0}r_- = -\frac{r_-^4}{3(r_+-r_-)}\Big|_{\Lambda=0}. \tag{1.6.26}$$

Part II: $\nu \neq 0$. In this case we choose $\ell = 0$ such that $P_\ell^{(0)} = 1$ and $\frac{\mathrm{d}P_\ell^{(0)}}{\mathrm{d}x} = 0$. Hence,

$$
\begin{aligned}
p(r(1))\partial_\Lambda|_{\Lambda=0}\tilde{B}(\ell,\Lambda) &= \partial_\Lambda|_{\Lambda=0}\int_{-1}^{1} r(x)^2\nu\Lambda\mathrm{d}x \\
&= \nu\partial_\Lambda|_{\Lambda=0}\int_{-1}^{1}\left(-\frac{r_+ - r_-}{2}x + \frac{r_+ + r_-}{2}\right)^2\Lambda\mathrm{d}x && (1.6.27)\\
&= \nu\left(\frac{1}{6}(r_+ - r_-)^2 + \frac{1}{2}(r_+ + r_-)^2\right)\Big|_{\Lambda=0} \neq 0. && (1.6.28)
\end{aligned}
$$

This shows that $\mathcal{P}_{\mathrm{se}} \ni (M,Q,\Lambda) \mapsto B(\ell_0(\nu), M, Q, \Lambda)$ is a non-trivial real analytic function which zero set $D(\nu)$ has zero measure. The proof also shows that $\mathcal{P}_{\mathrm{se}}^{\Lambda=0} \subset D(\nu)$ and that there exists an open set $U \subset \mathcal{P}_{\mathrm{se}}$ with $\mathcal{P}_{\mathrm{se}}^{\Lambda=0} \subset U$ and $D(\nu) \cap U = \mathcal{P}_{\mathrm{se}}^{\Lambda=0}$. $\square$

Proposition 1.6.2. *Let $\nu \in \mathbb{R}$ be fixed. Let $\omega \neq 0$, $(M,Q,\Lambda) \in \mathcal{P}_{\mathrm{se}}$, and $\ell \in \mathbb{N}_0$. Then, define completely analogously to Definition 1.2.4 transmission and reflection coefficients $\mathfrak{T}(\omega,\ell,\Lambda)$ and $\mathfrak{R}(\omega,\ell,\Lambda)$ as the unique coefficients such that*

$$u_1^{(\Lambda)} = \mathfrak{T}(\omega,\ell,\Lambda)v_1^{(\Lambda)} + \mathfrak{R}(\omega,\ell,\Lambda)v_2^{(\Lambda)} \tag{1.6.29}$$

holds.

Now, assume further that $(M,Q,\Lambda) \in \mathcal{P}_{\mathrm{se}} \setminus D(\nu)$, where $D(\nu)$ is defined in Proposition 1.6.1. Then, there exists an $\ell_0 = \ell_0(\nu)$ such that

$$\lim_{\omega\to 0}|\mathfrak{R}(\omega,\ell_0)| = \lim_{\omega\to 0}|\mathfrak{T}(\omega,\ell_0)| = +\infty. \tag{1.6.30}$$

This shows that $\mathfrak{T}$ and $\mathfrak{R}$ have a simple pole at $\omega = 0$.

Proof. Fix $\ell_0 = \ell_0(\nu)$ from Proposition 1.6.1 and $(M,Q,\Lambda) \in \mathcal{P}_{\mathrm{se}}$ such that

$$B(\ell_0,\Lambda,M,Q) \neq 0.$$

Now, note that the o.d.e. implies that $\frac{\mathrm{d}}{\mathrm{d}r_*}\mathrm{Im}(\bar{u}u') = 0$ which shows that $1 = |\mathfrak{T}|^2 - |\mathfrak{R}|^2$. In particular, either $|\mathfrak{T}|$ and $|\mathfrak{R}|$ are both bounded or both unbounded as $\omega \to 0$. Also note that as $\omega \to 0$, we have that $u_1^{(\Lambda)} \to \tilde{u}_1^{(\Lambda)}$ pointwise.

Now, assume for a contradiction that there exists a sequence $\omega_n \to 0$ such that $|\mathfrak{T}(\omega_n)|$

and $|\Re(\omega_n)|$ remain bounded. Thus,

$$\begin{aligned}\limsup_{\omega_n\to 0}\|u_1^{(\Lambda)}\|_{L^\infty(\mathbb{R})} \leq &\limsup_{\omega_n\to 0}\|u_1^{(\Lambda)}\|_{L^\infty((-\infty,0))}\\ &+\limsup_{\omega_n\to 0}\|\Re v_1^{(\Lambda)}+\Im v_2^{(\Lambda)}\|_{L^\infty((0,\infty))}\leq C \end{aligned}\tag{1.6.31}$$

for some constant $C>0$. Now, using that $B(\ell_0,\Lambda,M,Q)\neq 0$ in Proposition 1.6.1, we can choose a $r_0^*\in\mathbb{R}$ such that $|\tilde{u}_1^{(\Lambda)}(r_0^*)|>C$ which contradicts the fact that $u_1^{(\Lambda)}\to\tilde{u}_1^{(\Lambda)}$ pointwise as $\omega_n\to 0$. $\square$

Finally, this allows us to prove Theorem 1.6 which we restate in the following for the convenience of the reader.

Theorem 1.6. *Let $\nu\in\mathbb{R}$ be a fixed Klein–Gordon mass parameter. (In particular, we may choose $\nu=\frac{3}{2}$ to cover the conformal invariant case or $\nu=0$ for the wave equation* (1.1.1)*.) Consider the interior of a subextremal (anti-) de Sitter–Reissner–Nordström black hole with generic parameters $(M,Q,\Lambda)\in\mathcal{P}_{\mathrm{se}}\setminus D(\nu)$. (Here, $D(\nu)\subset\mathcal{P}_{\mathrm{se}}$ is a set with measure zero defined in Proposition 1.6.1 (see Section 1.6). Moreover $D(\nu)$ satisfies $\mathcal{P}_{\mathrm{se}}^{\Lambda=0}\subset D(\nu)$ and $U\cap D(\nu)=\mathcal{P}_{\mathrm{se}}^{\Lambda=0}$ for some open set $U\subset\mathcal{P}_{\mathrm{se}}$.)*

Then, there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of purely ingoing and compactly supported data on $\mathcal{H}_A$ with

$$\|\Psi_n\|_{\mathcal{E}^T_{\mathcal{H}}}=1 \text{ for all } n \tag{1.3.41}$$

such that the solution ψ_n to the Klein–Gordon equation with mass $\mu=\nu\Lambda$

$$\Box_{g_{M,Q,\Lambda}}\psi-\mu\psi=0 \tag{1.3.42}$$

arising from Ψ_n has unbounded T energy at the Cauchy horizon

$$\|\psi_n\restriction_{\mathcal{CH}}\|_{\mathcal{E}^T_{\mathcal{CH}}}\to\infty \text{ as } n\to\infty. \tag{1.3.43}$$

Proof. Fix $\ell_0=\ell_0(\nu)$ from Proposition 1.6.2 such that the reflection and transmission coefficients blow up as $\omega\to 0$. Define a sequence of compactly supported functions Ψ_n on $\mathcal{H}_A$ by $\Psi_n(v,\theta,\varphi)=f_n(v)Y_{0\ell}(\theta,\varphi)$, such that $f_n\in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}}\omega^2|\hat{f}_n(\omega)|^2\mathrm{d}\omega=1 \text{ and } \int_{-\frac{1}{n}}^{\frac{1}{n}}\omega^2|\hat{f}_n(\omega)|^2\mathrm{d}\omega\geq\epsilon\int_{\mathbb{R}}\omega^2|\hat{f}_n(\omega)|^2\mathrm{d}\omega=\epsilon \tag{1.6.32}$$

for some $\epsilon > 0$.[6] Imposing vanishing data on $\mathcal{H}_B$, this gives rise to a unique smooth solutions ψ_n up to but excluding the Cauchy horizon. Arguments completely analogous to those given in the proof of Proposition 1.5.1 show that

$$\|\psi_n \restriction_{\mathcal{CH}} \|^2_{\mathcal{E}^T_{\mathcal{CH}}} = \frac{r_+^2}{r_-^2} \int_{\mathbb{R}} \omega^2 (|\mathfrak{R}(\omega,\ell)|^2 + |\mathfrak{T}(\omega,\ell)|^2) |\hat{f}_n(\omega)|^2 \mathrm{d}\omega. \tag{1.6.34}$$

Thus,

$$\begin{aligned} \|\psi_n \restriction_{\mathcal{CH}} \|^2_{\mathcal{E}^T_{\mathcal{CH}}} &\geq \frac{r_+^2}{r_-^2} \int_{-\frac{1}{n}}^{\frac{1}{n}} \omega^2 (|\mathfrak{R}(\omega,\ell)|^2 + |\mathfrak{T}(\omega,\ell)|^2) |\hat{f}_n(\omega)|^2 \mathrm{d}\omega \\ &\geq \epsilon \frac{r_+^2}{r_-^2} \inf_{\omega \in [-\frac{1}{n}, \frac{1}{n}]} \left(|\mathfrak{R}|^2 + |\mathfrak{T}|^2 \right). \end{aligned} \tag{1.6.35}$$

Since $|\mathfrak{R}|, |\mathfrak{T}| \to \infty$ as $\omega \to 0$, also $\inf_{\omega \in [\frac{1}{2n}, \frac{1}{n}]} |\mathfrak{R}| \to \infty$ and $\inf_{\omega \in [\frac{1}{2n}, \frac{1}{n}]} |\mathfrak{T}| \to \infty$ as $n \to \infty$. Thus, as $n \to \infty$, we have

$$\|\psi_n \restriction_{\mathcal{CH}} \|^2_{\mathcal{E}^T_{\mathcal{CH}}} \to \infty. \tag{1.6.36}$$

□

1.7 Proof of Theorem 1.7: Breakdown of T energy scattering for the Klein–Gordon equation

In this last section we will prove that for a generic set of Klein–Gordon masses, there does not exist a T scattering theory on the interior of Reissner–Nordström for the Klein–Gordon equation. For the convenience of the reader, we have restated Theorem 1.7.

Theorem 1.7. *Consider the interior of a subextremal Reissner–Nordström black hole. There exists a discrete set $\tilde{D}(M,Q) \subset \mathbb{R}$ with $0 \in \tilde{D}$ such that the following holds true. For any $\mu \in \mathbb{R} \setminus \tilde{D}$ there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of purely ingoing and compactly*

[6] Such a function can be constructed by setting $f_n(v) := \frac{c}{\sqrt{n}} f(\frac{v}{n})$ for smooth $f \colon \mathbb{R} \to [0,1]$ with $\mathrm{supp}(f) \subset [-2,2]$, $f \restriction_{[-1,1]} = 1$ and some normalization constant $c > 0$. Indeed,

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} \omega^2 |\hat{f}_n(\omega)|^2 \mathrm{d}\omega = \int_{-\frac{1}{n}}^{\frac{1}{n}} \omega^2 |\sqrt{n} \hat{f}(n\omega)|^2 \mathrm{d}\omega = \int_{-1}^{1} \omega^2 |\hat{f}(\omega)|^2 =: \epsilon > 0 \tag{1.6.33}$$

in view of $\hat{f}(0) = \int_{\mathbb{R}} f(v) \mathrm{d}v > 0$.

supported data on $\mathcal{H}_A$ with

$$\|\Psi_n\|_{\mathcal{E}^T_{\mathcal{H}}} = 1 \text{ for all } n \tag{1.3.44}$$

such that the solution ψ_n to the Klein–Gordon equation with mass μ

$$\Box_{g_{M,Q,\Lambda}}\psi - \mu\psi = 0 \tag{1.3.45}$$

arising from Ψ_n has unbounded T energy at the Cauchy horizon

$$\|\psi_n \restriction_{\mathcal{CH}} \|_{\mathcal{E}^T_{\mathcal{CH}}} \to \infty \text{ as } n \to \infty. \tag{1.3.46}$$

Proof. The proof of this statement is easier than and similar to the proof of Theorem 1.6 and the proofs of the propositions leading up to it. More precisely, similar to Section 1.6 we define asymptotic states $\tilde{u}_1^{(\mu)}$, $\tilde{v}_1^{(\mu)}$ and $\tilde{v}_2^{(\mu)}$ and define $A(\ell,\mu)$ and $B(\ell,\mu)$ by $\tilde{u}_1^{(\mu)} = A(\ell,\mu)\tilde{v}_1^{(\mu)} + B(\ell,\mu)\tilde{v}_2^{(\mu)}$. As in Section 1.6, $\mathbb{R} \ni \mu \mapsto B(\ell,\mu)$ is real analytic and from the o.d.e. $-u'' + V_{\ell,\mu}u = 0$ we obtain

$$\left.\frac{\partial B(\ell,\mu)}{\partial \mu}\right|_{\mu=0} = \int_{-\infty}^{\infty} \left.\frac{\partial V_{\ell,\mu}}{\partial \mu}\right|_{\mu=0} \tilde{u}_1^2 \mathrm{d}r_*, \tag{1.7.1}$$

where

$$V_{\ell,\mu} = h\left(\frac{hh'}{r} + \frac{\ell(\ell+1)}{r^2} - \mu\right) = h\left(\frac{\frac{\mathrm{d}h}{\mathrm{d}r}}{r} + \frac{\ell(\ell+1)}{r^2} - \mu\right) \tag{1.7.2}$$

and

$$h = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{1.7.3}$$

as in (1.2.5). Now, note that

$$\left.\frac{\partial V_{\ell,\mu}}{\partial \mu}\right|_{\mu=0} = -h > 0 \tag{1.7.4}$$

which is manifestly positive from which we can infer, by analyticity, that $B(\ell,\mu) \neq 0$ for all $\mu \in \mathbb{R} \setminus \tilde{D}$, where $\tilde{D} = \tilde{D}(M,Q) \subset \mathbb{R}$ is a discrete set. This proves the analogous statements to Proposition 1.6.1 and Proposition 1.6.2. The claim of Theorem 1.7 follows now as in the proof of Theorem 1.6. $\square$

1.8 Appendix

Energy estimates in the interior.

Lemma 1.8.1. *Let $\Psi \in C_c^\infty(\mathcal{H})$ and denote by ψ its evolution in the interior. Then, the non-degenerate N energy of Ψ decays exponentially towards i_+ on every $\{r = r_0\}$ hypersurface for $r_{red} < r_0 < r_+$. Here, r_{red} only depends on the black hole parameters.*

Proof. This argument is very similar to [51, Proposition 4.2]. We only prove it for the right component of i^+ and clearly only have to look at a neighborhood of i^+. First, recall the existence of the celebrated redshift vector field N satisfying $K^N[\psi] \geq bJ^N_\mu[\psi]n^\mu_v$ for $r_+ \geq r \geq r_{red}$, where n_v is the normal to a $v = const.$ hypersurface.[7]

We set

$$E(v_0) = \int_{v=v_0, r_{red}\leq r\leq r_+} J^N_\mu n^\mu_v \mathrm{dvol}, \tag{1.8.1}$$

and apply the energy identity with the redshift vector field N in the region $\mathcal{R} = \{r \in [r_{red}, r_+], v \in [v_0, v_1]\}$, where v_0 is large enough such that $v_0 > \sup\operatorname{supp}(\Psi)$. This gives in view of the coarea formula that

$$E(v_1) - E(v_0) + \tilde{b}\int_{v_0}^{v_1} E(v)\mathrm{d}v \leq 0 \tag{1.8.2}$$

for every $v_1 \geq v_0 > \sup\operatorname{supp}(\Psi)$. Inequality (1.8.2), smoothness of $v \mapsto E(v)$ and a further application of the energy identity in the region $\{v \geq v_0, r_+ \geq r \geq r_{red}\}$ finally shows

$$\int_{v\geq v_0, r=r_{red}} J^N_\mu n^\mu_r \mathrm{dvol} \leq C\exp(-\tilde{b}v_0), \tag{1.8.3}$$

where C is a constant depending on Ψ. This concludes the proof. $\square$

Remark 1.8.1. *By cutting off smoothly we can clearly approximate Ψ on a $\{r = const.\}$ hypersurface with compactly supported functions for any fixed $r \in (r_{red}, r_+)$.*

Lemma 1.8.2. *Let ψ be a smooth solution of the wave equation on $\mathcal{M}_{\mathrm{RN}}$ such that its restriction to the event horizon has compact support and let $r_0 \in (r_{red}, r_+)$. Then,*

$$\int_{\mathcal{H}} J^T_\mu n^\mu \mathrm{dvol} \lesssim \int_{\{r=r_0\}} J^N_\mu n^\mu \mathrm{dvol}. \tag{1.8.4}$$

[7] The normal is fixed by making a choice of a volume form on the null hypersurface

Proof. We shall use the vector field $S = r^{-2}\partial_{r_*}$. By potentially making r_{red} larger, we can assure that the bulk term $K^S := \nabla^\mu J^S_\mu$ of the vector field S has a fixed negative sign in $r_0 \in (r_{red}, r_+)$. This current is analogous to the current introduced in [51, par. 4.1.3.2]. Moreover, applying the energy identity in the region $\mathcal{R} = \{r_0 \le r \le r_+\}$ and noting that $J^N[\psi]_\mu n^\mu|_{r=r_0} \sim J^S[\psi]_\mu n^\mu|_{r=r_0}$ as well as $J^T[\psi]_\mu n^\mu|_{\mathcal{H}} \sim J^S[\psi]_\mu n^\mu|_{\mathcal{H}}$ yields

$$\int_{\{r=r_0\}} J^N[\psi]_\mu n^\mu \mathrm{dvol} + \int_{\mathcal{R}} K^S \mathrm{dvol} \gtrsim \int_{\mathcal{H}} J^T_\mu n^\mu \mathrm{dvol}. \tag{1.8.5}$$

This concludes the proof. $\square$

Analytic properties of the potential and the scattering coefficients. In the following we would like to summarize analytic properties of the potential $V_\ell(r)$ and u_1,u_2, v_1 and v_2 as functions of ω. This is similar to parts of [14].

First, however we will show the the exponential decay of the potential V_ℓ as $r_* \to \pm\infty$.

Lemma 1.8.3. *We have*

$$|\Delta(r_*)| \lesssim e^{2k_+ r_*} \text{ for } r_* \le 0 \tag{1.8.6}$$

and

$$|\Delta(r_*)| \lesssim e^{2k_- r_*} \text{ for } r_* \ge 0. \tag{1.8.7}$$

Moreover, we have

$$|V_\ell(r_*)|, |V_\ell'(r_*)|, |V_\ell''(r_*)| \lesssim (1 + \ell(\ell+1))e^{2k_+ r_*} \text{ for } r_* \le 0 \tag{1.8.8}$$

and

$$|V_\ell(r_*)|, |V_\ell'(r_*)|, |V_\ell''(r_*)| \lesssim (1 + \ell(\ell+1))e^{2k_- r_*} \text{ for } r_* \ge 0. \tag{1.8.9}$$

Proof. Note that

$$r_+ - r = \tilde{C}\,(r - r_-)^{\frac{k_-}{k_+}}\, e^{-2k_+ r} e^{2k_+ r_*} \tag{1.8.10}$$

for a constant $\tilde{C}$ only depending on the black hole parameters. Thus, for $r_* \le 0$, we have

$$r_+ - r(r_*) = f(r_*)e^{2k_+ r_*} \tag{1.8.11}$$

for a smooth function $f(r_*)$, which is uniformly bounded below and above for $r_* \leq 0$. Moreover, we have $f'(r_*), f''(r_*) \to 0$ exponentially fast as $r_* \to -\infty$. The estimates (1.8.8) and (1.8.9) are now straightforward applications of the chain rule and the fact that $\frac{\mathrm{d}r}{\mathrm{d}r_*} = \frac{\Delta}{r^2}$ and $\Delta = (r - r_-)(r - r_+)$. □

Proposition 1.8.1. *The potential V_ℓ can be expanded as*

$$V_\ell(r_*) = \sum_{m \in \mathbb{N}} C_m e^{2\kappa_+ m r_*}, \tag{1.8.12}$$

where $|C_m| \lesssim_\ell e^{-\sigma m}$ for a $\sigma > 0$.

Proof. Define the variable

$$z(r) := e^{2\kappa_+ r_*(r)} = C e^{2\kappa_+ r}(r_+ - r)(r - r_-)^{\frac{\kappa_+}{\kappa_-}}, \tag{1.8.13}$$

where $C > 0$ is such that $z(\frac{r_+ + r_-}{2}) = 1$. From the inverse function theorem it follows that $V_\ell(z) = V_\ell(r(z))$ can be analytically continued in a neighborhood of $z = 0$ and thus, there exists a Taylor expansion around $z = 0$ such that

$$V_\ell(z) = \sum_{n=1}^{\infty} C_m z^m. \tag{1.8.14}$$

Hence,

$$V_\ell(r_*) = \sum_{n=1}^{\infty} C_m e^{2\kappa_+ m r_*}, \tag{1.8.15}$$

where

$$\begin{aligned} C_1 = \left.\frac{\mathrm{d}V_\ell}{\mathrm{d}z}\right|_{z=0} &= \left.\frac{\mathrm{d}V_\ell}{\mathrm{d}r}\right|_{r=r_+} \left.\frac{\mathrm{d}r}{\mathrm{d}z}\right|_{z=0} \\ &= \frac{r_+ - r_-}{r_+^4}\left(r_+^2(r_+ - 3r_-) + \ell(\ell+1)\right). \end{aligned} \tag{1.8.16}$$

Note that the coefficients C_m decay exponentially fast in m. To see this, remark that we can re-define $\tilde{r}_* := r_* - \rho$ for some constant $\rho > 0$. Similarly to (1.8.15), we expand V_ℓ as

$$V_\ell = \sum_{m=1}^{\infty} D_m e^{2\kappa_+ m \tilde{r}_*} \tag{1.8.17}$$

which shows $C_m = D_m e^{-2\kappa_+ m\rho}$. By analyticity we have $|D_m| \leq |\tilde{C}|^{m+1}$ for some $\tilde{C} > 0$

and thus,

$$|C_m| \lesssim_\ell e^{-\sigma m} \tag{1.8.18}$$

for a fixed $\sigma > 0$. $\square$

Proposition 1.8.2. *Let $\ell \in \mathbb{N}$ be fixed. Then,*

$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |\mathfrak{R}(\omega,\ell)| + |\mathfrak{T}(\omega,\ell)| \lesssim_\ell 1. \tag{1.8.19}$$

Moreover, $\mathfrak{T}(\omega,\ell)$ has a pole of order one at $\omega = i\kappa_+$ given that $\ell(\ell+1) \neq r_+^2(r_+ - 3r_-)$.

Proof. Recall, that u_1 is the unique solution to

$$u_1(r_*) = e^{i\omega r_*} + \int_{-\infty}^{r_*} \frac{\sin(\omega(r_* - y))}{\omega} V(y) u_1(y) \mathrm{d}y. \tag{1.8.20}$$

In [14] it is shown that the Volterra iteration has the form

$$u_1(r_*) = e^{i\omega r_*} \left(1 + \sum_{n=1}^{\infty} u_1^{(n)}(r_*) \right), \tag{1.8.21}$$

where

$$u_1^{(n)}(r_*) = \sum_{\substack{m_n \dots m_1 \in \mathbb{N} \\ m_n > \dots > m_1}} C_{m_n - m_{n-1}} C_{m_{n-1} - m_{n-2}} \dots C_{m_1} d_{m_n} \dots d_{m_1} e^{2\kappa_+ m_n r_*} \tag{1.8.22}$$

with $d_m = -(4m\kappa_+(m\kappa_+ + i\omega))^{-1}$. Note that in view of the bound in (1.8.18) one can check that the Volterra iteration for u_1 converges on $\omega \in \mathbb{C} \backslash \{im\kappa_+ : m \in \mathbb{N}\}$ and moreover,

$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |u_1(r_* = 0)| \lesssim_\ell 1, \tag{1.8.23}$$

$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |u_1'(r_* = 0)| \lesssim_\ell |\omega|. \tag{1.8.24}$$

Analogously, we have that v_1 is analytic on $\omega \in \mathbb{C} \setminus \{im\kappa_- : m \in \mathbb{N}\}$ and v_2 is analytic on

$\omega \in \mathbb{C} \setminus \{-im\kappa_- : m \in \mathbb{N}\}$. Moreover,

$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |v_1(r_* = 0)| \lesssim_\ell 1, \tag{1.8.25}$$
$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |v_1'(r_* = 0)| \lesssim_\ell |\omega|. \tag{1.8.26}$$

and

$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |v_2(r_* = 0)| \lesssim_\ell 1, \tag{1.8.27}$$
$$\sup_{\{|\operatorname{Re}(\omega)|>1\}} |v_2'(r_* = 0)| \lesssim_\ell |\omega|. \tag{1.8.28}$$

This finally shows (1.8.19) in view of the definition of the transmission and reflection coefficients $\mathfrak{T}$ and $\mathfrak{R}$ using Wronskians, cf. Definition 1.2.4.

Now, we prove that $\mathfrak{T}(\omega,\ell)$ has a pole of order one at $\omega = i\kappa_+$ assuming that $\ell(\ell+1) \neq r_+^2(r_+ - 3r_-)$. First note that

$$u_1^{(1)}(r_*) = \sum_{m_1 \in \mathbb{N}} C_{m_1} d_{m_1} e^{2\kappa_+ m_1 r_*} \tag{1.8.29}$$

has a pole of order one at $\omega = i\kappa_+$ since $C_1 \neq 0$, see (1.8.16). Since for $n \neq 1$ there is no term of the form $e^{2\kappa r_*}$ in (1.8.22) as $m_n \geq n$, the pole at $\omega = i\kappa_+$ cannot be canceled by the other terms and must occur in u_1. Moreover, this pole of u_1 at $\omega = i\kappa_+$ is not of higher order that one since d_1 does not occur at higher powers than one in the Volterra iteration. This implies that $\mathfrak{T}(\omega,\ell)$ has a pole of order one at $\omega = i\kappa_+$. □

Connectedness of the subextremal parameter range.

Proposition 1.8.3. *Let the subextremal parameter space $\mathcal{P}_{\mathrm{se}}^{\Lambda>0}$ and $\mathcal{P}_{\mathrm{se}}^{\Lambda<0}$ be defined as in* (1.3.39) *and* (1.3.40)*, respectively. Then, $\mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cap \{Q > 0\}$, $\mathcal{P}_{\mathrm{se}}^{\Lambda<0} \cap \{Q > 0\}$, $\mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cap \{Q < 0\}$ and $\mathcal{P}_{\mathrm{se}}^{\Lambda<0} \cap \{Q < 0\}$ are path-connected.*

Proof. The claim follows for $\mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cap \{Q > 0\}$ and $\mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cap \{Q > 0\}$ from the following continuous parametrizations

$$
\begin{aligned}
\mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cap \{Q>0\} = \Big\{ & (M,Q,\Lambda) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\colon \\
& \Lambda = 3(r_+^2 + r_-^2 + r_c^2 + r_+ r_c + r_c r_- + r_+ r_-)^{-1}, \\
& 6M = \Lambda(r_+ + r_-)(r_+ + r_c)(r_- + r_c), \\
& Q = \left(\frac{\Lambda}{3}(r_+ + r_- + r_c)(r_- r_+ r_c) \right)^{\frac{1}{2}} \\
& \text{for } 0 < r_- < r_+ < r_c \Big\}
\end{aligned} \tag{1.8.30}
$$

and

$$
\begin{aligned}
\mathcal{P}_{\mathrm{se}}^{\Lambda<0} \cap \{Q>0\} = \Big\{ & (M,Q,\Lambda) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\colon \Lambda = 3\left(\frac{3}{4}(r_+ + r_-)^2 - r_+ r_- - \xi_i \right)^{-1}, \\
& 6M = -\Lambda \left(\frac{1}{4}(r_+ + r_-)^2 + \xi_i - r_+ r_- \right)(r_+ + r_-), \\
& Q = \left(-\frac{\Lambda}{3} r_+ r_- \left(\frac{3}{4}(r_+ + r_-)^2 + \xi_i \right) \right)^{\frac{1}{2}}, \\
& \text{for } 0 < r_- < r_+ \text{ and } \xi_i > \left(\frac{3}{4}(r_+ + r_-)^2 - r_+ r_- \right)^{\frac{1}{2}} \Big\}
\end{aligned} \tag{1.8.31}
$$

in view of the fact that $\{0 < r_- < r_+ < r_c\}$ and $\{0 < r_- < r_+, \xi_i > (\frac{3}{4}(r_+ + r_-)^2 - r_+ r_-)^{\frac{1}{2}}\}$ are path-connected as subsets of $\mathbb{R}^3$. In the following we will show (1.8.30) and (1.8.31).

First, in the case $\Lambda > 0$, note that (1.8.30) follows from comparing coefficients of

$$
\frac{-3}{\Lambda}(r^2 - 2Mr + Q^2 - \frac{1}{3}\Lambda r^4) = (r - r_-)(r - r_+)(r - r_c)(r - r_0)
$$

for $r_0 < 0 < r_- < r_+ < r_c$. Indeed, we obtain $r_0 = -(r_- + r_+ + r_c)$ and (1.8.30) can be deduced.

In the case $\Lambda < 0$, note that $\frac{-3}{\Lambda}(r^2 - 2Mr + Q^2 - \frac{1}{3}\Lambda r^4)$ only has two real roots $0 < r_- < r_+$ such that we compare coefficients of

$$
\frac{-3}{\Lambda}(r^2 - 2Mr + Q^2 - \frac{1}{3}\Lambda r^4) = (r - r_-)(r - r_+)(r - \xi)(r - \bar{\xi})
$$

with $\xi = \xi_r + i\xi_i$. We obtain $2\xi_r = -(r_+ + r_-)$ and $\xi_i > \left(\frac{3}{4}(r_+ + r_-)^2 - r_+ r_-\right)^{\frac{1}{2}}$ to guarantee $\Lambda < 0$. Now, a direct computation shows (1.8.31).

Completely analogously we can show path-connectedness for $\mathcal{P}_{\mathrm{se}}^{\Lambda>0} \cap \{Q < 0\}$ and $\mathcal{P}_{\mathrm{se}}^{\Lambda<0} \cap \{Q < 0\}$. $\square$

Chapter 2

Uniform boundedness and continuity at the Cauchy horizon for linear waves on Reissner–Nordström–AdS black holes

2.1 Introduction

We initiate the study of (massive) linear waves satisfying on the *interior* of asymptotically Anti-de Sitter (AdS) black holes $(\mathcal{M}, g)$. We will consider Reissner–Nordström–AdS (RN–AdS) black holes [13] which can be viewed as the simplest model in the context of the question of stability of the Cauchy horizon. We consider the massive linear wave equation

$$\Box_{g_{\mathrm{RNAdS}}}\psi + \frac{\alpha}{l^2}\psi = 0 \tag{2.1.1}$$

for AdS radius $l^2 := -\frac{3}{\Lambda}$ on a fixed subextremal Reissner–Nordström–AdS black hole with mass parameter $M > 0$ and charge parameter $0 < |Q| < M$. Moreover, we assume the so-called Breitenlohner–Freedman bound [12] for the Klein–Gordon mass parameter $\alpha < \frac{9}{4}$, which includes the conformally invariant case for $\alpha = 2$. This bound is required to obtain well-posedness [73, 144, 142] of (2.1.1).

Recall from the discussion in the introduction of the thesis that solutions with fixed angular momentum ℓ actually decay exponentially in the exterior region. For such solutions with fixed ℓ, uniform boundedness with upper bound $C = C_\ell$ in the interior and continuity at the Cauchy horizon can be shown using the methods involving the vector field $S = |u|^p\partial_u + |v|^p\partial_v$. Note however that this does not imply that a general solution remains bounded in the interior as the constant C_ℓ is not summable: $\sum_{\ell=0}^{L} C_\ell \sim e^L \to +\infty$ as $L \to \infty$. Note in particular that, as a result of this, one cannot study the new non-trivial aspect of this problem restricted to spherical symmetry. (Nevertheless, see [8] for a discussion of the Ori model for Reissner–Nordström–AdS black holes.)

Main theorem: Uniform boundedness and continuity at the Cauchy horizon. We now state a rough version of our main result. See Theorem 2.3 for the precise statement.

Theorem 2.1 (Rough version of Theorem 2.3)**.** *Let ψ be a solution to* (2.1.1) *arising from smooth and compactly supported initial data (ψ_0, ψ_1) posed on a spacelike hypersurface Σ_0 as depicted in Fig. 1. Then, ψ remains* ***uniformly bounded*** *in the black hole interior*

$$|\psi| \leq C,$$

where C is constant depending on the parameters M, Q, l, α, the choice of Σ_0 and on some higher order Sobolev norm of the initial data (ψ_0, ψ_1). Moreover, ψ can be extended ***continuously*** *across the Cauchy horizon.*

As we have explained in the introduction of the thesis, the main difficulty compared to the asymptotically flat case, where the analysis was carried out entirely in physical space and requires inverse polynomial decay in the exterior [51], is the slow decay of ψ along the event horizon. Our strategy is to decompose the solution ψ in a low and high frequency part $\psi = \psi_\flat + \psi_\sharp$ with respect to the Killing field $T = \frac{\partial}{\partial t}$ and treat each term separately.

For the low frequency part $\psi_\flat$, we will show a superpolynomial decay rate in the exterior, see already Proposition 2.4.7. For this part we also use integrated energy decay estimates for bounded angular momenta ℓ established in [75]. This superpolynomial decay in the exterior is sufficient so as to follow the method of [51] with vector fields of the form like $S == |u|^p\partial_u + |v|^p\partial_v$ to show boundedness and continuity at the Cauchy horizon, up to the additional difficulty caused by the fact that we allow a possibly negative Klein–Gordon mass parameter. The violation of the dominant energy condition due to the presence of a negative mass term can be overcome with twisted derivatives [12, 144, 78], which provide a useful framework to replace Hardy inequalities for the lower order terms in this context.

For the high frequency part $\psi_\sharp$, which is exposed to stable trapping and does in general only decay at a sharp logarithmic rate in the exterior, the key ingredient is the scattering theory developed in [82] (see discussion above). More specifically, the uniform bounds for the transmission and reflections coefficients $\mathfrak{T}$ and $\mathfrak{R}$ for $|\omega| \geq \omega_0$ proved in [82] turn out to be useful for the high frequency part $\psi_\sharp$. These bounds allow us to control $|\psi_\sharp|$ at the Cauchy horizon by the T-energy norm on the event horizon commuted with angular derivatives. The T-energy flux on the event horizon is in turn bounded from initial data by a simple application of the T-energy identity in the exterior. *In particular, no quantitative decay along the event horizon is used for the high frequency part* $\psi_\sharp$. This is what allows us to overcome the problem of slow logarithmic decay.

Outline. This chapter is organized as follows. In Section 2.2 we set up the spacetime and summarize relevant previous work. In Section 2.3 we state and prove our main result Theorem 2.3. Parts of the proof require a separate analysis which are treated in Section 2.4 and Section 2.5.

2.2 Preliminaries

We start by setting up the Reissner–Nordström–AdS spacetime (see [13]) and defining relevant norms and energies. We will also introduce useful coordinate systems.

2.2.1 The Reissner–Nordström–AdS black hole

We are ultimately interested in the behavior of solutions to (2.1.1) to the future of a spacelike hypersurface Σ_0 as depicted in Fig. 1. For technical reasons (Fourier space decompositions are non-local operations) we will however construct also parts to the past of Σ_0. In the following will define the spacetime pictured in Fig. 2.1.

2.2.1.1 Construction of the spacetime $(\mathcal{M}_{\mathrm{RNAdS}}, g_{\mathrm{RNAdS}})$

First, for black hole parameters $M > 0, Q \neq 0, l^2 \neq 0$ define the polynomial

$$\Delta_{M,Q,l}(r) := r^2 - 2Mr + \frac{r^4}{l^2} + Q^2 \tag{2.2.1}$$

and define the non-degenerate set

$$\begin{aligned}\mathcal{P} := \{(M,Q,l) \in (0,\infty) \times \mathbb{R} \times (0,\infty)\colon \\ \Delta_{M,Q,l}(r) \text{ has two postive roots satisfying } 0 < r_- < r_+\}.\end{aligned} \tag{2.2.2}$$

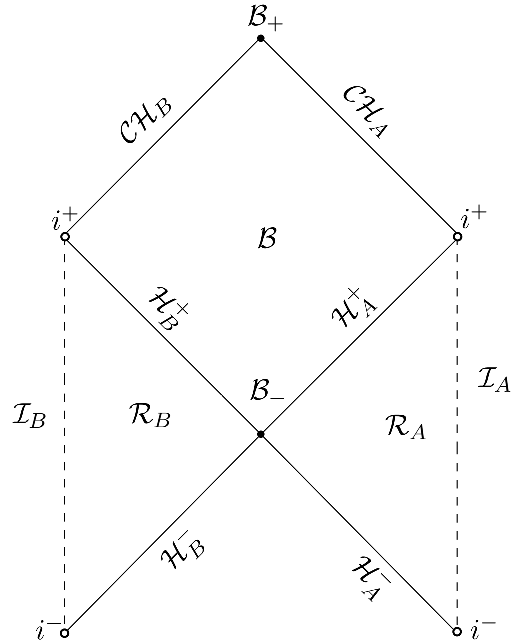

Figure 2.1: Penrose diagram of the constructed spacetime $(\mathcal{M}_{\text{RNAdS}}, g_{\text{RNAdS}})$

Note that $\mathcal{P}$ defines black hole parameters in the subextremal range. From now on, we will consider ***fixed*** parameters M, Q, l, α, where

$$(M, Q, l) \in \mathcal{P} \text{ and } \alpha < \frac{9}{4}. \tag{2.2.3}$$

Note that M is the mass parameter, Q the charge parameter of the black hole and $l = \sqrt{-\frac{3}{\Lambda}}$ is the Anti-de Sitter radius. For this specific choice of parameters we will also write $\Delta(r) := \Delta_{M,Q,l}(r)$ and denote by $0 < r_- < r_+$ the positive roots of Δ.

Now, let the two exterior regions $\mathcal{R}_A$, $\mathcal{R}_B$ and the black hole region $\mathcal{B}$ be smooth four dimensional manifolds diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$. On $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{B}$ we introduce global[1] coordinate charts:

$$\begin{aligned}
&(r_{\mathcal{R}_A}, t_{\mathcal{R}_A}, \theta_{\mathcal{R}_A}, \varphi_{\mathcal{R}_A}) \in (r_+, \infty) \times \mathbb{R} \times \mathbb{S}^2, \\
&(r_{\mathcal{R}_B}, t_{\mathcal{R}_B}, \theta_{\mathcal{R}_B}, \varphi_{\mathcal{R}_B}) \in (r_+, \infty) \times \mathbb{R} \times \mathbb{S}^2, \\
&(r_{\mathcal{B}}, t_{\mathcal{B}}, \theta_{\mathcal{B}}, \varphi_{\mathcal{B}}) \in (r_-, r_+) \times \mathbb{R} \times \mathbb{S}^2.
\end{aligned} \tag{2.2.4}$$

If it is clear from the context which coordinates are being used, we will omit their subscripts throughout the chapter. Again, on the manifolds $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{B}$ we define—using

[1]Up to the known degeneracy of spherical coordinates at the poles of the sphere.

the coordinates (t, r, θ, φ) on each of the patches—the Reissner–Nordström–Anti-de Sitter metric

$$g := -\frac{\Delta(r)}{r^2}\mathrm{d}t \otimes \mathrm{d}t + \frac{r^2}{\Delta(r)}\mathrm{d}r \otimes \mathrm{d}r + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\varphi \otimes \mathrm{d}\varphi). \tag{2.2.5}$$

On each of $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{B}$, we define time orientations using the vector field $\partial_{t_{\mathcal{R}_A}}$ on $\mathcal{R}_A$, $-\partial_{t_{\mathcal{R}_B}}$ on $\mathcal{R}_B$ and $-\partial_{r_{\mathcal{B}}}$ on $\mathcal{B}$.

We will also define the tortoise coordinate r_* by

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} := \frac{r^2}{\Delta} \tag{2.2.6}$$

in $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$ independently. This defines r_* up to an unimportant constant. Then, in each of the regions $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$, we define null coordinates by

$$v = r_* + t \text{ and } u = r_* - t, \tag{2.2.7}$$

where for example for the v coordinate on $\mathcal{R}_A$, we will use the notation $v_{\mathcal{R}_A}$ and analogously for the other regions. Note that throughout the chapter we will use the notation $'$ for derivatives $\frac{\partial}{\partial r_*}$.

Patching the regions $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{B}$ together. Now, we patch the regions $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$ together. We begin by attaching the future (resp. past) event horizon $\mathcal{H}_A^+$ (resp. $\mathcal{H}_A^-$) to $\mathcal{R}_A$ by formally[2] setting

$$\mathcal{H}_A^+ := \{u_{\mathcal{R}_A} = -\infty\} \text{ and } \mathcal{H}_A^- := \{v_{\mathcal{R}_A} = -\infty\}. \tag{2.2.8}$$

Similarly, we attach $\mathcal{H}_B^+ := \{v_{\mathcal{R}_B} = -\infty\}$ and $\mathcal{H}_B^- := \{u_{\mathcal{R}_B} = -\infty\}$ to $\mathcal{R}_B$. In the $(u_{\mathcal{B}}, v_{\mathcal{B}})$ coordinates associated to $\mathcal{B}$ we make the identifications $\mathcal{H}_A^+ = \{u_{\mathcal{B}} = -\infty\}$ and $\mathcal{H}_B^+ = \{v_{\mathcal{B}} = -\infty\}$. Then, we attach the Cauchy horizon $\mathcal{CH}_A := \{v_{\mathcal{B}} = +\infty\}$ and $\mathcal{CH}_B := \{u_{\mathcal{B}} = +\infty\}$ to $\mathcal{B}$.

Finally, we attach the past (resp. future) bifurcation sphere $\mathcal{B}_-$ (resp. $\mathcal{B}_+$) to $\mathcal{B}$ as

$$\mathcal{B}_- := \{u_{\mathcal{B}} = -\infty, v_{\mathcal{B}} = -\infty\} \text{ and } \mathcal{B}_+ := \{u_{\mathcal{B}} = +\infty, v_{\mathcal{B}} = +\infty\}. \tag{2.2.9}$$

We shall also set $\mathcal{CH} := \mathcal{CH}_A \cup \mathcal{CH}_B \cup \mathcal{B}_+$. Note that all horizons $\mathcal{H}_A^+, \mathcal{H}_A^-, \mathcal{H}_B^+, \mathcal{H}_B^-$, $\mathcal{CH}_A$, and $\mathcal{CH}_B$ are diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ and the past (future) bifurcation sphere $\mathcal{B}_-$ ($\mathcal{B}_+$) is

[2]This can be made rigorous using ingoing Eddington–Finkelstein coordinates (r, v, φ, θ) adapted to the event horizon. Since this is well-known, we avoid introducing yet another coordinate system.

diffeomorphic to $\mathbb{S}^2$. Moreover, we identify $\mathcal{B}_-$ with $\{u_{\mathcal{R}_A} = -\infty, v_{\mathcal{R}_A} = -\infty\}$ and also with $\{u_{\mathcal{R}_B} = -\infty, v_{\mathcal{R}_B} = -\infty\}$. The resulting manifold will be called $\mathcal{M}_{\mathrm{RNAdS}}$. Note that, g extends to a smooth Lorentzian metric on $\mathcal{M}_{\mathrm{RNAdS}}$ which we will call g_{RNAdS} and in particular, $(\mathcal{M}_{\mathrm{RNAdS}}, g_{\mathrm{RNAdS}})$ is a time oriented smooth Lorentzian manifold with corners. We illustrate the constructed spacetime as a Penrose diagram in Fig. 2.1. Note that the vector field ∂_t defined on $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$, respectively, extends to a smooth Killing field on $\mathcal{M}_{\mathrm{RNAdS}}$, which we will from now on call T. Moreover, the standard angular momentum operators $\mathcal{W}_i$ for $i = 1, 2, 3$, the generators of $\mathfrak{so}(3)$ defined as

$$\mathcal{W}_1 = \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi, \mathcal{W}_2 = -\cos\varphi\partial_\theta + \cot\theta\sin\varphi\partial_\varphi, \mathcal{W}_3 = -\partial_\varphi \tag{2.2.10}$$

are Killing vector fields. It shall be noted that $\mathcal{W}_i$ for $i = 1, 2, 3$ are spacelike everywhere, whereas T is future-directed timelike on $\mathcal{R}_A$, spacelike on $\mathcal{B}$ and past-directed timelike on $\mathcal{R}_B$. Moreover, T is future-directed null on $\mathcal{H}_A^-, \mathcal{H}_A^+, \mathcal{CH}_B$, past-directed null on $\mathcal{H}_B^-, \mathcal{H}_B^+, \mathcal{CH}_A$ and vanishes on $\mathcal{B}_-, \mathcal{B}_+$. Finally, note that one can attach conformal timelike boundaries $\mathcal{I}_A$ and $\mathcal{I}_B$ corresponding to $\{r_{\mathcal{R}_A} = +\infty\}$ and $\{r_{\mathcal{R}_B} = +\infty\}$, respectively.[3]

2.2.1.2 Initial hypersurface Σ_0

We will impose initial data on a spacelike hypersurface Σ_0 to be made precise in the following. Note that we can choose for convenience that the spacelike hypersurface Σ_0 lies to the future of the past bifurcation sphere $\mathcal{B}_-$. Indeed, by general theory (an energy estimate in a compact region) this can be assumed without loss of generality [27]. More precisely, let Σ_0 be a 3 dimensional connected, complete and spherically symmetric spacelike hypersurface extending to the conformal infinity $\mathcal{I} = \mathcal{I}_A \cup \mathcal{I}_B$. Moreover, assume that $\mathcal{B}_- \subset J^-(\Sigma_0) \setminus \Sigma_0$.

A possible choice of Σ_0 is denoted in Fig. 2.2. We are ultimately interested in the shaded region to the future of Σ_0. For the rest of the chapter, we will consider such a Σ_0 to be ***fixed***.

2.2.2 Conventions

With $a \lesssim b$ for $a \in \mathbb{R}$ and $b \geq 0$ we mean that there exists a constant $C(M, Q, l, \alpha, \Sigma_0)$ with $a \leq Cb$. If $C(M, Q, l, \alpha, \Sigma_0)$ depends on an additional parameter, say ℓ, we will write $a \lesssim_\ell b$. We also use $a \sim b$ for some $a, b \geq 0$ if there exist constants $C_1(M, Q, l, \alpha, \Sigma_0) > 0$

[3]Note that $\mathcal{I}_A$ and $\mathcal{I}_B$ are not contained in $\mathcal{M}_{\mathrm{RNAdS}}$.

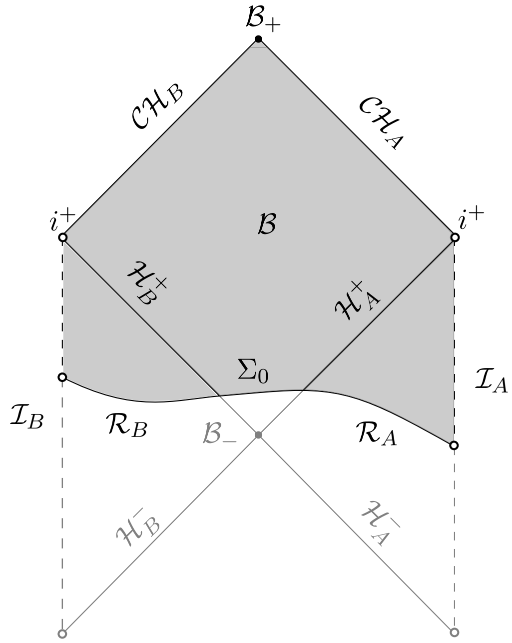

Figure 2.2: The shaded region of interest lies in the future of Σ_0.

$C_2(M, Q, l, \alpha, \Sigma_0) > 0$ with $C_1 a \leq b \leq C_2 a$. We shall also make use of the standard Landau notation O and o [119]. To be more precise, let X be a point set (e.g. $X = \mathbb{R}, [a, b], \mathbb{C}$) with limit point c. As $x \to c$ in X, $f(x) = O(g(x))$ means $\frac{|f(x)|}{|g(x)|} \leq C(M, Q, l, \alpha)$ holds in a fixed neighborhood of c. We write $O_\ell(g(x))$ if the constant C depends on an additional parameter ℓ. For the standard volume form in spherical coordinates (φ, θ) on the sphere $\mathbb{S}^2$ we will use the notation $\mathrm{d}\sigma_{\mathbb{S}^2} := \sin\theta \mathrm{d}\varphi \mathrm{d}\theta$. Finally, let the Japanese symbol be defined as $\langle x \rangle := \sqrt{1 + x^2}$ for $x \in \mathbb{R}$.

2.2.3 Norms and Energies

We are interested in solutions to the massive wave equation (2.1.1) associated to the metric g_{RNAdS} on a subextremal Reissner–Nordström AdS black hole with black hole parameters M, Q, l as in (2.2.3). In view of the timelike boundaries $\mathcal{I}_A$ and $\mathcal{I}_B$, we need to specify boundary conditions on $\mathcal{I}_A$ and $\mathcal{I}_B$ in addition to prescribing data on the spacelike hypersurface Σ_0, cf. Fig. 2.2. We will use Dirichlet (reflecting) boundary conditions which can be viewed as the most natural conditions in the context of stability of the Cauchy horizon. In principle, however, in view of [144], we could also use more general boundary conditions like Neumann or Robin conditions. We will now introduce an appropriate foliation and norms in order to state the well-posedness statement in Section 2.2.4.

We will foliate $\mathcal{R}_A \cup \mathcal{R}_B \cup \mathcal{H}_A^+ \cup \mathcal{H}_B^+ \cup \mathcal{B}$ with spacelike hypersurfaces. To do so, we let

$\mathcal{T}$ be a smooth future-directed causal vector field on $\mathcal{R}_A \cup \mathcal{R}_B \cup \mathcal{H}_A^+ \cup \mathcal{H}_B^+ \cup \mathcal{B}$ with the properties that

$$\mathcal{T} = \begin{cases} T & \text{on } \mathcal{R}_A \cup \mathcal{H}_A^+ \\ -T & \text{on } \mathcal{R}_B \cup \mathcal{H}_B^+ \end{cases}$$

and that $\mathcal{T}$ is a future-directed timelike vector field on $\mathcal{B}$. Now, define the leaves

$$\Sigma_{t^*} := \Phi^{\mathcal{T}}(t^*)[\Sigma_0], \tag{2.2.11}$$

where $\Phi^{\mathcal{T}}$ is the flow generated by $\mathcal{T}$ and $t^* \in \mathbb{R}$ is its affine parameter. We have illustrated some leaves in Fig. 2.3.

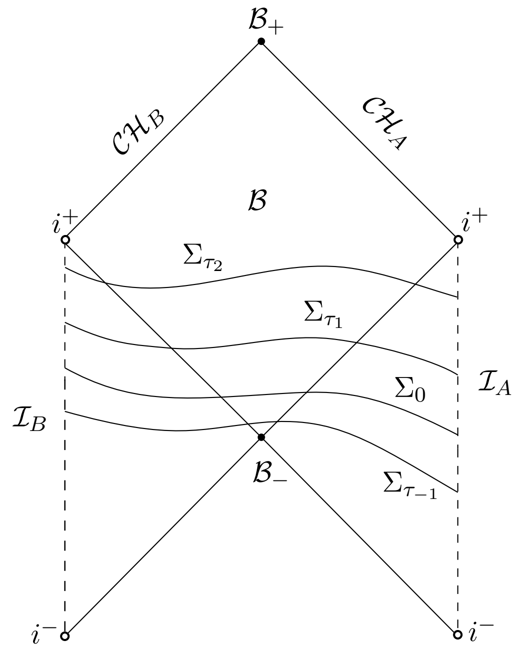

Figure 2.3: Illustration of the foliation with leaves Σ_τ defined in (2.2.11).

2.2.3.1 Further coordinates in the exterior region

In the region $\mathcal{R}_A \cup \mathcal{H}_A^+$, we moreover define a global (up to the well-known degeneracy on $\mathbb{S}^2$) coordinate system $(t^*, r, \varphi, \theta)$, where t^* is the affine parameter of the flow generated by $\mathcal{T}$. Note that on $\mathcal{R}_A \cup \mathcal{H}_A^+$ we have $\partial_{t^*} = T$ such that $t^*(t_2, r) - t^*(t_1, r) = t_2 - t_1$ and $t(t_2^*, r) - t(t_1^*, r) = t_2^* - t_1^*$. Similarly, we can define such a coordinate system on $\mathcal{R}_B$.

2.2.3.2 Norms on hypersurfaces Σ_{t^*}

By construction Σ_{t^*} intersects $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$. We will now define norms on Σ_{t^*} which are adaptations of the norms introduced in [73]. We define

$$\|\psi\|^2_{H^{k,s}_{\mathrm{RNAdS}}(\Sigma_{t^*})} := \|\psi\|^2_{H^k(\Sigma_{t^*}\cap\mathcal{B})} + \|\psi\|^2_{H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap(\mathcal{R}_A\cup\mathcal{H}^+_A))} + \|\psi\|^2_{H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap(\mathcal{R}_B\cup\mathcal{H}^+_B))} \tag{2.2.12}$$

and

$$CH^2_{\mathrm{RNAdS}} := C^2(\mathbb{R}_{t^*}; H^{0,-2}_{\mathrm{RNAdS}}(\Sigma_{t^*})) \cap C^1(\mathbb{R}_{t^*}; H^{1,0}_{\mathrm{RNAdS}}(\Sigma_{t^*})) \cap C^0(\mathbb{R}_{t^*}; H^{2,0}_{\mathrm{RNAdS}}(\Sigma_{t^*})), \tag{2.2.13}$$

where each of the terms appearing in (2.2.12) will be defined in the following.

Norms in the interior region. We begin by defining the first term in (2.2.12). We define $\|\cdot\|^2_{H^k(\Sigma_{t^*}\cap\mathcal{B})}$ as the standard Sobolev norm of order k on the Riemannian manifold $(\Sigma_{t^*}\cap\mathcal{B}, g_{\mathrm{RNAdS}}\restriction_{\Sigma_{t^*}\cap\mathcal{B}})$.

Norms in the exterior region. Due to the symmetry of the regions $\mathcal{R}_A$ and $\mathcal{R}_B$, we will only define the norms on $\mathcal{R}_A$ in the following. The norms on $\mathcal{R}_B$ are be constructed analogously. We use the coordinates $(t^*, r, \theta, \varphi)$ in $\mathcal{R}_A$ to define the norms

$$\begin{aligned}
\|\psi\|^2_{H^{0,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)} &:= \int_{\Sigma_{t^*}\cap\mathcal{R}_A} r^s|\psi|^2 r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta\mathrm{d}\varphi \\
\|\psi\|^2_{H^{1,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)} &:= \|\psi\|^2_{H^{0,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)} + \int_{\Sigma_{t^*}\cap\mathcal{R}_A} r^s\left(r^2|\partial_r\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2\right) r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \\
\|\psi\|^2_{H^{2,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)} &:= \|\psi\|^2_{H^{1,s}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)} \\
&\quad + \int_{\Sigma_{t^*}\cap\mathcal{R}_A} r^s\big(r^4|\partial_r^2\psi|^2 + r^2|\nabla\!\!\!\!/\,\partial_r\psi|^2 + |\nabla\!\!\!\!/\,\nabla\!\!\!\!/\,\psi|^2\big)r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\varphi
\end{aligned}$$

and similarly for higher order norms. Here and in the following we denote with $\nabla\!\!\!\!/$ and $g\!\!\!/$ the induced covariant derivative and the induced metric, respectively, on spheres of constant (t^*, r). We will also use the notation $|\nabla\!\!\!\!/\,\psi|^2 := g\!\!\!/(\nabla\!\!\!\!/\,\psi, \nabla\!\!\!\!/\,\psi)$. Now having defined (2.2.12), we will define energies in the following.

2.2.3.3 Energies on hypersurfaces Σ_{t^*}

We set

$$E_i[\psi](t^*) := E_i^A[\psi](t^*) + E_i^B[\psi](t^*) + E_i^{\mathcal{B}}[\psi](t^*) \tag{2.2.14}$$

for $i = 1, 2$, where all terms in (2.2.14) will be defined in the following.

Energies in the interior region. In the interior region we are not concerned with r-weights and define the energies as

$$E_1^{\mathcal{B}}[\psi](t^*) := \|\psi\|^2_{H^1(\Sigma_{t_*}\cap\mathcal{B})} + \|\partial_{t^*}\psi\|^2_{L^2(\Sigma_{t_*}\cap\mathcal{B})}, \tag{2.2.15}$$

$$E_2^{\mathcal{B}}[\psi](t^*) := \|\psi\|^2_{H^2(\Sigma_{t_*}\cap\mathcal{B})} + \|\partial_{t^*}\psi\|^2_{H^1(\Sigma_{t_*}\cap\mathcal{B})} + \|\partial^2_{t^*}\psi\|^2_{L^2(\Sigma_{t_*}\cap\mathcal{B})}. \tag{2.2.16}$$

Energies in the exterior region. To define the energies in the exterior region, it is convenient to start with defining the following energy densities

$$\begin{aligned} e_1[\psi] &:= \frac{1}{r^2}|\partial_{t^*}\psi|^2 + r^2|\partial_r\psi|^2 + |\nabla\!\!\!/\psi|^2 + |\psi|^2 \\ e_2[\psi] &:= e_1[\psi] + e_1[\partial_{t^*}\psi] + \sum_{i=1}^{3} e_1[\mathcal{W}_i\psi] + r^4|\partial_r\partial_r\psi|^2 + r^2|\nabla\!\!\!/\partial_r\psi|^2 + |\nabla\!\!\!/\nabla\!\!\!/\psi|^2 \end{aligned}$$

and their integrals as

$$E_i^A[\psi](t^*) := \int_{\Sigma_{t^*}\cap(\mathcal{R}_A\cup\mathcal{H}_A^+)} e_i[\psi] r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \tag{2.2.17}$$

for $i = 1, 2$. Note that we will write E_i^B for the analogous energy restricted to $\mathcal{R}_B$.

Also remark the following relation between the norms and energies defined above

$$E_1^A[\psi] = \|\psi\|^2_{H^{1,0}_{\mathrm{Ads}}(\Sigma_{t^*}\cap\mathcal{R}_A)} + \|\partial_{t^*}\psi\|^2_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)},$$

$$\begin{aligned} E_2^A[\psi] \sim \sum_i \|\mathcal{W}_i\psi\|^2_{H^{1,0}_{\mathrm{Ads}}(\Sigma_{t^*}\cap\mathcal{R}_A)} &+ \|\partial_{t^*}\psi\|^2_{H^{1,0}_{\mathrm{Ads}}(\Sigma_{t^*}\cap\mathcal{R}_A)} \\ &+ \|\psi\|^2_{H^{2,0}_{\mathrm{Ads}}(\Sigma_{t^*}\cap\mathcal{R}_A)} + \|\partial^2_{t^*}\psi\|^2_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t^*}\cap\mathcal{R}_A)}. \end{aligned}$$

2.2.4 Well-posedness and mixed boundary value Cauchy problem

Having set up the spacetime and the norms, we will restate the well-posedness result for (2.1.1) as a mixed boundary value-Cauchy problem. For asymptotically AdS spacetimes, well-posedness was first proved in [73].

Theorem 2.1 ([73])**.** *Let the Reissner–Nordström–AdS parameters* (M, Q, l) *and the Klein–Gordon mass* $\alpha < \frac{9}{4}$ *be as in* (2.2.3)*. Let initial data* $(\psi_0, \psi_1) \in C_c^\infty(\Sigma_0) \times C_c^\infty(\Sigma_0)$ *be prescribed on the spacelike hypersurface* Σ_0 *and impose Dirichlet (reflecting) boundary conditions on* $\mathcal{I} = \mathcal{I}_A \cup \mathcal{I}_B$*.*

Then, there exists a smooth solution $\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ *of* (2.1.1) *such that* $\psi \restriction_{\Sigma_0} = \psi_0$*,* $\mathcal{T}\psi \restriction_{\Sigma_0} = \psi_1$*. The solution* ψ *is also unique in the class* $C(\mathbb{R}_{t^*}; H^{1,0}_{\mathrm{RNAdS}}(\Sigma_{t^*})) \cap C^1(\mathbb{R}_{t^*}; H^{0,-2}(\Sigma_{t^*}))$*.*

Remark 2.2.1. *The well-posedness statement in Theorem 2.1 holds true for a more general class of initial data, called a* H^2_{AdS} *initial data triplet which give rise to a solution in* CH^2_{RNAdS}*, see [73].*

2.2.5 Energy identities and estimates

In order to prove energy estimates, it turns out to be useful to introduce two types of energy-momentum tensors. Besides the standard energy-momentum tensor associated to (2.1.1), a suitable *twisted* energy-momentum tensor plays an important role in our estimates. Indeed, due to the negative mass term, the standard energy-momentum tensor does not satisfy the dominant energy condition. However, the dominant energy condition can be restored for the twisted energy-momentum tensor introduced in [12, 144]. In particular, these twisted energies will be used in the interior region, whereas in the exterior region we will work with the standard energy-momentum tensor. We will first review the energy estimates in the exterior.

2.2.5.1 Energy estimates in the exterior region

Energy-momentum tensor. For a smooth function ϕ we define

$$\mathbf{T}_{\mu\nu}[\phi] := \mathrm{Re}(\partial_\mu \phi \overline{\partial_\nu \phi}) - \frac{1}{2} g_{\mu\nu} \left(\overline{\partial_\alpha \phi} \partial^\alpha \phi - \frac{\alpha}{l^2} |\phi|^2 \right). \tag{2.2.18}$$

For a smooth vector field X we also define

$$J^X[\phi] := \mathbf{T}[\phi](X, \cdot) \text{ and } K^X[\phi] := {}^X\pi_{\mu\nu} \mathbf{T}^{\mu\nu}[\phi], \tag{2.2.19}$$

where ${}^X\pi := \mathcal{L}_X g$ is the deformation tensor. The term K^X is often referred to as the "bulk term" and satisfies

$$K^X[\phi] = \nabla^\mu J^X_\mu[\phi] \tag{2.2.20}$$

if ϕ is a solution to (2.1.1). Note that if X is Killing, then K^X vanishes. More generally, integrating (2.2.20) one obtains an energy identity relating boundary and bulk terms. For more details about the energy-momentum tensor and its usage for standard energy estimates we refer to [27].

Boundedness and decay in the exterior region. In the exterior regions $\mathcal{R}_A$ and $\mathcal{R}_B$ we have energy decay and boundedness results which have been proved in [73, 72, 75, 77][4]. To state them we make the following choice of volume forms and normals on the event horizon. We set $\mathrm{dvol}_{\mathcal{H}_A^+} = r^2 \mathrm{d}t^* \mathrm{d}\sigma_{\mathbb{S}^2}$ and $n_{\mathcal{H}_A^+} = T$ and similarly for $\mathcal{H}_B^+$. Moreover, we denote by $\mathrm{dvol}_{\Sigma_{t^*}} \sim r \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ the induced volume form on the spacelike hypersurface $\Sigma_{t^*} \cap \mathcal{R}_A$ and by $n^\mu_{\Sigma^*_t}$ its future-directed unit normal. We summarize these energy identities and estimates in the following.

Proposition 2.2.1 ([73])**.** *A solution ψ to* (2.1.1) *arising from smooth and compactly supported data on Σ_0 as in Theorem 2.1 satisfies*

$$\int_{\Sigma_{t_2^*}\cap\mathcal{R}_A} J^T_\mu[\psi] n^\mu_{\Sigma_{t_2^*}} \mathrm{dvol}_{\Sigma_{t_2^*}} + \int_{\mathcal{H}_A^+(t_1^*,t_2^*)} J^T_\mu[\psi] n^\mu_{\mathcal{H}_A^+} \mathrm{dvol}_{\mathcal{H}_A^+} = \int_{\Sigma_{t_1^*}\cap\mathcal{R}_A} J^T_\mu[\psi] n^\mu_{\Sigma_{t_1^*}} \mathrm{dvol}_{\Sigma_{t_1^*}}, \tag{2.2.21}$$

where $t_1^* \le t_2^*$ and $\mathcal{H}_A^+(t_1^*, t_2^*) := \mathcal{H}_A^+ \cap \{t_1^* \le t^* \le t_2^*\}$. The analogous energy identity holds in $\mathcal{R}_B$. In particular, (2.2.21) *shows that the T-energy flux through $\mathcal{I} = \mathcal{I}_A \cup \mathcal{I}_B$ vanishes.*

Moreover, the T-energy flux through the event horizon is bounded by initial data

$$\int_{\mathcal{H}_A^+} J^T_\mu[\psi] n^\mu_{\mathcal{H}_A^+} \mathrm{dvol}_{\mathcal{H}_A^+} + \int_{\mathcal{H}_B^+} J^T_\mu[\psi] n^\mu_{\mathcal{H}_B^+} \mathrm{dvol}_{\mathcal{H}_B^+} \lesssim E_1[\psi](0). \tag{2.2.22}$$

Finally, note that

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\psi] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \sim \int_{\Sigma_{t_*}\cap\mathcal{R}_A} [r^{-2}|\partial_{t^*}\psi|^2 + \frac{\Delta}{r^2}|\partial_r\psi|^2 + |\nabla\!\!\!/\psi|^2 + |\psi|^2] r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi. \tag{2.2.23}$$

[4]Strictly speaking, in [75] this has been only explicitly proved for Kerr–AdS which includes Schwarzschild–AdS. However, the same proof as for Schwarzschild–AdS works *completely analogously* for Reissner–Nordström–AdS and we shall not repeat these arguments here.

Remark that (2.2.23) follows from a Hardy inequality (see [72, Equation (50)]) which is used to absorb the (possibly) negative contribution from the Klein–Gordon mass term.

Theorem 2.2 ([77, Theorem 1.1], [75, Section 12])**.** *A solution ψ to* (2.1.1) *arising from smooth and compactly supported data on Σ_0 as in Theorem 2.1 satisfies*

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A} e_1[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \lesssim \int_{\Sigma_0\cap\mathcal{R}_A} e_1[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \tag{2.2.24}$$

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A} e_2[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \lesssim \int_{\Sigma_0\cap\mathcal{R}_A} e_2[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \tag{2.2.25}$$

and similarly for higher order norms. Moreover, we have the energy decay statements

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A} e_1[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \lesssim \frac{1}{[\log(2+t^*)]^2} \int_{\Sigma_0\cap\mathcal{R}_A} e_2[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \tag{2.2.26}$$

for $t^* \geq 0$ and the pointwise decay

$$\sup_{\Sigma_{t^*}\cap\mathcal{R}_A} |\psi|^2 \lesssim \frac{1}{[\log(2+t^*)]^2} \int_{\Sigma_0\cap\mathcal{R}_A} (e_2[\psi] + e_2[\partial_{t^*}\psi]) r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \tag{2.2.27}$$

for $t^* \geq 0$ in the exterior region $\mathcal{R}_A$ and similarly in $\mathcal{R}_B$. Moreover, just like for Schwarzschild–AdS (cf. [75]), fixed angular frequencies decay exponentially. More precisely, let $Y_{\ell m}$ denote the spherical harmonics and let ψ be a solution to (2.1.1) *arising from smooth and compactly supported data on Σ_0. If there exists an $L \in \mathbb{N}$ with $\langle \psi, Y_{m\ell}\rangle_{L^2(\mathbb{S}^2)} = 0$ for $\ell \geq L$, then*

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A} e_1[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \lesssim \exp\left(-e^{-C(M,Q,l,\alpha)L} t^*\right) \int_{\Sigma_0\cap\mathcal{R}_A} e_1[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \tag{2.2.28}$$

for $t^* \geq 0$ and a constant $C(M,Q,l,\alpha) > 0$ only depending on the parameters M, Q, l, α.

Remark 2.2.2. *Note that* (2.2.28) *also implies pointwise exponential decay for ψ (assuming $\langle \psi, Y_{\ell m}\rangle_{L^2(\mathbb{S}^2)} = 0$ for $\ell \geq L$) and all higher derivatives of ψ using standard techniques like commuting with T and $\mathcal{W}_i$, elliptic estimates as well as applying a Sobolev embedding. Moreover, the previous estimates above also hold true for a the more general class of solutions CH^2_{RNAdS}. See [73] or [75, Theorem 4.1] for more details.*

Remark 2.2.3. *The previous decay estimates have only been stated to the future of Σ_0 in the region $\mathcal{R}_A$, nevertheless, they also hold in $\mathcal{R}_B$. Moreover, they also hold true to the past of Σ_0 for an appropriate foliation for which the leaves intersect $\mathcal{H}_A^-$ and $\mathcal{H}_B^-$, and are* *transported along the flow of $-T$ for $\mathcal{R}_A \cup \mathcal{H}_A^-$ and along the flow of T for $\mathcal{R}_B \cup \mathcal{H}_B^-$.*

We now turn to the energy estimates in the interior region $\mathcal{B}$.

2.2.5.2 Energy estimates in the interior region

Twisted energy-momentum tensor. We begin by defining twisted derivatives.

Definition 2.2.1 (Twisted derivative)**.** *For a smooth and nowhere vanishing function f we define the **twisted derivative***

$$\tilde{\nabla}_\mu := f \nabla_\mu \left(\frac{\cdot}{f} \right) \tag{2.2.29}$$

and its formal adjoint

$$\tilde{\nabla}_\mu^* := -\frac{1}{f} \nabla_\mu (f \cdot). \tag{2.2.30}$$

*We shall refer to f as the **twisting function**.*

Remark 2.2.4. *Note that we can rewrite the Klein–Gordon equation* (2.1.1) *in terms of the twisted derivatives as*

$$-\tilde{\nabla}_\mu^* \tilde{\nabla}^\mu \psi - \mathcal{V}\psi = 0, \tag{2.2.31}$$

where the potential $\mathcal{V}$ is given by

$$\mathcal{V} = -\left(\frac{\alpha}{l^2} + \frac{\Box_g f}{f} \right). \tag{2.2.32}$$

Now, we also associate a twisted energy-momentum tensor to the twisted derivatives.

Definition 2.2.2 (Twisted energy-momentum tensor)**.** *Let f be smooth and nowhere vanishing and $\tilde{\nabla}$ as defined in Definition 2.2.1. We define the **twisted energy-momentum tensor** associated to* (2.1.1) *and f as*

$$\tilde{\mathbf{T}}_{\mu\nu}[\phi] := \operatorname{Re}\left(\overline{\tilde{\nabla}_\mu \phi} \tilde{\nabla}_\nu \phi \right) - \frac{1}{2} g_{\mu\nu} (\overline{\tilde{\nabla}_\sigma \phi} \tilde{\nabla}^\sigma \phi + \mathcal{V}|\phi|^2), \tag{2.2.33}$$

where $\mathcal{V}$ is as in (2.2.32) *and ϕ is any smooth function.*

We will now compute the divergence of the twisted energy-momentum tensor.

Proposition 2.2.2 ([78, Proposition 3]). *Let ϕ be a smooth function and f be a smooth nowhere vanishing twisting function. Then,*

$$\nabla_\mu \tilde{\mathbf{T}}^\mu_\nu[\phi] = \mathrm{Re}\left(\left(-\tilde{\nabla}^*_\mu \tilde{\nabla}^\mu \phi - \mathcal{V}\phi\right) \tilde{\nabla}_\nu \phi\right) + \tilde{S}_\nu[\phi], \tag{2.2.34}$$

where

$$\tilde{S}_\nu[\phi] = \frac{\tilde{\nabla}^*_\nu (f\mathcal{V})}{2f}|\phi|^2 + \frac{\tilde{\nabla}^*_\nu f}{2f}\overline{\tilde{\nabla}_\sigma \phi} \tilde{\nabla}^\sigma \phi. \tag{2.2.35}$$

Now, assume that ϕ moreover satisfies (2.1.1) *and X is a smooth vector field. Set*

$$\tilde{J}^X_\mu[\phi] := \tilde{\mathbf{T}}_{\mu\nu}[\phi] X^\nu \ \text{and}\ \tilde{K}^X[\phi] := \ {}^X\pi_{\mu\nu}\tilde{\mathbf{T}}^{\mu\nu}[\phi] + X^\nu \tilde{S}_\nu[\phi]. \tag{2.2.36}$$

Then,

$$\nabla^\mu \tilde{J}^X_\mu[\phi] = \tilde{K}^X[\phi]. \tag{2.2.37}$$

Finally, note that if the twisting function f associated to $\tilde{\nabla}$ is chosen such that $\mathcal{V} \geq 0$, then $\tilde{\mathbf{T}}_{\mu\nu}$ satisfies the dominant energy condition, i.e. if X is a future pointing causal vector field, then so is $-\tilde{J}^X$.

We will make use of the twisted energy-momentum tensor in the interior region $\mathcal{B}$ for which we use null coordinates $(u_{\mathcal{B}}, v_{\mathcal{B}})$ introduced in Section 2.2.1. For the rest of the subsection we will drop the index $\mathcal{B}$. Then, setting

$$\Omega^2(u,v) := -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}\right), \tag{2.2.38}$$

where $r = r(u,v)$, we write the metric in the interior region $\mathcal{B}$ as

$$g_{\mathrm{RNAdS}} = -\frac{\Omega^2(u,v)}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + r^2(u,v)\mathrm{d}\sigma_{\mathbb{S}^2}. \tag{2.2.39}$$

Note that in the interior we have $r_- < r(u,v) < r_+$ and $\mathrm{d}r_* = \frac{r^2}{\Delta}\mathrm{d}r$. In Proposition 2.6.1 in the appendix we have written out the components of the twisted energy-momentum tensor, the twisted 1-jets $\tilde{J}^X$ and the twisted bulk term $\tilde{K}^X$ in null components. We will use the notation $\mathcal{C}_{u_1} := \{u = u_1\}$, $\underline{\mathcal{C}}_{v_1} = \{v = v_1\}$ for null cones and $\Sigma_{r_1} = \{r = r_1\}$ for spacelike hypersurfaces in the interior. Furthermore, we set (in mild abuse of notation)

$$\mathcal{C}_{u_1}(v_1, v_2) := \{u = u_1\} \cap \{v_1 \le v \le v_2\}, \tag{2.2.40}$$

$$\mathcal{C}_{u_1}(r_1, r_2) := \{u = u_1\} \cap \{r_1 \le r \le r_2\} \tag{2.2.41}$$

and analogously for Σ and $\underline{\mathcal{C}}$. We will also make use of the following notation. For any $\tilde{r} \in (r_-, r_+)$ we set

$$v_{\tilde{r}}(u) := 2r_*(\tilde{r}) - u,$$
$$u_{\tilde{r}}(v) := 2r_*(\tilde{r}) - v$$

and for hypersurfaces with constant u, v, r we denote $n_{\mathcal{C}_u}, n_{\underline{\mathcal{C}}_v}, n_{\Sigma_r}$ as their normals.[5]

Red-shift vector field.

Proposition 2.2.3. *There exist a $r_{\mathrm{red}} \in (r_-, r_+)$, a constant $b(M, Q, l, \alpha) > 0$, a nowhere vanishing smooth function f associated to the twisted energy momentum tensor and a future directed timelike vector field N such that*

$$0 \le \tilde{J}^N_\mu[\phi] n^\mu_{\underline{\mathcal{C}}_v} \le b \tilde{K}^N[\phi] \tag{2.2.42}$$

for $\mathcal{R}_{\mathrm{red}} := \{r_{\mathrm{red}} \le r \le r_+\} \cap \{v \ge 1\}$ and any smooth solution ϕ to (2.1.1)*.*

Proof. This is proven in Section 2.6.2. $\square$

We will now prove the main estimate which we will use in the red-shift region in the interior.

Proposition 2.2.4. *Let ϕ be a smooth solution to* (2.1.1) *and let $r_0 \in [r_{\mathrm{red}}, r_+)$. Then, for any $1 \le v_1 \le v_2$ we obtain*

$$\begin{aligned}\int_{\underline{\mathcal{C}}_{v_2}(r_0, r_+)} \tilde{J}^N_\mu[\phi] n^\mu_{\underline{\mathcal{C}}_v} \mathrm{dvol}_{\underline{\mathcal{C}}_v} + \int_{\Sigma_{r_0}(v_1, v_2)} \tilde{J}^N_\mu[\phi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} + \int_{v_1}^{v_2} \int_{\underline{\mathcal{C}}_v(r_0, r_+)} \tilde{J}^N_\mu[\phi] n^\mu_{\underline{\mathcal{C}}_v} \mathrm{dvol}_{\underline{\mathcal{C}}_v} \mathrm{d}v \\ \lesssim \int_{\underline{\mathcal{C}}_{v_1}(r_0, r_+)} \tilde{J}^N_\mu[\phi] n^\mu_{\underline{\mathcal{C}}_v} \mathrm{dvol}_{\underline{\mathcal{C}}_v} + \int_{\mathcal{H}(v_1, v_2)} \tilde{J}^N_\mu[\phi] n^\mu_{\mathcal{H}^+} \mathrm{dvol}_{\mathcal{H}^+}.\end{aligned} \tag{2.2.43}$$

[5] For null hypersurfaces there does not exist a unit norm normal vector, however, for a fixed volume form, there exists a canonical normal vector which we will choose here. Our choice of volume forms and the corresponding normals can be found in Section 2.6.1.

Proof. We apply the energy identity (spacetime integral of (2.2.37)) in the region $\mathcal{R}(v_1, v_2) := \{r_0 \leq r \leq r_+\} \cap \{v_1 \leq v \leq v_2\}$ to obtain

$$\int_{\underline{\mathcal{C}}_{v_2}(r_0,r_+)} \tilde{J}^N_\mu[\phi] n^\mu_{\underline{\mathcal{C}}_v} \mathrm{dvol}_{\underline{\mathcal{C}}_v} + \int_{\Sigma_{r_0}(v_1,v_2)} \tilde{J}^N_\mu[\phi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} + \int_{\mathcal{R}(v_1,v_2)} \tilde{K}^N[\phi] \mathrm{dvol}$$
$$= \int_{\underline{\mathcal{C}}_{v_1}(r_0,r_+)} \tilde{J}^N_\mu[\phi] n^\mu_{\underline{\mathcal{C}}_v} \mathrm{dvol}_{\underline{\mathcal{C}}_v} + \int_{\mathcal{H}(v_1,v_2)} \tilde{J}^N_\mu[\phi] n^\mu_{\mathcal{H}^+} \mathrm{dvol}_{\mathcal{H}^+}. \tag{2.2.44}$$

Finally, the claim follows from Proposition 2.2.3. □

No-shift vector field. In this region we propagate estimates towards i^+ from the red-shift region to the blue-shift region using a $T = \partial_t$ invariant vector field X and a t-independent twisting function f. Take r_{red} fixed from Proposition 2.2.3 and let $r_{\mathrm{blue}} > r_-$ be close to r_-. We will use the no-shift vector field in two different parts of the chapter: First, we will use it in the proof of Proposition 2.6.2 in the appendix in order to prove well-definedness of the Fourier projections. In this case we will choose r_{blue} in principle arbitrarily close to r_-. The estimate degenerates as we take $r_{\mathrm{blue}} \to r_-$, however for the purpose of Proposition 2.6.2 such an estimate is sufficient. Our second application of the no-shift vector field is to propagate decay of the low-frequency part $\psi_\flat$ in the interior (see already Section 2.4.2). Here, we will take $r_{\mathrm{blue}} = r_{\mathrm{blue}}(M, Q, l)$ only depending on the black hole parameters as determined in Proposition 2.4.12.

In either case, we will choose

$$X = X_{\mathrm{ns}} := \partial_u + \partial_v \tag{2.2.45}$$

as our vector field. (Indeed, any future directed and T invariant vector field would work.) We define our twisting function as

$$f_{\mathrm{ns}}(r) = e^{\beta_{\mathrm{ns}} r} \tag{2.2.46}$$

for some $\beta_{\mathrm{ns}} = \beta_{\mathrm{ns}}(r_{\mathrm{blue}}) > 0$ large enough such that

$$\mathcal{V} = -\frac{\Box_g f_{\mathrm{ns}}}{f_{\mathrm{ns}}} - \frac{\alpha}{l^2} = \Omega^2 \beta_{\mathrm{ns}}^2 + \beta_{\mathrm{ns}} \partial_r(\Omega^2) + \frac{2\beta_{\mathrm{ns}}}{r}\Omega^2 - \frac{\alpha}{l^2} \gtrsim 1 \tag{2.2.47}$$

uniformly in $[r_{\mathrm{blue}}, r_{\mathrm{red}}]$. In particular, since $r \in [r_{\mathrm{blue}}, r_{\mathrm{red}}]$ is bounded away from r_+, r_-, we have

$$\tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_r} \gtrsim |\tilde{\nabla}_u \phi|^2 + |\tilde{\nabla}_v \phi|^2 + |\nabla\!\!\!/ \phi|^2 + |\phi|^2 \tag{2.2.48}$$

for a smooth function ϕ. Our main estimate in the no-shift region is

Proposition 2.2.5. *Let ϕ be a smooth solution to (2.1.1) and $r_0 \in [r_{\text{blue}}, r_{\text{red}}]$. Then for any $v_* \geq 1$ we have*

$$\int_{\Sigma_{r_0}(v_*,2v_*)} \tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim \int_{\Sigma_{r_{\text{red}}}(v_{r_{\text{red}}}(u_{r_0}(v_*)),2v_*)} \tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r}, \tag{2.2.49}$$

where we remark that $v_* - v_{r_{\text{red}}}(u_{r_0}(v_*))) = const.$

Proof. We apply the energy identity (spacetime integral of (2.2.37)) with $X = \partial_u + \partial_v$ (cf. (2.2.45)) and f_{ns} as in (2.2.46) in the region $\{r_0 \leq r \leq r_{\text{red}}\} \cap \{u < u_{r_{\text{blue}}}(v_*)\} \cap \{v \leq 2v_*\}$. The choice of f_{ns} guarantees the twisted dominated energy condition for the twisted energy-momentum tensor. Together with the coarea formula as well as the facts that $[r_*(r_0), r_*(r_{\text{red}})]$ is compact and X is T invariant, we conclude

$$\begin{aligned}\int_{\Sigma_{r_0}(v_*,2v_*)} \tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \leq & B_1 \int_{r_0 \leq \bar{r} \leq r_{\text{red}}} \int_{\Sigma_{\bar{r}}(v_{\bar{r}}(u_{r_0}(v_*)),2v_*)} \tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_{\bar{r}}} \mathrm{dvol}_{\Sigma_{\bar{r}}} \mathrm{d}\bar{r} \\ & + \int_{\Sigma_{r_{\text{red}}}(v_{r_{\text{red}}}(u_{r_0}(v_*)),2v_*)} \tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_{r_{\text{red}}}} \mathrm{dvol}_{r_{\text{red}}}\end{aligned} \tag{2.2.50}$$

for a constant $B_1 = B_1(M, Q, l, \alpha, \Sigma_0, r_{\text{red}}, r_{\text{blue}})$. Similarly, after setting

$$E(\tilde{v}, \tilde{r}) := \int_{\Sigma_{\tilde{r}}(\tilde{v},2v_*)} \tilde{J}^X_\mu[\phi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \tag{2.2.51}$$

for $\tilde{r} \in [r_0, r_{\text{red}}]$, we also have

$$E(v_{\tilde{r}}(u_{r_0}(v_*)), \tilde{r}) \leq \tilde{B}_1 \int_{\tilde{r} \leq \bar{r} \leq r_{\text{red}}} E(v_{\bar{r}}(u_{r_0}(v_*)), \bar{r}) \mathrm{d}\bar{r} + E(v_{r_{\text{red}}}(u_{r_0}(v_*)), r_{\text{red}}) \tag{2.2.52}$$

for a constant $\tilde{B}_1 = \tilde{B}_1(M, Q, l, \alpha, \Sigma_0)$. An application of Grönwall's inequality yields

$$E(v_{\tilde{r}}(u_{r_0}(v_*)), \tilde{r}) \lesssim E(v_{r_{\text{red}}}(u_{r_0}(v_*)), r_{\text{red}}) \tag{2.2.53}$$

which implies the result. $\square$

We will use an additional vector field in the interior in the blue-shift region $(r_-, r_{\text{blue}}]$. We will however only define it later in the chapter in Section 2.4.2.3 when we actually use it to propagate estimates for the low-frequency part $\psi_\flat$ all the way to the Cauchy horizon.

Notation. In the main part of the chapter we will makes use of the Fourier transform and convolution associated to the coordinate t in (t, r, θ, φ) coordinates as in (2.2.4). We denote $\mathcal{F}_T$ as the Fourier transform (and $\mathcal{F}_T^{-1}$ as its inverse) defined as

$$\mathcal{F}_T[f](\omega, r, \theta, \varphi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} f(t, r, \theta, \varphi) \mathrm{d}t \tag{2.2.54}$$

in the coordinates (t, r, φ, θ) of $\mathcal{R}_A, \mathcal{R}_B$ and $\mathcal{B}$, respectively. Here, we assume that $t \mapsto f(t, r, \theta, \varphi)$ is (at least) a tempered distribution and (2.2.54), in general, is to be understood in the distributional sense. Moreover, the convolution $*$ associated to the coordinate t is defined as

$$(f * g)(t, r, \theta, \varphi) := \int_{\mathbb{R}} f(t - s, r, \theta, \varphi) g(s, r, \theta, \varphi) \mathrm{d}s, \tag{2.2.55}$$

where we again assume that $t \mapsto f(t, r, \theta, \varphi)$ is a tempered distribution and $t \mapsto g(t, r, \theta, \varphi)$ is a Schwartz function. Here, (2.2.55), in general, is to be understood in the distributional sense.

2.3 Main theorem and frequency decomposition

Now, we are in the position to state our main result

Theorem 2.3. *Let the Reissner–Nordström–AdS parameters (M, Q, l) and the Klein–Gordon mass $\alpha < \frac{9}{4}$ be as in (2.2.3). Let $\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ be a solution to (2.1.1) arising from smooth and compactly supported initial data $(\psi, \mathcal{T}\psi) \restriction_{\Sigma_0} = (\psi_0, \psi_1) \in C_c^\infty(\Sigma_0) \times C_c^\infty(\Sigma_0)$ on Σ_0 with Dirichlet (reflecting) boundary conditions imposed at $\mathcal{I}_A$ and $\mathcal{I}_B$ (cf. Theorem 2.1). Then, ψ is **uniformly bounded** in the interior region $\mathcal{B}$ satisfying*

$$\sup_{\mathcal{B}} |\psi| \lesssim D[\psi]^{\frac{1}{2}}, \tag{2.3.1}$$

where $D[\psi]$ is defined as

$$D[\psi] := E_1[\psi](0) + \sum_{i,j=1}^{3} E_1[\mathcal{W}_i \mathcal{W}_j \psi](0). \tag{2.3.2}$$

*Moreover, ψ extends **continuously** to the Cauchy horizon, i.e. $\psi \in C^0(\mathcal{M}_{\mathrm{RNAdS}})$.*

Remark 2.3.1. *The data term $D[\psi]$ in (2.3.2) can be controlled by the initial data (ψ_0, ψ_1)*

such that (2.3.1) *can be written in terms of initial data as*

$$\begin{aligned}\sup_{\mathcal{B}} |\psi| \leq C(M,Q,l,\alpha,\Sigma_0)\Big(\|\psi_0\|_{H^{1,0}_{\mathrm{RNAdS}(\Sigma_0)}} + \|\psi_1\|_{H^{0,-2}_{\mathrm{RNAdS}(\Sigma_0)}} \\ + \sum_{i,j=1}^{3} \|\mathcal{W}_i\mathcal{W}_j\psi_0\|_{H^{1,0}_{\mathrm{RNAdS}(\Sigma_0)}} + \sum_{i,j=1}^{3} \|\mathcal{W}_i\mathcal{W}_j\psi_1\|_{H^{0,-2}_{\mathrm{RNAdS}(\Sigma_0)}}\Big)\end{aligned} \tag{2.3.3}$$

for a constant $C(M,Q,l,\alpha,\Sigma_0)$ only depending on the parameters M,Q,l,α and the choice of initial hypersurface Σ_0.

Remark 2.3.2. *Theorem 2.3 can be extended to a more general class of initial data using standard density arguments. In the context of uniform boundedness and continuity at the Cauchy horizon, it is enough to consider smooth and localized initial data. Nevertheless, note that for more general initial data in appropriate Sobolev spaces, already well-posedness becomes more delicate [73].*

Proof of Theorem 2.3. We split up the proof in four steps, where *Step 3* and *Step 4* are the main parts relying on Section 2.4 and Section 2.5.

Step 1: Decomposition into low and high frequencies. Let

$$\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH}) \tag{2.3.4}$$

be as in the assumption of Theorem 2.3. Now, in $\mathcal{R}_A$, $\mathcal{R}_B$ and in $\mathcal{B}$, define the low frequency part $\psi_\flat$ and the high frequency part $\psi_\sharp$ as

$$\psi_\flat := \frac{1}{\sqrt{2\pi}}\mathcal{F}_T^{-1}\left[\chi_{\omega_0}\right] * \psi \text{ and } \psi_\sharp := \psi - \psi_\flat, \tag{2.3.5}$$

where

$$\chi_{\omega_0} \in C_c^\infty(\mathbb{R}) \text{ such that } \chi_{\omega_0}(\omega) = 0 \text{ for } |\omega| \geq \omega_0 \text{ and } \chi_{\omega_0}(\omega) = 1 \text{ for } |\omega| \leq \frac{1}{2}\omega_0. \tag{2.3.6}$$

From Proposition 2.6.3 in the appendix we know that the low and high frequency parts $\psi_\flat$ and $\psi_\sharp$ in (2.3.5) are well-defined and $\psi_\flat$ and $\psi_\sharp$ extend to smooth solutions of (2.1.1) on $\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH}$. The cut-off frequency $\omega_0 = \omega_0(M,Q,l,\alpha) > 0$ will be chosen in the proof of Proposition 2.4.4 only depending on M,Q,l,α. For convenience we can also assume that χ_{ω_0} is a symmetric function which implies that $\psi_\flat$ and $\psi_\sharp$ will be real-valued as long as ψ was real valued. This concludes Step 1.

Having decomposed the solution in low and high frequency parts $\psi_\flat$ and $\psi_\sharp$, we shall now see how the initial data $D[\psi_\flat]$ and $D[\psi_\sharp]$, respectively, can be bounded by the initial data $D[\psi]$ of ψ.

Step 2: Estimating the initial data of the decomposed solution. This step is the content of the following proposition.

Proposition 2.3.1. *Let ψ be as in* (2.3.4) *and $\psi_\flat, \psi_\sharp$ be as in* (2.3.5) *and recall the definition of $D[\cdot]$ from* (2.3.2). *Then,*

$$D[\psi_\flat] \lesssim D[\psi] \text{ and } D[\psi_\sharp] \lesssim D[\psi]. \tag{2.3.7}$$

Proof. Since $\psi = \psi_\flat + \psi_\sharp$, it suffices to obtain a bound of the type $D[\psi_\flat] \lesssim D[\psi]$, where $D[\cdot]$ is defined in (2.3.2). Because of the Dirichlet conditions imposed at infinity, the energy fluxes through $\mathcal{I}_A$ and $\mathcal{I}_B$ vanish (see (2.2.21)), and we estimate

$$D[\psi_\flat] \lesssim \tilde{D}[\psi_\flat],$$

where $\tilde{D}[\psi_\flat]$ is a higher order energy on the hypersurface

$$\tilde{\Sigma}_0 := (\mathcal{R}_A \cap \{t_{\mathcal{R}_A} = 0\}) \cup \mathcal{B}_- \cup (\mathcal{R}_B \cap \{t_{\mathcal{R}_B} = 0\})$$

to be made precise in the following. Note also that the normal vector field on $\mathcal{R}_A \cap \tilde{\Sigma}_0$ is $n_{\tilde{\Sigma}_0} = \frac{r}{\sqrt{\Delta}} \partial_t$.

More precisely, due to the support properties of the initial data, there exists a relatively compact 3-dimensional spherically symmetric submanifold[6] $K \subset \tilde{\Sigma}_0$ with $\mathcal{B}_- \subset K$ and such that

$$\begin{aligned} D[\psi_\flat] \lesssim \tilde{D}[\psi_\flat] := {} & \|\psi_\flat\|^2_{H^1(K)} + \|n_{\tilde{\Sigma}_0}\psi_\flat\|^2_{L^2(K)} + \sum_{i,j=1}^{3} \|\mathcal{W}_i\mathcal{W}_j\psi_\flat\|^2_{H^1(K)} \\ & + \sum_{i,j=1}^{3} \|\mathcal{W}_i\mathcal{W}_j n_{\tilde{\Sigma}_0}\psi_\flat\|^2_{L^2(K)} + \int_{\tilde{\Sigma}_0 \cap \mathcal{R}_A \setminus K} \left(e_1[\psi_\flat] + \sum_{i,j=1}^{3} e_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat] \right) r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\ & + \int_{\tilde{\Sigma}_0 \cap \mathcal{R}_B \setminus K} \left(e_1[\psi_\flat] + \sum_{i,j=1}^{3} e_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat] \right) r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi. \end{aligned} \tag{2.3.8}$$

Estimate (2.3.8) follows from general theory [27], that is a (higher order) energy estimate

[6] We introduce K just for a technical reason: The energy density $e_1[\cdot]$ defined on $\tilde{\Sigma}_0 \cap \mathcal{R}_A$ degenerates at the bifurcation sphere $\mathcal{B}_-$.

followed by an application of Grönwall's lemma. In order to estimate the energy on the compact hypersurface K we decompose K in $K \cap \mathcal{R}_A$ and $K \cap \mathcal{R}_B$ and estimate the energy on each of those slices independently. Again, in view of the fact that $\mathcal{R}_A$ and $\mathcal{R}_B$ can be treated analogously, we only show the estimate in $\mathcal{R}_A$. Note that all the terms of

$$\|\psi_\flat\|^2_{H^1(K\cap\mathcal{R}_A)} + \|n_{\tilde{\Sigma}_0}\psi_\flat\|^2_{L^2(K\cap\mathcal{R}_A)} + \sum_{i,j=1}^{3} \|\mathcal{W}_i\mathcal{W}_j\psi_\flat\|^2_{H^1(K\cap\mathcal{R}_A)}$$
$$+ \sum_{i,j=1}^{3} \|\mathcal{W}_i\mathcal{W}_j n_{\tilde{\Sigma}_0}\psi_\flat\|^2_{L^2(K\cap\mathcal{R}_A)} + \int_{\tilde{\Sigma}_0\cap\mathcal{R}_A} \left(e_1[\psi_\flat] + \sum_{i,j=1}^{3} e_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat] \right) r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

are of the form

$$\int_{\{t=0\}\cap\mathcal{R}_A} f|\partial^k\psi_\flat|^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

for appropriate T invariant weight functions $f \geq 0$ and T invariant coordinate derivatives $\partial \in \{\partial_t, \partial_r, \partial_\theta, \partial_\varphi\}$ of order $k = 0, 1, 2, 3$. Using that

$$\psi_\flat = \frac{1}{\sqrt{2\pi}} \mathcal{F}_T^{-1}\left[\chi_{\omega_0}\right] * \psi,$$

where $\mathcal{F}_T^{-1}\left[\chi_{\omega_0}\right] =: \eta$ is a fixed Schwartz function, we conclude—again since T is Killing—that

$$\begin{aligned}
\int_{\{t=0\}\cap\mathcal{R}_A} f(r)|\partial^k\psi_\flat|^2(0,r,\varphi,\theta)\mathrm{d}r\mathrm{d}\sigma_{\mathbb{S}^2} &= \int_{\{r\geq r_+\}\times\mathbb{S}^2} f(r)|\eta * \partial^k\psi|^2(0,r,\varphi,\theta)\mathrm{d}r\mathrm{d}\sigma_{\mathbb{S}^2} \\
&= \int_{\{r\geq r_+\}\times\mathbb{S}^2} f(r) \left| \int_{\mathbb{R}} \eta(-s)\partial^k\psi(s,r,\varphi,\theta)\mathrm{d}s \right|^2 \mathrm{d}r\mathrm{d}\sigma_{\mathbb{S}^2} \\
&\leq \int_{\mathbb{R}} |\eta(s)|\mathrm{d}s \int_{\mathbb{R}} |\eta(-s)| \int_{\{r\geq r_+\}\times\mathbb{S}^2} f(r)|\partial^k\psi(s,r,\varphi,\theta)|^2 \mathrm{d}r\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}s \\
&\lesssim \sup_{s\in\mathbb{R}} \int_{\mathbb{R}} f(r)|\partial^k\psi(s,r,\varphi,\theta)|^2\mathrm{d}\sigma_{\mathbb{S}^2} \\
&\lesssim \int_{\{t=0\}\cap\mathcal{R}_A} f(r)|\partial^k\psi|^2(0,r,\varphi,\theta)\mathrm{d}r\mathrm{d}\sigma_{\mathbb{S}^2} \lesssim \tilde{D}[\psi],
\end{aligned}$$

where we have used boundedness of higher order energies in the exterior which are proved in [72] and restated in Theorem 2.2. Also note that we can interchange the derivatives with the convolution since T is a Killing vector field. Thus, we conclude that $\tilde{D}[\psi_\flat] \lesssim \tilde{D}[\psi]$ and again by Cauchy stability and the vanishing of the energy flux at $\mathcal{I}$ (see (2.2.21)), we can bound $\tilde{D}[\psi] \lesssim D[\psi]$ which finally shows $D[\psi_\flat] \lesssim D[\psi]$. Hence, $D[\psi_\sharp] \lesssim D[\psi]$ also holds true. $\square$

The previous analysis in Step 1 and Step 2 allows us to treat the low and high frequency parts $\psi_\flat$ and $\psi_\sharp$ completely independently.

Step 3: Uniform boundedness for $\psi_\flat$ and $\psi_\sharp$. This step is at the heart of the chapter and will be proved in Section 2.4 and Section 2.5. According to Proposition 2.4.13 and Proposition 2.5.1,

$$\sup_{\mathcal{B}} |\psi_\flat|^2 \lesssim D[\psi_\flat] \tag{2.3.9}$$

and

$$\sup_{\mathcal{B}} |\psi_\sharp|^2 \lesssim D[\psi_\sharp]. \tag{2.3.10}$$

Thus, in view of *Step 2*, we conclude

$$\sup_{\mathcal{B}} |\psi|^2 \lesssim \sup_{\mathcal{B}} |\psi_\flat|^2 + \sup_{\mathcal{B}} |\psi_\sharp|^2 \lesssim D[\psi_\flat] + D[\psi_\sharp] \lesssim D[\psi] \tag{2.3.11}$$

which shows (2.3.1).

Step 4: Continuous extendibility beyond the Cauchy horizon. Again, this is proved Section 2.4 and Section 2.5. In particular, in Proposition 2.4.14 and Proposition 2.5.2 it is proved that $\psi_\flat$ and $\psi_\sharp$, respectively, are continuously extendible beyond the Cauchy horizon. Thus, $\psi = \psi_\flat + \psi_\sharp$ can be continuously extended beyond the Cauchy horizon which concludes the proof. $\square$

2.4 Low frequency part $\psi_\flat$

We will begin this section by showing that $\psi_\flat$ decays superpolynomially in the exterior regions $\mathcal{R}_A$ and $\mathcal{R}_B$ (Section 2.4.1). This strong decay in the exterior regions then leads to uniform boundedness of $\psi_\flat$ in the interior $\mathcal{B}$ and continuous extendibility of $\psi_\flat$ beyond the Cauchy horizon. This will be shown in Section 2.4.2. In the following, it suffices to only consider $\mathcal{R}_A$ because the region $\mathcal{R}_B$ can be treated completely analogously.

2.4.1 Exterior estimates

We will now consider $\psi_\flat$ in the exterior region $\mathcal{R}_A$ and show an integrated energy decay estimate which will eventually lead to the superpolynomial decay for $\psi_\flat$. First, however, we review the separation of variables for solutions to (2.1.1).

Definition 2.4.1. *Let $\phi \in CH^2_{\mathrm{RNAdS}}$ be a solution to* (2.1.1) *satisfying*

$$\sum_{0\leq i,j\leq 2} \int_{\mathbb{R}} |\partial_t^i \partial_r^j \langle \phi, Y_{\ell m}\rangle_{\mathbb{S}^2}(r,t)| \mathrm{d}t < \infty \tag{2.4.1}$$

for $r \in (r_-, r_+)$, $r \in (r_+, \infty)$ and every $|m| \leq \ell$. In the regions $\mathcal{B}$ and $\mathcal{R}_A$, respectively, set

$$u[\phi](r,\omega,\ell,m) := \frac{r}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} \langle \phi, Y_{\ell m}\rangle_{L^2(\mathbb{S}^2)} \mathrm{d}t, \tag{2.4.2}$$

where $(Y_{\ell m})_{|m|\leq \ell}$ are the standard spherical harmonics.

Proposition 2.4.1. *Let ψ be as in* (2.3.4) *and $\psi_\flat$, $\psi_\sharp$ be as in* (2.3.5)*. Then, $u[\psi](r,\omega,\ell,m)$, $u[\psi_\flat](r,\omega,\ell,m)$ and $u[\psi_\sharp](r,\omega,\ell,m)$ as in Definition 2.4.1 are well-defined and smooth functions of r,ω in $\mathcal{R}_A$ and $\mathcal{B}$.*

Proof. First, note that $\psi^{\ell m} := \langle \psi, Y_{\ell m}\rangle Y_{\ell m}$ is a solution to (2.1.1), supported on the fixed angular parameter tuple (ℓ, m). Thus, in view of Theorem 2.2 and Proposition 2.6.4, $\psi^{\ell m}(t,r,\theta,\varphi)$ and all its derivatives decay exponentially in t in $\mathcal{R}_A$ and in $\mathcal{B}$ on any $\{r = \mathit{const.}\}$ slice. $\square$

Proposition 2.4.2. *Let $\phi \in CH^2_{\mathrm{RNAdS}}$ be a C^2-solution to* (2.1.1) *satisfying* (2.4.1)*. Let $u[\phi]$ be defined as in* (2.4.2)*. Then, $u[\phi]$ solves the* ***radial o.d.e.*** *(in $\mathcal{B}$ and $\mathcal{R}_A$)*

$$-u'' + (V_\ell - \omega^2)u = 0, \tag{2.4.3}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}r_*}$,

$$V_\ell(r) = h\left(\frac{\frac{\mathrm{d}h}{\mathrm{d}r}}{r} + \frac{\ell(\ell+1)}{r^2} - \frac{\alpha}{l^2}\right) \tag{2.4.4}$$

and

$$h = \frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{r^2}{l^2} + \frac{Q^2}{r^2}. \tag{2.4.5}$$

Moreover, in the exterior region $\mathcal{R}_A$ *we have* $\lim_{r\to\infty}|r^{\frac{1}{2}}u[\phi]| = 0$, $\lim_{r\to\infty}|r^{-\frac{1}{2}}u[\phi]'| = 0$. *Finally, note that*

$$\frac{\mathrm{d}V_\ell}{\mathrm{d}r} = \frac{\mathrm{d}h}{\mathrm{d}r}\left(\frac{\frac{\mathrm{d}h}{\mathrm{d}r}}{r} + \frac{\ell(\ell+1)}{r^2} - \frac{\alpha}{l^2}\right) + h\left(-\frac{\frac{\mathrm{d}h}{\mathrm{d}r}}{r^2} + \frac{\frac{d^2h}{\mathrm{d}r^2}}{r} - \frac{2\ell(\ell+1)}{r^3}\right). \tag{2.4.6}$$

Proof. The fact that $u[\phi]$ solves the radial o.d.e. is a direct computation. For the decay statement as $r \to \infty$, note that $u[\phi](r,\omega,\ell,m) = u[\phi_{\ell m}](r,\omega,\ell,m)$, where $\phi_{\ell m} := \langle\phi, Y_{\ell m}\rangle_{\mathbb{S}^2} Y_{\ell m}$. In particular, (2.2.28) (together with Remark 2.2.2) then implies

$$\int_{-\infty}^{\infty}\left(\int_{r_+}^{\infty} r^2|\langle\phi, Y_{\ell m}\rangle_{\mathbb{S}^2}|^2\mathrm{d}r\right)^{\frac{1}{2}}\mathrm{d}t < \infty. \tag{2.4.7}$$

Thus,

$$\begin{aligned}\left(\int_{r_+}^{\infty}|u[\phi]|^2\mathrm{d}r\right)^{\frac{1}{2}} &\lesssim \left(\int_{r_+}^{\infty}\left(\int_{-\infty}^{\infty} r^2|\langle\phi, Y_{\ell m}\rangle_{\mathbb{S}^2}|\mathrm{d}t\right)^2\mathrm{d}r\right)^{\frac{1}{2}}\\ &\leq \int_{-\infty}^{\infty}\left(\int_{r_+}^{\infty} r^2|\langle\phi, Y_{\ell m}\rangle_{\mathbb{S}^2}|^2\mathrm{d}r\right)^{\frac{1}{2}}\mathrm{d}t < \infty.\end{aligned} \tag{2.4.8}$$

Since $u[\phi]$ solves (2.4.3), analyzing the indicial equation at the regular singularity $r = \infty$ (see [40, Section 2.2.2]), shows that $|r^{\frac{1}{2}}u[\phi]| = O(r^{-\sqrt{\frac{9}{4}-\alpha}})$ and $|r^{-\frac{1}{2}}u[\phi]'| = O(r^{-\sqrt{\frac{9}{4}-\alpha}})$ as $r \to \infty$ in order to satisfy (2.4.8).[7] □

Next, we prove that the potential V_ℓ has a local maximum for large enough angular parameter ℓ_0.

Proposition 2.4.3. *There exists an* $\tilde{\ell}_0(M,Q,l,\alpha) \in \mathbb{N}$ *such that for all* $\ell \geq \tilde{\ell}_0$*, the potential* V_ℓ *has a local maximum* $r_{\ell,\max} > r_+$ *and* $V_\ell' \geq 0$ *for* $r_+ \leq r \leq r_{\ell,\max}$*. Moreover,* $r_{\ell,\max} \to r_{\max} := \frac{3}{2}M + \sqrt{\frac{9}{4}M^2 - 2Q^2}$ *as* $\ell \to \infty$.

Proof. Note that for ℓ large enough, V_ℓ is non-negative in a neighborhood of r_+ with $r \geq r_+$. Also, V_ℓ vanishes at $r = r_+$. Hence, it suffices to show that $\frac{\mathrm{d}V_\ell}{\mathrm{d}r}$ is negative somewhere for $r \geq r_+$. But note that

$$\frac{\mathrm{d}V_\ell}{\mathrm{d}r} = F(r) + r^{-3}\ell(\ell+1)\left(r\frac{\mathrm{d}h}{\mathrm{d}r} - 2h\right) = F(r) + 2r^{-3}\ell(\ell+1)\left(\frac{3M}{r} - 1 - \frac{2Q^2}{r^2}\right) \tag{2.4.9}$$

[7] The integrability condition (2.4.8) corresponds to the Dirichlet boundary condition at infinity on the level of the o.d.e.

for some function $F(r)$ which is independent of ℓ. Now, first choose $r > r_+$ large enough only depending on M, Q such that the last term is negative. Then, choose ℓ large enough such that it dominates the first term which proves that a $r_{\ell,\max}$ as in the statement exists. The limiting behavior $r_{\ell,\max} \to \frac{3}{2}M + \sqrt{\frac{9}{4}M^2 - 2Q^2}$ as $\ell \to \infty$ also follows from (2.4.9). This concludes the proof. □

Now, we are in the position to prove a frequency localized integrated decay estimate in the exterior region for the bounded frequencies $|\omega| \leq 2\omega_0$.

Proposition 2.4.4. *Let $u(r_*) = u^{(\omega,m,\ell)}(r_*)$ solve the radial o.d.e. (2.4.3) in the exterior $\mathcal{R}_A$ and assume that $\lim_{r\to\infty} |r^{\frac{1}{2}} u| = 0$ and $\lim_{r\to\infty} |r^{-\frac{1}{2}} u'| = 0$. Moreover, let $|\omega| \leq 2\omega_0$, where $\omega_0(M, Q, \ell, \alpha) > 0$ small enough will be fixed in the following proof. Then, we have*

$$\int_{R_*^{-\infty}}^{r=\infty} \frac{\Delta}{r^4} \left(|u'|^2 + |u|^2(\ell(\ell+1) + r^2)\right) \mathrm{d}r_* \lesssim -\tilde{Q}(R_*^{-\infty}) \tag{2.4.10}$$

for all $R_*^{-\infty}$ small enough such that $r(R_*^{-\infty}) < r_0$, where $r_0 = r_0(M, Q, l, \alpha) > r_+$ is determined in the following proof. Here, the boundary term $\tilde{Q}(R_*^{-\infty})$ satisfies

$$|\tilde{Q}(R_*^{-\infty})| \lesssim (|\omega|^2|u|^2 + |u'|^2)(1 + O_\ell(r - r_+)) \text{ as } R_*^{-\infty} \to -\infty. \tag{2.4.11}$$

Proof. We will first argue that it suffices to prove (2.4.10) for $\ell \geq \ell_0(M, Q, l, \alpha)$ for some fixed $\ell_0(M, Q, l, \alpha) \in \mathbb{N}_0$. Note that (2.4.10) for $\ell \leq \ell_0$ is an easier variant of []. Indeed, we perform the same steps in [] but instead take $a = 0$, $\omega_+ = 0$ and $H = 0$ throughout []. This leads to [] with L replaced by ℓ_0. The estimate on the boundary term follows from [].

We will now consider $\ell \geq \ell_0$, where ℓ_0 is determined below. Let r_0, r_1 depending only on M, Q, l, α be such that $r_+ < r_0 < r_1 < r_{\max} - \delta$, where $r_{\max}$ is defined in Proposition 2.4.3. Here, $\delta = \delta(\ell_0) > 0$ is such that $V' \geq 0$ for all $r_+ \leq r \leq r_{\max} - \delta$, cf. Proposition 2.4.3. We can make $\delta(\ell_0)$ as small as we want by choosing ℓ_0 sufficiently large. Now, we choose $\omega_0(M, Q, l, \alpha) > 0$ small enough and ℓ_0 large enough such that

$$\begin{aligned} &V - \omega^2 + \frac{\Delta}{4l^2r^2} \gtrsim \ell(\ell+1) + \frac{\Delta}{r^2} \quad \text{for } r \geq r_0, \\ &V - \omega^2 \geq 0 \quad \text{for } r_0 \leq r \leq r_1, \end{aligned} \tag{2.4.12}$$

and for all $|\omega| \le 2\omega_0$, $\ell \ge \ell_0$. For smooth $f(r_*)$ and $\tilde{h}(r_*)$, we define the currents

$$Q^f := f\left[|u'|^2 + (\omega^2 - V)|u|^2\right] + f' \operatorname{Re}(u'\bar{u}) - \frac{1}{2}f''|u|^2, \tag{2.4.13}$$

$$Q^{\tilde{h}} := \tilde{h} \operatorname{Re}(\bar{u}u') - \frac{1}{2}\tilde{h}'|u|^2 \tag{2.4.14}$$

with

$$Q^{f'} = \frac{\mathrm{d}Q^f}{\mathrm{d}r_*} = 2f'|u'|^2 - fV'|u|^2 - \frac{1}{2}f'''|u|^2, \tag{2.4.15}$$

$$Q^{\tilde{h}'} = \frac{\mathrm{d}Q^{\tilde{h}}}{\mathrm{d}r_*} = \tilde{h}\left[|u'|^2 + (V - \omega^2)|u|^2\right] - \frac{1}{2}\tilde{h}''|u|^2, \tag{2.4.16}$$

where we recall that $'$ denotes the derivative $\frac{\mathrm{d}}{\mathrm{d}r_*}$. Thus,

$$Q^{f'} + Q^{\tilde{h}'} = |u'|^2(2f' + \tilde{h}) + |u|^2\left(-fV' - \frac{1}{2}f''' + \tilde{h}(V - \omega^2) - \frac{1}{2}\tilde{h}''\right).$$

We choose a smooth $f \le 0$ such that

- f is monotonically increasing,
- $f = -1/r^2$ in a neighborhood of $r = r_+$,
- $f \le -c_1$ for $r_+ \le r \le r_1$ and some $c_1(M, Q, l) > 0$,
- $\Delta \lesssim f' \lesssim \Delta$ for $r_+ \le r \le r_1$,
- $|f'''| \lesssim \Delta$,
- $f = 0$ for $r \ge r_{\max} - \delta$.

and a smooth $\tilde{h} \ge 0$ such that

- $\tilde{h} = 0$ for $r \le r_0$,
- $|\tilde{h}''| \lesssim 1$ for $r_0 < r_1$,
- $\tilde{h} = 1$ for $r \ge r_1$.

Then, we have

$$Q^{f'} + Q^{\tilde{h}'} \ge \begin{cases} 2f'|u'|^2 + |u|^2(-fV' - \frac{1}{2}f''') & \text{for } r_+ \le r \le r_0, \\ 2f'|u'|^2 + |u|^2(-fV' - \frac{1}{2}f''' - \frac{1}{2}\tilde{h}'') & \text{for } r_0 \le r \le r_1, \\ |u'|^2 + |u|^2(-\frac{1}{2}f''' + (V - \omega^2)) & \text{for } r \ge r_1. \end{cases} \tag{2.4.17}$$

Thus, choosing $\ell_0(M,Q,l,\alpha)$ large enough (and $\omega_0(M,Q,l,\alpha) > 0$ possibly smaller) and using (2.4.17), (2.4.12), (2.4.9) and the properties of f and $\tilde{h}$, we have

$$Q^{f'} + Q^{\tilde{h}'} \gtrsim \frac{\Delta}{r^4}\left(|u'|^2 + |u|^2(\ell(\ell+1) + r^2)\right) \tag{2.4.18}$$

for $r_+ \leq r \leq r_{\max} - \delta$ and

$$Q^{f'} + Q^{\tilde{h}'} \gtrsim |u'|^2 + (V-\omega^2)|u|^2 \geq |u'|^2 - |u|^2 \frac{\Delta}{4l^2r^2} + \tilde{c}\left(\ell(\ell+1) + \frac{\Delta}{r^2}\right)|u|^2 \tag{2.4.19}$$

for $r \geq r_{\max} - \delta$ and some $\tilde{c}(M,Q,l,\alpha) > 0$. Integrating $Q^{f'} + Q^{\tilde{h}'}$ in the region $r_* \in (R_*^{-\infty}, r_*(r=+\infty))$ and applying the following Hardy inequality (see [72, Lemma 7.1])

$$\int_{r=r_{\max}-\delta}^{r=\infty} |u'|^2 \mathrm{d}r_* \geq \int_{r=r_{\max}-\delta}^{r=\infty} \frac{\Delta}{4l^2r^2}|u|^2 \mathrm{d}r_* \tag{2.4.20}$$

to control the negative signed term in (2.4.19), yields

$$\int_{R_*^{-\infty}}^{r=+\infty} \frac{\Delta}{r^4}\left(|u'|^2\chi_{\{r\leq r_{\max}-\delta\}} + |u|^2(\ell(\ell+1)+r^2)\right)\mathrm{d}r_* \lesssim -Q^f(R_*(-\infty)). \tag{2.4.21}$$

Note that we use $\lim_{r\to\infty}|r^{\frac{1}{2}}u| = 0$ and $\lim_{r\to\infty}|r^{-\frac{1}{2}}u'| = 0$ to apply the Hardy inequality. To obtain control of $|u'|^2$ in the region $r \geq r_{\max} - \delta$ in (2.4.21) we just add a small portion of the integral over (2.4.19). This proves

$$\int_{R_*^{-\infty}}^{r=+\infty} \frac{\Delta}{r^4}\left(|u'|^2 + |u|^2(\ell(\ell+1)+r^2)\right)\mathrm{d}r_* \lesssim -Q^f(R_*(-\infty)), \tag{2.4.22}$$

where $|Q^f(R_*^{-\infty})| \lesssim (|\omega|^2|u|^2 + |u'|^2)(1 + O_\ell(r - r_+)$ as $R_*^{-\infty} \to -\infty$ is satisfied by the construction of f. $\square$

With the frequency localized integrated energy decay estimate of Proposition 2.4.4 we will now prove a *local* integrated energy decay estimate in physical space. Indeed, a naive application of Plancherel's theorem to (2.4.10) gives a *global* integrated energy estimate. However, localizing this energy decay requires some sort of cut-off which does not respect the compact frequency support. Nevertheless, by carefully choosing a localization, we can show that the error term decays superpolynomially in time. At this point we shall remark that we do expect $\psi_\flat$ to decay exponentially. However, for our problem, superpolynomial decay in the exterior is (more than) sufficient.

Proposition 2.4.5. *Let $\psi_\flat$ be as in* (2.3.5)*. Then, for any $q > 1$, $\tau_1 \geq 0$ and in view of* (2.2.23)*, we have the integrated energy decay estimate*

$$\begin{aligned}
&\int_{\mathcal{R}_A \cap \{t^* \geq 2\tau_1\}} \left[r^{-2} |\partial_{t^*} \psi_\flat|^2 + r^{-2} |\partial_{r_*} \psi_\flat|^2 + |\nabla\!\!\!/ \psi_\flat|^2 + |\psi_\flat|^2 \right] r^2 \mathrm{d}t^* \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \\
&\lesssim \int_{\Sigma_{\tau_1} \cap \mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu + \frac{C(q)}{1+\tau_1^q} \int_{\Sigma_0} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0},
\end{aligned} \tag{2.4.23}$$

where $C(q) > 0$ is a constant only depending on q. Moreover, for any $\tau_2 \geq 2\tau_1$, this directly implies

$$\begin{aligned}
&\int_{\Sigma_{\tau_2} \cap \mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} \\
&+ \int_{\mathcal{R}_A \cap \{t^* \geq 2\tau_1\}} \left[r^{-2} |\partial_{t^*} \psi_\flat|^2 + r^{-2} |\partial_{r_*} \psi_\flat|^2 + |\nabla\!\!\!/ \psi_\flat|^2 + |\psi_\flat|^2 \right] r^2 \mathrm{d}t^* \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \\
&\lesssim \int_{\Sigma_{\tau_1} \cap \mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu + \frac{C(q)}{1+\tau_1^q} \int_{\Sigma_0 \cap \mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}
\end{aligned} \tag{2.4.24}$$

for the T-energy.

Proof. In order to show (2.4.23) we will first construct an auxiliary solution Ψ of (2.1.1). We set initial data for Ψ on Σ_{τ_1} as $(\Psi_0, \Psi_1) := (\psi_\flat, \mathcal{T}\psi_\flat) \restriction_{\Sigma_{\tau_1} \cap \mathcal{R}_A}$. Then, we will define data Ψ_2 on $\mathcal{H}^+_A \cap \{t^* \leq \tau_1\}$ such that the data can be extended to a C^k function in a neighborhood of $\mathcal{H}^+_A \cap \{t^* = \tau_1\}$ for some finite regularity k. Choosing the regularity k large enough will guarantee well-posedness. More precisely, in local coordinates $(t^*, r, \theta, \varphi)$ and for $r = r_+$, we define

$$\Psi_2(t^*, r_+, \varphi, \theta) := \sum_{j=1}^{k} \lambda_j \psi_\flat \restriction_{\{t^* \geq \tau_1\}} (-j(t^* - \tau_1) + \tau_1, r_+, \varphi, \theta) \tag{2.4.25}$$

for $t^* \leq \tau_1$ and some uniquely determined $(\lambda_j)_{1 \leq j \leq k}$ such that

$$\mathbb{R} \times \mathbb{S}^2 \ni (t^*, \varphi, \theta) \mapsto \begin{cases} \Psi_2(t^*, r_+, \varphi, \theta) & \text{for } t^* \leq \tau_1 \\ \psi_\flat(t^*, r_+, \varphi, \theta) & \text{for } t^* > \tau_1 \end{cases} \tag{2.4.26}$$

is C^k. Indeed, the function is smooth everywhere except at $t^* = \tau_1$. Now, we consider the mixed boundary value-Cauchy-characteristic problem, where we impose data as follows. On the null hypersurface $\mathcal{H}^+_A \cap \{t^* \leq \tau_1\}$ we impose Ψ_2. This null cone intersects the spacelike hypersurface Σ_{τ_1} on which we have prescribed (Ψ_0, Ψ_1) as data. As before, we

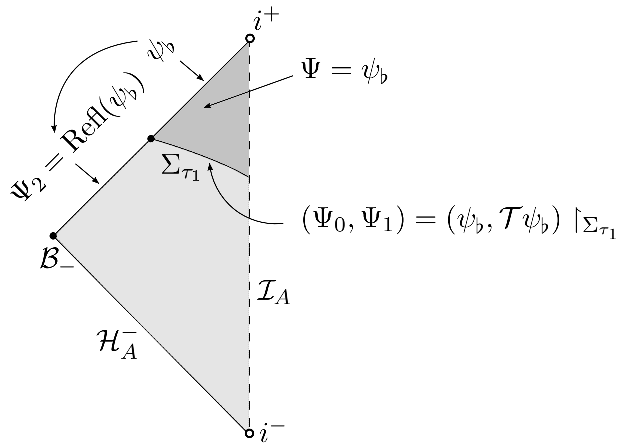

Figure 2.4: In the darker shaded region $J^+(\Sigma_{\tau_1}) \cap \mathcal{R}_A$ we have that $\Psi = \psi_\flat$ and in the lighter shaded region we can estimate the energy of Ψ in terms of $\psi_\flat$. This holds true as Ψ_2 is the C^k reflection of $\psi_\flat$ from $\mathcal{H}_A^+ \cap \{t^* \geq \tau_1\}$ to $\mathcal{H}_A^+ \cap \{t^* < \tau_1\}$.

assume the Dirichlet condition on $\mathcal{I}_A$. For fixed $k > 0$ large enough, this is a well-posed problem and can be solved backwards and forwards in $\mathcal{R}_A$ [111, Theorem 2]. We will call the arising solution Ψ and by uniqueness note that $\Psi = \psi_\flat$ on $(\mathcal{R}_A \cup \mathcal{H}_A^+) \cap J^+(\Sigma_{\tau_1})$. Indeed, analogously to $\psi_\flat$, we have $\Psi \in CH^2_{\mathrm{RNAdS}}$ and by choosing k large enough, we can make Ψ arbitrarily regular, in particular C^2. Moreover, Ψ decays logarithmically and $\langle \Psi, Y_{\ell m}\rangle Y_{\ell m}$ decays exponentially towards i^+ and i^- on a $\{r = \mathit{const.}\}$ hypersurface.[8] Refer to Fig. 2.4 for a visualization of the Cauchy-characteristic problem with Dirichlet boundary conditions.

Analogously to $\psi = \psi_\flat + \psi_\sharp$, we decompose the new solution Ψ in low and high frequencies $\Psi = \Psi_\flat + \Psi_\sharp$: We define

$$\Psi_\flat := \frac{1}{\sqrt{2\pi}} \mathcal{F}_T^{-1}[\chi_{2\omega_0}] * \Psi, \text{ and } \Psi_\sharp := \Psi - \Psi_\flat, \tag{2.4.27}$$

where $\chi_{2\omega_0}$ is a smooth cutoff function such that $\chi_{2\omega_0} = 1$ for $|\omega| \leq \omega_0$ and $\chi_{2\omega_0} = 0$ for $|\omega| \geq 2\omega_0$. Now, note that from the T-energy identity (2.2.21) we have

$$\int_{\mathcal{H}_A^+(\tau_1,\infty)} J^T_\mu[\psi_\flat] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}} = \int_{\Sigma_{\tau_1} \cap \mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}} \tag{2.4.28}$$

as the flux through $\mathcal{I}_A$ vanishes in view of the Dirichlet boundary condition at $\mathcal{I}_A$. Here, we use the notation $\mathcal{H}_A^+(a,b) := \mathcal{H}_A^+ \cap \{a < t^* < b\}$. Moreover, from the T energy identity,

[8] We will use this statement only in a qualitative way such that $u[\Psi_\flat]$ is well-defined in (2.4.31) and satisfies (2.4.10).

we have

$$\begin{aligned}\int_{\mathcal{H}_A^-} J^T_\mu[\Psi] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}} &= \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}} + \int_{\mathcal{H}^+_A(-\infty,\tau_1)} J^T_\mu[\Psi] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}}\\ &\lesssim \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}} + \int_{\mathcal{H}^+_A(\tau_1,\infty)} J^T_\mu[\psi_\flat] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}}\\ &\lesssim \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}. \end{aligned} \tag{2.4.29}$$

We have used the estimate

$$\int_{\mathcal{H}^+_A(-\infty,\tau_1)} J^T_\mu[\Psi] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}} \lesssim \int_{\mathcal{H}^+_A(\tau_1,\infty)} J^T_\mu[\psi_\flat] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}}$$

which follows from our construction of the initial data. Thus,

$$\int_{\mathcal{H}_A^-} J^T_\mu[\Psi] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}} + \int_{\mathcal{H}_A^+} J^T_\mu[\Psi] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}} \lesssim \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\Psi] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}. \tag{2.4.30}$$

Now, note that $u[\Psi_\flat]$ defined as

$$u[\Psi_\flat](r,\omega,\ell,m) = \frac{r}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega t} \langle \Psi_\flat, Y_{\ell m}\rangle_{L^2(\mathbb{S}^2)} \mathrm{d}t \tag{2.4.31}$$

satisfies the assumptions of Proposition 2.4.4 such that (2.4.10) holds true for $u[\Psi_\flat]$. We now integrate the frequency localized energy estimate (2.4.10) associated to $u[\Psi_\flat]$ in ω and sum over all spherical harmonics. There are two main terms appearing and we will estimate them in the following. This step is similar to [] so we will be rather brief. An application of Plancherel's theorem for the integrated left hand side of (2.4.10) yields

$$\begin{aligned}&\int_{\mathcal{R}_A} \left[|\partial_t\Psi_\flat|^2 + |\partial_{r_*}\Psi_\flat|^2 + r^2|\nabla\!\!\!/\,\Psi_\flat|^2 + r^2|\Psi_\flat|^2\right] \mathrm{d}t^*\mathrm{d}r \sin\theta \mathrm{d}\theta\mathrm{d}\varphi\\ &\lesssim \lim_{R_*^{-\infty}\to-\infty} \sum_{m\ell} \int_{\mathbb{R}} \mathrm{d}\omega \int_{R_*^{-\infty}}^{r=\infty} \mathrm{d}r_* \frac{\Delta}{r^4} \left[\omega^2 |u[\Psi_\flat]|^2 + |u[\Psi_\flat]'|^2 + \ell(\ell+1)|u[\Psi_\flat]|^2 + r^2|u[\Psi_\flat]|^2\right]. \end{aligned} \tag{2.4.32}$$

To estimate the boundary term on the right hand side of (2.4.10), we first decompose $u[\Psi_\flat]$ as $u[\Psi_\flat] = a(\omega,m,\ell)u_1 + b(\omega,m,\ell)u_2$, where u_1, u_2 are defined as the unique solutions to the radial o.d.e. (2.4.3) in the exterior satisfying $u_1 = e^{i\omega r_*} + O_\ell(r-r_+)$ and $u_2 = e^{-i\omega r_*} + O_\ell(r-r_+)$ as $r\to r_+$ ($r_*\to-\infty$). Here, $a = a(\omega,\ell,m)$ and $b = b(\omega,\ell,m)$ are the unique coefficients of the decomposition. Then, in view of (2.4.11) and $u_1' = i\omega u_1 + O_\ell(r-r_+)$,

$u_2' = -i\omega u_2 + O_\ell(r-r_+)$, we estimate

$$\begin{aligned}|\tilde{Q}| &\lesssim \left(|\omega|^2|a(\omega)|^2|u_1|^2 + |\omega|^2|b(\omega)|^2|u_2|^2\right)(1+O_\ell(r-r_+))\\ &= \left(|\omega|^2|a(\omega)|^2 + |\omega|^2|b(\omega)|^2\right)(1+O_\ell(r-r_+))\end{aligned} \tag{2.4.33}$$

as $r \to r_+$. Now, using that $\omega a(\omega)$, $\omega b(\omega)$ are in $L^1_\omega(\mathbb{R})$ and in $L^2_\omega(\mathbb{R})$ (note that they have compact support), an application of the Riemann–Lebesgue Lemma, the Fourier inversion theorem and Plancherel's theorem shows that $\sum_{m\ell}\int_{\mathbb{R}}|\omega|^2(|a(\omega,\ell,m)|^2+|b(\omega,\ell,m)|^2)\mathrm{d}\omega \lesssim \int_{\mathcal{H}_A^+}|T\Psi_\flat|^2 + \int_{\mathcal{H}_A^-}|T\Psi_\flat|^2 \leq 2\int_{\mathcal{H}_A^+}|T\Psi_\flat|^2$, where the last inequality follows from the T energy identity $\int_{\mathcal{H}_A^+}|T\Psi_\flat|^2 = \int_{\mathcal{H}_A^-}|T\Psi_\flat|^2$ in the region $\mathcal{R}_A$. Thus, we conclude the global integrated energy decay statement

$$\int_{\mathcal{R}_A}\left[\frac{1}{r^2}|\partial_t\Psi_\flat|^2 + \frac{1}{r^2}|\partial_{r_*}\Psi_\flat|^2 + |\nabla\!\!\!\!/\,\Psi_\flat|^2 + |\Psi_\flat|^2\right]\mathrm{dvol} \lesssim \int_{\mathcal{H}_A^+}|T\Psi_\flat|^2. \tag{2.4.34}$$

Hence, in view of $\psi_\flat = \Psi$ in $\{t^* \geq \tau_1\}\cap\mathcal{R}_A$ we have

$$\begin{aligned}&\int_{\mathcal{R}_A\cap\{t^*\geq 2\tau_1\}}\left[\frac{1}{r^2}|\partial_t\psi_\flat|^2 + \frac{1}{r^2}|\partial_{r_*}\psi_\flat|^2 + |\nabla\!\!\!\!/\,\psi_\flat|^2 + |\psi_\flat|^2\right]\mathrm{dvol}\\ &= \int_{\mathcal{R}_A\cap\{t^*\geq 2\tau_1\}}\left[\frac{1}{r^2}|\partial_t\Psi|^2 + \frac{1}{r^2}|\partial_{r_*}\Psi|^2 + |\nabla\!\!\!\!/\,\Psi|^2 + |\Psi|^2\right]\mathrm{dvol}\\ &\lesssim \int_{\mathcal{R}_A\cap\{t^*\geq 2\tau_1\}}\left[\frac{1}{r^2}|\partial_t\Psi_\flat|^2 + \frac{1}{r^2}|\partial_{r_*}\Psi_\flat|^2 + |\nabla\!\!\!\!/\,\Psi_\flat|^2 + |\Psi_\flat|^2\right]\mathrm{dvol}\\ &\quad + \int_{\mathcal{R}_A\cap\{t^*\geq 2\tau_1\}}\left[\frac{1}{r^2}|\partial_t\Psi_\sharp|^2 + \frac{1}{r^2}|\partial_{r_*}\Psi_\sharp|^2 + |\nabla\!\!\!\!/\,\Psi_\sharp|^2 + |\Psi_\sharp|^2\right]\mathrm{dvol}\\ &\lesssim \int_{\mathcal{H}_A^+}|T\Psi_\flat|^2 + \int_{t^*\geq 2\tau_1}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\Psi_\sharp]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\mathrm{d}t^*\\ &\lesssim \int_{\mathcal{H}_A^+}|T\Psi|^2 + \int_{t^*\geq 2\tau_1}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\Psi_\sharp]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\mathrm{d}t^*\\ &\lesssim \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\Psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}} + \int_{t^*\geq 2\tau_1}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\Psi_\sharp]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\mathrm{d}t^*\\ &= \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\psi_\flat]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}} + \int_{t^*\geq 2\tau_1}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\Psi_\sharp]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\mathrm{d}t^*.\end{aligned} \tag{2.4.35}$$

Here, we have also used (2.4.34), (2.2.23) and the fact that $\int_{\mathcal{H}_A^+}|T\Psi_\flat|^2 \lesssim \int_{\mathcal{H}_A^+}|T\Psi|^2$. Moreover, the estimate $\int_{\mathcal{H}_A^+}|T\Psi|^2 \lesssim \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^T_\mu[\Psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}}$ follows from (2.4.30).

Finally, we are left with the term $\int_{t^*\geq 2\tau_1}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\Psi_\sharp]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\mathrm{d}t^*$. We will show that this term decays at a superpolynomial rate. First, introduce $\chi_\sharp := 1 - \chi_{2\omega_0}$ and set $\check{\chi_{2\omega_0}} := \mathcal{F}_T^{-1}(\chi_{2\omega_0})$, $\check{\chi}_\sharp := \mathcal{F}_T^{-1}(\chi_\sharp)$, which are well-defined in the distributional sense. Then,

$$\Psi_\sharp = \frac{1}{\sqrt{2\pi}}\check{\chi}_\sharp * \Psi = \frac{1}{\sqrt{2\pi}}\check{\chi}_\sharp * (\Psi - \psi_\flat) \tag{2.4.36}$$

since $\check{\chi}_\sharp * \psi_\flat = 0$ in view of their disjoint Fourier support. In particular, for $t^* \geq \tau_1$ we have

$$\Psi_\sharp = \frac{1}{\sqrt{2\pi}}\check{\chi}_\sharp * (\Psi - \psi_\flat) = \frac{1}{\sqrt{2\pi}}(\sqrt{2\pi}\delta - \check{\chi_{2\omega_0}}) * (\Psi - \psi_\flat) = -\frac{1}{\sqrt{2\pi}}\check{\chi_{2\omega_0}} * (\Psi - \psi_\flat) \tag{2.4.37}$$

as $\delta * (\Psi - \psi_\flat) = \Psi - \psi_\flat = 0$ for $t^* \geq \tau_1$. To make notation easier we define $\phi := -\frac{1}{\sqrt{2\pi}}(\Psi - \psi_\flat)$ which is only supported for $t^* \leq \tau_1$ and satisfies $\Psi_\sharp = \check{\chi_{2\omega_0}} * \phi$. Now, as a result of the T invariance of $\mathrm{dvol}_{\Sigma_{t^*}}$ and $J^T_\mu[\cdot]n^\mu_{\Sigma_{t^*}}$, as well as (2.2.23), we have that

$$\begin{aligned}
&\int_{t^*\geq 2\tau_1}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^T_\mu[\Psi_\sharp]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\mathrm{d}t^*\\
&\lesssim \int_{t^*\geq 2\tau_1}\int_{(r_+,\infty)\times\mathbb{S}^2}\left(\frac{1}{r^2}|\partial_{t^*}\Psi_\sharp|^2+\frac{\Delta}{r^2}|\partial_r\Psi_\sharp|^2+|\nabla\!\!\!/\Psi_\sharp|^2\right)r^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}r\mathrm{d}t^*\\
&\leq \int_{t^*\geq 2\tau_1}\int_{(r_+,\infty)\times\mathbb{S}^2}\left[r^{-2}\left|\int_{-\infty}^{t(\tau_1,r)}\check{\chi_{2\omega_0}}(t(t^*,r)-s)(\partial_{t^*}\phi)(s)\mathrm{d}s\right|^2\right.\\
&\qquad+\frac{\Delta}{r^2}\left|\int_{-\infty}^{t(\tau_1,r)}\check{\chi_{2\omega_0}}(t(t^*,r)-s)(\partial_r\phi)(s)\mathrm{d}s\right|^2\\
&\qquad+\left.\left|\int_{-\infty}^{t(\tau_1,r)}|\check{\chi_{2\omega_0}}(t(t^*,r)-s)||\nabla\!\!\!/\phi|(s)\mathrm{d}s\right|^2\right]r^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}r\mathrm{d}t^*\\
&\leq \int_{-\infty}^{\infty}|\check{\chi_{2\omega_0}}(s)|\mathrm{d}s\int_{t^*\geq 2\tau_1}\int_{(r_+,\infty)\times\mathbb{S}^2}\left[\int_{-\infty}^{\tau_1}|\check{\chi_{2\omega_0}}(t^*-s^*)|r^{-2}|\partial_{t^*}\phi|^2(s^*)\mathrm{d}s^*\right.\\
&\qquad\left.+\int_{-\infty}^{\tau_1}|\check{\chi_{2\omega_0}}(t^*-s^*)|\frac{\Delta}{r^2}|\partial_r\phi|^2(s^*)\mathrm{d}s^*+\int_{-\infty}^{\tau_1}|\check{\chi_{2\omega_0}}(t^*-s^*)||\nabla\!\!\!/\phi|^2(s^*)\mathrm{d}s^*\right]r^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}r\mathrm{d}t^*\\
&\lesssim \int_{t^*\geq 2\tau_1}\int_{-\infty}^{\tau_1}|\check{\chi_{2\omega_0}}(t^*-s^*)|\\
&\qquad\times\left(\int_{(r_+,\infty)\times\mathbb{S}^2}\left[r^{-2}|\partial_{t^*}\phi|^2(s^*)+\frac{\Delta}{r^2}|\partial_r\phi|^2(s^*)+|\nabla\!\!\!/\phi|^2(s^*)\right]r^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}r\right)\mathrm{d}s^*\mathrm{d}t^*\\
&\lesssim \int_{\Sigma_0\cap\mathcal{R}_A}J^T_\mu[\phi]n^\mu_{\Sigma_0}\mathrm{dvol}_{\Sigma_0}\int_{t^*\geq 2\tau_1}\int_{-\infty}^{\tau_1}|\check{\chi_{2\omega_0}}(t^*-s^*)|\mathrm{d}s^*\mathrm{d}t^*\\
&\lesssim_q \int_{\Sigma_0\cap\mathcal{R}_A}J^T_\mu[\psi_\flat]n^\mu_{\Sigma_0}\mathrm{dvol}_{\Sigma_0}\int_{t^*\geq 2\tau_1}\int_{-\infty}^{\tau_1}\frac{1}{|t^*-s^*|^{q+2}}\mathrm{d}s^*\mathrm{d}t^*\\
&\lesssim_q \frac{\int_{\Sigma_0\cap\mathcal{R}_A}J^T_\mu[\psi_\flat]n^\mu_{\Sigma_0}\mathrm{dvol}_{\Sigma_0}}{1+\tau_1^q}.
\end{aligned}$$

Here, we have used the boundedness of the T-energy (cf. (2.2.22)), i.e.

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A}J^T_\mu[\phi]n^\mu_{\Sigma_{t^*}}\mathrm{dvol}_{\Sigma_{t^*}}\leq\int_{\Sigma_0\cap\mathcal{R}_A}J^T_\mu[\phi]n^\mu_{\Sigma_0}\mathrm{dvol}_{\Sigma_0}\lesssim\int_{\Sigma_0\cap\mathcal{R}_A}J^T_\mu[\psi_\flat]n^\mu_{\Sigma_0}\mathrm{dvol}_{\Sigma_0}. \tag{2.4.38}$$

Finally, we have also used that the Schwartz function $\check{\chi_{2\omega_0}}$ decays superpolynomially at any power $q>1$. This concludes the proof in view of (2.4.35). □

In order to remove the degeneracy of the T-energy at the event horizon, we will use the by now standard red-shift vector field [27]. As usual, the red-shift vector field N is a future-directed T invariant timelike vector field which has a positive bulk term $K^N \geq 0$ near the event horizon. In a compact r region bounded away from the event horizon $\mathcal{H}^+_A$, the bulk term K^N of N is sign-indefinite but this will be absorbed in the spacetime integral of the T current in Proposition 2.4.5. Also, note that $N = T$ for large enough r. In the negative mass AdS setting, we refer to [72, Section 4.2] for an explicit construction of the red-shift vector field N. Note that the red-shift vector field N has the property that

$$\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \sim \int_{\Sigma_{t^*}\cap\mathcal{R}_A} e_1[\psi_\flat] r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\varphi \tag{2.4.39}$$

for $\psi_\flat$ as in (2.3.5).

Proposition 2.4.6. *Let $\psi_\flat$ be as in* (2.3.5)*. Then for any $\tau_2 \geq 2\tau_1 \geq 0$, we have*

$$\begin{aligned}
&\int_{\Sigma_{\tau_2}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\mathcal{H}^+_A\cap\{2\tau_1\leq t^*\leq\tau_2\}} \left(|\partial_{t^*}\psi_\flat|^2 + |\nabla\!\!\!/\psi_\flat|^2 + |\psi_\flat|^2\right) \mathrm{d}t^* \mathrm{d}\sigma_{\mathbb{S}^2} \\
&+ \int_{2\tau_1}^{\tau_2} \int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \mathrm{d}t^* \lesssim_q \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu + \frac{\int_{\Sigma_0\cap\mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}}{1+\tau_1^q}
\end{aligned} \tag{2.4.40}$$

and in particular,

$$\begin{aligned}
\int_{\Sigma_{\tau_2}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} &+ \int_{2\tau_1}^{\tau_2} \int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \mathrm{d}t^* \\
&\lesssim_q \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu + \frac{\int_{\Sigma_0\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}}{1+\tau_1^q} \\
&\lesssim \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu + \frac{E^A_1[\psi_\flat](0)}{1+\tau_1^q}.
\end{aligned} \tag{2.4.41}$$

Proof. We apply the energy identity (the spacetime integral of (2.2.19)) with the red-shift vector field N for $\psi_\flat$ in the region $\mathcal{R}_A \cap \{2\tau_1 \leq t^* \leq \tau_2\}$, where $2\tau_1 \leq \tau_2$. After taking care of the negative lower order term via a Hardy inequality and absorbing the sign-indefinite bulk of N away from the horizon (in the region $\{r \geq r_0\}$ for some $r_0 > r_+$) in the spacetime integral of J^T on the right hand side (see [72, Section 4] for further details), we arrive at

$$\begin{aligned}\int_{\Sigma_{\tau_2}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} &+ \int_{\mathcal{H}^+_A\cap\{2\tau_1\leq t^*\leq\tau_2\}} J^N_\mu[\psi_\flat] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}} \\ &+ \int_{2\tau_1}^{\tau_2}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \mathrm{d}t^* \\ \lesssim \int_{2\tau_1}^{\tau_2}\int_{\Sigma_{t^*}\cap\mathcal{R}_A\cap\{r\geq r_0\}} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \mathrm{d}t^* &+ \int_{\Sigma_{2\tau_1}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}.\end{aligned} \tag{2.4.42}$$

First, note that the integrated energy term

$$\int_{2\tau_1}^{\tau_2}\int_{\Sigma_{t^*}\cap\mathcal{R}_A\cap\{r\geq r_0\}} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \mathrm{d}t^*$$

on the right-hand side of (2.4.42) can be controlled by the left-hand side of Proposition 2.4.5. Then, remark that the integral along the horizon $\int_{\mathcal{H}^+_A\cap\{2\tau_1\leq t^*\leq\tau_2\}} J^N_\mu[\psi_\flat] n^\mu_{\mathcal{H}} \mathrm{dvol}_{\mathcal{H}}$ is sign-indefinite due to the (possible) negative mass. However, this can be absorbed in the bulk term using an ϵ of the integrated bulk term of the red-shift vector field N and some of the bulk term of the integrated energy estimate in Proposition 2.4.5, cf. [72, Equation (70)]. Finally, using the integrated energy estimate from Proposition 2.4.5 again, we conclude

$$\begin{aligned}\int_{\Sigma_{\tau_2}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} &+ \int_{\mathcal{H}^+_A\cap\{2\tau_1\leq t^*\leq\tau_2\}} \left(|\partial_{t^*}\psi_\flat|^2 + |\nabla\!\!\!/\,\psi_\flat|^2 + |\psi_\flat|^2\right) \mathrm{d}t^* \mathrm{d}\sigma_{\mathbb{S}^2} \\ + \int_{2\tau_1}^{\tau_2}\int_{\Sigma_{t^*}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{t^*}} \mathrm{dvol}_{\Sigma_{t^*}} \mathrm{d}t^* &\lesssim_q \int_{\Sigma_{\tau_1}\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_{\tau_1}} + \frac{\int_{\Sigma_0\cap\mathcal{R}_A} J^T_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}}{1+\tau_1^q}.\end{aligned} \tag{2.4.43}$$

□

Now we obtain

Proposition 2.4.7. *Let $\psi_\flat$ be defined as in* (2.3.5)*. Then, for any $q>1$ and $\tau\geq 0$ we have*

$$\int_{\Sigma_\tau\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_\tau} \lesssim_q \frac{1}{1+\tau^q}\int_{\Sigma_0\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0} \lesssim_q \frac{1}{1+\tau^q} E^A_1[\psi_\flat](0) \tag{2.4.44}$$

and

$$\int_{\mathcal{H}(\tau,+\infty)} |\partial_{t^*}\psi_\flat|^2 + (|\nabla\!\!\!/\psi_\flat|^2 + |\psi_\flat|^2) \lesssim_q \frac{1}{1+\tau^q}\int_{\Sigma_0\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}$$
$$\lesssim_q \frac{1}{1+\tau^q} E^A_1[\psi_\flat](0). \tag{2.4.45}$$

Proof. In view of Proposition 2.4.6 it suffices to prove (2.4.44). Upon setting

$$f(s) := \int_{\Sigma_s\cap\mathcal{R}_A} J^N_\mu[\psi_\flat] n^\mu_{\Sigma_s} \mathrm{dvol}_{\Sigma_s},$$

we have from Proposition 2.4.6 that

$$f(t_2) + \int_{2t_1}^{t_2} f(s)\mathrm{d}s \lesssim_q f(t_1) + \frac{f(0)}{1+t_1^q}$$

for any $t_2 \geq 2t_1 \geq 0$. The claim follows now from Lemma 2.4.1 below. □

Lemma 2.4.1. *Let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function satisfying*

$$f(t_2) + \int_{2t_1}^{t_2} f(s)\mathrm{d}s \leq \alpha(q)\left(f(t_1) + \frac{f(0)}{1+t_1^q}\right) \tag{2.4.46}$$

for any $q > 1$, $0 \leq 2t_1 \leq t_2$ and some $\alpha(q) > 0$ only depending on q. Then, for all $q > 1$, there exists a constant $C(\alpha(q), q) > 0$ only depending on α and q such that

$$f(t) \leq \frac{C(\alpha(q), q)}{1+t^q} f(0) \tag{2.4.47}$$

for all $t \geq 0$.

Proof. Fix $q > 1$. First, note that from (2.4.46) we have for any $t_2 \geq 2t_1 > 0$

$$f(t_2) \leq \alpha(q)\left(f(t_1) + \frac{f(0)}{1+t_1^q}\right).$$

Without loss of generality, let $t > 10$ be arbitrary. Then, take a dyadic sequence $\tau_{k+1} = 2\tau_k$, where $\tau_0 = 1$. Now, there exists a $n \in \mathbb{N}_0$ such that $t \in [\tau_{n+3}, \tau_{n+4}]$. Then, again from (2.4.46) we have

$$\int_{\tau_{n+1}}^{\tau_{n+2}} f(s)\mathrm{d}s \leq \alpha(q)\left(f(\tau_n) + \frac{f(0)}{1+\tau_n^q}\right)$$

from which we conclude that there exists a $\xi \in [\tau_{n+1}, \tau_{n+2}]$ such that

$$f(\xi) \leq \alpha(q) \left(\frac{f(\tau_n)}{\tau_{n+1}} + \frac{f(0)}{1+\tau_n^{q+1}} \right).$$

Hence, since $2\xi \leq \tau_{n+3} \leq t \leq \tau_{n+4}$,

$$f(t) \leq \alpha(q) \left(f(\xi) + \frac{f(0)}{1+\tau_{n+1}^q} \right) \leq \alpha(q) \left(\alpha(q) \left(\frac{f(\tau_n)}{\tau_{n+1}} + \frac{f(0)}{1+\tau_n^{q+1}} \right) + \frac{f(0)}{1+\tau_{n+1}^q} \right). \tag{2.4.48}$$

Now, note that $\tau_n \sim t$ and hence, $f(t) \leq C(1, \alpha(q)) \frac{1}{1+t}$. This improved decay can now be fed into (2.4.48) to obtain a decay of the form $f(t) \leq C(2, \alpha(q)) \frac{1}{1+t^2}$. This procedure can be iterated until one obtains

$$f(t) \leq \frac{C(q, \alpha(q))}{1+t^q} f(0). \tag{2.4.49}$$

□

2.4.2 Interior estimates

Having obtained the superpolynomial decay for $\psi_\flat$ in the exterior and in particular on the event horizon, we will now use this to show uniform boundedness in the black hole interior. We will first propagate the superpolynomial decay on the horizon established in Proposition 2.4.7 further into the interior. To do so we will make use of the red-shift.

2.4.2.1 Red-shift region

With the help of the constructed red-shift current in Proposition 2.2.3, we obtain

Proposition 2.4.8. *Let $r_0 \in [r_{\mathrm{red}}, r_+)$. Let $\psi_\flat$ defined as in* (2.3.5) *and recall that from Proposition 2.4.7 we have*

$$\int_{\mathcal{H}(v_1,v_2)} \tilde{J}^N_\mu[\psi_\flat] n^\mu_{\mathcal{H}^+} \mathrm{dvol}_{\mathcal{H}^+} \lesssim_q \frac{1}{1+v_1^q} E_1^A[\psi_\flat](0) \tag{2.4.50}$$

for $1 \leq v_1 \leq v_2$. Then,

$$\int_{\underline{\mathcal{C}}_{v_1}(r_0,r_+)} \tilde{J}^N_\mu[\psi_\flat] n^\mu_{\underline{\mathcal{C}}_v} \mathrm{dvol}_{\underline{\mathcal{C}}_v} \sim \int_{-\infty}^{u_{r_0}(v_1)} \int_{\mathbb{S}^2} \frac{1}{\Omega^2} |\tilde{\nabla}_u \psi_\flat|^2 + \Omega^2 (|\nabla\!\!\!/ \psi_\flat|^2 + \mathcal{V}|\psi_\flat|^2) \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}u$$
$$\lesssim_q \frac{1}{1+v_1^q} E_1[\psi_\flat](0), \tag{2.4.51}$$

$$\int_{\Sigma_{r_0}(v_1,v_2)} \tilde{J}^N_\mu[\psi_\flat] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \sim \int_{v_1}^{v_2} \int_{\mathbb{S}^2} \frac{1}{\sqrt{\Omega^2}} |\tilde{\nabla}_u \psi_\flat|^2 + \sqrt{\Omega^2}\Big(|\tilde{\nabla}_v \psi_\flat|^2 + |\nabla\!\!\!/ \psi_\flat|^2 + \mathcal{V}|\psi_\flat|^2 \Big) \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \lesssim_q \frac{E_1[\psi_\flat](0)}{1+v_1^q} \tag{2.4.52}$$

for any $1 \leq v_1 \leq v_2$.

Proof. From Proposition 2.2.4, estimate (2.4.45) in Proposition 2.4.7 and upon defining

$$\tilde{E}(v) := \int_{\underline{C}_v(r_0,r_+)} \tilde{J}^N_\mu[\psi_\flat] n^\mu_{\underline{C}_v} \mathrm{dvol}_{\underline{C}_v}, \tag{2.4.53}$$

we obtain

$$\tilde{E}(v_2) + \int_{v_1}^{v_2} \tilde{E}(v)\mathrm{d}v \lesssim_q \tilde{E}(v_1) + \frac{E_1^A[\psi_\flat](0)}{1+v_1^q}, \tag{2.4.54}$$

for any $1 \leq v_1 \leq v_2$. This implies

$$\tilde{E}(v) \lesssim_q (\tilde{E}(v=1) + E_1^A[\psi_\flat](0)) \frac{1}{1+v^q} \tag{2.4.55}$$

for any $v \geq 1$. This follows from an argument very similar to Lemma 2.4.1. Note that we have by general theory [27] that $\tilde{E}(v=1) \lesssim E_1[\psi_\flat](0)$. Thus,

$$\tilde{E}(v) \lesssim_q E_1[\psi_\flat](0) \frac{1}{1+v^q} \tag{2.4.56}$$

for $v \geq 1$ which proves (2.4.51). The estimate (2.4.52) now follows from (2.4.51) and Proposition 2.2.4. □

2.4.2.2 No-shift region

Now, we will propagate the decay towards i^+ further into the black hole for $r \in [r_{\text{red}}, r_{\text{blue}}]$, where $r_{\text{blue}} > r_-$ is determined in the proof of Proposition 2.4.12.

Proposition 2.4.9. *Let* $\psi_\flat$ *defined as in* (2.3.5). *For any* $r_0 \in [r_{\text{blue}}, r_{\text{red}}]$, $q > 1$ *and any* $v_* \geq 1$ *we have*

$$\int_{\Sigma_{r_0}(v_*,2v_*)} \tilde{J}^X_\mu[\psi_\flat] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim_q \frac{E_1[\psi_\flat](0)}{1+v_*^q}. \tag{2.4.57}$$

Moreover, for any $1 < p < q$ we also have

$$\int_{\Sigma_{r_0}(v_*,+\infty)} (\langle v\rangle^p + \langle u\rangle^p)\tilde{J}^X_\mu[\psi_\flat] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim_{q,p} E_1[\psi_\flat](0). \tag{2.4.58}$$

Proof. Applying Proposition 2.2.5 with $\phi = \psi_\flat$ we have (2.2.49) for $\psi_\flat$. To estimate the right-hand side of (2.2.49) we use Proposition 2.4.8 and the fact that the difference $v_* - v_{r_{\text{red}}}(u_{r_0}(v_*))) = const.$ to obtain

$$\int_{\Sigma_{r_{\text{red}}}(v_{r_{\text{red}}}(u_{r_0}(v_*)),2v_*)} \tilde{J}^X_\mu[\psi_\flat] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim_q \frac{E_1[\psi_\flat](0)}{1+v_*^q} \tag{2.4.59}$$

from which (2.4.57) follows. Finally, (2.4.58) is a consequence of the fact that $\langle v\rangle^p \sim \langle u\rangle^p$ (using $r_{\text{blue}} \leq r \leq r_{\text{red}}$) and the following well-known lemma. □

Lemma 2.4.2. *Let $f\colon [1,\infty) \to \mathbb{R}_{\geq 0}$ be continuous and assume that there exists a $q \in \mathbb{R}$, $q > 1$ such that $\int_x^{2x} f(s)\mathrm{d}s \leq \frac{D}{x^q}$ for all $x \geq 1$ and some constant $D > 0$. Let $1 < p < q$ be fixed. Then, $\int_1^\infty s^p f(s)\mathrm{d}s < C(q,p)D$ for a constant $C(p,q) > 0$ only depending on p and q.*

Proof. Set $x_i := 2^i$. Then, $\int_1^\infty s^p f(s)\mathrm{d}s = \sum_{i=0}^\infty \int_{x_i}^{x_{i+1}} s^p f(s)\mathrm{d}s \leq 2^p D \sum_{i=0}^\infty 2^{ip-iq} < C(q,p)D$. □

Remark 2.4.1. *From now on we will consider p and q as* ***fixed*** *and constants appearing in $\lesssim$, $\gtrsim$ and $\sim$ can additionally depend on $1 < p < q$.*

By doing the analogous analysis in the neighborhood of the left component of i^+ we obtain

Proposition 2.4.10. *Let $\psi_\flat$ defined as in (2.3.5). Then, for any $r_0 \in [r_{\text{blue}}, r_+)$ we have*

$$\int_{\Sigma_{r_0}} (\langle v\rangle^p + \langle u\rangle^p)\left(|\tilde{\nabla}_u\psi_\flat|^2 + |\tilde{\nabla}_v\psi_\flat|^2 + |\nabla\!\!\!/\psi_\flat|^2 + |\psi_\flat|^2\right)\mathrm{dvol}_{\Sigma_r} \lesssim E_1[\psi_\flat](0). \tag{2.4.60}$$

Commuting with angular momentum operators $(\mathcal{W}_i)_{1\leq i\leq 3}$, an application of the Sobolev embedding $H^2(\mathbb{S}^2) \hookrightarrow L^\infty(\mathbb{S}^2)$ and using the fact that $p > 1$, we also conclude

Proposition 2.4.11. *Let $\psi_\flat$ defined as in (2.3.5). Then,*

$$\sup_{\mathcal{B}\cap\{r_{\text{blue}}\leq r<r_+\}} |\psi_\flat|^2 \lesssim E_1[\psi_\flat](0) + \sum_{i,j=1}^3 E_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat](0). \tag{2.4.61}$$

Finally, we will use the decay towards i^+ to show uniform boundedness in the interior and continuity all the way up to and including the Cauchy horizon for $\psi_\flat$.

2.4.2.3 Blue-shift region

We will now introduce the twisting function and vector field which we will use in the blue-shift region. Recall that we look for a twisting function f which satisfies $\mathcal{V} \gtrsim 1$, where

$$\mathcal{V} = -\left(\frac{\Box_g f}{f} + \frac{\alpha}{l^2}\right). \tag{2.4.62}$$

To do so, we set $f := e^{\beta_{\text{blue}} r}$ and obtain

$$\mathcal{V} = -\frac{\Box_g f}{f} - \frac{\alpha}{l^2} = \beta_{\text{blue}}^2 \Omega^2 + \beta_{\text{blue}} \partial_r(\Omega^2) + \frac{2}{r}\beta_{\text{blue}}\Omega^2 - \frac{\alpha}{l^2}. \tag{2.4.63}$$

Note that for $r_{\text{blue}} > r_-$ close enough to r_-, we have

$$\partial_r \Omega^2 \geq c_{\text{blue}} \tag{2.4.64}$$

for all $r_{\text{blue}} \geq r \geq r_-$ and some constant $c_{\text{blue}} > 0$ only depending on the black hole parameters. Thus, we obtain $\mathcal{V} \gtrsim 1$ uniformly in the blue-shift region $r_{\text{blue}} \geq r \geq r_-$ by choosing $\beta_{\text{blue}} > 0$ large enough and r_{blue} close enough to r_-. In the blue-shift region we define the vector field

$$S_N := r^N (\langle u \rangle^p \partial_u + \langle v \rangle^p \partial_v) \tag{2.4.65}$$

for some potentially large $N > 0$ and $p > 1$ as in Remark 2.4.1. We will show in the following that $\sup_{\theta,\varphi} |\psi_\flat(u_0, v_0, \theta, \varphi)|$ is uniformly bounded from initial data $D[\psi_\flat]$ independently of $(u_0, v_0) \in J^+(\Sigma_{r_{\text{blue}}}) \cap \mathcal{B}$. To do so, we will apply the energy identity (spacetime integral of (2.2.37)) in the region

$$\mathcal{R}_f = \mathcal{R}_f(u_0, v_0) = J^+(\Sigma_{r_{\text{blue}}}) \cap J^-(v_0, u_0) = J^+(\Sigma_{r_{\text{blue}}}) \cap \{u \leq u_0\} \cap \{v \leq v_0\} \tag{2.4.66}$$

which we depict in Fig. 2.5.

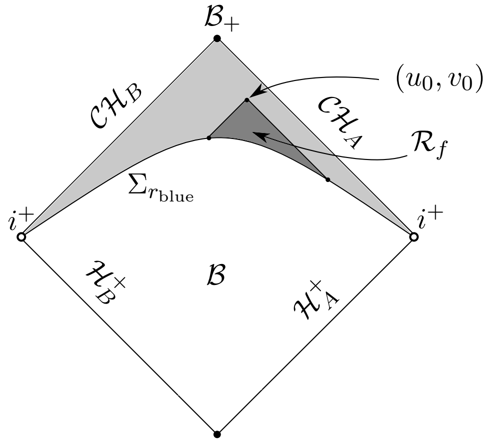

Figure 2.5: Illustration of the region $\mathcal{R}_f$ as the darker shaded region in the Penrose diagram of the interior $\mathcal{B}$. The lighter shaded region is the blue-shift region.

This leads to

$$\int_{\mathcal{C}_{u_0}(v_{r_{\mathrm{blue}}}(u_0),v_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\mathcal{C}_{u_0}} \mathrm{dvol}_{\mathcal{C}_{u_0}} + \int_{\underline{\mathcal{C}}_{v_0}(u_{r_{\mathrm{blue}}}(v_0),u_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\underline{\mathcal{C}}_{v_0}} \mathrm{dvol}_{\underline{\mathcal{C}}_{v_0}} + \int_{\mathcal{R}_f} \tilde{K}^{S_N}[\psi_\flat] \mathrm{dvol} = \int_{\Sigma_{r_{\mathrm{blue}}}\cap J^-(v_0,u_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\Sigma_{r_{\mathrm{blue}}}} \mathrm{dvol}_{\Sigma_{r_{\mathrm{blue}}}}, \quad (2.4.67)$$

where $\psi_\flat$ is defined in (2.3.5). In the following we will show, that after choosing $N > 0$ large enough and an appropriate integration by parts to control error terms, we can control the flux terms by initial data. This gives

Proposition 2.4.12. *Let $\psi_\flat$ defined as in* (2.3.5)*. Then,*

$$\int_{\mathcal{C}_{u_0}(v_{r_{\mathrm{blue}}}(u_0),v_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\mathcal{C}_{u_0}} \mathrm{dvol}_{\mathcal{C}_{u_0}} + \int_{\underline{\mathcal{C}}_{v_0}(u_{r_{\mathrm{blue}}}(v_0),u_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\underline{\mathcal{C}}_{v_0}} \mathrm{dvol}_{\underline{\mathcal{C}}_{v_0}} \lesssim \int_{\Sigma_{r_{\mathrm{blue}}}\cap J^-(v_0,u_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\Sigma_{r_{\mathrm{blue}}}} \mathrm{dvol}_{\Sigma_{r_{\mathrm{blue}}}} \lesssim E_1[\psi_\flat](0) \quad (2.4.68)$$

and

$$\int_{\mathcal{C}_{u_0}(v_{r_{\mathrm{blue}}}(u_0),v_0)} \left(\langle v\rangle^p |\partial_v \psi_\flat|^2 + (|\nabla\!\!\!/ \psi_\flat|^2 + |\psi_\flat|^2)\Omega^2\right) \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\underline{\mathcal{C}}_{v_0}(u_{r_{\mathrm{blue}}}(v_0),u_0)} \left(\langle u\rangle^p |\partial_u \psi_\flat|^2 + (|\nabla\!\!\!/ \psi_\flat|^2 + |\psi_\flat|^2)\Omega^2\right) \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \lesssim \int_{\Sigma_{r_{\mathrm{blue}}}\cap J^-(v_0,u_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\Sigma_{r_{\mathrm{blue}}}} \mathrm{dvol}_{\Sigma_{r_{\mathrm{blue}}}} \lesssim E_1[\psi_\flat](0) \quad (2.4.69)$$

for any $(u_0, v_0) \in J^+(\Sigma_{r_{\text{blue}}})$. *Commuting with the angular momentum operators* $(\mathcal{W}_i)_{1\leq i\leq 3}$ *also gives*

$$\int_{\mathcal{C}_{u_0}(v_{r_{\text{blue}}}(u_0),v_0)} \langle v\rangle^p \big(|\partial_v\psi_\flat|^2 + \sum_{i,j}|\partial_v\mathcal{W}_i\mathcal{W}_j\psi_\flat|^2\big)\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \lesssim E_1[\psi_\flat](0) + \sum_{i,j=1}^{3} E_1[\mathcal{W}_j\mathcal{W}_i\psi_\flat](0). \tag{2.4.70}$$

Proof. The general strategy of the proof is to apply (2.4.67) and to show that

$$\int_{\mathcal{R}_f} \tilde{K}^{S_N}\mathrm{dvol} \geq 0 + \text{boundary terms}, \tag{2.4.71}$$

where the boundary terms are small (lower orders in Ω) and by choosing r_{blue} closer to r_-, can be absorbed in the positive flux terms on the left hand side of (2.4.67). In the first part, we compute the flux terms for our vector field S^N defined in (2.4.65). Then, in the second part, we will estimate the bulk term and indeed show (2.4.71). From this we will then deduce (2.4.68).

Part I: Flux terms of S_N. We obtain three flux terms from (2.4.67). The future flux terms read (cf. Proposition 2.6.1)

$$\begin{aligned}\int_{\mathcal{C}_{u_0}(v_{r_{\text{blue}}}(u_0),v_0)} & \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\mathcal{C}_{u_0}}\mathrm{dvol}_{\mathcal{C}_{u_0}} \\ &= \int_{\mathcal{C}_{u_0}(v_{r_{\text{blue}}}(u_0),v_0)} \left(\langle v\rangle^p|\tilde{\nabla}_v\psi_\flat|^2 + \Omega^2\frac{\langle u\rangle^p}{4}(|\nabla\!\!\!/\psi_\flat|^2 + \mathcal{V}|\psi_\flat{}^2|)\right) r^{N+2}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\end{aligned} \tag{2.4.72}$$

and

$$\begin{aligned}\int_{\underline{\mathcal{C}}_{v_0}(u_{r_{\text{blue}}}(v_0),u_0)} & \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\mathcal{C}_{v_0}}\mathrm{dvol}_{\underline{\mathcal{C}}_{v_0}} \\ &= \int_{\underline{\mathcal{C}}_{v_0}(u_{r_{\text{blue}}}(v_0),u_0)} \left(\langle u\rangle^p|\tilde{\nabla}_u\psi_\flat|^2 + \Omega^2\frac{\langle v\rangle^p}{4}(|\nabla\!\!\!/\psi_\flat|^2 + \mathcal{V}|\psi_\flat|^2)\right) r^{N+2}\mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2}.\end{aligned} \tag{2.4.73}$$

The past flux term on the spacelike hypersurface $\Sigma_{r_{\text{blue}}}$ is uniformly bounded by initial data from Proposition 2.4.10:

$$\int_{\Sigma_{r_{\text{blue}}}\cap J^-(v_0,u_0)} \tilde{J}^{S_N}_\mu[\psi_\flat] n^\mu_{\Sigma_{r_{\text{blue}}}}\mathrm{dvol}_{\Sigma_{r_{\text{blue}}}} \lesssim E_1[\psi_\flat](0). \tag{2.4.74}$$

Part II: Bulk term of S_N. We will now estimate the bulk term

$$\int_{\mathcal{R}_f} \tilde{K}^{S_N} \mathrm{dvol}$$

appearing in the energy identity (2.4.67). The terms appearing in $\tilde{K}^{S_N}$ can be read off in (2.6.4) with $S_N^u = X^u = r^N \langle u \rangle^p$ and $S_N^v = X^v = r^N \langle v \rangle^p$. To estimate all terms, we will also integrate by parts and substitute terms of the form $\partial_u \partial_v \psi_\flat$ using the equation $\Box_g \psi_\flat = 0$. The boundary terms arising from the integration by parts will then be absorbed in the future flux terms appearing in *Part I: Flux terms of* S_N. In the following we shall treat each terms of $\tilde{K}^X$ as in (2.6.4) with $X = S_N$ individually.

***First term of* (2.6.4).** The first term of (2.6.4) is non-negative:

$$-\frac{2}{\Omega^2}\Big(\langle v \rangle^p \partial_u (r^N) |\tilde{\nabla}_v \psi_\flat|^2 + \langle u \rangle^p \partial_v (r^N) |\tilde{\nabla}_u \psi_\flat|^2 \Big) = N r^{N-1} (\langle v \rangle^p |\tilde{\nabla}_v \psi_\flat|^2 + \langle u \rangle^p |\tilde{\nabla}_u \psi_\flat|^2). \tag{2.4.75}$$

This means that—by choosing $N > 0$ large enough—we will be able to absorb sign-indefinite terms of the form $r^{N-1} \langle v \rangle^p |\tilde{\nabla}_v \psi_\flat|^2$ and $r^{N-1} \langle u \rangle^p |\tilde{\nabla}_u \psi_\flat|^2$. This will be used in the following.

Before we treat the second term appearing in (2.6.4), which is sign-indefinite, we look at the angular and potential term in the second line of (2.6.4).

***Angular and potential term: Second line of* (2.6.4).** Now, we look at the term involving angular derivatives. In the region $\mathcal{R}_f$ we have

$$\begin{aligned} &-\left(\frac{1}{2} (\partial_v (r^N \langle v \rangle^p) + \partial_u (r^N \langle u \rangle^p)) - \frac{r^N}{4} \left(\partial_r \Omega^2 \right) (\langle v \rangle^p + \langle u \rangle^p) \right) \left(|\nabla\!\!\!/ \psi_\flat|^2 + \mathcal{V} |\psi_\flat|^2 \right) \\ &\gtrsim r^N (\langle v \rangle^p + \langle u \rangle^p) \left(|\nabla\!\!\!/ \psi_\flat|^2 + \mathcal{V} |\psi_\flat|^2 \right). \end{aligned} \tag{2.4.76}$$

The terms arising when ∂_v hits $\langle v \rangle^p$ and when ∂_u hits $\langle u \rangle^p$ are sign-indefinite and of the form

$$-\frac{p}{2} r^N \left(\langle v \rangle^{p-2} v + \langle u \rangle^{p-2} u \right) \left(|\nabla\!\!\!/ \psi_\flat|^2 + \mathcal{V} |\psi_\flat|^2 \right). \tag{2.4.77}$$

They are absorbed in $r^N (\langle v \rangle^p + \langle u \rangle^p) \left(|\nabla\!\!\!/ \psi_\flat|^2 + \mathcal{V} |\psi_\flat|^2 \right)$. Indeed, for any fixed $\epsilon = \epsilon(p) > 0$, we can choose r_{blue} even closer to r_- (depending on ϵ) such that $|v| \langle v \rangle^{p-2} \leq \langle v \rangle^{p-1} \leq \epsilon (\langle v \rangle^p + \langle v - 2r_* \rangle^p)$ holds in $\mathcal{R}_f$ and similarly for $|u| \langle u \rangle^{p-2}$. Also recall that we have chosen the twisting function such that $\mathcal{V} \gtrsim 1$.

***Second, sign-indefinite term of* (2.6.4).** Now, note that the second term in the first line of (2.6.4)

$$-2r^{N-1}(\langle v\rangle^p + \langle u\rangle^p)\,\mathrm{Re}\left(\overline{\tilde{\nabla}_v\psi_\flat}\tilde{\nabla}_u\psi_\flat\right) \tag{2.4.78}$$

is sign-indefinite, however, we can absorb it in other positive terms after integrating by parts in the region $\mathcal{R}_f$ as we will see in the following. In order to integrate by parts, it is useful to express the twisted derivatives with ordinary derivatives. The integration by parts will generate boundary terms. As mentioned above, we estimate these boundary terms with the fluxes in the energy identity. This will be done later in (2.4.84) and we will not write the boundary terms explicitly in the following. We will also have to control (sign-indefinite) ordinary derivatives by positive terms in (2.4.75) and (2.4.76). Note that this is possible since

$$\langle v\rangle^p|\partial_v\psi_\flat|^2 = \langle v\rangle^p|\tilde{\nabla}_v\psi|^2 - \langle v\rangle^p\Omega^2\,\mathrm{Re}\left(\overline{\psi_\flat}\partial_v\psi_\flat\right) - \frac{1}{4}\langle v\rangle^p\Omega^4|\psi_\flat|^2, \tag{2.4.79}$$

where the right hand side of (2.4.79) is controlled by (2.4.75), (2.4.76) and potentially choosing r_{blue} closer to r_-. The analogous statement holds true for $\langle u\rangle^p|\partial_u\psi_\flat|^2$.

The integrated term we have to estimate reads

$$\int_{\mathcal{R}_f} -2r^{N-1}(\langle v\rangle^p + \langle u\rangle^p)\frac{1}{f^2}\,\mathrm{Re}\left(\overline{\partial_v(f\psi_\flat)}\partial_u(f\psi_\flat)\right)\Omega^2r^2\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}. \tag{2.4.80}$$

We only look at

$$\Big|\int_{\mathcal{R}_f} r^{N+1}\langle v\rangle^p\frac{1}{f^2}\,\mathrm{Re}\left(\overline{\partial_v(f\psi_\flat)}\partial_u(f\psi_\flat)\right)\Omega^2\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big|$$

as the term in (2.4.80) involving $\langle u\rangle^p$ is estimated in an analogous manner. Using the explicit form of f and noting that we have control over $(\langle v\rangle^p + \langle u\rangle^p)\Omega^4|\psi_\flat|^2$ from (2.4.76), it suffices to estimate

$$\begin{aligned}\Big|\int_{\mathcal{R}_f} r^{N+1}\langle v\rangle^p\,\mathrm{Re}\left(\overline{\partial_v\psi_\flat}\partial_u\psi_\flat\right)\Omega^2\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| + \Big|\int_{\mathcal{R}_f}\Omega^2\langle v\rangle^p\,\mathrm{Re}\left(\overline{\psi_\flat}(\partial_v\psi_\flat)\right)\Omega^2\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| \\ + \Big|\int_{\mathcal{R}_f}\Omega^2\langle v\rangle^p\,\mathrm{Re}\left(\overline{\psi_\flat}(\partial_u\psi_\flat)\right)\Omega^2\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big|.\end{aligned} \tag{2.4.81}$$

Now, note that the second term of (2.4.81) (excluding the factor Ω^2 appearing in the volume form) reads $r^{-2}\Omega^2\langle v\rangle^p\,\mathrm{Re}\left(\overline{\psi_\flat}(\partial_v\psi_\flat)\right)$ and is controlled by (2.4.75) and (2.4.76) using Cauchy's inequality and by potentially choosing r_{blue} even closer to r_-. Now, in both terms, the first and third term of (2.4.81), we integrate by parts in u. We also use $\operatorname{Re}\left(\overline{\psi_\flat}\partial_u\psi_\flat\right) = \frac{1}{2}\partial_u(|\psi_\flat|^2)$. Then, it follows that—up to boundary contributions which will be dealt with below in (2.4.84)—we have to control the terms

$$\Big|\int_{\mathcal{R}_f} N r^N \langle v\rangle^p \operatorname{Re}\left(\overline{\psi_\flat}\partial_v\psi_\flat\right)\Omega^4 \mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| + \Big|\int_{\mathcal{R}_f} r^{N+1}\langle v\rangle^p \operatorname{Re}\left(\overline{\psi_\flat}(\partial_u\partial_v\psi_\flat)\right)\Omega^4\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| + \Big|\int_{\mathcal{R}_f}\langle v\rangle^p|\psi_\flat|^2\Omega^4\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big|. \tag{2.4.82}$$

The first and third term (excluding Ω^2 as above) of (2.4.82) are controlled by (2.4.75), (2.4.76) and by potentially choosing r_{blue} even closer to r_-. For the second term of (2.4.82) we will use (2.1.1) which reads

$$0 = \Box_{g_{\text{RNAdS}}}\psi_\flat + \frac{\alpha}{\ell^2}\psi_\flat = \frac{-4}{\Omega^2}(\partial_u\partial_v\psi_\flat) + \frac{2}{r}\left(\partial_v\psi_\flat + \partial_u\psi_\flat\right) + \frac{1}{r^2}\slashed{\Delta}_{\mathbb{S}^2}\psi_\flat + \frac{\alpha}{\ell^2}\psi_\flat$$

to substitute $\partial_u\partial_v\psi_\flat$. Replacing $\partial_u\partial_v\psi_\flat$ and integrating by parts on the sphere, we estimate all but one term of (2.4.82) using (2.4.76) and (2.4.75). The term which we cannot estimate with (2.4.76) and (2.4.75) is of the form

$$\Big|\int_{\mathcal{R}_f} r^N\langle v\rangle^p \operatorname{Re}\left(\overline{\psi_\flat}(\partial_u\psi_\flat)\right)\Omega^6\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| = \frac{1}{2}\Big|\int_{\mathcal{R}_f} r^N\langle v\rangle^p\partial_u(|\psi_\flat|^2)\Omega^6\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big|. \tag{2.4.83}$$

This is of a similar form as the third term in (2.4.81), which we control—as before—via an integration by parts in u. Finally we have controlled all terms except for boundary terms arising from the integration by parts.

The first boundary terms arose from integrating by parts the first term in (2.4.81). It consists of two parts and is of the form

$$\Big|\int_{\mathcal{C}_{u_0}\cap\{v_{r_{\text{blue}}}(u_0)\leq v\leq v_0\}} r^{N+1}\langle v\rangle^p\operatorname{Re}\left(\overline{\psi_\flat}(\partial_v\psi_\flat)\right)\Omega^2\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| \tag{2.4.84}$$

$$+ \Big|\int_{\Sigma_{r_{\text{blue}}}\cap J^-(v_0,u_0)} r^{N+1}\langle v\rangle^p\operatorname{Re}\left(\overline{\psi_\flat}(\partial_v\psi_\flat)\right)\Omega^2\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big|. \tag{2.4.85}$$

The second term (2.4.85) is absorbed in the past flux term on the spacelike hypersurface $\Sigma_{r_{\text{blue}}}$ by choosing r_{blue} possibly closer to r_- and noting that $\mathrm{dvol}_{\Sigma_{r_{\text{blue}}}} = \sqrt{\Omega^2}r^2\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}$.

The first term (2.4.84) is controlled as follows

$$\begin{aligned}
&\Big| \int_{\mathcal{C}_{u_0}\cap\{v_{r_{\mathrm{blue}}}(u_0)\leq v\leq v_0\}} r^{N+1}\langle v\rangle^p \operatorname{Re}\left(\overline{\psi_\flat}(\partial_v\psi_\flat)\right) \Omega^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| \\
&\leq \Big| \int_{\mathcal{C}_{u_0}\cap\{v_{r_{\mathrm{blue}}}(u_0)\leq v\leq v_0\}} r^{N+1}\langle v\rangle^p |\partial_v\psi_\flat|^2\sqrt{\Omega^2}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big| \\
&+ \Big| \int_{\mathcal{C}_{u_0}\cap\{v_{r_{\mathrm{blue}}}(u_0)\leq v\leq v_0\}} r^{N+1}\langle v\rangle^p |\psi_\flat|^2(\Omega^2)^{\frac14}(\Omega^2)^{\frac14}\Omega^2\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\Big|. \qquad (2.4.86)
\end{aligned}$$

Now, note that

$$\langle v\rangle^p(\Omega^2)^{\frac14} \lesssim \langle r_* - u\rangle^p(\Omega^2)^{\frac14} \lesssim 1 + \langle u\rangle^p(\Omega^2)^{\frac14}, \qquad (2.4.87)$$

where we have used that $r_*^p(\Omega^2)^{\frac14} \lesssim 1$ for $r_* \geq r_*(r_{\mathrm{blue}})$ which holds true since Ω^2 decays exponentially as $r_* \to \infty$. Using (2.4.87) we absorb (2.4.86) in the flux term (2.4.72) by potentially choosing r_{blue} closer to r_- such that Ω^2 is uniformly small in the blue-shift region. Completely analogously, we control the other boundary terms which arose from integrating by parts.

Now, we are left with the terms of the last two lines in (2.6.4).

Terms from last two lines of (2.6.4)**.** We will only look at the terms with v weights as the terms involving u weights are estimated completely analogously. It suffices to estimate the terms

$$r^N\Big|\frac{\Omega^2}{2r}\langle v\rangle^p\mathcal{V}|\psi_\flat|^2\Big| + r^N\Big|\langle v\rangle^p\frac{\partial_v(f^2\mathcal{V})}{2f^2}|\psi_\flat|^2\Big| \qquad (2.4.88)$$

and

$$-r^N\langle v\rangle^p\frac{\partial_v f^2}{2f^2}\tilde{\nabla}_\sigma\psi_\flat\overline{\tilde{\nabla}^\sigma\psi_\flat}. \qquad (2.4.89)$$

Since $\left|\frac{\partial_v(f^2\mathcal{V})}{2f^2}\right| \lesssim \Omega^2$, we control the terms in (2.4.88) using (2.4.76) and by potentially choosing r_{blue} closer to r_-. Expanding (2.4.89) yields

$$-r^N\langle v\rangle^p\frac{\partial_v f^2}{2f^2}\tilde{\nabla}_\sigma\psi_\flat\overline{\tilde{\nabla}^\sigma\psi_\flat} = -2\beta_{\mathrm{blue}}r^N\langle v\rangle^p\operatorname{Re}\left(\overline{\tilde{\nabla}_u\psi_\flat}\tilde{\nabla}_v\psi_\flat\right) + \frac{\beta_{\mathrm{blue}}}{2}r^N\langle v\rangle^p\Omega^2|\not\nabla\psi_\flat|^2. \qquad (2.4.90)$$

The second term on the right-hand side is estimated by (2.4.76) and potentially choosing r_{blue} closer to r_-. The first term on the right-hand side of (2.4.90) has the same from as

(2.4.78) and is estimated in the same way as (2.4.78).

Finally, we have estimated and absorbed all sign-indefinite terms in the energy identity to obtain (2.4.71). Thus, we have proved (2.4.68), which concludes the first part of the proof.

***Part III: Proof of* (2.4.69) *and* (2.4.70).** Now, observe that the estimate (2.4.69) follows from (2.4.68) and (2.4.79). More precisely, the error arising from interchanging the twisted derivatives with partial derivatives on $\mathcal{C}_u$ are estimated as

$$\begin{aligned}\langle v\rangle^p|\partial_v\psi_\flat|^2 &= \langle v\rangle^p|\tilde{\nabla}_v\psi|^2 + \langle v\rangle^p\Omega^2\operatorname{Re}\left(\overline{\psi_\flat}\partial_v\psi_\flat\right) - \frac{1}{4}\langle v\rangle^p\Omega^4|\psi_\flat|^2\\ &\leq \langle v\rangle^p|\tilde{\nabla}_v\psi|^2 + |\langle v\rangle^p\Omega^2\operatorname{Re}\left(\overline{\psi_\flat}\partial_v\psi_\flat\right)|.\end{aligned}$$

Finally, note that the error term on the right hand side is controlled as in (2.4.84). This works for $\underline{\mathcal{C}}_v$ completely analogously which concludes the proof. □

2.4.2.4 Uniform boundedness and continuity at the Cauchy horizon for bounded frequencies

Now, Proposition 2.4.12 allows us to prove the uniform boundedness.

Proposition 2.4.13. *Let* $\psi_\flat$ *be as defined in* (2.3.5)*. Then,*

$$\sup_{\mathcal{B}\cap J^+(\Sigma_0)}|\psi_\flat|^2 \lesssim E_1[\psi_\flat](0) + \sum_{i,j=1}^{3}E_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat](0) \lesssim D[\psi_\flat]. \tag{2.4.91}$$

Proof. In view of Proposition 2.4.11, it suffices to prove (2.4.91) only in $J^+(\Sigma_{r_{\text{blue}}})\cap\mathcal{B}$. Let $(u_0,v_0)\in J^+(\Sigma_{r_{\text{blue}}})\cap\mathcal{B}$ be arbitrary. Then, by Proposition 2.4.11, Proposition 2.4.12 and the Sobolev embedding on the sphere $H^2(\mathbb{S}^2)\hookrightarrow L^\infty(\mathbb{S}^2)$, we have

$$\begin{aligned}|\psi_\flat(u_0,v_0,\varphi,\theta)|^2 &\lesssim \left(\int_{v_{r_{\text{blue}}}(u_0)}^{v_0}|\partial_v\psi_\flat(u_0,v,\varphi,\theta)|\mathrm{d}v\right)^2 + |\psi_\flat(u_0,v_{r_{\text{blue}}}(u_0),\varphi,\theta)|^2\\ &\lesssim \int_{\mathcal{C}_{u_0}(v_{r_{\text{blue}}}(u_0),v_0)}\langle v\rangle^p|\partial_v\psi_\flat|^2\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\\ &\quad+\sum_{i,j}\int_{\mathcal{C}_{u_0}(v_{r_{\text{blue}}}(u_0),v_0)}\langle v\rangle^p|\partial_v\mathcal{W}_i\mathcal{W}_j\psi_\flat|^2\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}\\ &\quad+E_1[\psi_\flat](0)+\sum_{i,j=1}^{3}E_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat] \lesssim E_1[\psi_\flat](0)+\sum_{i,j=1}^{3}E_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat](0),\end{aligned} \tag{2.4.92}$$

where $(\mathcal{W}_i)_{i=1,2,3}$ are the angular momentum operators. This shows (2.4.91). $\square$

Proposition 2.4.14. *Let* $\psi_\flat$ *be as defined in* (2.3.5). *Then,* $\psi_\flat$ *is continuously extendible beyond the Cauchy horizon* $\mathcal{CH}$.

Proof. Similarly to (2.4.92) we have

$$\begin{aligned}|\psi_\flat(u_0,v_2,\varphi,\theta)-\psi_\flat(u_0,v_1,\varphi,\theta)|^2 &\lesssim \int_{v_1}^{v_2}\langle v\rangle^{-p}\mathrm{d}v\int_{v_1}^{v_2}\langle v\rangle^{p}|\partial_v\psi_\flat(u_0,v,\varphi,\theta)|^2\mathrm{d}v\\ &\lesssim \int_{v_1}^{v_2}\langle v\rangle^{-p}\mathrm{d}v\left(E_1[\psi_\flat]+\sum_{i,j=1}^{3}E_1[\mathcal{W}_i\mathcal{W}_j\psi_\flat]\right)\end{aligned} \tag{2.4.93}$$

uniformly in u_0,φ,θ. The same estimate holds after interchanging the roles of u and v. After commuting the equation with $\mathcal{W}_3$, we have from (2.4.91)

$$\sup_{\mathcal{B}}|\partial_\varphi\psi|^2 \lesssim E_1[\partial_\varphi\psi_\flat](0)+\sum_{i,j=1}^{3}E_1[\mathcal{W}_i\mathcal{W}_j\partial_\varphi\psi_\flat](0)<\tilde{C}<\infty \tag{2.4.94}$$

for some constant $\tilde{C}<\infty$ depending on the initial data. (Recall that we assumed our initial data to be smooth and compactly supported.) Thus, for $\varphi_1\leq\varphi_2$, we have

$$|\psi_\flat(u_0,v_0,\varphi_2,\theta)-\psi_\flat(u_0,v_0,\varphi_1,\theta_0)|^2\lesssim\int_{\varphi_1}^{\varphi_2}\sup_{\mathcal{B}}|\partial_\varphi\psi_\flat|\leq\tilde{C}|\varphi_2-\varphi_1| \tag{2.4.95}$$

uniformly in u_0,v_0,θ_0. A similar estimate holds true for θ. Applications of the fundamental theorem of calculus and a triangle inequality finally yield the continuity result for $\psi_\flat$. $\square$

2.5 High frequency part $\psi_\sharp$

In the previous section we have shown the uniform boundedness for the low frequency part $\psi_\flat$. Now, we turn to $\psi_\sharp$, the high frequency part. The key ingredient for the proof of the uniform boundedness for $|\psi_\sharp|$ in the interior is *(a)* the uniform boundedness of transmission and reflection coefficients associated to the radial o.d.e. (2.4.3) which is proved in [82] for $\Lambda=0$, together with *(b)* the finiteness of the (commuted) T-energy flux on the event horizon given by (2.2.22).

Now, recall the radial o.d.e. (2.4.3) which reads $-u''+V_\ell u=\omega^2 u$ in the interior, where V_ℓ decays exponentially as $r_*\to+\infty(r\to r_-)$ and $r_*\to-\infty(r\to r_+)$. For $\omega\neq 0$, so in particular for $|\omega|>\frac{\omega_0}{2}$, the radial o.d.e. admits the following pairs of mode solutions (u_1,u_2) and (v_1,v_2), where u_1 and u_2 are solutions to (2.4.3) satisfying $u_1=$

$e^{i\omega r_*} + O_\ell(r-r_+)$ and $u_2 = e^{-i\omega r_*} + O_\ell(r-r_+)$ as $r_* \to -\infty$. Similarly, v_1 and v_2 satisfy $v_1 = e^{i\omega r_*} + O_\ell(r-r_-)$ and $v_2 = e^{-i\omega r_*} + O_\ell(r-r_-)$ as $r_* \to +\infty$. Now, for $\omega \neq 0$, the transmission and reflection coefficients $\mathfrak{T}(\omega,\ell)$ and $\mathfrak{R}(\omega,\ell)$ are defined as the unique coefficients satisfying

$$u_1 = \mathfrak{T}(\omega,\ell)v_1 + \mathfrak{R}(\omega,\ell)v_2. \tag{2.5.1}$$

See [82] for more details. In the following we will state the uniform boundedness of $\mathfrak{T}(\omega,\ell)$ and $\mathfrak{R}(\omega,\ell)$ for $|\omega| \geq \frac{\omega_0}{2}$. In [82, Proposition 4.7, Proposition 4.8] this has been proven for $\Lambda = 0$. However, the proof of Proposition 4.7 and Proposition 4.8 in [82] also applies if we include a non-vanishing cosmological constant.[9]

Lemma 2.5.1 ([82, Proposition 4.7, Proposition 4.8]). *Fix subextremal Reissner–Nordström–AdS black hole parameters (M,Q,l), a constant $\omega_0 > 0$ and a Klein–Gordon mass parameter $\alpha < \frac{9}{4}$. Then, the scattering coefficients $\mathfrak{T}(\omega,\ell)$ and $\mathfrak{R}(\omega,\ell)$ as defined above satisfy*

$$\sup_{|\omega|\geq\frac{\omega_0}{2},\ell\in\mathbb{N}_0} (|\mathfrak{T}(\omega,\ell)| + |\mathfrak{R}(\omega,\ell)|) \lesssim_{M,Q,l,\omega_0,\alpha} 1 \tag{2.5.2}$$

and the mode solutions u_1, u_2 and v_1, v_2 are uniformly bounded

$$\sup_{|\omega|\geq\frac{\omega_0}{2},\ell\in\mathbb{N}_0} \|u_1\|_{L^\infty(\mathbb{R})} \lesssim_{M,Q,l,\omega_0,\alpha} 1, \quad \sup_{|\omega|\geq\frac{\omega_0}{2},\ell\in\mathbb{N}_0} \|u_2\|_{L^\infty(\mathbb{R})} \lesssim_{M,Q,l,\omega_0,\alpha} 1, \tag{2.5.3}$$

$$\sup_{|\omega|\geq\frac{\omega_0}{2},\ell\in\mathbb{N}_0} \|v_1\|_{L^\infty(\mathbb{R})} \lesssim_{M,Q,l,\omega_0,\alpha} 1, \quad \sup_{|\omega|\geq\frac{\omega_0}{2},\ell\in\mathbb{N}_0} \|v_2\|_{L^\infty(\mathbb{R})} \lesssim_{M,Q,l,\omega_0,\alpha} 1. \tag{2.5.4}$$

Proof. Since we are the regime $|\omega| \geq \frac{\omega_0}{2}$, the proof for $\Lambda < 0$ works exactly as for $\Lambda = 0$ as shown in [82, Proposition 4.7, Proposition 4.8]. Thus, we will be very brief.

We first consider the case $\ell \leq \ell_0$, where ℓ_0 is chosen sufficiently large later in the second part. Note that u_1 solves the Volterra equation

$$u_1(r_*) = e^{i\omega r_*} + \int_{-\infty}^{r_*} \frac{\sin(\omega(r_*-y))}{\omega} V(y)u_1(y)\mathrm{d}y. \tag{2.5.5}$$

As $|\omega| \geq \frac{\omega_0}{2}$ and since the potential V is uniformly bounded (in the regime $\ell \leq \ell_0$) and decays exponentially as $r_* \to \pm\infty$, standard estimates for Volterra integral equations (see [82, Proposition 2.3]) yield (2.5.3) for u_1 and similarly for u_2, v_1 and v_2.

[9]Note that for $\Lambda \neq 0$ the scattering coefficients $\mathfrak{R}$ and $\mathfrak{T}$ have a pole at $\omega = 0$. However, for frequencies bounded away from $\omega = 0$, so in particular for $|\omega| \geq \frac{\omega_0}{2}$ as in the present case, $\mathfrak{T}$ and $\mathfrak{R}$ are uniformly bounded for both cases $\Lambda = 0$ *and* $\Lambda \neq 0$. See [82] for more details.

For the regime $\ell \geq \ell_0$, we will use a WKB approximation. Indeed, choosing ℓ_0 sufficiently large, we have that $p := \omega^2 - V$ is positive for $r_* \in \mathbb{R}$ and smooth. Now, u_1 is a solution of the radial o.d.e. $u'' = -pu$. Just like in [82, Equation (4.149)] we control the error term $F(r_*) = \int_{-\infty}^{r_*} p^{-\frac{1}{4}} |\frac{\mathrm{d}^2}{\mathrm{d}y^2} p^{-\frac{1}{4}}| \mathrm{d}y$ of the WKB approximation and conclude that u_1 remains uniformly bounded. Similarly, this holds true for u_2, v_1 and v_2 and for the scattering coefficients $\mathfrak{R}$ and $\mathfrak{T}$ which concludes the proof. $\square$

Another result which we will use from [82] is the representation formula for $\psi_\sharp$ in the separated picture. It is essential that $|\omega| \geq \frac{\omega_0}{2}$ to apply the same steps as in [82, Proof of Proposition 5.1].

Lemma 2.5.2 ([82, Proof of Proposition 5.1])**.** *Let $\psi_\sharp$ as in (2.3.5). Then, we have*

$$\begin{aligned}\psi_\sharp(t,r,\varphi,\theta) =& \frac{1}{\sqrt{2\pi} r} \sum_{\ell \in \mathbb{N}_0} \sum_{|m| \leq \ell} Y_{\ell m}(\theta,\varphi) \int_{|\omega| \geq \frac{\omega_0}{2}} \mathcal{F}_{\mathcal{H}_A^+} \left[\psi_\sharp \restriction_{\mathcal{H}_A^+} \right] (\omega, m, \ell) u_1(\omega,\ell,r) e^{i\omega t} \mathrm{d}\omega \\ &+ \frac{1}{\sqrt{2\pi} r} \sum_{\ell \in \mathbb{N}_0} \sum_{|m| \leq \ell} Y_{\ell m}(\theta,\varphi) \int_{|\omega| \geq \frac{\omega_0}{2}} \mathcal{F}_{\mathcal{H}_B^+} \left[\psi_\sharp \restriction_{\mathcal{H}_B^+} \right] (\omega, m, \ell) u_2(\omega,\ell,r) e^{i\omega t} \mathrm{d}\omega,\end{aligned} \tag{2.5.6}$$

where

$$\mathcal{F}_{\mathcal{H}_A^+}[\phi](\omega,m,\ell) := \frac{r_+}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega v} \langle \phi, Y_{\ell m} \rangle_{\mathbb{S}^2} \mathrm{d}v \tag{2.5.7}$$

and

$$\mathcal{F}_{\mathcal{H}_B^+}[\phi](\omega,m,\ell) := \frac{r_+}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega u} \langle \phi, Y_{\ell m} \rangle_{\mathbb{S}^2} \mathrm{d}u. \tag{2.5.8}$$

Proof of Lemma 2.5.2. This proof is very similar to [82, Proof of Proposition 5.1] so we will be rather brief.

Let $\psi_\sharp$ as in (2.3.5). Since the expansion in spherical harmonics converges pointwise, it suffices to prove (2.5.6) for $\psi_\sharp^{\ell m} := \langle \psi_\sharp, Y_{\ell m} \rangle_{\mathbb{S}^2} Y_{\ell m}$ for fixed m, ℓ. Now, define $u[\psi_\sharp^{\ell m}]$ as in (2.4.2) such that

$$\psi_\sharp^{\ell m} = \frac{1}{\sqrt{2\pi} r} Y_{\ell m} \int_{|\omega| \geq \frac{\omega_0}{2}} u[\psi_\sharp^{\ell m}] e^{i\omega t} \mathrm{d}\omega. \tag{2.5.9}$$

This is well-defined in the interior in view of Proposition 2.4.1. Moreover, $u[\psi_\sharp^{\ell m}]$ solves the radial o.d.e. and can be expanded in the basis u_1 and u_2 ($|\omega| > \frac{\omega_0}{2}$):

$$u[\psi_\sharp^{\ell m}](r_*, \omega, m, \ell) = a(\omega, m, \ell) u_1(r_*, \ell, \omega) + b(\omega, m, \ell) u_2(r_*, \ell, \omega). \tag{2.5.10}$$

Now, first note Proposition 2.6.4 implies that $\omega \mapsto u[\psi_\sharp^{\ell m}](r, \omega)$ is a Schwartz function for $r \in (r_-, r_+)$. Since

$$|a(\omega, m, \ell)| = \left| \frac{\mathfrak{W}(u[\psi_\sharp^{\ell m}], u_2)}{\mathfrak{W}(u_1, u_2)} \right| = \left| \frac{\mathfrak{W}(u[\psi_\sharp^{\ell m}], u_2)}{2\omega} \right| \lesssim \left| \mathfrak{W}(u[\psi_\sharp^{\ell m}], u_2) \right| \tag{2.5.11}$$

in view of $|\omega| \geq \frac{\omega_0}{2}$, we conclude that $\omega \mapsto a(\omega, m, \ell)$ is in $L^1(\mathbb{R})$ for fixed ℓ, m. Recall that the Wronskian $\mathfrak{W}(f, g) := f'g - fg'$ is independent of r_* for two solutions of the radial o.d.e. (2.4.3). We have also used that $\|u_2\|_{L^\infty} \lesssim 1$ and $\|u_2'\|_{L^\infty} \lesssim_\ell 1 + |\omega|$ for $|\omega| \geq \frac{\omega_0}{2}$ (cf. [82, Proposition 4.7 and Proposition 4.8]). Similarly, we have that $\omega \mapsto b(\omega, m, l)$ is in $L^1(\mathbb{R})$. Using

$$\psi_\sharp^{\ell m} = Y_{\ell m} \frac{1}{\sqrt{2\pi} r} \int_{|\omega| \geq \frac{\omega_0}{2}} \left(a(\omega, m, \ell) u_1(r, \omega, \ell) + b(\omega, m, \ell) u_2(r, \omega, \ell) \right) e^{i\omega t} \mathrm{d}\omega \tag{2.5.12}$$

and a direct adaptation of [82, Proof of Proposition 5.1] finally shows $a(\omega, m, \ell) = \mathcal{F}_{\mathcal{H}_A^+}[\psi_\sharp^{\ell m} \restriction_{\mathcal{H}_A^+}](\omega, m, \ell)$, $b(\omega, m, \ell) = \mathcal{F}_{\mathcal{H}_B^+}[\psi_\sharp^{\ell m} \restriction_{\mathcal{H}_B^+}](\omega, m, \ell)$.[10] This shows the representation formula (2.5.6) for $\psi_\sharp$. $\square$

We will now prove the uniform boundedness for $\psi_\sharp$.

Proposition 2.5.1. *Let $\psi_\sharp$ be as defined in* (2.3.5). *Then,*

$$\sup_{\mathcal{B} \cap J^+(\Sigma_0)} |\psi_\sharp|^2 \lesssim E_1[\psi_\sharp](0) + \sum_{i,j=1}^{3} E_1[\mathcal{W}_i \mathcal{W}_j \psi_\sharp](0) \lesssim D[\psi_\sharp]. \tag{2.5.13}$$

Proof. We start with the representation of $\psi_\sharp$ as in (2.5.6). For convenience, we will only estimate the term involving $\mathcal{F}_{\mathcal{H}_A^+}[\phi](\omega, m, \ell)$ and assume without loss of generality that $\mathcal{F}_{\mathcal{H}_B^+}[\phi](\omega, m, \ell) = 0$. Indeed the term $\mathcal{F}_{\mathcal{H}_B^+}[\phi](\omega, m, \ell)$ can be treated analogously. Now,

[10] More precisely, following the lines starting from equation (5.20) in [82, Proof of Proposition 5.1] which contain an application of Lebesgue's dominated convergence, the Riemann–Lebesgue lemma and the inverse Fourier transform yields the result.

in view of (2.5.3), we conclude

$$
\begin{aligned}
|\psi_\sharp(r,t,\varphi,\theta)|^2 &\lesssim \left| \sum_{\ell\in\mathbb{N}_0} \sum_{m\in\mathbb{Z},|m|\le\ell} Y_{\ell m}(\varphi,\theta) \int_{|\omega|\ge\frac{\omega_0}{2}} \mathcal{F}_{\mathcal{H}_A}\left[\psi_\sharp \restriction_{\mathcal{H}_A^+}\right](\omega,m,\ell)\mathrm{d}\omega \right|^2 \\
&\le \sum_{\ell\in\mathbb{N}_0} \sum_{m\in\mathbb{Z},|m|\le\ell} \int_{|\omega|\ge\frac{\omega_0}{2}} (1+\ell)^3\omega^2 \left|\mathcal{F}_{\mathcal{H}_A}\left[\psi_\sharp \restriction_{\mathcal{H}_A^+}\right](\omega,m,\ell)\right|^2 \mathrm{d}\omega \\
&\cdot \sum_{\ell\in\mathbb{N}_0} \sum_{m\in\mathbb{Z},|m|\le\ell} \frac{|Y_{\ell m}(\varphi,\theta)|^2}{(1+\ell)^3} \int_{|\omega|\ge\frac{\omega_0}{2}} \frac{1}{\omega^2}\mathrm{d}\omega \\
&\lesssim \sum_{\ell\in\mathbb{N}_0} \sum_{m\in\mathbb{Z},|m|\le\ell} \int_{|\omega|\ge\frac{\omega_0}{2}} (1+\ell)^3\omega^2 \left|\mathcal{F}_{\mathcal{H}_A}\left[\psi_\sharp \restriction_{\mathcal{H}_A^+}\right](\omega,m,\ell)\right|^2 \mathrm{d}\omega \\
&\lesssim \int_{\mathcal{H}_A^+} |T\psi_\sharp|^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} + \sum_{i,j=1}^{3} \int_{\mathcal{H}_A^+} |T\mathcal{W}_i\mathcal{W}_j\psi_\sharp|^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}. \qquad (2.5.14)
\end{aligned}
$$

Here, we have used that

$$\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\varphi,\theta)|^2 = \frac{2\ell+1}{4\pi} \qquad (2.5.15)$$

which is known as Unsöld's Theorem [138, Eq. (69)].

Finally, on the right hand side of (2.5.14) we only see the commuted T-energy flux. An application of the T-energy identity in the exterior and an energy estimate in a compact spacetime region shows that the commuted T-energy flux on the event horizon is controlled from the initial data (cf. (2.2.22) in Theorem 2.1). Thus, in view of (2.5.14) we conclude

$$|\psi_\sharp(r,t,\varphi,\theta)|^2 \lesssim E_1[\psi_\sharp](0) + \sum_{i,j=1}^{3} E_1[\mathcal{W}_i\mathcal{W}_j\psi_\sharp](0). \qquad (2.5.16)$$

□

Proposition 2.5.2. *Let $\psi_\sharp$ be as defined in* (2.3.5)*. Then, $\psi_\sharp$ is continuously extendible across the Cauchy horizon $\mathcal{CH}$.*

Proof. Let $(u_n, v_n, \theta_n, \varphi_n) \to (\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{\varphi})$ be a convergent sequence. We will also allow $\tilde{u} = +\infty$ and $\tilde{v} = +\infty$ as limits which correspond to limits to the Cauchy horizon. We represent $\psi_\sharp$ again as in (2.5.6). Similar to the proof of Proposition 2.5.1, it is enough to consider the case where $\mathcal{F}_{\mathcal{H}_B^+}[\psi_\sharp\restriction_{\mathcal{H}_B^+}]$ vanishes. Hence,

$$\psi_\sharp(t,r,\varphi,\theta) = \frac{1}{\sqrt{2\pi}r}\sum_{\ell\in\mathbb{N}_0}\sum_{|m|\leq\ell} Y_{\ell m}(\theta,\varphi)\int_{|\omega|\geq\frac{\omega_0}{2}} \mathcal{F}_{\mathcal{H}_A^+}\left[\psi_\sharp\restriction_{\mathcal{H}_A^+}\right](m,\ell,\omega)u_1(\omega,\ell,r)e^{i\omega t}\mathrm{d}\omega. \tag{2.5.17}$$

First from (2.5.15) we have $\sup_{\varphi,\theta}|Y_{\ell m}(\varphi,\theta)| \lesssim 1+\ell$ and from (2.5.3) we have that

$$\sup_{u,v}|u_1 e^{i\omega t(u,v)}| = \sup_{t,r}|u_1 e^{i\omega t}| \lesssim 1.$$

Then, a similar estimate as in (2.5.14) and an application of Lebesgue's dominated convergence theorem allow us to interchange the limit $n\to\infty$ with the sum $\sum_{\ell\in\mathbb{N}_0}\sum_{|m|\leq\ell}$. Since $Y_{\ell m}(\theta_n,\varphi_n)\to Y_{\ell m}(\tilde\theta,\tilde\varphi)$ pointwise as $n\to\infty$, it remains to show that

$$\begin{aligned}
&\int_{|\omega|\geq\frac{\omega_0}{2}} \mathcal{F}_{\mathcal{H}_A^+}\left[\psi_\sharp\restriction_{\mathcal{H}_A^+}\right](m,\ell,\omega)u_1(\omega,\ell,r(u_n,v_m))e^{i\omega t(u_n,v_n)}\mathrm{d}\omega\\
&\qquad = \int_{|\omega|\geq\frac{\omega_0}{2}} \mathcal{F}_{\mathcal{H}_A^+}\left[\psi_\sharp\restriction_{\mathcal{H}_A^+}\right](m,\ell,\omega)\Big(\mathfrak{T}(\omega,\ell)v_1(\omega,\ell,r(u_n,v_n))\\
&\qquad\quad + \mathfrak{R}(\omega,\ell)v_2(\omega,\ell,r(u_n,v_n))\Big)e^{i\omega t(u_n,v_n)}\mathrm{d}\omega
\end{aligned}$$

converges as $n\to\infty$ for fixed angular parameters m,ℓ. But, in view of (2.5.2), depending on whether $\tilde v=+\infty$ or $\tilde u=+\infty$, we can deduce the continuity using Lebesgue's dominated convergence and the Riemann–Lebesgue lemma. Both are justified by a slight adaptation of the steps which resulted in (2.5.12). This concludes the proof. □

2.6 Appendix

2.6.1 Twisted energy-momentum tensor in null coordinates in the interior

We will write out the components of the twisted energy-momentum tensor in the interior.

Proposition 2.6.1. *Consider null coordinates (u,v,θ,φ) in the interior region $\mathcal{B}$. Recall that the metric is given by* (2.2.39). *Let $f\in C^\infty(\mathcal{B})$ be a spherically symmetric nowhere vanishing real valued function and X be a smooth vector field of the form $X = X^u\partial_u + X^v\partial_v$.*

The components of the twisted energy-momentum tensor (2.2.33) *associated to f are* *given by*

$$\begin{aligned}
\tilde{\mathbf{T}}_{uu} &= |\tilde{\nabla}_u\phi|^2 = f^2\left|\partial_u\left(\frac{\phi}{f}\right)\right|^2, \tilde{T}_{vv} = |\tilde{\nabla}_v\phi|^2 = f^2\left|\partial_v\left(\frac{\phi}{f}\right)\right|^2,\\
\tilde{\mathbf{T}}_{uv} &= \tilde{\mathbf{T}}_{vu} = \frac{\Omega^2}{4}\left(|\nabla\!\!\!/\phi|^2 + \mathcal{V}|\phi|^2\right),\\
\tilde{\mathbf{T}}_{\theta\theta} &= |\partial_\theta\phi|^2 + \frac{2r^2}{\Omega^2}\operatorname{Re}\left(\overline{\tilde{\nabla}_u\phi}\tilde{\nabla}_v\phi\right) - \frac{r^2}{2}\left(|\nabla\!\!\!/\phi|^2 + \mathcal{V}|\phi|^2\right),\\
\tilde{\mathbf{T}}_{\varphi\varphi} &= |\partial_\varphi\phi|^2 + \frac{2r^2\sin^2\theta}{\Omega^2}\operatorname{Re}\left(\overline{\tilde{\nabla}_u\phi}\tilde{\nabla}_v\phi\right) - \frac{r^2\sin^2\theta}{2}\left(|\nabla\!\!\!/\phi|^2 + \mathcal{V}|\phi|^2\right).
\end{aligned}$$

The deformation tensor ${}^X\pi := \frac{1}{2}\mathcal{L}_X g$ *is given by*

$$\begin{aligned}
{}^X\pi^{vv} &= -\frac{2}{\Omega^2}\partial_u X^v, {}^X\pi^{uu} = -\frac{2}{\Omega^2}\partial_v X^u,\\
{}^X\pi^{uv} &= -\frac{1}{\Omega^2}\left(\partial_u X^u + \partial_v X^v\right) - \frac{2}{\Omega^2}\left(\frac{\partial_v\sqrt{\Omega^2}}{\sqrt{\Omega^2}}X^v + \frac{\partial_u\sqrt{\Omega^2}}{\sqrt{\Omega^2}}X^u\right),\\
{}^X\pi^{\theta\theta} &= -\frac{\Omega^2}{2r^3}(X^v + X^u), {}^X\pi^{\varphi\varphi} = -\frac{\Omega^2}{2r^3\sin^2\theta}(X^v + X^u).
\end{aligned}$$

In the following we explicitly write down future-directed normals and induced volume forms for hypersurfaces of constant r *values* Σ_r *and for null cones* $\mathcal{C}_u$ *and* $\underline{\mathcal{C}}_v$ *of constant* u *and* v *values, respectively.*

$$\begin{aligned}
n_{\Sigma_r} &= \frac{1}{\sqrt{\Omega^2}}(\partial_u + \partial_v), \mathrm{dvol}_{\Sigma_r} = r^2\sqrt{\Omega^2}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}u = r^2\sqrt{\Omega^2}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v,\\
n_{\underline{\mathcal{C}}_v} &= \frac{2}{\Omega^2}\partial_u, \mathrm{dvol}_{\underline{\mathcal{C}}_v} = \frac{r^2}{2}\Omega^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}u,\\
n_{\mathcal{C}_u} &= \frac{2}{\Omega^2}\partial_v, \mathrm{dvol}_{\mathcal{C}_u} = \frac{r^2}{2}\Omega^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v.
\end{aligned}$$

Then, the fluxes of X *are given by*

$$\tilde{J}^X_\mu[\phi]n^\mu_{\mathcal{C}_u} = \frac{2X^v}{\Omega^2}|\tilde{\nabla}_v\phi|^2 + \frac{X^u}{2}\left(|\nabla\!\!\!/\phi|^2 + \mathcal{V}|\phi|^2\right), \tag{2.6.1}$$

$$\tilde{J}^X_\mu[\phi]n^\mu_{\underline{\mathcal{C}}_v} = \frac{2X^u}{\Omega^2}|\tilde{\nabla}_u\phi|^2 + \frac{X^v}{2}\left(|\nabla\!\!\!/\phi|^2 + \mathcal{V}|\phi|^2\right), \tag{2.6.2}$$

$$\tilde{J}^X_\mu[\phi]n^\mu_{\Sigma_r} = \frac{1}{\sqrt{\Omega^2}}\left(X^u|\tilde{\nabla}_u\phi|^2 + X^v|\tilde{\nabla}_v\phi|^2 + \frac{\Omega^2}{4}(X^u + X^v)(|\nabla\!\!\!/\phi|^2 + \mathcal{V}|\phi|^2)\right). \tag{2.6.3}$$

The twisted bulk term associated to the twisting function f reads (cf. [144])

$$\tilde{K}^X = {}^X\pi_{\mu\nu}\tilde{T}^{\mu\nu} + X^\nu \tilde{S}_\nu,$$

where

$$\tilde{S}_\nu = \frac{\tilde{\nabla}^*_\nu (f\mathcal{V})}{2f}|\phi|^2 + \frac{\tilde{\nabla}^*_\nu f}{2f}\overline{\tilde{\nabla}_\sigma\phi}\tilde{\nabla}^\sigma\phi.$$

In coordinates we have

$$\begin{aligned}\tilde{K}^X =& -\frac{2}{\Omega^2}\left(\partial_u X^v|\tilde{\nabla}_v\phi|^2 + \partial_v X^u|\tilde{\nabla}_u\phi|^2\right) - \frac{2}{r}(X^u + X^v)\operatorname{Re}(\overline{\tilde{\nabla}_u\phi}\tilde{\nabla}_v\phi)\\ &- \left(\frac{1}{2}(\partial_v X^v + \partial_u X^u) - \frac{\partial_r\Omega^2}{4}(X^v + X^u)\right)\left(|\tilde{\nabla}\!\!\!/\,\phi|^2 + \mathcal{V}|\phi|^2\right)\\ &+ \frac{\Omega^2}{2r}(X^v + X^u)\mathcal{V}|\phi|^2 + X^u\left(-\frac{\partial_u(f^2\mathcal{V})}{2f^2}|\phi|^2 - \frac{\partial_u f^2}{2f^2}\overline{\tilde{\nabla}_\sigma\phi}\tilde{\nabla}^\sigma\phi\right)\\ &+ X^v\left(-\frac{\partial_v(f^2\mathcal{V})}{2f^2}|\phi|^2 - \frac{\partial_v f^2}{2f^2}\overline{\tilde{\nabla}_\sigma\phi}\tilde{\nabla}^\sigma\phi\right).\end{aligned} \tag{2.6.4}$$

2.6.2 Construction of the red-shift vector field

In this section we will give the proof of Proposition 2.2.3.

Proof of Proposition 2.2.3. We choose the ansatz $N = N^u\partial_u + N^v\partial_v$ for our red-shift vector field. We will first estimate the twisted 1-jet $\tilde{J}$ and then the twisted bulk term $\tilde{K}$.

$\tilde{J}$ ***current.*** From (2.6.2), we have

$$\tilde{J}^N_\mu[\phi]n^\mu_{\underline{\mathcal{C}}_v} = \frac{2N^u}{\Omega^2}|\tilde{\nabla}_u\phi|^2 + \frac{N^v}{2}\left(|\tilde{\nabla}\!\!\!/\,\phi|^2 + \mathcal{V}|\phi|^2\right), \tag{2.6.5}$$

where

$$\mathcal{V} = -\left(\frac{\Box_g f}{f} + \frac{\alpha}{l^2}\right). \tag{2.6.6}$$

First, if $f = f(r)$ we have

$$-\frac{\Box_g f}{f} = \Omega^2\frac{\ddot{f}}{f} + \left(\frac{2\Omega^2}{r} + \partial_r(\Omega^2)\right)\frac{\dot{f}}{f}, \tag{2.6.7}$$

where $\dot{f} := \frac{\mathrm{d}f}{\mathrm{d}r}$. Thus, choosing $f = e^{-\beta_{\mathrm{red}} r}$ gives

$$\mathcal{V} = -\left(\frac{\Box_g f}{f} + \frac{\alpha}{l^2}\right) = \beta_{\mathrm{red}}^2 \Omega^2 - \partial_r(\Omega^2)\beta_{\mathrm{red}} - \frac{2\beta_{\mathrm{red}}}{r}\Omega^2 - \frac{\alpha}{l^2}. \tag{2.6.8}$$

Note that for $r_{\mathrm{red}} < r_+$ close enough to r_+, we have

$$-\partial_r \Omega^2 \geq c_{\mathrm{red}} \tag{2.6.9}$$

for all $r_{\mathrm{red}} \leq r \leq r_+$ and some constant $c_{\mathrm{red}} > 0$ only depending on the black hole parameters. The constant $c_{\mathrm{red}} > 0$ does not decrease, when we choose r_{red} even closer r_+. Now, by choosing $\beta_{\mathrm{red}} > 0$ large enough to absorb the negative contribution from $-\frac{\alpha}{l^2}$ and by choosing r_{red} close enough to r_+, we ensure that $\mathcal{V} \gtrsim 1$ in $r_{\mathrm{red}} \leq r \leq r_+$. This finally shows that if we take N as a future directed vector field, the 1-jet $\tilde{J}^N_\mu n^\mu_{\underline{C}_v}$ is positive definite. We will construct the explicit form of N in the bulk term estimate.

Bulk term $\tilde{K}^N$. Now, we will estimate the bulk term. We will choose the components of the timelike vector field $N = N^u \partial_u + N^v \partial_v$ as

$$N^u := \frac{1}{\Omega^2} - \frac{1}{\delta_1} \text{ and } N^v := 1 - \frac{\Omega^2}{\delta_2}. \tag{2.6.10}$$

Note that N is smooth in $\mathcal{R}_{\mathrm{red}}$. Moreover, for fixed $\delta_1, \delta_2 > 0$ (only depending on the black hole parameters), we can choose r_{red} close enough to r_+ such that N is future directed in $\mathcal{R}_{\mathrm{red}}$. Then, note that

$$\tilde{K}^N[\phi] = \left(-\partial_r \Omega^2\right)\left(\frac{1}{\delta_2}|\tilde{\nabla}_v \phi|^2 + \frac{1}{\Omega^4}|\tilde{\nabla}_u \phi|^2\right) - \frac{2}{r}\left(\frac{1}{\Omega^2} - \frac{1}{\delta_1} + 1 - \frac{1}{\delta_2}\Omega^2\right)\mathrm{Re}(\overline{\tilde{\nabla}_u \phi}\tilde{\nabla}_v \phi) \tag{2.6.11}$$

$$+ \frac{1}{4}\left(-\frac{\mathrm{d}\Omega^2}{\mathrm{d}r}\right)\left(\frac{1}{\delta_1} - 1 + \frac{2\Omega^2}{\delta_2}\right)(|\nabla\!\!\!/ \phi|^2 + \mathcal{V}|\phi|^2) \tag{2.6.12}$$

$$+ \frac{1}{2r}\left(1 + \left(1 - \frac{1}{\delta_1}\right)\Omega^2 - \frac{1}{\delta_2}\Omega^4\right)\mathcal{V}|\phi|^2 \tag{2.6.13}$$

$$+ \left(\frac{1}{\Omega^2} - \frac{1}{\delta_1}\right)\frac{-\partial_u(f^2\mathcal{V})}{2f^2}|\phi|^2 + \left(\frac{1}{\Omega^2} - \frac{1}{\delta_1}\right)\frac{-\partial_u(f^2)}{2f^2}\overline{\tilde{\nabla}_\sigma \phi}\tilde{\nabla}^\sigma \phi \tag{2.6.14}$$

$$+ \left(1 - \frac{\Omega^2}{\delta_2}\right)\frac{-\partial_v(f^2\mathcal{V})}{2f^2}|\phi|^2 + \left(1 - \frac{\Omega^2}{\delta_2}\right)\frac{-\partial_v(f^2)}{2f^2}\overline{\tilde{\nabla}_\sigma \phi}\tilde{\nabla}^\sigma \phi. \tag{2.6.15}$$

In the following we will show that

$$\tilde{K}^N[\phi] \gtrsim \frac{1}{\Omega^4}|\tilde{\nabla}_u \phi|^2 + |\tilde{\nabla}_v \phi|^2 + (|\nabla\!\!\!/ \phi|^2 + \mathcal{V}|\phi|^2). \tag{2.6.16}$$

We will start with the sign-indefinite term appearing in (2.6.11). We estimate it as follows

$$\Big| -\frac{2}{r}\left(\frac{1}{\Omega^2} - \frac{1}{\delta_1} + 1 - \frac{1}{\delta_2}\Omega^2\right) \mathrm{Re}(\overline{\tilde{\nabla}_u \phi} \tilde{\nabla}_v \phi)\Big| \lesssim \frac{\epsilon}{\Omega^4}|\tilde{\nabla}_u \phi|^2 + \frac{1}{\epsilon}|\tilde{\nabla}_v \phi|^2, \tag{2.6.17}$$

where we have applied an ϵ-weighted Young's inequality. We have also used that—by choosing r_{red} closer to r_+—we can make Ω^2 uniformly smaller than any constant, in particular smaller than δ_1 and δ_2 once those are fixed. Choosing ϵ small enough, we absorb the term $\frac{\epsilon}{\Omega^4}|\tilde{\nabla}_u \phi|^2$ of (2.6.17) in the first term of (2.6.11). Then, choosing $\delta_2(\delta_1, \epsilon)$ small enough, we can also absorb the term $\frac{1}{\epsilon}|\tilde{\nabla}_v \phi|^2$ in the first term of (2.6.11). Completely analogously and by potentially choosing δ_2 and δ_1 even smaller, we estimate the terms of the form $\frac{1}{\Omega^2}\mathrm{Re}(\overline{\tilde{\nabla}_u \phi} \tilde{\nabla}_v \phi)$ arising from (2.6.14) and (2.6.15).

Next, note that, in view of $\mathcal{V} \gtrsim 1$ and $\left|\frac{-\partial_v(f^2\mathcal{V})}{2f^2}\right| \lesssim \Omega^2$, we choose δ_1 small enough such that we absorb error terms coming from (2.6.14) and (2.6.15) in the term with the good sign in (2.6.12). By doing so we also have to make $\delta_2(\epsilon, \delta_1) > 0$ small enough. Finally, once δ_1 and δ_2 are fixed, note that we can make terms involving higher orders of Ω^2 arbitrarily small by choosing r_{red} close to r_+. This finally shows (2.6.16) and concludes the proof. $\square$

2.6.3 Well-definedness of the Fourier projections $\psi_\flat$ and $\psi_\sharp$

Proposition 2.6.2. *Let $\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ be as in (2.3.4) and let $r \in (r_-, r_+)$, $(\varphi, \theta) \in \mathbb{S}^2$ be fixed. Then, $t \mapsto \psi(t, r, \theta, \varphi)$ is a tempered distribution. Moreover, higher derivatives $t \mapsto \partial^k \psi(t, r, \theta, \varphi)$, where $\partial \in \{\partial_t, \partial_r, \partial_\theta, \partial_\varphi\}$ are also tempered distributions.*

Proof. Fix $r \in (r_-, r_+)$, $(\varphi, \theta) \in \mathbb{S}^2$. We will first prove that $t \mapsto \psi(t, r, \varphi, \theta)$ is slowly growing.[11] Since $\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ and in view of the facts that $\Box_g$ commutes with $T = \partial_t$ and our initial data are smooth and compactly supported, it suffices to obtain a polynomial bound for $\psi(t, r, \varphi, \theta)$. To do this we will propagate mild polynomial growth from the exterior region in the interior. (Note that this growth is far from being sharp but it will be sufficient for the purpose of proving well-definedness of $\psi_\flat$ and $\psi_\sharp$.)

From Theorem 2.2 and Remark 2.2.2 we infer that ψ and its derivatives remain bounded

[11] With *slowly growing* we mean that $t \mapsto \psi(t, r, \varphi, \theta)$ and all its ∂_t derivatives have at most polynomial growth as $|t| \to \infty$.

along the event horizon $\mathcal{H}$. A direct integration yields

$$\int_{\mathcal{H}(v_1,v_2)} \tilde{J}^N_\mu[\psi] n^\mu_{\mathcal{H}^+} \mathrm{dvol}_{\mathcal{H}^+} \lesssim_{\psi_0,\psi_1} \langle v_2 \rangle, \tag{2.6.18}$$

where $\langle v_2 \rangle$ denotes the Japanese bracket and $0 \leq v_1 \leq v_2$. The constant appearing in $\lesssim_{\psi_0,\psi_1}$ depends on some higher Sobolev norm of the initial data.

Then, using the red-shift vector field (more precisely, applying Proposition 2.2.4) yields

$$\int_{\Sigma_{r_0}(v_1,v_2)} \tilde{J}^N_\mu[\psi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim_{\psi_0,\psi_1} \langle v_2 \rangle \tag{2.6.19}$$

for any $r_0 \in [r_{\mathrm{red}}, r_+)$. If $r \in (r_-, r_+)$ as fixed above lies in the red-shift region $[r_{\mathrm{red}}, r_+)$, we directly conclude (2.6.21) after commuting with the angular momentum operators $\mathcal{W}_i$ and a Sobolev embedding on $\mathbb{S}^2$. If however $r \in (r_-, r_{\mathrm{red}})$, we choose $r_{\mathrm{blue}} = r_{\mathrm{blue}}(r)$ small enough such that $r \in [r_{\mathrm{blue}}, r_{\mathrm{red}}]$, i.e. r lies in the no-shift region. Then, Proposition 2.2.5 yields

$$\int_{\Sigma_r(v,2v)} \tilde{J}^X_\mu[\psi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim_r \int_{\Sigma_{r_{\mathrm{red}}}(v_{r_{\mathrm{red}}}(u_r(v)),2v)} \tilde{J}^X_\mu[\psi] n^\mu_{\Sigma_r} \mathrm{dvol}_{\Sigma_r} \lesssim_{\psi_0,\psi_1,r} \langle v \rangle \tag{2.6.20}$$

for any $v \geq 1$. After commuting with angular momentum operators $\mathcal{W}_i$ and a Sobolev embedding on $\mathbb{S}^2$ we obtain

$$\int_0^t |\psi(t,r,\varphi,\theta)|^2 + |\partial_t \psi(t,r,\varphi,\theta)|^2 \mathrm{d}t \lesssim_{\psi_0,\psi_1,r} \langle t \rangle \tag{2.6.21}$$

from which we can deduce that $t \mapsto \psi(t,r,\varphi,\theta)$ is slowly growing (where we recall that r,φ,θ are fixed). Similarly, as $t \to -\infty$, we obtain the same conclusion.

Now, commuting with ∂_t, the angular momentum operators $\mathcal{W}_i$ and using elliptic estimates it follows that higher order derivatives are also slowly growing which concludes the proof. $\square$

Corollary 2.6.1. *The Fourier projections $\psi_\flat$ and $\psi_\sharp$ in the interior $\mathcal{B}$ as in* (2.3.5) *are well-defined and are smooth solutions of* (2.1.1)*.*

Proof. From Proposition 2.6.2 we know that $t \mapsto \psi(t,r,\varphi,\theta)$ is a tempered distribution in the interior for fixed r,φ,θ. Thus, $\psi_\flat$ defined in (2.3.5) is well defined as $\mathcal{F}_T^{-1}[\chi_{\omega_0}]$ is a Schwartz function. Moreover, $\psi_\flat$ is smooth because ψ is smooth itself and by Proposition 2.6.2 we have that all higher derivatives $t \mapsto \partial^k \psi(t,r,\varphi,\theta)$ are tempered distributions, too. Now, this also implies that $\psi_\flat \in C^\infty(\mathcal{B})$ solves (2.1.1) which concludes the proof in view of $\psi = \psi_\flat + \psi_\sharp$. $\square$

Proposition 2.6.3. *Let $\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ be defined as in* (2.3.4). *Then, there exist $\psi_\flat \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ and $\psi_\sharp \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$, two solutions of* (2.1.1) *with*

$$\psi_\flat = \frac{1}{\sqrt{2\pi}} \mathcal{F}_T^{-1} \left[\chi_{\omega_0}\right] * \psi \text{ and } \psi_\sharp = \psi - \psi_\flat, \tag{2.6.22}$$

where χ_{ω_0} is defined in (2.3.6) *and*

$$\psi_\flat(t, r, \varphi, \theta) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \mathcal{F}_T^{-1}[\chi_{\omega_0}](s)\psi(t - s, r, \varphi, \theta)\mathrm{d}s \tag{2.6.23}$$

in all coordinate patches $(t_{\mathcal{R}_A}, r_{\mathcal{R}_A}, \theta_{\mathcal{R}_A}, \varphi_{\mathcal{R}_A})$, $(t_{\mathcal{R}_B}, r_{\mathcal{R}_B}, \theta_{\mathcal{R}_B}, \varphi_{\mathcal{R}_B})$ and $(t_{\mathcal{B}}, r_{\mathcal{B}}, \theta_{\mathcal{B}}, \varphi_{\mathcal{B}})$ in the regions $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$, respectively.

Proof. First, from Theorem 2.2 we know that ψ and all higher derivatives decay logarithmically on the exterior regions $\mathcal{R}_A$ and $\mathcal{R}_B$.[12] Hence, ψ and all higher derivatives are smooth tempered distributions (for fixed r, φ, θ) in the exterior regions $\mathcal{R}_A$ and $\mathcal{R}_B$ as functions of $t_{\mathcal{R}_A}$ and $t_{\mathcal{R}_B}$, respectively. Thus, the Fourier projections $\psi_\flat$ (2.6.23) is well-defined in $\mathcal{R}_A$ and $\mathcal{R}_B$ and it follows by Lebesgue's dominated convergence that $\psi_\flat$ is a smooth solution of (2.1.1). Moreover, from Corollary 2.6.1 we deduce that $\psi_\flat$ is also a well-defined smooth solution of (2.1.1) in the interior $\mathcal{B}$.

Finally, $\psi_\flat$, defined a priori only in $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{B}$, extends to a smooth solution of (2.1.1) on $\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH}$. This follows from using regular coordinates near the respective event horizons (outgoing Eddington–Finkelstein coordinates (v, r, θ, φ), where $v(t, r) = t + r_*, r(t, r) = r, \theta = \theta, \varphi = \varphi$ near $\mathcal{H}_A$ and ingoing Eddington–Finkelstein coordinates near $\mathcal{H}_B$) and writing $\psi_\flat$ again as a convolution in this coordinate system $\psi_\flat = \frac{1}{\sqrt{2\pi}}\mathcal{F}_T^{-1}[\chi_{\omega_0}] * \psi$. Note that $T = \partial_v$ in this coordinate system. This concludes the proof in view of $\psi = \psi_\flat + \psi_\sharp$. $\square$

Proposition 2.6.4. *Assume that $\psi \in C^\infty(\mathcal{M}_{\mathrm{RNAdS}} \setminus \mathcal{CH})$ is a solution of* (2.1.1) *arising from smooth and compactly supported initial data as in Theorem 2.1. Assume further that there exists an $L \in \mathbb{N}$ with $\langle \psi, Y_{m\ell}\rangle_{L^2(\mathbb{S}^2)} = 0$ for $\ell \geq L$. Then, for every $r \in (r_-, r_+)$ and $(\theta, \varphi) \in \mathbb{S}^2$, the function $t \mapsto \psi(t, r, \varphi, \theta)$ is a Schwartz function. Moreover, higher derivatives $t \mapsto \partial^k \psi(t, r, \theta, \varphi)$, where $\partial \in \{\partial_t, \partial_r, \partial_\theta, \partial_\varphi\}$ are also Schwartz functions.*

Proof. The proof follows the same lines as the proof Proposition 2.6.2 with the difference

[12] This decay is only used in a qualitative way.

that we have exponential decay on the event horizon

$$\int_{v_1}^{v_2} \tilde{J}^N_\mu[\psi] n^\mu_{\mathcal{H}^+_A} \mathrm{dvol}_{\mathcal{H}^+_A} \lesssim D[\psi] \exp\left(-e^{-C(M,Q,l,\alpha)L} v_1\right), \tag{2.6.24}$$

where $D[\psi]$ is as in (2.3.2). Note that (2.6.24) follows from []. Analogously to the proof of Proposition 2.6.2 we can propagate this decay to any $\{r = const.\}$ hypersurface in the interior. This is very similar to [20]. As before, by commuting with ∂_t and $\mathcal{W}_i$ as well as using elliptic estimates, we see that on $\{r = const.\}$, ψ and higher derivatives $\partial^k \psi$ decay exponentially towards both components of i^+. This concludes the proof. $\square$

Chapter 3

Diophantine approximation as cosmic censor for AdS black holes

3.1 Introduction

We consider perturbations ψ solving the conformal scalar wave equation

$$\Box_g \psi - \frac{2}{3}\Lambda\psi = 0 \tag{3.1.1}$$

on Kerr–AdS black holes $(\mathcal{M}, g)$, see already (3.2.11) for the metric. We restrict to subextremal parameters satisfying the Hawking–Reall bound (3.2.8) and assume $a \neq 0$. We further consider reflecting boundary conditions at infinity. Our main result Theorem 3.1 shows that perturbations ψ solving (3.1.1) ***blow up*** everywhere at the Cauchy horizon on Kerr–AdS if the dimensionless black hole parameters mass $\mathfrak{m} = M\sqrt{-\Lambda}$ and angular momentum $\mathfrak{a} = a\sqrt{-\Lambda}$ satisfy certain Diophantine properties. We show that such black holes are ***Baire-generic*** but ***Lebesgue-exceptional***. This is in sharp contrast to the analogous result [81] on Reissner–Nordström–AdS black holes in Chapter 2, where it was shown that such perturbations remain bounded and extend continuously across the Cauchy horizon.

We also conjecture that, if the dimensionless black hole parameters mass $\mathfrak{m} = M\sqrt{-\Lambda}$ and angular momentum $\mathfrak{a} = a\sqrt{-\Lambda}$ do not satisfy the Diophantine conditions, linear perturbations remain ***bounded*** at the Cauchy horizon. This would then hold for ***Lebesgue-generic*** but ***Baire-exceptional*** black hole parameters.

Since the black hole parameters satisfy the Hawking–Reall bound, superradiance is absent. In particular, the instability in Theorem 3.1 originates from an intricate resonance phenomenon of stable trapping in the *exterior* coupled to the zero-frequency resonances associated to the Killing generator of the Cauchy horizon in the *interior*. In order to present our main theorem and the connection of the linear scalar analog of the Strong Cosmic Censorship conjecture (Conjecture 3) on Kerr–AdS to Diophantine approximation, we will first outline the behavior of waves on the black hole exterior in Section 3.1.1 and then focus on the interior in Section 3.1.2. Finally, putting both insights together, the

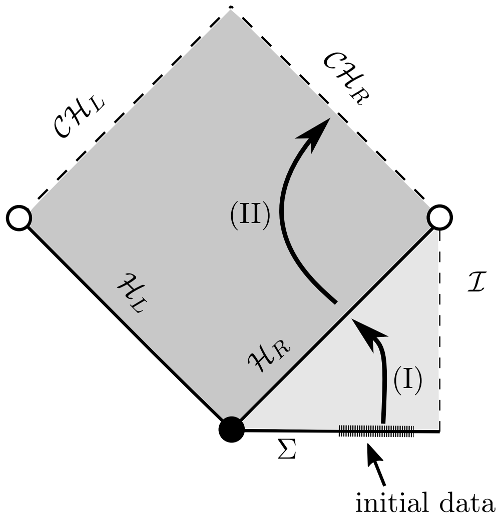

Figure 3.1: (I): Exterior propagation, (II): Interior propagation

connection to Diophantine approximation becomes transparent in Section 3.1.3. This will lead to a new expectation that transcends Conjecture 3 and Conjecture 4 which we formulate in Section 3.1.4 in terms of Conjecture 5 and Conjecture 6. In Section 3.1.5 we state the our main result Theorem 3.1, which resolves Conjecture 5 in the affirmative. Then, in Section 3.1.6 we give an outlook on Conjecture 6. We briefly describe our proof of Theorem 3.1 in Section 3.1.7 and give an outline of Chapter 3 in Section 3.1.8.

3.1.1 Exterior: log-decay, resonances and semi-classical heuristics

We recall from the discussion in the introduction of the thesis the result by Holzegel–Smulevici [75, 77] that perturbations ψ solving (3.1.1) decay at a sharp inverse logarithmic rate

$$|\psi| \leq \frac{C}{\log(t)} \tag{3.1.2}$$

on the Kerr–AdS exterior. The reason for the slow decay is a stable trapping phenomenon near infinity. One manifestation of this phenomenon is the existence of so called *quasi-*

modes and *resonances* (*quasinormal modes*) which are "converging exponentially fast" to the real axis. While our proof of Theorem 3.1 does not make use of a quasinormal mode construction or decomposition, they do provide good intuition—paired with the interior analysis in Section 3.1.2—how the relation to Diophantine approximation arises. Our discussion of quasi(normal) modes starts with the property that (3.1.1) is formally separable as shown in [13].

Separation of Variables. With the fixed-frequency ansatz

$$\psi = \frac{u(r)}{\sqrt{r^2+a^2}} S_{m\ell}(a\omega, \cos\theta) e^{im\phi} e^{-i\omega t}, \tag{3.1.3}$$

the wave equation (3.1.1) reduces to a coupled system of o.d.e's (see already (3.2.36)). The radial o.d.e. reads

$$-u''(r^*) + V(r^*, \omega, \lambda_{m\ell})u = 0 \tag{3.1.4}$$

for a rescaled radial variable $r^* \in (-\infty, \frac{\pi}{2})$ with $r^*(r = r_+) = -\infty$, $r^*(r = +\infty) = \frac{\pi}{2}$. The radial o.d.e (3.1.4) couples to the angular o.d.e. through the potential V which depends on the eigenvalues $\lambda_{m\ell}(a\omega)$ of the angular o.d.e.

$$P(a\omega) S_{m\ell}(a\omega, \cos\theta) = \lambda_{m\ell}(a\omega) S_{m\ell}(a\omega, \cos\theta), \tag{3.1.5}$$

where $P(a\omega)$ a self-adjoint Sturm–Liouville operator. The radial o.d.e. (3.1.4) is equipped with suitable boundary conditions at $r^* = -\infty$ and $r^* = \frac{\pi}{2}$ which stem from imposing regularity for ψ at the event horizon and Dirichlet boundary conditions at infinity. This leads to the concept of a *mode solution* ψ of (3.1.1) defined to be of the form (3.1.3) such that u solves (3.1.4) and $S_{m\ell}$ solves (3.1.5) with the appropriate boundary conditions imposed. If such a solution ψ were to exist for $\omega \in \mathbb{R}$, this would correspond to a time-periodic solution. Such a solutions are however incompatible with the fact that all admissible solutions decay. Nevertheless, there exist "almost solutions" which are time-periodic. This leads us to the concept of

Quasimodes. In [77] it was shown that there exists a set of real and axi-symmetric frequencies $(\omega_n, m_n = 0, \ell_n)_{n\in\mathbb{N}}$ such that the corresponding functions ψ_n "almost" solve (3.1.1) in the sense that $\Box_g \psi_n + \frac{2}{3}\Lambda\psi_n = F_n$ with $|F_n| \lesssim \exp(-n)$. These almost-solutions are called *quasimodes* and their existence actually implies that the logarithmic decay is sharp as shown in [77]. These quasimodes are equivalently characterized through the condition that the Wronskian $\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]$ of solutions $u_{\mathcal{H}^+}, u_\infty$ of (3.1.4) adapted to the boundary conditions satisfies

$$|\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty](\omega_n, m_n, \ell_n)| \lesssim e^{-cn}. \tag{3.1.6}$$

The reason why there exist such quasimodes is the fact that in the high frequency limit, the potential in (3.1.4) admits a region of stable trapping, see already Fig. 3.2.

Quasinormal modes. Although the Wronskian $\mathfrak{W}[u_{\mathcal{H}}, u_\infty]$ does not have any real zeros, $\mathfrak{W}[u_{\mathcal{H}}, u_\infty] \neq 0$, it might very well have zeros in the lower half-plane with $\operatorname{Im}(\omega) < 0$. These zeros correspond to so-called *quasinormal modes*, i.e. finite energy solutions of the form (3.1.3) which decay at an exponential rate. For a more precise definition, construction and a more detailed discussion of quasinormal modes in general we refer to the introduction of [58]. Turning back to Kerr–AdS, we note that the bound (3.1.6) implies the existence of zeros of $\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]$ exponentially close to the real axis as shown in [59], see also [145]. More precisely, it was shown that there exist axisymmetric, finite energy solutions to (3.1.1) of the form (3.1.3) with frequencies $m = 0$ and $(\omega, \ell) = (\omega_n, \ell_n)_{n\in\mathbb{N}}$ satisfying

$$c\ell_n \leq |\operatorname{Re}(\omega_n)| \leq C\ell_n, \tag{3.1.7}$$

$$0 < -\operatorname{Im}(\omega_n) \leq C\exp(-c\ell_n). \tag{3.1.8}$$

While the previous results were proved in axisymmetry to simplify the analysis, in principle, they also extend to non-axisymmetric solutions as remarked in [59].

Semi-classical heuristics. We turn to the heuristic distribution of quasinormal mode frequencies in the semi-classical (high frequency) limit. For large $m \in \mathbb{Z}$, $\ell \geq |m|$, we expect a quasinormal mode with frequencies $m, \ell, \omega = \omega_R + i\omega_I$ to exists, if the potential $V(r^*, \omega_R, m, \lambda_{m\ell}(a\omega))$ appearing in the radial o.d.e. (3.1.4) satisfies (see Fig. 3.2)

- $V(r^*, \omega_R, m, \lambda_{m\ell}(a\omega_R)) > 0$ for $r_1 < r^* < r_2$,
- $V(r^*, \omega_R, m, \lambda_{m\ell}(a\omega_R) < 0$ for $r_2 < r^*$.

Note that the conditions above are satisfied for a range of ω_R of the form $c\ell < |\omega_R| < C\ell$. In addition, the potential has to satisfy the Bohr–Sommerfeld quantization condition, i.e. the phase space volume

$$\frac{1}{2\pi}\operatorname{vol}\left\{(r^*, \xi) \colon \xi^2 + V(r^*, \omega_R, m, \lambda_{m\ell}(a\omega_R)) < 0, r^* > r_2\right\} \tag{3.1.9}$$

should be an integer multiple modulo the Maslov index. Heuristically, we expect that for given but large $|m|, \ell \geq |m|$, there exist $N(m, \ell)$ quasinormal modes with $N(m, \ell) \sim \ell$ and

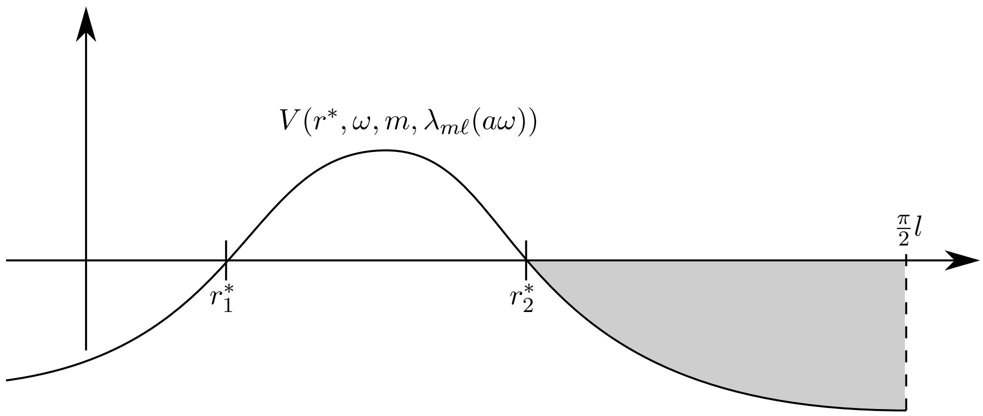

Figure 3.2: Potential V with parameters for which we expect quasimodes. The grey area is a suitable projection of the phase space volume.

$(\omega_{m\ell n})_{n=1,\cdots,N(m,\ell)}$ satisfying

$$c\ell \leq |\operatorname{Re}(\omega_{m\ell n})| \leq C\ell, \tag{3.1.10}$$

$$0 < -\operatorname{Im}(\omega_{m\ell n}) \leq C\exp(-\ell). \tag{3.1.11}$$

For our heuristic analysis we will now consider a solution ψ of (3.1.1) which consists of a sum of quasinormal modes. (Warning: A general solution cannot be written as a sum of quasinormal modes.) We denote with $\psi_{m\ell n}$ the quasinormal mode associated to the frequencies $(\omega_{m\ell n}, m, \ell)$ and formally consider the sum

$$\psi(t, r, \theta, \phi) = \sum_{m\in\mathbb{Z}} \sum_{\ell\geq|m|} \sum_{n=1}^{N(m,\ell)} \psi_{m\ell n}(t, r, \theta, \phi). \tag{3.1.12}$$

Restricting ψ to the event horizon yields

$$\psi \restriction_{\mathcal{H}} (v, \theta, \tilde{\phi}_+) = \sum_{m\in\mathbb{Z}} \sum_{\ell\geq|m|} \sum_{n=1}^{N(m,\ell)} a(m, \ell, n) e^{-i\omega_{m\ell n} v} S_{m\ell}(a\omega_{m\ell n}, \cos\theta) e^{im\tilde{\phi}_+} \tag{3.1.13}$$

for suitable weights $a(m, \ell, n)$. Since $\psi \restriction_{\mathcal{H}}$ has finite energy and finite L^2 norm along the horizon, we infer that $\sum_{m\in\mathbb{Z}} \sum_{\ell\geq|m|} \sum_{n=1}^{N(m,\ell)} |a(m,\ell,n)|^2 \frac{1}{|2\operatorname{Im}(\omega_{m\ell n})|} < \infty$. This is true if the weights satisfy (see (3.1.11))

$$|a(m, \ell, n)|^2 \sim \exp(-\ell). \tag{3.1.14}$$

3.1.2 Interior: scattering from event to Cauchy horizon

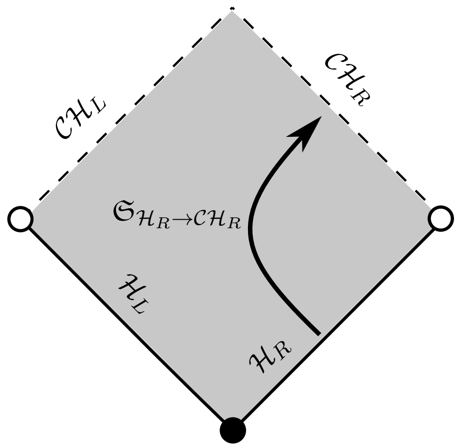

Figure 3.3: Interior scattering $\mathfrak{S}_{\mathcal{H}_R \to \mathcal{CH}_R}$ from event horizon $\mathcal{H}$ to Cauchy horizon $\mathcal{CH}$

We now turn to the interior problem and we will view some aspects of the propagation of ψ from the event horizon to the Cauchy horizon as a scattering problem as visualized in Fig. 3.3. We refer to Chapter 1 for a detailed discussion of the scattering problem on black hole interiors. Unlike Chapter 1, we will not develop a scattering theory for Kerr–AdS, but rather make use of a key insight from Chapter 1 adapted to our context. Recall from Proposition 1.6.2 that on Reissner–Nordström–AdS, the scattering operator $\mathfrak{S}_{\mathcal{H}_R \to \mathcal{CH}_R}$ in the interior has a zero-frequency resonance. This zero frequency, however, has to measured with respect to the Killing generator of the Cauchy horizon. In the present case for Kerr–AdS, the vector field $K_- := T + \omega_- \Phi$ generates the Cauchy horizon. One thus expects the analog of the scattering operator $\mathfrak{S}_{\mathcal{H}_R \to \mathcal{CH}_R}$ on the interior of Kerr–AdS to have the form

$$\mathfrak{S}_{\mathcal{H}_R \to \mathcal{CH}_R} = \mathcal{F}^{-1} \circ \mathfrak{R}(\omega, m, \ell) \circ \mathcal{F} = \mathcal{F}^{-1} \circ \frac{\mathfrak{r}(\omega, m, \ell)}{\omega - \omega_- m} \circ \mathcal{F} \tag{3.1.15}$$

with the zero frequency resonance $\omega - \omega_- m = 0$. At this point, we also refer to Fig. 3.4 for an illustration of the main difference of the behavior of linear perturbations on Reissner–Nordström–AdS and Kerr–AdS.

3.1.3 Heuristics and relation to Diophantine approximation

We now connect the exterior analysis from Section 3.1.1 to the interior analysis in Section 3.1.2. The following analysis will be purely formal but illustrates the connection to Diophantine approximation. From the exterior analysis, we assume that our solution ψ is a sum of quasinormal modes as in (3.1.13). Now, we have to apply the scattering operator

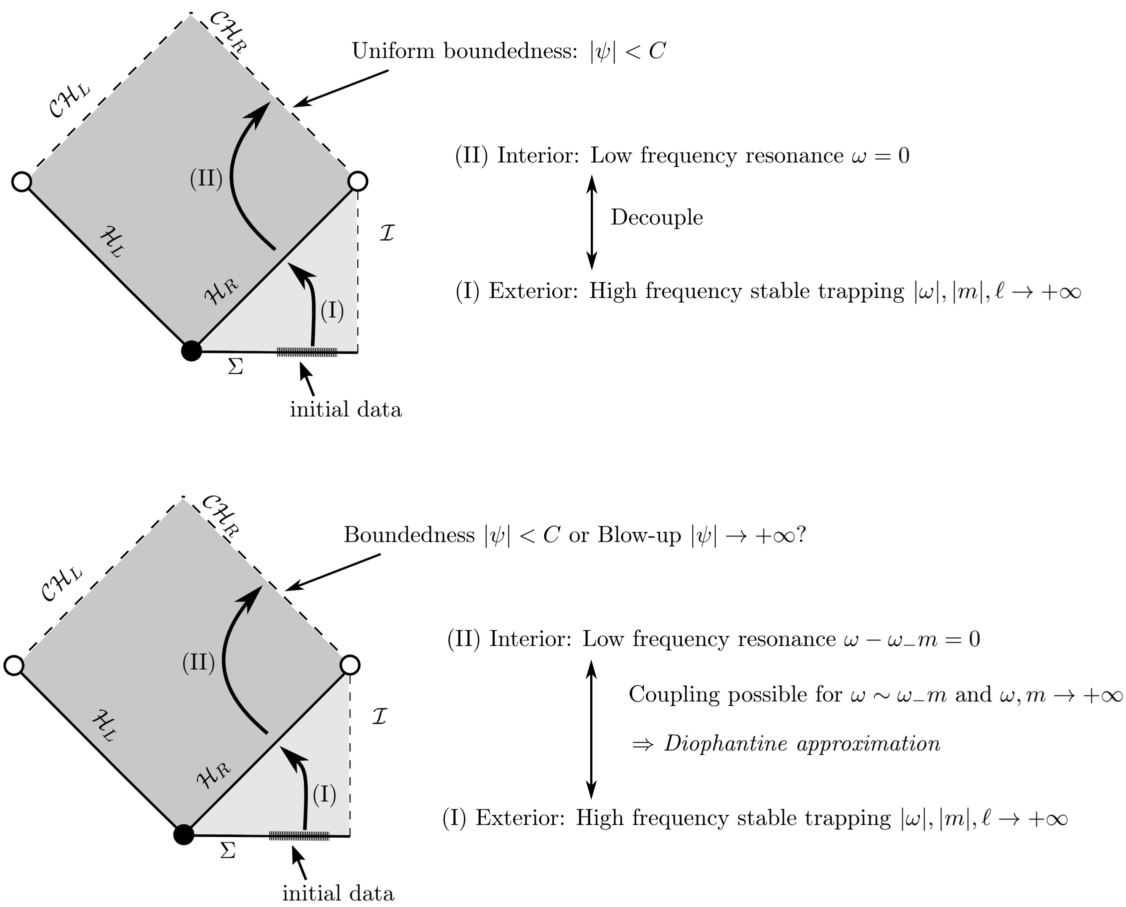

Figure 3.4: Reissner–Nordström–AdS (top): High frequency stably trapped perturbations ***decouple*** from low frequency resonance $\omega = 0$.
Kerr–AdS (bottom): High frequency stably trapped perturbations ***couple*** to low frequency resonance $\omega = \omega_- m$.

(3.1.15) to obtain the behavior at the Cauchy horizon. First, we may formally think of the Fourier transform along the event horizon to be only supported on the quasinormal frequencies to obtain

$$\mathcal{F}[\psi \restriction_{\mathcal{H}}] \sim \sum_{m\in\mathbb{Z}} \sum_{\ell \geq |m|} \sum_{n=1}^{N(m,\ell)} a(m,\ell,n)\delta(\omega - \omega_{m\ell n}). \tag{3.1.16}$$

Now, we multiply the reflection coefficient

$$\mathfrak{R}(\omega, m, \ell) = \frac{\mathfrak{r}(\omega, m, \ell)}{\omega - \omega_- m} \tag{3.1.17}$$

in Fourier space. Then, taking the inverse Fourier transform and neglecting $\mathfrak{r}$ yields formally

$$\begin{aligned}\psi \restriction_{\mathcal{CH}} &\sim \mathcal{F}^{-1} \circ \mathfrak{R} \circ \mathcal{F}[\psi \restriction_{\mathcal{H}}] \\ &\sim \sum_{m\in\mathbb{Z}} \sum_{\ell \geq |m|} \sum_{n=1}^{N(m,\ell)} \int_{\mathbb{R}} \frac{a(m,\ell,n)\delta(\omega - \omega_{m\ell n})}{\omega - \omega_{-} m} e^{-i\omega u} e^{im\tilde{\phi}_{-}} S_{m\ell}(a\omega) \mathrm{d}\omega. \qquad (3.1.18)\end{aligned}$$

Finally, we consider the $L^2(\mathbb{S}^2)$-norm of the sphere such that we formally obtain

$$\|\psi \restriction_{\mathcal{CH}} (u_0)\|^2_{L^2(\mathbb{S}^2)} \sim \sum_{m\in\mathbb{Z}} \sum_{\ell \geq |m|} \sum_{n=1}^{N(m,\ell)} \frac{|a(m,\ell,n)|^2}{|\omega_{m\ell n} - \omega_{-} m|^2}, \qquad (3.1.19)$$

where we recall that $a(m,\ell,n)$ decay exponentially as in (3.1.14). To resolve Conjecture 3, we have to determine whether for such exponentially decaying coefficients $a(m,\ell,n)$, the sum (3.1.19) remains uniformly bounded or whether this sum can become infinite.

Small divisors and Diophantine approximation. The convergence of (3.1.19) is an example *par excellence* of a ***small divisor problem***. Indeed, if $|\omega_{m\ell n} - \omega_{-} m|$ is also exponentially small in m, ℓ, n, the sum in (3.1.19) becomes infinite for general $a(m,\ell,n)$. More precisely, for the sum in (3.1.19) to become infinite, it suffices that there exist infinitely many (m,ℓ,n) such that $|\omega_{m\ell n} - \omega_{-} m|$ decays exponentially. Thus, we conjecture blow-up if

$$|\omega_{m\ell n} - \omega_{-} m| < \exp(-\ell) \text{ for infinitely many admissible } (m,\ell,n), \qquad (3.1.20)$$

where (m,ℓ,n) are admissible if $m \in \mathbb{Z}, \ell \geq |m|, n = 1, \dots, N(m,\ell)$.

Conditions like (3.1.20) lie at the heart of ***Diophantine approximation***. Indeed, semi-classical heuristics suggest that $\omega_{m\ell n}$ are uniformly distributed and we assume for a moment that $\omega_{m\ell n} = c(\ell + \frac{n}{\ell})$ for $n = 0, 1, \dots, \ell$ for a constant $c = c(M, a, \Lambda)$. Then, the ratio $r(\mathfrak{m}, \mathfrak{a}) := \frac{\omega_{-}}{c}$, which is dimensionless and only depends on the dimensionless black hole parameters ($\mathfrak{m} = M\sqrt{-\Lambda}, \mathfrak{a} = a\sqrt{-\Lambda}$), has to satisfy the Diophantine condition

$$r(M, a, \Lambda) \in \mathscr{R} := \left\{ x \in \mathbb{R} \colon \left| \frac{\ell + \frac{n}{\ell}}{m} - x \right| < \exp(-\ell) \text{ for } \infty\text{-many admissible } (m,\ell,n) \right\}. \qquad (3.1.21)$$

Thus, from our heuristic derivation, we conjecture that linear perturbations ***blow up*** on the Cauchy horizon of Kerr–AdS with mass $M = \mathfrak{m}/\sqrt{-\Lambda}$ and angular momentum

$a = \mathfrak{a}/\sqrt{-\Lambda}$ if the ratio $r = r(\mathfrak{m}, \mathfrak{a})$ satisfies the Diophantine condition (3.1.21).

The set $\mathscr{R}$ is Baire-generic and Lebesgue-exceptional. The set $\mathscr{R}$ can be written as a lim sup set as

$$\mathscr{R} = \bigcap_{m_0 \in \mathbb{N}} \bigcup_{|m| \geq m_0} \bigcup_{\ell \geq |m|} \bigcup_{0 \leq n \leq \ell} \left\{ x \in \mathbb{R} \colon \left| \frac{\ell + \frac{n}{\ell}}{m} - x \right| < \exp(-\ell) \right\}. \tag{3.1.22}$$

It is a countable intersection of open and dense sets such that $\mathscr{R}$ is of second category in view of Baire's theorem. Thus, the set $\mathscr{R}$ is generic from a topological point of view, which we refer to as ***Baire-generic***. On the other hand, from a measure-theoretical point of view, the set $\mathscr{R}$ is exceptional. Indeed, an application of the Borel–Cantelli lemma shows that the Lebesgue measure of $\mathscr{R}$ vanishes. This is the easy part of the famous theorem by Khintchine [84] stating that for a decreasing function ϕ, the set

$$W[\phi] := \left\{ x \in \mathbb{R} \colon \left| x - \frac{p}{q} \right| < \frac{\phi(q)}{q} \text{ for } \infty\text{-many rationals } \frac{p}{q} \right\} \tag{3.1.23}$$

has full Lebesgue measure if and only if the sum $\sum_q \phi(q)$ diverges. Thus, $\mathscr{R}$ is ***Lebesgue-exceptional***.

More refined measure: The Hausdorff and packing measures. This naturally leads us to consider the more refined versions of measure, the so-called ***Hausdorff*** and ***packing measures*** H^f, P^f together with their associated ***dimensions*** $\dim_H$, $\dim_P$ (see Section 3.2.1). The Hausdorff and packing measure generalize the Lebesgue measure to non-integers. In a certain sense, they can be considered to be dual to each-other: The Hausdorff measure approximates and measures sets by a most economical covering, whereas the packing measure packs as many disjoint balls with centers inside the set. While for all sufficiently nice sets these notions agree, they indeed turn out to give different results in our context.

We first consider the Hausdorff dimension. A version of the Borell–Cantelli lemma (more precisely the Hausdorff–Cantelli lemma) and using the natural cover for $\mathscr{R}$ shows that the set $\mathscr{R}$ is of ***Hausdorff dimension zero***. This again can be seen as a consequence of a theorem going back to Jarník [79] and Besicovitch [7] which states the set $W[\phi]$ as in (3.1.23) has Hausdorff measure

$$H^s(W[\phi]) = \begin{cases} 0 & \text{if } \sum_q q^{1-s} \psi^s(q) < \infty \\ +\infty & \text{if } \sum_q q^{1-s} \psi^s(q) = \infty \end{cases} \tag{3.1.24}$$

for $s \in (0,1)$. However, measuring also logarithmic scales, i.e. considering the Hausdorff measure H^f for $f = \log^t(r)$ for some $t > 0$, it follows that the set $\mathscr{R}$ is of ***logarithmic Hausdorff dimension***. On the other hand, using the dual notion of packing dimension, it turns out that $\mathscr{R}$ has ***full packing dimension***, a consequence of the fact that it is a set of second category (Baire-generic) [46].

Summary of properties of $\mathscr{R}$. To summarize, we obtain that

- $\mathscr{R}$ is Baire-generic,
- $\mathscr{R}$ is Lebesgue-exceptional,
- $\mathscr{R}$ has zero Hausdorff dimension $\dim_H(\mathscr{R}) = 0$,
- $\mathscr{R}$ is of logarithmic Hausdorff dimension,
- $\mathscr{R}$ has full packing dimension $\dim_P(\mathscr{R}) = 1$.

The above heuristics will enter in our revised conjectures, Conjecture 5 and Conjecture 6, which transcend Conjecture 3 and Conjecture 4 for $\Lambda < 0$. Before we turn to that in Section 3.1.4, we briefly discuss other aspects of PDEs and dynamical systems for which Diophantine approximation plays a crucial role.

Diophantine approximation in dynamical systems and PDEs. Most prominently, Diophantine approximation and the small divisor problem are intimately tied to the problem of the stability of the solar system [110] and more generally, the stability of Hamiltonian systems in classical mechanics. We refer to the discussion in the prologue of the thesis. The small divisor problem and Diophantine approximation are ubiquitous in modern mathematics and arise naturally in many other aspects of PDEs and dynamical systems. We refer to [47, 114] and for a connection to wave equations with periodic boundary conditions and to the more general results in [61] as well as the monograph [130]. Similar results have been obtained for the Schrödinger equation on the torus in [86, 80, 45]. Further applications of Diophantine approximation include the characterization of homeomorphisms on $\mathbb{S}^1$ by the Diophantine properties of their rotation numbers or analyzing the Lyapunov stability of vector fields, see the discussion in [87].

3.1.4 Conjecture 5 and Conjecture 6 replace Conjecture 3 and Conjecture 4 for AdS black holes

With the above heuristics at hand, we now transcend Conjecture 3 and Conjecture 4 for subextremal Kerr–AdS black holes below the Hawking–Reall bound in terms of the following two conjectures.

Conjecture 5. *Linear scalar perturbations ψ satisfying* (3.1.1)*, and arising from generic smooth data on a spacelike hypersurfaces with Dirichlet boundary conditions at infinity,* ***blow up***

$$\|\psi\|_{L^2(\mathbb{S}^2)}(u,r) \to +\infty \tag{3.1.25}$$

at the Cauchy horizon of Kerr–AdS for a set $\mathscr{P}_{\text{Blow-up}}$ of dimensionless black hole parameters mass $\mathfrak{m} = M\sqrt{-\Lambda}$ and angular momentum $\mathfrak{a} = a\sqrt{-\Lambda}$ with the following properties

- *$\mathscr{P}_{\text{Blow-up}}$ is* ***Baire-generic*** *(of second category),*
- *$\mathscr{P}_{\text{Blow-up}}$ is* ***Lebesgue-exceptional*** *(zero Lebesgue measure).*

Remark 3.1.1. *Remark that Conjecture 5 is an instability result for a linear equation. Thus, it suffices to show that there exists one admissible set of initial data (ψ_0, ψ_1) leading to a solution ψ which blows up in the sense of* (3.1.25)*. Indeed, if true, this shows that data $(\tilde{\psi}_0, \tilde{\psi}_1)$ for which the arising solution does not blow up are exceptional in the sense that they obey the following co-dimension 1 property: The solution arising from the perturbed data $(\psi_0 + c\tilde{\psi}_0, \psi_1 + c\tilde{\psi}_1)$ blows up for each $c \in \mathbb{R} \setminus \{0\}$. This is analogous to the notion of genericity used by Christodoulou in his proof of weak cosmic censorship for the spherically symmetric Einstein-scalar-field system [16, 15]. It is an interesting question to find further, more refined genericity conditions for the set of initial data leading to solutions which blow up as in* (3.1.25)*.*

Remark 3.1.2. *Moreover, we conjecture that the set $\mathscr{P}_{\text{Blow-up}}$ has*

- *Hausdorff dimension $\dim_H(\mathscr{P}_{\text{Blow-up}}) = 1 + \log$,*
- *full packing dimension $\dim_P(\mathscr{P}_{\text{Blow-up}}) = 2$.*

Moreover, in view of our discussion we additionally conjecture

Conjecture 6. (A) *Linear perturbations ψ satisfying* (3.1.1)*, and arising from generic smooth data on a spacelike hypersurfaces with Dirichlet boundary conditions at infinity,* ***remain uniformly bounded***

$$\|\psi\|_{L^2(\mathbb{S}^2)}(u,r) \leq C, \tag{3.1.26}$$

yet, blow up in energy

$$\|\psi\|_{H^1_{\text{loc}}} \to +\infty \tag{3.1.27}$$

at the Cauchy horizon of Kerr–AdS for a set $\mathscr{P}_{\text{Bounded}}$ *of dimensionless black hole parameters mass* $\mathfrak{m} = M\sqrt{-\Lambda}$ *and angular momentum* $\mathfrak{a} = a\sqrt{-\Lambda}$ *with the following properties*

- $\mathscr{P}_{\text{Bounded}}$ *is* ***Baire-exceptional*** *(of first category),*
- $\mathscr{P}_{\text{Bounded}}$ *is* ***Lebesgue-generic*** *(full Lebesgue measure).*

(B) *Linear perturbations* ψ *satisfying* (3.1.1)*, and arising from generic smooth data on a spacelike hypersurfaces with Dirichlet boundary conditions at infinity* ***blow up in energy***

$$\|\psi\|_{H^1_{\text{loc}}} \to +\infty \tag{3.1.28}$$

at the Cauchy horizon of Kerr–AdS for ***all*** *subextremal parameters below the Hawking–Reall bound.*

Blow-up in amplitude. Note that in Conjecture 5 (and Conjecture 6) we have replaced the statement of blow-up in amplitude from Conjecture 3 with statement about the blow-up of the L^2-norm on the sphere. Indeed, the blow of the L^2-norm in Conjecture 5, if true, implies that $\|\psi\|_{L^\infty(\mathbb{S}^2)} \to +\infty$. In this sense, if Conjecture 5 is true, the amplitude also blows up.

It is however an interesting and open question of whether one may actually replace the $L^\infty(\mathbb{S}^2)$ blow-up statement in (3.1.25) with the pointwise blow-up

$$\lim_{r\to r_-} |\psi(u, r, \theta, \phi^*_-)| \to +\infty \tag{3.1.29}$$

for every $(\theta, \phi^*_-) \in \mathbb{S}^2$. One may even speculate about the geometry of the set of $(\theta, \phi^*_-) \in \mathbb{S}^2$ for which pointwise blow-up holds. It appears that ultimately one has to quantitatively understand the nodal domains associated to the generalized spheroidal harmonics $S_{m\ell}(a\omega_- m, \cos\theta)$ at the resonant frequency.

More general boundary conditions. The above conjectures are both stated for Dirichlet conditions at infinity. Neumann conditions are also natural to consider and indeed well-posedness was proved in [144, 78]. For Neumann conditions we also expect the same behavior as for the case of Dirichlet boundary conditions. For other more general conditions, it may be the case that linear waves grow exponentially (as for suitable Robin boundary conditions [78]) or on the other hand even decay superpolynomially as in the case for purely outgoing conditions [74]. For even more general boundary conditions, even well-posedness may be open.

3.1.5 Main result: Conjecture 5 is true

Our main result of Chapter 3 is the following resolution of Conjecture 5.

Theorem 3.1. *Conjecture 5 is true.*

The proof of Theorem 3.1 will be given in Section 3.8.3.

Remark 3.1.3. *We also prove in Section 3.8.3 the statement about the packing dimension of $\mathscr{P}_{\text{Blow-up}}$ as conjectured in Remark 3.1.2. The statement concerning the Hausdorff dimension, however, remains open.*

Remark 3.1.4. *While we only consider Dirichlet boundary conditions at infinity, in principle, our proof is expected to also apply to Neumann boundary conditions.*

3.1.6 Outlook on Conjecture 6

We also expect that our methods provide a possible framework for the resolution of Conjecture 6.

We have already remarked in the introduction to the present thesis that our methods may in principle also show the statement of energy blow-up Conjecture 6(B). Indeed, we expect that a quasinormal mode which decays at sufficiently slow exponential decay rate compared to the surface gravity of the Cauchy horizon will indeed lead to blow up in energy at the Cauchy horizon.

Towards Conjecture 6(A), we note that our proof, particularly the formula (3.8.100), reveals the main obstruction for boundedness which can serve as a starting point for a resolution of Conjecture 6(A).

3.1.7 Brief description of the proof

We will give a brief description of the key ideas of our proof. First, we mention that compared to the heuristic discussion above, we will not make use of quasinormal modes. Our proof will be based on frequency analysis on the real axis, i.e. with $\omega \in \mathbb{R}$.

We start with the interior analysis. We recall from our previous discussion that the analog of a scattering operator (3.1.15) from the event to the Cauchy horizon has a singularity at the resonant frequency $\omega - \omega_- m$. In reality, this singularity becomes evident in the formula (3.8.100) which roughly translates to the statement that, as $r \to r_-$, we have

$$\|\psi(u_0, r)\|^2_{L^2(\mathbb{S}^2)} \sim \sum_{m\ell} |m|^2 \left|\mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}](\omega = \omega_- m)\right|^2 + \mathrm{Err}(D), \tag{3.1.30}$$

where $\mathrm{Err}(D)$ is uniformly bounded by an (higher order) energy of the initial data. Both, the proof and the use of formula (3.8.100) lie at the heart of the proof of Theorem 3.1. The proof of (3.8.100) is technical and combines physical space methods with techniques from harmonic analysis. One of the key technical steps (see Proposition 3.3.3) is a quantitative bound (see already (3.3.67)) on the derivative of the generalized spheroidal harmonics

$$\sup_{|a\omega - a\omega_- m| < \frac{1}{m}} \|\partial_\omega S_{m\ell}(a\omega)\|^2_{L^2} \lesssim m$$

near the resonant frequency. This is shown in Section 3.3.3 and the proof relies on uniform bounds (in m, ℓ and $\omega \approx \omega_- m$) on the resolvent of the associated singular Sturm–Liouville operator, see the discussion in Section 3.3.3. These bounds are shown by constructing the associated integral kernel using suitable approximations with parabolic cylinder functions and Airy functions. The analogous resolvent bounds and estimates for solutions of the radial o.d.e. in the interior are shown in Section 3.8.1. The proofs in Section 3.8.1 rely on WKB approximations and estimates on Volterra integral equations.

In what follows, we will connect the aforementioned interior analysis and in particular the formula (3.1.30) to the exterior. A key step is to characterize the generalized Fourier transform along the event horizon $\mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}]$ in terms of the initial data which is the content of Section 3.7. While in the actual proof (see already Proposition 3.7.1), we will make use of suitable cut-offs in time and space, we may think of $\mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}]$ as having the form

$$\mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}](\omega, m, \ell) \sim \frac{1}{\mathfrak{W}[u_{\mathcal{H}}, u_\infty]} \int_{\Sigma_0} u_\infty H(\psi_0, \psi_1) \mathrm{dvol}_{\Sigma_0}, \tag{3.1.31}$$

where $H(\psi_0, \psi_1)$ depends on the initial data which will be chosen to be smooth and compactly supported. A consequence of the work by Holzegel–Smulevici [75] is that for our choice of initial data, higher order energy fluxes along the event horizon are bounded. Thus, $|\partial_v^i \nabla\!\!\!\!/^j \psi \restriction_{\mathcal{H}}|_{g\!\!\!/} \in L^2(\mathcal{H})$ for all $i, j \geq 0$ which corresponds in Fourier space to the statement that

$$|\omega|^i \ell^{j_1} |m|^{j_2} \mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}] \in L^2(\mathbb{R}_\omega \times \mathbb{Z}_m \times \mathbb{Z}_{\ell \geq |m|}) \text{ for all } i, j_1, j_2 \geq 0. \tag{3.1.32}$$

In view of the above and (3.1.30), in order to show blow-up, it is necessary that the Wronskian (cf. Section 3.1.1 before) evaluated at the resonant frequency $\mathfrak{W}[u_{\mathcal{H}}, u_\infty](\omega = \omega_- m)$ decays (at least) superpolynomially for infinitely many (m, ℓ). In our proof, we actually require that the Wronskian even decays exponentially $|\mathfrak{W}[u_{\mathcal{H}}, u_\infty](\omega = \omega_- m)| \leq e^{-\ell} e^{-|m|}$ for infinitely many (m, ℓ). (Connecting this to our previous heuristic discussion with quasinormal modes in Sections 3.1.1–3.1.3, this can be interpreted as the statement that there exist infinitely many quasinormal mode frequencies exponentially close to the resonant frequencies $\omega = \omega_- m$.)

Before we address this question of whether the Wronskian actually decays exponentially $|\mathfrak{W}[u_{\mathcal{H}}, u_\infty](\omega = \omega_- m)| \leq e^{-\ell} e^{-|m|}$ for infinitely many (m, ℓ), we will assume for a moment that this is indeed the case. Then, with carefully chosen initial data (see already Section 3.6), the decay of the Wronskian $|\mathfrak{W}[u_{\mathcal{H}}, u_\infty](\omega = \omega_- m)| \leq e^{-\ell} e^{-|m|}$ for infinitely many (m, ℓ) corresponds to peaks in $\mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}]$ at the resonant frequencies (see already Lemma 3.6.2) which are however consistent with the integrability properties of $\mathcal{F}_{\mathcal{H}}[\psi \restriction_{\mathcal{H}}]$ as in (3.1.32). Since these peaks appear for infinitely many (m, ℓ), infinitely many summands of (3.1.30) are greater than $e^{m^{\frac{1}{4}}}$, see already (3.8.137), from which the blow-up result follows.

Finally, this leaves us to address the question of whether the Wronskian satisfies $|\mathfrak{W}[u_{\mathcal{H}}, u_\infty](\omega = \omega_- m)| \leq e^{-\ell} e^{-m}$ for infinitely many (m, ℓ). Similar to our previous heuristic discussion, we will show that this Diophantine conditions holds true for a set of dimensionless black hole parameters $(\mathfrak{m}, \mathfrak{a}) = (M\sqrt{-\Lambda}\ , a\sqrt{-\Lambda}) \in \mathscr{P}_{\text{Blow-up}}$ which is Baire-generic but Lebesgue-exceptional. We specify this set $\mathscr{P}_{\text{Blow-up}}$ in Definition 3.5.3.

3.1.8 Outline of Chapter 3

In Section 3.2 we set up the Kerr–AdS spacetime and recall the decay statement on the exterior as well as Carter's separation of variables. Section 3.3 and Section 3.4 are devoted to the analysis of the angular and radial o.d.e., respectively. Then, in Section 3.5 we define the set $\mathscr{P}_{\text{Blow-up}}$ and show its topological and metric properties. Then, for fixed parameters in $\mathscr{P}_{\text{Blow-up}}$ we define suitable compactly supported initial data in Section 3.6. In Section 3.7 we treat the exterior problem and estimate the behavior of the solution along the horizon. Finally, in Section 3.8 we propagate the solution from the event to the Cauchy horizon and eventually show the blow-up result.

3.2 Preliminaries

3.2.1 Fractal measures and dimensions

3.2.1.1 Hausdorff and Packing measures

We begin by introducing the Hausdorff and packing measure. We refer to the monograph [46] for a more detailed discussion. For an increasing *dimension function* $f\colon [0, \infty) \to$

$[0, \infty)$ we define the Hausdorff measure $H^f(A)$ of a set A as

$$H^f(A) := \sup_{\delta>0} H^f_\delta(A), \tag{3.2.1}$$

where

$$H^f_\delta(A) := \inf\{\sum_{i=1}^{\infty} f(\mathrm{diam}(U_i))\colon \{U_i\}_{i=1}^{\infty} \text{ countable cover of } A, \mathrm{diam}(U_i) \le \delta\}. \tag{3.2.2}$$

If $f(r) = r^s$, we write $H^s = H^{r^s}$ and for $s \in \mathbb{N}$, the measure H^s reduces to the Lebesgue measure up to some normalization. While the Hausdorff measure quantifies the size of a set by approximation it from outside via efficient coverings, we also recall the dual notation: The *packing measure* quantifies the size of sets by placing as many disjoint balls with centers contained in the set. Again, for a dimension function f, we define the pre-measure

$$\begin{aligned} P^f_0(A) := \limsup_{\delta\to 0} \Big\{ \sum_{i=1}^{\infty} f(\mathrm{diam}(B_i))\colon \{B_i\}_{i=1}^{\infty} &\text{ collection of closed,} \\ &\text{pairwise disjoint balls with } \mathrm{diam}(B_i) \le \delta \text{ and centers in } A\Big\} \end{aligned} \tag{3.2.3}$$

and finally the packing measure as

$$P^f(A) := \inf\left\{\sum_{i=1}^{\infty} P^f_0(A_i)\colon A \subset \bigcup_{i=1}^{\infty} A_i\right\}. \tag{3.2.4}$$

3.2.1.2 Hausdorff and Packing dimensions

For $f(r) = r^s$ Hausdorff and Packing dimensions $\dim_H$ and $\dim_P$ are now characterized as the jump value, where the respective measure jumps from 0 to ∞, more precisely

$$\dim_H(A) = \sup\{s\colon H^s(A) = 0\}, \quad \dim_P(A) = \sup\{s\colon P^s(A) = 0\}. \tag{3.2.5}$$

We also say that a set A has Hausdorff dimension $\dim_H(A) = s + \log$ if the jump appears for the dimension function $f(r) = r^s \log^t(r)$ for some $t > 0$.

3.2.2 Kerr–AdS spacetime

3.2.2.1 Parameter space

We let the value of the cosmological constant $\Lambda < 0$ be *fixed* throughout the paper. For convenience and as it is convention, we re-parametrize the cosmological constant by the

AdS radius

$$l := \sqrt{\frac{-3}{\Lambda}}. \tag{3.2.6}$$

We consider Kerr–AdS black holes which are parametrized by their mass $M > 0$ and their angular momentum $a \neq 0$. Moreover, without loss of generality we will only consider $a > 0$ and require $0 < a < l$ for the spacetime to be regular. For $M > 0$, $0 < a < l$, we consider the polynomial

$$\Delta(r) := (a^2 + r^2)\left(1 + \frac{r^2}{l^2}\right) - 2Mr. \tag{3.2.7}$$

We are interested in spacetimes without naked singularities. To ensure this, we define a parameter tuple $(M, a) \in \mathbb{R}^2_{>0}$ to be *non-degenerate* if $0 < a < l$ and $\Delta(r)$ defined in (3.2.7) has two real roots satisfying $0 < r_- < r_+$. Finally, to exclude growing mode solutions (see [40]) we assume the Hawking–Reall (non-superradiant) bound

$$r_+^2 > al. \tag{3.2.8}$$

This leads us to the definition of the dimensionless black hole parameter space

$$\mathscr{P} := \{(\mathfrak{m}, \mathfrak{a}) \in \mathbb{R}^2_{>0} \colon (M, a) := (\mathfrak{m}l/\sqrt{3}, \mathfrak{a}l/\sqrt{3}) \text{ is non-degenerate and } r_+^2 > al\}.$$

Note that in view of (3.2.6), we have $M = \mathfrak{m}/\sqrt{-\Lambda} = \mathfrak{m}l/\sqrt{3}$ and $a = \mathfrak{a}/\sqrt{-\Lambda} = \mathfrak{a}l/\sqrt{3}$.

Finally, remark that $\mathscr{P}$ is a Baire space as a (non-empty) open subset of $\mathbb{R}^2$. In particular, this allows us to speak about the notion of *Baire-exceptional* and *Baire-generic* subsets. Recall that a subset is Baire-meager if it is a countable union of nowhere dense sets and a subset is called Baire-generic if it is a countable intersection of open and dense sets. Note that if a subset is Baire-generic then its complement is meager and vice versa. Finally, in a Baire space every Baire-generic subset is dense.

3.2.2.2 Kerr–AdS spacetime

Fixed manifold. We begin by constructing the Kerr–AdS spacetime. We define the exterior region $\mathcal{R}$ and the black hole interior $\mathcal{B}$ as smooth four dimensional manifolds diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$. On $\mathcal{R}$ and on $\mathcal{B}$ we assume to have global (up to the well-known degeneracy on $\mathbb{S}^2$) coordinate charts

$$(t_{\mathcal{R}}, r_{\mathcal{R}}, \theta_{\mathcal{R}}, \phi_{\mathcal{R}}) \in \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2, \tag{3.2.9}$$

$$(t_{\mathcal{B}}, r_{\mathcal{B}}, \theta_{\mathcal{B}}, \phi_{\mathcal{B}}) \in \mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2. \tag{3.2.10}$$

These coordinates (t, r, ϕ, θ) are called Boyer–Lindquist coordinates. If it is clear from the context which coordinates are being used, we will omit their subscripts throughout the chapter.

The Kerr–AdS metric. For $(\mathfrak{m}, \mathfrak{a}) \in \mathscr{P}$ and $M = \mathfrak{m}l/\sqrt{3}$ and $a = \mathfrak{a}l/\sqrt{3}$, we define the Kerr–AdS metric on $\mathcal{R}$ and $\mathcal{B}$ in terms of the Boyer–Lindquist coordinates as

$$\begin{aligned} g_{\mathrm{KAdS}} :=& -\frac{\Delta - \Delta_\theta a^2 \sin^2\theta}{\Sigma} \mathrm{d}t \otimes \mathrm{d}t + \frac{\Sigma}{\Delta} \mathrm{d}r \otimes \mathrm{d}r + \frac{\Sigma}{\Delta_\theta} \mathrm{d}\theta \otimes \mathrm{d}\theta \\ &+ \frac{\Delta_\theta (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Xi^2 \Sigma} \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi \\ &- \frac{\Delta_\theta (r^2 + a^2) - \Delta}{\Xi \Sigma} a \sin^2\theta (\mathrm{d}t \otimes \mathrm{d}\phi + \mathrm{d}\phi \otimes \mathrm{d}t), \end{aligned} \tag{3.2.11}$$

where

$$\Sigma := r^2 + a^2 \cos\theta, \;\; \Delta_\theta := 1 - \frac{a^2}{l^2} \cos^2\theta, \;\; \Xi := 1 - \frac{a^2}{l^2} \tag{3.2.12}$$

and Δ is as in (3.2.7). We will also write $\Delta_x := 1 - \frac{a^2}{l^2} x^2$ which arises from the substitution $x = \cos\theta$ in Δ_θ. At this point we define

$$\omega_+ := \frac{a\Xi}{r_+^2 + a^2}, \;\; \omega_- := \frac{a\Xi}{r_-^2 + a^2}, \;\; \omega_r := \frac{a\Xi}{r^2 + a^2}. \tag{3.2.13}$$

Now, we time-orient the patches $\mathcal{R}$ and $\mathcal{B}$ with $-\nabla t_{\mathcal{R}}$ and $-\nabla r_{\mathcal{B}}$, respectively. We also note that ∂_t and ∂_ϕ are Killing fields in each of the patches. The inverse metric reads

$$\begin{aligned} g_{\mathrm{KAdS}}^{-1} =& \left(-\frac{(r^2 + a^2)^2}{\Sigma \Delta} + \frac{a^2 \sin^2\theta}{\Sigma \Delta_\theta} \right) \partial_t \otimes \partial_t + \frac{\Delta}{\Sigma} \partial_r \otimes \partial_r + \frac{\Delta_\theta}{\Sigma} \partial_\theta \otimes \partial_\theta \\ &+ \left(\frac{\Xi^2}{\Sigma \Delta_\theta \sin^2\theta} - \frac{\Xi^2 a^2}{\Sigma \Delta} \right) \partial_\phi \otimes \partial_\phi + \left(\frac{\Xi a (r^2 + a^2)}{\Delta \Sigma} - \frac{\Xi}{\Delta_\theta \Sigma} \right) (\partial_t \otimes \partial_\phi + \partial_\phi \otimes \partial_t). \end{aligned} \tag{3.2.14}$$

On $\mathcal{R}$ and $\mathcal{B}$, we define the tortoise coordinate $r^*(r)$ by

$$\frac{\mathrm{d}r^*}{\mathrm{d}r}(r) := \frac{r^2+a^2}{\Delta(r)}, \tag{3.2.15}$$

where Δ is as in (3.2.7). For definiteness we set $r^*(r=+\infty) := \frac{\pi}{2}l$ on $\mathcal{R}$ and $r^*(\frac{1}{2}(r_+ + r_-)) = 0$ on $\mathcal{B}$.

Eddington–Finkelstein-like coordinates. We also define outgoing Eddington–Finkelstein-like coordinates $(v, r, \theta, \tilde{\phi}_+)$ in the exterior $\mathcal{R}$ as

$$v(t,r) := t + r^*\chi_v(r), \;\; \tilde{\phi}_+(\phi, r) := \phi + \omega_+ r^*(r)\chi_v(r) \bmod 2\pi, \tag{3.2.16}$$

where $\chi_v(r)$ is a smooth monotone cut-off function with $\chi_v(r) = 1$ for $r \leq r_+ + \eta$ and $\chi_v(r) = 0$ for $r \geq r_+ + 2\eta$ for some $\eta > 0$ small enough such that $J^+(\{r > 2r_+\} \cap \{t_{\mathcal{R}} = 0\}) \cap \{v = 0\} = \emptyset$ [1] and $\eta < \frac{r_+}{4}$. In these coordinates the spacetime $(\mathcal{R}, g_{\mathrm{KAdS}})$ can be extended (see [75] for more details) to a time-oriented Lorentzian manifold $(\mathcal{D}, g_{\mathrm{KAdS}})$ defined as $\mathcal{D} := \{(r, v, \theta, \tilde{\phi}_+) \in (r_-, \infty) \times \mathbb{R} \times \mathbb{S}^2\}$. Moreover, the Lorentzian submanifold $(\mathcal{D} \cap \{r_- < r < r_+\}, g_{\mathrm{KAdS}})$ coincides (up to time-orientation preserving isometry) with $(\mathcal{B}, g_{\mathrm{KAdS}})$. We identify these regions and denote the (right) event horizon as $\mathcal{H}_R := \{r = r_+\}$. The Killing null generator of the event horizon is

$$K_+ := \partial_v + \omega_+ \partial_{\tilde{\phi}_+}. \tag{3.2.17}$$

The Killing field K_+ is called the Hawking vector field and is future-directed and timelike in $\mathcal{R}$, a consequence the Hawking–Reall bound $r_+ > al$.

To attach the (left) Cauchy horizon $\mathcal{CH}_L$ we introduce in $\mathcal{B}$ further coordinates $(v, r, \theta, \tilde{\phi}_-)$, as

$$v = t + r^*, \;\; \tilde{\phi}_-(\phi, r) := \phi + \omega_- r^* \bmod 2\pi, \;\; r = r, \;\; \theta = \theta. \tag{3.2.18}$$

In these coordinates, the Lorentzian manifold extends smoothly to $r = r_-$ and the null hypersurface $\mathcal{CH}_L := \{r = r_-\}$ is the left Cauchy horizon with null generator

$$K_- := \partial_v + \omega_- \partial_{\tilde{\phi}_-}. \tag{3.2.19}$$

Note that $\partial_v = \partial_t$ and $\partial_{\tilde{\phi}_-} = \partial_\phi$ in $\mathcal{B}$.

[1]Note that ∇v is not timelike everywhere on $\mathcal{R}$, in particular $g(\nabla v, \nabla v) = a^2 \sin^2\theta \Sigma^{-1} \Delta_\theta^{-1}$ for $r \in [r_+, r_+ + \eta]$.

To attach the left event horizon $\mathcal{H}_L$ we introduce new coordinates on $\mathcal{B}$ defined as $(u, r, \theta, \phi_+^*) \in \mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2$ by

$$u(t, r) := -t + r^*, \quad \phi_+^* := \phi - \omega_+ r^* \bmod 2\pi, r = r, \theta = \theta \tag{3.2.20}$$

on $\mathcal{B}$ and attach $\mathcal{H}_L$ as $\mathcal{H}_L = \{r = r_+\}$. Similarly, introduce (u, r, θ, ϕ_-^*) as

$$u(t, r) := -t + r^*, \quad \phi_-^* := \phi - \omega_- r^* \bmod 2\pi, r = r, \theta = \theta \tag{3.2.21}$$

on $\mathcal{B}$ and attach the right Cauchy horizon $\mathcal{CH}_R$ as $\mathcal{CH}_R = \{r = r_-\}$ in this coordinate system. Indeed, K_+ and K_- extend to Killing vector fields expressed as $K_+ := -\partial_u + \omega_+ \partial_{\phi_+^*}$ and $K_- := -\partial_u + \omega_- \partial_{\phi_-^*}$. They are past directed Killing generators of $\mathcal{H}_L$ and $\mathcal{CH}_R$, respectively. Finally, we attach the past and future bifurcation spheres $\mathcal{B}_{\mathcal{H}}$ and $\mathcal{B}_{\mathcal{CH}}$. Formally, they are defined as $\mathcal{B}_{\mathcal{H}} := \{v = -\infty\} \times \{r = r_+\} \times \mathbb{S}^2 = \{u = -\infty\} \times \{r = r_+\} \times \mathbb{S}^2$ respectively in the coordinates systems $(v, r, \theta, \tilde{\phi}_+)$ and (u, r, θ, ϕ_+^*). Similarly, we have $\mathcal{B}_{\mathcal{CH}} := \{v = +\infty\} \times \{r = r_-\} \times \mathbb{S}^2 = \{u = +\infty\} \times \{r = r_-\} \times \mathbb{S}^2$. Finally, we define the Cauchy horizon $\mathcal{CH} := \mathcal{CH}_L \cup \mathcal{CH}_R \cup \mathcal{B}_{\mathcal{CH}}$. This is standard and we refer to the preliminary section of [30] for more details. The metric g_{KAdS} extends to a smooth Lorentzian metric on $\mathcal{B}_{\mathcal{H}}$, $\mathcal{B}_{\mathcal{CH}}$ and we define $(\mathcal{M}_{\mathrm{KAdS}}, g_{\mathrm{KAdS}})$ as the Lorentzian manifold constructed above. Moreover, $T := \partial_t$ and $\Phi := \partial_\phi$ extend to smooth Killing vector fields on $\mathcal{M}_{\mathrm{KAdS}}$ with $K_+ = T + \omega_+ \Phi$ and $K_- = T + \omega_- \Phi$.

Kerr–AdS-star coordinates. On the exterior region $\mathcal{R}$ we define an additional system of coordinates (t^*, r, θ, ϕ^*) from the Boyer–Lindquist coordinates through

$$t^* := t + A(r) \ \ , r = r, \theta = \theta, \phi^* := \phi + B(r) \tag{3.2.22}$$

where $\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{2Mr}{\Delta(1+\frac{r^2}{l^2})}$ and $\frac{\mathrm{d}B}{\mathrm{d}r} = \frac{a\Xi}{\Delta}$ and $A = B = 0$ at $r = +\infty$. As shown in [75, Section 2.6], these coordinates extend smoothly to the event horizon $\mathcal{H}_R$ and we call the coordinates (t^*, r, θ, ϕ^*) covering $\mathcal{R} \cup \mathcal{H}_R$ Kerr–AdS-star coordinates. Note that the event horizon is characterized as $\mathcal{H}_R = \{r = r_+\}$ in these coordinates.

Foliations and Initial Hypersurface. We foliate the region $\mathcal{R} \cup \mathcal{H}_R$ with constant t^* hypersurfaces Σ_{t^*} which are spacelike and intersect the event horizon at $r = r_+$. We also foliate the region $\mathcal{R}$ with constant t hypersurfaces Σ_t which are also spacelike and terminate at the bifurcation sphere $\mathcal{B}_{\mathcal{H}}$ as $r \to r_+$. For the initial data we will consider the axially symmetric spacelike hypersurface

$$\Sigma_0 := \Sigma_{t=0} = \mathcal{R} \cap \{t_{\mathcal{R}} = 0\}. \tag{3.2.23}$$

Note that Σ_0 does not contain the bifurcation sphere $\mathcal{B}_{\mathcal{H}}$. We will impose initial data on $\Sigma_0 \cup \mathcal{B}_{\mathcal{H}} \cup \mathcal{H}_L$. We will choose the support of our initial data to lie in a compact subset $K \subset \Sigma_0 \cap \{r \geq 2r_+\}$. Thus, we assume vanishing data on $\mathcal{H}_L \cup \mathcal{B}_{\mathcal{H}}$. This will be made precise in Section 3.6.

Boundary conditions. Note that the conformal boundary $\mathcal{I}$, expressed formally as $\{r = +\infty\}$, is timelike such that $(\mathcal{M}_{\mathrm{KAdS}}, g_{\mathrm{KAdS}})$ is not globally hyperbolic. Additional to Cauchy data for (3.1.1), we will also impose Dirichlet boundary conditions at $\mathcal{I} = \{r = +\infty\}$.

3.2.3 Conventions

If X and Y are two (typically non-negative) quantities, we use $X \lesssim Y$ of $Y \gtrsim X$ to denote that $X \leq C(M,a,l)Y$ for some constant $C(M,a,l) > 0$ only depending on the black hole parameters (M,a,l) unless stated explicitly otherwise. We also use $X = O(Y)$ for $|X| \lesssim Y$. We use $X \sim Y$ for $X \lesssim Y \lesssim X$ and if the constants appearing in $\lesssim, \gtrsim, \sim$ or O depend on additional parameters a_i we include those as a subscript, e.g. $X \lesssim_{a_1 a_2} Y$.

In Section 3.6 we will *fix* parameter $(\mathfrak{m}, \mathfrak{a}) \in \mathscr{P}_{\text{Blow-up}}$ and all constants appearing in $\lesssim$ and $\gtrsim$ throughout Section 3.6 Section 3.7, Section 3.8 will only depend on this particular choice and on $l > 0$ as defined in (3.2.6).

3.2.4 Norms and energies

To state the well-posedness result of (3.1.1) and the logarithmic decay result on the Kerr–AdS exterior, we define the following norms and energies in the exterior region $\mathcal{R} \cup \mathcal{H}_R$. These are based on the works [73, 75, 77], where more details can be found. In the region $\mathcal{R} \cup \mathcal{H}_R$ we let $\not{g}$ and $\not{\nabla}$ be the induced metric and induced connection of the spheres $\mathbb{S}^2_{t^*,r}$ of constant t^* and r. For a smooth function ψ we denote $|\not{\nabla} \ldots \not{\nabla}\psi|^2 = \not{g}^{AA'} \cdots g^{BB'} \not{\nabla}_A \ldots \not{\nabla}_B \bar{\psi} \not{\nabla}_{A'} \ldots \not{\nabla}_{B'} \psi$. Now, we define energy densities in Kerr–AdS-star coordinates as

$$e_1[\psi] := \frac{1}{r^2}|\partial_{t^*}\psi|^2 + r^2|\partial_r\psi|^2 + |\not{\nabla}\psi|^2 + |\psi|^2, \tag{3.2.24}$$

$$e_2[\psi] := e_1[\psi] + e_1[\partial_{t^*}\psi] + \sum_{i=1}^{3} e_1[\Omega_i\psi] + r^4|\partial_r\partial_r\psi|^2 + r^2|\partial_r\not{\nabla}\psi|^2 + |\not{\nabla}\not{\nabla}\psi|^2, \tag{3.2.25}$$

and similarly for higher order energy densities. Here, $(\Omega_i)_{i=1,2,3}$ denote the angular momentum operators on the unit sphere in θ, ϕ^* coordinates. We also define the energy norms on constant t^* hypersurfaces as

$$\|\psi\|^2_{H^{0,s}_{\mathrm{AdS}}(\Sigma_{t^*})} = \int_{\Sigma_{t^*}} r^s |\psi|^2 r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi^*, \tag{3.2.26}$$

$$\|\psi\|^2_{H^{1,s}_{\mathrm{AdS}}(\Sigma_{t^*})} = \int_{\Sigma_{t^*}} r^s \left(r^2 |\partial_r \psi|^2 + |\nabla\!\!\!/ \psi|^2 + |\psi|^2 \right) r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi^*, \tag{3.2.27}$$

$$\begin{aligned}\|\psi\|^2_{H^{2,s}_{\mathrm{AdS}}(\Sigma_{t^*})} =& \|\psi\|^2_{H^{1,s}_{\mathrm{AdS}}(\Sigma_{t^*})} + \int_{\Sigma_{t^*}} r^s \big(r^4 |\partial_r \partial_r \psi|^2 + r^2 |\nabla\!\!\!/ \partial_r \psi|^2 \\ &+ |\nabla\!\!\!/ \nabla\!\!\!/ \psi|^2 \big) r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi^*. \end{aligned}\tag{3.2.28}$$

We now denote the space $H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^*})$ as the space of functions with $\nabla^i \psi \in L^2_{loc}(\Sigma_{t^*})$ for $i = 0, \dots, k$ and such that $\|\psi\|^2_{H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^*})} < \infty$ and we denote with CH^k_{AdS} the space of functions ψ on $\mathcal{R} \cup \mathcal{H}_R$ such that $\psi \in \bigcap_{q=0,\dots,k} C^q(\mathbb{R}_{t^*}; H^{k-q,s_q}_{\mathrm{AdS}}(\Sigma_{t^*}))$, where $s_k = -2, s_{k-1} = 0$ and $s_j = 0$ for $j = 0, \dots, k-2$.

3.2.5 Well-posedness and log-decay on the exterior region

In the following we state well-posedness and decay for (3.1.1) with Dirichlet boundary conditions. The following theorem is a summary of several results by Holzegel, Smulevici and Warnick shown in [73, 75, 77, 78].

Theorem 3.2 ([73, 75, 77, 78])**.** *Let the initial data $\Psi_0, \Psi_1 \in C^\infty_c(\Sigma_0)$. Assume Dirichlet boundary conditions at $\mathcal{I}$ and vanishing incoming data on $\mathcal{H}_L \cup \mathcal{B}_{\mathcal{H}}$. Then, there exists a unique solution $\psi \in C^\infty(\mathcal{M}_{\mathrm{KAdS}} \setminus \mathcal{CH})$ of* (3.1.1) *such that $\psi|_{\Sigma_0} = \Psi_0, n_{\Sigma_0}\psi|_{\Sigma_0} = \Psi_1$, $\psi \restriction_{\mathcal{H}_L \cup \mathcal{B}_{\mathcal{H}}} = 0$. The solutions satisfies $\psi \restriction_{\mathcal{R} \cup \mathcal{H}_R} \in CH^k_{\mathrm{AdS}}$ for every $k \in \mathbb{N}$. We also have boundedness of energy as*

$$\int_{\Sigma_{t_2^*}} e_1[\psi] r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi^* \lesssim \int_{\Sigma_{t_1^*}} e_1[\psi] r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi^* \lesssim \int_{\Sigma_0} e_1[\psi] r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi \tag{3.2.29}$$

for $t_2^* \geq t_1^* \geq 0$ as well as for all higher order energies. Similarly, the energy along the event horizon is bounded by the initial energy as

$$\sum_{0 \leq i_1 + i_2 \leq k} \int_{\mathcal{H}_R} |\nabla\!\!\!/^{i_1} \partial_v^{i_2} \psi|^2 r^2 \sin\theta \mathrm{d}v \mathrm{d}\theta \mathrm{d}\tilde{\phi}_+ \lesssim_k \int_{\Sigma_0} e_k[\psi] r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi. \tag{3.2.30}$$

Moreover, the energy of ψ decays

$$\int_{\Sigma_{t^*}} e_1[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi^* \lesssim \frac{1}{[\log(2+t^*)]^2} \int_{\Sigma_0} e_2[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{3.2.31}$$

for all $t^* \geq 0$ and similar estimates hold for all higher order energies. Similarly, by commuting and applications of the Soboelv embeddings, ψ and its derivatives also decay pointwise

$$\sum_{0\leq i_1+i_2+i_3\leq k} |\nabla\!\!\!/^{i_1} \partial_{t^*}^{i_2} \partial_r^{i_3} \psi|^2 \lesssim_k \frac{1}{[\log(2+t^*)]^2} \int_{\Sigma_0} e_{k+3}[\psi] r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi. \tag{3.2.32}$$

3.2.6 Separation of variables

The wave equation (3.1.1) is formally separable [13] and we can consider pure mode solution in the Boyer–Lindquist coordinates of the form

$$\psi(t,r,\theta,\phi) = \frac{u(r)}{\sqrt{r^2+a^2}} e^{-i\omega t} S_{m\ell}(a\omega, \cos\theta) e^{im\phi} \tag{3.2.33}$$

for two unknown functions $u(r)$ and $S_{m\ell}(a\omega,\cos\theta)$. Plugging this ansatz into (3.1.1) leads to a coupled system of o.d.e's. The angular o.d.e. is the eigenvalue equation of the operator $P(a\omega)$ which reads

$$P(a\omega) S_{m\ell}(a\omega,\cos\theta) = \lambda_{m\ell}(a\omega) S_{m\ell}(a\omega,\cos\theta), \tag{3.2.34}$$

where

$$\begin{aligned} P(\xi)f = P_m(\xi)f = &- \frac{1}{\sin\theta}\partial_\theta(\Delta_\theta \sin\theta \partial_\theta f) + \frac{\Xi^2 m^2}{\Delta_\theta \sin^2\theta} - \Xi\xi^2 \Delta_\theta^{-1}\cos^2\theta f \\ &- 2m\xi \frac{\Xi}{\Delta_\theta}\frac{a^2}{l^2}\cos^2\theta + \frac{2}{l^2}a^2\sin^2\theta. \end{aligned} \tag{3.2.35}$$

The operator (3.2.35) is realized as a self-adjoint operator on some suitable domain in $L^2((0,\pi);\sin\theta\mathrm{d}\theta)$. As a Sturm–Liouville operator, the spectrum of $P(a\omega)$ consists of simple eigenvalues $\lambda_{m\ell}(a\omega)$, where $\ell \in \mathbb{Z}_{\geq|m|}$ labels the eigenvalue in ascending order. The eigenvalue $\lambda_{m\ell}(a\omega)$ of $P(a\omega)$ couples the angular o.d.e. to the radial o.d.e.

$$-u'' + (V-\omega^2)u = 0, \tag{3.2.36}$$

where $' := \frac{\mathrm{d}}{dr^*}$. Here, the potential is given by

$$V = V_0 + V_1 \tag{3.2.37}$$

with purely radial part

$$V_1 := \frac{-\Delta^2 3r^2}{(r^2+a^2)^4} - \Delta \frac{5\frac{r^4}{l^2} + 3r^2\left(1+\frac{a^2}{l^2}\right) - 4Mr + a^2}{(r^2+a^2)^3} - \frac{2\Delta}{l^2}\frac{1}{r^2+a^2} \tag{3.2.38}$$

and frequency dependent part

$$V_0 := \frac{\Delta(\lambda_{m\ell}(a\omega) + \omega^2 a^2) - \Xi^2 a^2 m^2 - 2m\omega a \Xi(\Delta - (r^2+a^2))}{(r^2+a^2)^2}. \tag{3.2.39}$$

We will be particularly interested in the case where the frequency parameter ω excites the resonance at the Cauchy horizon at $\omega = \omega_- m$. Moreover, in order to be in the regime of trapping in the exterior we also want ω and m to be large. Hence, we will think of $\frac{1}{m}$ as a small semiclassical parameter. In particular, setting $\omega = \omega_- m$ in (3.2.36) and separating out the m^2 we obtain

$$-u'' + (m^2 V_{\text{main}} + V_1)u = 0, \tag{3.2.40}$$

where V_1 is as in (3.2.38) and

$$V_{\text{main}} := \frac{V_0(\omega = \omega_- m)}{m^2} = \frac{\Delta}{(r^2+a^2)^2}\left(\frac{\lambda_{m\ell}(a\omega_- m)}{m^2} + \omega_-^2 a^2 - 2a\omega_- \Xi\right) - (\omega_- - \omega_r)^2. \tag{3.2.41}$$

We will prove the main theorem, Theorem 3.1, in Section 3.8.3. Before that, we first have to show various properties of the angular o.d.e. and the radial o.d.e. for fixed frequencies (ω, m, ℓ). We start by considering the angular o.d.e. (3.2.34).

3.3 The angular o.d.e.

For the operator $P(\xi)$ as in (3.2.35) we change variables to $x = \cos\theta$. This is a unitary transformation and thus, the eigenvalues of $P(\xi)$ are equal to the eigenvalues of P_x given by

$$P_x(\xi) := -\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\cdot\right) + \frac{\Xi^2 m^2}{\Delta_x(1-x^2)} - \Xi\xi^2\frac{x^2}{\Delta_x} - 2m\xi\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2 + \frac{2}{l^2}a^2(1-x^2). \tag{3.3.1}$$

The Sturm–Liouville operator P_x is realized as a self-adjoint operator acting on a domain $\mathcal{D} \subset L^2(-1,1)$ which can be explicitly characterized as

$$\mathcal{D} = \{f \in L^2(-1,1) \colon f \in AC^1(-1,1), P_x f \in L^2(-1,1), \lim_{x\to\pm 1}(1-x^2)f'(x) = 0 \text{ if } m = 0\}. \tag{3.3.2}$$

Having the same spectrum as P, the operator P_x has eigenvalues $(\lambda_{m\ell})_{\ell\geq|m|}$ with corresponding real-analytic eigenfunctions $S_{m\ell} = S_{m\ell}(\xi, x)$ which satisfy

$$P_x S_{m\ell} = \lambda_{m\ell} S_{m\ell} \quad \text{and} \quad \|S_{m\ell}(\xi)\|_{L^2(-1,1)} = 1. \tag{3.3.3}$$

We note that for $\xi = a = 0$, the eigenvalues $(\lambda_{m\ell})_{\ell\geq|m|}$ reduce to the eigenvalues of the Laplacian on the sphere $\lambda_{m\ell}(a = \xi = 0) = \ell(\ell+1)$. We also define the shifted eigenvalues

$$\Lambda_{m\ell}(\xi) := \lambda_{m\ell}(\xi) + \xi^2. \tag{3.3.4}$$

A computation (see [75, Proof of Lemma 5.1]) shows that

$$P_x(\xi) + \xi^2 - \frac{2}{l^2}a^2(1-x^2) \geq \Xi^2 P_x(\xi = 0, a = 0) \tag{3.3.5}$$

in the sense of self-adjoint operators acting on $\mathcal{D} \subset L^2(-1,1)$. Hence,

$$\Lambda_{m\ell}(\xi) \geq \Xi^2\ell(\ell+1) \geq \Xi^2|m|(|m|+1). \tag{3.3.6}$$

Having recalled basic properties of the angular problem we now focus on the resonant frequency $\omega = \omega_- m$. We assume that $m \neq 0$ for the rest of Section 3.3. This will simplify the notation as $\frac{1}{m}$ is well-defined.

3.3.1 Analysis of the angular potential W_1 at resonant frequency in semi-classical limit

In the current Section 3.3.1 and in Section 3.3.2 we consider the operator

$$P_{\omega_-} := P_x(\xi = am\omega_-) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\cdot\right) + \frac{\Xi^2 m^2}{\Delta_x(1-x^2)} - \Xi a^2 m^2 \omega_-^2 \frac{x^2}{\Delta_x}$$
$$- 2m^2 a\omega_- \frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2 + \frac{2}{l^2}a^2(1-x^2) \tag{3.3.7}$$

with corresponding eigenvalues $\lambda_{m\ell} := \lambda_{m\ell}(a\omega_- m)$. We re-write the eigenvalue problem

$$P_{\omega_-} f = \lambda f \tag{3.3.8}$$

as

$$\tilde{P}_{\omega_-} f = 0, \tag{3.3.9}$$

where

$$\tilde{P}_{\omega_-} := \Delta_x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\cdot\right) + m^2 W_1(x) + P_{\mathrm{error}}, \tag{3.3.10}$$

$$P_{\mathrm{error}} := \Delta_x(1-x^2)\frac{2}{l^2}a^2(1-x^2) \tag{3.3.11}$$

and

$$W_1 := \Xi^2 - \left[\Xi a^2 \omega_-^2 + 2a\omega_- \Xi \frac{a^2}{l^2}\right] x^2(1-x^2) - \tilde{\lambda}\Delta_x(1-x^2), \tag{3.3.12}$$

with

$$\tilde{\lambda} := \frac{\lambda}{m^2}. \tag{3.3.13}$$

In the semi-classical limit $m^2 \to \infty$ we consider P_{error} as a perturbation and W_1 determines the leading order terms of the eigenvalues and eigenfunctions. Consequently, our analysis focuses on W_1 which we analyze in the following lemma.

Lemma 3.3.1. *Let W_1 be the potential defined in* (3.3.12).

1. *For $\tilde{\lambda} < \Xi^2$, we have $W_1 > 0$ for $x \in [0,1]$.*

2. *For $\tilde{\lambda} = \Xi^2$, we have $W_1 > 0$ on $(0,1]$ and $W_1(x=0) = 0$.*

3. For $\tilde{\lambda} > \Xi^2$, the potential W_1 has exactly one root in $x \in [0,1]$ and satisfies

$$\frac{\mathrm{d}W_1}{\mathrm{d}x} \gtrsim \tilde{\lambda} x \tag{3.3.14}$$

for $x \in [0,1]$. We call this root x_0 which also satisfies $x_0 \in (0,1)$.

Proof. We start by expanding W_1 and obtain

$$W_1(x) = \Xi^2 - \tilde{\lambda} + a_1 x^2 + a_2 x^4 \tag{3.3.15}$$

with

$$\begin{aligned} a_1 &= \tilde{\lambda}\left(1 + \frac{a^2}{l^2}\right) - \frac{a^4(a^2 - l^2)^2(a^2 + l^2 + 2r_-^2)}{l^6(a^2 + r_-^2)^2} \\ &= \Xi^2\left(1 + \frac{a^2}{l^2}\right) - \frac{a^4(a^2 - l^2)^2(a^2 + l^2 + 2r_-^2)}{l^6(a^2 + r_-^2)^2} + (\tilde{\lambda} - \Xi^2)\left(1 + \frac{a^2}{l^2}\right) \\ &= \frac{(a-l)^2(a+l)^2(2a^2 l^2 r_-^2 + (a^2 + l^2) r_-^4)}{l^6(a^2 + r_-^2)^2} + (\tilde{\lambda} - \Xi^2)\left(1 + \frac{a^2}{l^2}\right) \end{aligned} \tag{3.3.16}$$

and

$$a_2 = \frac{a^4(a^2 - l^2)^2(a^2 + l^2 + 2r_-^2)}{l^6(a^2 + r_-^2)^2} - \frac{a^2}{l^2}\tilde{\lambda}. \tag{3.3.17}$$

We also note that

$$W_1(x = 0) = \Xi^2 - \tilde{\lambda}. \tag{3.3.18}$$

We now consider the case $\tilde{\lambda} \geq \Xi^2$ and remark that

$$\frac{\mathrm{d}W_1}{\mathrm{d}x} = 2a_1 x + 4a_2 x^3. \tag{3.3.19}$$

We look at two cases now, $a_2 \geq 0$ and $a_2 < 0$. If $a_2 \geq 0$, then we directly infer that $\frac{\mathrm{d}W_1}{\mathrm{d}x} \geq 2a_1 x$. If $a_2 < 0$, then we use that $x^3 < x$ and estimate

$$\frac{\mathrm{d}W_1}{\mathrm{d}x} = 2a_1 x + 4a_2 x^3 \geq (2a_1 + 4a_2)x. \tag{3.3.20}$$

Now, a direct computation yields

$$2a_1 + 4a_2 = 2\Xi\left(\Xi \frac{a^2}{l^2} \frac{a^2 + l^2 + 2r_-^2}{(a^2 + r_-^2)^2} + \tilde{\lambda}\right) \gtrsim \tilde{\lambda}. \tag{3.3.21}$$

Note that this shows (3.3.14) for $x \in [0,1]$ and we conclude *3*. Together with (3.3.18), this also shows that $W_1(x) > 0$ for $x \in (0,1]$ and $\tilde{\lambda} = \Xi^2$ such that we have *2*.

Finally for $\tilde{\lambda} < \Xi^2$, we have $W_1 > 0$ everywhere because for each fixed $x \in [0,1)$, the function $\tilde{\lambda} \mapsto W_1(x)$ is strictly decreasing and $W_1(x=1) = \Xi^2 > 0$. $\square$

3.3.2 Existence of sequence of angular eigenvalues at resonant frequency with $\lambda_{m_i\ell_i} = \tilde{\lambda} m_i^2 + O(1)$

For our proof later on, we will use that there exists a sequence of eigenvalues of the form $\lambda_{m_i\ell_i} = \tilde{\lambda} m_i^2 + O(1)$ at the resonant frequency. To show such a result we state the following theorem on semi-classical distribution of eigenvalues. This is also referred to as Bohr–Sommerfeld quantization. Its proof relies on suitable connection formulas of Airy functions and can be found in [119, Chapter 13, §9.1]. We denote the total variation of a function $f\colon \mathbb{R} \to \mathbb{R}$ in the interval (a,b) with $\mathcal{V}_{a,b}(f)$.

Proposition 3.3.1 ([119, Chapter 13, §9.1]). *Let $f, g \in C^2(\mathbb{R})$ and assume that $f(x)/[(x-\hat{x}_0)(x-x_0)]$ is positive, i.e. f has two simple roots at $\hat{x}_0 < x_0$. Define the error-control function*

$$F(x) := \int \frac{1}{|f|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{|f|^{\frac{1}{4}}}\right) - \frac{g}{|f|^{\frac{1}{2}}} \mathrm{d}x \tag{3.3.22}$$

for some irrelevant normalization and further assume that for some large $c > 0$,

- *the error-control function satisfies $\mathcal{V}_{-\infty,-c}(F), \mathcal{V}_{c,+\infty}(F) < \infty$,*
- *$\int_c^x \sqrt{f}$ diverges as $x \to +\infty$ and $\int_x^{-c} \sqrt{f}$ diverges as $x \to -\infty$.*

Then, there exists an error function ϑ satisfying $|\vartheta| \lesssim_{f,g} u^{-2}$ such that for all u large enough the following holds true. There exists a bound state w (i.e. a solution which is recessive at both ends $x \to \pm\infty$) of the differential equation

$$w'' = (u^2 f + g) w, \tag{3.3.23}$$

if and only there exists a positive integer $n \in \mathbb{N}$ such that

$$\frac{2}{\pi} \int_{\hat{x}_0}^{x_0} \sqrt{-f} \mathrm{d}x + \vartheta = \frac{2n+1}{u}. \tag{3.3.24}$$

With the above proposition at hand we proceed to the main proposition of this subsection, where we recall that we still consider the case $\omega = \omega_- m$.

Proposition 3.3.2. *Let $\mathfrak{p}_0 \in \mathscr{P}$ be arbitrary but fixed. Then, for almost every $\tilde{\lambda}_0 \in (\Xi^2, \infty)$ (more precisely, for every $\tilde{\lambda}_0 \in (\Xi^2, \infty) \setminus \mathcal{N}_{\mathfrak{p}_0}$ for some Lebesgue null set $\mathcal{N}_{\mathfrak{p}_0}$), there exists a strictly increasing sequence of natural numbers $(m_i)_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, the operator P_{ω_-} admits an eigenvalue $\lambda_i := \lambda_{m_i \ell_i} = \lambda_{m_i \ell_i}(\omega = \omega_- m)$ satisfying*

$$\tilde{\lambda}_i := \lambda_i m_i^{-2} = \tilde{\lambda}_0 + \lambda_{\text{error}}^{(i)} m_i^{-2}, \tag{3.3.25}$$

where $|\lambda_{\text{error}}^{(i)}| \lesssim_{\tilde{\lambda}_0, \mathfrak{p}_0} 1$ as $m_i \to \infty$. Here, $m_i \sim \ell_i$ and particularly $\ell_i \leq m_i^2$ for i sufficiently large.

Proof. We consider the equivalent formulation of the angular o.d.e. in (3.3.9) and moreover change coordinates

$$y(x) = \int_0^x \frac{1}{\Delta_{\tilde{x}}(1-\tilde{x}^2)} \mathrm{d}\tilde{x} \tag{3.3.26}$$

such that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \Delta_x (1 - x^2). \tag{3.3.27}$$

This yields the equivalent eigenvalue problem

$$-\frac{1}{m^2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} g + (W_1 + \frac{1}{m^2} P_{\text{error}}) g = 0 \tag{3.3.28}$$

for g in a dense domain of $L^2(\mathbb{R}, w(y)\mathrm{d}y)$ with weight

$$w(y) := \Delta_x(1-x^2) = \Delta_{x(y)}(1 - x(y)^2). \tag{3.3.29}$$

From Lemma 3.3.1 we infer that W_1 has a unique positive root for $\tilde{\lambda} > \Xi^2$ which we denote with $y_0(\tilde{\lambda}) := y(x_0(\tilde{\lambda}))$. We also define

$$\xi(\tilde{\lambda}) := \int_{-y_0(\tilde{\lambda})}^{y_0(\tilde{\lambda})} \sqrt{-W_1} \mathrm{d}y, \tag{3.3.30}$$

where we recall that W_1 is symmetric around the origin. For the potential W_1, we have (e.g. [48, p. 118]) that $\xi \colon (\Xi^2, \infty) \to \mathbb{R}, \tilde{\lambda} \mapsto \xi(\tilde{\lambda})$ is a strictly increasing smooth (even real-analytic) function. Further note that

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tilde{\lambda}} = \int_{-y_0(\tilde{\lambda})}^{y_0(\tilde{\lambda})} \frac{\Delta_\theta (1 - x(y)^2)}{2\sqrt{-W_1}} \mathrm{d}y > 0 \tag{3.3.31}$$

so by the inverse function theorem, ξ has a smooth inverse.

Now, by a standard result on Diophantine approximation (see e.g. [66, Theorem 6.2]), we have that for each $x \in \mathbb{R} \setminus \mathcal{N}$, where $\mathcal{N}$ is a Lebesgue null set, there exist sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ with $n_{i+1} > n_i$ and $m_{i+1} > m_i$ such that

$$\left|\frac{2}{\pi}x - \frac{2n_i+1}{m_i}\right| \leq \frac{1}{m_i^2} \tag{3.3.32}$$

for all $i \in \mathbb{N}$. Now, since ξ has a smooth inverse, there exists a Lebesgue null set $\mathcal{N}_{\mathfrak{p}_0} := \xi^{-1}(\mathcal{N}) \subset (\Xi^2, \infty)$ such that for each $\tilde{\lambda}_0 \in (\Xi^2, \infty) \setminus \mathcal{N}_{\mathfrak{p}_0}$ we have

$$\left|\frac{2}{\pi}\xi(\tilde{\lambda}_0) - \frac{2n_i+1}{m_i}\right| \leq \frac{1}{m_i^2} \tag{3.3.33}$$

for a sequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ with $n_{i+1} > n_i$ and $m_{i+1} > m_i$.

Now, we will apply Proposition 3.3.1. First, we see that for all $\tilde{\lambda}$ in a small neighborhood of $\tilde{\lambda}_0 \in (\Xi^2, \infty) \setminus \mathcal{N}_{\mathfrak{p}_0}$, the potential $W_1(y)$ and P_{error} satisfy the assumptions of the proposition: Indeed, both W_1 and P_{error} are smooth. Moreover, W_1 has two simple roots which do not coalesce and $\int_0^y \sqrt{|W_1|}\mathrm{d}\tilde{y}$ diverges as $y \to \pm\infty$. Finally, for $|y| \to \infty$, we have

$$W_1 \geq \frac{\Xi^2}{2}, \text{ and } \left|\frac{\mathrm{d}W_1}{\mathrm{d}y}\right|, \left|\frac{\mathrm{d}^2 W_1}{\mathrm{d}y^2}\right| \lesssim \frac{\mathrm{d}x}{\mathrm{d}y} \tag{3.3.34}$$

as well as

$$|P_{\text{error}}| \lesssim \frac{\mathrm{d}x}{\mathrm{d}y}. \tag{3.3.35}$$

Thus, we infer

$$\mathcal{V}_{c,\infty}(F), \mathcal{V}_{-\infty,-c}(F) < \infty \tag{3.3.36}$$

for some $c > 0$ large enough, where

$$F(y) := \int \frac{1}{|W_1|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{1}{|W_1|^{\frac{1}{4}}}\right) - \frac{P_{\text{error}}}{|W_1|^{\frac{1}{2}}}\mathrm{d}y. \tag{3.3.37}$$

From Proposition 3.3.1 we now conclude that the eigenvalues $\lambda = \tilde{\lambda}m^2$ for $\tilde{\lambda}$ in a neighborhood of $\tilde{\lambda}_0$ are characterized by

$$\frac{2}{\pi}\xi(\tilde{\lambda}) + \vartheta_{\tilde{\lambda}_0} = \frac{2n+1}{m} \tag{3.3.38}$$

for $n \in \mathbb{N}$, where $|\vartheta_{\tilde{\lambda}_0}| \lesssim_{\tilde{\lambda}_0} m^{-2}$.

Now, for fixed $\tilde{\lambda}_0 \in (\Xi^2, \infty)$, let the sequence $(m_i, n_i)_{i\in\mathbb{N}}$ as above be such that (3.3.33) holds. Then, we obtain associated eigenvalues from (3.3.38) which satisfy

$$\tilde{\lambda}_i = \xi^{-1}\left(\frac{\pi}{2}\frac{2n_i+1}{m_i} - \frac{\pi}{2}\vartheta_{\tilde{\lambda}_0}\right) = \xi^{-1}\left(\xi(\tilde{\lambda}_0) + O_{\tilde{\lambda}_0}(m_i^{-2})\right) = \tilde{\lambda}_0 + O_{\tilde{\lambda}_0}(m_i^{-2}). \tag{3.3.39}$$

The last equality holds due Taylor's theorem and (3.3.31). $\square$

3.3.3 Bounds on $\partial_\xi \lambda_{m\ell}$ and $\partial_\xi S_{m\ell}$ near resonant frequency

For the main proof in Section 3.8.3 we need to bound the quantities $\partial_\omega \lambda_{m\ell}(a\omega)$ and $\partial_\omega S_{m\ell}(a\omega)$ near the resonant frequency, i.e. for $\omega \approx \omega_- m$ with m sufficiently large. We will choose our initial data in Section 3.6 to be supported on angular modes $m > 0$ which are large and positive. This is just done to make the notation easier. Thus, for the rest of the subsection, we assume that $m > 0$.

We first note that a direct computation shows that

$$\partial_\xi S_{m\ell} = \frac{\partial S_{m\ell}(\xi, x)}{\partial \xi} \tag{3.3.40}$$

solves the inhomogeneous o.d.e.

$$(P_x - \lambda_{m\ell})\partial_\xi S_{m\ell} = (\partial_\xi P_x - \partial_\xi \lambda_{m\ell})S_{m\ell} \tag{3.3.41}$$

with Dirichlet boundary conditions at $x = \pm 1$, where

$$\partial_\xi P_x = \frac{\partial P_x(\xi)}{\partial \xi} = -2\Xi\xi\frac{x^2}{\Delta_x} - 2m\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2. \tag{3.3.42}$$

We will first consider $\partial_\xi \lambda_{m\ell}$.

Lemma 3.3.2. *The eigenvalues $\lambda_{m\ell}(\xi)$ satisfy*

$$|\partial_\xi \lambda_{m\ell}(\xi)| \leq |\langle S_{m\ell}, \partial_\xi P_x S_{m\ell}\rangle_{L^2(-1,1)}| \tag{3.3.43}$$

and thus,

$$\sup_{\xi \in (am\omega_- - \frac{1}{m}, am\omega_- + \frac{1}{m})} |\partial_\xi \lambda_{m\ell}(\xi)| \lesssim |m|. \tag{3.3.44}$$

Proof. Taking the L^2-inner product of (3.3.41) with $S_{m\ell}$ and using that P_x is self-adjoint, as well as $\xi \mapsto \langle S_{m\ell}, S_{m\ell} \rangle_{L^2(-1,1)} = 1$, we see that

$$|\partial_\xi \lambda| \leq |\langle S_{m\ell}, \partial_\xi P_x S_{m\ell} \rangle_{L^2(-1,1)}| \leq \|\partial_\xi P_x\|. \tag{3.3.45}$$

Now, the claim follows from the fact that $\|\partial_\xi P_x\| \lesssim |\xi| + |m|$. $\square$

It is far more difficult to estimate $\partial_\xi S_{m\ell}$ which we express as

$$\partial_\xi S_{m\ell} = \mathrm{Res}(\lambda_{m\ell}; P_x) \Pi^{\perp}_{S_{m\ell}} H,$$

where $H = (\partial_\xi P_x - \partial_\xi \lambda_{m\ell}) S_{m\ell}$ is the inhomogeneous term of (3.3.41), $\mathrm{Res}(\lambda_{m\ell}; P_x)$ is the resolvent and $\Pi^{\perp}_{S_{m\ell}}$ is the orthogonal projection on the orthogonal complement of $S_{m\ell}$. A possible way to control the resolvent operator $\mathrm{Res}(\lambda_{m\ell}; P_x)\Pi^{\perp}_{S_{m\ell}}$, is to show lower bounds on the spectral gaps $|\lambda_{m,\ell}(a\omega) - \lambda_{m,\ell+1}(a\omega)|$ *uniformly* in $m, \ell \to \infty$ and $\omega \approx \omega_- m$. Our formally equivalent approach is based on an explicit construction of the resolvent kernel via suitable approximations with parabolic cylinder functions and Airy functions.

We begin by noting that from standard results on solutions to Sturm–Liouville problems, each eigenfunction $S_{m\ell}$ is either symmetric or anti-symmetric around $x = 0$. Moreover, $\partial_\xi S_{m\ell}$ admits the same symmetries as $S_{m\ell}$. If $S_{m\ell}$ is antisymmetric around $x = 0$ we have $S_{m\ell}(x = 0) = 0$, i.e. Dirichlet boundary conditions at $x = 0$. Similarly, if $S_{m\ell}$ is symmetric, we have Neumann boundary conditions at $x = 0$, i.e. $\frac{\mathrm{d}}{\mathrm{d}x} S_{m\ell}(x = 0) = 0$. Hence, the problem reduces to studying the interval $x \in [0, 1)$ with Dirichlet/Neumann boundary conditions at $x = 0$ and Dirichlet boundary conditions at $x = 1$. We moreover recall that

$$\langle S_{m\ell}, S_{m\ell} \rangle_{L^2(-1,1)} = \int_{-1}^{1} S_{m\ell}^2 \mathrm{d}x = 1 \tag{3.3.46}$$

such that

$$\int_0^1 S_{m\ell}^2 \mathrm{d}x = \frac{1}{2}. \tag{3.3.47}$$

Now, any solution of the inhomogeneous o.d.e. (3.3.41) can be written as

$$\partial_\xi S_{m\ell} = S_p + c_1 S_{m\ell} + c_2 \tilde{S}_{m\ell}, \tag{3.3.48}$$

where $\tilde{S}_{m\ell}$ is a solution to the homogeneous o.d.e. which is linearly independent from $S_{m\ell}$ and S_p is any particular solution to (3.3.41). Also, c_1 and c_2 are constants depending on the choice of $\tilde{S}_{m\ell}$ and the choice of particular solution S_p.

Now, we remark that

$$\tilde{S}_{m\ell} \notin L^2 \tag{3.3.49}$$

as $\tilde{S}_{m\ell}$ is linearly independent of $S_{m\ell}$. Indeed, analyzing the singularity $x = 1$ with the Frobenius method, we directly infer that all solutions which are in L^2 at $x = 1$ are multiples of $S_{m\ell}$. (Here, we recall that $m \neq 0$.)

In Lemma 3.3.3 below we will construct a particular solution which is bounded at $x = 1$. Since $\partial_\xi S_{m\ell}$ is bounded at $x = 1$, we have that for such a choice of particular solution, necessarily $c_2 = 0$. Hence, for S_p as in Lemma 3.3.3 we have

$$\partial_\xi S_{m\ell} = S_p + c_1 S_{m\ell}. \tag{3.3.50}$$

Now, as $\partial_\xi S_{m\ell}$ is orthogonal to $S_{m\ell}$ in $L^2(-1,1)$ and both of them are either symmetric or anti-symmetric around $x = 0$, we conclude that they are also orthogonal with respect to $L^2(0,1)$. Hence, multiplying (3.3.50) with $\partial_\xi S_{m\ell}$ and applying the Cauchy–Schwarz inequality yields

$$\|\partial_\xi S_{m\ell}\|_{L^2(0,1)} \leq \|S_p\|_{L^2(0,1)}. \tag{3.3.51}$$

In the language of spectral theory and in view of our previous discussion, the bound (3.3.51) shows that

$$\|\partial_\xi S_{m\ell}\|_{L^2} = \|\operatorname{Res}(\lambda_{m\ell}; P_x)\Pi^{\perp}_{S_{m\ell}} H\|_{L^2} \leq \|S_p\|_{L^2}.$$

Lemma 3.3.3. *Let $m \in \mathbb{N}$ be sufficiently large. In the parameter range $\xi \in (a\omega_- m - \frac{1}{m}, a\omega_- m + \frac{1}{m})$ there exists a particular solution S_p to* (3.3.41) *satisfying*

$$\|S_p\|_{L^2(0,1)} \lesssim |m|^{\frac{1}{2}}. \tag{3.3.52}$$

Proof. We let $|\epsilon| < 1$ such that $\xi = am\omega_- + \frac{\epsilon}{m}$. We now construct a particular solution

S_p of (3.3.41) which satisfies

$$\Big[-\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\cdot\right)+\frac{\Xi^2m^2}{\Delta_x(1-x^2)}-\Xi m^2a^2\omega_-^2\frac{x^2}{\Delta_x}-2m^2a\omega_-\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2$$
$$+\frac{2}{l^2}a^2(1-x^2)-\Xi(2\epsilon a\omega_-+\epsilon^2m^{-2})\frac{x^2}{\Delta_x}-2\epsilon\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2-\lambda\Big]S_p$$
$$=\left[\partial_\xi\lambda+2\Xi(am\omega_-+\frac{\epsilon}{m})\frac{x^2}{\Delta_x}+2m\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right]S_{m\ell}. \tag{3.3.53}$$

Again we introduce new variables $y(0)=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\Delta(1-x^2)}$ which can be computed explicitly as

$$y(x)=\frac{1}{2\Xi}\left(\log(1+x)-\log(1-x)+\frac{a}{l}\log(1-\frac{a}{l}x)-\frac{a}{l}\log(1+\frac{a}{l}x)\right). \tag{3.3.54}$$

Note that

$$e^{2\Xi y}=\frac{1+x}{1-x}\left(\frac{1-\frac{a}{l}x}{1+\frac{a}{l}x}\right)^{\frac{a}{l}}. \tag{3.3.55}$$

We define

$$s_1(y):=S_{m\ell}(x(y)), \tag{3.3.56}$$

such that

$$\int_0^\infty s_1^2(y)\Delta_x(1-x^2(y))\mathrm{d}y=\int_0^1 S_{m\ell}^2\mathrm{d}x=\frac{1}{2}. \tag{3.3.57}$$

We also write

$$g_p(y)=S_p(x(y)). \tag{3.3.58}$$

Then, in these variables we re-write (3.3.53) as

$$-\frac{\mathrm{d}^2}{\mathrm{d}y^2}g+m^2\left(\Xi^2-\left[\Xi a^2\omega_-^2+2a\omega_-\Xi\frac{a^2}{l^2}\right]x^2(1-x^2)-\tilde{\lambda}\Delta_x(1-x^2)\right)g_p$$
$$+\Delta_x(1-x^2)\left(\frac{2}{l^2}a^2(1-x^2)-\Xi(2\epsilon\omega_-+\epsilon^2m^{-2})\frac{x^2}{\Delta_x}-2\epsilon\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right)g_p$$
$$=\Delta_x(1-x^2)\left[\partial_\xi\lambda+2\Xi(am\omega_-+\frac{\epsilon}{m})\frac{x^2}{\Delta_x}+2m\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right]s_1. \tag{3.3.59}$$

We recall the definition of W_1 in (3.3.12) and define

$$W_2(y) := \Delta_x(1-x^2)\left(\frac{2}{l^2}a^2(1-x^2) - \Xi(2\epsilon a\omega_- + \epsilon^2 m^{-2})\frac{x^2}{\Delta_x} - 2\epsilon\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right) \tag{3.3.60}$$

and

$$F(y) := \Delta_x(1-x^2)\left[\partial_\xi\lambda + 2\Xi(am\omega_- + \frac{\epsilon}{m})\frac{x^2}{\Delta_x} + 2m\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right]. \tag{3.3.61}$$

Thus, we are interested in construction a particular solution g_p satisfying the inhomogeneous second order o.d.e.

$$-\frac{\mathrm{d}^2}{\mathrm{d}y^2}g_p + (m^2W_1 + W_2)g_p = Fs_1. \tag{3.3.62}$$

A direct computation shows that g_p, which we define as

$$g_p := s_2(y)\int_y^\infty s_1^2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y} + s_1(y)\int_0^y s_2(\tilde{y})s_1(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}, \tag{3.3.63}$$

solves (3.3.62), where s_2 is a suitable solution of

$$-\frac{\mathrm{d}^2}{\mathrm{d}y^2}g + (m^2W_1 + W_2)g = 0 \tag{3.3.64}$$

which we will construct in Lemma 3.3.4 below. Moreover, we will choose s_2 such that the integrals in (3.3.63) converge, $\mathfrak{W}(s_1, s_2) = 1$, and

$$\int_0^\infty g_p(y)^2(1-x(y))\Delta_x \mathrm{d}y \lesssim m. \tag{3.3.65}$$

We state the existence of such a particular solution g_p in the following and refer to Section 3.3.4 for its proof. More precisely, Lemma 3.3.4 follows from Lemma 3.3.13 and Lemma 3.3.18.

Remark 3.3.1. *By construction, the solution to the inhomogeneous o.d.e. satisfies* $g_p(y(x)) = \partial_\xi S_{m\ell}$ *as* $g_p(y(x)) = \mathrm{Res}(\lambda_{m\ell}; P_x)\Pi^\perp_{S_{m\ell}}(H)$.

Lemma 3.3.4. *Let* $m \in \mathbb{N}$ *sufficiently large as in Section 3.3.4. For* $\xi \in (a\omega_- m - \frac{1}{m}, a\omega_- m + \frac{1}{m})$*, there exists a solution* s_2 *to* (3.3.64) *with* $\mathfrak{W}(s_1, s_2) = 1$ *such that* g_p *as in*

(3.3.63) *satisfies*

$$\int_0^\infty g_p(y)^2(1-x(y)^2)\Delta_x \mathrm{d}y \lesssim m. \tag{3.3.66}$$

With Lemma 3.3.4 at hand, we conclude the proof of Lemma 3.3.3. $\square$

Finally, combing (3.3.51) and Lemma 3.3.3 we have proved the main proposition of this subsection.

Proposition 3.3.3. *For all $m \in \mathbb{N}$ sufficiently large, the eigenfunctions $S_{m\ell}(\xi, \cos\theta)$ of the operator P defined in* (3.2.35) *satisfy*

$$\sup_{\omega \in (\omega_- m - \frac{1}{am}, \omega_- m + \frac{1}{am})} \|\partial_\omega S_{m\ell}(a\omega, \cdot)\|_{L^2([0,\pi];\sin\theta \mathrm{d}\theta)} \lesssim m^{\frac{1}{2}}. \tag{3.3.67}$$

3.3.4 Proof of Lemma 3.3.4

Throughout this subsection (Section 3.3.4) we assume that

$$\xi \in \left(a\omega_- m - \frac{1}{m}, a\omega_- m + \frac{1}{m}\right) \tag{3.3.68}$$

and $m > 0$. We first argue that for sufficiently large m, we only need to consider the cases $\tilde{\lambda} > \Xi^2$.

Lemma 3.3.5. *For sufficiently large m, we have $\inf_{y \in \mathbb{R}} \left(m^2 W_1(y) + W_2(y)\right) > 0$ for any $\tilde{\lambda} \leq \Xi^2$.*

Proof. Choosing m sufficiently large, recalling (3.3.68) and in view of Lemma 3.3.1, the result follows from

$$W_2(x=0) = \frac{2a^2}{l^2} > 0. \tag{3.3.69}$$

$\square$

Lemma 3.3.6. *For ξ as in* (3.3.68) *and for sufficiently large m as in Lemma 3.3.5, any eigenvalue $\lambda = m^2\tilde{\lambda}$ of P_x satisfies $\tilde{\lambda} > \Xi^2$.*

Proof. This is immediate as for $\tilde{\lambda} \leq \Xi^2$ and sufficiently large m, the operator $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + m^2 W_1 + W_2$ is strictly positive in view of Lemma 3.3.5. $\square$

Thus, it suffices to show Lemma 3.3.4 for $\tilde{\lambda} > \Xi^2$ and we consider the case $\tilde{\lambda} \in (\Xi^2, \Xi^2 + 1]$ in Section 3.3.4.1 and the case $\tilde{\lambda} \in (\Xi^2 + 1, \infty)$ in Section 3.3.4.2.

3.3.4.1 The case $\Xi^2 < \tilde{\lambda} \leq \Xi^2 + 1$

Let $\tilde{\lambda} \in (\Xi^2, \Xi^2 + 1]$. In this range, $\tilde{\lambda}$ can be arbitrarily close to Ξ^2. As $\tilde{\lambda} \to \Xi^2$, the root $y_0 > 0$ of the potential $W_1(y)$ coalesces with $y = 0$. Thus, our estimates need to be uniform in this limit and the appropriate approximation is given by parabolic cylinder functions. To do so we will introduce the following Liouville transform which is motivated by [118]. We define a new variable[2]

$$\xi = \xi(y) \tag{3.3.70}$$

to satisfy

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}y}\right)^2 = \frac{W_1(y)}{\xi^2 - \alpha^2}, \tag{3.3.71}$$

where we choose $\alpha > 0$ such that $\xi(y_0) = \alpha > 0$ and $\xi(y = 0) = 0$. By construction, this defines $\xi = \xi(y)$ as a smooth (even real-analytic) increasing function with values in $[0, \infty)$, see also [118, Section 2.2]. Note that this holds true as the right hand side satisfies

$$\frac{W_1(y)}{\xi^2 - \alpha^2} > 0 \tag{3.3.72}$$

for $y > 0$. Equivalently, the function $\xi(y)$ can be expressed as

$$\int_y^{y_0} (-W_1)^{\frac{1}{2}} \mathrm{d}\tilde{y} = \int_{\xi(y)}^{\alpha} (\alpha^2 - \tau^2)^{\frac{1}{2}} \mathrm{d}\tau \text{ for } y \leq y_0, \tag{3.3.73}$$

$$\int_{y_0}^{y} W_1^{\frac{1}{2}} \mathrm{d}\tilde{y} = \int_{\alpha}^{\xi(y)} (\tau^2 - \alpha^2)^{\frac{1}{2}} \mathrm{d}\tau \text{ for } y_0 \leq y < \infty. \tag{3.3.74}$$

We also consider $y = y(\xi)$ as a function ξ and define

$$\sigma_1 := \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^{-\frac{1}{2}} s_1. \tag{3.3.75}$$

In this new variable ξ, the function $\sigma_1 = \sigma_1(\xi)$ satisfies

$$-\frac{\mathrm{d}^2\sigma}{\mathrm{d}\xi^2} + \left[m^2(\xi^2 - \alpha^2) + \Psi\right] \sigma = 0, \tag{3.3.76}$$

[2] Here and in the following, ξ is not to mixed up with ξ appearing in (3.3.1).

where the error function Ψ is given by

$$\Psi = \frac{\mathrm{d}y}{\mathrm{d}\xi} W_2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^{\frac{1}{2}} \frac{\mathrm{d}^2(y^{-\frac{1}{2}})}{\mathrm{d}\xi^2}. \tag{3.3.77}$$

Since W_1 is analytic and non-increasing in $\tilde{\lambda}$, we apply [118, Lemma 1] to conclude that Ψ is continuous for $(\xi, \tilde{\lambda}) \in [0, \infty) \times [\Xi^2, \Xi^2 + 1]$. Now, we define the error-control function (see (6.3) of [118])

$$F_1 := \int_0^\xi \frac{\Psi}{\Omega(\xi\sqrt{2m})} \mathrm{d}\xi \tag{3.3.78}$$

with $\Omega(x) = |x|^{\frac{1}{3}}$. We will now bound the total variation of the error-control function F_1 in (3.3.78). To do so we first show

Lemma 3.3.7. *The smooth and monotonic functions $\xi = \xi(y)$ and $y = y(\xi)$ as defined in* (3.3.71) *satisfy*

$$\xi^2(y) \sim y \tag{3.3.79}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} \sim \xi \tag{3.3.80}$$

$$\left|\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2}\right| \lesssim 1 \tag{3.3.81}$$

for all ξ sufficiently large.

Proof. We estimate

$$\frac{\mathrm{d}\xi}{\mathrm{d}y} \lesssim \sqrt{\frac{\Xi^2}{\xi^2 - \alpha^2}} \lesssim \frac{1}{\xi} \tag{3.3.82}$$

for all ξ large enough, where we have used that $W_1 \sim \Xi^2$ for large ξ. Similarly,

$$\frac{\mathrm{d}\xi}{\mathrm{d}y} \gtrsim \frac{1}{\xi} \tag{3.3.83}$$

for ξ large which shows (3.3.80). Upon integrating the inequalities, we obtain (3.3.79).

For (3.3.81), we differentiate (3.3.71) to obtain

$$\left|\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}\xi}\sqrt{\frac{\xi^2 - \alpha^2}{W_1(y(\xi))}}\right| \lesssim \sqrt{\frac{W_1}{\xi^2 - \alpha^2}} \left|\frac{\xi}{W_1} + \frac{\xi^2}{W_1^2}\frac{\mathrm{d}W_1}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\xi}\right| \lesssim 1, \tag{3.3.84}$$

where we have used that

$$W_1 \sim 1, \frac{\mathrm{d}W_1}{\mathrm{d}x} \lesssim 1, \frac{\mathrm{d}x}{\mathrm{d}y} \lesssim e^{-2\Xi y} \lesssim e^{-\xi}, \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\xi} \lesssim \xi \tag{3.3.85}$$

for ξ large enough. $\square$

This allows us now to estimate the total variation of the error control function F_1.

Lemma 3.3.8. *The error control function* F_1 *satisfies*

$$\mathcal{V}_{0,\infty}(F_1) \lesssim \frac{1}{m^{\frac{1}{6}}}. \tag{3.3.86}$$

Proof. As Ψ is continuous on $[0, \infty)$, it suffices to control the integral for large ξ. We control both terms of

$$\Psi = \frac{\mathrm{d}y}{\mathrm{d}\xi} W_2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^{\frac{1}{2}} \frac{\mathrm{d}^2(y^{-\frac{1}{2}})}{\mathrm{d}\xi^2} \tag{3.3.87}$$

independently. For large ξ, we estimate the first term as

$$\left|\frac{\mathrm{d}y}{\mathrm{d}\xi} W_2\right| \leq |W_2| \sqrt{\frac{\xi^2 - \alpha^2}{|W_1|}} \lesssim \xi |W_2| \tag{3.3.88}$$

in view of

$$W_1 \geq \frac{\Xi^2}{2} \tag{3.3.89}$$

for ξ large enough. Moreover, we note that

$$|W_2| \lesssim e^{2\Xi y} \tag{3.3.90}$$

for y large enough. Hence, for ξ large enough we have

$$\xi |W_2| \lesssim \xi e^{-2\Xi y(\xi)} \lesssim e^{-\xi} \tag{3.3.91}$$

in view of Lemma 3.3.7. For the second term, we have

$$\left| \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^{\frac{1}{2}} \frac{\mathrm{d}^2(y^{-\frac{1}{2}})}{\mathrm{d}\xi^2} \right| \lesssim \left| \frac{\frac{\mathrm{d}y}{\mathrm{d}\xi}}{y} \right|^{\frac{5}{2}} + \left| \frac{\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} (\frac{\mathrm{d}y}{\mathrm{d}\xi})^{\frac{1}{2}}}{y^{\frac{3}{2}}} \right| \lesssim \frac{1}{\xi^{\frac{5}{2}}} \tag{3.3.92}$$

for ξ sufficiently large. Hence,

$$|\Psi| \lesssim (1+\xi)^{-\frac{5}{2}} \tag{3.3.93}$$

for ξ sufficiently large. Recall that Ψ is continuous everywhere and $\Omega = |x|^{\frac{1}{3}}$ such that

$$\mathcal{V}_{0,\infty}(F_1) \lesssim \int_0^\infty \frac{|\Psi|}{\xi^{\frac{1}{3}} m^{\frac{1}{6}}} \mathrm{d}\xi \lesssim m^{-\frac{1}{6}}. \tag{3.3.94}$$

□

Having controlled the error terms we now proceed to the definition of our fundamental solutions based on appropriate parabolic cylinder functions.

Proposition 3.3.4. *There exist solutions w_1 and w_2 of* (3.3.76) *satisfying*

$$w_1 = U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) + \tilde{\eta}_1 \tag{3.3.95}$$

$$w_2 = \bar{U}\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) + \tilde{\eta}_2, \tag{3.3.96}$$

where U and $\bar{U}$ are parabolic cylinder functions defined in Definition 3.9.2. The error terms satisfy

$$\tilde{\eta}_1 = E_U^{-1}\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) M_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) O(m^{-\frac{2}{3}}) \tag{3.3.97}$$

$$\tilde{\eta}_2 = E_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) M_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) O(m^{-\frac{2}{3}}) \tag{3.3.98}$$

$$\partial_\xi\tilde{\eta}_1 = E_U^{-1}\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) N_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) O(m^{-\frac{1}{6}}) \tag{3.3.99}$$

$$\partial_\xi\tilde{\eta}_2 = E_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) N_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) O(m^{-\frac{1}{6}}) \tag{3.3.100}$$

uniformly in $\tilde{\lambda} \in [\Xi^2, \Xi^2+1]$ and $\xi \in [0,\infty)$, where the weight function E_U, the modulus functions M_U and N_U are defined in Section 3.9.2.

Proof. This follows from [118, Theorem 1]. The error bounds hold in view of Lemma 3.3.8 and [118, Section 6.3]. □

Remark 3.3.2. *For large $x > 0$, the function U is recessive, whereas $\bar{U}$ is dominant. Hence, w_1 is recessive and w_2 is dominant.*

Lemma 3.3.9. *The Wronskian $\mathfrak{W}(w_1, w_2)$ satisfies*

$$|\mathfrak{W}_\xi(w_1, w_2)| \sim \sqrt{m}\Gamma\left(\frac{1}{2} + \frac{1}{2}m\alpha^2\right). \tag{3.3.101}$$

for m sufficiently large.

Proof. For $U(b,x)$ and $\bar{U}(b,x)$ we have the Wronskian identity $\mathfrak{W}(U,\bar{U}) = \sqrt{\frac{2}{\pi}}\Gamma(\frac{1}{2}-b)$, see [118, Equation (5.8)]. The result follows now from the chain rule and the error estimates in Proposition 3.3.4. $\square$

Lemma 3.3.10. *The function σ_1 defined in* (3.3.75) *has the form*

$$\sigma_1 = A_1 w_1, \tag{3.3.102}$$

where w_1 is as in Proposition 3.3.4 and $A_1 \neq 0$ is a real constant.

Proof. Both functions σ_1 and w_1 are non-trivial solutions to (3.3.76) which are recessive as $\xi \to \infty$ (or $y \to \infty$). The claim follows now as the space of solutions of (3.3.76) which are recessive as $\xi \to \infty$ is one-dimensional. $\square$

Using the parabolic cylinder functions, we now define a solution σ_2 which is linearly independent of σ_1.

Definition 3.3.1. *We define the solution σ_2 of* (3.3.76) *as*

$$\sigma_2 := \frac{1}{A_1 \mathfrak{W}(w_1, w_2)} w_2 \tag{3.3.103}$$

and the solution s_2 to (3.3.64) *as*

$$s_2(y) := \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^{\frac{1}{2}} \sigma_2(\xi(y)). \tag{3.3.104}$$

A direct computation shows

Lemma 3.3.11. *We have*

$$\mathfrak{W}_y(s_1, s_2) = \mathfrak{W}_\xi(\sigma_1, \sigma_2) = 1. \tag{3.3.105}$$

Here, $\mathfrak{W}_y$ and $\mathfrak{W}_\xi$ denote the Wronskians with respect to the y and ξ variable.

Lemma 3.3.12. *With σ_1 and σ_2 as defined in* (3.3.75) *and* (3.3.103) *we have*

$$|\sigma_1| \lesssim |A_1| E_U^{-1}\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) M_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) \tag{3.3.106}$$

$$|\sigma_2| \lesssim \left|\frac{1}{A_1 \mathfrak{W}(w_1, w_2)}\right| E_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) M_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) \tag{3.3.107}$$

$$|\sigma_1(\xi)\sigma_2(\xi)| \lesssim \frac{1}{|\mathfrak{W}(w_1, w_2)|} M_U^2\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right). \tag{3.3.108}$$

Proof. We estimate

$$\begin{aligned} |\sigma_1| = |A_1 w_1| &= |A_1| \left| U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) + \tilde{\eta}_1 \right| \\ &\lesssim |A_1| E_U^{-1}\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) M_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) \end{aligned} \tag{3.3.109}$$

and

$$\begin{aligned} |\sigma_2| &= \left|\frac{1}{A_1 \mathfrak{W}(w_1, w_2)}\right| |w_2| \leq \left|\frac{1}{A_1 \mathfrak{W}(w_1, w_2)}\right| \left| \bar{U}\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) + \tilde{\eta}_2 \right| \\ &\lesssim \left|\frac{1}{A_1 \mathfrak{W}(w_1, w_2)}\right| E_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right) M_U\left(-\frac{1}{2}m\alpha^2, \xi\sqrt{2m}\right). \end{aligned} \tag{3.3.110}$$

□

Now, we recall the definition of g_p in (3.3.63) as

$$g_p := s_2(y) \int_y^\infty s_1^2(\tilde{y}) F(\tilde{y}) \mathrm{d}\tilde{y} + s_1(y) \int_0^y s_1(\tilde{y}) s_2(\tilde{y}) F(\tilde{y}) \mathrm{d}\tilde{y}. \tag{3.3.111}$$

for s_1 as in (3.3.56) and where we take s_2 as in (3.3.104). Now, we are in the position to show the main lemma of Section 3.3.4.1.

Lemma 3.3.13. *Let $\tilde{\lambda} \in (\Xi^2, \Xi^2 + 1]$ and let s_2 as in* (3.3.104)*. Then, g_p satisfies*

$$\int_0^\infty g_p(y)^2 (1 - x(y)^2) \Delta_x \mathrm{d}y \lesssim m. \tag{3.3.112}$$

Proof. We plug (3.3.111) into (3.3.112) and we will estimate both terms which appear independently.

For the first term, we change variables from y to ξ, use that $x \mapsto E_U(b, x)$ is non-decreasing, as well as Lemma 3.3.12 to estimate

$$\begin{aligned}
&\int_0^\infty s_2^2(y)\left(\int_y^\infty s_1^2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x\mathrm{d}y\\
&=\int_0^\infty \sigma_2^2(\xi)\left(\int_\xi^\infty \sigma_1^2(\tilde{\xi})F(\tilde{\xi})\mathrm{d}\tilde{\xi}\right)^2(1-x(\xi)^2)\Delta_x(\xi)\mathrm{d}\xi\\
&\lesssim\int_0^\infty\left[\frac{\left|M_U\left(-\frac{1}{2}m\alpha^2,\xi\sqrt{2m}\right)\right|^2}{|\mathfrak{W}(w_1,w_2)|^2}\Big(\int_\xi^\infty|\sigma_1(\tilde{\xi})F(\tilde{\xi})|\right.\\
&\qquad\qquad\qquad\left.|M_U(-\frac{1}{2}m\alpha^2,\tilde{\xi}\sqrt{2m})|\mathrm{d}\tilde{\xi}\Big)^2(1-x(\xi)^2)\Delta_x(\xi)\right]\mathrm{d}\xi.
\end{aligned}\tag{3.3.113}$$

Now, we use the bounds on M_U and $\mathfrak{W}(w_1,w_2)$ from Proposition 3.9.1 and Lemma 3.3.9 to deduce

$$\begin{aligned}
&\int_0^\infty s_2^2(y)\left(\int_y^\infty s_1^2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x\mathrm{d}y\\
&\qquad\lesssim\frac{1}{m}\int_0^\infty(1-x(\xi)^2)\Delta_x(\xi)\mathrm{d}\xi\left(\int_0^\infty|\sigma_1(\tilde{\xi})||F(\tilde{\xi})|\mathrm{d}\tilde{\xi}\right)^2\\
&\qquad\lesssim\frac{1}{m}\int_0^\infty|s_1|^2(1-x(y)^2)\Delta_x(y)\mathrm{d}y\int_0^\infty\frac{|F|^2}{\Delta_x(1-x(y)^2)}\frac{\mathrm{d}\xi}{\mathrm{d}y}\mathrm{d}y\\
&\qquad\lesssim\frac{1}{m}\int_0^\infty\Delta_x(1-x^2)\left[\partial_\xi\lambda+2\Xi(am\omega_-+\frac{\epsilon}{m})\frac{x^2}{\Delta_x}+2m\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right]^2\frac{\mathrm{d}\xi}{\mathrm{d}y}\mathrm{d}y\\
&\qquad\lesssim m,
\end{aligned}\tag{3.3.114}$$

where we used the Cauchy–Schwarz inequality and the fact that s_1 satisfies (3.3.57) as well as (3.3.44).

For the second term we argue similarly and obtain

$$
\begin{aligned}
&\left|\int_0^\infty s_1^2(y)\left(\int_0^y s_2(\tilde{y})s_1(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x\mathrm{d}y\right| \\
&\lesssim \int_0^\infty s_1^2(1-x(y)^2)\Delta_x\mathrm{d}y\left(\int_0^\infty s_2(\tilde{y})s_1(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2 \\
&\lesssim \left(\int_0^\infty \frac{M_U^2\left(-\frac{1}{2}m\alpha^2,\xi\sqrt{2m}\right)}{|\mathfrak{W}(w_1,w_2)|}|F(\xi)|\mathrm{d}\xi\right)^2 \\
&\lesssim \frac{1}{m}\left(\int_0^\infty \Delta_x(1-x(y)^2)\left|\partial_\xi\lambda+2\Xi(am\omega_-+\frac{\epsilon}{m})\frac{x^2}{\Delta_x}+2m\frac{\Xi}{\Delta_x}\frac{a^2}{l^2}x^2\right|\frac{\mathrm{d}\xi}{\mathrm{d}y}\mathrm{d}y\right)^2 \\
&\lesssim m. \qquad (3.3.115)
\end{aligned}
$$

□

3.3.4.2 The case $\tilde{\lambda}\in(\Xi^2+1,\infty)$

For the parameter range $\tilde{\lambda}\in(\Xi^2+1,\infty)$ we consider $\lambda=m^2\tilde{\lambda}$ as a large parameter and re-write the o.d.e. (3.3.62) as

$$-\frac{\mathrm{d}^2}{\mathrm{d}y^2}g_p+m^2\tilde{\lambda}\tilde{W}_1g_p+W_2g_p=Fs_1, \qquad (3.3.116)$$

where

$$\tilde{W}_1=\tilde{W}_1(y):=\frac{W_1}{\tilde{\lambda}}=\frac{\Xi^2}{\tilde{\lambda}}-\left[\Xi a^2\omega_-^2+2a\omega_-\Xi\frac{a^2}{l^2}\right]\frac{x^2(1-x^2)}{\tilde{\lambda}}-\Delta_x(1-x^2). \qquad (3.3.117)$$

We also recall the homogeneous o.d.e. (3.3.64)

$$-\frac{d^2}{\mathrm{d}y^2}g+m^2\tilde{\lambda}\tilde{W}_1g+W_2g=0. \qquad (3.3.118)$$

Recall also that s_1 as defined in (3.3.56) is a solution of (3.3.118). As before, we define y_0 as the unique non-negative root of $\tilde{W}_1(y)$. It satisfies

$$y_0\sim\frac{1}{2\Xi}\log(\tilde{\lambda}). \qquad (3.3.119)$$

which becomes arbitrarily large for $\tilde{\lambda}\to\infty$. Our estimates have to take care of this limit.

Lemma 3.3.14. *In the region* $0 \le y \le y_0 - 1$ *we have*

$$\frac{1}{\tilde{\lambda}} \lesssim -\tilde{W}_1 \lesssim 1, \quad \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} \lesssim |\tilde{W}_1|, \quad \left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right| \lesssim \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} + (1 - x(y)^2)|\tilde{W}_1|. \tag{3.3.120}$$

For $y_0 - 1 \le y \le y_0 + 1$*, we have*

$$\frac{|y - y_0|}{\tilde{\lambda}} \lesssim |\tilde{W}_1| \lesssim \frac{1}{\tilde{\lambda}}, \quad \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} \sim \frac{1}{\tilde{\lambda}}, \quad \left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right| \lesssim \frac{1}{\tilde{\lambda}}, \quad \left|\frac{\mathrm{d}^3\tilde{W}_1}{\mathrm{d}y^3}\right| \lesssim \frac{1}{\tilde{\lambda}}. \tag{3.3.121}$$

For $y_0 + 1 \le y < \infty$*, we have*

$$\tilde{W}_1 \sim \frac{1}{\tilde{\lambda}} \text{ and } \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}, \left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right| \lesssim \frac{\mathrm{d}x}{\mathrm{d}y} \lesssim \frac{1}{\tilde{\lambda}}. \tag{3.3.122}$$

Proof. From Lemma 3.3.1 we have that $\tilde{W}_1$ is increasing on $y \in [0, \infty)$ and moreover, for $y \in [y_0 - 1, y_0 + 1]$ we have that

$$\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} = \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} \gtrsim x(y)\frac{\mathrm{d}x}{\mathrm{d}y} \gtrsim \frac{1}{\tilde{\lambda}}. \tag{3.3.123}$$

Thus, for $y \le y_0 - 1$ we have

$$-\tilde{W}_1(y) \ge -\tilde{W}_1(y_0 - 1) \ge \int_{y_0-1}^{y_0} \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}\mathrm{d}\tilde{y} \gtrsim \frac{1}{\tilde{\lambda}}. \tag{3.3.124}$$

Moreover, for $0 \le y \le y_0 - 1$ we have

$$\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} \lesssim \frac{\mathrm{d}x}{\mathrm{d}y} = \Delta_x(1 - x^2) \lesssim |\tilde{W}_1| + \frac{1}{\tilde{\lambda}} \lesssim |\tilde{W}_1| \tag{3.3.125}$$

using the definition of $\tilde{W}_1$ and $|\tilde{W}_1| \gtrsim \frac{1}{\tilde{\lambda}}$. Similarly, we obtain

$$\left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right| \lesssim (1 - x(y)^2)\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}x} + (1 - x(y)^2)^2\left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}x^2}\right| \lesssim \frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} + (1 - x(y)^2)|\tilde{W}_1|. \tag{3.3.126}$$

In the region $y \in [y_0 - 1, y_0 + 1]$, recall from (3.3.123) that $\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} \gtrsim \frac{1}{\tilde{\lambda}}$. Moreover, just as in (3.3.126), we obtain

$$\left|\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}\right|, \left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right|, \left|\frac{\mathrm{d}^3\tilde{W}_1}{\mathrm{d}y^3}\right| \lesssim \frac{\mathrm{d}x}{\mathrm{d}y} \lesssim \frac{1}{\tilde{\lambda}}. \tag{3.3.127}$$

In the region $y \in (y_0 + 1, \infty)$, analogous to (3.3.124), we have

$$\frac{1}{\tilde{\lambda}} \lesssim \tilde{W}_1 \lesssim \frac{1}{\tilde{\lambda}} \tag{3.3.128}$$

and moreover,

$$\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}, \left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right| \lesssim \frac{\mathrm{d}x}{\mathrm{d}y} \lesssim \frac{1}{\tilde{\lambda}}. \tag{3.3.129}$$

□

With the estimates of Lemma 3.3.14 at hand we will define the variable ς as

$$\frac{2}{3}\varsigma^{\frac{3}{2}} = \int_{y_0}^{y} \sqrt{\tilde{W}_1(y)}\mathrm{d}y \tag{3.3.130}$$

for $y \geq y_0$ and

$$\frac{2}{3}(-\varsigma)^{\frac{3}{2}} = \int_{y}^{y_0} \sqrt{-\tilde{W}_1(y)}\mathrm{d}y \tag{3.3.131}$$

for $y \leq y_0$. We denote

$$\varsigma_0 := \varsigma(y=0) = -\left(\frac{3}{2}\int_0^{y_0} \sqrt{-\tilde{W}_1(y)}\mathrm{d}y\right)^{\frac{2}{3}}. \tag{3.3.132}$$

We further introduce the error control function

$$H(y) := \int_{y_0}^{y} \frac{1}{|\tilde{W}_1|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(|\tilde{W}_1|^{-\frac{1}{4}}\right) - \frac{W_2}{|\tilde{W}_1|^{\frac{1}{2}}} - \frac{5|\tilde{W}_1|^{\frac{1}{2}}}{16\varsigma(y)^3}\mathrm{d}y. \tag{3.3.133}$$

The fact that H is absolutely continuous is a standard result and follows from [117, Lemma, Section 4], see also [119, Lemma 3.1, Chapter 11]. In the following we establish a quantitative version of this.

Lemma 3.3.15. *The error-function H defined in* (3.3.133) *satisfies*

$$\mathcal{V}_{0,\infty}(H) \lesssim \tilde{\lambda}^{\frac{1}{2}}. \tag{3.3.134}$$

Proof. Since H is absolutely continuous we compute

$$\begin{aligned}\mathcal{V}_{0,\infty}(H) &= \int_0^{y_0-1}\left|\frac{1}{|\tilde{W}_1|^{\frac14}}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(|\tilde{W}_1|^{-\frac14}\right)-\frac{W_2}{|\tilde{W}_1|^{\frac12}}-\frac{5|\tilde{W}_1|^{\frac12}}{16\varsigma(y)^3}\right|\mathrm{d}y\\
&+\int_{y_0-1}^{y_0+1}\left|\frac{1}{|\tilde{W}_1|^{\frac14}}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(|\tilde{W}_1|^{-\frac14}\right)-\frac{W_2}{|\tilde{W}_1|^{\frac12}}-\frac{5|\tilde{W}_1|^{\frac12}}{16\varsigma(y)^3}\right|\mathrm{d}y\\
&+\int_{y_0+1}^{+\infty}\left|\frac{1}{|\tilde{W}_1|^{\frac14}}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(|\tilde{W}_1|^{-\frac14}\right)-\frac{W_2}{|\tilde{W}_1|^{\frac12}}-\frac{5|\tilde{W}_1|^{\frac12}}{16\varsigma(y)^3}\right|\mathrm{d}y\\
&=: I+II+III \end{aligned}\tag{3.3.135}$$

and estimate each term independently relying on Lemma 3.3.14.

Term $\boldsymbol{I}$**.** We estimate term I as

$$I\lesssim\int_0^{y_0-1}\frac{1}{|\tilde{W}_1|^{\frac52}}\left(\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}\right)^2+\frac{1}{|\tilde{W}_1|^{\frac32}}\left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right|+\frac{|W_2|}{|\tilde{W}_1|^{\frac12}}+\frac{|\tilde{W}_1|^{\frac12}}{\varsigma^3}\mathrm{d}y. \tag{3.3.136}$$

We consider the first term appearing in (3.3.136) and in view of Lemma 3.3.14 we obtain

$$\int_0^{y_0-1}\frac{1}{|\tilde{W}_1|^{\frac52}}\left(\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}\right)^2\mathrm{d}y\lesssim\int_0^{y_0-1}\frac{\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}}{|\tilde{W}_1|^{\frac32}}\mathrm{d}y\lesssim\left.\frac{1}{|\tilde{W}_1|^{\frac12}}\right|_0^{y_0-1}\lesssim\tilde{\lambda}^{\frac12}. \tag{3.3.137}$$

For the second term involving the second derivative, we use (3.3.120) to conclude that

$$\int_0^{y_0-1}\frac{1}{|\tilde{W}_1|^{\frac32}}\left|\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right|\mathrm{d}y\lesssim\int_0^{y_0-1}\frac{\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}}{|\tilde{W}_1|^{\frac32}}+\frac{1-x(y)^2}{|\tilde{W}_1|^{\frac12}}\mathrm{d}y\lesssim\tilde{\lambda}^{\frac12}. \tag{3.3.138}$$

For the third term we use that $|W_2|\lesssim 1-x(y)^2$ such that

$$\int_0^{y_0-1}\frac{|W_2|}{|\tilde{W}_1|^{\frac12}}\lesssim\tilde{\lambda}^{\frac12}\int_0^{\infty}(1-x(y)^2)\mathrm{d}y\lesssim\tilde{\lambda}^{\frac12}. \tag{3.3.139}$$

For the last term in (3.3.136), we have

$$\begin{aligned}\int_0^{y_0-1}\frac{|\tilde{W}_1|^{\frac12}}{\varsigma^3}\mathrm{d}y&\lesssim\int_0^{y_0-1}\frac{\sqrt{-\tilde{W}_1}}{\left(\int_y^{y_0}\sqrt{-\tilde{W}_1}\mathrm{d}\tilde{y}\right)^2}\mathrm{d}y\lesssim\frac{1}{\int_{y_0-1}^{y_0}\sqrt{-\tilde{W}_1}\mathrm{d}\tilde{y}}\\
&\lesssim\frac{1}{\int_{y_0-1}^{y_0}\sqrt{|\tilde{y}-y_0|\tilde{\lambda}^{-1}}\mathrm{d}\tilde{y}}\lesssim\tilde{\lambda}^{\frac12}.\end{aligned}\tag{3.3.140}$$

***Term** II.* For this term, we use Taylor's theorem around the point $y = y_0$ and a lengthy but direct computation shows that

$$\left| \frac{1}{|\tilde{W}_1|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(|\tilde{W}_1|^{-\frac{1}{4}} \right) - \frac{W_2}{|\tilde{W}_1|^{\frac{1}{2}}} - \frac{5|\tilde{W}_1|^{\frac{1}{2}}}{16\varsigma(y)^3} \right| \lesssim \frac{1}{|y-y_0|^{\frac{1}{2}}} \sup_{y\in(y_0-1,y_0+1)} \left(\frac{\left| 9\left(\frac{\mathrm{d}^2\tilde{W}_1}{\mathrm{d}y^2}\right)^2 - 10\frac{\mathrm{d}^3\tilde{W}_1}{\mathrm{d}y^3}\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y} \right|}{140\left(\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}\right)^{\frac{5}{2}}} + \frac{|W_2|}{\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}} \right) \lesssim \frac{\tilde{\lambda}^{\frac{1}{2}}}{|y-y_0|^{\frac{1}{2}}} \tag{3.3.141}$$

uniformly in $y \in [y_0 - 1, y_0 + 1]$ from which we conclude that $|II| \lesssim \tilde{\lambda}^{\frac{1}{2}}$.

***Term** III.* In the region $y \in [y_0 + 1, \infty)$ we first have

$$\varsigma^3(y) \gtrsim \left(\int_{y_0}^{y_0+1} \sqrt{\tilde{W}_1}\mathrm{d}\tilde{y} \right)^2 + \left(\int_{y_0+1}^{y} \sqrt{\tilde{W}_1}\mathrm{d}\tilde{y} \right)^2 \gtrsim \frac{1}{\tilde{\lambda}} + (y-y_0-1)^2\frac{1}{\tilde{\lambda}} \tag{3.3.142}$$

such that

$$\frac{|\tilde{W}_1|^{\frac{1}{2}}}{\varsigma^3} \lesssim \frac{\tilde{\lambda}^{\frac{1}{2}}}{1+(y-y_0-1)^2} \tag{3.3.143}$$

which is integrable at $y = \infty$. Moreover, we have

$$\left| \frac{1}{|\tilde{W}_1|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(|\tilde{W}_1|^{-\frac{1}{4}} \right) - \frac{W_2}{|\tilde{W}_1|^{\frac{1}{2}}} \right| \lesssim \frac{\mathrm{d}x}{\mathrm{d}y} \left(\frac{\frac{\mathrm{d}\tilde{W}_1}{\mathrm{d}y}}{\tilde{W}_1^{\frac{3}{2}}} + \frac{1}{\tilde{W}_1^{\frac{1}{2}}} \right) \lesssim \frac{\mathrm{d}x}{\mathrm{d}y}\tilde{\lambda}^{\frac{1}{2}}. \tag{3.3.144}$$

Combining the estimates (3.3.143) and (3.3.144) we obtain that $|III| \lesssim \tilde{\lambda}^{\frac{1}{2}}$ which concludes the proof. □

Finally, we also introduce

$$\hat{W}_1 := \frac{\tilde{W}_1}{\varsigma} \text{ or equivalently } \hat{W}_1 = \left(\frac{\mathrm{d}\varsigma}{\mathrm{d}y}\right)^2 \tag{3.3.145}$$

which we will bound from below in the following.

Lemma 3.3.16. *We have*

$$\frac{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))}{\hat{W}_1^{\frac{1}{4}}} \lesssim \tilde{\lambda}^{\frac{1}{6}}. \tag{3.3.146}$$

Proof. First, for $y_0 - 1 \leq y \leq y_0$ we have

$$\frac{2}{3}(-\varsigma)^{\frac{3}{2}} = \int_y^{y_0} \sqrt{|\tilde{W}_1|} \mathrm{d}\tilde{y} \leq (y_0 - y)\sqrt{-\tilde{W}_1(y)} \tag{3.3.147}$$

and for $y_0 \leq y \leq y_0 + 1$ we have

$$\frac{2}{3}\varsigma^{\frac{3}{2}} = \int_{y_0}^{y} \sqrt{\tilde{W}_1} \mathrm{d}\tilde{y} \leq (y - y_0)\sqrt{\tilde{W}_1(y)}, \tag{3.3.148}$$

where we have used the monotonicity of $\tilde{W}_1$. Hence,

$$\hat{W}_1 = \frac{\tilde{W}_1}{\varsigma} \gtrsim \left(\frac{\tilde{W}_1}{y - y_0}\right)^{\frac{2}{3}} \gtrsim \tilde{\lambda}^{\frac{2}{3}} \tag{3.3.149}$$

for $|y - y_0| \leq 1$. Now, using $M_{\mathrm{Ai}}(x) \lesssim 1$ we conclude that for $|y - y_0| \leq 1$ we have

$$\frac{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))}{\hat{W}_1^{\frac{1}{4}}} \lesssim \frac{1}{\hat{W}_1^{\frac{1}{4}}} \lesssim \tilde{\lambda}^{\frac{1}{6}}. \tag{3.3.150}$$

For $|y - y_0| \geq 1$, we use that $M_{\mathrm{Ai}}(x) \lesssim |x|^{-\frac{1}{4}}$ to obtain

$$\frac{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))}{\hat{W}_1^{\frac{1}{4}}} \lesssim \frac{|\varsigma|^{\frac{1}{4}}}{|\varsigma|^{\frac{1}{4}}\lambda^{\frac{1}{12}}|\tilde{W}_1|^{\frac{1}{4}}} \lesssim \frac{\tilde{\lambda}^{\frac{1}{4}}}{\lambda^{\frac{1}{12}}} \lesssim \tilde{\lambda}^{\frac{1}{6}} m^{-\frac{1}{6}} \lesssim \tilde{\lambda}^{\frac{1}{6}}. \tag{3.3.151}$$

□

Now, we are in the position to define the following fundamental solutions.

Proposition 3.3.5. *There exist solutions w_1 and w_2 of* (3.3.118) *satisfying*

$$w_1 = \frac{1}{\hat{W}_1^{\frac{1}{4}}}\left(\mathrm{Ai}(\lambda^{\frac{1}{3}}\varsigma(y)) + \eta_{\mathrm{Ai}}(\lambda, y)\right) \tag{3.3.152}$$

$$w_2 = \frac{1}{\hat{W}_1^{\frac{1}{4}}}\left(\mathrm{Bi}(\lambda^{\frac{1}{3}}\varsigma(y)) + \eta_{\mathrm{Bi}}(\lambda, y)\right), \tag{3.3.153}$$

where

$$\frac{|\eta_{\mathrm{Ai}}(\lambda,y)|}{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma)}, \frac{|\partial_y\eta_{\mathrm{Ai}}(\lambda,y)|}{\lambda^{\frac{1}{3}}N_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma)\hat{W}_1^{\frac{1}{2}}} \lesssim E_{\mathrm{Ai}}^{-1}(\lambda^{\frac{1}{3}}\varsigma)m^{-1}, \tag{3.3.154}$$

$$\frac{|\eta_{\mathrm{Bi}}(\lambda,y)|}{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma)}, \frac{|\partial_y\eta_{\mathrm{Bi}}(\lambda,y)|}{\lambda^{\frac{1}{3}}N_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma)\hat{W}_1^{\frac{1}{2}}} \lesssim E_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma)m^{-1}. \tag{3.3.155}$$

Moreover, the Wronskian of w_1 and w_2 satisfies

$$|\mathfrak{W}(w_1,w_2)| \sim \lambda^{\frac{1}{3}}. \tag{3.3.156}$$

Proof. This follows from [119, Chapter 11, Theorem 3.1] and the error bounds follow from the bounds on $\mathcal{V}_{0,\infty}(H)$ in Lemma 3.3.15. The Wronskian identity is a direct consequence of the chain rule. □

Lemma 3.3.17. *There exists a constant $A_2 \neq 0$ such that $s_1 = A_2 w_1$, where w_1 is defined in* (3.3.152) *and s_1 is defined in* (3.3.56).

Proof. Note that both, w_1 and s_1 are recessive as $y \to \infty$. Since the space of solutions which are recessive at $y \to \infty$ is one-dimensional, we conclude that s_1 and w_1 are linearly dependent. □

In view of Lemma 3.3.17 we define

$$s_2 := \frac{1}{A_2\mathfrak{W}(w_1,w_2)}w_2, \tag{3.3.157}$$

where w_2 is as in (3.3.153). Note that this implies that

$$\mathfrak{W}(s_1,s_2) = 1. \tag{3.3.158}$$

Lemma 3.3.18. *Let $\tilde{\lambda} \in [\Xi^2+1,\infty)$ and let s_2 as in* (3.3.157)*. Then, g_p defined as*

$$g_p := s_2(y)\int_y^\infty s_1^2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y} + s_1(y)\int_0^y s_1(\tilde{y})s_2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}. \tag{3.3.159}$$

satisfies

$$\int_0^\infty g_p(y)^2(1-x(y)^2)\Delta_x\mathrm{d}y \lesssim m. \tag{3.3.160}$$

Proof. Analogous to the proof of Lemma 3.3.13 we first estimate

$$
\begin{aligned}
&\int_0^\infty s_2^2(y)\left(\int_y^\infty s_1^2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x \mathrm{d}y \\
&\qquad = \int_0^\infty \frac{w_2^2(y)}{\mathfrak{W}(w_1,w_2)^2}\left(\int_y^\infty w_1(\tilde{y})s_1(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x(y)\mathrm{d}y. \quad (3.3.161)
\end{aligned}
$$

Now, we use Proposition 3.3.5 and standard bounds on Airy functions from Section 3.9.1, as well as Lemma 3.3.16 to obtain

$$
|w_1(y)| \lesssim \left| E_{\mathrm{Ai}}^{-1}(\lambda^{\frac{1}{3}}\varsigma(y))\frac{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))}{\hat{W}_1^{\frac{1}{4}}(y)}\right| \lesssim E_{\mathrm{Ai}}^{-1}(\lambda^{\frac{1}{3}}\varsigma(y))\tilde{\lambda}^{\frac{1}{6}}, \quad (3.3.162)
$$

$$
|w_2(y)| \lesssim \left| E_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))\frac{M_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))}{\hat{W}_1^{\frac{1}{4}}(y)}\right| \lesssim E_{\mathrm{Ai}}(\lambda^{\frac{1}{3}}\varsigma(y))\tilde{\lambda}^{\frac{1}{6}}. \quad (3.3.163)
$$

Now, plugging these estimates into (3.3.161) and using that $E^{-1}(\lambda^{\frac{1}{3}}\varsigma(y))$ is a decreasing function, we conclude

$$
\begin{aligned}
&\int_0^\infty s_2^2(y)\left(\int_y^\infty s_1^2(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x \mathrm{d}y \\
&\qquad \lesssim \int_0^\infty \frac{\tilde{\lambda}^{\frac{2}{3}}}{\mathfrak{W}(w_1,w_2)^2}\left(\int_y^\infty s_1(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x \mathrm{d}y. \\
&\qquad \lesssim \frac{\tilde{\lambda}^{\frac{2}{3}}}{\mathfrak{W}(w_1,w_2)^2}\int_0^\infty (1-x(y)^2)\Delta_x \mathrm{d}y \\
&\qquad\quad \times \int_0^\infty s_1^2(y)(1-x(y)^2)\Delta_x \mathrm{d}y \int_0^\infty F^2(y)\frac{1}{(1-x(y)^2)\Delta_x}\mathrm{d}y \\
&\qquad \lesssim \frac{\tilde{\lambda}^{\frac{2}{3}}m^2}{\mathfrak{W}(w_1,w_2)^2}. \quad (3.3.164)
\end{aligned}
$$

For the second term, we argue similarly and estimate

$$\begin{aligned}
&\int_0^\infty s_1^2(y)\left(\int_0^y s_2(\tilde{y})s_1(\tilde{y})F(\tilde{y})\mathrm{d}\tilde{y}\right)^2(1-x(y)^2)\Delta_x \mathrm{d}y\\
&\quad\leq \left(\int_0^\infty |s_2(\tilde{y})s_1(\tilde{y})F(\tilde{y})|\mathrm{d}\tilde{y}\right)^2\int_0^\infty s_1^2(y)(1-x(y)^2)\Delta_x\mathrm{d}y\\
&\quad= \frac{1}{2}\left(\int_0^\infty |s_2(\tilde{y})s_1(\tilde{y})F(\tilde{y})|\mathrm{d}\tilde{y}\right)^2\\
&\quad\lesssim \frac{\tilde{\lambda}^{\frac{2}{3}}}{\mathfrak{W}(w_1,w_2)^2}\left(\int_0^\infty |F|\mathrm{d}\tilde{y}\right)^2 \lesssim \frac{\tilde{\lambda}^{\frac{2}{3}}m^2}{\mathfrak{W}(w_1,w_2)^2}.
\end{aligned} \tag{3.3.165}$$

Thus, we conclude

$$\int_0^\infty g_p(y)^2(1-x(y)^2)\Delta_x\mathrm{d}y \lesssim \frac{\tilde{\lambda}^{\frac{2}{3}}m^2}{\mathfrak{W}(w_1,w_2)^2} \lesssim \frac{\tilde{\lambda}^{\frac{2}{3}}m^2}{\lambda^{\frac{2}{3}}} \lesssim m^{\frac{2}{3}} \lesssim m. \tag{3.3.166}$$

□

3.4 The radial o.d.e. on the exterior

We will now derive for which frequency parameters (ω, m, ℓ) the resonant frequencies $\omega = \omega_- m$ in the interior are excited by frequency parameters which are exposed to stable trapping in the black hole exterior. This allows us then to define the set $\mathscr{P}_{\text{Blow-up}}$ in Section 3.5.1. Thus, we will analyze the radial o.d.e. at frequency $\omega = \omega_- m$.

3.4.1 Radial o.d.e. at resonant frequency admits stable trapping

We now construct the range of angular eigenvalues $\lambda_{m\ell}(am\omega_-)$ at the resonant frequency $\omega = \omega_- m$ for which the radial o.d.e. admits trapping. Recall from (3.2.41) that the normalized high frequency part of the potential is given by

$$V_{\text{main}} = \frac{\Delta}{(r^2+a^2)^2}\left(\frac{\lambda_{m\ell}(a\omega_- m)}{m^2} + \omega_-^2 a^2 - 2a\omega_-\Xi\right) - (\omega_- - \omega_r)^2. \tag{3.4.1}$$

Note that

$$V_{\text{main}} \to -(\omega_- - \omega_+)^2 < 0 \text{ as } r^* \to -\infty. \tag{3.4.2}$$

For the potential V_{main} to admit stable trapping, we require that V_{main} has two roots $r_1 < r_2$, see already Fig. 3.5. This is the case if the angular eigenvalues $\lambda_{m\ell}(a\omega_- m)m^{-2}$

lie in an interval $(\sigma_1(\mathfrak{p}), \sigma_2(\mathfrak{p}))$, where σ_1 and σ_2 depend continuously on the black hole parameters $\mathfrak{p} = (\mathfrak{m}, \mathfrak{a})$. In order to define σ_1 and σ_2, we set

$$E := \bigsqcup_{\mathfrak{p}\in\mathscr{P}} \{\mathfrak{p}\} \times E_{\mathfrak{p}} = \bigsqcup_{\mathfrak{p}\in\mathscr{P}} \{\mathfrak{p}\} \times (\mu_0(\mathfrak{p}), \mu_1(\mathfrak{p})), \tag{3.4.3}$$

where

$$E_{\mathfrak{p}} := \Big\{ \mu > \Xi^2 \colon \left(\mu + \omega_-^2 a^2 - 2a\omega_-\Xi\right) - l^2\omega_-^2 < 0, \\ \left(1 + 3\frac{M^2}{\Xi} l^2 \Big(\frac{9M^2}{\Xi^2} + a^2\Big)^{-2}\right) \left(\mu + \omega_-^2 a^2 - 2a\omega_-\Xi\right) - l^2\omega_-^2 > 0 \Big\} \tag{3.4.4}$$

is a bounded open interval and $\mu_0(\mathfrak{p}) := \inf E_{\mathfrak{p}}$, $\mu_1(\mathfrak{p}) := \sup E_{\mathfrak{p}}$. Again, $E_{\mathfrak{p}}$ is well-defined as the conditions only depend on $\mathfrak{p}$. We will show that E is a fiber bundle. To do so we first show that $E_{\mathfrak{p}}$ is indeed non-empty.

Lemma 3.4.1. *For any $\mathfrak{p} \in \mathscr{P}$, the set $E_{\mathfrak{p}}$ defined in* (3.4.4) *is non-empty, hence a bounded open interval.*

Proof. By continuity, it suffices to show that

$$\frac{1}{l^2}\left(\Xi^2 + a^2\omega_-^2 - 2a\omega_-\Xi\right) - \omega_-^2 < 0 \tag{3.4.5}$$

which in turn follows from

$$r_-^2 < al. \tag{3.4.6}$$

To see that $r_-^2 < al$ holds true, we write $\Delta(r)$ in terms of r_- as

$$\Delta(r) = l^{-2}(r - r_-)(r^3 + r^2 r_- + r(r_-^2 + a^2 + l^2) - a^2 l^2 r_-^{-1}). \tag{3.4.7}$$

Hence,

$$0 > \partial_r \Delta(r_-) = l^{-2} r_-^{-1}(3r_-^4 + r_-^2 a^2 + r_-^2 l^2 - a^2 l^2) \tag{3.4.8}$$

implies

$$3r_-^4 < a^2 l^2 \tag{3.4.9}$$

from which

$$r_-^2 < al \tag{3.4.10}$$

follows. $\square$

From (3.4.4) it also follows that μ_0 and μ_1 are manifestly continuous functions on $\mathscr{P}$. Hence, E is a (topological) fiber bundle. Now, also note that E is trivial with global trivialization $\varphi_E\colon E \to \mathscr{P} \times (0,1), (\mathfrak{p}, \mu) \mapsto (\mathfrak{p}, \frac{\mu}{\mu_1 - \mu_0} - \frac{\mu_0}{\mu_1 - \mu_0})$ and we find (using this trivialization) two global sections

$$\sigma_1 \in \Gamma(E) \text{ and } \sigma_2 \in \Gamma(E) \text{ with } \sigma_1(\mathfrak{p}) < \sigma_2(\mathfrak{p}) \tag{3.4.11}$$

for all $\mathfrak{p} \in \mathscr{P}$ (in mild abuse of notation). Having constructed σ_1 and σ_2, we will now show the existence of turning points $r_1 < r_2$ of V_{main}.

Lemma 3.4.2. *Let $m^{-2}\lambda_{m\ell}(a\omega_- m) \in (\sigma_1, \sigma_2)$ as chosen in* (3.4.11)*. Then, V_{main} has a maximum $r_{max} \in (r_+, \infty)$, and two roots r_1, r_2 with $r_+ < r_1 < r_{max} < r_2 < \infty$ such that*

$$V_{\text{main}} > 0 \textit{ for } r \in (r_1, r_2), \tag{3.4.12}$$

$$V_{\text{main}} < 0 \textit{ for } r \in [r_+, \infty) \setminus [r_1, r_2] \tag{3.4.13}$$

and $r_2 - r_1 \gtrsim_{\sigma_1,\sigma_2} 1$.

Proof. By construction of σ_1 and σ_2, for any $m^{-2}\lambda_{m\ell} \in (\sigma_1, \sigma_2)$, the potential V_{main} has a maximum and satisfies

$$\lim_{r\to\infty} V_{\text{main}} < 0, V_{\text{main}}(r = r_+) < 0 \text{ and } V_{\text{main}}(r = r_{\text{cut}}) > 0, \tag{3.4.14}$$

where $r_{\text{cut}} = \frac{3M}{\Xi}$. See also [77, Lemma 3.1].

We will show now that V_{main} has exactly two roots in $[r_+, \infty)$ from which (3.4.12) and (3.4.13) follow. Indeed, in view of the above, V_{main} either has two or four roots in $[r_+, \infty)$. To exclude the case of four roots, it suffices to exclude the case of three critical points in $[r_+, \infty)$. To see this, note that

$$\frac{\mathrm{d}V_{\text{main}}}{\mathrm{d}r} = \frac{(-2\Xi r^3 + 6Mr^2 - 2\Xi a^2 r - 2Ma^2)m^{-2}\lambda_{m\ell}(a\omega_- m) + 4a\Xi(\omega_r - \omega_-)(r^2 + a^2)}{(r^2 + a^2)^3} \tag{3.4.15}$$

has at most three real roots, one of which is in $[r_+, \infty)$ in view of the construction above. Indeed, one other root has to lie in $(-\infty, r_-]$ as

$$\lim_{r\to-\infty} \frac{\mathrm{d}V_{\text{main}}}{\mathrm{d}r} > 0 \text{ and } \frac{\mathrm{d}V_{\text{main}}}{\mathrm{d}r}(r = r_-) = \frac{\partial_r \Delta(r_-)}{(r_-^2 + a^2)^2} < 0. \tag{3.4.16}$$

Thus, V_{main} has at least one and at most two critical points in $[r_+, \infty)$ from which we deduce (3.4.12) and (3.4.13). □

3.4.2 Fundamental pairs of solutions

We will now define various solutions to the radial o.d.e. associated to the boundary and to the turning points. Recall that the turning points define the transition from the trapping region to the semiclassically forbidden region.

3.4.2.1 Solutions associated to the boundary

We first define the associated solution to the radial o.d.e. (3.2.36) which satisfies the Dirichlet boundary conditions at $r^* = \frac{\pi}{2} l$. Throughout Section 3.4.2.1 we consider all frequency parameters ω, m, ℓ.

Definition 3.4.1. *We define the solution u_∞ as the unique solution to* (3.2.36) *satisfying*

$$u_\infty \left(\frac{\pi}{2} l \right) = 0 \tag{3.4.17}$$

$$u_\infty' \left(\frac{\pi}{2} l \right) = 1, \tag{3.4.18}$$

where we recall that $' = \frac{\mathrm{d}}{\mathrm{d}r^*}$.

Definition 3.4.2. *We also define $u_{\mathcal{H}^+}$ and $u_{\mathcal{H}^-}$ as the unique solutions to* (3.2.36) *satisfying*

$$u_{\mathcal{H}^+} = e^{-i(\omega - \omega_+ m) r^*} \text{ as } r^* \to -\infty, \tag{3.4.19}$$

$$u_{\mathcal{H}^-} = e^{i(\omega - \omega_+ m) r^*} \text{ as } r^* \to -\infty. \tag{3.4.20}$$

3.4.2.2 Solutions associated to turning points at resonant frequency

For the solutions associated to the turning points we are only interested in the resonant frequency so we are now considering the radial o.d.e. for $\omega = \omega_- m$. We define solutions associated to the turning point r_1^* and r_2^* as illustrated in Fig. 3.5. In view of Lemma 3.4.2, we define solutions to the radial o.d.e. as follows.

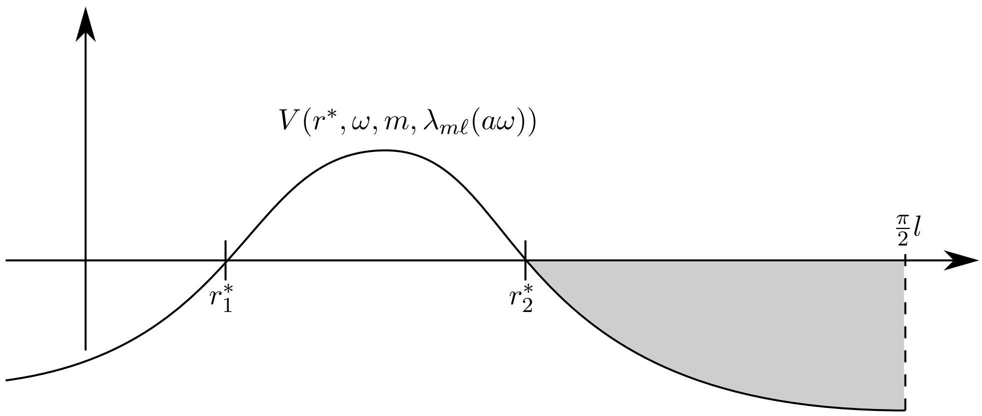

Figure 3.5: Rough shape of the potential and the turning points r_1^* and r_2^*.

Definition 3.4.3. *Assume that $\sigma_1 \leq \lambda_{m\ell} m^{-2} \leq \sigma_2$ and denote the turning points of V_{main} with $r_1^* := r^*(r_1) < r^*(r_2) =: r_2^*$. Then, for some fixed $\epsilon > 0$ sufficiently small only depending on the black hole parameters, define*

$$\xi_1(r^*, m) := \begin{cases} -\left(\frac{3}{2}\int_{r^*}^{r_1^*} |V_{\mathrm{main}}|^{\frac{1}{2}} \mathrm{d}y\right)^{\frac{2}{3}} & r^* \in (-\infty, r_1^*) \\ \left(\frac{3}{2}\int_{r_1^*}^{r^*} V_{\mathrm{main}}^{\frac{1}{2}} \mathrm{d}y\right)^{\frac{2}{3}} & r^* \in [r_1^*, r_2^* - \epsilon] \end{cases} \tag{3.4.21}$$

$$\xi_2(r^*, m) := \begin{cases} \left(\frac{3}{2}\int_{r^*}^{r_2^*} V_{\mathrm{main}}^{\frac{1}{2}} \mathrm{d}y\right)^{\frac{2}{3}} & r^* \in (r_1^* + \epsilon, r_2^*) \\ -\left(\frac{3}{2}\int_{r_2^*}^{r^*} |V_{\mathrm{main}}|^{\frac{1}{2}} \mathrm{d}y\right)^{\frac{2}{3}} & r^* \in [r_2^*, \frac{\pi}{2}l] \end{cases} \tag{3.4.22}$$

$$\hat{f}_1 := \frac{V_{\mathrm{main}}}{\xi_1} \text{ for } r^* \in (-\infty, r_2^* - \epsilon], \tag{3.4.23}$$

$$\hat{f}_2 := \frac{V_{\mathrm{main}}}{\xi_2} \text{for } r^* \in [r_1^* + \epsilon, \frac{\pi}{2}l] \tag{3.4.24}$$

and the error control functions

$$H_1(r^*) := \int_{r_1^*}^{r^*} \left\{ \frac{1}{|V_{\mathrm{main}}|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} \left(\frac{1}{|V_{\mathrm{main}}|^{\frac{1}{4}}} \right) - \frac{V_1}{|V_{\mathrm{main}}|^{\frac{1}{2}}} - \frac{5|V_{\mathrm{main}}|^{\frac{1}{2}}}{16|\xi_1|^3} \right\} \mathrm{d}y, \tag{3.4.25}$$

$$H_2(r^*) := \int_{r_2^*}^{r^*} \left\{ \frac{1}{|V_{\mathrm{main}}|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} \left(\frac{1}{|V_{\mathrm{main}}|^{\frac{1}{4}}} \right) - \frac{V_1}{|V_{\mathrm{main}}|^{\frac{1}{2}}} - \frac{5|V_{\mathrm{main}}|^{\frac{1}{2}}}{16|\xi_2|^3} \right\} \mathrm{d}y. \tag{3.4.26}$$

Lemma 3.4.3. *The error control functions H_1 and H_2 satisfy*

$$\mathcal{V}_{-\infty,r_2^*-\epsilon}(H_1) \lesssim_{\epsilon,\sigma_1,\sigma_2} 1 \tag{3.4.27}$$

$$\mathcal{V}_{r_1^*+\epsilon,l\frac{\pi}{2}}(H_2) \lesssim_{\epsilon,\sigma_1,\sigma_2} 1. \tag{3.4.28}$$

Proof. We will use [119, Chapter 11, Lemma 3.1] to show the bounds on the total variation of H_1 and H_2.

We begin with H_2. From

$$\sigma_1 < \lambda_{m\ell}(a\omega_-m) < \sigma_2 \tag{3.4.29}$$

we have that

$$\frac{V_{\text{main}}}{r^* - r_2^*} \tag{3.4.30}$$

is a positive smooth function on $[r_1 + \epsilon, \frac{\pi}{2}l]$. Moreover, V_1 is a smooth function. Thus, we can apply [119, Chapter 11, Lemma 3.1] and since the interval $[r_1 + \epsilon, \frac{\pi}{2}l]$ is compact, we conclude that (3.4.28) holds true.

For H_1, we have to deal with the unbounded region $r^* \in (-\infty, r_2^* + \epsilon)$. We decompose

$$\mathcal{V}_{-\infty,r_2^*-\epsilon}(H_1) = \mathcal{V}_{-\infty,r_1^*-1}(H_1) + \mathcal{V}_{r_1^*-1,r_2^*-\epsilon}(H_1). \tag{3.4.31}$$

Completely analogous to the proof of the bound on H_2 we have

$$\mathcal{V}_{r_1^*-1,r_2^*-\epsilon}(H_1) \lesssim_{\epsilon,\sigma_1,\sigma_2} 1. \tag{3.4.32}$$

For the term $\mathcal{V}_{-\infty,r_1^*-1}(H_1)$ we remark that

$$-V_{\text{main}} \sim_{\sigma_1,\sigma_2} 1, \;\; |V'_{\text{main}}|, |V''_{\text{main}}| \lesssim_{\sigma_1,\sigma_2} e^{2\kappa_+ r^*} \text{ and } |V_1| \lesssim e^{2\kappa_+ r^*} \tag{3.4.33}$$

for $r^* \in (-\infty, r_1^* - 1)$. Using the lower bound $-V_{\text{main}}$, we infer from (3.4.21) that

$$-\xi_1(r^*) \gtrsim (-r^*)^{\frac{2}{3}} \tag{3.4.34}$$

for $r^* \in (-\infty, r_1^* - 1)$. Hence,

$$\begin{aligned}
\mathcal{V}_{-\infty, r_1^*-1}(H_1) &\lesssim_{\sigma_1,\sigma_2} \int_{-\infty}^{r_1^*-1} \left| \frac{V''_{\mathrm{main}}}{|V_{\mathrm{main}}|^{\frac{3}{2}}} \right| + \left| \frac{V'^2_{\mathrm{main}}}{|V_{\mathrm{main}}|^{\frac{5}{2}}} \right| + \left| \frac{V_1}{|V_{\mathrm{main}}|^{\frac{1}{2}}} \right| + \frac{|V_{\mathrm{main}}|^{\frac{1}{2}}}{|\xi_1|^3} \mathrm{d}r^* \\
&\lesssim_{\sigma_1,\sigma_2} \int_{-\infty}^{r_1^*-1} e^{2\kappa_+ r^*} + \frac{1}{|r^*|^2} \mathrm{d}r^* \lesssim_{\sigma_1,\sigma_2} 1. \qquad (3.4.35)
\end{aligned}$$

□

With the bounds in Lemma 3.4.3 at hand, we apply [119, Chapter 11, Theorem 3.1] which allow us to define the following. In particular, the error bounds (3.4.40)–(3.4.46) hold due to Lemma 3.4.3.

Definition 3.4.4. *We define solutions to the radial o.d.e.* (3.2.36) *for* $\omega = \omega_- m$ *as*

$$u_{\mathrm{Ai1}}(r^*, m) := \hat{f}_1^{\frac{1}{4}}(r_1^*) \hat{f}_1^{-\frac{1}{4}}(r^*) \left\{ \mathrm{Ai}(m^{\frac{2}{3}} \xi_1) + \epsilon_{\mathrm{Ai1}}(m, r^*) \right\} \text{ for } r^* \in (-\infty, r_2^* - \epsilon], \qquad (3.4.36)$$

$$u_{\mathrm{Bi1}}(r^*, m) := \hat{f}_1^{\frac{1}{4}}(r_1^*) \hat{f}_1^{-\frac{1}{4}}(r^*) \left\{ \mathrm{Bi}(m^{\frac{2}{3}} \xi_1) + \epsilon_{\mathrm{Bi1}}(m, r^*) \right\} \text{ for } r^* \in (-\infty, r_2^* - \epsilon], \qquad (3.4.37)$$

$$u_{\mathrm{Ai2}}(r^*, m) := \hat{f}_2^{\frac{1}{4}}(r_2^*) \hat{f}_2^{-\frac{1}{4}}(r^*) \left\{ \mathrm{Ai}(m^{\frac{2}{3}} \xi_2) + \epsilon_{\mathrm{Ai2}}(m, r^*) \right\} \text{ for } r^* \in [r_1^* + \epsilon, \frac{\pi}{2} l], \qquad (3.4.38)$$

$$u_{\mathrm{Bi2}}(r^*, m) := \hat{f}_2^{\frac{1}{4}}(r_2^*) \hat{f}_2^{-\frac{1}{4}}(r^*) \left\{ \mathrm{Bi}(m^{\frac{2}{3}} \xi_2) + \epsilon_{\mathrm{Bi2}}(m, r^*) \right\} \text{ for } r^* \in [r_1^* + \epsilon, \frac{\pi}{2} l]. \qquad (3.4.39)$$

Moreover,

$$\begin{aligned}
|\epsilon_{\mathrm{Ai1}}| &\lesssim_\epsilon M_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_1) E_{\mathrm{Ai}}^{-1}(m^{\frac{2}{3}} \xi_1) m^{-1}, && (3.4.40) \\
|\epsilon'_{\mathrm{Ai1}}| &\lesssim_\epsilon \hat{f}_1^{\frac{1}{2}} N_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_1) E_{\mathrm{Ai}}^{-1}(m^{\frac{2}{3}} \xi_1) m^{-\frac{1}{3}}, && (3.4.41) \\
|\epsilon_{\mathrm{Bi1}}| &\lesssim_\epsilon M_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_1) E_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_1) m^{-1}, && (3.4.42) \\
|\epsilon'_{\mathrm{Bi1}}| &\lesssim_\epsilon \hat{f}_1^{\frac{1}{2}} N_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_1) E_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_1) m^{-\frac{1}{3}}, && (3.4.43) \\
|\epsilon_{\mathrm{Ai2}}| &\lesssim_\epsilon M_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_2) E_{\mathrm{Ai}}^{-1}(m^{\frac{2}{3}} \xi_2) m^{-1}, && (3.4.44) \\
|\epsilon'_{\mathrm{Ai2}}| &\lesssim_\epsilon \hat{f}_2^{\frac{1}{2}} N_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_2) E_{\mathrm{Ai}}^{-1}(m^{\frac{2}{3}} \xi_2) m^{-\frac{1}{3}}, && (3.4.45) \\
|\epsilon_{\mathrm{Bi2}}| &\lesssim_\epsilon M_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_2) E_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_2) m^{-1}, && (3.4.46) \\
|\epsilon'_{\mathrm{Bi2}}| &\lesssim_\epsilon \hat{f}_2^{\frac{1}{2}} N_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_2) E_{\mathrm{Ai}}(m^{\frac{2}{3}} \xi_2) m^{-\frac{1}{3}}, && (3.4.47)
\end{aligned}$$

where (3.4.40)–(3.4.43) *hold uniformly in* $r^* \in (-\infty, r_2^* - \epsilon]$ *and* (3.4.44)–(3.4.47) *hold* *uniformly in* $r^* \in [r_1^* + \epsilon, \frac{\pi}{2} l]$. *Finally, we choose the initialization such that*

$$|\epsilon_{\mathrm{Ai2}}(r^*)| \lesssim_\epsilon M_{\mathrm{Ai}}(m^{\frac{2}{3}}\xi_2) E_{\mathrm{Ai}}^{-1}(m^{\frac{2}{3}}\xi_2) \left(\exp\left[2\mathcal{V}_{r_1^*+\epsilon, r^*}(H_2) m^{-1}\right] - 1\right), \tag{3.4.48}$$

$$|\epsilon_{\mathrm{Bi2}}(r^*)| \lesssim_\epsilon M_{\mathrm{Ai}}(m^{\frac{2}{3}}\xi_2) E_{\mathrm{Ai}}(m^{\frac{2}{3}}\xi_2) \left(\exp\left[2\mathcal{V}_{r^*, l\frac{\pi}{2}}(H_2) m^{-1}\right] - 1\right) \tag{3.4.49}$$

and in particular, $|\epsilon_{\mathrm{Ai2}}(\frac{\pi}{2} l)| \lesssim_\epsilon m^{-\frac{7}{6}}, \epsilon_{\mathrm{Bi2}}(\frac{\pi}{2} l) = 0$.

3.5 Definition and properties of $\mathscr{P}_{\text{Blow-up}}$

With the fundamental solutions from Section 3.4.2, we are now in the position to define the set of black hole parameters $\mathscr{P}_{\text{Blow-up}}$.

3.5.1 Definition of $\mathscr{P}_{\text{Blow-up}}$

We first define Wronskians of solutions of the radial o.d.e. which will play a fundamental role in the estimates.

Definition 3.5.1. *We define* $\mathfrak{W}_1 \colon \mathscr{P} \times \mathbb{Z}_m \times \mathbb{Z}_{\ell \geq |m|} \to \mathbb{C}$ *and* $\mathfrak{W}_2 \colon \mathscr{P} \times \mathbb{Z}_m \times \mathbb{Z}_{\ell \geq |m|} \to \mathbb{C}$ *as*

$$\mathfrak{W}_1(\mathfrak{p}, m, \ell) := \mathfrak{W}[u_{\mathcal{H}^+}, u_\infty](m, \ell, \omega = \omega_- m, \mathfrak{p}), \tag{3.5.1}$$

$$\mathfrak{W}_2(\mathfrak{p}, m, \ell) := \mathfrak{W}[u_{\mathrm{Ai2}}, u_\infty](m, \ell, \omega = \omega_- m, \mathfrak{p}). \tag{3.5.2}$$

Note that this is well-defined as the Wronskians $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *only depend on* $\mathscr{P}$ *(by construction). Moreover, they are manifestly continuous functions on* $\mathscr{P}$ *for fixed* m, ℓ.

Remark 3.5.1. *Note that the Wronskian* $\mathfrak{W}_1$ *does not vanish as discussed in the introduction. Nevertheless,* $\mathfrak{W}_1$ *can be very small (as* $m \to \infty$*) which corresponds to frequency parameters associated to stable trapping. On the other hand,* $\mathfrak{W}_2$ *may vanish and this indeed corresponds to stable trapping. In particular, if* $\mathfrak{W}_2$ *vanishes, then the solution* u_∞ *is a multiple of the* u_{Ai2} *which is exponentially damped in the semi-classical forbidden region. In this case, we infer that* $\mathfrak{W}_1$ *is exponentially small and indeed, we are in the situation of stable trapping. This intuition leads to the definition of* $\mathscr{P}_{\text{Blow-up}}$ *in Definition 3.5.2 and Definition 3.5.3 below.*

Before we define the set $\mathscr{P}_{\text{Blow-up}}$, we parametrize the level sets $\{\mathfrak{a} = \mathit{const.}\} \subset \mathscr{P}$ by

$$\vartheta = \vartheta(\mathfrak{p}) := a\omega_-(\mathfrak{p}).$$

That is, for each $\mathfrak{a}_0$, there exists an interval $(\vartheta_1, \vartheta_2)$ and a real-analytic embedding $\gamma_{\mathfrak{a}_0} \colon (\vartheta_1, \vartheta_2) \to \mathscr{P}, \gamma_{\mathfrak{a}_0}(\vartheta) = (\mathfrak{m}(\vartheta), \mathfrak{a}_0)$ with $\gamma_{\mathfrak{a}_0}((\vartheta_1, \vartheta_2)) = \{\mathfrak{a} = \mathfrak{a}_0\} \cap \mathscr{P}$. This embedding depends smoothly on $\mathfrak{a}_0$ and thus, the vector field defined by

$$\Gamma(\mathfrak{p}) := \dot{\gamma}_{\mathfrak{a}(\mathfrak{p})} \tag{3.5.3}$$

is smooth and tangential to the level sets $\{\mathfrak{a} = const.\}$. The flow generated by Γ will be denoted by Φ^Γ_τ.

Definition 3.5.2. *For $m_0 \in \mathbb{N}$ we define*

$$U_{m_0} := \bigcup_{\substack{m \geq m_0 \\ m \in \mathbb{N}}} \bigcup_{\substack{m \leq \ell \leq m^2 \\ \ell \in \mathbb{N}}} U(m, \ell), \tag{3.5.4}$$

where

$$\begin{aligned} U(m,\ell) := \Big\{ \mathfrak{p} \in \mathscr{P} \colon |\mathfrak{W}_1| < e^{-\sqrt{m}}, \sigma_1(\mathfrak{p}) < \lambda_{m\ell}(a\omega_-(\mathfrak{p})m) < \sigma_2(\mathfrak{p}), |\mathfrak{W}_2| < e^{-\ell}e^{-m}, \\ |\Gamma\mathfrak{W}_2| > 1, |\mathfrak{W}_2(\Phi^\Gamma_\tau(\mathfrak{p}))| > e^{-\ell}e^{-m} \text{ for all } |\tau| \in [e^{-\ell}e^{-m}, \frac{1}{m^2}] \Big\}. \end{aligned} \tag{3.5.5}$$

Definition 3.5.3. *We define*

$$\mathscr{P}_{\text{Blow-up}} := \bigcap_{m_0 \in \mathbb{N}} U_{m_0} \tag{3.5.6}$$

While a priori the set $\mathscr{P}_{\text{Blow-up}}$ could be empty, we will show in the following that it is *dense* in $\mathscr{P}$ and *Baire-generic*, i.e. a countable intersection of open and dense sets.

3.5.2 Topological genericity: $\mathscr{P}_{\text{Blow-up}}$ is Baire-generic

We will first show that each U_{m_0} is dense. To do so, we let m_0 and $\mathfrak{p}_0 = (\mathfrak{m}_0, \mathfrak{a}_0) \in \mathscr{P}$ be arbitrary and fixed. Also, let $\mathcal{U} \subset \mathscr{P}$ be an open neighborhood of $\mathfrak{p}_0$. We will show that there exists an element of U_{m_0} which is contained in $\mathcal{U}$. We now define a curve of parameters through $\mathfrak{p}_0$ as follows.

Definition 3.5.4. *For $\delta = \delta(\mathfrak{p}_0, \mathcal{U}) > 0$ sufficiently small, we define the real-analytic embedded curve $\gamma_\delta(\mathfrak{p}_0) \subset \mathcal{U}$ through $\mathfrak{p}_0$ as*

$$\gamma_\delta(\mathfrak{p}_0) := \{\mathfrak{p} = (\mathfrak{m}, \mathfrak{a}) \in \mathscr{P} \colon \mathfrak{a} = \mathfrak{a}_0, |\vartheta(\mathfrak{p}) - \vartheta(\mathfrak{p}_0)| \leq \delta\}. \tag{3.5.7}$$

Throughout Section 3.5.2 we will only consider

$$\mathfrak{p} \in \gamma_\delta(p_0). \tag{3.5.8}$$

We parameterize $\gamma_\delta(\mathfrak{p}_0)$ with $\vartheta \in (\vartheta_0 - \delta, \vartheta_0 + \delta)$, where $\vartheta_0 = \vartheta(\mathfrak{p}_0)$.

Remark 3.5.2. *Note that the expression Ξ is (by construction) constant on $\gamma_\delta(p_0)$. A direct computation shows that the eigenvalues satisfy $\Gamma(\lambda_{m\ell}(a\omega_- m)+a^2\omega_-^2 m^2-2a\omega_- m^2\Xi) \in (-2m^2, 0)$.*

Remark 3.5.3. *From Proposition 3.3.2 and Remark 3.5.2 we have that for almost every $\tilde{\lambda}_0 > \Xi^2$, there exist sequences $(m_i)_{i\in\mathbb{N}}$ and $(\ell_i)_{i\in\mathbb{N}}$ ($m_i \leq \ell_i \leq m_i^2$) with $m_i \to \infty, \ell_i \to \infty$ as $i \to \infty$ such that the angular eigenvalues satisfy*

$$\lambda_i = \lambda_{m_i\ell_i}(\omega = \omega_- m) = \tilde{\lambda}_i m^2 = \tilde{\lambda}_0 m_i^2 + \lambda_{\text{error}}^{(i)}, \tag{3.5.9}$$

where $|\lambda_{\text{error}}^{(i)}| \lesssim_{\tilde{\lambda}_0, \mathfrak{p}_0} 1$ at $\mathfrak{p}_0$ and $|\lambda_{\text{error}}^{(i)}| \lesssim_{\tilde{\lambda}_0, \mathfrak{p}_0} 1 + |\vartheta_0 - \vartheta| m_i^2 \lesssim 1 + \delta m_i^2$ uniformly on $\gamma_\delta(\mathfrak{p}_0)$ as $m_i \to \infty$. Moreover, we assume without loss of generality that $m_{i+1} > m_i$ and note that the choice of subsequence m_i, ℓ_i depends on $\mathfrak{p}_0$.

Lemma 3.5.1. *Let $\lambda_1 := \sup_{\mathfrak{p}\in\gamma_\delta(\mathfrak{p}_0)} \sigma_1(\mathfrak{p})$ and $\lambda_2 := \inf_{\mathfrak{p}\in\gamma_\delta(\mathfrak{p}_0)} \sigma_2(\mathfrak{p})$ and choose $\delta > 0$ potentially smaller such that $\lambda_1 < \lambda_2$. Then, there exists a $\tilde{\lambda}_0 \in (\lambda_1, \lambda_2) \setminus \mathcal{N}_{\mathfrak{p}_0}$ (see Remark 3.5.3) with the following properties.*

Let $\tilde{\lambda}_i = \tilde{\lambda}_0 + \lambda_{\text{error}}^{(i)} m^{-2}$ be the associated angular eigenvalues from Proposition 3.3.2. Then, for all $\mathfrak{p} \in \gamma_\delta(\mathfrak{p}_0)$, and for all $i \in \mathbb{N}$ sufficiently large, we have

$$|\Gamma \xi_\infty| \geq c(\delta, \mathfrak{p}_0), \tag{3.5.10}$$

where $\xi_\infty : \gamma_\delta(\mathfrak{p}_0) \to \mathbb{R}$ is defined as

$$\xi_\infty(\vartheta) := \int_{r_2^*}^{\frac{\pi}{2}l} \sqrt{|V_{\text{main}}|}\mathrm{d}r^* = \int_{r_2}^{\infty} \sqrt{-\frac{(r^2+a^2)^2}{\Delta^2} V_{\text{main}}}\mathrm{d}r \tag{3.5.11}$$

and $c(\delta, \mathfrak{p}_0) > 0$ only depends on $\delta > 0$ and $\mathfrak{p}_0$.

Proof. Note that to highest order in r/l, the function $\Gamma\left(\frac{(r^2+a^2)^2}{l^2\Delta^2} V_{\text{main}}\right)$ has a sign. Thus, by choosing $\delta > 0$ sufficiently small and $r_2/l = r_2/l(\mathfrak{p}_0, \tilde{\lambda}_i)$ sufficiently large (choose $\tilde{\lambda}_i \in (\lambda_1, \lambda_2) \setminus \mathcal{N}_{\mathfrak{p}_0}$ sufficiently close to σ_2 and potentially increase σ_2), we have that for all $i \in \mathbb{N}$, $|\Gamma\xi_\infty| > c(\delta, \mathfrak{p}_0)$ for all parameters in $\gamma_\delta(\mathfrak{p}_0)$. $\square$

Recall the definition of $\mathfrak{W}_1$ and $\mathfrak{W}_2$ from Definition 3.5.1.

Proposition 3.5.1. *Let $m_0 \in \mathbb{N}$. Then, there exist a parameters*

$$\mathfrak{p}_{\text{Blow-up}} \in \gamma_\delta(\mathfrak{p}_0) \subset \mathcal{U}$$

and an $i \in \mathbb{N}$ such that $m_0 \leq m_i \leq \ell_i \leq m_i^2$ with

$$\sigma_1(\mathfrak{p}_{\text{Blow-up}}) < \lambda_{m_i \ell_i} < \sigma_2(\mathfrak{p}_{\text{Blow-up}}) \tag{3.5.12}$$

and

$$|\mathfrak{W}_1(\vartheta_{\text{Blow-up}})| < e^{-\sqrt{m}_i} \tag{3.5.13}$$

as well as

$$\begin{aligned} &|\mathfrak{W}_2(\vartheta_{\text{Blow-up}})| = 0 \text{ and } |\Gamma\mathfrak{W}_2(\vartheta_{\text{Blow-up}})| > 1, \\ &|\mathfrak{W}_2(\vartheta)| > e^{-\ell_i} e^{-m_i} \text{ for all } e^{-\ell_i} e^{-m_i} < |\vartheta_{\text{Blow-up}} - \vartheta| < \frac{1}{m_i^2}. \end{aligned} \tag{3.5.14}$$

The proof of Proposition 3.5.1 relies on the following two lemmata and will given thereafter. First, we will start by showing that for every $m_i \geq m_0$ large enough, there exists a $\mathfrak{p}_{\text{Blow-up}} \in \gamma_\delta(\mathfrak{p}_0)$ such that $\mathfrak{W}_2 = 0$ and $|\Gamma\mathfrak{W}_2| > 1$. We will state this as the following Lemma.

Lemma 3.5.2. *For every $\tilde{m}_0 > 0$ there exists an $i \in \mathbb{N}$ with $m_i > \tilde{m}_0$ and a parameter $\vartheta_{\text{Blow-up}}$ with $|\vartheta_{\text{Blow-up}} - \vartheta(\mathfrak{p}_0)| \leq \delta$ such that*

1. $\mathfrak{W}_2(\vartheta_{\text{Blow-up}}, m_i) = 0$,

2. $u_{\text{Ai2}} = \alpha_\infty \hat{f}_2^{\frac{1}{2}}(\frac{\pi}{2} l) u_\infty$ *for* $\vartheta = \vartheta_{\text{Blow-up}}$ *with* $|\alpha_\infty(\vartheta_{\text{Blow-up}})| \sim m_i^{\frac{5}{6}}$,

3. $|\Gamma\mathfrak{W}_2(\vartheta_{\text{Blow-up}}, m_i)| > 1$,

4. *For all ϑ with $e^{-\ell_i} e^{-m_i} < |\vartheta - \vartheta_{\text{Blow-up}}| < \frac{1}{m_i^2}$, we have $|\mathfrak{W}_2(\vartheta)| > e^{-\ell_i} e^{-m_i}$.*

Proof. Throughout the proof we use the convention that all constants appearing in $\lesssim$, $\gtrsim$, $\sim$ and O only depend on $\mathfrak{p}_0, l$ and $\delta > 0$.

Let $\tilde{m}_0 > 0$. We begin by showing *1.* From Lemma 3.4.3 and (3.5.11) we have

$$\begin{aligned}\mathfrak{W}[u_{\mathrm{Ai}2}, u_\infty](m_i, \vartheta) = u_{\mathrm{Ai}2}(r^* = l\pi/2, m_i) &= \hat{f}_2^{\frac{1}{4}}(r_2^*)\hat{f}_2^{-\frac{1}{4}}(\pi/2)\left\{\mathrm{Ai}(m_i^{\frac{2}{3}}\xi_2(\pi/2)) + \epsilon_{\mathrm{Ai}2}(m_i, l\pi/2)\right\} \\ &= \hat{f}_2^{\frac{1}{4}}(r_2^*)\hat{f}_2^{-\frac{1}{4}}(\pi/2)\left\{\mathrm{Ai}\left(-\left(\frac{3}{2}m_i\xi_\infty\right)^{\frac{2}{3}}\right) + O(m_i^{-\frac{7}{6}})\right\}\end{aligned} \tag{3.5.15}$$

on $\gamma_\delta(\mathfrak{p}_0)$. Now, for all $m_i > \tilde{m}_0$ sufficiently large, we use the asymptotics for the Airy functions as shown in Lemma 3.9.1 to conclude that

$$\mathrm{Ai}\left(-\left(\frac{3}{2}m_i\xi_\infty\right)^{\frac{2}{3}}\right) + O(m_i^{-\frac{7}{6}}) = \frac{1}{\sqrt{\pi}}\left(\frac{3}{2}m_i\xi_\infty\right)^{-\frac{1}{6}}\left(\cos\left(m_i\xi_\infty - \frac{\pi}{4}\right) + O(m_i^{-1})\right). \tag{3.5.16}$$

Thus, in order to conclude that $\mathfrak{W}[u_{\mathrm{Ai}2}, u_\infty](m_i, \vartheta) = 0$ for some value on $\gamma_\delta(\mathfrak{p}_0)$, we have to vary $\mathfrak{p}(\vartheta) \in \gamma_\delta(\mathfrak{p}_0)$ such that $m_i\xi_\infty$ goes through a period of 2π. Thus, it suffices to let ξ_∞ go through a period of $2\pi m_i^{-1}$. From (3.5.10) we have

$$|\Gamma\xi_\infty| \geq c(\delta, \mathfrak{p}_0) > 0 \tag{3.5.17}$$

on $\gamma_\delta(\mathfrak{p}_0)$. Thus, by potentially choosing $m_i > \tilde{m}_0$ even larger, there exists a parameter $\vartheta_{\text{Blow-up}}$ with

$$|\vartheta_{\text{Blow-up}} - \vartheta(\mathfrak{p}_0)| \lesssim \frac{1}{m_i} \tag{3.5.18}$$

such that

$$\mathfrak{W}_2(\vartheta_{\text{Blow-up}}, m_i) = 0 \text{ and } \mathfrak{p}(\vartheta_{\text{Blow-up}}) \in \gamma_\delta(\mathfrak{p}_0). \tag{3.5.19}$$

Having found m_i and $\vartheta_{\text{Blow-up}}$, we will now prove *2.* We again use Lemma 3.9.1, (3.4.45) and an analogous computation as in (3.5.15) to conclude that for $\vartheta = \vartheta_{\text{Blow-up}}$ we have

$$|u'_{\mathrm{Ai}2}(r^* = l\pi/2, m_i)| \sim \hat{f}_2^{\frac{1}{2}}(l\pi/2)m_i^{\frac{5}{6}}. \tag{3.5.20}$$

Thus, for $\vartheta = \vartheta_{\text{Blow-up}}$ we have

$$u_{\text{Ai2}} = \hat{f}_2^{\frac{1}{2}}(l\pi/2)\alpha_\infty u_\infty \text{ with } |\alpha_\infty| \sim m_i^{\frac{5}{6}}. \tag{3.5.21}$$

To show *3*, we recall that

$$|\Gamma\xi_\infty| \geq c(\delta, \mathfrak{p}_0) \tag{3.5.22}$$

on $\gamma_\delta(\mathfrak{p}_0)$ in view of (3.5.10). Now, we take the derivative of (3.5.15) with respect to Γ. Since $|l^2\Gamma V_{\text{main}}| \lesssim 1$ and $|l^2\Gamma V_1| \lesssim 1$, we have that $\Gamma O(m_i^{-\frac{7}{6}}) = O(m_i^{-\frac{1}{6}})$ in (3.5.15). Note that this follows from [119, Chapter 11, Proof of Theorem 3.1]. Thus, for $\vartheta = \vartheta_{\text{Blow-up}}$ we infer

$$|\Gamma\mathfrak{W}_2| \sim m_i^{\frac{5}{6}} \tag{3.5.23}$$

which shows *3*. We also have the estimate (3.5.23) for all ϑ with $|\vartheta - \vartheta_{\text{Blow-up}}| < m_i^{-2}$ for m_i large enough which upon integration shows *4*. $\square$

Lemma 3.5.3. *There exists a constant $c > 0$ (only depending on $\mathfrak{p}_0$ and $\delta > 0$) such that for $\mathfrak{p} \in \gamma_\delta(\mathfrak{p}_0)$ we have*

$$|\mathfrak{W}(u_{\text{Ai2}}, u_{\text{Bi1}})| \lesssim \hat{f}_1^{\frac{1}{2}}(r_1^*)e^{-cm_i} \text{ and } |\mathfrak{W}(u_{\text{Ai2}}, u_{\text{Ai1}})| \lesssim \hat{f}_1^{\frac{1}{2}}(r_1^*)e^{-cm_i} \tag{3.5.24}$$

for all m_i sufficiently large. Moreover, there exist constants $\alpha_1 = \alpha_1(m_i) \in \mathbb{R}$ and $\beta_1 = \beta_1(m_i) \in \mathbb{R}$ satisfying $|\alpha_1| \lesssim e^{-cm_i}$ and $|\beta_1| \lesssim e^{-cm_i}$ such that $u_{\text{Ai2}} = \alpha_1 u_{\text{Ai1}} + \beta_1 u_{\text{Bi1}}$.

Proof. We start by proving $|\mathfrak{W}(u_{\text{Ai2}}, u_{\text{Bi1}})| \lesssim \hat{f}_1^{\frac{1}{2}}(r_1^*)e^{-cm_i}$. Choose $\epsilon > 0$ sufficiently small but fixed and evaluate the Wronskian at $r^* := r_1^* + \epsilon$. Then, using standard bounds on Airy functions from Lemma 3.9.1 we obtain

$$|u_{\text{Ai2}}(r_1^* + \epsilon)| \lesssim_\epsilon \frac{1}{m_i^{\frac{1}{6}}\xi_2^{\frac{1}{4}}(r_1^* + \epsilon)} e^{-\frac{2}{3}m_i\xi_2^{\frac{3}{2}}(r_1^*+\epsilon)}, \tag{3.5.25}$$

$$|u'_{\text{Ai2}}(r_1^* + \epsilon)| \lesssim_\epsilon m_i^{\frac{1}{6}}\xi_2^{\frac{1}{4}}(r_1^* + \epsilon)\hat{f}_2^{\frac{1}{2}}(r_2^*)e^{-\frac{2}{3}m_i\xi_2^{\frac{3}{2}}(r_1^*+\epsilon)}, \tag{3.5.26}$$

$$|u_{\text{Bi1}}(r_1^* + \epsilon)| \lesssim_\epsilon \frac{1}{m_i^{\frac{1}{6}}\xi_1^{\frac{1}{4}}(r_1^* + \epsilon)} e^{\frac{2}{3}m_i\xi_1^{\frac{3}{2}}(r_1^*+\epsilon)}, \tag{3.5.27}$$

$$|u'_{\text{Bi1}}(r_1^* + \epsilon)| \lesssim_\epsilon m_i^{\frac{1}{6}}\xi_1^{\frac{1}{4}}(r_1^* + \epsilon)\hat{f}_1^{\frac{1}{2}}(r_1^*)e^{\frac{2}{3}m_i\xi_1^{\frac{3}{2}}(r_1^*+\epsilon)}. \tag{3.5.28}$$

Now, choosing $\epsilon > 0$ sufficiently small only depending on $\mathfrak{p}_0$ and δ, there exists a constant $c > 0$ such that

$$\xi_2^{\frac{3}{2}}(r_1^* + \epsilon) - \xi_1^{\frac{3}{2}}(r_1^* + \epsilon) > c \tag{3.5.29}$$

from which the first estimate follows by evaluating the Wronskian $\mathfrak{W}(u_{\text{Ai2}}, u_{\text{Bi1}})$ at $r^* = r_1^* + \epsilon$ and the fact that $\hat{f}_1(r_1^*)/\hat{f}_2(r_2^*) \sim 1$. The second estimate of (3.5.24) follows in the same manner but it is easier as u_{Ai2} is already exponentially small in the region between the turning points r_1^* and r_2^* since

$$|u_{\text{Ai1}}(r_1^* + \epsilon)| \lesssim_\epsilon \frac{1}{m_i^{\frac{1}{6}} \xi_1^{\frac{1}{4}}(r_1^* + \epsilon)} e^{-\frac{2}{3} m_i \xi_1^{\frac{3}{2}}(r_1^* + \epsilon)}, \tag{3.5.30}$$

$$|u'_{\text{Ai1}}(r_1^* + \epsilon)| \lesssim_\epsilon m_i^{\frac{1}{6}} \xi_1^{\frac{1}{4}}(r_1^* + \epsilon) \hat{f}_1^{\frac{1}{2}}(r_1^*) e^{-\frac{2}{3} m_i \xi_1^{\frac{3}{2}}(r_1^* + \epsilon)}. \tag{3.5.31}$$

For the second part we first note that

$$\alpha_1 = \frac{\mathfrak{W}(u_{\text{Ai2}}, u_{\text{Bi1}})}{\mathfrak{W}(u_{\text{Ai1}}, u_{\text{Bi1}})}, \beta_1 = \frac{\mathfrak{W}(u_{\text{Ai2}}, u_{\text{Ai1}})}{\mathfrak{W}(u_{\text{Bi1}}, u_{\text{Ai1}})}. \tag{3.5.32}$$

To conclude it suffices to show that

$$\mathfrak{W}(u_{\text{Ai1}}, u_{\text{Bi1}}) \sim \hat{f}_1^{\frac{1}{2}}(r_1^*) m_i^{\frac{2}{3}}. \tag{3.5.33}$$

In view of the error bounds from (3.4.40)–(3.4.43) and the chain rule, we infer that

$$|\mathfrak{W}_{r^*}(u_{\text{Ai1}}, u_{\text{Bi1}})| \sim \hat{f}_1^{\frac{1}{2}}(r_1^*) m_i^{\frac{2}{3}} \mathfrak{W}_x(\text{Ai}(x), \text{Bi}(x)) \sim \hat{f}_1^{\frac{1}{2}}(r_1^*) m_i^{\frac{2}{3}} \tag{3.5.34}$$

for all m_i sufficiently large. $\square$

Now, we are in the position to prove Proposition 3.5.1.

Proof of Proposition 3.5.1. Let $m_0 \in \mathbb{N}$ be arbitrary. Using Lemma 3.5.2, we let $m_i > m_0$ and fix $\mathfrak{p}_{\text{Blow-up}} \in \gamma_\delta(p_0) \subset \mathcal{U}$ such that $\mathfrak{W}_2 = 0$ and $|\Gamma \mathfrak{W}_2| > 1$ as well as $|\mathfrak{W}_2(\vartheta)| > e^{-\ell_i} e^{-m_i}$ for $e^{-\ell_i} e^{-m_i} < |\vartheta - \vartheta_{\text{Blow-up}}| < \frac{1}{m_i^2}$. We moreover have

$$u_\infty = \alpha_\infty^{-1} \hat{f}_2^{-\frac{1}{2}}(l\pi/2) u_{\text{Ai2}} = \alpha_\infty^{-1} \hat{f}_2^{-\frac{1}{2}}(l\pi/2) \left(\alpha_1 u_{\text{Ai1}} + \beta_1 u_{\text{Bi1}}\right), \tag{3.5.35}$$

where $|\alpha_\infty| \sim m_i^{\frac{5}{6}}$. Thus, in view of Lemma 3.5.3 we have

$$\begin{aligned} |\mathfrak{W}[u_\infty, u_{\mathcal{H}^+}]| &= |\alpha_\infty^{-1} \hat{f}_2^{-\frac{1}{2}}(l\pi/2)\left(\alpha_1 \mathfrak{W}[u_{\mathrm{Ai1}}, u_{\mathcal{H}^+}] + \beta_1 \mathfrak{W}[u_{\mathrm{Bi1}}, u_{\mathcal{H}^+}]\right)| \\ &\lesssim \hat{f}_2^{-\frac{1}{2}}(l\pi/2) m_i^{-\frac{5}{6}} e^{-cm_i}\left(|\mathfrak{W}[u_{\mathrm{Ai1}}, u_{\mathcal{H}^+}]| + |\mathfrak{W}[u_{\mathrm{Bi1}}, u_{\mathcal{H}^+}]|\right) \end{aligned} \tag{3.5.36}$$

for some constant $c = c(\mathfrak{p}_0) > 0$. To estimate $\mathfrak{W}[u_{\mathrm{Ai1}}, u_{\mathcal{H}^+}]$ and $\mathfrak{W}[u_{\mathrm{Bi1}}, u_{\mathcal{H}^+}]$ we infer from Lemma 3.9.1 and (3.4.36), (3.4.37) together with the associated error bounds, that

$$|u_{\mathrm{Ai1}}| \lesssim m_i^{-\frac{1}{6}}, \quad |u_{\mathrm{Ai1}}'| \lesssim \hat{f}_1^{\frac{1}{2}}(r_1^*) m_i^{\frac{5}{6}}, \quad |u_{\mathrm{Bi1}}| \lesssim m_i^{-\frac{1}{6}}, \quad |u_{\mathrm{Bi1}}'| \lesssim \hat{f}_1^{\frac{1}{2}}(r_1^*) m_i^{\frac{5}{6}} \tag{3.5.37}$$

for all r^* sufficiently small and particularly as $r^* \to -\infty$. Moreover, as $r^* \to -\infty$, we have that

$$u_{\mathcal{H}^+} \sim e^{-i(\omega_- - \omega_+)mr^*} \tag{3.5.38}$$

such that

$$|\mathfrak{W}[u_{\mathrm{Ai1}}, u_{\mathcal{H}^+}]|, |\mathfrak{W}[u_{\mathrm{Bi1}}, u_{\mathcal{H}^+}]| \lesssim \hat{f}_1^{\frac{1}{2}}(r_1^*) m_i^{\frac{5}{6}}. \tag{3.5.39}$$

Thus, by potentially choosing m_i even larger (i.e. choose $\tilde{m}_0$ larger in Lemma 3.5.2) and noting that $\hat{f}_2^{\frac{1}{2}}(\frac{\pi}{2}l) \sim (\omega_- - \omega_+) \sim \hat{f}_1^{\frac{1}{2}}(r_1^*)$, we have

$$|\mathfrak{W}[u_\infty, u_{\mathcal{H}^+}]| \lesssim m_i^{-\frac{5}{6}} m_i^{\frac{5}{6}} e^{-cm_i} = e^{-cm_i} < e^{-\sqrt{m_i}}. \tag{3.5.40}$$

□

Now, we can conclude

Proposition 3.5.2. *The set $\mathscr{P}_{\text{Blow-up}}$ is a Baire-generic subset of $\mathscr{P}$.*

Proof. Since $\mathfrak{p}_0 \in \mathscr{P}$ and $\mathcal{U} \subset \mathscr{P}$, $\mathcal{U} \ni \mathfrak{p}_0$ were arbitrary, Proposition 3.5.1 shows that for any $m_0 \in \mathbb{N}$ large enough, the set U_{m_0} as defined in Definition 3.5.2 is dense in $\mathscr{P}$. Since $\mathfrak{W}_1$, $\mathfrak{W}_2$, σ_1 and σ_2 are continuous, U_{m_0} is manifestly open. Thus, in view of Baire's theorem, $\mathscr{P}_{\text{Blow-up}}$ is Baire-generic and in particular dense. □

3.5.3 Metric genericity: $\mathscr{P}_{\text{Blow-up}}$ is Lebesgue-exceptional and 2-packing dimensional

Proposition 3.5.3. *The set $\mathscr{P}_{\text{Blow-up}}$ is a Lebesgue null set.*

Proof. It suffices to show that $\mathscr{P}_{\text{Blow-up}} \cap C$ has vanishing Lebesgue measure (denoted by $|\cdot|$) for any closed square C contained in $\mathscr{P}$ with side length less than unity. Throughout the proof, all constants appearing in $\lesssim, \gtrsim, \sim$ and O will only depend on the cube C. We start by estimating $U(m,\ell) \cap C$ with the co-area formula: We have

$$|U(m,\ell) \cap C| = \int_{\tilde{\mathfrak{a}} \in (\mathfrak{a}_1, \mathfrak{a}_2)} H^1(U(m,\ell) \cap C \cap \{\mathfrak{a} = \tilde{\mathfrak{a}}\}) \mathrm{d}\tilde{\mathfrak{a}}. \tag{3.5.41}$$

We recall that H^1 denotes the one dimensional Hausdorff measure. As $|\mathfrak{a}_2 - \mathfrak{a}_1| \leq 1$, it suffices to estimate $H^1(U(m,\ell) \cap C \cap \{\mathfrak{a} = \tilde{\mathfrak{a}}\})$ uniformly for $\tilde{\mathfrak{a}} \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

For each $\tilde{\mathfrak{a}} \in (\mathfrak{a}_1, \mathfrak{a}_2)$ we claim that $U(m,\ell) \cap C \cap \{\mathfrak{a} = \tilde{\mathfrak{a}}\}$ can be decomposed into at most $O(m^2)$ many subsets, each of which with length at most $O(e^{-\ell}e^{-m})$. More precisely, for $\vartheta_1 < \vartheta_2$ let $\gamma_{\tilde{\mathfrak{a}}}(\vartheta_1)$, $\gamma_{\tilde{\mathfrak{a}}}(\vartheta_2)$ be elements of $U(m,\ell) \cap C \cap \{\mathfrak{a} = \tilde{\mathfrak{a}}\}$. Then, we claim that either, $|\vartheta_2 - \vartheta_1| \leq 2e^{-\ell}e^{-m}$ or $|\vartheta_2 - \vartheta_1| > \frac{1}{m^2}$.

Indeed, note that $\gamma_{\tilde{\mathfrak{a}}}(\vartheta_2) = \Phi^{\Gamma}_{|\vartheta_2 - \vartheta_1|}(\gamma_{\tilde{\mathfrak{a}}}(\vartheta_1))$. Thus, from the definition of $U(m,\ell)$ and since both, $\gamma_{\tilde{\mathfrak{a}}}(\vartheta_1), \gamma_{\tilde{\mathfrak{a}}}(\vartheta_2) \in U(m,\ell)$, we conclude that

$$|\vartheta_2 - \vartheta_1| < 2e^{-\ell}e^{-m} \text{ or } \frac{1}{m^2} < |\vartheta_2 - \vartheta_1|. \tag{3.5.42}$$

Hence, we decompose $U(m,\ell) \cap C \cap \{\mathfrak{a} = \tilde{\mathfrak{a}}\}$ into $O(m^2)$ many subsets, each of which has length $O(e^{-m}e^{-\ell})$. Thus,

$$H^1(U(m,\ell) \cap C \cap \{\mathfrak{a} = \tilde{\mathfrak{a}}\}) \lesssim m^2 e^{-\ell} e^{-m} \tag{3.5.43}$$

which implies

$$|U(m,\ell) \cap C| \lesssim m^2 e^{-\ell} e^{-m}. \tag{3.5.44}$$

Now,

$$U_{m,C} := \bigcup_{m \leq \ell \leq m^2} U(m,\ell) \cap C \tag{3.5.45}$$

satisfies $|U_{m,C}| \lesssim e^{-m}$. Since $\mathscr{P}_{\text{Blow-up}} \cap C = \limsup_{m\to\infty} U_{m,C}$, we conclude

$$|\mathscr{P}_{\text{Blow-up}} \cap C| = 0 \tag{3.5.46}$$

in view of Borel–Cantelli lemma. $\square$

Proposition 3.5.4. *The set $\mathscr{P}_{\text{Blow-up}}$ has full packing dimension, i.e.* $\dim_P(\mathscr{P}_{\text{Blow-up}}) =$

2.

Proof. This follows from Proposition 3.5.2 and [46, Corollary 3.10]. $\square$

3.6 Construction of the initial data

We will now define our initial data and show first properties of the corresponding solution, particularly a quantitative description of the (generalized) Fourier transform of the solution along the event horizon. We begin by fixing an arbitrary parameter

$$\mathfrak{p} = (\mathfrak{m}, \mathfrak{a}) \in \mathscr{P}_{\text{Blow-up}}. \tag{3.6.1}$$

which we keep fixed through the rest of the chapter, i.e. throughout Section 3.6, Section 3.7, Section 3.8. This also fixes the mass $M = \mathfrak{m}l/\sqrt{3}$ and angular momentum $a = \mathfrak{a}l/\sqrt{3}$. As stated in the conventions in Section 3.2.3, all constants appearing in $\lesssim$, $\gtrsim$, $\sim$ and O will now depend on $\mathfrak{p}$ as fixed in (3.6.1) (and on $l > 0$ as fixed in (3.2.6)) throughout Section 3.6, Section 3.7, Section 3.8.

By construction of $\mathscr{P}_{\text{Blow-up}}$ and since $\mathfrak{p} \in \mathscr{P}_{\text{Blow-up}}$, there exists an infinite sequence

$$m_i \to \infty, \ell_i \to \infty \tag{3.6.2}$$

with

$$|\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty](\omega = \omega_- m_i, m_i, \ell_i)| < e^{-\sqrt{m_i}}, \tag{3.6.3}$$

$$\frac{\lambda_{m_i \ell_i}(a\omega_- m_i)}{m_i^2} \in (\sigma_1(\mathfrak{p}), \sigma_2(\mathfrak{p})). \tag{3.6.4}$$

Lemma 3.6.1. *There exists a compact interval $K =\subset (-\infty, \frac{\pi}{2}l)$, an $\epsilon > 0$ and a constant $c > 0$ such that for every $i \in \mathbb{N}$ large enough, there exists a subinterval $K_i = [r_i^* - \frac{c}{m_i}, r_i^* + \frac{c}{m_i}] \subset K$ with*

$$u_\infty^{\omega_-} := u_\infty(\omega = \omega_- m_i, m_i, \ell_i, r^*) > \frac{\epsilon}{m_i}. \tag{3.6.5}$$

Moreover, we choose K such that $\inf K > 3r_+$.

Proof. By Definition 3.4.1, $u_\infty^{\omega_-} = u_\infty(\omega = \omega_- m_i, m_i, \ell_i, r^*)$ is a solution to (3.2.40), i.e. a solution to

$$-u'' + (m_i^2 V_{\text{main}} + V_1)u = 0, \tag{3.6.6}$$

where

$$V_{\text{main}} = \frac{\Delta}{(r^2+a^2)^2}\left(\frac{\lambda_{m\ell}(a\omega_- m_i)}{m_i^2} + \omega_-^2 a^2 - 2a\omega_-\Xi\right) - (\omega_- - \omega_r)^2, \tag{3.6.7}$$

$$V_1 = \frac{-\Delta^2 3r^2}{(r^2+a^2)^4} - \Delta\frac{5\frac{r^4}{l^2} + 3r^2\left(1+\frac{a^2}{l^2}\right) - 4Mr + a^2}{(r^2+a^2)^3} - \frac{2\Delta}{l^2}\frac{1}{r^2+a^2} \tag{3.6.8}$$

with $u_\infty^{\omega_-}(l\pi/2) = 0$ and $u_\infty^{\omega_-}{}'(l\pi/2) = 1$. Since

$$\frac{\lambda_{m_i\ell_i}(a\omega_- m_i)}{m_i^2} \in (\sigma_1(\mathfrak{p}), \sigma_2(\mathfrak{p})), \tag{3.6.9}$$

there exists an $\delta > 0$ and a $\tilde{r}_2^*$ such that

$$V_{\text{main}} < -\delta \text{ for } r^* \in \left[\tilde{r}_2^*, l\frac{\pi}{2}\right), \tag{3.6.10}$$

see Lemma 3.4.2. Without loss of generality we can assume that $\tilde{r}_2^* > r^*(r = 3r_+)$. In particular, for m_i sufficiently large, we have that

$$V_{\text{main}} + V_1 m_i^{-2} < -\delta \tag{3.6.11}$$

for $r^* \in [\tilde{r}_2^*, \frac{\pi}{2}l)$. Now, let $K =:= [\tilde{r}_2^*, r_3^*] \subset (\tilde{r}_2^*, \frac{\pi}{2}l)$ be a compact subinterval. In the region $[\tilde{r}_2^*, \frac{\pi}{2}l)$, the smooth potential V_{main} satisfies

$$V_{\text{main}} < -\delta, |V'_{\text{main}}| \lesssim 1 \text{ and } |V''_{\text{main}}| \lesssim 1. \tag{3.6.12}$$

Moreover, $|V_1| \lesssim 1$ uniformly in $[\tilde{r}_2^*, \frac{\pi}{2}l)$. This allows us to approximate $u_\infty^{\omega_-}$ via a WKB approximation. First, we introduce the error-control function

$$F_\infty(r^*) := \int_{r^*}^{\frac{\pi}{2}l} |V_{\text{main}}|^{-\frac{1}{4}}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(|V_{\text{main}}|^{-\frac{1}{4}}\right) - \frac{V_1}{|V_{\text{main}}|^{\frac{1}{2}}}\mathrm{d}y \tag{3.6.13}$$

and note that $F_\infty(\frac{\pi}{2}l) = 0$. In view of the above bounds on V_{main} and V_1 we obtain

$$\mathcal{V}_{\tilde{r}_2^*, \frac{\pi}{2}l}(F_\infty) \lesssim 1. \tag{3.6.14}$$

Hence, from [119, Chapter 6, §5] the solution $u_\infty^{\omega_-}$ is given as

$$u_\infty^{\omega_-} = \frac{A}{m_i |V_{\text{main}}|^{\frac{1}{4}}} \sin\left(-m_i \int_{r^*}^{\frac{\pi}{2}l} \sqrt{|V_{\text{main}}|} \mathrm{d}y \right) (1 + \epsilon_{u_\infty}), \tag{3.6.15}$$

where $A = |V_{\text{main}}^{-\frac{1}{4}}(r^* = l\frac{\pi}{2})|$ satisfies $|A| \sim 1$ and ϵ_{u_∞} satisfies $\epsilon_{u_\infty}\left(\frac{\pi}{2}l\right) = 0$ as well as

$$|\epsilon_{u_\infty}|, \frac{|\epsilon'_{u_\infty}|}{2m_i|V_{\text{main}}|^{\frac{1}{2}}} \lesssim \frac{\mathcal{V}_{\tilde{r}_2^*, \frac{\pi}{2}l}(F_\infty)}{m_i} \lesssim \frac{1}{m_i}. \tag{3.6.16}$$

Now, since $u_\infty^{\omega_-}$ oscillates with period proportional to m_i, there exists a compact subinterval $K_i \subset K$ of the form $K_i = [r_i^* - \frac{c}{m_i}, r_i^* + \frac{c}{m_i}]$ for some $c > 0$ such that for all $r^* \in K_i$, we have

$$u_\infty^{\omega_-}(r^*, m_i, \ell_i,) > \frac{\epsilon}{m_i}. \tag{3.6.17}$$

□

We are now in the position to define our initial data which will be supported in the compact set K as defined in Lemma 3.6.1. We assume without loss of generality that all m_i are large enough such that we can apply Lemma 3.6.1. First, let $\chi \colon \mathbb{R} \to [0,1]$ be a smooth bump function satisfying $\chi = 0$ for $|x| \geq 1$ and $\chi = 1$ for $|x| \leq \frac{1}{2}$. Then, for $i \in \mathbb{N}$ we set

$$\chi_i \colon (-\infty, \pi/2) \to [0,1], r^* \mapsto \chi(c^{-1} m_i (r^* - r_i^*)). \tag{3.6.18}$$

Definition 3.6.1. *Let m_i, ℓ_i be as in* (3.6.2)*. For each $i \geq i_0$ for some $i_0 \in \mathbb{N}$ sufficiently large, let $K_i \subset K$ be the associated subinterval as specified in Lemma 3.6.1 and let χ_i defined as in* (3.6.18)*. Then, we define initial data on Σ_0 as*

$$\psi \restriction_{\Sigma_0} = \Psi_0 := 0, \tag{3.6.19}$$

$$n_{\Sigma_0} \psi \restriction_{\Sigma_0} (r, \theta, \phi) = \Psi_1(r, \theta, \phi) := \sum_{i \geq i_0} e^{-m_i^{\frac{1}{5}}} \psi_i(r, \theta, \phi), \tag{3.6.20}$$

where

$$\psi_i(r, \theta, \phi) = \frac{\sqrt{r^2 + a^2} \chi_i(r^*(r))}{-\Sigma \sqrt{-g^{tt}}(r, \theta) u_\infty^{\omega_-}(r^*(r))} S_{m_i \ell_i}(a\omega_- m_i, \cos\theta) e^{i m_i \phi}. \tag{3.6.21}$$

Having set up the initial data we proceed to

Definition 3.6.2. *Throughout the rest of Section 3.7 and Section 3.8 we define* $\psi \in C^\infty(\mathcal{M}_{\text{Kerr-AdS}} \setminus \mathcal{CH})$ *to be the unique smooth solution to* (3.1.1) *of the mixed Cauchy-boundary value problem with vanishing data on* $\mathcal{H}_L \cup \mathcal{B}_{\mathcal{H}}$*, Dirichlet boundary conditions at infinity and the initial data* $(\Psi_0, \Psi_1) \in C_c^\infty(\Sigma_0)$ *posed on* Σ_0 *specified in Definition 3.6.1. This is well-posed in view of Theorem 3.2.*

Remark 3.6.1. *We note that our initial data are only supported on the positive azimuthal frequencies* $m = m_i$ *for* $i \geq i_0$*. The same will apply to the arising solution* ψ*.*

In the following we define the quantity $a_{\mathcal{H}}$ which will turn out to be the (generalized) Fourier transform of the solution ψ along the event horizon.

Definition 3.6.3. *For the initial data* Ψ_0, Ψ_1 *as in Definition 3.6.1 we define*

$$\begin{aligned} a_{\mathcal{H}}(\omega, m, \ell) := \frac{1}{\sqrt{2\pi}\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} \Bigg\{ \frac{\Sigma}{\sqrt{r^2+a^2}} u_\infty e^{-im\phi} S_{m\ell}(a\omega, \cos\theta) \\ \times \left(-\sqrt{-g^{tt}}\Psi_1 - i\omega g^{tt}\Psi_0 + g^{t\phi}\partial_\phi \Psi_0 \right) \Bigg\} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}r. \end{aligned} \tag{3.6.22}$$

Now, we show that the (generalized) Fourier transform of our initial data has "peaks" at the resonant frequencies $\omega = \omega_- m$ for infinitely many m. This is a consequence of our choice of initial data. We formulate this in

Lemma 3.6.2. *For* $a_{\mathcal{H}}$ *as in Definition 3.6.1 we have*

$$a_{\mathcal{H}}(\omega = \omega_- m, m, \ell) = a_{\mathcal{H}}(\omega = \omega_- m_i, m_i, \ell_i)\delta_{mm_i}\delta_{\ell\ell_i}, \tag{3.6.23}$$

where

$$|a_{\mathcal{H}}(\omega = \omega_- m_i, m_i, \ell_i)| \gtrsim e^{m_i^{\frac{1}{4}}} \tag{3.6.24}$$

for (m_i, ℓ_i) *large enough as in* (3.6.2)*.*

Proof. As $\Psi_0 = 0$, we compute

$$\begin{aligned} &\int_{r_+}^{\infty} \int_{\mathbb{S}^2} \frac{\Sigma}{\sqrt{r^2+a^2}} u_\infty^{\omega_-} e^{-im\phi} S_{m\ell}(a\omega_- m, \cos\theta)(-\sqrt{-g^{tt}})\Psi_1 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}r \\ &= e^{-m_i^{\frac{1}{5}}} \delta_{mm_i}\delta_{\ell\ell_i} \int_{r_+}^{\infty} \chi_i(r) \mathrm{d}r \sim e^{-m_i^{\frac{1}{5}}} m_i^{-1} \delta_{mm_i}\delta_{\ell\ell_i}. \end{aligned} \tag{3.6.25}$$

To conclude we use that from (3.6.3) we have

$$|\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty](\omega = \omega_- m_i, m_i, \ell_i)| < e^{-\sqrt{m_i}}. \tag{3.6.26}$$

□

3.7 Exterior analysis

3.7.1 Cut-off in time and inhomogeneous equation

We will now consider the ψ as defined in Section 3.6. The goal of this section is to determine the Fourier transform of ψ along the event horizon. To do so we will first take of a time cut-off of ψ. Let $\chi\colon \mathbb{R} \to [0,1]$ be a smooth and monotone cut-off function with $\chi(x) = 0$ for $x \leq 0$, $\chi(x) = 1$ for $x \geq 1$. Now, define $\chi_\epsilon^R(v) := \chi(v/\epsilon)\chi(R-v)$ such $\chi_\epsilon^R \to \mathbb{1}_{(0,\infty)}$ pointwise as $\epsilon \to 0$ and $R \to \infty$. Moreover,

$$\partial_v(\chi(v/\epsilon)) \to \delta_0(v) \text{ and } \partial_v^2(\chi(v/\epsilon)) \to \delta_0'(v) \tag{3.7.1}$$

as $\epsilon \to 0$ in the sense of distributions. On $\mathcal{R} \cup \mathcal{H}_R$ we set

$$\psi_\epsilon^R(v, r, \theta, \tilde{\phi}_+) := \psi(v, r, \theta, \tilde{\phi}_+)\chi_\epsilon^R(v) \text{ and } \psi^R := \psi(v, r, \theta, \tilde{\phi}_+)\chi(R-v) \tag{3.7.2}$$

and note that ψ_ϵ^R is smooth and compactly supported in v and satisfies the inhomogeneous equation

$$\Box_{g_{\mathrm{Kerr-AdS}}}\psi_\epsilon^R + \frac{2}{l^2}\psi_\epsilon^R = F_\epsilon^R := 2(\partial_v\chi_\epsilon^R)(\nabla v)\psi + \psi\Box_{g_{\mathrm{Kerr-AdS}}}\chi_\epsilon^R. \tag{3.7.3}$$

Analogously, ψ^R satisfies the inhomogeneous equation with

$$\Box_{g_{\mathrm{Kerr-AdS}}}\psi^R + \frac{2}{l^2}\psi^R = F^R := 2(\partial_v\chi^R)(\nabla v)\psi + \psi\Box_{g_{\mathrm{Kerr-AdS}}}\chi^R. \tag{3.7.4}$$

As in [75, Section 5.1] we have

$$|F^R|^2 r^2 \lesssim \frac{1}{r^2}|\partial_v\psi|^2 + r^2|\partial_r\psi|^2 + |\nabla\!\!\!/\psi|^2 + |\psi|^2, \tag{3.7.5}$$

$$|F_\epsilon^R|^2 r^2 \lesssim \frac{1}{\epsilon^2 r^2}|\partial_v\psi|^2 + r^2|\partial_r\psi|^2 + |\nabla\!\!\!/\psi|^2 + |\psi|^2. \tag{3.7.6}$$

In view of our coordinates, we also have that $\psi_\epsilon^R(t, r, \theta, \phi)$ is a compactly supported function in $\mathbb{R}_t$ with values in $C^\infty((r_+,\infty) \times \mathbb{S}^2)$. This allows us to apply Carter's separation of variables to express ψ_ϵ^R as

$$\psi_\epsilon^R(t,r,\theta,\phi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathfrak{F}[\psi_\epsilon^R](\omega,r,\theta,\phi)e^{-i\omega t}\mathrm{d}\omega, \tag{3.7.7}$$

where

$$\mathfrak{F}[\psi_\epsilon^R](\omega,r,\theta,\phi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \psi_\epsilon^R(t,r,\theta,\phi)e^{i\omega t}\mathrm{d}t \tag{3.7.8}$$

is smooth and a Schwartz function on $\mathbb{R}_\omega$ with values in $C^\infty((r_+,\infty)\times\mathbb{S}^2)$. We further decompose $\mathfrak{F}[\psi_\epsilon^R](\omega,r,\theta,\phi)$ in (generalized) spheroidal harmonics

$$\mathfrak{F}[\psi_\epsilon^R](\omega,r,\theta,\phi) = \sum_{m\ell} \hat{\psi_\epsilon^R}(\omega,m,\ell,r)S_{m\ell}(a\omega,\cos\theta)e^{im\phi}, \tag{3.7.9}$$

where

$$\hat{\psi_\epsilon^R}(\omega,m,\ell,r) := \int_{\mathbb{S}^2} \mathfrak{F}[\psi_\epsilon^R](\omega,r,\theta,\phi)e^{-im\phi}S_{m\ell}(a\omega,\cos\theta)\mathrm{d}\sigma_{\mathbb{S}^2} \tag{3.7.10}$$

is smooth in ω and r for fixed m and ℓ and moreover

$$\hat{\psi_\epsilon^R} \in L^2(\mathbb{R}_\omega\times\mathbb{Z}_m\times\mathbb{Z}_{\ell\geq|m|};C^\infty(r_+,\infty)) \tag{3.7.11}$$

in view of Plancherel's theorem. Equivalently, we have

$$\hat{\psi_\epsilon^R}(\omega,m,\ell,r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{S}^2} \psi_\epsilon^R(t,r,\theta,\phi)e^{i\omega t}e^{-im\phi}S_{m\ell}(a\omega,\cos\theta)\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}t. \tag{3.7.12}$$

Note that $\psi_\epsilon^R(v,r,\theta,\tilde{\phi}_+)$ is smooth and compactly supported on $\mathbb{R}_v$ and takes values in $C^\infty([r_+,\infty)_r\times\mathbb{S}^2_{\theta,\tilde{\phi}_+})$ which shows after a change of coordinates in (3.7.12) that

$$\hat{\psi}_\epsilon^R(\omega,r,m,\ell)e^{i(\omega-\omega_+m)r^*} \tag{3.7.13}$$

extends smoothly to $r=r_+$ ($r^*\to-\infty$). Similarly to the above, we have

$$\widehat{\Sigma F_\epsilon^R}(\omega,m,\ell,r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{S}^2} \Sigma F_\epsilon^R(t,r,\theta,\phi)e^{i\omega t}e^{-im\phi}S_{m\ell}(a\omega,\cos\theta)\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}t. \tag{3.7.14}$$

Now, we define

$$u_\epsilon^R = u_\epsilon^R(\omega,m,\ell,r) := (r^2+a^2)^{\frac{1}{2}}\hat{\psi_\epsilon^R}(\omega,m,\ell,r) \tag{3.7.15}$$

and

$$H_\epsilon^R(\omega, m, \ell, r) := \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}} \widehat{\Sigma F_\epsilon^R}(\omega, m, \ell, r). \tag{3.7.16}$$

Then, since ψ_ϵ^R satisfies (3.7.3), we have that u_ϵ^R satisfies the inhomogeneous radial o.d.e.

$$-u_\epsilon^{R\prime\prime} + (V - \omega^2) u_\epsilon^R = H_\epsilon^R \tag{3.7.17}$$

pointwise for each ω, m, ℓ on $r^* \in (-\infty, \frac{\pi}{2}]$, where we recall that $' = \frac{\mathrm{d}}{\mathrm{d}r^*}$.

3.7.2 Estimates for the inhomogeneous radial o.d.e.

Lemma 3.7.1. *The solution u_ϵ^R as defined in* (3.7.15) *satisfies the boundary conditions*

$$u_\epsilon^R \to 0 \text{ as } r^* \to \frac{\pi}{2}, \tag{3.7.18}$$

$$u_\epsilon^{R\prime} + i(\omega - \omega_+ m) u_\epsilon^R = O_\psi(\Delta) \text{ as } r^* \to -\infty \tag{3.7.19}$$

and in the inhomogeneity H_ϵ^R defined in (3.7.17) *also satisfies*

$$H_\epsilon^R = 0 \text{ as } r^* \to \infty, \quad H_\epsilon^{R\prime} + i(\omega - \omega_+ m) H_\epsilon^R = O_\psi(\Delta) \text{ as } r^* \to -\infty. \tag{3.7.20}$$

Proof. To see (3.7.18) note that

$$|u_\epsilon^R| \leq (r^2+a^2)^{\frac{1}{2}} |\hat{\psi_\epsilon^R}| = \left| \int_{\mathbb{R}} \int_{\mathbb{S}^2} (r^2+a^2)^{\frac{1}{2}} \psi_\epsilon^R(r, t, \theta, \phi) e^{i\omega t} S_{m\ell}(a\omega, \cos\theta) e^{-im\phi} \mathrm{d}\sigma \mathrm{d}t \right|. \tag{3.7.21}$$

In view of the compact support of ψ_ϵ^R in t, it suffices to show that the pointwise limit

$$\lim_{r\to\infty} r \psi_\epsilon^R(t, r, \theta, \phi) = 0 \tag{3.7.22}$$

holds true. But this follows from the fact that $\psi_\epsilon^R \in CH^1_{\mathrm{AdS}}$.

For (3.7.19), we use (3.7.13) to see that $\partial_r(\hat{\psi}_\epsilon^R(\omega, r, m, \ell) e^{i(\omega - \omega_+ m) r^*})$ extends smoothly to $r = r_+$. Thus, using $\partial_r = \frac{r^2+a^2}{\Delta} \partial_{r^*}$, we infer that

$$u_\epsilon^{R\prime} + i(\omega - \omega_+ m) u_\epsilon^R = O_\psi(\Delta) \text{ as } r^* \to -\infty. \tag{3.7.23}$$

Analogously, we obtain (3.7.20). □

Lemma 3.7.2. *We represent u_ϵ^R as*

$$u_\epsilon^R(r^*) = \frac{1}{\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]} \left\{ u_{\mathcal{H}^+} \int_{r^*}^{\frac{\pi}{2}} u_\infty H_\epsilon^R \mathrm{d}y + u_\infty \int_{-\infty}^{r^*} u_{\mathcal{H}^+} H_\epsilon^R \mathrm{d}y \right\}. \tag{3.7.24}$$

Moreover,

$$\lim_{r_* \to -\infty} u_\epsilon^R(r^*) e^{i(\omega - \omega_+ m) r^*} = a_{\epsilon,\mathcal{H}}^R, \tag{3.7.25}$$

where $a_{\epsilon,\mathcal{H}}^R$ is defined as

$$a_{\epsilon,\mathcal{H}}^R := \frac{1}{\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]} \int_{-\infty}^{\frac{\pi}{2}} u_\infty H_\epsilon^R \mathrm{d}y. \tag{3.7.26}$$

Proof. First, since there do not exist pure mode solutions as shown in [75, Theorem 1.3], the Wronskian $\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]$ never vanishes. Thus, (3.7.24) is well-defined and in view of the boundary conditions of u_ϵ^R and H_ϵ^R as shown in Lemma 3.7.1, a direct computation shows (3.7.24). To show (3.7.25) we note that

$$\begin{aligned} &\limsup_{r^* \to -\infty} \left| \int_{-\infty}^{r^*} u_{\mathcal{H}^+} H_\epsilon^R \mathrm{d}y \right|^2 \\ &\quad \leq \limsup_{r^* \to -\infty} \left(\int_{r_+}^{r(r^*)} \frac{|\widehat{\Sigma F_\epsilon^R}|^2}{r^2 + a^2} \mathrm{d}r \int_{-\infty}^{r_*} |u_{\mathcal{H}^+}|^2 \frac{\Delta}{r^2 + a^2} \mathrm{d}y \right) = 0 \end{aligned} \tag{3.7.27}$$

because

$$\int_{r_+}^{r(r^*)} \frac{|\widehat{\Sigma F_\epsilon^R}|^2}{r^2 + a^2} \mathrm{d}r < \infty \quad \text{and} \quad \sup_{r^* \in (-\infty, r_1)} |u_{\mathcal{H}^+}| < \infty. \tag{3.7.28}$$

□

Lemma 3.7.3. *The inhomogeneous term H_ϵ^R has the pointwise limit*

$$\begin{aligned} H^R &:= \lim_{\epsilon \to 0} H_\epsilon^R \\ &= \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^2} \Sigma e^{-im\phi} S_{m\ell}(a\omega) \left(-\sqrt{-g^{tt}} \Psi_1 - i\omega g^{tt} \Psi_0 + g^{t\phi} \partial_\phi \Psi_0 \right) \mathrm{d}\sigma_{\mathbb{S}^2} \\ &+ \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}} \frac{e^{-i(\omega - \omega_+ m) r^*}}{\sqrt{2\pi}} \int_{R-1}^{R} \int_{\mathbb{S}^2} \Sigma F_R(v, r, \theta, \tilde{\phi}_+) e^{i\omega v} e^{-im\tilde{\phi}_+} S_{m\ell}(a\omega) \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}v. \end{aligned} \tag{3.7.29}$$

In addition,

$$a^R_{\epsilon,\mathcal{H}} \to a^R_{\mathcal{H}} := \frac{1}{\mathfrak{W}[u_{\mathcal{H}^+}, u_\infty]} \int_{-\infty}^{\frac{\pi}{2}} u_\infty H^R \mathrm{d}r^* \tag{3.7.30}$$

pointwise as $\epsilon \to 0$.

Moreover, we have

$$H^R \to H := \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^2} \Sigma e^{-im\phi} S_{m\ell}(a\omega) \\ \left(-\sqrt{-g^{tt}}\Psi_1 - i\omega g^{tt}\Psi_0 + g^{t\phi}\partial_\phi \Psi_0\right) \mathrm{d}\sigma_{\mathbb{S}^2} \tag{3.7.31}$$

and

$$a^R_{\mathcal{H}} \to a_{\mathcal{H}} \tag{3.7.32}$$

pointwise as $R \to \infty$.

Proof. We start with the decomposition $F^R_\epsilon = F_\epsilon + F_R$, where the support of F_ϵ is in $\{0 \le t \le \epsilon\} \cap \{r \ge 2r_+\}$ for $\epsilon > 0$ small enough and the support of F_R is in the set $\{R-1 \le v \le R\}$. This decomposition is well-defined in view of the finite speed of propagation.

We first consider F_ϵ and write its (generalized) Fourier transform as

$$\widehat{\Sigma F_\epsilon}(\omega, m, \ell, r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^2} \int_{\mathbb{R}} F_\epsilon(t, r, \theta, \phi) e^{i\omega t} \mathrm{d}t \; \Sigma e^{-im\phi} S_{m\ell}(a\omega, \cos\theta) \mathrm{d}\sigma_{\mathbb{S}^2}. \tag{3.7.33}$$

Recall that for all $\epsilon > 0$ sufficiently small, we have that $v(t, r, \theta, \phi) = t$ on the support of F_ϵ. Thus,

$$F_\epsilon \to F_0 := -\sqrt{-g^{tt}}\delta_{t=0}\Psi_1 + g^{tt}\delta'_{t=0}\Psi_0 + g^{t\phi}\delta_{t=0}\partial_\phi\Psi_0 \tag{3.7.34}$$

as $\epsilon \to 0$ in the sense of distributions (compactly supported distributions) on $\mathbb{R}_t$ with values in $C^\infty((r_+, \infty) \times \mathbb{S}^2)$. Hence,

$$\widehat{\Sigma F_\epsilon}(\omega, m, \ell, r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^2} \int_{\mathbb{R}} e^{-im\phi} S_{m\ell}(a\omega, \cos\theta) F_\epsilon e^{i\omega t} e^{-i\omega\phi} \mathrm{d}t \mathrm{d}\sigma_{\mathbb{S}^2} \\ \to \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^2} e^{-im\phi} S_{m\ell}(a\omega, \cos\theta) \left(-\sqrt{-g^{tt}}\Psi_1 - i\omega g^{tt}\Psi_0 + g^{t\phi}\partial_\phi\Psi_0\right) \mathrm{d}\sigma_{\mathbb{S}^2} \tag{3.7.35}$$

pointwise. Thus,

$$
\begin{aligned}
H^R &= \lim_{\epsilon\to 0} H^R_\epsilon = \\
&= \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{S}^2}\Sigma e^{-im\phi}S_{m\ell}(a\omega)\left(-\sqrt{-g^{tt}}\Psi_1 - i\omega g^{tt}\Psi_0 + g^{t\phi}\partial_\phi\Psi_0\right)\mathrm{d}\sigma_{\mathbb{S}^2} \\
&+ \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}}\frac{e^{-i(\omega-\omega_+ m)r^*}}{\sqrt{2\pi}}\int_{R-1}^{R}\int_{\mathbb{S}^2}\Sigma F_R(v,r,\theta,\tilde{\phi}_+)e^{i\omega v}e^{-im\tilde{\phi}_+}S_{m\ell}(a\omega)\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v
\end{aligned}
\tag{3.7.36}
$$

pointwise.

Now, to show that $a^R_{\epsilon,\mathcal{H}} \to a^R_{\mathcal{H}}$ it suffices to show

$$
\int_{-\infty}^{\frac{\pi}{2}} u_\infty \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}}\widehat{\Sigma F_\epsilon}\mathrm{d}y \to \int_{-\infty}^{\frac{\pi}{2}} u_\infty \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}}\widehat{\Sigma F_0}\mathrm{d}y \tag{3.7.37}
$$

pointwise as $\epsilon\to 0$. But recall that F_ϵ is compactly supported in (r_+,∞) for all $0<\epsilon<\epsilon_0$ sufficiently small and $\sup_{0<\epsilon<\epsilon_0}\sup_{r^*}|\widehat{\Sigma F_\epsilon}|<\infty$ so we can interchange the integral with the limit $\epsilon\to 0$.

Now, we will show that $H^R\to H$ as $R\to\infty$. As ψ and its derivatives decay pointwise at a logarithmic rate (see Theorem 3.2), we infer that

$$
\sup_{r\in(r_+,\infty),\theta,\tilde{\phi}_+\in\mathbb{S}^2} |F_r|(v,r,\theta,\tilde{\phi}_+)\to 0 \tag{3.7.38}
$$

as $R\to\infty$. Thus, we have

$$
\begin{aligned}
&\left|\int_{R-1}^{R}\int_{\mathbb{S}^2}\Sigma F_R(v,r,\theta,\tilde{\phi}_+)e^{i\omega v}e^{-im\tilde{\phi}_+}S_{m\ell}(a\omega)\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v\right| \\
&\qquad\lesssim r^2 \sup_{\substack{v\in(R-1,R)\\ r\in(r_+,\infty),\theta,\tilde{\phi}_+\in\mathbb{S}^2}} |F_r|(v,r,\theta,\tilde{\phi}_+)\to 0
\end{aligned}
\tag{3.7.39}
$$

pointwise as $R\to\infty$. This shows $H^R\to H$ pointwise.

Finally, to show that $a_{\mathcal{H}}^R \to a_{\mathcal{H}}$ as $R \to \infty$, we estimate

$$
\begin{aligned}
&\left| \int_{-\infty}^{\frac{\pi}{2}} u_\infty \frac{\Delta}{(r^2+a^2)^{\frac{3}{2}}} \frac{e^{-i(\omega-\omega_+ m)r^*}}{\sqrt{2\pi}} \int_R^{R+1} \int_{\mathbb{S}^2} \Sigma F_R(v,r,\theta,\tilde{\phi}_+) e^{i\omega v} e^{-im\tilde{\phi}_+} S_{m\ell}(a\omega) \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}v \mathrm{d}r^* \right|^2 \\
&\lesssim \int_{-\infty}^{\frac{\pi}{2}} |u_\infty|^2 \frac{1}{r^2} \frac{\Delta}{r^2+a^2} \mathrm{d}r^* \sup_{v\in(R,R+1)} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} \Sigma^2 |F_R|^2 \frac{1}{r^2+a^2} r^2 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}r \\
&\lesssim \int_{-\infty}^{\frac{\pi}{2}} C_{m\ell\omega}^2 |r^*|^2 \frac{1}{r^2} \frac{\Delta}{r^2+a^2} \mathrm{d}r^* \sup_{v\in(R,R+1)} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} r^2 |F_R|^2 r^2 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}r \\
&\lesssim C_{m\ell\omega}^2 \sup_{v\in(R,R+1)} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} e_1[\psi] r^2 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}r \to 0
\end{aligned}
\tag{3.7.40}
$$

as $R \to \infty$. Here, we have used that $|u_\infty| \leq C_{m\ell\omega}|r^*|$ which holds true as for each ω, m, ℓ there exist constants a_s, a_c only depending on the ω, m, ℓ such that $u_\infty = a_s u_s + a_c u_c$, where u_s and u_c are solutions to the radial o.d.e. satisfying $u_s \sim \frac{\sin((\omega-\omega_+ m)r^*)}{\omega-\omega_+ m}$ and $u_c \sim \cos((\omega-\omega_+ m)r^*)$ as $r^* \to -\infty$. In the case $\omega = \omega_+ m$, this reduces to $u_s \sim r^*$ and $u_c \sim 1$ as $r^* \to -\infty$. Hence, $a_{\mathcal{H}}^R \to a_{\mathcal{H}}$ as $R \to \infty$ pointwise for each ω, m, ℓ. $\square$

3.7.3 Representation formula for ψ at the event horizon

Proposition 3.7.1. *Let $a_{\mathcal{H}}^R$ be as defined in* (3.7.30)*. Then, on the event horizon $\mathcal{H}_R$ we have*

$$
\psi^R(v, r_+, \theta, \tilde{\phi}_+) = \frac{1}{\sqrt{2\pi(r_+^2+a^2)}} \sum_{m\ell} \int_{\mathbb{R}} a_{\mathcal{H}}^R S_{m\ell}(a\omega, \cos\theta) e^{im\tilde{\phi}_+} e^{-i\omega v} \mathrm{d}\omega, \tag{3.7.41}
$$

in $L^2(\mathbb{R}_v \times \mathbb{S}^2)$. Moreover,

$$
a_{\mathcal{H}}^R = \sqrt{\frac{r_+^2+a^2}{2\pi}} \int_{\mathbb{R}\times\mathbb{S}^2} \psi^R(v, r_+, \theta, \tilde{\phi}_+) S_{m\ell}(a\omega, \cos\theta) e^{-im\tilde{\phi}_+} e^{i\omega v} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}v \tag{3.7.42}
$$

pointwise and in $L^2(\mathbb{R}_\omega \times \mathbb{Z}_m \times \mathbb{Z}_{\ell\geq|m|})$.

Proof. We have

$$
\psi_\epsilon^R(v, r, \theta, \tilde{\phi}_+) = \frac{1}{\sqrt{2\pi(r^2+a^2)}} \sum_{m\ell} \int_{\mathbb{R}} e^{i(\omega-\omega_+ m)r^*} u_\epsilon^R S_{m\ell}(a\omega, \cos\theta) e^{im\tilde{\phi}_+} e^{-i\omega v} \mathrm{d}\omega \tag{3.7.43}
$$

and

$$e^{i(\omega-\omega_+ m)r^*}u_\epsilon^R = \sqrt{\frac{r^2+a^2}{2\pi}}\int_{\mathbb{R}\times\mathbb{S}^2}\psi_\epsilon^R(v,r,\theta,\tilde{\phi}_+)S_{m\ell}(a\omega,\cos\theta)e^{-im\tilde{\phi}_+}e^{i\omega v}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v \tag{3.7.44}$$

for $r_+ < r < r_+ + \eta$. Now, since ψ_ϵ^R is compactly supported in v uniformly as $r_* \to -\infty$, we can interchange the limit $r^* \to -\infty$ with the integral over v. Thus, sending $r \to r_+$ ($r^* \to -\infty$) yields in view Lemma 3.7.2

$$a_{\epsilon,\mathcal{H}}^R = \sqrt{\frac{r_+^2+a^2}{2\pi}}\int_{\mathbb{R}\times\mathbb{S}^2}\psi_\epsilon^R(v,r_+,\theta,\tilde{\phi}_+)S_{m\ell}(a\omega,\cos\theta)e^{-im\tilde{\phi}_+}e^{i\omega v}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v, \tag{3.7.45}$$

where $a_{\epsilon,\mathcal{H}}^R$ is given in (3.7.26). Now we will perform the limit $\epsilon \to 0$. First, from Lemma 3.7.3 we have that

$$a_{\epsilon,\mathcal{H}}^R \to a_{\mathcal{H}}^R = \frac{1}{\mathfrak{W}[u_1,u_\infty]}\int_{-\infty}^{\frac{\pi}{2}} u_\infty H^R \mathrm{d}y \tag{3.7.46}$$

as $\epsilon \to 0$ pointwise. Moreover, ψ_ϵ^R has compact support uniformly as $\epsilon \to 0$ and $\psi_\epsilon^R \to \psi^R$ pointwise and in $L^2(\mathbb{R}_v \times \mathbb{S}^2)$ as $\epsilon \to 0$. Thus, the right hand side of (3.7.45) converges pointwise and due to Plancherel also in $L^2(\mathbb{R}_\omega \times \mathbb{Z}_m \times \mathbb{Z}_{\ell\geq|m|})$ as $\epsilon \to 0$. Hence, $a_{\epsilon,\mathcal{H}}^R \to a_{\mathcal{H}}^R$ also holds in $L^2(\mathbb{R}_\omega \times \mathbb{Z}_m \times \mathbb{Z}_{\ell\geq|m|})$ and we conclude

$$a_{\mathcal{H}}^R = \sqrt{\frac{r_+^2+a^2}{2\pi}}\int_{\mathbb{R}\times\mathbb{S}^2}\psi^R(v,r_+,\theta,\tilde{\phi}_+)S_{m\ell}(a\omega,\cos\theta)e^{-im\tilde{\phi}_+}e^{i\omega v}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v \tag{3.7.47}$$

which holds pointwise and in $L^2(\mathbb{R}_\omega \times \mathbb{Z}_m \times \mathbb{Z}_{\ell\geq|m|})$. And by Plancherel we also have

$$\psi^R(v,r_+,\theta,\tilde{\phi}_+) = \frac{1}{\sqrt{2\pi(r_+^2+a^2)}}\sum_{m\ell}\int_{\mathbb{R}} a_{\mathcal{H}}^R S_{m\ell}(a\omega,\cos\theta)e^{im\tilde{\phi}_+}e^{-i\omega v}\mathrm{d}\omega \tag{3.7.48}$$

in $L^2(\mathbb{R}_v \times \mathbb{S}^2)$. □

3.8 Interior analysis

Having established the behavior of our solution ψ in the exterior $\mathcal{R}$, we will now consider the interior region $\mathcal{B}$ characterized by $r \in (r_-, r_+)$. We first consider the radial o.d.e. in the interior which allow us to represent our solution ψ as a suitable generalized Fourier transform. We also recall that in the interior region the tortoise coordinate is defined in (3.2.15) as

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{\Delta}{r^2 + a^2}, \tag{3.8.1}$$

where $r^*(\frac{r_+ + r_-}{2}) = 0$ and that $\Delta < 0$ in the whole interior region.

Remark 3.8.1. *As our initial data are only supported on azimuthal modes m which are large and positive, we only need to consider this frequency range in the following estimates.*

3.8.1 Definitions and estimates for the radial o.d.e. in the interior

We recall the radial o.d.e. (3.2.36) and write it in the interior $r_- < r < r_+$ as

$$-u'' + \left(\frac{\Delta L}{(r^2 + a^2)^2} - (m\omega_r - \omega)^2 + V_1 \right) u = 0, \tag{3.8.2}$$

where

$$L := \lambda_{m\ell} + a^2\omega^2 - 2m\omega a \Xi \tag{3.8.3}$$

and V_1 is defined in (3.2.38). Note that $L \geq 0$ follows from [75, Lemma 5.4]. Also note that $V_1 = O(|\Delta|)$ uniformly for $r^* \in (-\infty, \infty)$. We will mainly treat V_1 as a perturbation and recall that the high-frequency part of the potential is given by

$$V_\sharp := \frac{\Delta L}{(r^2 + a^2)^2} - (m\omega_r - \omega)^2. \tag{3.8.4}$$

We now define fundamental pairs of solutions to the radial o.d.e. corresponding to the event and Cauchy horizon, respectively.

Definition 3.8.1. *We define solutions $u_{\mathcal{H}_R}, u_{\mathcal{H}_L}$ to* (3.8.2) *in the interior as*

$$u_{\mathcal{H}_R} = e^{-i(\omega - \omega_+ m) r^*} \tag{3.8.5}$$
$$u_{\mathcal{H}_L} = e^{i(\omega - \omega_+ m) r^*} \tag{3.8.6}$$

for $r^* \to -\infty$. For $\omega \neq \omega_+ m$, they form a fundamental pair. For $\omega = \omega_+ m$ the solutions $u_{\mathcal{H}_L}$ and $u_{\mathcal{H}_R}$ are linear dependent. Analogously, we define

$$u_{\mathcal{CH}_L} = e^{-i(\omega - \omega_- m) r^*} \tag{3.8.7}$$
$$u_{\mathcal{CH}_R} = e^{i(\omega - \omega_- m) r^*} \tag{3.8.8}$$

as $r^* \to +\infty$. For $\omega \neq \omega_- m$, they form a fundamental pair. For $\omega = \omega_- m$ the solutions $u_{\mathcal{CH}_L}$ and $u_{\mathcal{CH}_R}$ are linearly dependent.

We moreover define reflection and transmission coefficients.

Definition 3.8.2. *For $\omega \neq \omega_- m$ define the transmission coefficient $\mathfrak{T} = \mathfrak{T}(\omega, m, \ell)$ and the reflection coefficient $\mathfrak{R} = \mathfrak{R}(\omega, m, \ell)$ as the unique coefficients such that*

$$u_{\mathcal{H}_R} = \mathfrak{T} u_{\mathcal{CH}_L} + \mathfrak{R} u_{\mathcal{CH}_R}. \tag{3.8.9}$$

Equivalently, we have

$$\mathfrak{T} = \frac{\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]}{\mathfrak{W}[u_{\mathcal{CH}_L}, u_{\mathcal{CH}_R}]} = \frac{\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]}{2i(\omega - \omega_- m)}, \tag{3.8.10}$$

$$\mathfrak{R} = \frac{\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}]}{\mathfrak{W}[u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L}]} = \frac{\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}]}{-2i(\omega - \omega_- m)}. \tag{3.8.11}$$

They satisfy the Wronskian identity

$$|\mathfrak{T}|^2 = |\mathfrak{R}|^2 + \frac{\omega - \omega_+ m}{\omega - \omega_- m}. \tag{3.8.12}$$

Further, we define the renormalized transmission and reflection coefficient

$$\mathfrak{t} := (\omega - \omega_- m)\mathfrak{T} = \frac{1}{2i}\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}], \tag{3.8.13}$$

$$\mathfrak{r} := (\omega - \omega_- m)\mathfrak{R} = -\frac{1}{2i}\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}] \tag{3.8.14}$$

which satisfy

$$\mathfrak{t}^{\omega_-} := \mathfrak{t}(\omega = \omega_- m) = -\mathfrak{r}(\omega = \omega_- m) =: \mathfrak{r}^{\omega_-}. \tag{3.8.15}$$

Lemma 3.8.1. *There exists a constant $\epsilon_{\mathrm{cut}} > 0$ only depending on the black hole parameters such that the following holds true. Assume that $|\omega - \omega_r m| \leq \epsilon_{\mathrm{cut}} m$ for some $r \in [r_-, r_+]$, then $L \gtrsim m^2$.*

Proof. Note that L is larger than the lowest eigenvalue of the operator $P(a\omega) + a^2\omega^2 - 2a\Xi\omega m$, where P is as in (3.2.34). Since

$$\begin{aligned} P(a\omega) + a^2\omega^2 - 2a\Xi\omega m = &- \frac{1}{\sin\theta}\partial_\theta(\Delta_\theta \sin\theta\partial_\theta \cdot) \\ &+ \frac{1}{\Delta_\theta}\left(m\frac{\Xi}{\sin\theta} - a\omega\sin\theta\right)^2 + 2\frac{a^2}{l^2}\sin^2\theta, \end{aligned} \tag{3.8.16}$$

it suffices to show that the second term is bounded from below by $O(m^2)$. To do so, let $r \in [r_-, r_+]$ such that $|\omega - \omega_r m| \leq \epsilon_{\text{cut}} m$. Then, in view of

$$a\omega_r = \Xi \frac{a^2}{r^2 + a^2}, \tag{3.8.17}$$

we conclude

$$\begin{aligned} \left(m\frac{\Xi}{\sin\theta} - a\omega \sin\theta \right)^2 &= \left(m\frac{\Xi}{\sin\theta} - a\omega_r m \sin\theta + a(\omega_r m - \omega)\sin\theta \right)^2 \\ &\geq \frac{m^2 \Xi^2}{\sin^2\theta} \left(1 - \frac{a^2}{a^2 + r^2}\sin^2\theta - \left| \frac{a}{\Xi} \frac{\omega_r m - \omega}{m} \sin^2\theta \right| \right)^2 \gtrsim m^2 \end{aligned} \tag{3.8.18}$$

for sufficiently small $\epsilon_{\text{cut}} > 0$ only depending on the black hole parameters M, a, l. □

Lemma 3.8.2. *Assume that $|\omega - \omega_r m| \geq \epsilon_{\text{cut}} m$ for all $r \in [r_-, r_+]$. Then,*

$$\|u_{\mathcal{H}_R}\|_{L^\infty(\mathbb{R})} \lesssim 1, \|u_{\mathcal{H}_R}{}'\|_{L^\infty(\mathbb{R})} \lesssim |\omega| + |m| + L^{\frac{1}{2}}, \tag{3.8.19}$$

$$\|u_{\mathcal{CH}_L}\|_{L^\infty(\mathbb{R})} \lesssim 1, \|u_{\mathcal{CH}_L}{}'\|_{L^\infty(\mathbb{R})} \lesssim |\omega| + |m| + L^{\frac{1}{2}}, \tag{3.8.20}$$

$$\|u_{\mathcal{CH}_R}\|_{L^\infty(\mathbb{R})} \lesssim 1, \|u_{\mathcal{CH}_R}{}'\|_{L^\infty(\mathbb{R})} \lesssim |\omega| + |m| + L^{\frac{1}{2}}. \tag{3.8.21}$$

Proof. We first assume that $\omega - \omega_r m \geq \epsilon_{\text{cut}} m$ for all $r \in [r_-, r_+]$. Moreover, from our assumption, we have that the principle part of the potential $V_\sharp$ satisfies

$$-V_\sharp \gtrsim m^2 \text{ and } \left| \frac{V_\sharp'}{V_\sharp} \right|, \left| \frac{V_\sharp''}{V_\sharp} \right| \lesssim |\Delta| \tag{3.8.22}$$

and the error term satisfies $|V_1| \lesssim |\Delta|$. Thus, the error control function

$$F_{u_{\mathcal{H}_{R1}}}(r^*) := \int_{-\infty}^{r^*} \frac{1}{|V_\sharp|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(|V_\sharp|^{-\frac{1}{4}} \right) - \frac{V_1}{|V_\sharp|^{\frac{1}{2}}} \mathrm{d}y \tag{3.8.23}$$

satisfies $\mathcal{V}_{-\infty,\infty}(F_{u_{\mathcal{H}_{R1}}}) \lesssim \frac{1}{m}$. This allows us to apply standard estimates on WKB approximation such as [119, Chapter 6, Theorem 2.2] and deduce that

$$u_{\mathcal{H}_R} = A_{u_{\mathcal{H}_R}} \frac{|V_\sharp(-\infty)|^{\frac{1}{4}}}{|V_\sharp(r^*)|^{\frac{1}{4}}} \exp\left(-i \int_0^{r^*} |V_\sharp(y)|^{\frac{1}{2}} \mathrm{d}y \right) \left(1 + \epsilon_{u_{\mathcal{H}_R}}(r^*) \right), \tag{3.8.24}$$

for some $A_{u_{\mathcal{H}_R}}$ with $|A_{u_{\mathcal{H}_R}}| = 1$. Moreover,

$$\sup_{r^*\in\mathbb{R}} |\epsilon_{u_{\mathcal{H}_R}}(r^*)| \lesssim \frac{1}{m}, \quad \sup_{r^*\in\mathbb{R}} \left| \frac{\epsilon'_{u_{\mathcal{H}_R}}(r^*)}{|V_\sharp|^{\frac{1}{2}}} \right| \lesssim \frac{1}{m} \text{ and } \epsilon_{u_{\mathcal{H}_R}}(-\infty) = \epsilon'_{u_{\mathcal{H}_R}}(-\infty) = 0. \tag{3.8.25}$$

This shows that

$$\|u_{\mathcal{H}_R}\|_{L^\infty(\mathbb{R})} \lesssim 1 \text{ and } \|u_{\mathcal{H}_R}{}'\|_{L^\infty(\mathbb{R})} \lesssim \||V_\sharp|^{\frac{1}{2}}\|_{L^\infty(\mathbb{R})} \lesssim |\omega| + |m| + L^{\frac{1}{2}}. \tag{3.8.26}$$

Similarly, we show that the above holds for $\omega_r m - \omega \geq \epsilon_{\text{cut}} m$. This shows (3.8.19). The bounds (3.8.20) and (3.8.21) are shown completely analogously and their proofs are omitted. $\square$

In the rest of the section we will make use of

Definition 3.8.3. *We define*

$$u_{\tilde{\mathcal{H}}_R} := e^{i(\omega-\omega_- m)r^*} u_{\mathcal{H}_R}, \tag{3.8.27}$$
$$u_{\tilde{\mathcal{H}}_L} := e^{-i(\omega-\omega_- m)r^*} u_{\mathcal{H}_L}, \tag{3.8.28}$$
$$u_{\tilde{\mathcal{CH}}_R} := e^{-i(\omega-\omega_- m)r^*} u_{\mathcal{CH}_R}, \tag{3.8.29}$$
$$u_{\tilde{\mathcal{CH}}_L} := e^{i(\omega-\omega_- m)r^*} u_{\mathcal{CH}_L}. \tag{3.8.30}$$

Lemma 3.8.3. *Assume that $|\omega - \omega_r m| \leq \epsilon_{\text{cut}} m$ for some $r \in [r_-, r_+]$ and assume that $m \in \mathbb{N}$ is sufficiently large. Let $R_1 := -\frac{1}{2\kappa_+}\log(L)$ and $R_2 := \frac{1}{2|\kappa_-|}\log(L)$. Then,*

$$\|u_{\mathcal{H}_R}\|_{L^\infty(-\infty,R_2]} \lesssim 1, \|u_{\mathcal{H}_R}{}'\|_{L^\infty(-\infty,R_2]} \lesssim |\omega| + |m| + L^{\frac{1}{2}}, \tag{3.8.31}$$
$$\|u_{\mathcal{CH}_L}\|_{L^\infty[R_1,\infty)} \lesssim 1, \|u_{\mathcal{CH}_L}{}'\|_{L^\infty[R_1,\infty)} \lesssim |\omega| + |m| + L^{\frac{1}{2}}, \tag{3.8.32}$$
$$\|u_{\mathcal{CH}_R}\|_{L^\infty[R_1,\infty)} \lesssim 1, \|u_{\mathcal{CH}_R}{}'\|_{L^\infty[R_1,\infty)} \lesssim |\omega| + |m| + L^{\frac{1}{2}}, \tag{3.8.33}$$

and

$$|\partial_\omega u_{\mathcal{H}_R}|(R_1) \lesssim \log(L), |\partial_\omega u_{\mathcal{H}_R}{}'|(R_1) \lesssim \log(L)(|\omega| + |m|), \tag{3.8.34}$$
$$|\partial_\omega u_{\mathcal{CH}_R}|(R_2) \lesssim \log(L), |\partial_\omega u_{\mathcal{CH}_R}{}'|(R_2) \lesssim \log(L)(|\omega| + |m|), \tag{3.8.35}$$
$$|\partial_\omega u_{\mathcal{CH}_L}|(R_2) \lesssim \log(L), |\partial_\omega u_{\mathcal{CH}_L}{}'|(R_2) \lesssim \log(L)(|\omega| + |m|). \tag{3.8.36}$$

Moreover,

$$\|\partial_\omega u_{\tilde{\mathcal{H}}_R}\|_{L^\infty(-\infty,R_1)} \lesssim 1,\ \|\partial_\omega u_{\tilde{\mathcal{H}}_R}{}'\|_{L^\infty(-\infty,R_1)} \lesssim 1, \tag{3.8.37}$$
$$\|\partial_\omega u_{\tilde{\mathcal{CH}}_R}\|_{L^\infty(R_2,\infty)} \lesssim 1,\ \|\partial_\omega u_{\tilde{\mathcal{CH}}_R}{}'\|_{L^\infty(R_2,\infty)} \lesssim 1, \tag{3.8.38}$$
$$\|\partial_\omega u_{\tilde{\mathcal{CH}}_L}\|_{L^\infty(R_2,\infty)} \lesssim 1,\ \|\partial_\omega u_{\tilde{\mathcal{CH}}_L}{}'\|_{L^\infty(R_2,\infty)} \lesssim 1. \tag{3.8.39}$$

Proof. From Lemma 3.8.1 we know that $L \gtrsim m^2$. Now, we write $u_{\mathcal{H}_R}$ as the solution to the Volterra equation

$$u_{\mathcal{H}_R} = e^{-i(\omega-\omega_+ m)r^*} + \int_{-\infty}^{r^*} K(r^*,y)(1+R_1-y)\tilde{V}(y)u_{\mathcal{H}_R}(y)\mathrm{d}y, \tag{3.8.40}$$

where the kernel is given by

$$K(r^*,y) = \frac{1}{1+R_1-y}\frac{\sin((\omega-\omega_+ m)(r^*-y))}{\omega-\omega_+ m} \tag{3.8.41}$$

and $\tilde{V} := V_\sharp + V_1 + (\omega-\omega_+ m)^2$. For $y \in (-\infty, R_1)$, a direct computation shows

$$(1+R_1-y)|\tilde{V}(y)| \lesssim (1+R_1-y)Le^{2\kappa_+ y}, \quad \int_{-\infty}^{R_1}(1+R_1-y)|\tilde{V}(y)|\mathrm{d}y \lesssim 1$$

and

$$\sup_{y\leq r^*\leq R_1} |K(r^*,y)| \lesssim 1. \tag{3.8.42}$$

Standard estimates on Volterra integral equations (apply [119, Chapter 6, Theorem 10.1] to the term $u_{\mathcal{H}_R} - e^{-i(\omega-\omega_+ m)r^*}$) yield

$$\|u_{\mathcal{H}_R}\|_{L^\infty(-\infty,R_1)} \lesssim 1 \tag{3.8.43}$$
$$\begin{aligned}\|u_{\mathcal{H}_R}{}'\|_{L^\infty(-\infty,R_1)} &\lesssim 1 + |\omega-\omega_+ m| \lesssim 1 + |\omega-\omega_{R_1} m| + m|\omega_{R_1}-\omega_+| \\ &\lesssim 1 + |\omega-\omega_{R_1} m| + m|\Delta(R_1)| \lesssim 1 + |\omega-\omega_{R_1} m|. \end{aligned}\tag{3.8.44}$$

Now, for the region $r^* \in [R_1, R_2]$ we approximate $u_{\mathcal{H}_R}$ with a WKB approximation. To do so we remark that for $r^* \in [R_1, R_2]$ we have

$$-V_\sharp \gtrsim 1 \text{ and } \left|\frac{V_\sharp'}{V_\sharp}\right|, \left|\frac{V_\sharp''}{V_\sharp}\right| \lesssim |\Delta| \tag{3.8.45}$$

and the error term satisfies $|V_1| \lesssim |\Delta|$. Thus, the error control function

$$F_{u_{\mathcal{H}_{R2}}}(r^*) := \int_{R_1}^{r^*} \frac{1}{|V_\sharp|^{\frac{1}{4}}} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(|V_\sharp|^{-\frac{1}{4}}\right) - \frac{V_1}{|V_\sharp|^{\frac{1}{2}}} \mathrm{d}y \tag{3.8.46}$$

is bounded as $\mathcal{V}_{R_1,R_2}(F_{u_{\mathcal{H}_{R2}}}) \lesssim 1$. This allows us to apply [119, Chapter 6, Theorem 2.2] to deduce that

$$\begin{aligned} u_{\mathcal{H}_R} &= A_{u_{\mathcal{H}_R}} u_{\mathrm{WKB}_A} + B_{u_{\mathcal{H}_R}} u_{\mathrm{WKB}_B} \\ &= A_{u_{\mathcal{H}_R}} \frac{|V_\sharp(R_1)|^{\frac{1}{4}}}{|V_\sharp(r^*)|^{\frac{1}{4}}} \exp\left(-i \int_{R_1}^{r^*} |V_\sharp(y)|^{\frac{1}{2}} \mathrm{d}y\right) \left(1 + \epsilon_{u_{\mathcal{H}_{RA}}}(r^*)\right) \\ &+ B_{u_{\mathcal{H}_R}} \frac{|V_\sharp(R_1)|^{\frac{1}{4}}}{|V_\sharp(r^*)|^{\frac{1}{4}}} \exp\left(i \int_{R_1}^{r^*} |V_\sharp(y)|^{\frac{1}{2}} \mathrm{d}y\right) \left(1 + \epsilon_{u_{\mathcal{H}_{RB}}}(r^*)\right), \end{aligned} \tag{3.8.47}$$

for

$$A_{u_{\mathcal{H}_R}} = \frac{\mathfrak{W}(u_{\mathrm{WKB}_B}, u_{\mathcal{H}_R})}{\mathfrak{W}(u_{\mathrm{WKB}_B}, u_{\mathrm{WKB}_A})} \text{ and } B_{u_{\mathcal{H}_R}} = \frac{\mathfrak{W}(u_{\mathrm{WKB}_A}, u_{\mathcal{H}_R})}{\mathfrak{W}(u_{\mathrm{WKB}_A}, u_{\mathrm{WKB}_B})}. \tag{3.8.48}$$

Moreover,

$$\sup_{r^* \in [R_1, R_2]} |\epsilon_{u_{\mathcal{H}_{RA}}}(r^*)| \lesssim 1, \tag{3.8.49}$$

$$\sup_{r^* \in [R_1, R_2]} |\epsilon'_{u_{\mathcal{H}_{RA}}}(r^*)| \lesssim \sup_{r^* \in [R_1, R_2]} |V_\sharp|^{\frac{1}{2}} \lesssim L^{\frac{1}{2}} + |m| + |\omega|, \tag{3.8.50}$$

$$\epsilon_{u_{\mathcal{H}_{RA}}}(R_1) = \epsilon'_{u_{\mathcal{H}_{RA}}}(R_1) = 0, \tag{3.8.51}$$

and analogously for $\epsilon_{u_{\mathcal{H}_{RB}}}$. Evaluating the Wronskians at $r^* = R_1$, we obtain

$$|A_{u_{\mathcal{H}_R}}|, |B_{u_{\mathcal{H}_R}}| \lesssim 1 \tag{3.8.52}$$

in view of (3.8.43) and (3.8.44). This shows that

$$\|u_{\mathcal{H}_R}\|_{L^\infty(-\infty, R_2)} \lesssim 1, \tag{3.8.53}$$

$$\|u_{\mathcal{H}_R}{}'\|_{L^\infty(-\infty, R_2)} \lesssim |\omega| + |m| + L^{\frac{1}{2}}. \tag{3.8.54}$$

To show the bound on $\partial_\omega u_{\mathcal{H}_R}$ we consider $u_{\tilde{\mathcal{H}}_R}$. Then, $u_{\tilde{\mathcal{H}}_R}$ satisfies the Volterra equation

$$u_{\tilde{\mathcal{H}}_R} = 1 + \int_{-\infty}^{r^*} \frac{\tilde{K}(r^*, y)}{1 + R_1 - y} (1 + R_1 - y) \tilde{V}(y) u_{\tilde{\mathcal{H}}_R}(y) \mathrm{d}y, \tag{3.8.55}$$

where $\tilde{K}(r^*,y) = e^{i(\omega-\omega_+ m)(r^*-y)}\frac{\sin((\omega-\omega_+ m)(r^*-y))}{\omega-\omega_+ m}$. Completely analogous to before, it follows that

$$\|u_{\tilde{\mathcal{H}}_R}\|_{L^\infty(-\infty,R_1)} \lesssim 1 \text{ and } \|u_{\tilde{\mathcal{H}}_R}{}'\|_{L^\infty(-\infty,R_1)} \lesssim 1. \tag{3.8.56}$$

Now $\partial_\omega u_{\tilde{\mathcal{H}}_R}$ solves

$$\begin{aligned}\partial_\omega u_{\tilde{\mathcal{H}}_R} = &\int_{-\infty}^{r^*} \left(\partial_\omega \tilde{K}(r^*,y)\tilde{V}(y) + \tilde{K}(r^*,y)\partial_\omega \tilde{V}(y)\right) u_{\tilde{\mathcal{H}}_R}(y)\mathrm{d}y \\ &+ \int_{-\infty}^{r^*} \frac{\tilde{K}(r^*,y)}{1+R_1-y}(1+R_1-y)\tilde{V}(y)\partial_\omega u_{\tilde{\mathcal{H}}_R}(y)\mathrm{d}y. \end{aligned} \tag{3.8.57}$$

As $|\partial_\omega \lambda_{m\ell}(a\omega)| \lesssim |m|$ from Lemma 3.3.2, we conclude that

$$|\partial_\omega \tilde{V}| \lesssim |\Delta| m \text{ and } |\partial_\omega \tilde{K}(r^*,y)| \lesssim (r^*-y)^2 \tag{3.8.58}$$

such that

$$\int_{-\infty}^{R_1} \left| \left(\partial_\omega \tilde{K}(r^*,y)\tilde{V}(y) + \tilde{K}(r^*,y)\partial_\omega \tilde{V}(y)\right) u_{\tilde{\mathcal{H}}_R}(y)\right| \mathrm{d}y \lesssim 1. \tag{3.8.59}$$

Again, by standard bounds on Volterra integral equations [119, Chapter 6, §10] and using (3.8.53), (3.8.53), we obtain

$$\|\partial_\omega u_{\tilde{\mathcal{H}}_R}\|_{L^\infty(-\infty,R_1)} \lesssim 1 \tag{3.8.60}$$

and

$$\|\partial_\omega u_{\tilde{\mathcal{H}}_R}{}'\|_{L^\infty(-\infty,R_1)} \lesssim 1. \tag{3.8.61}$$

This shows (3.8.37)–(3.8.39). Now, we write

$$\partial_\omega u_{\mathcal{H}_R} = \partial_\omega (e^{-i(\omega-\omega_+ m)r^*} u_{\tilde{\mathcal{H}}_R}) = -ir^* u_{\mathcal{H}_R} + e^{-i(\omega-\omega_+ m)r^*}\partial_\omega u_{\tilde{\mathcal{H}}_R} \tag{3.8.62}$$

and

$$\partial_\omega u_{\mathcal{H}_R}{}' = -iu_{\mathcal{H}_R} - ir^* u_{\mathcal{H}_R}{}' - i(\omega-\omega_+ m)e^{-i(\omega-\omega_+ m)r^*}\partial_\omega u_{\tilde{\mathcal{H}}_R} + e^{-i(\omega-\omega_+ m)r^*}\partial_\omega u_{\tilde{\mathcal{H}}_R}{}'. \tag{3.8.63}$$

Evaluating this at $r^* = R_1$ we obtain

$$|\partial_\omega u_{\mathcal{H}_R}|(R_1) \lesssim |R_1| \lesssim \log(L). \tag{3.8.64}$$

and

$$|\partial_\omega u_{\mathcal{H}_R}{}'|(R_1) \lesssim \log(L)(|m| + |\omega|). \tag{3.8.65}$$

This shows (3.8.34)–(3.8.36). The proofs for $u_{\mathcal{CH}_L}$ and $u_{\mathcal{CH}_R}$ are completely analogous. $\square$

Lemma 3.8.4. *The renormalized transmission and reflection coefficients satisfy*

$$2|\mathfrak{t}| = |\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]| \lesssim |m| + |\omega| + L^{\frac{1}{2}} \tag{3.8.66}$$

$$2|\mathfrak{r}| = |\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}]| \lesssim |m| + |\omega| + L^{\frac{1}{2}} \tag{3.8.67}$$

and

$$\sup_{\omega\in(\omega_- m-1, \omega_- m+1)} 2\,|\partial_\omega \mathfrak{t}| = \sup_{\omega\in(\omega_- m-1, \omega_- m+1)} |\partial_\omega \mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]| \lesssim (|m| + L^{\frac{1}{2}})\log(L) \tag{3.8.68}$$

$$\sup_{\omega\in(\omega_- m-1, \omega_- m+1)} 2\,|\partial_\omega \mathfrak{r}| = \sup_{\omega\in(\omega_- m-1, \omega_- m+1)} |\partial_\omega \mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}]| \lesssim (|m| + L^{\frac{1}{2}})\log(L). \tag{3.8.69}$$

Moreover, we have

$$|\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](r^*)| + |\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}](r^*)| \lesssim \log(L)(|\omega| + |m| + L^{\frac{1}{2}}), \tag{3.8.70}$$

$$|\mathfrak{W}[u_{\mathcal{H}_R}, \partial_\omega u_{\mathcal{CH}_R}](r^*)| + |\mathfrak{W}[u_{\mathcal{H}_R}, \partial_\omega u_{\mathcal{CH}_L}](r^*)| \lesssim \log(L)(|\omega| + |m| + L^{\frac{1}{2}}) \tag{3.8.71}$$

uniformly for $r^* \in [R_1, R_2]$.

Proof. The bounds (3.8.66) and (3.8.67) follow directly from Lemma 3.8.2 and Lemma 3.8.3. To show (3.8.68) we evaluate the Wronskian at $r^* = 0$ and write

$$\begin{aligned}\partial_\omega \mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}] &= \partial_\omega \mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](r^* = 0)\\ &= \mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](r^* = 0) + \mathfrak{W}[u_{\mathcal{H}_R}, \partial_\omega u_{\mathcal{CH}_R}](r^* = 0).\end{aligned} \tag{3.8.72}$$

Hence, (3.8.68) follows from (3.8.70). To show (3.8.70), we apply the fundamental theorem of calculus for $R_1 \leq r^* \leq R_2$ and obtain

$$|\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](r^*)| \leq \int_{R_1}^{r^*} |\partial_{r^*}\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]|\mathrm{d}r^* + |\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](R_1)|. \tag{3.8.73}$$

Since $\partial_{r^*}\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}] = -u_{\mathcal{H}_R}u_{\mathcal{CH}_R}\partial_\omega(V_\sharp + V_1)$, we conclude in view of Lemma 3.8.3 that

$$\sup_{r^*\in[R_1,R_2]} |\partial_{r^*}\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]| \lesssim |m| + |\omega|. \tag{3.8.74}$$

From the proof of Lemma 3.8.3 we also have

$$|\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](R_1)| \lesssim \log(L)(|\omega| + |m| + L^{\frac{1}{2}}) \tag{3.8.75}$$

such that

$$\sup_{r^*\in[R_1,R_2]} |\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](r^*)| \lesssim \log(L)(|\omega| + |m| + L^{\frac{1}{2}}) \tag{3.8.76}$$

follows. Similarly, we obtain

$$\sup_{r^*\in[R_1,R_2]} |\mathfrak{W}[u_{\mathcal{H}_R}, \partial_\omega u_{\mathcal{CH}_R}](r^*)| \lesssim \log(L)(|\omega| + |m|) \tag{3.8.77}$$

leading to (3.8.68) and (3.8.70). Completely analogously we obtain (3.8.69) as well as (3.8.71). $\square$

With the above lemma in hand we conclude

Lemma 3.8.5. *Let $m \in \mathbb{N}$ be sufficiently large and let $\epsilon > 0$ be sufficiently small only depending on the black hole parameters. Then,*

$$\sup_{|\omega-\omega_+ m|\leq\epsilon} \|\partial_\omega u_{\tilde{\mathcal{H}}_R}\|_{L^\infty(-\infty,0)} \lesssim L^{\frac{1}{2}}\log(L), \tag{3.8.78}$$

$$\sup_{|\omega-\omega_- m|\leq\epsilon} \|\partial_\omega u_{\tilde{\mathcal{CH}}_R}\|_{L^\infty(0,\infty)} \lesssim L^{\frac{1}{2}}\log(L), \tag{3.8.79}$$

$$\sup_{|\omega-\omega_- m|\leq\epsilon} \|\partial_\omega u_{\tilde{\mathcal{CH}}_L}\|_{L^\infty(0,\infty)} \lesssim L^{\frac{1}{2}}\log(L). \tag{3.8.80}$$

Proof. We again only show the claim for $u_{\tilde{\mathcal{H}}_R}$ as the other cases are completely analogous. Assume that $|\omega - \omega_+ m| \leq \epsilon$ for some $\epsilon > 0$ sufficiently small. In view of Lemma 3.8.3 it suffices to consider the region $r^* \in [R_1, 0]$. Now, note that

$$\begin{aligned}\partial_\omega u_{\mathcal{H}_R} =& \frac{1}{\mathfrak{W}[u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L}]}\Big(u_{\mathcal{CH}_L}\int_{R_1}^{r^*} u_{\mathcal{CH}_R} u_{\mathcal{H}_R}\partial_\omega(-V_\sharp - V_1) \\ & - u_{\mathcal{CH}_R}\int_{R_1}^{r^*} u_{\mathcal{H}_R} u_{\mathcal{CH}_L}\partial_\omega(-V_\sharp - V_1)\Big) \\ & + \frac{\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_L}](R_1)}{\mathfrak{W}[u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L}]} u_{\mathcal{CH}_R} + \frac{\mathfrak{W}[\partial_\omega u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}](R_1)}{\mathfrak{W}[u_{\mathcal{CH}_L}, u_{\mathcal{CH}_R}]} u_{\mathcal{CH}_L}.\end{aligned} \tag{3.8.81}$$

Hence, using Lemma 3.8.3, Lemma 3.8.4,

$$\sup_{r^* \in [R_1, R_2]} |\partial_\omega(V_\sharp + V_1)| \lesssim |m|, \tag{3.8.82}$$

as well as the lower bound $|\mathfrak{W}[u_{\mathcal{CH}_R}, u_{\mathcal{CH}_L}]| \gtrsim |m|$, we obtain

$$\sup_{r^* \in [R_1, 0]} |\partial_\omega u_{\mathcal{H}_R}| \lesssim L^{\frac{1}{2}} \log(L). \tag{3.8.83}$$

In view of $u_{\tilde{\mathcal{H}}_R} = e^{i(\omega - \omega_+ m) r^*} u_{\mathcal{H}_R}$ and the chain rule, the claim follows. □

Lemma 3.8.6. *The renormalized transmission and reflection coefficients satisfy*

$$|\mathfrak{t}^{\omega_-}| \gtrsim |m| \text{ and } |\mathfrak{r}^{\omega_-}| \gtrsim |m|. \tag{3.8.84}$$

Proof. Throughout the proof we assume that $\omega = \omega_- m$. As $u_{\mathcal{CH}_R} = u_{\mathcal{CH}_L}$ for $\omega = \omega_- m$, it suffices to bound the Wronskian $|\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]|$ from below. To do so, let $\mathfrak{A}$ and $\mathfrak{B}$ be the unique coefficients satisfying $u_{\mathcal{CH}_R} = \mathfrak{A} u_{\mathcal{H}_R} + \mathfrak{B} u_{\mathcal{H}_L}$. From $u_{\mathcal{CH}_R} = \overline{u_{\mathcal{CH}_R}}$ it follows that $u_{\mathcal{CH}_R} = 2\mathrm{Re}(\mathfrak{A} u_{\mathcal{H}_R})$. Now, for $\epsilon > 0$ to be chosen later, define

$$R_2^\epsilon := \frac{1}{2|\kappa_-|}\log(L) + \frac{1}{\epsilon}. \tag{3.8.85}$$

Now, $u_{\mathcal{CH}_R} - 1$ is a solution to the Volterra equation

$$u_{\mathcal{CH}_R} - 1 = \int_{r^*}^{\infty} \frac{y - r^*}{y - R_2^\epsilon}(y - R_2^\epsilon)\tilde{V}(y)\left[(u_{\mathcal{CH}_R} - 1) + 1\right] \mathrm{d}y, \tag{3.8.86}$$

where $\tilde{V} = V_1 + V_\sharp(\omega = \omega_- m)$. We have

$$\int_{R_2^\epsilon}^{\infty} (y - R_2^\epsilon)\tilde{V}(y) \lesssim L e^{-2|\kappa_-| R_2^\epsilon} \lesssim e^{-\frac{2|\kappa_-|}{\epsilon}}. \tag{3.8.87}$$

Using bounds on solutions to Volterra integral equations as before (see [119, Chapter 6, §10]), we obtain that

$$\|u_{\mathcal{CH}_R} - 1\|_{L^\infty(R_2^\epsilon,\infty)} < \frac{1}{2} \tag{3.8.88}$$

for $\epsilon > 0$ small enough depending on the choice of parameters M, a, l. Thus,

$$\frac{1}{2} < u_{\mathcal{CH}_R}(R_2^\epsilon) = 2\mathrm{Re}(\mathfrak{A} u_{\mathcal{H}_R}(R_2^\epsilon)) \lesssim 2|\mathfrak{A}|\|u_{\mathcal{H}_R}\|_{L^\infty(-\infty,R_2^\epsilon)}. \tag{3.8.89}$$

Note that (3.8.31) also holds if we replace R_2 by R_2^ϵ for some fixed value of $\epsilon > 0$. Thus, we conclude that $|\mathfrak{B}| = |\mathfrak{A}| \gtrsim 1$ which shows

$$|\mathfrak{W}[u_{\mathcal{H}_R}, u_{\mathcal{CH}_R}]| \gtrsim (\omega_- - \omega_+)|m| \gtrsim |m|. \tag{3.8.90}$$

This concludes the proof. □

3.8.2 Representation formula for ψ on the interior

Proposition 3.8.1. *Let $\psi_0 \in C_c^\infty(\mathcal{H}_R)$ and let $\tilde\psi \in C^\infty(\mathcal{B})$ be the arising solution of* (3.1.1) *with vanishing data on $\mathcal{H}_L$. Then,*

$$\tilde\psi(v,r,\theta,\tilde\phi_+) = \frac{1}{\sqrt{2\pi(r^2+a^2)}} \int_{\mathbb{R}} \sum_{m\ell} e^{-i\omega(v-r^*)} e^{im(\tilde\phi_+ - \omega_+ r^*)} S_{m\ell}(a\omega,\cos\theta) \mathcal{F}_{\mathcal{H}}[\psi_0] u_{\mathcal{H}_R} \mathrm{d}\omega, \tag{3.8.91}$$

where $u_{\mathcal{H}_R}$ is defined in (3.8.1) *and*

$$\mathcal{F}_{\mathcal{H}}[\psi_0](\omega,m,\ell) := \frac{\sqrt{r_+^2+a^2}}{\sqrt{2\pi}} \int_{\mathbb{S}^2}\int_{\mathbb{R}} \psi_0(v,\theta,\tilde\phi_+) e^{i\omega v} e^{-im\tilde\phi_+} S_{m\ell}(a\omega,\cos\theta) \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}\omega. \tag{3.8.92}$$

Moreover, in $\mathcal{B}$ we have

$$\begin{aligned}\tilde{\psi}(v,r,\theta,\tilde{\phi}_-) = &\frac{1}{\sqrt{2\pi(r^2+a^2)}}\sum_{m\ell} \text{p.v.}\int_{\mathbb{R}} e^{-i\omega(v-r^*)}e^{im(\tilde{\phi}_- - \omega_- r^*)}\\ &\qquad S_{m\ell}(a\omega,\cos\theta)\mathcal{F}_{\mathcal{H}}[\psi_0]\frac{\mathfrak{t}(\omega,m,\ell)}{\omega-\omega_- m}u_{\mathcal{CH}_L}\mathrm{d}\omega\\ &+\frac{1}{\sqrt{2\pi(r^2+a^2)}}\sum_{m\ell} \text{p.v.}\int_{\mathbb{R}} e^{-i\omega(v-r^*)}e^{im(\tilde{\phi}_- - \omega_- r^*)}\\ &\qquad S_{m\ell}(a\omega,\cos\theta)\mathcal{F}_{\mathcal{H}}[\psi_0]\frac{\mathfrak{r}(\omega,m,\ell)}{\omega-\omega_- m}u_{\mathcal{CH}_R}\mathrm{d}\omega\end{aligned} \tag{3.8.93}$$

as well as

$$\begin{aligned}\tilde{\psi}(u,r,\theta,\phi^*_-) = &\frac{1}{\sqrt{2\pi(r^2+a^2)}}\sum_{m\ell} \text{p.v.}\int_{\mathbb{R}} e^{i\omega(u-r^*)}e^{im(\phi^*_- + \omega_- r^*)}\\ &\qquad S_{m\ell}(a\omega,\cos\theta)\mathcal{F}_{\mathcal{H}}[\psi_0]\frac{\mathfrak{t}(\omega,m,\ell)}{\omega-\omega_- m}u_{\mathcal{CH}_L}\mathrm{d}\omega\\ &+\frac{1}{\sqrt{2\pi(r^2+a^2)}}\sum_{m\ell} \text{p.v.}\int_{\mathbb{R}} e^{i\omega(u-r^*)}e^{im(\phi^*_- + \omega_- r^*)}\\ &\qquad S_{m\ell}(a\omega,\cos\theta)\mathcal{F}_{\mathcal{H}}[\psi_0]\frac{\mathfrak{r}(\omega,m,\ell)}{\omega-\omega_- m}u_{\mathcal{CH}_R}\mathrm{d}\omega.\end{aligned} \tag{3.8.94}$$

Proof. Note that $\mathcal{F}_{\mathcal{H}}[\psi_0](\omega,m,\ell)$ is rapidly decaying and smooth which follows from the fact that $\psi_0 \in C_c^\infty(\mathcal{H})$. Moreover, in view of Lemma 3.8.2 and Lemma 3.8.3, we have that the right hand side of (3.8.91) is a smooth solution to (3.1.1) in the interior region $\mathcal{B}$. Now, the claims follows from the uniqueness of the characteristic problem and the fact that the right hand side of (3.8.91) converges to ψ_0 as $r \to r_+$. The other formulas follow from a change of coordinates. □

Before we prove the blow-up result, we need one more final ingredient which is a consequence of the domain of dependence.

Lemma 3.8.7. *Let $\tilde{\psi} \in C^\infty(\mathcal{B})$ be a solution to* (3.1.1) *arising from vanishing data on $\mathcal{H}_L$ and smooth data $\psi_0 \in C^\infty(\mathcal{H}_R)$ posed on $\mathcal{H}_R$. Then, $\tilde{\psi}(u_0,r_0,\theta_0,\phi^*_-)$ only depends on $\psi_0 \restriction_{\{v\leq 2r^*(r_0)-u_0+\tilde{c}\}}$, where $\tilde{c}>0$ is a constant only depending on the black hole parameters.*

Proof. In coordinates $(v,r,\theta,\tilde{\phi}_-)$ (or equivalently in coordinates $(v,r,\theta,\tilde{\phi}_+)$) define the function $\tilde{v} := v + f(r)$ on $\mathcal{B}$ and choose f to satisfy $\frac{\mathrm{d}f}{\mathrm{d}r} = -\sqrt{\frac{a^2}{\Xi}\frac{1}{|\Delta|}}$ with initial condition $f(r_+)=0$. This is well defined as $\frac{1}{\sqrt{|\Delta|}}$ is integrable at the event and Cauchy horizons. Now f is non-negative and satisfies $\sup_{r\in(r_-,r_+)} f \leq \tilde{c}$ for a constant $\tilde{c}>0$ only depending on the black hole parameters. A computation also shows that, uniformly on $\mathcal{B}$, we have

$$g_{\mathrm{KAdS}}(\nabla\tilde{v},\nabla\tilde{v}) = \frac{a^2\sin^2}{\Sigma\Delta_\theta} - \frac{a^2}{\Sigma\Xi} < 0 \text{ and } g_{\mathrm{KAdS}}(\nabla\tilde{v},-\nabla r) < 0. \tag{3.8.95}$$

This means that $\nabla\tilde{v}$ is a future-directed timelike vector field. Thus, level sets of the function $\tilde{v}$ are spacelike.

Now, consider

$$\tilde{\psi}(u_0, r_0, \theta_0, \phi_-^*). \tag{3.8.96}$$

Since $\nabla\tilde{v}$ is future directed and timelike, it follows from the domain of dependence that (3.8.96) only depends on

$$\psi_0 \restriction_{\{\tilde{v}(v,r_+)\leq\tilde{v}(v(r_0,u_0),r_0)\}} = \psi_0 \restriction_{\{v\leq 2r^*(r_0)-u_0+f(r_0)\}}, \tag{3.8.97}$$

since $\tilde{v}(v(r_0,u_0),r_0) = 2r^*(r_0) - u_0 + f(r_0)$. This concludes the proof. $\square$

3.8.3 Proof of Theorem 3.1

We recall that the cosmological constant $\Lambda < 0$ (and thus $l = \sqrt{-3/\Lambda} > 0$) was arbitrary but fixed as in (3.2.6).

Theorem 3.1. *Conjecture 5 holds true.*

More precisely, let the dimensionless black hole parameters $(\mathfrak{m},\mathfrak{a}) \in \mathscr{P}_{\text{Blow-up}}$ *be arbitrary but fixed as in* (3.6.1)*, where* $\mathscr{P}_{\text{Blow-up}}$ *is defined in Definition 3.5.3.*

Let $\psi \in C^\infty(\mathcal{M}_{\mathrm{KAdS}} \setminus \mathcal{CH})$ *be the unique solution to* (3.1.1) *arising from the smooth and compactly supported data specified in Definition 3.6.2 on Kerr–AdS with parameters* $(M,a) = (\mathfrak{m}/\sqrt{-\Lambda}, \mathfrak{a}/\sqrt{-\Lambda})$.

Then, for each $u_0 \in \mathbb{R}$*, the solution* ψ *blows up at the Cauchy horizon* $\mathcal{CH}_R$ *as*

$$\lim_{r\to r_-} \|\psi(u_0,r)\|^2_{L^2(\mathbb{S}^2)} = +\infty. \tag{3.8.98}$$

Moreover, $\mathscr{P}_{\text{Blow-up}} \subset \mathscr{P}$ *has the following properties:*

- $\mathscr{P}_{\text{Blow-up}}$ *is Baire-generic,*
- $\mathscr{P}_{\text{Blow-up}}$ *is a Lebesgue-exceptional* ($\mathscr{P}_{\text{Blow-up}}$ *has zero Lebesgue measure),*
- $\mathscr{P}_{\text{Blow-up}}$ *has full packing dimension* $\dim_P(\mathscr{P}_{\text{Blow-up}}) = 2$.

Remark 3.8.2. *The above statement shows that generic data (i.e. data which do not satisfy the co-dimension 1 property as described in Remark 3.1.1) lead to solutions which blow up as in* (3.8.98). *Thus, the above statement indeed proves that Conjecture 5 holds true.*

Proof of Theorem 3.1. The stated properties of $\mathscr{P}_{\text{Blow-up}}$ on the Baire-genericity, the zero Lebesgue measure and the full packing dimension follow from Proposition 3.5.2, Proposition 3.5.3 and Proposition 3.5.4, respectively.

We now turn to the proof of (3.8.98). First, we write $\psi_0 := \psi \restriction_{\mathcal{H}_R}$ and note that

$$D := \sum_{0 \leq i+j \leq 4} \int_{\mathbb{R} \times \mathbb{S}^2} |\slashed{\nabla}^i K_+^j \psi_0(v, \theta, \tilde{\phi}_+)|^2 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}v < \infty \tag{3.8.99}$$

in view of Theorem 3.2. Now, let $u_0 \in \mathbb{R}$ be fixed and let $r_n^* \to \infty$ be a sequence with $r_n^* > r_0^*$ for sufficiently large r_0^*. We will first prove that as $r_n^* \to \infty$, we have that

$$\|\psi(u_0, r_n^*)\|^2_{L^2(\mathbb{S}^2)} = \sum_{m\ell} \left| \pi \frac{\mathfrak{t}^{\omega_-} u_{\tilde{\mathcal{CH}_L}}{}^{\omega_-}(r_n^*)}{\sqrt{r_n^2 + a^2}} a_{\mathcal{H}}^{R_n}(\omega = \omega_- m) \right|^2 + \mathrm{Err}(D), \tag{3.8.100}$$

where $|\mathrm{Err}(D)| \lesssim_{u_0} D$ uniformly for all $r_n^* \geq r_0^*$ and $R_n := 2r_n^* - u_0 + \tilde{c}$. Also recall the definition of $a_{\mathcal{H}}^R$ in (3.7.30). Here we also introduced the notation $u_{\tilde{\mathcal{CH}_L}}{}^{\omega_-} := u_{\tilde{\mathcal{CH}_L}}(\omega = \omega_- m)$. Once we have established (3.8.100), the blow-up result of (3.8.98) will be proved.

Thus, we now turn to the proof of (3.8.100). In view of Lemma 3.8.7 we have that $\psi(u_0, r_n^*, \theta, \phi_-^*)$ only depends on $\psi_0 \restriction_{\{v \leq 2r_n^* - u_0 + \tilde{c}\}}$. Consider now

$$\psi_0^n(v, \theta, \tilde{\phi}_+) := \psi_0^{R_n}(v, \theta, \tilde{\phi}_+), \tag{3.8.101}$$

where $\psi_0^{R_n}(v, \theta, \tilde{\phi}_+) = \psi_0(v, \theta, \tilde{\phi}_+)\chi(R_n - v)$ is defined in (3.7.2) with $R_n = 2r_n^* - u_0 + \tilde{c}$. Now, $\psi(u_0, r_n^*, \theta, \phi_-^*)$ only depends on ψ_0^n.

Using the representation formula (3.8.94) in Proposition 3.8.1 we write

$$\begin{aligned}\psi(u_0, r_n^*, \theta, \phi_-^*) = {} & \frac{1}{\sqrt{2\pi(r_n^2+a^2)}} \sum_{m\ell} \text{p.v.} \int_{\mathbb{R}} e^{i\omega(u_0 - r_n^*)} e^{im(\phi_-^* + \omega_- r_n^*)} \\ & \qquad \times S_{m\ell}(a\omega, \cos\theta) \mathcal{F}_{\mathcal{H}}[\psi_0^n] \frac{\mathfrak{t}(\omega, m, \ell)}{\omega - \omega_- m} u_{\mathcal{CH}_L} \mathrm{d}\omega \\ & + \frac{1}{\sqrt{2\pi(r_n^2+a^2)}} \sum_{m\ell} \text{p.v.} \int_{\mathbb{R}} e^{i\omega(u_0 - r_n^*)} e^{im(\phi_-^* + \omega_- r_n^*)} \\ & \qquad \times S_{m\ell}(a\omega, \cos\theta) \mathcal{F}_{\mathcal{H}}[\psi_0^n] \frac{\mathfrak{r}(\omega, m, \ell)}{\omega - \omega_- m} u_{\mathcal{CH}_R} \mathrm{d}\omega \\ & =: I + II. \end{aligned} \tag{3.8.102}$$

We consider both terms individually and start with the term I. Moreover, we split the term I into $|a\omega - a\omega_- m| < \frac{1}{m}$ and $|a\omega - a\omega_- m| \geq \frac{1}{m}$ and call the terms I_{res} and $I_{\text{non-res}}$, respectively, such that $I = I_{\text{res}} + I_{\text{non-res}}$. First, we claim that the spherical L^2-norm of the term

$$\begin{aligned} I_{\text{non-res}} = {} & \frac{1}{\sqrt{2\pi(r_n^2+a^2)}} \sum_{m\ell} \text{p.v.} \int_{|\omega_- m - \omega| \geq \frac{1}{am}} e^{i\omega(u_0 - r_n^*)} e^{im(\phi_-^* + \omega_- r_n^*)} \\ & \qquad \times S_{m\ell}(a\omega, \cos\theta) \mathcal{F}_{\mathcal{H}}[\psi_0^n] \frac{\mathfrak{t}(\omega, m, \ell)}{\omega - \omega_- m} u_{\mathcal{CH}_L} \mathrm{d}\omega \end{aligned} \tag{3.8.103}$$

is controlled by D uniformly as $r_n^* \to \infty$.

Lemma 3.8.8. *We have $\|I_{\text{non-res}}\|^2_{L^2(\mathbb{S}^2)}(r_n^*, u_0) \lesssim D$ for all r_n^* large enough.*

Proof. Using $|\frac{1}{\omega - \omega_- m}| \leq am$ in the integrand of (3.8.103) and $\int_0^{2\pi} e^{i(m-\tilde{m})\phi} \mathrm{d}\phi = 2\pi \delta_{m\tilde{m}}$ we estimate

$$\|I_{\text{non-res}}\|^2_{L^2(\mathbb{S}^2)} \lesssim \sum_m m^2 \int_0^\pi \left| \sum_{\ell \geq |m|} \int_{\mathbb{R}} |S_{m\ell}(a\omega, \cos\theta) \mathcal{F}_{\mathcal{H}}[\psi_0^n] \, \mathfrak{t} \, u_{\mathcal{CH}_L}| \mathrm{d}\omega \right|^2 \sin\theta \mathrm{d}\theta. \tag{3.8.104}$$

From the Cauchy–Schwarz inequality as well as Lemma 3.8.2, Lemma 3.8.3 and Lemma 3.8.4, we obtain

$$\|I_{\text{non-res}}\|^2_{L^2(\mathbb{S}^2)} \lesssim \sum_m \Big[\int_0^\pi \sum_{\tilde\ell\geq|m|} \int_{\mathbb{R}} \frac{|S_{m\tilde\ell}(a\omega,\cos\theta)|^2}{(1+\omega^2)(1+\Lambda_{m\tilde\ell})} \mathrm{d}\omega \sin\theta \mathrm{d}\theta \\ \times \sum_{\ell\geq|m|} \int_{\mathbb{R}} (1+\omega^2)(1+\Lambda_{m\ell}) m^2 (1+\omega^2+\Lambda_{m\ell}) |\mathcal{F}_{\mathcal{H}}[\psi_0^n]|^2 \mathrm{d}\omega \Big] \\ \lesssim \sum_m \sum_{\ell\geq|m|} \int_{\mathbb{R}} (1+\omega^2)(1+\Lambda_{m\ell}) m (1+\omega^2+\Lambda_{m\ell}) |\mathcal{F}_{\mathcal{H}}[\psi_0^n]|^2 \mathrm{d}\omega \lesssim D, \tag{3.8.105}$$

where we have used that $\Lambda_{m\ell} \geq \Xi^2 \ell(\ell+1)$ such that $\sum_{\tilde\ell\geq|m|} \frac{1}{1+\Lambda_{m\tilde\ell}} \lesssim \frac{1}{m}$. □

Now, we turn to the term I_{res}:

$$I_{\text{res}} = \frac{1}{\sqrt{2\pi(r_n^2+a^2)}} \sum_{m\ell} \text{p.v.} \int_{\omega_- m - \frac{1}{am}}^{\omega_- m + \frac{1}{am}} e^{i\omega(u_0 - r_n^*)} e^{im(\phi_-^* + \omega_- r_n^*)} \\ \times S_{m\ell}(a\omega,\cos\theta) \mathcal{F}_{\mathcal{H}}[\psi_0^n] \frac{\mathfrak{t}(\omega,m,\ell)}{\omega - \omega_- m} u_{\mathcal{CH}_L} \mathrm{d}\omega \tag{3.8.106}$$

and write $u_{\mathcal{CH}_L} = e^{-i(\omega-\omega_- m)r^*} \tilde{u_{\mathcal{CH}_L}}$. Then,

$$I_{\text{res}} = I_{\text{res}}^a + I_{\text{res}}^b = \frac{1}{\sqrt{2\pi(r_n^2+a^2)}} \sum_{m\ell} \text{p.v.} \int_{\omega_- m - \frac{1}{am}}^{\omega_- m + \frac{1}{am}} \frac{e^{-2i(\omega-\omega_- m)r_n^*} e^{i\omega u_0} \mathcal{F}_{\mathcal{H}}[\psi_0^n]}{\omega - \omega_- m} \mathrm{d}\omega \\ \times e^{im\phi_-^*} S_{m\ell}(a\omega_- m,\cos\theta) \mathfrak{t}(\omega_- m, m,\ell) \tilde{u_{\mathcal{CH}_L}}^{\omega_-} \\ + \frac{1}{\sqrt{2\pi(r_n^2+a^2)}} \sum_{m\ell} \int_{\omega_- m - \frac{1}{am}}^{\omega_- m + \frac{1}{am}} e^{-2i(\omega-\omega_- m)r_n^*} e^{i\omega u_0} \mathcal{F}_{\mathcal{H}}[\psi_0^n] e^{im\phi_-^*} \\ \times \Big[S_{m\ell}(a\omega_- m,\cos\theta) \partial_\omega \left(\mathfrak{t}(\omega,m,\ell) \tilde{u_{\mathcal{CH}_L}}\right)(\tilde\xi) \\ + \mathfrak{t}(\omega,m,\ell) \tilde{u_{\mathcal{CH}_L}} \frac{S_{m\ell}(a\omega,\cos\theta) - S_{m\ell}(a\omega_- m,\cos\theta)}{\omega - \omega_- m} \Big] \mathrm{d}\omega \tag{3.8.107}$$

for some $\tilde\xi(\omega) \in (\omega_- m - \frac{1}{am}, \omega_- m + \frac{1}{am})$. We also use the notation

$$\tilde{u_{\mathcal{CH}_L}}^{\omega_-}(r^*) = \tilde{u_{\mathcal{CH}_L}}(\omega = \omega_- m, r^*). \tag{3.8.108}$$

Again, we consider both terms I_{res}^a and I_{res}^b individually and begin with term I_{res}^b.

Lemma 3.8.9. *We have* $\|I_{\text{res}}^b\|^2_{L^2(\mathbb{S}^2)}(r_n^*, u_0) \lesssim D$ *for all* $r_n^* \geq r_0^*$.

Proof. We decompose the term $I^b_{\text{res}} = I^{b1}_{\text{res}} + I^{b2}_{\text{res}}$ further into the two summands appearing in the ω-integral. We will estimate each of them individually. We begin with I^{b1}_{res} and estimate

$$\begin{aligned}
\|I^{b1}_{\text{res}}\|^2_{L^2(\mathbb{S}^2)} &\lesssim \sum_m \int_0^\pi \left| \sum_{\ell\geq|m|} \int_{\omega_-m-\frac{1}{am}}^{\omega_-m+\frac{1}{am}} |\mathcal{F}_{\mathcal{H}}[\psi_0^n] S_{m\ell}(a\omega_-m)\partial_\omega\left(\mathrm{t}u_{\tilde{\mathcal{CH}}_L}\right)(\tilde{\xi})|\mathrm{d}\omega\right|^2 \sin\theta\mathrm{d}\theta \\
&\lesssim \sum_m \left(\sum_{\ell\geq|m|}\int_0^\pi\int_{\mathbb{R}}(1+\Lambda^3_{m\ell})|\mathcal{F}_{\mathcal{H}}[\psi_0^n]|^2|S_{m\ell}(a\omega_-m)|^2\mathrm{d}\omega\sin\theta\mathrm{d}\theta\right) \\
&\quad\times\left(\frac{1}{m}\sum_{\ell\geq|m|}\sup_{|\tilde{\xi}-a\omega_-m|<\frac{1}{m}}\frac{|\partial_\omega(\mathrm{t}u_{\tilde{\mathcal{CH}}_L})(\tilde{\xi})|^2}{1+\Lambda^3_{m\ell}(\tilde{\xi})}\right) \\
&\lesssim \sum_m\left(\sum_{\ell\geq|m|}\int_{\mathbb{R}}(1+\Lambda^3_{m\ell})|\mathcal{F}_{\mathcal{H}}[\psi_0^n]|^2\mathrm{d}\omega\right)\left(\sum_{\ell\geq|m|}\frac{\Lambda^2_{m\ell}(a\omega_-m)\log^2(\Lambda_{m\ell}(a\omega_-m))}{1+\Lambda^3_{m\ell}(a\omega_-m)}\right)\lesssim D.
\end{aligned} \tag{3.8.109}$$

Here we have used Lemma 3.8.3, Lemma 3.8.4, Lemma 3.8.5 and the fact that

$$\Lambda_{\omega_-,m\ell} := \Lambda_{m\ell}(a\omega_-m) \sim \Lambda_{m\ell}(a\xi) \tag{3.8.110}$$

for all $|\xi-\omega_-m|<\frac{1}{m}$ which in turn is a consequence of Lemma 3.3.2.

We now control the second term I^{b2}_{res} and estimate

$$\begin{aligned}
\|I^{b2}_{\text{res}}\|^2_{L^2(\mathbb{S}^2)} &\lesssim \sum_m\int_0^\pi\left|\sum_{\ell\geq|m|}\int_{\omega_-m-\frac{1}{am}}^{\omega_-m+\frac{1}{am}}\left|\mathcal{F}_{\mathcal{H}}[\psi_0^n]\frac{S_{m\ell}(a\omega)-S_{m\ell}(a\omega_-m)}{\omega-\omega_-m}\mathrm{t}u_{\tilde{\mathcal{CH}}_L}\right|\mathrm{d}\omega\right|^2\sin\theta\mathrm{d}\theta \\
&\lesssim\sum_m\left(\sum_{\ell\geq|m|}\int_{\mathbb{R}}(1+\Lambda^3_{m\ell})|\mathcal{F}_{\mathcal{H}}[\psi_0^n]|^2\mathrm{d}\omega\right) \\
&\quad\times\left(\sum_{\ell\geq|m|}\int_{\omega_-m-\frac{1}{am}}^{\omega_-m+\frac{1}{am}}\frac{|\mathrm{t}u_{\tilde{\mathcal{CH}}_L}|^2}{1+\Lambda^3_{m\ell}}\int_0^\pi\left|\frac{S_{m\ell}(a\omega)-S_{m\ell}(a\omega_-m)}{\omega-\omega_-m}\right|^2\sin\theta\mathrm{d}\theta\mathrm{d}\omega\right) \\
&\lesssim\sum_m\left(\sum_{\ell\geq|m|}\int_{\mathbb{R}}(1+\Lambda^3_{m\ell})|\mathcal{F}_{\mathcal{H}}[\psi_0^n]|^2\mathrm{d}\omega\right) \\
&\quad\times\left(\sum_{\ell\geq|m|}\frac{\Lambda_{\omega_-,m\ell}}{1+\Lambda^3_{\omega_-,m\ell}}\int_{\omega_-m-\frac{1}{am}}^{\omega_-m+\frac{1}{am}}\sup_{|\tilde{\xi}-\omega_-m|\leq\frac{1}{m}}\int_0^\pi|\partial_\omega S_{m\ell}|^2(a\tilde{\xi})\sin\theta\mathrm{d}\theta\mathrm{d}\omega\right)\lesssim D,
\end{aligned} \tag{3.8.111}$$

where we have used the mean value property, Lemma 3.8.3, Lemma 3.8.4 and Proposition 3.3.3. $\square$

Now, we proceed with I^a_{res}, i.e. the first term in (3.8.107). We begin by recalling the definition of $\mathcal{F}_{\mathcal{H}}[\psi_0^n]$:

$$\mathcal{F}_{\mathcal{H}}[\psi_0^n] = \frac{\sqrt{r_+^2+a^2}}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{S}^2} e^{i\omega v}\psi_0^n(v,\theta,\tilde{\phi}_+)S_{m\ell}(a\omega,\cos\theta)e^{-im\tilde{\phi}_+}\mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2}. \tag{3.8.112}$$

Similar to Lemma 3.8.9 we will replace the $S_{m\ell}(a\omega)$ appearing in (3.8.112) with $S_{m\ell}(a\omega_-m)$. In order to do so, we introduce

$$\hat{I}^a_{\text{res}} := \frac{1}{\sqrt{2\pi(r_n^2+a^2)}}\sum_{m\ell}\text{p.v.}\int_{\omega_-m-\frac{1}{am}}^{\omega_-m+\frac{1}{am}}\frac{e^{-2i(\omega-\omega_-m)r_n^*}e^{i\omega u_0}\tilde{\mathcal{F}_{\mathcal{H}}}[\psi_0^n]}{\omega-\omega_-m}\mathrm{d}\omega$$
$$\times e^{im\phi_-^*}S_{m\ell}(am\omega_-,\cos\theta)\mathfrak{t}(\omega_-m,m,\ell)u_{\tilde{\mathcal{CH}_L}}{}^{\omega_-}, \tag{3.8.113}$$

$$\tilde{\mathcal{F}_{\mathcal{H}}}[\psi_0^n] := \frac{\sqrt{r_+^2+a^2}}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi_{0\,m\ell}^n(v)e^{i\omega v}\mathrm{d}v = \sqrt{r_+^2+a^2}\mathfrak{F}[\psi^n_{0\,m\ell}], \tag{3.8.114}$$

and[3]

$$\psi^n_{0\,m\ell}(v) := \int_{\mathbb{S}^2}\psi_0^n(v,\theta,\tilde{\phi}_+)S_{m\ell}(a\omega_-m,\cos\theta)e^{-im\tilde{\phi}_+}\mathrm{d}\sigma_{\mathbb{S}^2}. \tag{3.8.115}$$

Lemma 3.8.10.

$$\|\hat{I}^a_{\text{res}} - I^a_{\text{res}}\|^2_{L^2(\mathbb{S}^2)} \lesssim D, \tag{3.8.116}$$

Proof. Similarly to the proof of Lemma 3.8.9, we write

$$S_{m\ell}(a\omega) = S_{m\ell}(a\omega_-m) + (\omega-\omega_-m)\frac{S_{m\ell}(a\omega)-S_{m\ell}(a\omega_-m)}{\omega-\omega_-m}. \tag{3.8.117}$$

for frequencies $|\omega-\omega_-m| \leq \frac{1}{am}$ in (3.8.112). Then, using a Cauchy–Schwarz inequality on the sphere, $\sup_{|\xi-a\omega_-m|\leq\frac{1}{m}}|\partial_\xi\Lambda(\xi)| \lesssim |m|$, $\sup_{|\xi-a\omega_-m|\leq\frac{1}{m}}|\partial_\xi P(\xi)| \lesssim |m|$ (see (3.3.42)),

[3]Recall that $\mathfrak{F}$ denotes the standard Fourier transform $\mathfrak{F}[f](\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)e^{i\xi x}\mathrm{d}x$.

Proposition 3.3.3 as well as elliptic estimates, we control the error term as

$$(1+m^2\Lambda^2_{\omega_-,m\ell})\left|\frac{\mathcal{F}_{\mathcal{H}}-\tilde{\mathcal{F}}_{\mathcal{H}}}{\omega-\omega_- m}\right|^2 \lesssim m\Big[\int_{\mathbb{S}^2}\left|\int_{\mathbb{R}} e^{i\omega v}\psi_0^n(v,\theta,\tilde{\phi}_+)\mathrm{d}v\right|^2 \mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\mathbb{S}^2}\left|\int_{\mathbb{R}} e^{i\omega v}\nabla\!\!\!\!/^3\psi_0^n(v,\theta,\tilde{\phi}_+)\mathrm{d}v\right|^2 \mathrm{d}\sigma_{\mathbb{S}^2}\Big]. \tag{3.8.118}$$

Now, from Lemma 3.8.3 and Lemma 3.8.4 we conclude after an application of the Cauchy–Schwarz inequality and Plancherel's theorem that

$$\|\hat{I}^a_{\text{res}} - I^a_{\text{res}}\|^2_{L^2(\mathbb{S}^2)} \lesssim D. \tag{3.8.119}$$

□

Note that the function $\omega \mapsto \mathfrak{F}[\psi^n_{0\,m\ell}](\omega)$ is a $L^2(\mathbb{Z}_m\times\mathbb{Z}_{\ell\geq|m|})$-valued Schwartz function since $v\mapsto \psi^n_{0\,m\ell}(v)$ is a $L^2(\mathbb{Z}_m\times\mathbb{Z}_{\ell\geq|m|})$-valued Schwartz function. We also define

$$\tilde{I}^a_{\text{res}} := \frac{1}{\sqrt{2\pi(r_n^2+a^2)}}\sum_{m\ell}\text{p.v.}\int_{\mathbb{R}}\frac{e^{-2i(\omega-\omega_- m)r^*_n}e^{i\omega u_0}\tilde{\mathcal{F}}_{\mathcal{H}}[\psi_0^n]}{\omega-\omega_- m}\mathrm{d}\omega \times e^{im\phi^*_-}S_{m\ell}(am\omega_-,\cos\theta)\mathfrak{t}(\omega_- m,m,\ell)u_{\tilde{\mathcal{CH}_L}}{}^{\omega_-}. \tag{3.8.120}$$

Lemma 3.8.11. *We have $\|\hat{I}^a_{\text{res}} - \tilde{I}^a_{\text{res}}\|^2_{L^2(\mathbb{S}^2)} \lesssim D$ for all $r^*_n \geq r^*_0$.*

Proof. We use that the spheroidal harmonics $S_{m\ell}(am\omega_-,\cos\theta)e^{im\phi^*_-}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$ to estimate

$$\begin{aligned}\|\tilde{I}^a_{\text{res}} - \hat{I}^a_{\text{res}}\|^2_{L^2(\mathbb{S}^2)} &\lesssim \sum_{m\ell}|m|^2\left|\left(\int_{-\infty}^{\omega_- m-\frac{1}{am}} + \int_{\omega_- m+\frac{1}{am}}^{+\infty}\right)|\tilde{\mathcal{F}_{\mathcal{H}}}[\psi_0^n][\psi_0^n]|\mathrm{d}\omega|u_{\tilde{\mathcal{CH}_L}}{}^{\omega_-}\mathfrak{t}^{\omega_-}|\right|^2\\ &\lesssim \sum_{m\ell}|m|^2\Lambda_{m\ell}(a\omega_- m)\left|\int_{\mathbb{R}}|\tilde{\mathcal{F}_{\mathcal{H}}}[\psi_0^n]|\mathrm{d}\omega|\right|^2 \lesssim D,\end{aligned} \tag{3.8.121}$$

where we used the Cauchy–Schwarz inequality in the last step. □

Now, we turn to $\tilde{I}^a_{\text{res}}$ as defined in (3.8.120) and first only consider the ω-integral

$$\text{Int}^a_{\text{res}} := \frac{1}{\sqrt{2\pi(r_n^2+a^2)}}\text{p.v.}\int_{\mathbb{R}}\frac{e^{-2i(\omega-\omega_- m)r^*_n}e^{i\omega u_0}\tilde{\mathcal{F}_{\mathcal{H}}}[\psi_0^n]}{\omega-\omega_- m}\mathrm{d}\omega. \tag{3.8.122}$$

We have

$$\begin{aligned}
\mathrm{Int}^a_{\mathrm{res}} &= \frac{\sqrt{r_+^2+a^2}}{\sqrt{r_n^2+a^2}}\frac{1}{\sqrt{2\pi}}\mathrm{p.v.}\int_{\mathbb{R}}\frac{\mathfrak{F}[\psi^n_{0\,m\ell}(\cdot-u_0+2r_n^*)e^{i\omega_- m\cdot}]}{\omega}e^{2i\omega_- m r_n^*}\mathrm{d}\omega\\
&= \frac{\sqrt{r_+^2+a^2}}{\sqrt{r_n^2+a^2}}\frac{1}{\sqrt{2\pi}}e^{2i\omega_- m r_n^*}\mathrm{p.v.}\left(\frac{1}{\omega}\right)\left[\mathfrak{F}[\psi^n_{0\,m\ell}(\cdot-u_0+2r_n^*)e^{i\omega_- m\cdot}]\right]\\
&= \frac{\sqrt{r_+^2+a^2}}{\sqrt{r_n^2+a^2}}\frac{1}{\sqrt{2\pi}}e^{2i\omega_- m r_n^*}i\pi\,\mathrm{sgn}\Big[\psi^n_{0\,m\ell}(\cdot-u_0+2r_n^*)e^{i\omega_- m\cdot}\Big],
\end{aligned} \tag{3.8.123}$$

where sgn has to be understood as a Schwartz distribution. We have used that $\mathfrak{F}\left[\mathrm{p.v.}\left(\frac{1}{\omega}\right)\right] = i\pi\mathrm{sgn}$ in the sense of distributions. Now, since ψ_0 is smooth, the function $v\mapsto \psi^n_{0\,m\ell}$ is a Schwartz function with values in the space of superpolynomially decaying sequences in m,ℓ as a subspace of $L^2(\mathbb{Z}_m\times\mathbb{Z}_{\ell\geq|m|})$. Particularly, this implies that

$$\mathfrak{t}^{\omega_-}u_{\tilde{\mathcal{CH}}_L}{}^{\omega_-}\mathrm{Int}^a_{\mathrm{res}}\in L^2(\mathbb{Z}_m\times\mathbb{Z}_{\ell\geq|m|};L^\infty(r_0^*,\infty)), \tag{3.8.124}$$

so we can project $\tilde{I}^a_{\mathrm{res}}$ on $e^{im\phi^*_-}S_{m\ell}(am\omega_-,\cos\theta)$. Indeed, this yields

$$\begin{aligned}
&\langle e^{im\tilde{\phi}^*_-}S_{m\ell}(am\omega_-,\cos\theta),\tilde{I}^a_{\mathrm{res}}\rangle_{L^2(\mathbb{S}^2)}\\
&\quad= \frac{\sqrt{r_+^2+a^2}}{\sqrt{r_n^2+a^2}}\frac{\mathfrak{t}^{\omega_-}\,u_{\tilde{\mathcal{CH}}_L}{}^{\omega_-}}{\sqrt{2\pi}}e^{2i\omega_- m r_n^*}i\pi\,\mathrm{sgn}\Big[\psi^n_{0\,m\ell}(\cdot-u_0+2r_n^*)e^{i\omega_- m\cdot}\Big].
\end{aligned} \tag{3.8.125}$$

To summarize, we have decomposed I as

$$I = I_{\mathrm{res}}+I_{\text{non-res}} = I_{\mathrm{res}}+I_{\text{non-res}} = \tilde{I}^a_{\mathrm{res}}+(I^a_{\mathrm{res}}-\hat{I}^a_{\mathrm{res}})+(\hat{I}^a_{\mathrm{res}}-\tilde{I}^a_{\mathrm{res}})+I^b_{\mathrm{res}}+I_{\text{non-res}}, \tag{3.8.126}$$

where $\tilde{I}^a_{\mathrm{res}}$ satisfies (3.8.125) and

$$\|(I^a_{\mathrm{res}}-\hat{I}^a_{\mathrm{res}})+(\hat{I}^a_{\mathrm{res}}-\tilde{I}^a_{\mathrm{res}})+I^b_{\mathrm{res}}+I_{\text{non-res}}\|_{L^2(\mathbb{S}^2)}\lesssim D^{\frac{1}{2}}. \tag{3.8.127}$$

Completely analogous to the the analysis before, we also decompose II as

$$\begin{aligned}
II = II_{\mathrm{res}}+II_{\text{non-res}} &= II_{\mathrm{res}}+II_{\text{non-res}}\\
&= \tilde{II}^a_{\mathrm{res}}+(II^a_{\mathrm{res}}-\hat{II}^a_{\mathrm{res}})+(\hat{II}^a_{\mathrm{res}}-\tilde{II}^a_{\mathrm{res}})+II^b_{\mathrm{res}}+II_{\text{non-res}},
\end{aligned} \tag{3.8.128}$$

where

$$\|(II^a_{\text{res}} - \hat{II}^a_{\text{res}}) + (\hat{II}^a_{\text{res}} - \tilde{II}^a_{\text{res}}) + II^b_{\text{res}} + II_{\text{non-res}}\|_{L^2(\mathbb{S}^2)} \lesssim D^{\frac{1}{2}} \tag{3.8.129}$$

and $\tilde{II}^a_{\text{res}}$ satisfies

$$\langle e^{im\phi^*_-} S_{m\ell}(am\omega_-, \cos\theta), \tilde{II}^a_{\text{res}} \rangle_{L^2(\mathbb{S}^2)} = \frac{\sqrt{r_+^2 + a^2}}{\sqrt{r_n^2 + a^2}} \frac{\mathfrak{r}^{\omega_-} \tilde{u_{\mathcal{C}\mathcal{H}_R}}^{\omega_-}}{\sqrt{2\pi}} i\pi \operatorname{sgn}\Big[\psi^n_{0\,m\ell}(\cdot - u_0) e^{i\omega_- m\cdot}\Big]. \tag{3.8.130}$$

Hence, using

$$\mathfrak{r}^{\omega_-} = -\mathfrak{t}^{\omega_-} \text{ and } \tilde{u_{\mathcal{C}\mathcal{H}_R}}^{\omega_-} = \tilde{u_{\mathcal{C}\mathcal{H}_L}}^{\omega_-}, \tag{3.8.131}$$

we obtain

$$\begin{aligned} &\langle e^{im\phi^*_-} S_{m\ell}(am\omega_-, \cos\theta), \tilde{I}^a_{\text{res}} + \tilde{II}^a_{\text{res}} \rangle_{L^2(\mathbb{S}^2)} \\ &\qquad = i\pi \frac{\sqrt{r_+^2 + a^2}}{\sqrt{r_n^2 + a^2}} \frac{\mathfrak{t}^{\omega_-} \tilde{u_{\mathcal{C}\mathcal{H}_L}}^{\omega_-}}{\sqrt{2\pi}} e^{i\omega_- m u_0} \int_{-u_0}^{2r_n^* - u_0} \psi^n_{0\,m\ell}(v) e^{i\omega_- m v} \mathrm{d}v. \end{aligned} \tag{3.8.132}$$

Now, by construction of ψ^n_0, we have that $\psi^n_{0\,m\ell}(v) = 0$ for $v \geq 2r_n^* - u_0 + \tilde{c}$, where $\tilde{c}$ is a constant only depending on the black hole parameters. In particular, this implies that

$$\sum_{m\ell} \left| i\pi \frac{\sqrt{r_+^2 + a^2}}{\sqrt{r_n^2 + a^2}} \frac{\mathfrak{t}^{\omega_-} \tilde{u_{\mathcal{C}\mathcal{H}_L}}^{\omega_-}}{\sqrt{2\pi}} e^{i\omega_- m u_0} \left(\int_{-\infty}^{-u_0} + \int_{2r_n^* - u_0}^{+\infty} \right) \psi^n_{0\,m\ell}(v) e^{i\omega_- m v} \mathrm{d}v \right|^2 \lesssim_{u_0} D \tag{3.8.133}$$

which allows us to—up to a term bounded by $D^{\frac{1}{2}}$—replace the integral in (3.8.132) with an integral on the whole real line $v \in \mathbb{R}$. Finally, from Proposition 3.7.1 (more precisely (3.7.42)), we obtain

$$\begin{aligned} \|\psi\|^2_{L^2(\mathbb{S}^2)}(u_0, r_n^*) &= \sum_{m\ell} \left| \pi \frac{\sqrt{r_+^2 + a^2}}{\sqrt{r_n^2 + a^2}} \frac{\mathfrak{t}^{\omega_-} \tilde{u_{\mathcal{C}\mathcal{H}_L}}^{\omega_-}}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi^n_{0\,m\ell}(v) e^{i\omega_- m v} \mathrm{d}v \right|^2 + \mathrm{Err}(D) \\ &= \sum_{m\ell} \left| \pi \frac{\mathfrak{t}^{\omega_-} \tilde{u_{\mathcal{C}\mathcal{H}_L}}^{\omega_-}}{\sqrt{r_n^2 + a^2}} a^{R_n}_{\mathcal{H}}(\omega = \omega_- m) \right|^2 + \mathrm{Err}(D), \end{aligned} \tag{3.8.134}$$

where $|\mathrm{Err}(D)| \lesssim_{u_0} D$ uniformly for all $r_n^* \geq r_0^*$. This established the claim (3.8.100) at the beginning of the proof.

Now, from Lemma 3.7.3 we have that $a_{\mathcal{H}}^{R_n} \to a_{\mathcal{H}}$ pointwise for fixed ω, m, ℓ as $R_n \to \infty$. We also have the pointwise limit

$$u_{\tilde{\mathcal{CH}_L}}{}^{\omega_-} \to 1 \text{ as } r_n^* \to \infty. \tag{3.8.135}$$

Hence, applying Fatou's lemma yields

$$\liminf_{r_n^* \to \infty} \|\psi\|_{L^2(\mathbb{S}^2)}^2(u_0, r_n^*) \geq \frac{\pi^2}{r_-^2 + a^2} \sum_{m\ell} |\mathfrak{t}^{\omega_-}|^2 |a_{\mathcal{H}}(\omega = \omega_- m)|^2 - C_{u_0} D, \tag{3.8.136}$$

where $C_{u_0} > 0$ is a constant depending on u_0. Since

$$|\mathfrak{t}^{\omega_-}| \gtrsim |m| \text{ and } |a_{\mathcal{H}}(\omega = \omega_- m_i, m_i, \ell)| \gtrsim e^{m_i^{\frac{1}{4}}} \tag{3.8.137}$$

for infinitely many m_i as shown in Lemma 3.8.6 and Lemma 3.6.2, respectively, we obtain

$$\lim_{r_n^* \to \infty} \|\psi\|_{L^2(\mathbb{S}^2)}^2(u_0, r_n^*) = +\infty. \tag{3.8.138}$$

Since the sequence $r_n^* \to \infty$ was arbitrary we conclude (3.8.98). $\square$

3.9 Appendix

3.9.1 Airy functions

We recall the definition of the Airy functions of first and second kind Ai and Bi as follows.

Definition 3.9.1. *For $x \in \mathbb{R}$, we define* $\mathrm{Ai}(x)$ *and* $\mathrm{Bi}(x)$ *via the improper Riemann integrals*

$$\mathrm{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t, \tag{3.9.1}$$

$$\mathrm{Bi}(x) := \frac{1}{\pi} \int_0^\infty \left[\exp\left(-\frac{t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right]. \tag{3.9.2}$$

Equivalently, the Airy functions are the unique solutions of

$$u'' = xu \tag{3.9.3}$$

with

$$\mathrm{Ai}(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}, \mathrm{Ai}'(0) = \frac{-1}{3^{\frac{1}{2}}\Gamma(\frac{1}{3})}, \tag{3.9.4}$$

$$\mathrm{Bi}(0) = \frac{1}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})}, \mathrm{Bi}'(0) = \frac{3^{\frac{1}{6}}}{\Gamma(\frac{1}{3})} \tag{3.9.5}$$

such that $\mathfrak{W}_x(\mathrm{Ai}(x), \mathrm{Bi}(x)) = \frac{1}{\pi}$. Further, we define the constant c as the largest negative root of $\mathrm{Ai}(x) = \mathrm{Bi}(x)$. Then, we introduce the error-control functions

$$E_{\mathrm{Ai}}(x) := \begin{cases} (\mathrm{Bi}(x)/\mathrm{Ai}(x))^{\frac{1}{2}} & x \geq c \\ 1 & x \leq c \end{cases} \text{ and } M_{\mathrm{Ai}}(x) := \begin{cases} (2\mathrm{Ai}(x)\mathrm{Bi}(x))^{\frac{1}{2}} & x \geq c \\ (\mathrm{Ai}^2(x) + \mathrm{Bi}^2(x))^{\frac{1}{2}} & x \leq c \end{cases} \tag{3.9.6}$$

and $E_{\mathrm{Ai}}^{-1}(x) := \frac{1}{E_{\mathrm{Ai}}(x)}$. From [119, Chapter 11, §2] we remark that E_{Ai} is a monotonically increasing function of x which is never less than 1 and moreover,

$$|\mathrm{Ai}(x)| \leq \frac{M_{\mathrm{Ai}}(x)}{E_{\mathrm{Ai}}(x)} \text{ as well as } |\mathrm{Bi}(x)| \leq M_{\mathrm{Ai}}(x)E_{\mathrm{Ai}}(x). \tag{3.9.7}$$

The Airy functions obey the following asymptotics.

Lemma 3.9.1 ([119, Chapter 11, §1], [39, §9.7])**.** *For large $x > 0$, the asymptotic behaviors of the Airy functions are*

$$\mathrm{Ai}(-x) = \frac{1}{\sqrt{\pi}x^{\frac{1}{4}}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right) + \tilde{\epsilon}_{\mathrm{Ai}}(x), \quad \mathrm{Ai}'(-x) = \frac{x^{\frac{1}{4}}}{\sqrt{\pi}} \sin\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\pi\right) + \tilde{\epsilon}_{\mathrm{Ai}'}(x), \tag{3.9.8}$$

$$\mathrm{Bi}(-x) = \frac{-1}{\sqrt{\pi}x^{\frac{1}{4}}} \sin\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right) + \tilde{\epsilon}_{\mathrm{Ai}}(x), \quad \mathrm{Ai}'(-x) = \frac{x^{\frac{1}{4}}}{\sqrt{\pi}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\pi\right) + \tilde{\epsilon}_{\mathrm{Ai}'}(x), \tag{3.9.9}$$

where $|\tilde{\epsilon}_{\mathrm{Ai}}| \lesssim x^{-\frac{7}{4}}$ and $|\tilde{\epsilon}_{\mathrm{Ai}'}| \lesssim x^{-\frac{5}{4}}$. In particular, we have

$$|\mathrm{Ai}(-x)|, |\mathrm{Bi}(-x)| \lesssim \frac{1}{1 + x^{\frac{1}{4}}} \text{ and } |\mathrm{Ai}'(-x)|, |\mathrm{Bi}'(-x)| \lesssim 1 + x^{\frac{1}{4}} \tag{3.9.10}$$

for $x \geq 0$. Moreover, for $x > 0$ we have

$$\mathrm{Ai}(x) \leq \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}, \tag{3.9.11}$$

$$|\mathrm{Ai}'(x)| \leq \frac{x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}}\left(1+\frac{7}{48x^{\frac{3}{2}}}\right), \tag{3.9.12}$$

$$\mathrm{Bi}(x) \leq \frac{e^{\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{\pi}x^{\frac{1}{4}}}\left(1+\left(\chi_{\mathrm{Ai}}\left(\frac{7}{6}\right)+1\right)\frac{5}{48x^{\frac{3}{2}}}\right), \tag{3.9.13}$$

$$\mathrm{Bi}'(x) \leq \frac{x^{\frac{1}{4}}e^{\frac{2}{3}x^{\frac{3}{2}}}}{\sqrt{\pi}}\left(1+\left(\frac{\pi}{2}+1\right)\frac{7}{48x^{\frac{3}{2}}}\right), \tag{3.9.14}$$

where $\chi_{\mathrm{Ai}}(x) = \sqrt{\pi}\frac{\Gamma(\frac{1}{2}x+1)}{\Gamma(\frac{1}{2}x+\frac{1}{2})}$.

3.9.2 Parabolic cylinder functions

We define the parabolic cylinder functions U and $\bar{U}$ in the following.

Definition 3.9.2. *For $b \leq 0$ and $x \geq 0$ we recall the definition of the parabolic cylinder functions*

$$\begin{aligned} U(b,x) =& \frac{\pi^{\frac{1}{2}}e^{-\frac{1}{4}(2b+1)}e^{-\frac{1}{4}x^2}}{\Gamma(\frac{3}{4}+\frac{1}{2}b)}{}_1F_1\left(\frac{1}{2}b+\frac{1}{4};\frac{1}{2};\frac{1}{2}x^2\right) \\ &- \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b-1)}}{\Gamma(\frac{1}{4}+\frac{1}{2}b)}e^{-\frac{1}{4}x^2}x{}_1F_1\left(\frac{1}{2}b+\frac{3}{4};\frac{3}{2};\frac{1}{2}x^2\right), \end{aligned} \tag{3.9.15}$$

$$\begin{aligned} \bar{U}(b,x) =& \pi^{-\frac{1}{2}}2^{-\frac{1}{4}(2b+1)}\Gamma\left(\frac{1}{4}-\frac{1}{2}b\right)\sin\left(\frac{3}{4}\pi-\frac{1}{2}b\pi\right)e^{-\frac{1}{4}x^2}{}_1F_1\left(\frac{1}{2}b+\frac{1}{4};\frac{1}{2};\frac{1}{2}x^2\right) \\ &- \pi^{-\frac{1}{2}}2^{-\frac{1}{4}(2b-1)}\Gamma\left(\frac{3}{4}-\frac{1}{2}b\right)\sin\left(\frac{5}{4}\pi-\frac{1}{2}b\pi\right)e^{-\frac{1}{4}x^2}x{}_1F_1\left(\frac{1}{2}b+\frac{3}{4};\frac{3}{2};\frac{1}{2}x^2\right), \end{aligned} \tag{3.9.16}$$

where ${}_1F_1(a;b;z) := \sum_{n=0}^{\infty}\frac{a^{(n)}z^n}{b^{(n)}n!}$ denotes the confluent hypergeometric function. Here, we use the notation $a^{(n)} := x(x+1)(x+2)\cdots(x+n)$ for the rising factorial.

Remark that $\mathfrak{W}(U,\bar{U}) = \sqrt{\frac{2}{\pi}}\Gamma(\frac{1}{2}-b)$.

We define auxiliary functions to control error terms in terms of parabolic cylinder functions. We first define $\rho(b)$ as the largest real root of the equation $\bar{U}(b,x) = U(b,x)$. Note that $\rho(b) \geq 0$ for $b \leq 0$.

Definition 3.9.3. *For $b \leq 0$, we set*

$$E_U(b,x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \rho(b) \\ \sqrt{\frac{\bar{U}(b,x)}{U(b,x)}} & \text{for } x \geq \rho(b) \end{cases} \tag{3.9.17}$$

For fixed b, the function $E_U(b,x)$ is continuous and non-decreasing in $0 \leq x < \infty$. We denote $E_U^{-1} := \frac{1}{E_U}$.

Definition 3.9.4. *For $b \leq 0$, $x \geq 0$, we also define functions M_U and N_U by*

$$M_U(b,x) := \begin{cases} \sqrt{U^2 + \bar{U}^2} & \text{for } 0 \leq x \leq \rho(b) \\ \sqrt{2U\bar{U}} & \text{for } \rho(b) \leq x. \end{cases} \tag{3.9.18}$$

$$N_U(b,x) := \begin{cases} \sqrt{U'^2 + \bar{U}'^2} & \text{for } 0 \leq x \leq \rho(b) \\ \sqrt{\frac{U'^2\bar{U}^2 + \bar{U}'^2U^2}{U\bar{U}}} & \text{for } \rho(b) \leq x. \end{cases} \tag{3.9.19}$$

Definition 3.9.5. *We define the function ζ_U as*

$$\zeta_U(t) := \begin{cases} -\left(\frac{3}{2}\int_t^1 (1-\tau^2)^{\frac{1}{2}} \mathrm{d}\tau\right)^{\frac{2}{3}} & \textit{for } 0 \leq t \leq 1, \\ \left(\frac{3}{2}\int_1^t (\tau^2-1)^{\frac{1}{2}} \mathrm{d}\tau\right)^{\frac{2}{3}} & \textit{for } t \geq 1. \end{cases} \tag{3.9.20}$$

Note that we manifestly have

$$|U| \leq E^{-1}M_U, |\bar{U}| \leq EM_U \text{ and } |U\bar{U}| \leq M_U^2 \tag{3.9.21}$$

for $x \geq 0$ and $b \leq 0$.

Proposition 3.9.1. *The envelope function M_U satisfies*

$$M_U^2\left(-\frac{1}{2}\mu^2, \mu y\sqrt{2}\right) \lesssim \frac{1}{\mu^{\frac{1}{3}}} \frac{1}{1+|\zeta_U(y)|^{\frac{1}{4}}} \frac{1}{1+\mu^{\frac{2}{3}}|\zeta_U(y)|^{\frac{1}{2}}} \Gamma\left(\frac{1}{2}+\frac{1}{2}\mu^2\right) \tag{3.9.22}$$

uniformly in $\mu \geq 1$ and $y \geq 0$ and

$$M_U^2\left(-\frac{1}{2}\mu^2, \mu y\sqrt{2}\right) \lesssim \frac{1}{1+\sqrt{\mu y}} \tag{3.9.23}$$

uniformly in $0 \leq \mu \leq 1$ and $y \geq 0$. In particular, M_U satisfies

$$\left|M_U\left(-\frac{1}{2}\mu^2, \mu y\sqrt{2}\right)\right|^2 \lesssim \Gamma\left(\frac{1}{2}+\frac{1}{2}\mu^2\right). \tag{3.9.24}$$

Proof. These estimates follow from [118, Equation (5.23), (6.12) and Section 6.2]. □

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